3A. WHAT ARE MATHEMATICAL MODELS AND WHAT SHOULD THEY BE?

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No matter what the area of scientific research, whether social or physical, mathematical thinking is involved, explicitly or implicitly. At the least, the precise formulation of a problem entails some aspect of set theory and logic. Generally speaking, the working scientist uses the term 'mathematical model' for whatever branch of mathematics he may be applying to his present problem. On the other hand, the purist mathematician-logician insists strictly on the use of 'model' to mean a certain interpretation of an abstract axiom system in the real world.

We begin with a self-contained development of the concepts needed for the discussion of research processes. This leads to the distinction between the real and abstract world, and the interaction between them by interpretation and abstraction. A similar, but conceptually different bifurcation is proposed for the two levels of research: digging into the foundations versus extending the horizons of knowledge. These considerations are assembled into a comprehensive Research Schema which enables a concise analysis of scientific discovery. Classical illustrations are provided, including true stories about Newton, Darwin, Freud, and Einstein. We conclude with some subjective evaluations of acceptability of mathematical models.

1. What Are They?

We have just noted that the word 'model' has different meanings for the mathematician and the scientist. When a mathematician uses the word, he is referring to the physical or social realization of his theory. On the other hand, when a scientist speaks of a mathematical model, he means the area of mathematics which applies to his work. Thus one (following Abraham Kaplan, oral communication) could say as a mnemonic aid that a model is always the other fellow's system. Contrariwise it also appears to be customary by usage to refer to "research" as what goes on in your own domain.

In order to define a model rigorously, it is convenient to develop (as in Wilder [4] or in a more elementary presentation, Richardson [3]) several notions in the foundations of mathematics. Recall from high school geometry that Euclid's axioms are about as follows (depending on which book you read). The words "point" and "line" are undefined terms.

A_1 (Axiom 1) Every line is a collection of points.

A_2 There exist at least two points.

A_3 If u and v are points, then there exists one and only one line containing u and v.

A_4 If L is a line, then there exists a point not on L.

A_5 If L is a line, and v is a point not on L, then there exists one and only one line L' containing v which is parallel to L, i.e., $L \cap L' = \emptyset$.

Axiom 5 is the celebrated "Parallel Postulate" of Euclid.

An *axiom system* $\Sigma = (P, A)$ consists of two sets: a set P of primitives and a set A of axioms. *Primitives* are the deliberately undefined terms upon which all definitions in the system are based. *Axioms* are statements which are assumed to be true, and from which other statements called *theorems*, can be derived. Primitives and axioms serve to avoid so-called circular definitions and circular reasoning. Each axiom in the system is an assertion about the primitives.

Euclid's axiom system consists of two primitives, 'point' and 'line', and five axioms. When Euclid developed geometry, he made a distinction between axioms and postulates. Both were statements whose truth was assumed, but axioms were considered self-evident while postulates were not! This distinction eventually proved unnecessary and even undesirable, and today axiom and postulate are synonyms.

We shall denote by T or $T(\Sigma)$ the set of all theorems derivable from an axiom system Σ. Then a *mathematical system* (P, A, T) is an axiom system together with all theorems derivable from it.

An *independent axiom* A of Σ is one which cannot be derived from the remaining axioms. An *axiom system* is *independent* if every axiom is independent. In is called *consistent* if there are no two contradictory statements in $T(\Sigma)$.

One of the classical problems in 19th Century mathematics was to determine whether or not Euclid's Parallel Postulate, A_5, was independent. The consensus of opinion was that A_5 was dependent, that is, it could be derived from $A_1 - A_4$. Unsuccessful attempts to derive A_5 led to the discovery instead of non-euclidean geometry. The two types of non-euclidean geometry are now respectively called *hyperbolic geometry* (Bolyai-Lobachewski independently) in which there can be many parallels to a line through a point, and *elliptic geometry* (Riemann) in which there can be no such parallel.

An *interpretation* of an axiom system is an assignment of meanings to its primitives which makes the axioms become true statements. The results of an interpretation of Σ is called a *model* for Σ. This is the strict use of 'model' mentioned earlier.

An axiom system is called *satisfiable* if it has at least one model. Two models, M_1 and M_2 of Σ are *isomorphic* if there is a 1-1 correspondence between the elements of M_1 and those of M_2 which preserves every Σ-statement. In a *categorical* axiom system, any two models are isomorphic.

