APPENDIX: CHAOS AND TRANSITIONS

Previous research showed how to use elements of mathematical chaos theory to suggest points of irreversibility in a semidesert soil science setting (Arlinghaus, Nystuen, and Woldenberg, 1992 a and b). The same sort of general strategy might work when coupled with transitions. This possibility is explored below, using Feigenbaum's graphical analysis.

Feigenbaum's graphical analysis--geometric feedback.

Feigenbaum's graphical analysis employs the line y=x as an alternate axis from which to insert inputs into a mathematical function. In Figure A.1, the function y=4x(1-x) is graphed. Choose the input value x=0.2 and find the y output that corresponds to it. Graphically, the y-output is the height of the vertical line from x=0.2 to the curve. Use this output value as the next input or x-value. To do so, slide horizontally over to the line y=x--so that y becomes x, or output becomes input. Now move vertically from this location to the curve. Continue this process indefinitely to produce an easy-to-understand picture of how (possibly) complex geometric feedback operates within a system (Feigenbaum, 1980; Hofstadter, 1981; Gleick, 1987; Devaney, 1989). The feedback is traced out as a pattern of directed lines bouncing back and forth between points on the curve and points on y=x, much as singing might bounce back and forth from the crowd to the microphone in an auditorium. The pattern of lines is referred to as the "orbit" of the "seed" value (in this case, x=0.2). Different seed values give rise to different orbits.

Consider the seed value x=0.4 in the graphical analysis of Figure A.1. Figure A.2 shows the feedback pattern created using this seed value. A dictionary defines chaos as "a state of utter confusion" (Webster's Seventh Collegiate). Hence, it is a state which defies the prediction of pattern. The decimal expressions of numbers such as pi defy prediction of pattern; they exhibit a sort of arithmetic chaos. The feedback pattern in Figure A.2 is chaotic; Figure A.3 shows the detail of the pattern. Any seed value strictly between 0 and 1 will produce geometric chaos with respect to this parabola. The seed value of x=0 is a fixed point; it is the first intersection of y=x with the curve--there is no room to bounce back and forth between curve and line.

When other curves are used, different geometric dynamics of feedback arise. Indeed, because it is possible to examine the pattern of feedback purely graphically, it is not even necessary to know the equation of a given curve in order to consider the geometric dynamics associated with it. It is better to know the equation, of course, for then one can interpolate or extrapolate; however, in situations based on accumulating real-world data, there will likely be no curve that fits them. One can of course choose to fit such data with a prescribed curve, either exactly (using cubic spline interpolation, for example) or approximately (using a logistic curve, for example), but then forecasts are limited by this initial decision. Graphical analysis offers a systematic and replicable strategy for looking at geometric dynamics that is not necessarily biased by initial curve-fitting decisions.

Consider the S-shaped curve in Figure A.4. There are three intersection points of the line $y=x$ with the curve. Their x-coordinates partition the x-axis into four separate intervals. Choose a seed value in the interval labelled II. Its orbit follows a descending staircase that leads eventually to the first intersection point of $y=x$ with the curve. The orbit of any seed value in interval II has this property. The first intersection of $y=x$ with the curve is thus an "attractive" fixed point of the geometric configuration. The second intersection point of $y=x$ with the curve is a "repelling" fixed point of the geometric configuration--geometric process is driven away from it. Choose a seed value in interval III; its orbit climbs an ascending staircase toward the third intersection of $y=x$ with the curve. Again, the second intersection repels geometric process; the third intersection is an attracting fixed point. Any seed value in interval III has an orbit that eventually settles down to the third intersection of $y=x$ with the curve.

Because there is symmetry around the repelling fixed point--it functions as a repelling fixed point for orbits of seed values in both of the intervals for which it is a boundary point--it might serve as a threshold of irreversibility. To the left of the repeller (which may or may not be an inflection point, as well) the geometric dynamics are driven one way and to the right they are driven another. If the dynamic to the left of the repeller is regarded as "desirable" as it might be in issues that involve population and environment interaction, in which some increase is tolerable so long as it doesn't go beyond the threshold of irreversibility, then one would hope to be able to implement policies that would produce data sets that would follow curves, which if S-shaped, would have their repelling fixed point farther to the right than had been customary. For then, interval II, of seed values leading to desirable dynamics, would be enlarged. Indeed, alteration of policy to force the repeller--as a threshold of irreversibility--of a geometric dynamic in a favorable direction might help to guide decisions about a population-environment dynamic.

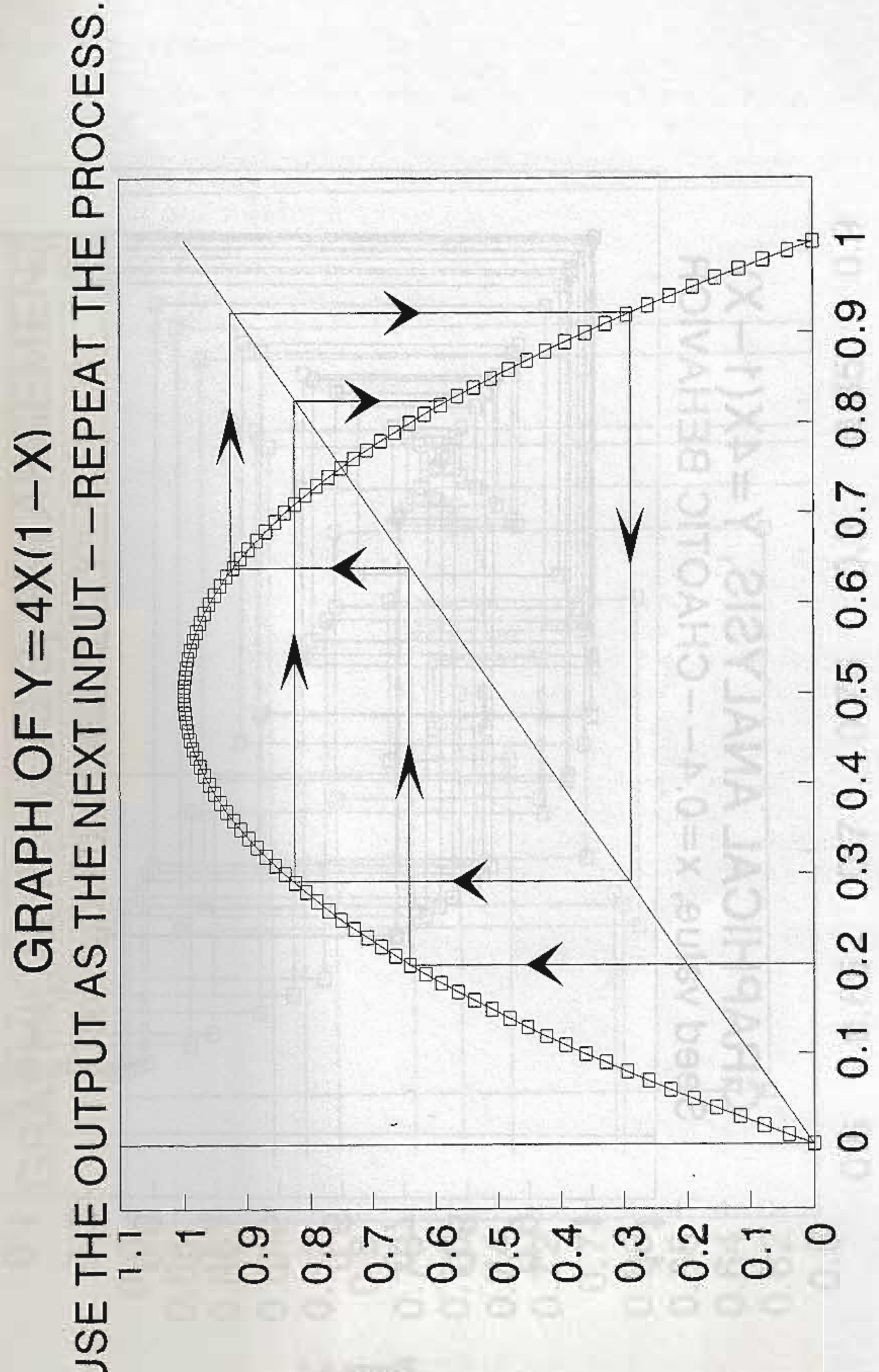
GRAPH OF Y=4X(1 – X)
USE THE OUTPUT AS THE NEXT INPUT – – REPEAT THE PROCESS.
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

Figure A.1

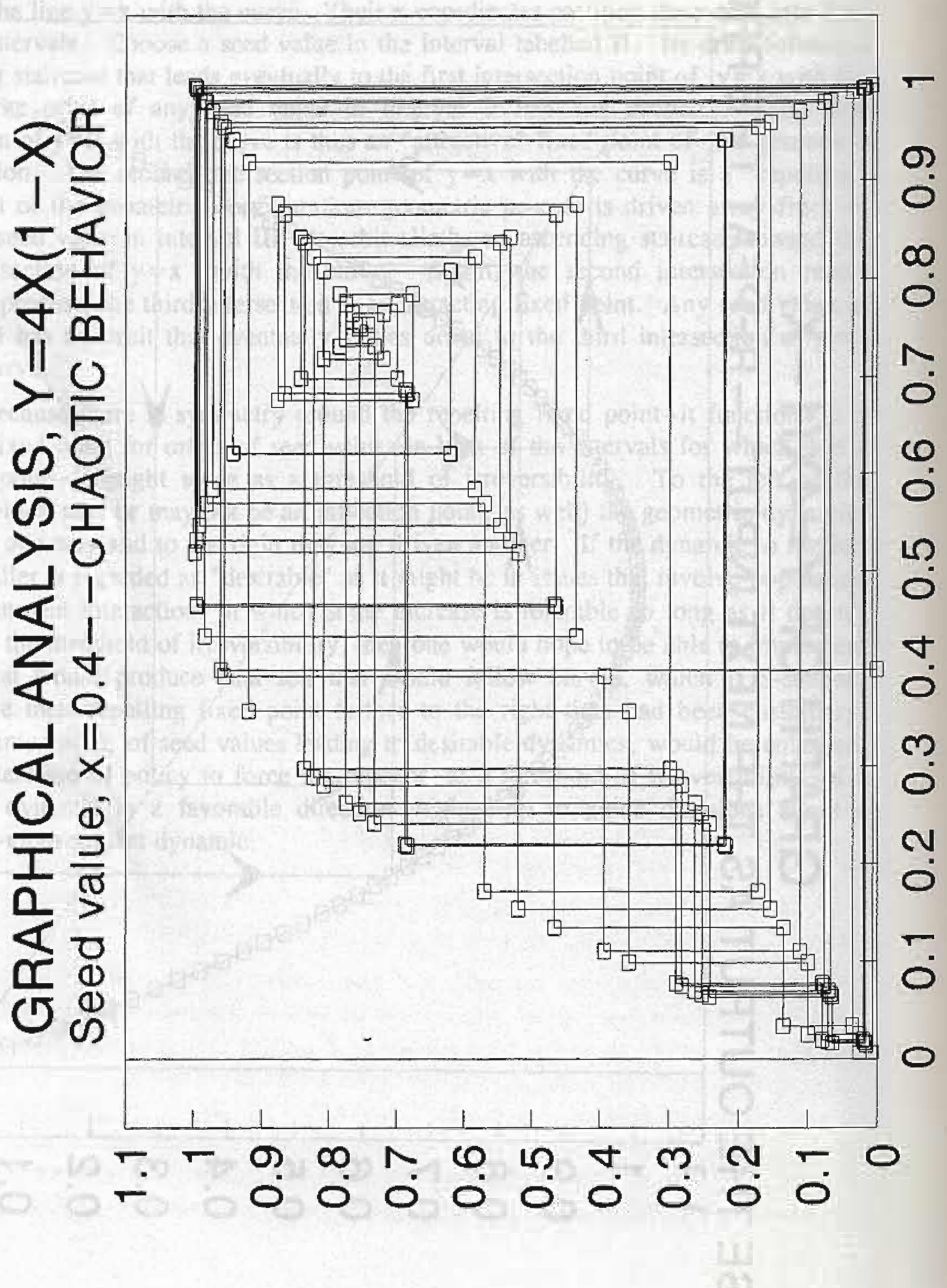
GRAPHICAL ANALYSIS, Y=4X(1 – X)
Seed value, x=0.4 -- CHAOTIC BEHAVIOR
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1

