The Quadratic World of Kinematic Waves

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Kinematic waves differ from "ordinary" waves insofar as it is the kinematics–the dynamic aspects of motion other than mass and force–that are the focus. Thus, Langbein and Leopold [1968, p. 1] define a kinematic wave as "a grouping of moving objects in zones along a flow path and through which the objects pass. These concentrations may be characterized by a simple relation between the speed of the moving objects and their spacing as a result of interaction between them." Flow in a channel is characteristically expressed as a function of concentration, be that as cars per hour as a function of cars per mile or as transport in cubic feet per minute of sand in a one inch tube as a function of linear concentration of sand in pounds per square foot [Langbein and Leopold 1968; Haight 1963; Lighthill and Whitham I and II 1955]. Examples of kinematic waves are abundant in physical and urban settings alike–in realms as disparate as sand transport in a flume or car movement on an Interstate Highway [Langbein and Leopold, 1968]. When empirical data are graphed, they often trace out a parabola (or a curve close to a parabola); thus, the relationship between concentration and flow is often a quadratic one [Langbein and Leopold, 1968].

The classical analysis of the parabolic graphs of these waves rests on considering what happens to flow as a result of minor perturbations in local concentrations–techniques are based in notions from the calculus [Langbein and Leopold 1968]. Consider a concave down parabola with its maximum in the first quadrant that passes through the origin. Flow is a function of concentration; thus, concentration appears on the x-axis and flow on the y-axis. Choose two points on the curve, one with coordinates (x_1, y) and the other with coordinates (x_2, y)–the x-coordinates are different and lead to the same y-coordinate. They are placed symmetrically on the x-axis about a vertical line through the curve's maximum (Figure 1; for electronic readers only, please draw this curve and subsequent ones as per text). Assuming that x_1 is to the left of the maximum, the traditional analysis notes that at x_1, a slight increase in concentration results in a slight increase in flow; a slight decrease in concentration at x_1 results in a slight decrease in flow. The channel is relatively sparsely congested–slight changes in concentration result in directly parallel changes in flow. Further, the closer one is to the x-coordinate of the maximum, the less difference these slight changes cause. On the other hand, at x_2 (to the right of the maximum) a slight increase in concentration results in a decrease in flow, suggesting a channel which cannot easily assimilate any extra traffic. Further, a slight decrease in concentration at x_2 results in an increase in flow, again reflecting a relatively congested condition of this channel. When the horizontal line suggested by x_1 and x_2 is tangent to the parabola, at its maximum, the kinematic wave is stationary relative to the channel; thus, as the distance of horizontal lines increases away from this tangent line, there is a corresponding increase in the amount of change caused by local perturbations. The origin, as a location for x_1, represents a completely uncrowded condition, while the second intersection of the curve with the x-axis represents the most crowded position within this interval [Langbein and Leopold 1968].

The traditional analysis, based merely on considering what slight changes in x_1 and x_2 might suggest, fits well with real-world travel experience. Consider the concentration on the x-axis to be density of automobiles as vehicles per mile; on the y-axis, consider flow to be vehicles per hour. Practical evidence does suggest that an improvement in the maximum capacity of the road does result in improved transmission of flow, but only up to a point. Thus, highway systems are widened around cities and endowed with limited access to increase the number of vehicles per hour that can move from origin to destination. Beyond about 1800 vehicles per hour, this "improvement" is no longer useful [Nystuen 1992]; indeed, congestion increases and flow per hour decreases toward the point of gridlock—the ultimate disaster that can affect millions of individuals. This sort of ceaseless "improvement," to the point of disaster, of what worked well in a less congested arena, appears in a variety of contexts; when an optical cable with the capacity to serve millions is cut, disaster comes to many rather than to few, and chaos in communication becomes a real possibility [Austin 1991].

The traditional analysis also allows for computation of various other features associated with the kinematics of the phenomenon it describes. For example, the average speed of particles in the channel, or wave celerity, can be measured at any point on the curve, simply by finding the slope of the chord joining that point to the origin [Langbein and Leopold 1968]. However, when a given density leads to a certain flow, which is then used to determine the next input to create a new density level, feedback occurs. Feedback is not measured in the traditional analysis. It also fits with travel experience and indeed is the sort of process that can get chaotic. Thus, it seems plausible to consider graphical analysis of kinematic curves, based in Feigenbaum's Graphical Analysis from the mathematics of Chaos Theory, as a supplement to the traditional analysis.

Consider the following set of parabolas as Figures 2 through 7: $y = 1.5x(1-x)$; $y = 2x(1-x)$; $y = 3x(1-x)$; $y = 3.75x(1-x)$; $y = 4x(1-x)$; and, $y = 5x(1-x)$. The e-reader should draw each of these curves, noting that each parabola is of the sort described above—consider the units on the axes, ranging from 0 to less than 1.5, as percentages. Thus, 0.5 on the x-axis represents a concentration of 50%. Also include in each graph the line $y = x$. Each parabola intersects this 45-degree line in two points–one at the origin and one that is either to the left or to the right of the curve's maximum. As the coefficient of the curve increases from 1.5 to 5, the curves become successively less flat, have a higher maximum, and have a second intersection with the line $y = x$ farther to the right.

To represent geometric feedback visually on Figures 2 to 7, proceed as follows [based on material from Feigenbaum 1980; Gleick 1987; Devaney and Keen 1989]. Locate the point 0.1 on the x-axis of each figure. Draw a vertical line from that point (as a "seed" value for the graphical analysis) to the parabola. Now draw a horizontal line from the curve to the line $y = x$; next read vertically from this location to the parabola. The effect here is to use output as input; for, 0.1 was the initial input. When that value was mapped to the parabola, an output resulted— when that output was mapped horizontally to $y = x$; it was then used as input when it was next sent to the curve. Successive iteration of this process should result in the following paths from the iteration ("orbits"): Figure 2—a staircase with shallow rises; Figure 3— a staircase with sharper rises than in Figure 2; Figure 4— a tightly bounded cyclical orbit closing in on the second intersection of the line with the parabola; Figure 5—an unpredictable, bounded orbit; Figure 6—a chaotic, bounded orbit; Figure 7—an orbit that escapes to negative infinity (from a curve whose maximum is beyond the 100% concentration level). Geometrically, control over the dynamics of the orbit becomes less stable as one proceeds from Figures 2 to 7. It makes little difference which initial seed is chosen; the dynamics of the orbit are invariant with respect to these curves (parabolas). Unlike the traditional analysis, in which there is considerable variation in the measures used, with respect to a single curve, the pattern of the orbit is constant throughout each figure—as a sort of a shape-invariant. Indeed, any of these curves might be employed equally for the traditional, but not for the graphical, analysis.

What determines the extent of stability in the geometric dynamics noted in these figures are the height of the parabola and the position of the second intersection of $y = x$ with that parabola. Higher parabolas have intersection point with $y = x$ farther to the right of the curve's maximum, producing more uncontrolled feedback. This fits well with traffic observations; increase of a road's maximum capacity beyond some critical level leads to disastrous congestion. The tool of graphical analysis looks promising as a tool in analyzing real-world phenomena [Feigenbaum 1980; Gleick 1987] that follow kinematic waves as well as those that follow more complicated curves [Arlinghaus, Nystuen, and Woldenberg 1992].

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