To illustrate, consider an axiom system with primitives $P = \{S, \circ\}$, where S is a set of integers, and $\circ$, is chosen as an undefined term for a binary operation denoted $a \circ b$, in order to avoid preconceived notions that a familiar symbol like $a+b$ would bring to mind. The following statements A_1 — A_4 are called *group axioms*, and any set S on which they hold under the operation $\circ$ is called a *group*.

A_1 (Closure Law) S is closed under $\circ$, that is, if a and b are in S, $a \circ b$ is in S.

A_2 (Associative Law) Operation $\circ$ is associative, that is, $a \circ (b \circ c) = (a \circ b) \circ c$ for all a, b, and c in S.

A_3 (Identity Law) There is a unique element i in S, called the identity element, such that $a \circ i = i \circ a = a$ for all a in S.

A_4 (Inverse Law) For every a in S, there is a unique element, written a^{-1} and called the *inverse* of a, such that $a \circ a^{-1} = a^{-1} \circ a = i$. Each of the four group axioms is independent, and so this is an independent axiom system. To verify that this axiom system is satisfiable, we now display a model.

One model for this system is the set $S_1 = \{1, -1\}$ under multiplication $\times$. Thus this is called a group of *order* 2, i.e., having just two elements. The identity element is 1, each element has itself as an inverse, and S is obviously closed and associative, as can be seen from the following multiplication table:

$\times$	1	-1
1	1	-1
-1	-1	1

Another model for this axiom system is the set $S_2 = \{0, 1\}$ under addition modulo 2. We define the sum of a and b mod 2 to be the remainder of $a+b$ after division by 2. Under this operation, we see at once from the next table that S_2 is closed and associative, 0 is the identity, and each element is again its own inverse. Thus S_2 is also a group of order 2.

+ mod 2	0	1
0	0	1
1	1	0

More generally, one can take S to be the set $\{0, 1, 2, \ldots, n-1\}$ and $a \circ b$ to mean $a + b \bmod n$. Then for each positive integer n, we get a distinct group of order n. Thus the above axiom system for groups is not categorical, since it has many non-isomorphic models.

These two groups, S_1 and S_2, are isomorphic since we can let operation $\times$ correspond with + mod 2 and set $\{1, -1\}$ to correspond with $\{0, 1\}$. All statements derivable from the axioms still hold. That the two models are isomorphic is also shown in the fact that their tables both have the following form:

$\circ$	a	b
a	a	b
b	b	a

In fact, any pair of groups with two elements are isomorphic, so it is customary to speak of "the group of order two."

The study of group theory was originally motivated by properties which are possessed by the symmetries of a configuration, whether it be geometric, algebraic, architectural, physical, or chemical. It is readily verified that symmetries satisfy the four group axioms. For example, the inverse of a symmetry of a configuration is the corresponding symmetry mapping done in reverse.

2. Two Worlds: Abstract and Empirical

The realm of research activity is naturally divided into two worlds: the abstract and the empirical. The abstract world is generally regarded as the domain of the mathematician, logician, or purely theoretical physicist, while the empirical world is inhabited by experimental scientists of many varieties: physical, social, and others. [It has been established empirically that the less scientific a subject, the more likely it is that its practitioners call it a science. Outstanding examples include (in alphabetical order): divinity science, library science, military science, political science, and secretarial science.] There is a growing tendency, however, for people to live in both worlds in these interdisciplinary times.

Those who work entirely in the abstract world are engaged in deriving new theorems either from axioms or from an existing theory or coherent body of theorems. Such results are usually expressed in symbols rather than numbers, and rarely touch upon the real world.

On the other hand, the inhabitants of the empirical world "work for a living." Some live in laboratories and perform experiments in order to collect meaningful data leading to a scientific theory.

The two worlds are shown in Figure 1. The two loops, called theory building and experimentation, represent purely theoretical and purely experimental research.

Figure 1 exhibits a symmetric pair of directed links between the worlds, the first of which can be called interpretation in accordance with the use of this word in the preceding section. In a confrontation between these two worlds, the mathematician's theorems become predictions about the real world, which can be tested by the scientist. If a prediction is verified by an appropriate experiment, the scientist feels that the theorem really works, and the mathematician has found a *realization*.

