SUM GRAPHS AND GEOGRAPHIC INFORMATION

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Abstract

We examine a new graph theoretic concept called a "sum graph," display a new sum graph construction, and prove a new theorem about sum graphs (sum graph unification theorem) verifying the construction. The sum graph is then generalized, ultimately as an augmented reversed logarithmic sum graph, so that it is useful in dealing with large sets of geographic information. The generalized form permits 1) the compression of large data sets, and 2) the simultaneous consideration of data sets at various levels of resolution.

The advantages of employing sum graph unification and the augmented reversed logarithmic sum graph to handle data sets are illustrated by hypothetical example; as a data structure, the various forms of sum graph data management provide compact handling of data and do so in a manner that permits variability of resolution, at multiple levels (unlike the quadtree), within a single layer of mathematical manipulation.

Our interest in creating, and exploring, this sort of data structure rests in searching for structures that are translation invariant. Data structures resting on geographic direction, such as the quadtree, seem destined not to be translation invariant; structures that are not tied to the ordering of the space in which they are embedded, but only to an ordering within the structure itself, have the potential to be translation invariant.

Geography and graph theory have a long history of interaction: the Four Color Problem (now Theorem) and the Königsberg Bridge Problem of graph theory arose as geographical questions. As geography has stimulated mathematical creation, so too has the body of theory developed by graph theorists stimulated careful analysis of various geographical networks. It is within this well-established spirit of interaction, and within the technological framework where electronic processing of data may be characterized using graph theory, that we examine a new graph theoretic concept, called a sum graph, as a theoretical data structure.

In this structure, the numerical pattern of the labels of the nodes in the "sum graph" will be dictated by the linkage pattern in the underlying data, rather than the other way around, which is more conventional. Thus, data that are "linear" (sequential), such as data streams in a raster mode, will be represented by a sum graph whose linkage pattern is linear, thereby forcing a certain style of label to be present on the associated nodes. We demonstrate the theoretical concepts in this paper using examples limited to the linear case because it is easy to express and because it has wide applicability.

Thus, the first section introduces the reader to elements of the abstract development of sum graphs, focusing only on those concepts that will actually be applied. The second section shows how to force "correct" labelling of "sum graphs" to permit the simultaneous consideration of data at multiple levels of resolution within subsets of a data set that is linear in character. The third section introduces the concept of "logarithmic sum graph," used to compress large data sets into subsets within bands of width of one unit — a critical strategy as the length of the linear sequence (data stream) increases. The fourth section introduces the "reversed sum graph" which also permits simultaneous consideration of data at more than one scale and does so with optimal labelling within bands of one unit. The fifth section introduces the "augmented reversed logarithmic sum graph," a graph combining the desirable elements of previously considered structures augmented by a set of linkages, induced by the numerical labelling of subsets, that permits inclusion of data at variable levels of resolution and offers a means to link that data between, in addition to within, subsets. Throughout, we show how to use these concepts in a small application derived from a set of data concerning North American cities.

1. Sum Graphs

Definition 1 (Harary, 1989)

Let S be a set of n distinct positive integers. Define *the sum graph* $G^+(S)$ as follows:

1. $G^+(S)$ has n nodes, each labelled with a different element (number) of S;
2. there is an edge between two nodes labelled a and b if and only if $a + b \in S$.

Example 1

Figure 1 shows the sum graph of $S_1 = \{1, 4, 5, 7, 8, 9\}$. S_1 is a set of arbitrarily chosen labels for the nodes. Because the label "9" is an element of S_1, it follows that the edge linking 4 and 5 ($4 + 5 = 9$) is present in the graph. Because the label "6" is *not* an element of S_1 there is an edge linking 1 and 5 ($1 + 5 = 6$). A number of theorems concerning sum graphs appear in the mathematics literature (Harary, 1990; Bergstrand *et al.*, 1990, 1991). We state those results without proof; others wishing to employ these methods should read with understanding the proofs in the mathematics literature, lest the methods be inappropriately applied in different situations. First note that the largest number in S cannot be the label of a node joined to any other node.

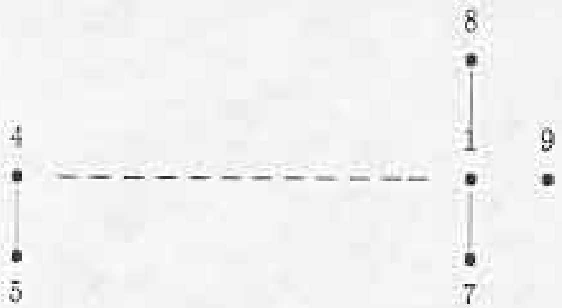

Figure 1.
$G^+(S_1)$: the sum graph of $\{1, 4, 5, 7, 8, 9\}$.
Reader is to solidify any dashed lines with a pencil

Lemma 1 (Harary, 1990)

Every sum graph contains at least one isolated node.

Example 2:

The sum graph of $S_2 = \{2, 3, 5, 6, 10\}$ is displayed in Figure 2.

Figure 2.
$G^+(S_2)$: the sum graph of $\{2, 3, 5, 6, 10\}$.

Lemma 1 assures that the node labelled 10 is isolated. Example 2 illustrates that more than one isolated node is possible; hence, the phrase "at least" in the statement of Lemma 1.

Definition 2 (Harary, 1969)

Two graphs G_1 and G_2 are *isomorphic* is there is a one-to-one correspondence f between their node sets such that, for any two nodes a and b in G_1, (a, b) is an edge in G_1 if and only if $(f(a), F(b))$ is an edge in G_2. Thus two graphs are isomorphic not only when they look the same, but perhaps have different labellings of the nodes, but they are also isomorphic when the graphs do not look alike but have the same connection pattern — as are views of the same digital terrain model from different vantage points. Figure 3 illustrates this phenomenon for the graph of the octahedron. Isomorphic structures are invariant under geometric translation.

Notation Given a set S of positive integers, write $kS = \{kx : x \in S\}$.

Theorem 1 (Harary, 1990)

If $G^+(S)$ is a sum graph and $S' = kS$, k a positive integer, then $G^+(S)$ and $G^+(S')$ are isomorphic.

Example 3

Consider the sum graph of Example 1, $G^+(S_1)$ with $S_1 = \{1, 4, 5, 7, 8, 9\}$. When $k = 3$, we have $S_1 = \{3, 12, 15, 21, 24, 27\}$. The distributive law of algebra guarantees that exactly the same edges will appear in $G^+(S_1')$ as in $G^+(S_1)$. For example, because $5 \in S_1$, 1 and 4 are adjacent in $G^+(S_1)$; because $3 \cdot 5 \in S_1'$, $3 \cdot 1$ and $3 \cdot 4$ are adjacent in $G^+(S_1')$, since $3{\cdot}1 {-} 3{\cdot}4 = 3{\cdot}(1{+}4)$. Thus, $G^+(S_1)$ and $G^+(S_1')$ have the same edge structure (but different node labellings, hence, perhaps, different geographic positions), so they are isomorphic.

One interesting structure a sum graph might have is a graph-theoretic path (Harary, 1969).

Definition 3 (Niven and Zuckerman, 1960)

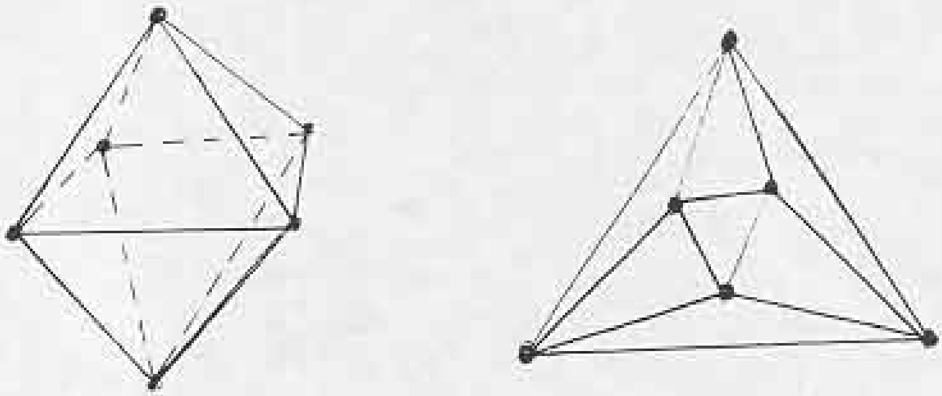

Figure 3.
The octahedron in two different views (View A on the left; View B on the right)
The reader should draw it

The sequence of Fibonacci numbers F_n is defined as follows: $F_1 = 1$, $F_2 = 2$, $F_n = F_{n-2} + F_{n-1}$ when $n \geq 2$. For example, the first nine elements of this sequence are 1, 2, 3, 5, 8, 13, 21, 34, 55.

Theorem 2 (Harary, 1990)

If $S = \{F_1, F_2, \ldots, F_p\}$ is the set consisting of the first p Fibonacci numbers, then $G^{+}(S)$ consists of a path connecting F_1 and F_{p-1} and the isolated node F_p.

Example 4

Let $S_3 = \{1, 2, 3, 5, 8, 13, 21, 34, 55\}$. Then $G^{+}(S_3)$ is the graph of Figure 4.

2. Sum Graph Unification: Construction

One of the characteristics that distinguishes a sum graph from other graphs is that the algebraic rule assigning edges forces the sum graph to have at least one isolated node. Thus, in aligning this graph-theoretic concept with geographic notions, one might, at the outset, be tempted to look for applications that require "isolating" one geographic location from a set of others, as in site-selection for toxic waste sites, for prisons, or for other similar societally-obnoxious facilities.

Further reflection suggests, however, that the power behind this "isolation" might be best exploited by considering the isolated node as one with linkages not visible at the graph-scale shown, much as inset maps generally do not reveal linkages to the larger-scale maps they modify. Thus, this cartographic conception of the isolated node as a node with invisible edges will provide a systematic method for shifting scale, or varying resolution, without disturbing the associated spatial structure. The isolated node acts as a "cataloging" node functioning at a scale different from the content it catalogues (the term "isolated" will therefore be reserved for the graph-theoretic case; when viewed in a geographic context, the "isolated" node will be referred to as a "cataloging" node to emphasize this role).

Consider three disjoint sets of nodes, A, B, and C, with a linear linkage pattern joining each (Figure 5). The linear linkage pattern of each path is based on some serial arrangement of data, such as data ordered by longitude from east to west.

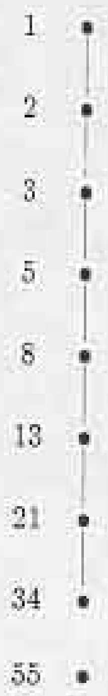

Figure 4.
$G^+(S_3)$: A Fibonacci sum graph containing a path and an isolate

It is not difficult to obtain the paths, P_3, P_4, P_5 of Figure 5 as three distinct sum graphs using Theorem 2. Fibonacci labelling of the nodes of Figure 5, shown in Figure 6, generates (as sum graphs) exactly the path-patterns of Figure 5; e.g., the edge joining 2 to 3 in A is present because $2 + 3 = 5$ is also a node label. An additional node, a cataloging one, is necessarily introduced in each sum-graph, A, B, and C of Figure 6. When the label of a cataloging node is used as a label for an entire configuration, this sum graph represents not only the linear linkage within the path, but also, at the same time, represents information (as a label) for the entire path. Information at different cartographic scales is displayed simultaneously.

In Figure 6, the simple Fibonacci labelling scheme of Theorem 2 produced three distinct sum graphs. Because the same labels are re-used, it would not be possible to compare information concerning these distinct sum graphs. Stronger theoretical results follow: results that will permit such comparison, while retaining the desirable asset of simultaneous display of data at different cartographic scales.

Consider, as a whole, the set of twelve nodes from Figure 5. Find a strategy for labelling these nodes that will produce exactly the three paths of Figure 5 as subgraphs of a single sum graph. Viewing the three parts of Figure 5 as subgraphs of a *single* sum graph will guarantee distinct labels for distinct nodes while retaining scale-shift characteristics.

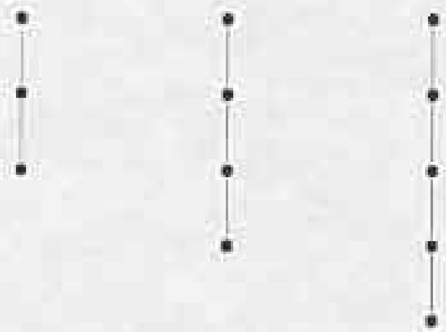

Figure 5.
Three graphs, Left, Middle, and Right, representing serial linkage of data.

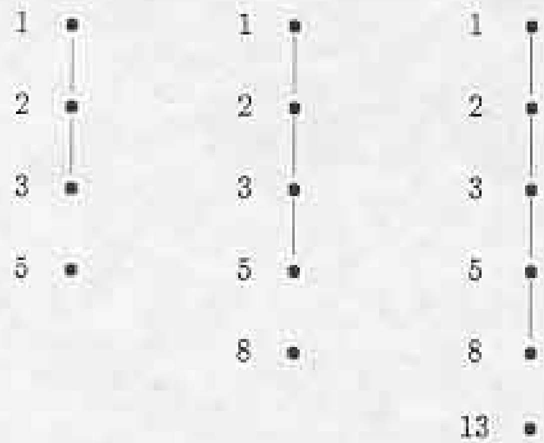

Figure 6.
The three distinct Fibonacci sum graphs showing the paths
P_3 (on the left), P_4 (middle), and P_5 (right).

One way to achieve such a labelling is as follows. Assign Fibonacci numbers consecutively (starting with 1) to the nodes of one subgraph (A, in Figure 7). Continue this scheme to a node of subgraph B; thus, in Figure 7, A has nodes with labels 1, 2, 3 and one node in B has label 5. It might be natural to label the next node in B with the next Fibonacci number — 8. However, this would introduce an unwanted edge between 3 and 5. So, label the next node with one more than the next Fibonacci number — in this case 9 — to remove the possibility of introducing unwanted edges. Label the remaining nodes in the Fibonacci-style with 5 and 9 as the first two elements. Continue this scheme through to one node of subgraph C (labels 14, 23, and 37 are thus introduced). The second node in the third subgraph must not be labelled 60, or else an unwanted edge is introduced linking 23 to 37. Call the label of the second node "61". Continue labelling in the Fibonacci style using 37 and 61 as the first two elements of a Fibonacci-style label-generating scheme. In the case of Figure 7, all nodes are now labelled; a single extra node, which is a cataloging one, is also labelled. All paths of this single sum graph are exactly those desired. The label associated with the cataloging node, 416, is the catalogue number for the entire configuration; other labels describe the local, linear linkage patterns. Distinct labels correspond to distinct nodes in such a way that only desired paths are introduced between nodes. A single added cataloging node permits associating information with a label for this node at the scale of the entire configuration — in the manner of object-oriented data structures.

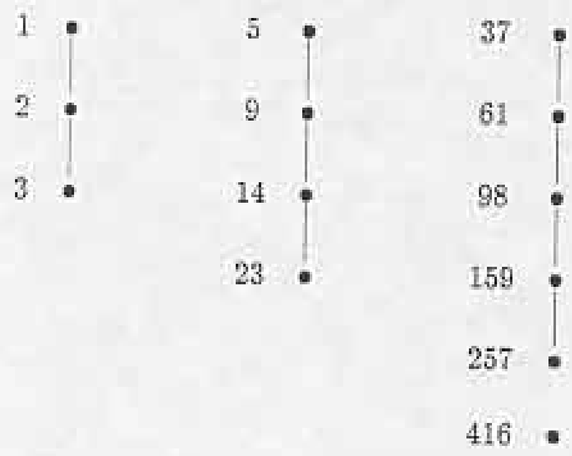

Figure 7.
A Fibonacci-style of labelling for a sum graph with one cataloging node (416) showing the paths P_3 (on the left), P_4 (middle), and P_5 (right) as subgraphs.

Thus, two levels of variability in resolution are displayed — that of the linkage pattern within individual subgraphs, and that of the weight of the entire graph, reflecting to some extent on the size of the data set, and the style of its subgraphs and their pattern of internal connection (had the subgraph in the middle terminated at 14, the subgraph on the right (with an added edge) would have begun with 23 and had an isolated node with label 419).

Stronger yet would be to construct a single sum graph from which desired paths would emerge (as in Figure 7) and in which distinct paths would correspond to distinctly-labelled cataloging nodes as in Figure 6. The notion of wanting one cataloging node per desired path, in order to ensure greater variability in resolution, motivates the following definition.

Definition 4

Suppose a set of n nodes is partitioned into t subsets. Further suppose k of these subsets contain more than one node. To each of these k subsets add a node. The resulting t subsets will be called "constellations" (Figure 8).

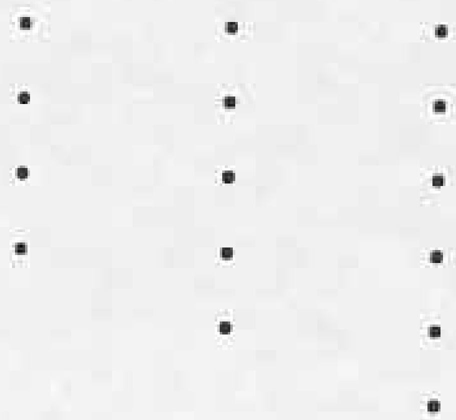

Figure 8.
Three constellations, Left, Middle, and Right, partition a distribution of nodes.

Now we return to the example of Figure 5, with three nodes added to make three constellations (all with more than one node, as in Figure 6). We seek some labelling for the entire set of constellation nodes (Figure 8), as nodes of a single sum graph, that will

1. produce the paths P_3, P_4, P_5;
2. produce cataloging nodes within the subgraphs containing P_3, P_4, P_5;
3. make retrieval of path structure simple.

Because there are paths that are to be retrieved as subgraphs of a single sum graph, some sort of Fibonacci or Fibonacci-style labelling will be needed (Theorem 2). The labels from Figure 5 cannot be chosen because under that circumstance distinct nodes do not have distinct labels. Theorem 1 suggests that distinctness in labelling as well as retention of path structure is achieved by multiplying Fibonacci numbers by constants. Thus, the issue is to know what values to choose as these "multipliers" so that distinctness of node labels (required by Definition 1) is ensured. Example 5, below, suggests a general construction that will satisfy these conditions. It will be proved in full generality in Theorem 3.

Example 5

1. To ensure path structure, give the underlying Fibonacci label pattern of 1,2,3,5; 1,2,3,5,8; 1,2,3,5,8,13 to, respectively, the left, middle, and right constellations (Definition 4) in the node pattern of Figure 8. To produce a set of suitable multipliers for these nodes, proceed to step 2.

2. Choose the smallest prime number greater than the sum of the largest and next largest

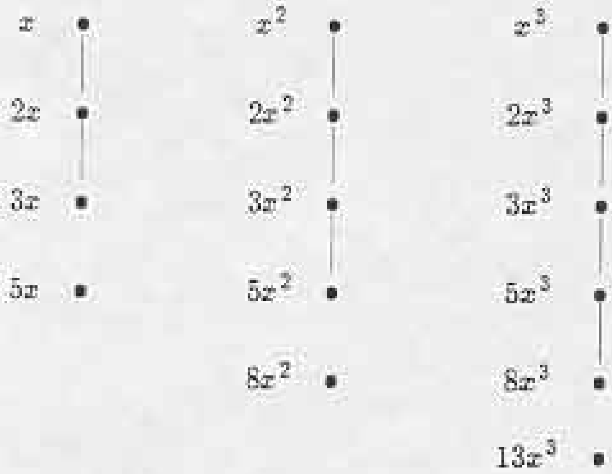

Figure 9.
Sum graph derived from Figure 6 using the base multiplier and its powers, writing $x = 23$ for brevity.

numbers used in the underlying Fibonacci pattern. In this case, 13 is the largest number in the underlying Fibonacci pattern and 8 is the next largest, so choose 23, the smallest prime number larger than 13+8=21 (choosing 21 would introduce an unwanted edge). This number will be the multiplier for one constellation (in this case, we arbitrarily choose to use it for the left-hand constellation).

3. Use successive powers of 23 (23 functions therefore as a base-multiplier) to label the nodes of successive constellations. In this case, 23^2 is used as the multiplier for the right-hand constellation. The nodes are now labelled as shown in Figure 9.

When this set of nodes is used as the set S of Definition 1, the resulting sum graph is isomorphic to the union of the three sum graphs in Figure 5. The fact that three cataloging nodes are introduced by this procedure gives an indication from each coefficient of the cataloging nodes of size, shape, and connection pattern of the subgraph it represents (as did the single cataloging node of 416 for the entire graph in Figure 7). The set of steps in Example 5 may be stated more generally as in the Construction below.

3. Cartographic Application of Sum Graph Unification

The following application will show how the labelling produced by the Sum Graph Unification Construction might be used. Consider a set of seven North American cities together with selected suburbs of those cities (Table 1.1). Column 1 in Table 1.1 lists these cities and suburbs in seven groups as metropolitan areas (the latter named in all upper case letters): constellations. To consider the east-west extent a proposed metropolitan mass transit system might need to cover, the longitude is also associated with each location (in column 2 of Table 1.1). The sequential ordering of cities and suburbs, by longitude from east to west, describes a path within each constellation linking these nodes. The metro area node is a cataloging

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Construction: Sum Graph Unification

Given a set of nodes partitioned into constellations. To ensure a prescribed path structure linking the nodes, that can be retrieved electronically entirely (only) from the numerical characteristics of the labels for the nodes, assign labels in the following manner.

1. Label the nodes of each constellation with Fibonacci numbers, in order, beginning with the label "1" in each constellation.

2. Find a base multiplier for each Fibonacci label. Form the sum of the two largest labels from step 1. The smallest prime number greater than this sum will serve as a multiplier. Use this prime base multiplier as the multiplier for labels of the nodes in one constellation.

3. Use successive powers of the prime in step 2 as multipliers for labels of the nodes in successive constellations.

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node not hooked into the path. Column 3 associates a Fibonacci number with each node of the entire distribution of nodes (step 1 in the Construction). Column 4 shows weights for the nodes by constellation; 37 is the base multiplier because it is the smallest prime greater than 21+13 (steps 2 and 3 in the Construction). Column 5 shows the product of columns 3 and 4; distinct nodes have distinct labels.

Suppose the entire list is rearranged by longitude, independent of constellation: positions of data within all but the New Orleans constellation remain the same. In the New Orleans constellation, the suburb of Metairie is shifted from the New Orleans constellation to the St. Louis constellation (between E. St. Louis and Lemay). That Metairie jumps metropolitan area is evident from the factored weight associated with it: it belongs to constellation 7, that of New Orleans, as its exponent in the factored weight shows (Table 1.2). Thus, the sum graph node label shows that it is out of regional order and provides a direct means to re-sort it back into regional order. Rank-ordering or other conventional means would not do so; rank ordering does not show which city belongs in which constellation. These sum graph node labels offer a way to organize data and to retrieve predetermined sequential order of information from a jumbled data set. The node labels are somewhat large in magnitude, but that is irrelevant in this particular application. It may be important in others, and thus it is to this issue and to the related one of data compression that the remainder of the material is directed.

4. Sum Graph Unification: Theory

The example above may prove a useful source of mental reference points on which to base the formal proof of the following lemmas needed to probe Theorem 3 below. The first Lemma will prove that there are no unwanted edges linking nodes within constellations and the second one will prove that there are no edges linking nodes between constellations.

For the most part, Theorem 3 is just a formalization of the method developed in the example based on Figure 9. However, additional details are necessary to allow for constellations of a single node (in these cases no new node is added). One might interpret such a node as a small city with no suburbs. (Readers wishing to examine the rigor of this method should read Theorem 3 and associated material with care; others might wish to skip to the next section.)

Lemma 3a

Let a, b, c, i, j be positive integers. If $p > a + b$, and $p > c$, it is impossible for $a \cdot p^i + b \cdot p^i = c \cdot p^j$ if $j \neq i$.

Proof

Note that $a \cdot p^i + b \cdot p^i = (a+b)p^i < p^{i+1} \leq c \cdot p^j$ if $j > i$. Similarly, if $j < i$, $c \cdot p^j < p^i < a \cdot p^i + b \cdot p^i$. Thus, in either case, the equation of the lemma is impossible.

NOTE: We will want to choose p greater than the sum of the largest two occurring Fibonacci numbers. For example, suppose 21 is the largest occurring Fibonacci number. Then $21 \cdot 23^a + 2 \cdot 23^a = 23^{a+1}$, so using 23 as the base multiplier would introduce an edge between the nodes $21 \cdot 23^a$ and $2 \cdot 23^a$.

Lemma 3b

Let a, b, c be positive integers, $p > a + b$. Let x, y, z be positive integers, $x \neq y$. Then $a \cdot p^x + b \cdot p^y = c \cdot p^z$ is impossible.

Proof:

Without loss of generality, assume $x < y$. Then, $p^y < a \cdot p^x + b \cdot p^y < (a+b)p^y < p^{y+1}$. Thus, for the equation to be possible, $z = y$. But then $a \cdot p^x \equiv 0 (\text{mod} p)$, which is impossible, since $ap^x < p^{x+1} \leq p^y$.

We now formalize the ideas exhibited in the construction of Example 3.

Definition 5 (Harary, 1970)

A linear tree is a path. A linear forest is a union of disjoint linear trees.

Theorem 3 (Fibonacci sum graph unification)

Suppose we are given a set of n nodes, which are partitioned into t subsets, k of which contain more than a single node. Then there is a set S of $n+k$ suitably chosen positive integers whose sum graph $G^+(S)$ consists of t isolates (k additional nodes and $t-k$ nodes from single-node subsets) together with a linear forest of k nontrivial paths.

Proof:

Suppose that the n original nodes are $a_1, a_2, \ldots, a_n$. Divide these into the t desired subsets

$$\{x_{11}, x_{12}, \ldots x_{1n_1}\}$$
$$\{x_{21}, x_{22}, \ldots x_{2n_2}\}$$
$$\cdots$$
$$\{x_{t1}, x_{t2}, \ldots x_{tn_t}\}$$

where $n_1 + n_2 + \cdots + n_t = n$. Let $N = 2 + \max\{n_1, n_2, \ldots, n_t\}$. Let p be the smallest prime greater than F_N, the Nth Fibonacci number. Now label $n+k$ nodes as follows:

1. If $n_i = 1$, label x_{i1} with p^i (subsets with exactly one node).
2. If $n_i \neq 1$, label x_{i1} with p^i, x_{i2} with $2p^i, \ldots x_{in_i}$ with $p^i F_{n_i}$, and a new node y_i with $p^i F_{(1+n_i)}$ (subsets with more than one node).

Follow this procedure for all i, $1 \leq i \leq t$. Let S consist of the original nodes together with the new y_is. Now consider constellations consisting of the nodes labelled x_i if $i = 1$ and the nodes $\{x_{i1}, \ldots, x_{in}, y_i\}$ is $i \neq 1$. Then Theorems 1 and 2 assure that there are Fibonacci paths $x_{i1}, x_{i2}, \ldots x_{in}$ and that y_i is not adjacent to x_{ia} for any a $(1 \leq a \leq n_i)$. Lemma 3a assures that there are no edges within a constellation other than the Fibonacci path. Lemma 3b assures that there are no edges between constellations. Thus, the theorem is proved.

5. Logarithmic Sum Graphs

The procedure displayed in the Construction, and proved in Theorem 3, meets the criteria of producing desired paths, from the labelling scheme alone, each with a corresponding cataloging node, as subgraphs of a single sum graph. In cases based on large data sets, the multipliers get very large very quickly. However, if the logarithm (using the base multiplier, x, as the base of the logarithm) of each label is taken, this issue of apparent significance vanishes (Table 2). In the example on which Figure 9 was based, the values of the multipliers transformed by the log base 23 display clearly the constellation structure. The nodes associated with all entries with integral part "1" are grouped in a constellation, all with integral part "2" in another, and all with integral part "3" in yet another. The integral values serve as a data "key" to this data structure. The fractional values are, of course, the same from subset to subset, exhibiting the same underlying Fibonacci linkage pattern from subset to subset. The largest value in each subset is the cataloging node; if other nodes were to be included in, for example, the third constellation, those also would have a logarithmic value greater than 3.8180367 but less than 4. Thus, independent of how many nodes there are in a single constellation, all the logarithmic labels are contained in a band of real numbers one unit wide: 3 is a greatest lower bound (which is attained), and 4 is an upper bound for labels in the third constellation. Further, the logarithmically - transformed labels increase additively: there are only as many different data keys as there are different constellations.

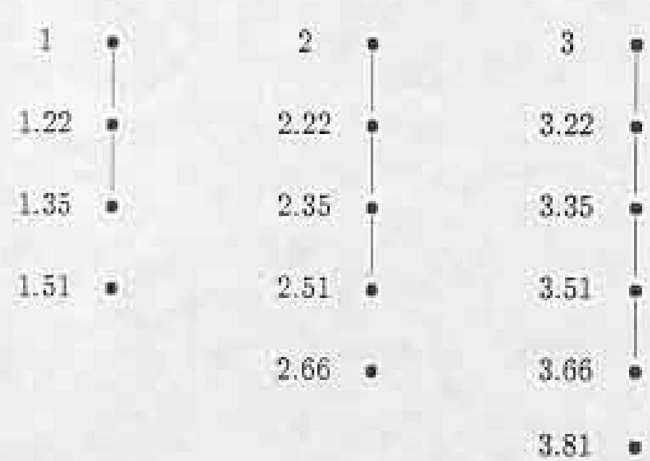

Figure 10.
Logarithmic sum graph

When these logarithmic labels are attached to the nodes of the graph in Figure 9 we refer to the resulting graph as a "logarithmic sum graph" (Figure 10). Note, however, that even though this graph is isomorphic to the sum graph of Figure 9, it is not itself a sum graph (in much the way that a truncated cone is not itself a cone, even though it is derived from a cone).

From a purely theoretical standpoint, it is possible to identify the constellation to which a node belongs very simply from its assigned multiplier. For, if p is the base multiplier, a node whose multiplier is $N = a \cdot p^k$ has $k \le \log_p N \le k+1$, since $a < p$. Thus, a node with multiplier N belongs to constellation k if and only if $[\log_p N] = k$ (where brackets denote the greatest integer function). (From a computer standpoint, one must be careful, since occasionally computational error might make $\log_p p^k < k$ computationally. Adding a suitably small amount to $\log_p N$ before determining its constellation should avert this difficulty.) In fact, it seems easier computationally to store $\log_p N$ rather than N as a multiplier, since then much smaller numbers can be stored. This motivates the following formal characterization of logarithmic sum graphs.

Definition 6

Let S be a set of n distinct positive integers, p a prime. Define the *logarithmic sum graph*, relative to p, $G^+(\log_p S)$ as follows:

1. $G^+(\log_p S)$ has n nodes, labelled with the n different labels $\{\log_p x \mid x \in S\}$.
2. there is an edge between two nodes labelled a and b if $p^a + p^b \in S$.

Logarithmic sum graphs retain all the advantages afforded by Theorem 3, and they make it possible to handle large data sets more easily.

6. Reversed Sum Graphs.

In the procedure of Theorem 3, and in the logarithmic modification of that procedure to accommodate large data sets, the cataloging nodes all have the largest labels within their subgraph. It might be useful, in some situations, for the cataloging nodes to have the smallest labels within their subgraphs. For this purpose, we define the notion of a "reversed" sum graph.

Definition 7

Let S be a set of positive integers such that the sum graph $G^+(S)$ [logarithmic sum graph $G^+(\log_p S)$] is partitioned into constellations such as those of Theorem 3. Define the *reversed sum graph* $^+G(S)$ [*reversed logarithmic sum graph* $^+G(\log_p S)$], isomorphic to $G^+(S)$ [$G^+(\log_p S)$], as follows. If the nodes in a given constellation have labels $a_1 < a_2 < \ldots < a_p$, relabel them $a_p, a_{p-1}, \ldots, a_1$. That is, the node labelled a_i is given the new label a_{p+1-i}. (Note that single-node constellations are not affected.)

Example 6

Let $S_4 = \{1, 2, 3, 5, 8, 13\}$. The graphs $G^+(S)$, $^+G(S)$ are displayed in Figure 11. (As in the case of the logarithmic sum graph, note that a reversed sum graph (Definition 7) is not itself a sum graph.)

As Definition 7 suggests, logarithmic sum graphs may also be reversed. Figure 12 shows the logarithmic sum graph of Figure 10 and its reversed logarithmic sum graph. Reversed sum graphs, logarithmic or not, always assign an integer, the data key, to the cataloging node.

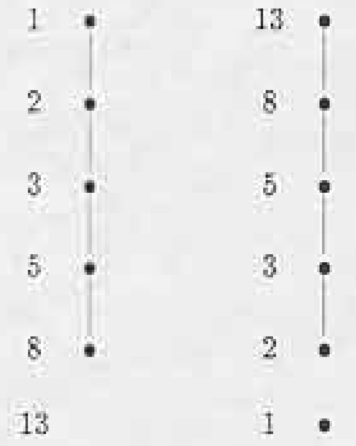

Figure 11.
A Fibonacci sum graph $G^+(S)$ (left)
and its reversed sum graph $^+G(S)$ (right).

This feature is particularly important in the case of the logarithmic representation, when data might be added to or deleted from a single subgraph, all with integral part of their labels identical to that of the cataloging label.

7. Augmented Reversed Logarithmic Sum Graphs

Reversed logarithmic sum graphs single out cataloging nodes as the only nodes with integral labels. It may be useful to consider linkages within the set of cataloging nodes and to "augment" the reversed logarithmic sum graph with edges displaying these linkages (Figure 13).

Definition 8

The *augmented reversed logarithmic sum graph, ARL sum graph,* denoted $^+A(\log_p S)$, consists of the nodes and edges of $^+G(\log_p S)$ together with all edges linking the nodes with integer labels in $^+G(\log_p S)$. Thus, $^+A(\log_p S) = {}^+G(\log_p S) \cup \{$ complete graph on nodes with integer labels in $^+G(\log_p S)\}$.

If m is the number of nodes with integer labels in $^+G(\log_p S)$, this augmentation adds $\binom{m}{2}$ edges to the reversed sum graph $^+G(\log_p S)$. The ARL sum graph is not itself a sum graph.

Augmented reversed logarithmic sum graphs retain all the characteristics of Theorem 3, have the computational advantage of logarithmic sum graphs in handling large data sets, permit the reverse sum graph strategy of integral labelling of the cataloging node, and have the added feature of displaying the complete linkage pattern among cataloging nodes. Linkage patterns emerge both at the local scale and at the more global cataloging scale.

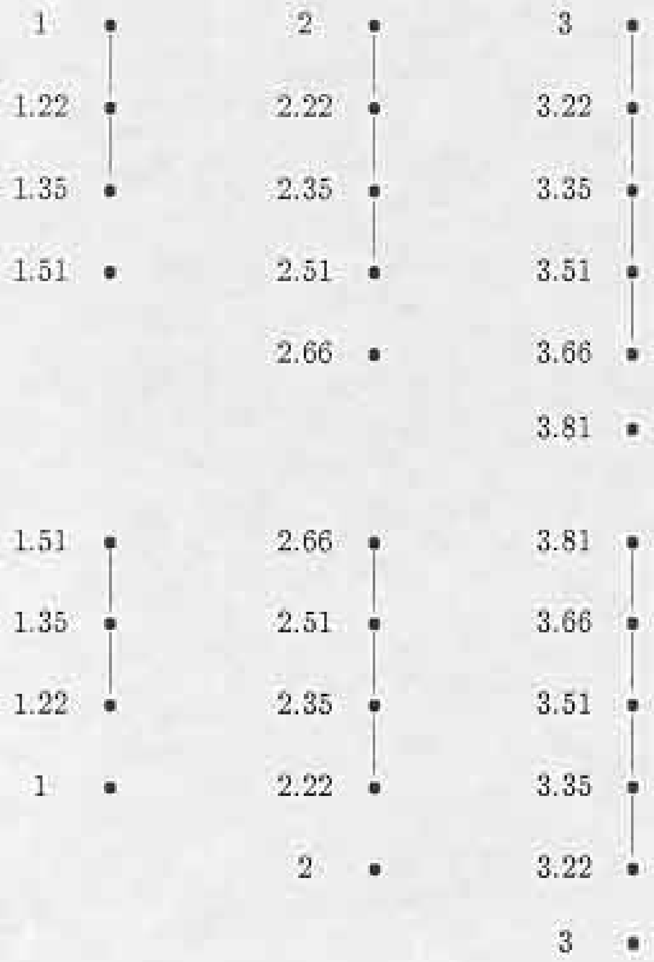

Figure 12.
Logarithmic sum graph (top) and reversed logarithmic sum graph (bottom).

8. Cartographic Application of ARL Sum Graphs

The labels of Table 1.1, derived from the Sum Graph Unification Construction, offer a way to organize data and to retrieve predetermined sequential order of information from a jumbled data set. The relative sizes of the weights for the nodes in Table 1.1 are, however, awkward. A simple way to overcome this awkwardness is to take the logarithm of the node weights (to the base of the base multiplier). Thus, in Table 3, column 6 shows the $\log_{37}$ of each node weight determined in Table 1.1 (listed in column 5 of Table 3). The constellation number is easily read off as the integral part of the logarithm and all entries for a single constellation are contained within a band of values one unit wide. When the labels are reversed, the integral label corresponds to the cataloging node. This reversed logarithmic sum graph (represented by Table 3) retains the favorable characteristics of Table 1.1 for

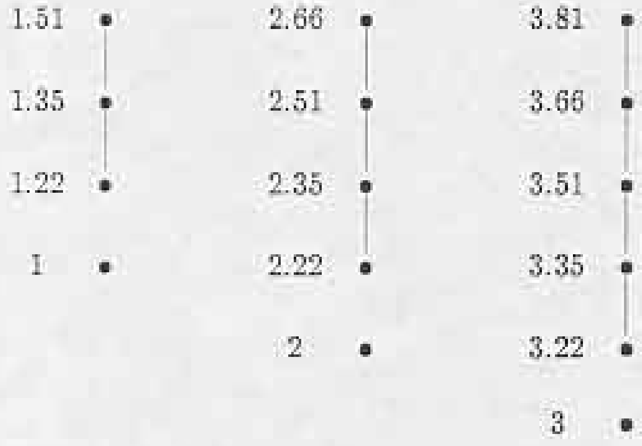

Figure 13.
ARL sum graph derived from a reversed logarithmic sum graph
Reader should draw edges joining nodes 1 and 2, 2 and 3, and 1 and 3

sorting of data; the node labelling scheme of Table 3 is, however, easy to handle.

The augmentation afforded by ARL sum graphs permits significant compression of data, particularly in large data sets, as it retains the favorable characteristics of the reversed logarithmic sum graph noted above. To illustrate this capability, we present the following application.

Consider the set of 39 cities and metropolitan regions labelled in Table 1.1. One set of data that is often stored is distances between places ("distance" is used as an example). Generally this set is stored in a square array, or better, sometimes in an upper- or lower-triangular matrix.

Sum graphs can reduce greatly the number of entries that need to be stored. Table 4.0 shows a complete set of great-circle distances between metropolitan areas. Each metropolitan area is assigned the latitude and longitude of the city for which it is named. Thus, particular sets of geographic coordinates are viewed simultaneously at two different scales. Tables 4.1 to 4.7 show complete sets of great-circle distances among the cities in each of the seven metropolitan areas (constellations).

The distance between Livonia and Scarborough (for example), which does not appear directly in any of the set of Tables in Table 4, may nonetheless be obtained by summing the distances from Livonia to DETROIT, from DETROIT to TORONTO, from TORONTO to Scarborough (Figure 14). The algorithm displayed in Figure 14 shows how to use the reversed logarithmic node label of two arbitrary nodes to determine the distance between them using only the entries in Table 4.0, between metropolitan areas (constellations), and in Tables 4.1-4.7 (showing local linkages within each constellation). The distance so-obtained is not itself a great-circle distance but it may well be a distance more realistically representing current air-travel circumstances.

DETROIT ⟶ TORONTO

| ↓

Livonia ⟶ Scarborough

Figure 14.

Commutative diagram showing distance calculation scheme using Table 4; algorithm showing how to find distance within Table 4 using the data key provided by the reversed logarithmic sum graph label.

Algorithm

1. Assumption: the cataloging city is also the city with the lowest non-integral label in its constellation.
2. Find the distance from a city with a node with reversed logarithmic sum graph label $j.x$ to one with label $k.y$, $j \leq k$ (and $x < y$ if $j = k$)

a. If $j = k$, use Table $4.j$ to find the distance from $j.x$ to $j.y$.

b. If $j < k$,

   i. use Table 4.0 to find distance between cataloging cities j and k.

   ii. use Table $4.j$ to find distance from j.lowest to $j.x$.

   iii. use Table $4.k$ to find distance from k.lowest to $k.y$.

Add the results of i, ii, and iii to find the required distance.

There are 32 different cities in this example. An upper-triangular 32 by 32 matrix of $\binom{32}{2} = 496$ different entries would normally be required to find between-city distances. Using the sum graph method, shown in the algorithm of Figure 14, requires the use of 8 smaller Tables: Table 4.0 for distances between cataloging node cities and Table $4.i$, $1 \leq i \leq 7$, for distances of cities in constellation i from cataloging city i. The latter procedure, composed of smaller matrices, requires storing (from each matrix) a total of

$$\binom{7}{2} + \binom{6}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} + \binom{5}{2} + \binom{3}{2} + \binom{3}{2}$$

$= 21 + 15 + 6 + 10 + 15 + 10 + 3 + 3 = 83$ separate entries. In this case, sum graph methods afford a compression ratio of about 6 to 1 over traditional methods.

With larger data sets, the compression ratio becomes much more substantial. Given a data set of 10,000 entries to be partitioned into 100 constellations of 100 entries each, traditional methods using an upper triangular matrix would require that $\binom{10{,}000}{2} = 49{,}995{,}000$ entries be stored. Sum graph methods would require storing $\binom{100}{2}$ entries for Table 4.0 and $\binom{100}{2}$ entries for each of Tables $4.i$, $1 \leq i \leq 100$, for a total of $101 \cdot \binom{100}{2} = 499{,}950$ entries.

In this case the compression ratio is 100 to 1. If instead the 10,000 entries are partitioned in a different manner, different compression ratios result. If 1000 constellations of 10 entries each are used, the corresponding compression ratio is 91.8 to 1; if 10 constellations of 1000 each are used, the compression ratio is 10.09 to 1. Clearly the manner in which the partition is selected is important. Larger data sets bring even larger compression ratios: if 1,000,000 data points are considered, and are partitioned into 1000 constellations of 1000 each, the corresponding compression ratio is 1000 to 1.

Any process of this sort also needs to accommodate the insertion of new data; when it does so without having to alter existing structure, it is "dynamic." The Sum Graph Unification Construction is dynamic to an extent. Table 5 shows part of the data set of Table 3 with Ann Arbor added to the Detroit metro area. Only the one constellation needs relabelling; all others remain undisturbed. If, however, enough new entries had been added to force an increase in the prime base multiplier, then a global change would have been required for that single entry (generally easy to achieve electronically). None of the formulæ would have required alteration.

"Dynamic" tables of this sort might see application as on-board mapping systems in cars or buses giving optimum route displays in an interactive mode (so-called IVHS or other commonly-used acronyms). So data becomes accurate more quickly in response to changing traffic patterns transmitted to the vehicle in some sort of interactive fashion. Advances in theory can bring advances in technology to the level of affordable cost and widespread application. The application of sum graphs might be one effort in that direction.

9. Summary

We have taken a tool from graph theory and specialized it in a number of directions in order to deal with various types of problems that often arise with data structures. Table 6 organizes these specializations in capsule format. Independent of how the sum graph is specialized to adapt to various difficulties in data management, however, the linkage pattern between nodes in a sum graph is determined by node weight alone, which is derived from whether or not one node is linked to another. There is no reliance on geographic direction or on any sort of other relative ordering based on the underlying space in which the nodes are embedded. Hence, the sum graph data structure has a theoretical base free from directional bias and is perhaps therefore, translation invariant. Determining whether or not this theoretical data structure offers a graphical application at the level of GIS theory–as in the quadtree) that permits translational invariance of the structure (independent of pixel shape) under GIS constraints, appears a significant next step in bringing theory into practice.

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TABLE 1.1:
Analysis according to sum graph unification construction

City Suburb METRO	LONG-ITUDE west	FIBO-NACCI LABEL	BASE MULTI-PLIER	FACTORED WEIGHT	NODE WEIGHT	RANK ORDER
Salem	70 54	1	37	$1 \cdot 37$	37	1
Lynn	70 57	2	37	$2 \cdot 37$	74	2
Quincy	71 00	3	37	$3 \cdot 37$	111	3
Brockton	71 01	5	37	$5 \cdot 37$	185	4
Cambridge	71 07	8	37	$8 \cdot 37$	296	5
Boston	71 07	13	37	$13 \cdot 37$	481	6
BOSTON		21	37	$21 \cdot 37$	777	7
Longueuil	73 30	1	37^2	$1 \cdot 37^2$	1369	8
Verdun	73 34	2	37^2	$2 \cdot 37^2$	2738	9
Montreal	73 35	3	37^2	$3 \cdot 37^2$	4107	10
Laval	73 44	5	37^2	$5 \cdot 37^2$	6845	11
MONTREAL		8	37^2	$8 \cdot 37^2$	10952	12
Camden	75 06	1	37^3	$1 \cdot 37^3$	50653	13
Philadelphia	75 13	2	37^3	$2 \cdot 37^3$	101306	14
Upper Darby	75 16	3	37^3	$3 \cdot 37^3$	151959	15
Norristown	75 21	5	37^3	$5 \cdot 37^3$	253265	16
Chester	75 22	8	37^3	$8 \cdot 37^3$	405224	17
PHILADELPHIA		13	37^3	$13 \cdot 37^3$	658489	18
Scarborough	79 12	1	37^4	$1 \cdot 37^4$	1874161	19
Toronto	79 23	2	37^4	$2 \cdot 37^4$	3738322	20
North York	79 25	3	37^4	$3 \cdot 37^4$	5622483	21
York	79 29	5	37^4	$5 \cdot 37^4$	9370805	22
Etobicoke	79 34	8	37^4	$8 \cdot 37^4$	14993288	23
Mississauga	79 37	13	37^4	$13 \cdot 37^4$	24364093	24
TORONTO		21	37^4	$21 \cdot 37^4$	39357381	25
Windsor	83 00	1	37^5	$1 \cdot 37^5$	69343957	26
Warren	83 03	2	37^5	$2 \cdot 37^5$	138687914	27
Detroit	83 10	3	37^5	$3 \cdot 37^5$	208031871	28
Dearborn	83 15	5	37^5	$5 \cdot 37^5$	346719785	29
Livonia	83 23	8	37^5	$8 \cdot 37^5$	554751656	30
DETROIT		13	37^5	$13 \cdot 37^5$	901471441	31
E. St. L.	90 10	1	37^6	$1 \cdot 37^6$	2565726409	32
St. Louis	90 15	2	37^6	$2 \cdot 37^6$	5131452818	33
Lemay	90 17	3	37^6	$3 \cdot 37^6$	7697179227	34
ST. LOUIS		5	37^6	$5 \cdot 37^6$	12828632045	35
New Orleans	90 05	1	37^7	$1 \cdot 37^7$	94931877133	36
Marrero	90 06	2	37^7	$2 \cdot 37^7$	189863754266	37
Metairie	90 11	3	37^7	$3 \cdot 37^7$	284795631399	38
NEW ORLEANS		5	37^7	$5 \cdot 37^7$	474659385665	39

TABLE 1.2:
Analysis according to sum graph unification construction
Two constellations ordered from east to west by longitude

City Suburb METRO	LONG-ITUDE west	FIBO-NACCI LABEL	BASE MULTI-PLIER	FACTORED WEIGHT	NODE WEIGHT	RANK ORDER
New Orleans	90 05	1	37^7	$1 \cdot 37^7$	94931877133	36
NEW ORLEANS		5	37^7	$5 \cdot 37^7$	474659385665	39
Marrero	90 06	2	37^7	$2 \cdot 37^7$	189863754266	37
E. St. Louis	90 10	1	37^6	$1 \cdot 37^6$	2565726409	32
Metairie	90 11	3	37^7	$3 \cdot 37^7$	284795631399	38
St. