PARALLELS BETWEEN PARALLELS

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"I have a little shadow that goes in and out with me,
And what can be the use of him is more than I can see."
Robert Louis Stevenson
"My Shadow" in *A Child's Garden of Verses*

Abstract:

The earth's sun introduces a symmetry in the perception of its trajectory in the sky that naturally partitions the earth's surface into zones of affine and hyperbolic geometry. The affine zones, with single geometric parallels, are located north and south of the geographic tropical parallels. The hyperbolic zone, with multiple geometric parallels, is located between the geographic tropical parallels. Evidence of this geometric partition is suggested in the geographic environment—in the design of houses and of gameboards.

1. Introduction.

Subtle influences shape our perceptions of the world. The breadth of a world-view is a function not only of "real"-world experience, but also of the "abstract"-world context within which that experience can be structured. As William Kingdon Clifford asked in his Postulates of the Science of Space [3], how can one recognize flatness when magnification of the landscape merely reveals new wrinkles to traverse?

Geometry is a "source of form" not only in mathematics [10], but also in the "real" world [2]. Street patterns are geometric; architectural designs are geometric; and, diffusion patterns are geometric. In this study, the geometric notion of parallelism is examined in relation to the manner in which the sun's trajectory in the earth's sky is observed by inhabitants at various latitudinal positions: from north and south of the tropics to between the tropical parallels of latitude. A fundamental geometrical notion is thus aligned with fundamental geographical and astronomical relationships; this alignment is interpreted in cultural contexts ranging from the design of rooflines to the design of board games.

2. Basic Geometric Background.

To understand how geometry might guide the perception of form, it is therefore important to understand what "geometry" might be. Projective geometry is totally symmetric and possesses a completely "dual" vocabulary: "points" and "lines," "collinear" and "concurrent," and a host of others, are interchangeable terms [6]. Indeed, a Principle of Duality serves as a linguistic axis, or mirror, halving the difficulty of proving theorems. Thus, because "two points determine a line" is true, it follows, dually, that "two lines determine a point" is also true. The corresponding situation does not hold in the Euclidean plane: two lines do not necessarily determine a point because parallel lines do not determine a point [6].

Coxeter classifies other geometries as specializations of projective geometry based on the notion of parallelism, depending on whether a geometry admits zero, one, or more than one lines parallel to a given line, through a point not on the given line [6]. In the "elliptic" geometry of Riemann, there are no parallel lines, much as there are none in the geometry

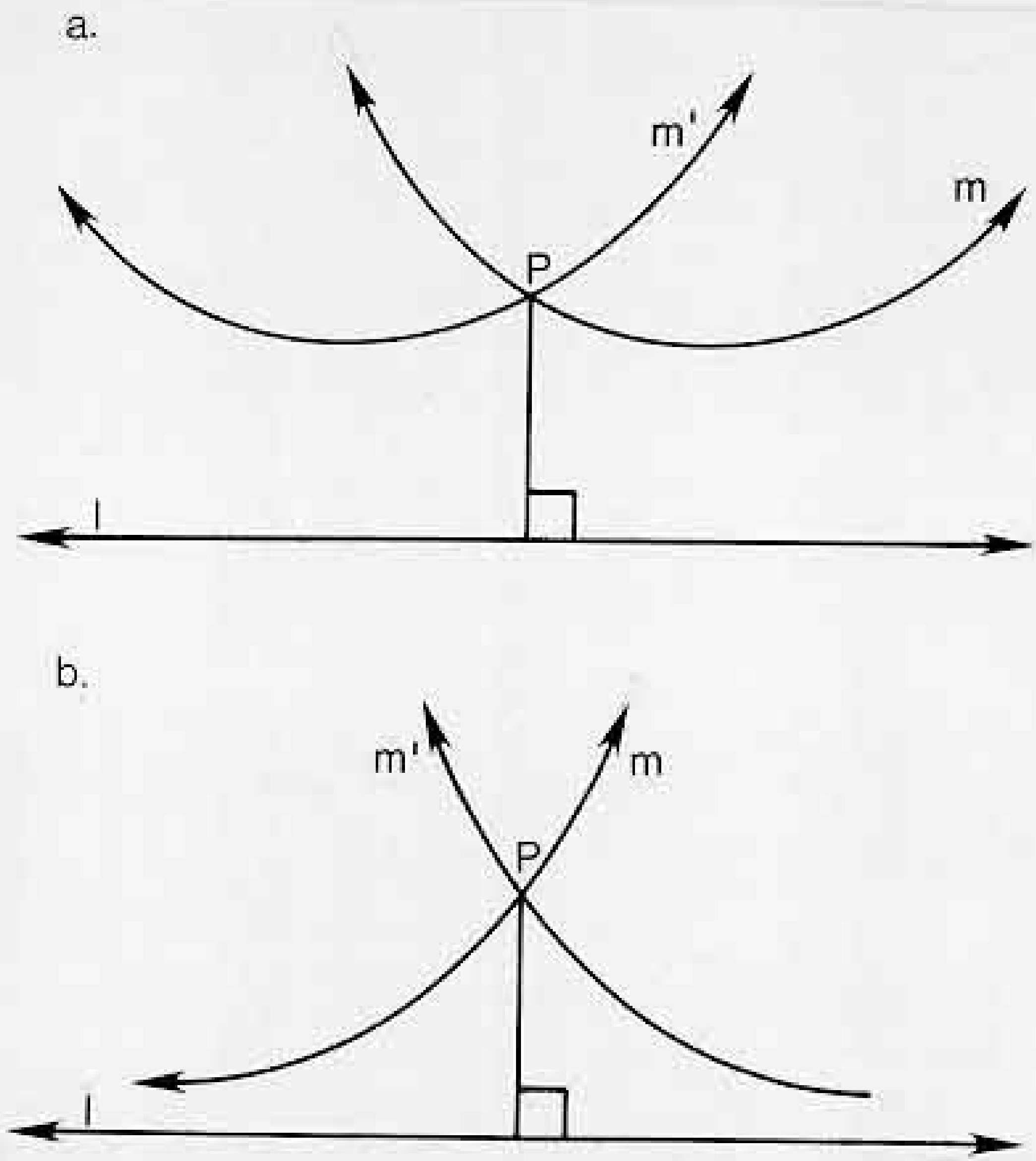

Figure 1. The hyperbolic plane.
a. Two lines m and m' (passing through P) are divergently parallel to line ℓ.
b. Two lines m and m' (passing through P) are asymptotically parallel to line ℓ.

of the sphere that includes great circles as the only lines, any two of which intersect at antipodal points. In "affine" geometry, there is exactly one line parallel to a given line, through a point not on that line. Affine geometry is further subdivided into Euclidean and Minkowskian geometries. Finally, in the "hyperbolic" geometry of Lobachevsky, there are at least two lines parallel to a given line through a point not on that line.

To visualize, intuitively, the possibility of more than one line parallel to a given line it is helpful to bend the lines, sacrificing "straightness" in order to retain the non- intersecting character of parallel lines. Thus, two upward- bending lines m and m' passing through a point P not on a given line ℓ never intersect ℓ; they are divergently parallel to ℓ (Figure 1.a). Or, one might imagine lines m and m' that are asymptotically parallel to ℓ (Figure 1.b) [8].

Elliptic geometry, with no parallels, and associated great-circle charts and maps have long been used as the basis for finding routes to traverse the surface of the earth. The suggestion here is that affine geometry, with single geometric parallels, captures fundamental elements of the earth-sun system outside the tropical parallels of latitude, and that hyperbolic geometry, with multiple geometric parallels does so between the tropical parallels of latitude.

3. Geographic and Geometric "Parallels".

As the Principle of Duality is a "meta" concept about symmetry in relation to projective geometry, so too is the earth-sun system in relation to terrestrial space. The changing seasons and the passing from daylight into darkness are straightforward facts of life on earth, often taken for granted. Some individuals appear to be more sensitive to observing this broad relationship, and to deriving information from it, than do others. Shadows may serve as markers of orientation as well as of the passing of time.

3.1 North and south of the tropical parallels.

Individuals north of 23.5° N. latitude and those south of 23.5° S. latitude always look in the same direction for the path of the sun: either to the south, or to the north (not both). Shadows give them linear information only, as to whether it is before or after noon; shadows never lie on the south side of an object north of the Tropic of Cancer. The perceived path of the sun in the sky does not intersect the expanse of the observer's habitat, from horizon to horizon. Thus, it is "parallel" to that habitat. North and South of the tropics there is but one such parallel, corresponding to the one basic direction an individual must look to follow the sun's trajectory across the sky.

3.2 Between the tropical parallels.

Between the tropics, however, the situation is entirely different. On the equator, for example, one must look half the year to the north and half the year to the south to follow the path of the sun. Thus, there are two distinct (asymptotic) parallels for the path of the sun through the observer's point of perception. Shadows can lie in any direction, providing a full compass-rose of straightforward information as to time of day as well as to time of year: apparently a broader "use" of shadow than Stevenson envisioned!

This population is thus surrounded, in its perception of the external environment of earth-sun relations, by the multiple parallel notion. (Those accustomed to primarily an Euclidean earth-sun trajectory might find this disconcerting.) This hyperbolic "vision" of the earth-sun system, suggests a consistency, for tropical inhabitants only, established in a natural correspondence of the perception of the external environment and the internal environment of the brain. For, it is the contention of R. K. Luneberg that hyperbolic geometry is the natural geometry of the mapping of visual images onto the brain [9].

4. The Poincaré Model of the Hyperbolic Plane.

To see how this variation in perception of the earth-sun system might be reflected in real-world settings, and to compare such settings between and outside the tropical parallels, it is necessary to understand one of these geometries in terms of the other. Both Euclidean and hyperbolic geometries are single, complete mathematical systems. They are not, themselves, composed of multiple subgeometries, nor can one of them be deduced from the other: they have the mathematical attributes of being categorical and consistent [6]. A mathematical system is categorical if all possible (mathematical) models of the system are structurally equivalent to one another (isomorphic) [13]; these models are, by definition, Euclidean and are therefore useful as tools of visualization. Because the hyperbolic plane is a categorical system, all models of it are isomorphic. Therefore, it will suffice to understand but a single one, and that one will then serve as an Euclidean model of the hyperbolic plane.

Henri Poincaré's conformal disk model (in the Euclidean plane) of the hyperbolic plane [8], was inspired by considering the path of a light ray (in a circle) whose velocity at an arbitrary point in the circle is equal to the distance of the point from the circular perimeter [4]. To understand how the model works, a "dictionary" that aligns basic shapes in the hyperbolic plane with corresponding Euclidean objects is useful (Table 1, Figure 2) [8].

The hyperbolic plane is represented as the disk, D, interior to an Euclidean circle C. Because the bounding circle, C, is not included, the notion of infinity is suggested by choosing points of D closer and closer to this unreachable boundary. Points in the hyperbolic plane correspond to points in D. Lines in the hyperbolic plane correspond to diameters of D or to arcs of circles orthogonal to C. These arcs and diameters are referred to as "Poincaré" lines. Because C is not included in the model, the endpoints of the Poincaré lines are not included, suggesting the notion of two points at infinity. Two Poincaré lines ℓ and m are parallel if and only if they have no common point. Thus, the disk diameter ℓ and the circular arc, m, orthogonal to C are parallel because they do not intersect; however, the disk diameter ℓ and the circular arc, m', orthogonal to C are not parallel because they do intersect (Figure 2a).

5. Hyperbolic Triangles and Quadrilaterals.

Any triangle in the hyperbolic plane is such that the sum of its angles is less than $180°$. When a triangle is drawn in the Poincaré model this becomes quite believable; draw Poincaré lines ℓ and m as disk diameters and draw Poincaré line n as an arc of a circle orthogonal to the disk boundary (Figure 2b) [8]. The triangle formed in this manner has one side that has "caved-in" suggesting how it happens that the angle sum can be less than $180°$ (note that three diameters cannot intersect in a triangle because all diameters are concurrent at the center of the disk). Triangles formed from more than one Poincaré line that is an arc of a circle would become even more concave.

Because all triangles have angle sum less than $180°$, there can be no rectangles (quadrilaterals with four right angles) in the hyperbolic plane. The idea that corresponds to that of a rectangle is a quadrilateral with three right angles, one acute angle, and pairs of opposite sides parallel (in the hyperbolic sense). The sides, OP, OQ, PR, and RQ, of this quadrilateral are drawn on Poincaré lines that are segments of disk diameters or arcs of circles orthogonal to the outer circle (Figure 2c; OQ is parallel to PR and RQ is parallel to PO). This quadrilateral is called a Lambert quadrilateral after Johann Heinrich Lambert [8], creator of the "Lambert" azimuthal equal area map projection (among others) [12]. When such a quadrilateral is drawn in the Poincaré model, the acute angle at R can be drawn to suggest that its sides are divergent, asymptotic, or intersecting. Here, these sides have been drawn to intersect (Figure 2c) and to evidently compress the angle at R as a suggestion of the angular compression [12] present in azimuthal map projections (including those of Lambert)

Table 1:
The Poincaré conformal model of the hyperbolic plane
(referenced to Figure 2—after Greenberg)

Term in hyperbolic geometry	Corresponding term in the Poincaré model in the Euclidean plane
Hyperbolic plane	A disk, D, interior to a Euclidean circle, C
Point	Point, P, in the disk, D.
Line	1. Disk diameter, ℓ, not including endpoints on C); or 2. Arcs, m, m', in D of circles orthogonal to C (tangent lines at points of intersection are mutually perpendicular).

around the projection center.

6. Tiling the Hyperbolic Plane.

If one views a map grid as a tiling by quadrilaterals of a portion of the Euclidean plane, then it might be instructive to consider a tiling of the "map" of the Poincaré disk model

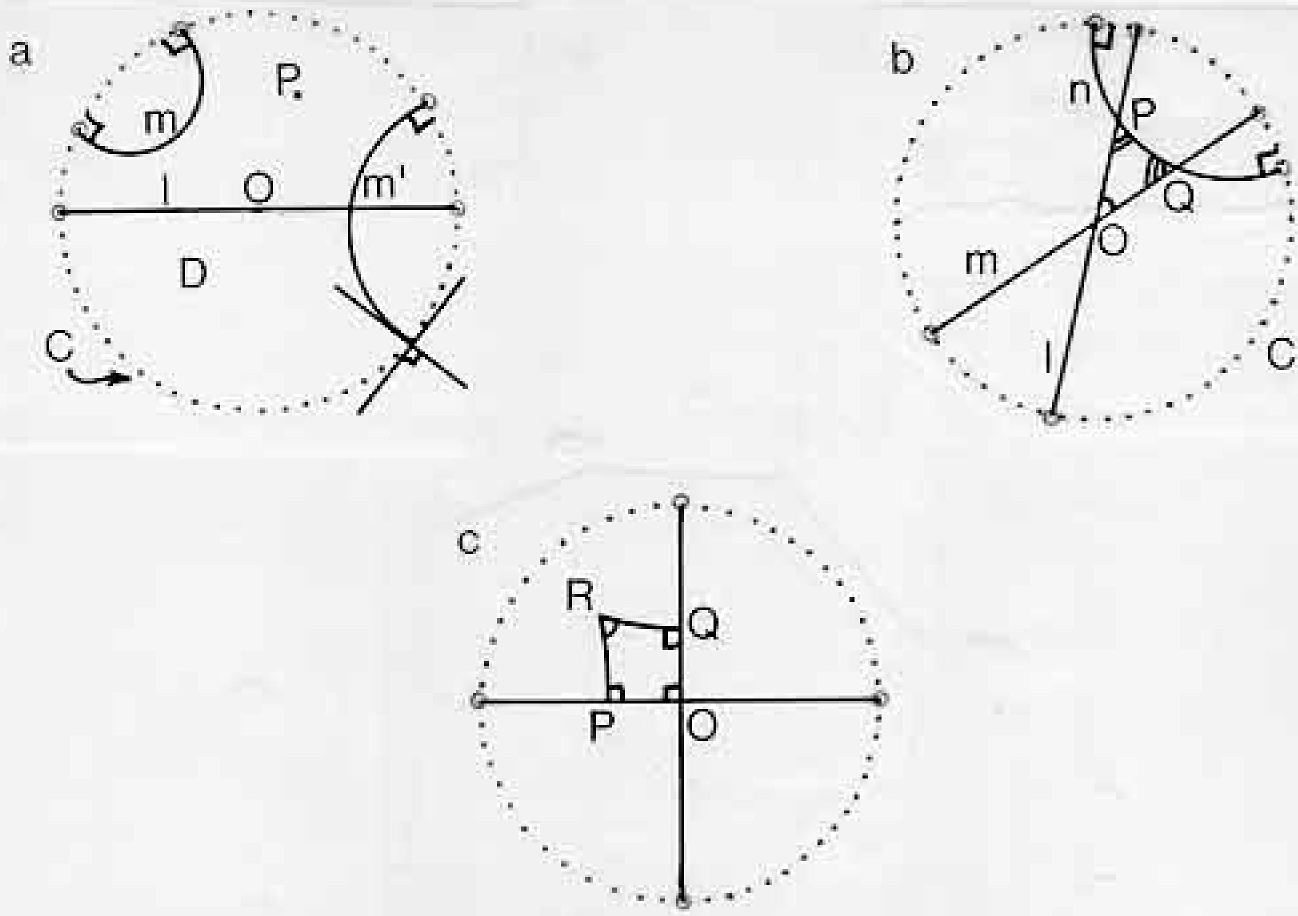

Figure 2. The Poincaré Disk Model of the hyperbolic plane.

a. The diameter, ℓ, is a Poincaré line of the model, as are arcs m and m' which are orthogonal to the boundary C. The Poincaré lines ℓ and m are parallel (do not intersect); the lines ℓ and m' are not parallel (do intersect).

b. The sum of the angles of ΔOPQ is less than 180°. The triangle is formed by sides ℓ, m, n; the Poincaré lines ℓ and m are diameters, and the Poincaré line n is an arc of a circle orthogonal to C.

c. A Lambert quadrilateral with three right angles and one acute angle (PRQ). Pairs of opposite sides are parallel.

by Lambert and other quadrilaterals [5]. Gluing quadrilaterals together along Poincaré lines produces a variety of quadrilaterals (Figure 3). All have pairs of opposite sides parallel; Poincaré lines represented as arcs are orthogonal to the outer circle. Naturally, the tiling can never completely cover the disk, because the disk boundary is not included. Thus, tilings of this map have quadrilaterals of shrinking dimensions as the outer circle is approached.

This permits hyperbolic "tilings" to suggest the infinite; indeed, they have served as artistic inspiration for the "limitless" art of M. C. Escher [7].

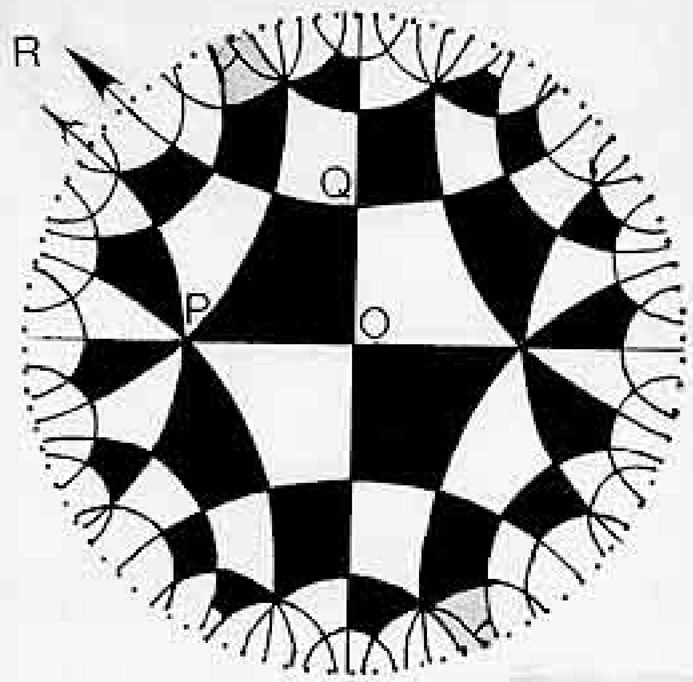

Figure 3. A partial tiling of the Poincaré Disk Model by quadrilaterals bounded by Poincaré lines. Quadrilateral $(OPQR)$ is a Lambert quadrilateral with two sides drawn asymptotic to each other.

7. Triangles, Quadrilaterals, and Tilings Between the Tropics.

Concern with home and family are universal human values. Typical American houses exhibit Euclidean cross sections: a rectangular one from a side view and a pentagonal one, as a triangular roofline atop a square base, from a head-on view. Western Sumatran Minangkabau house- types fit more naturally into a non-Euclidean framework than they do into the Euclidean one, exhibiting hyperbolic cross sections as a Saccheri quadrilateral (two Lambert quadrilaterals glued together along a "straight" edge (Figure 4a) [8]) when viewed from the side, and as a concave, hyperbolic, triangle atop a (possibly Euclidean) quadrilateral when viewed from the front (Figure 4b).

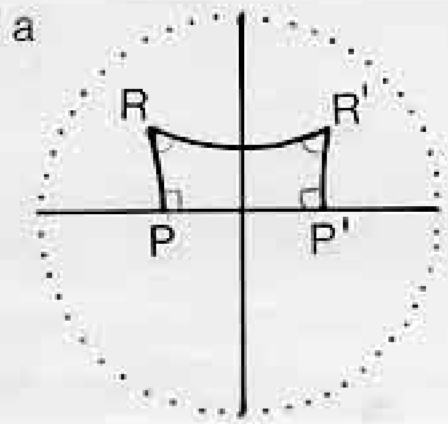

Figure 4.

a. A Saccheri quadrilateral, formed from two Lambert quadrilaterals. It has two right angles and two acute angles. Pairs of opposite sides are parallel, as drawn in the Poincaré Disk Model.

b. West Sumatran Minangkabau house. Roofline is suggestive of a Saccheri quadrilateral. Photograph by John D. Nystuen.

Games children play often reveal deeper traditions of an entire society. As the sun moves through its entire range of possible positions, shadows dance across the full range of

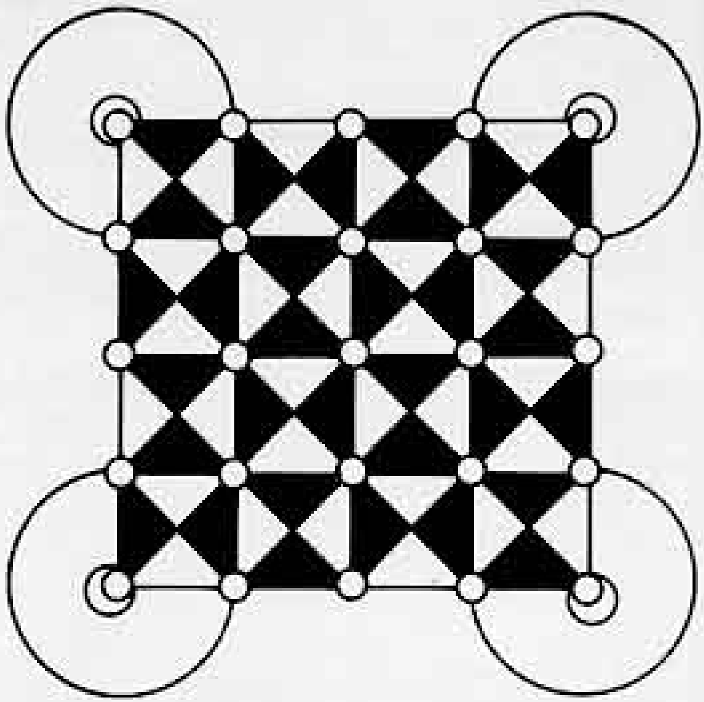

Figure 5.

a. Sodokan game board in Euclidean space. Markers travel along lines separating regions of contrasting color and along circular loops at the corners.

compass positions on Indonesian soil and come alive, as "shadow puppets," in Indonesian theatrical productions. Elegant cut–outs traced on goat skins and other hides are mounted on sticks and dance in a plane of light between a single point–source and a screen, casting their filigreed, shadowy outlines high enough for all to see. The motions of the Indonesian puppetteer are regulated by the world of projective geometry, with shadows stretching out diffuse arms toward the infinite.

A commonly played Indonesian board game is "Sodokan," a variant of checkers [1]. Two people play until all of an opponent's ten pieces, arranged initially on the intersection points of the last two lines of a 5×5 board (Figure 5a), have been captured. Pieces move across the board horizontally, vertically, or diagonally, one square at a time. What is unusual is the method of capture; to take an opponent's marker requires a "surprise" attack along the loops outside the apparent natural grid of the gameboard.

For example, with just two pieces remaining (so that there are no intervening pieces), black may capture white (Figure 5b). To do so, black must traverse at least one loop; in the act of capture, black can slide across as many open grid intersections as required to gain entry to a loop. Then, still in the same turn, black slides around the loop, re–enters the game board, and continues to slide across grid intersections and loops until an opponent's marker is reached, and therefore captured.

The name, "Sodokan," means "push out." Its name seems to apply only loosely to the

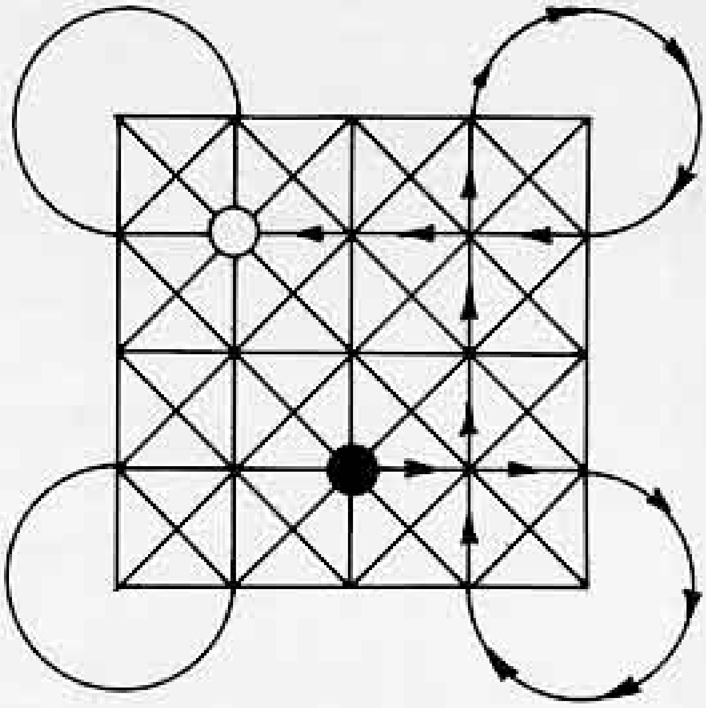

Figure 5.
b. Sample of capture. Black captures white—a single move.

5×5 Euclidean game board (Figure 5a) because the loops are not, themselves, "pushed out" from the natural gameboard grid. If they were, the corners of the Euclidean grid would disappear. However, when the game board is drawn on a grid in the Poincaré disk model of the hyperbolic plane (Figure 5c), the loops appear naturally from grid intersections outside the circular boundary. A marker engaged in a capture on this non–Euclidean (hyperbolic) board traverses the entire hyperbolic plane ("universe"), passes across the infinite and is provided a natural avenue within the system for return to the universe. The loops are naturally "pushed out" of the underlying grid, tiled partially by Lambert quadrilaterals; they might suggest paths along which gods [11], skipping across space, interrupt (sacrifice) elements within the predictable universe of the life–space in the disk. However, independent of speculation as to what such paths might mean, the fact remains that it is within the hyperbolic geometric framework, only, that this game board emerges as a part of a natural grid system. Thus, capture is no longer a mysterious event from "outside" the system; the change in theoretical framework, from an Euclidean to an hyperbolic viewpoint, made it a logical occurence.

A change in the underlying symmetry introduced order. The "meta" earth–sun system, when viewed as that which introduces a symmetric partition of the earth according to bands of sun–delivered affine and hyperbolic geometry, offered order in understanding roofline and gameboard shape where none had been apparent.

Sources of evidence for other similar interpretations are plentiful: from Indonesian calen-

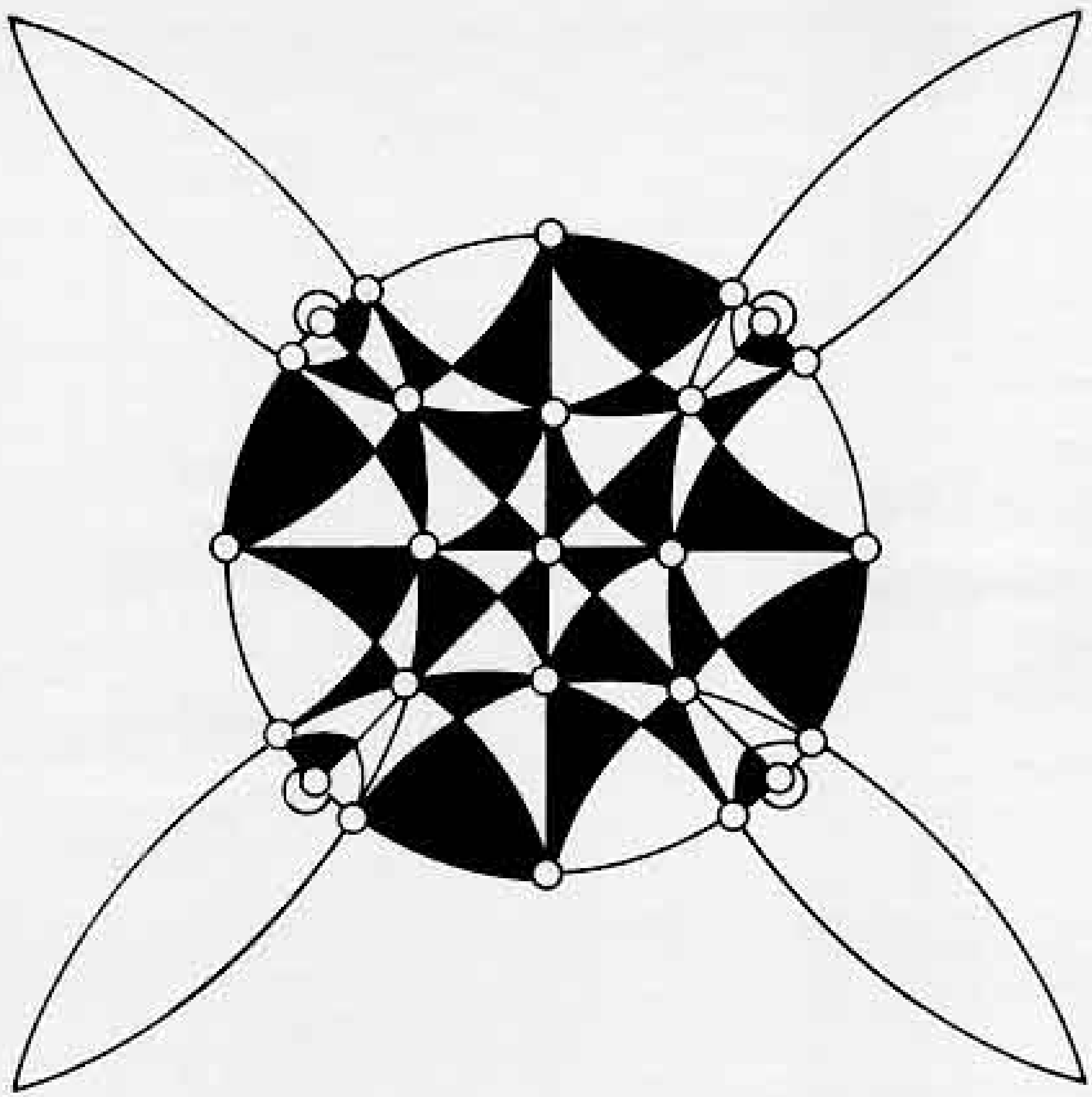

Figure 5.

c. Sodokan game board drawn on the Poincaré Disk Model of the hyperbolic plane. The four central quadrilaterals are Lambert quadrilaterals—the intersecting versions of quadrilateral $(OPQR)$ in Figure 3. When their sides are extended, the gameboard loops are formed naturally by these grid lines and their intersection points.

dars based on a nested hierarchy of cycles, to the loops within loops creating the syncopated forms characteristic of Indonesian gamelan music. Perhaps Indonesians and other between-the-parallels dwellers have escaped the asymmetric confines of Euclidean thought, enabling them to include a comfortable vision of infinity as part of the underlying symmetry of their daily circle of life.

8. References.

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