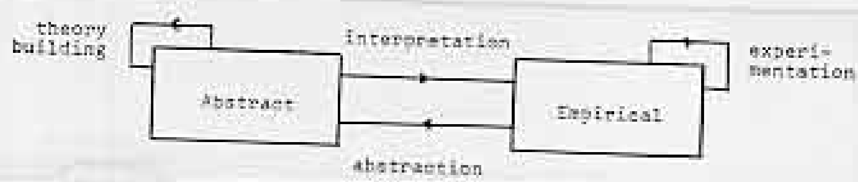

Figure 1 — Two worlds. [Two rectangles, representing the two worlds, are linked by a left arrow and a right arrow. The left rectangle is labelled "Abstract"; the right rectangle is labelled "Empirical." The right arrow, from the abstract world to the empirical world is labelled "interpretation." The left arrow, from the empirical world to the abstract world is labelled abstraction. A loop, labelled "theory building," is attached to the upper left of the abstract world. A loop, labelled "experimentation," is attached to the upper right of the empirical world. Inserted by Ed.]

If the predictions are entirely incorrect, the model cannot be used. However, in cases where the predictions are not verified, yet are "rather close" to correct, further abstraction is in order to construct a working model. This abstraction in the light of the experiment may suggest alternate hypotheses which should result in new theorems. These theorems hopefully will lead to better predictions than previously, and to a working model.

3. Two Worlds: Two Levels

Each of our two worlds may be divided into two levels. As we have indicated, the upper level of the abstract world deals with the development of mathematical systems by the derivation of theorems. We have discussed interaction between worlds at this level by means of interpretation and abstraction. In this section we shall observe that this same type of interaction can occur at the lower level.

The lower level of the abstract world deals with the foundations of mathematics, axioms, and logic. The research activities might involve trying to prove consistency or independence of an axiom system.

A rather esoteric and dramatic recent example of an important discovery at this level is given by the definitive work of Paul Cohen [1]. It is known (see Wilder [4], for example) that a $1-1$ correspondence can be constructed between the natural numbers $1, 2, \ldots$, and all the integers, $\ldots -3, -2, -1, 0, 1, 2, \ldots$, and between the integers and the rational numbers. These three sets of numbers are all said to have the same (infinite) *cardinality* which is conventionally denoted $\aleph_0$.

It is also known that there are more real numbers than integers. The real line is sometimes called the *continuum*, and so $\mathbf{c}$ is written for the number of reals. The *continuum hypothesis* states that there is no infinite set with cardinality between $\aleph_0$ and $\mathbf{c}$.

Cohen proved that the continuum hypothesis (as well as its negation) is consistent with the usual axioms of set theory. As a consequence, it is independent and can neither be proved nor disproved in that axiom system. Analogous to the development of non-euclidean geometry, two entirely different axiom systems have been created; one by assuming the continuum hypothesis, and the other by taking its negation. Cohen also proved the independence of the "axiom of choice."

On the other hand, the lower level of the empirical world also deals with foundations, but in the form of the basic laws of science. Kepler's Laws of Planetary Motion, Darwin's Law of Natural Selection, Newton's Laws of Motion, Kirchhoff's Laws of Electricity, and Einstein's Law of Special Relativity are all there.

The link between the two worlds at this lower level is quite analogous to that at the upper level. Thus interpretation of an axiom leads to a basic law about the real world, while an abstraction, a coherent set of scientific laws becomes an axiom system. The schematic representation of interaction between the two worlds is shown in Figure 2.

4. Two Levels: Derivation and Selection

Having discussed interaction between the two worlds, we shall now establish links between their upper and lower levels. The process of climbing from the lower level to the upper in the abstract world can be regarded as *derivation*. For we begin with an axiom system and then, sometimes painfully, derive progressively complicated theorems to obtain a mathematical system.

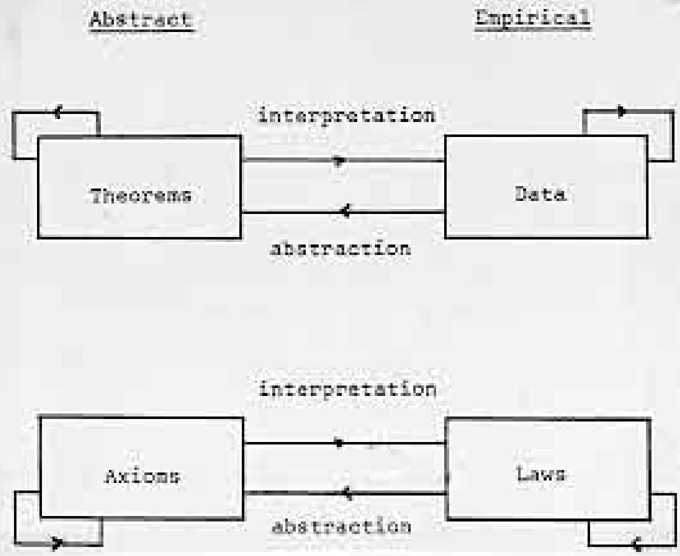

Figure 2 — Interaction between the two worlds. [There are four rectangles in this figure, arranged at the upper left, upper right, lower left, and lower right. The two uppers have a left arrow and right arrow linking them, as do the two lowers. The upper left rectangle is labelled "Theorems"; the upper right, "Data"; the lower left, "Axioms"; and, the lower right "Laws." The right arrow in each case is labelled "interpretation." The left arrow in each case is labelled "abstraction." The left hand side of the figure is labelled "Abstract"; the right, "Empirical." There is a loop attached to each of the four rectangles. Ed.]

Now consider how one goes from the upper level to the lower. From an existing body of theorems, an axiom system is to be built. To accomplish this, we select a body of particularly appropriate and fruitful theorems to use as axioms. This process of *selection* yields a small, more manageable and often more powerful system, which is conducive to the derivation of new theorems.

Selection in the empirical world involves collecting and studying vast amounts of data, and observing a pattern which may suggest a general law. Thus it is actually the *induction* process.

There appears to be no direct link in the empirical world from the lower level to the upper. Derivation does occur, and in fact uses the deduction process, but again and again we find that it takes the "long way around," as shown in Figure 3. One begins with several scientific laws (lower right), and abstracts them to formulas (lower left) from which theorems can be derived (upper left) which make predictions about the real world (upper right). It is convenient, however, to draw the link representing derivation directly as well, as we do later.

In general, innovative research is initiated in the upper level, and particularly in the upper right quadrant. This is due to the fact that the great majority of natural and fundamental questions arise from an attempt to observe or explain empirical phenomena. In fact, most research is done at the upper level, both right and left, while almost no one continuously remains at the lower level.

For example, in ancient Egypt, the discovery of geometric formulas was necessitated by the search for improved techniques in measuring and surveying. Problems in geometry were

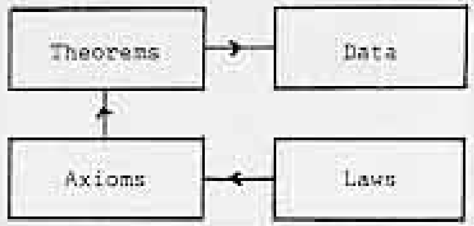

Figure 3 — Derivation in the empirical world [The four rectangles of Figure 2, are linked with three arrows from Laws to Axioms to Theorems to Data. Ed.]

solved long before Euclid organized the subject in an axiomatic formulation.

5. Research Schema

We contend that the above Research Schema represents all the types of interaction between the abstract and empirical worlds during the processes of research and discovery. Its two diagonal links are shortcuts which represent research processes that go directly to "opposite" quadrants. There do not seem to be any directly ascending diagonal links.

It is rarely but definitely possible to predict scientific laws from a body of theorems without actually working with experimental data. This is represented by the diagonal from upper left to lower right in the Research Schema. We shall see that Einstein took this route in his formulation of the theory of special relatively.

The shortcut from experimental data to axioms, skipping the formulation of laws, occasionally occurs in the social sciences when a careful analysis of data patterns produces a set of formulas that can be taken as axioms. These are then interpreted, and hopefully suggest an empirical law, without the selection process.

When considering routes between the two worlds, one must also allow for traversing loops at any quadrant one or more times. The upper right loop, for example, when traversed several times, indicates repeated efforts in observation and collection of data, before attempting to select corresponding laws.

One must also note that the most direct route is not often taken in research. This will become evident in the next section when we take a closer look at particular cases of discovery.

6. Sketches of Discovery

We shall illustrate the Research Schema with the work of several men who represent varied branches of science and mathematics. We begin with Euclid, whose work in the axiomatization and derivation of what we now call euclidean geometry is represented schematically in Figure 5. [It has been said that the ultimate recognition of a man's contribution is conferred when his name is made an adjective and not capitalized.]

Euclid: Although Euclid is the acknowledged father of geometry, his main contribution was to its organization rather than to its derivation. The early Egyptians already knew the rudiments of geometry, including a form of the pythagorean theorem, and formulas for the area and volume of many geometric figures. Thus we attribute the upper right quadrant in

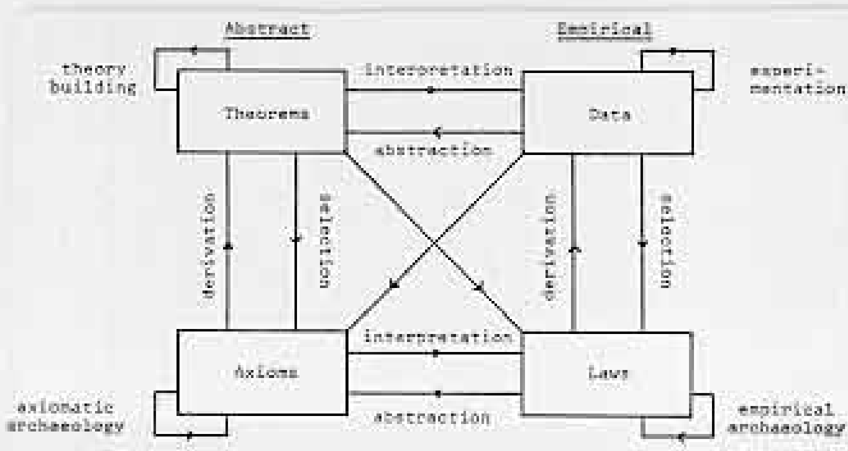

Figure 4. Research schema. [Draw Figure 2. Label the loop on "Theorems" as "theory building"; that on "Data" as "experimentation"; that on "Axioms" as "axiomatic archaeology"; and, that on "Laws" as "empirical archaeology." Add up and down vertical arrows joining the rectangles; label the downward arrow in each case as "selection"; the upward as "derivation." Draw the two diagonals – one with an arrow to suggest going from "Theorems" to "Laws" and the other from "Data" to "Axioms." Ed.]

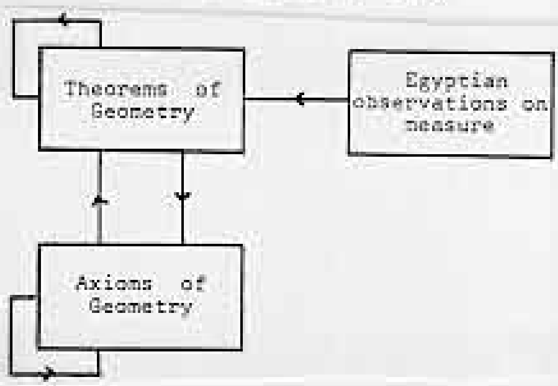

Figure 5 — Euclid's Research Schema. [Draw three rectangles: upper left, upper right, lower left. Label them, respectively, "Theorems of Geometry," "Egyptian observations on measure," "Axioms of Geometry." Add a loop to the two rectangles on the left. Join the upper left and lower left rectangles by an up arrow and a down arrow. Draw an arrow from the upper right to the upper left rectangle. Ed.]

Figure 5 to the Egyptians. The emphasis on proof, however, was introduced by the early Greeks and Euclid's contemporaries developed many of the theorems of geometry. Euclid selected the five axioms above from existing results. He then proved from these all the theorems of geometry then known and a few new ones, and presented a logical organization of the material in an exhaustive text. By today's standards, Euclid's axiomatic work is not rigorous, but it was an outstanding accomplishment for its time.

Newton: Unlike Euclid, Newton occupied every quadrant of the Research Schema. His

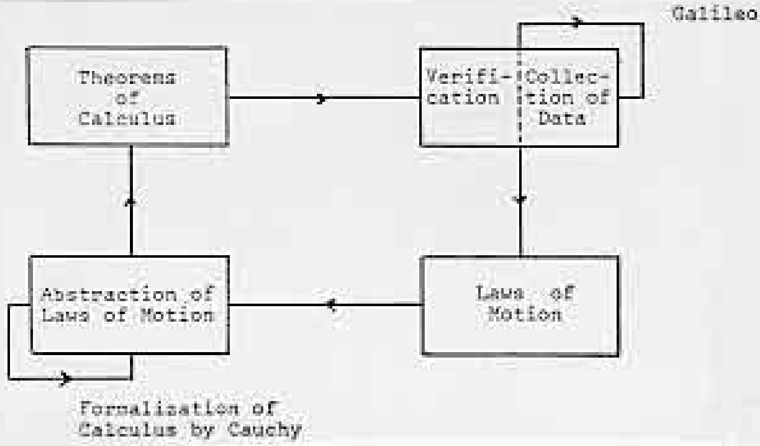

Figure 6 — Newton's Research Schema. [Draw four rectangles: upper left – "Theorems of Calculus"; upper right – left half labelled "Verification" right half labelled "Collection of Data"; lower left – Abstraction of Laws of Motion; lower right – "Laws of Motion." Join the rectangles with arrows forming a rectangular cycle oriented in a clockwise direction. Add a loop to the lower left rectangle; label the loop "Formalization of Calculus by Cauchy." Add a loop to the upper right rectangle; label the loop "Galileo." Link the "Galileo" loop to the down arrow as a dashed line separating "Verification" from "Collection of Data" in the upper right hand box. Ed.]

first work was on the upper level of the empirical world, where he experimented in chemistry and optics while still a student. Newton's most important results, however, were not derived from his own data, but from the work of those before him. His formulation of the Laws of Motion was induced from Galileo's extensive experimentation. Hence we credit the upper right loop in Newton's Research Schema to Galileo. Newton's Laws of Motion have been stated as follows:

1. Every body will continue in its state of rest or uniform motion in a straight line unless it is compelled to change that state by impressed force.
2. The rate of change of momentum is proportional to the impressed force and takes place in the line in which the force acts.
3. For every action, there is an equal and opposite reaction.

Newton left the empirical world and entered the abstract by expressing his laws symbolically as equations. His work with these resulted in the discovery of both differential and integral calculus. Others independently discovered these concepts, but it is believed that only Newton and Leibnitz (who discovered calculus independently) realized that differentiation and integration were inverse processes.

Calculus did not become mathematically precise until the next century when Cauchy introduced the necessary concepts of limit and infinite sequence. We draw a loop in the lower left quadrant of Figure 6 to represent Cauchy's work in the foundations of calculus.

This new branch of mathematics readily produced an abundant supply of theorems. The predictions which resulted were tested in the laboratory, and found to be entirely correct within the range of current measuring instruments.

Einstein: Eventually, more accurate measuring devices revealed that Newton's Laws of Motion could not explain the behavior of light on either the microscopic or astronomical level. Furthermore, the Michelson-Morley experiment proved conclusively that "ether" did

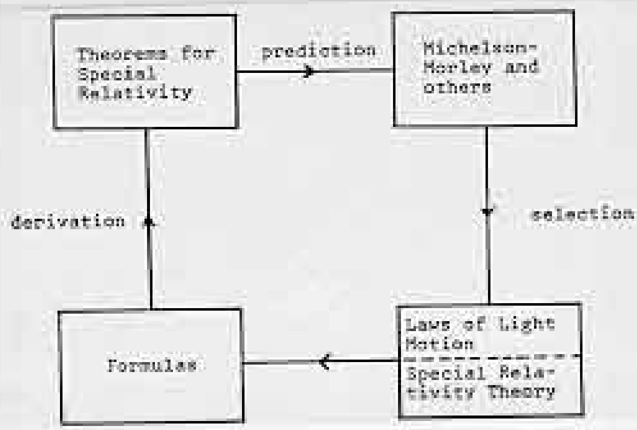

Figure 7 — Einstein's Research Schema. [Draw four rectangles. Label upper left: "Theorems for Special Relativity"; upper right – "Michelson-Morley and others"; lower left – "Formulas"; and, lower right is split (by a dashed line) - top half "Laws of Light Motion," bottom half "Special Relativity Theory." Arrows from upper left to upper right – "prediction"; from upper right to lower right – "selection"; from lower right to lower left; from lower left to upper left – "derivation." Ed.]

not exist. These discoveries led to a period of great activity in physics pioneered by Albert Einstein.

Like Newton, Einstein's major work resulted from data collected by scientists before him. Einstein was a purely theoretical physicist, and never worked in the upper right quadrant of the Research Schema himself. But he certainly stimulated an enormous number of experiments there. He proposed the following empirical axiom system as laws of light motion:

1. No physical object can travel faster than the speed of light.
2. The speed of light depends not at all on the relative positions of the source of light and the observer, or their relative speeds.
3. The mass at a velocity v of a particle equals its mass at velocity 0 divided by $\sqrt{1 - v^2/c^2}$, where c is the speed of light.

Einstein abstracted these three laws to an axiom system, from which he derived the body of theorems interpreted as the theory of special relativity. He found that in particular, his distance formulas for relativity theory were related to those of hyperbolic non-euclidean geometry; thus relativity theory provides a physical model for hyperbolic geometry. The Research Schema for this discovery is shown in Figure 7. We begin in the upper right with the Michelson-Morley experiment, and then go to the Laws of Motion of Light in the lower right, and their abstractions in the lower left. From there we go to the theorems of special relativity in the upper left, and finally to the experimental verification in the upper right where this cycle started. Einstein then went around this cycle again with his more refined theory of general relativity, which led to more precise predictions of physical measurements.

Darwin: Charles Darwin spent most of his life doing research in only one quadrant of the Research Schema, the upper right. His research career began when he became the official naturalist on the good ship Beagle, and embarked upon a five-year voyage. He made observations on all species of animals he could find, and took voluminous notes. During the remainder of his life, Darwin analyzed and classified these notes and all other available

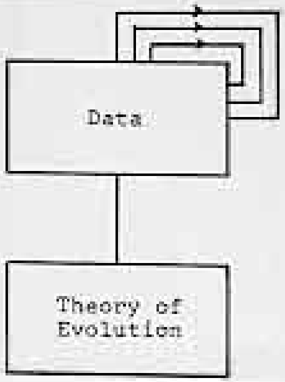

Figure 8 — Darwin's Research Schema. [Draw two rectangles, one above and one below. The top one is labelled "Data"; the bottom one is labelled "Theory of Evolution." There are three loops attached to the top one. There is a line linking the two rectangles. Ed.]

information. The climax of his work was the formulation of his Law of Natural Selection and his Theory of Evolution.

Darwin's theory asserts that all animal species have descended from a common origin. The variety of species results from "natural selection," in which those animals which are best adapted to their environment survive. Due to occasional mutations, certain animals in a species are better able to survive than others. These mutations may be passed on to their offspring who in turn will tend to survive and reproduce, eventually resulting in a new species which has been naturally selected.

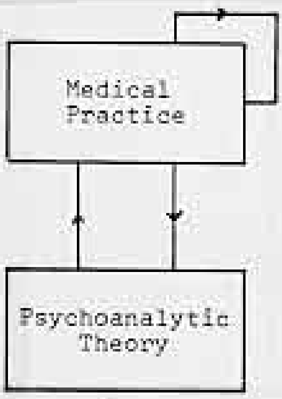

Figure 9 — Freud's Research Schema. [Draw two rectangles, one above and one below. Label the top one "Medical Practice." Label the bottom one "Psychoanalytic Theory." Join the two rectangles with an up arrow and a down arrow. There is a loop attached to the top rectangle. Ed.]

Freud: Sigmund Freud, like Darwin, stayed in the empirical world. In fact, their Research Schemata are quite alike, as seen in Figures 8 and 9. He began with a medical degree and turned from general practice to specialization. Freud (in collaboration with J. Breuer initially) did research in the treatment of "hysterical" patients who had physical symptoms for which no physical cause could be found. He inferred from the study of many cases that the symptoms could be traced back to some repressed childhood trauma, and went on to develop the concept of the subconscious together with the id, ego, and superego. First through hypnosis, and later through "free association," Freud was able to induce himself and his patients to recall these forgotten experiences, and alleviate their symptoms.

Much of the psychoanalytic theory which Freud developed is still highly controversial today, although it has made a lasting impact on the development of many modern theories in psychology.

There has been a highly publicized report of the proof of a deep and important theorem by a mathematician while boarding a bus in Paris. It may be just as true as the anecdote about Newton's finding his law of gravitational attraction when an apple fell off its tree and landed on his head. This sort of phenomenon does occur, but fortunately is not an intrinsic part of the discovery procedure. In the words of Hans Zinsser,

> It is an erroneous impression, fostered by sensational popular biography, that scientific discovery is often made by inspiration .... This is rarely the case. Even Archimedes' sudden inspiration in the bathtub; Descartes' geometrical discoveries in his bed; Darwin's flash of lucidity on reading a passage in Malthus; Kekule's vision of the closed carbon ring came to him on top of a London bus; and Einstein's brilliant solution of the Michelson puzzle in the patent office in Bern, were not messages out of the blue. They were the final co-ordinations, by minds of genius, of innumerable accumulated facts and impressions which lesser men could grasp only in their uncorrelated isolation, which — by them — were seen in entirety and integrated into general principles. The scientist takes off from the manifold observations of predecessors, and shows his intelligence, if any, by his ability to discriminate between the important and the negligible, by selecting here and there the significant steppingstones that will lead across the difficulties to new understanding. The one who places the last stone and steps across to the *terra firma* of accomplished discovery gets all the credit. Only the initiated know and honor those whose patient integrity and devotion to exact observation have made the last step possible.

When a researcher has become sufficiently steeped in his problem, he has amassed enough meaningful data (mathematicians also accumulate data via "thought-experiments") to perceive the proper pattern and conceive the correct conjecture. This is a necessary but not sufficient step toward establishing a theorem. A proof, which is valid, must be supplied; otherwise, the assertion remains a conjecture. The two talents of conjecture and proof appear to be quite separable.

7. What Should They Be?

It is becoming more fashionable to use mathematical models as a powerful analytic device for advancing scientific research in a remarkable variety of disciplines. This usage is certainly not unwarranted, since models, when used with care and discretion, can and should be of great value in the clarification of existing problems and the formulation of important new ones. Unfortunately, it seems that models are misused all too often. The word 'model' is sometimes bandied about by people with little conception of its real meaning simply because it is *au courant*. They don't even define 'model', but use the word to suit their own purposes.

A model need not be impressively confusing in order to be valuable. In fact, one of the main contributions of a model lies in its ability to simplify a problem, and so it should be no more complicated than necessary.

Neither should a model be symbol-rich but idea-poor. Models which hide miniscule content behind a mass of symbolic formulas tend to look impressive, but add nothing. "Mystery is no criterion of knowledge." For example, a recent paper in a leading psychological journal had only one abstract idea: the number of elements in the union of two sets is the sum of the number of elements in each minus the number they have in common. Alas, the author apparently did not recognize it as the simplest special case of the Principle of Inclusion and Exclusion.

Another unfortunate use of mathematical models occurred in a published paper in sociology in which there were ten axioms and zero theorems. However, an interpretation was then given which resulted in ten "empirical theorems," one for each axiom. This 1-1 correspondence between axioms and empirical theorems simply involves the preparation of axioms which will yield desired empirical assertions.

Furthermore, an axiom system should not be constructed for the artificial purpose of deriving just one theorem which has already been verified statistically. Clearly such a model only clutters the literature and does not involve genuine derivation.

We do not wish to lay all the blame for the misuse of mathematical models on scholars in the empirical world; it occurs in the abstract world as well. The following passage by the eminent linguist Gustave Herdan [2] shows the dual roles the two worlds can play in the misuse of models.

> Without going into details, I will only mention a certain quantitative relation known to linguists as the 'Zipf law'. Mathematicians believe in it as a law, because they think that linguists have established it as a relation of linguistic facts, and linguists believe in it because they, on their part, think that mathematicians have established it to be a mathematical law. As can be shown in five minutes, it is not a law at all in the sense in which we speak of natural laws.

Loosely stated, this law of Zipf proposes a high correlation between the frequency of use of words and their brevity.

Another typical superficial use of mathematical models involves the bland assumption that the most elementary parts of an existing branch of mathematics apply unchanged to a problem in social science. Typical examples include high school algebra, coordinate geometry, matrix manipulation, graph theory, and the probabilistic theory of Markov chains. In such models, the typical procedure is to assign empirical terms to the mathematical variables by way of interpretation at the lower level. Then the existing theorems and methods of calculation are translated at the upper level into statements which are claimed to be new empirical findings.

What, then, should mathematical models be? We have suggested that they should lead to new theorems, but this is not always necessary. The precise thinking involved in the careful formulation of an axiom system will lead to an improved conceptualization of the empirical phenomena at hand. This in turn can suggest the proper variables to measure, and perhaps an approach to the measurement problem.

Sometimes an existing area of mathematics can be quite useful as a mathematical model provided it is augmented by one or more new axioms suggested by the real world. The most productive models, however, have involved derivation. For it is only after the derivation of new theorems that unexpected and far-reaching predictions can be made. From a mathematician's viewpoint, it is best if derivation leads to nontrivial theorems, which actually qualify for publication in the mathematical literature. To summarize, it is our personal and perhaps controversial contention that mathematical models will lead to significant and natural growth in both the abstract and empirical worlds.

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