Figure A.2

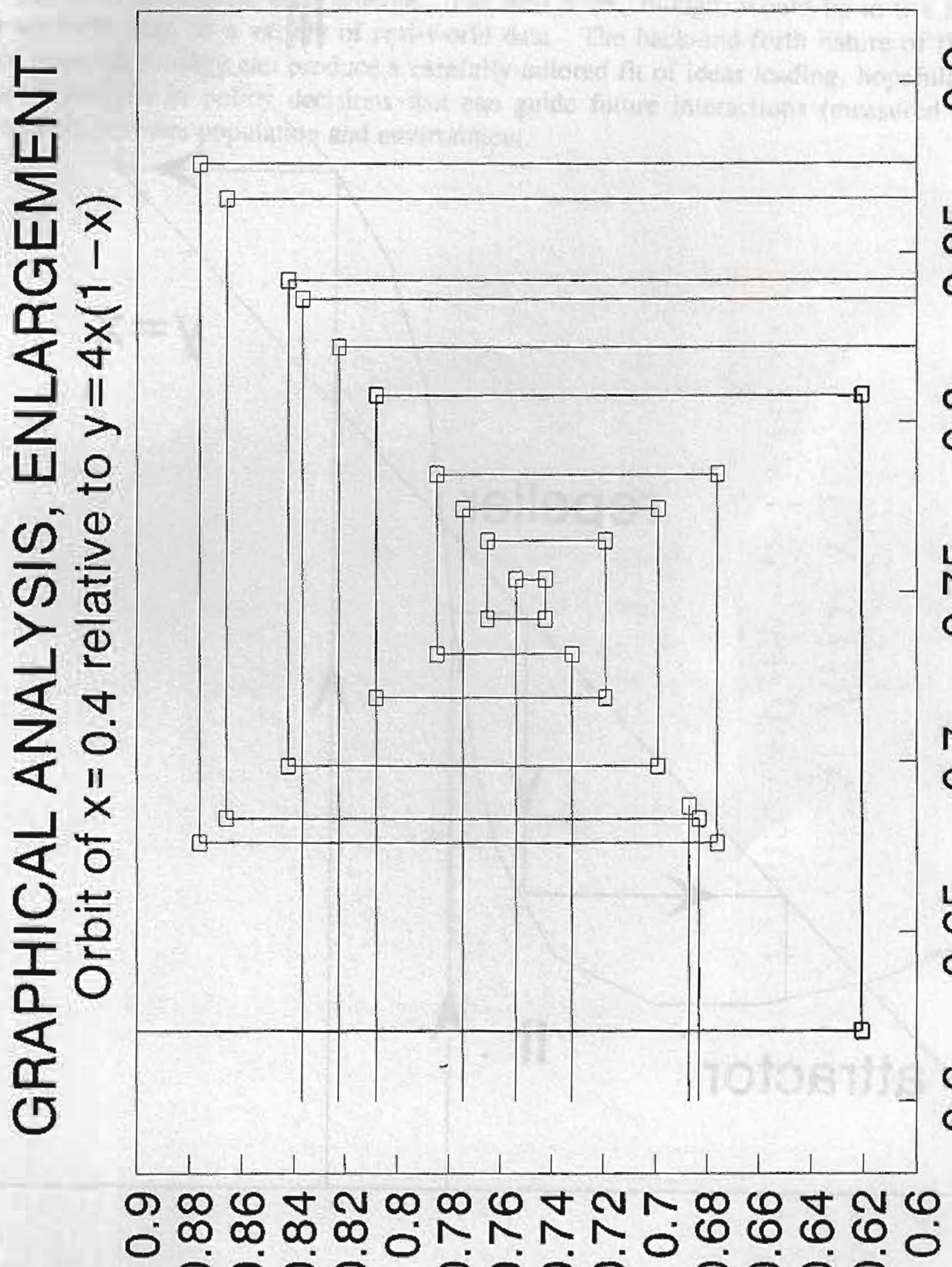
GRAPHICAL ANALYSIS, ENLARGEMENT
Orbit of x = 0.4 relative to y = 4x(1 − x)
0.9
0.88
0.86
0.84
0.82
0.8
0.78
0.76
0.74
0.72
0.7
0.68
0.66
0.64
0.62
0.6
0.6
0.65
0.7
0.75
0.8
0.85
0.9

Figure A.3

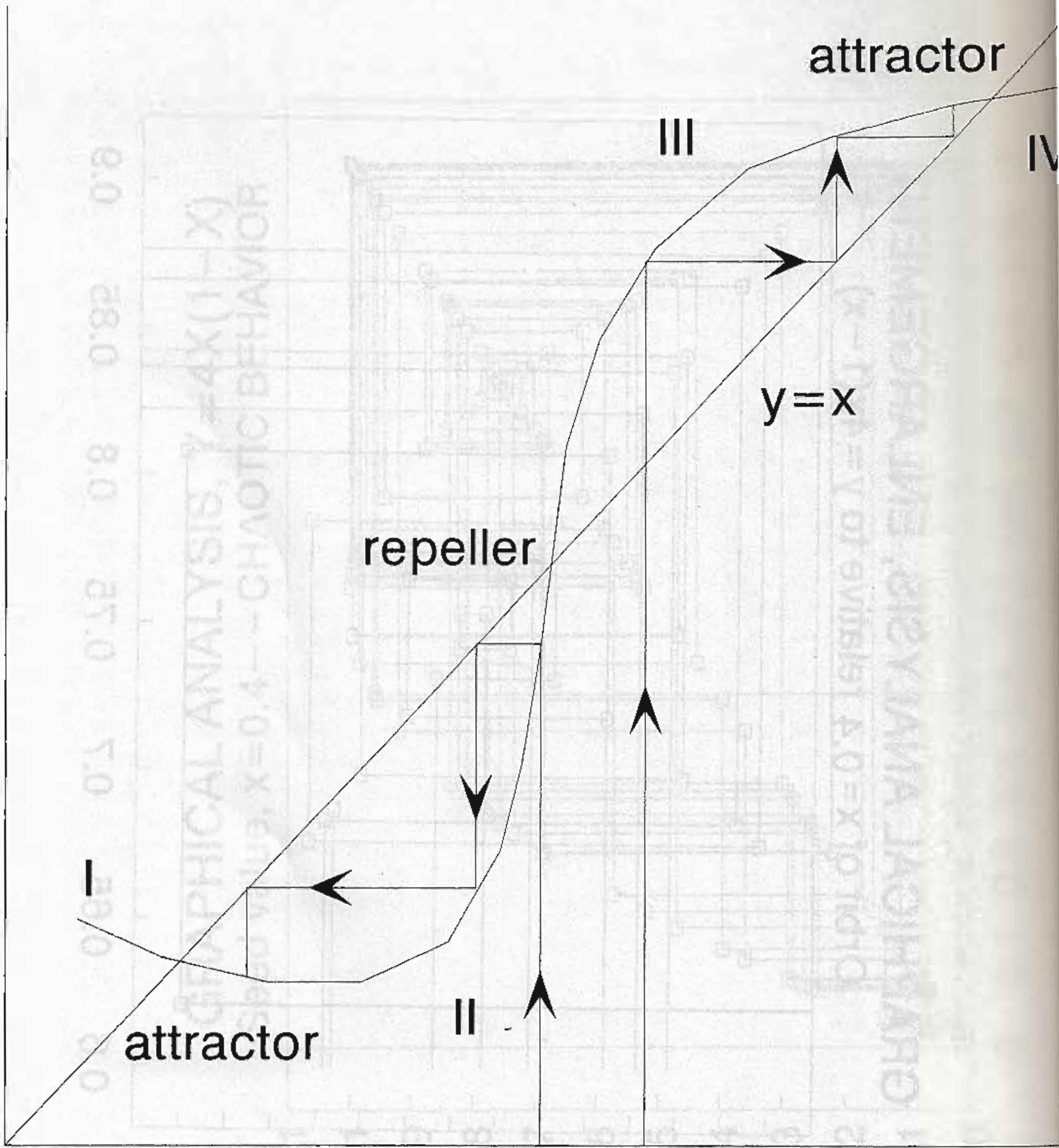

Figure A.4

Directions for further research.

There are a number of conceptual issues that remain--what types of curves give what patterns of irreversibility? What is the relation of the inflection point to irreversibility? When repellers and inflection points are linked, it may be that even more powerful conceptual tools emerge. The next stage, though, would be to use the ideas set forth here on a variety of real-world data. The back-and-forth nature of this sort of research strategy can produce a carefully-tailored fit of ideas leading, hopefully, to useful insights in policy decisions that can guide future interactions (measured as data output) between population and environment.

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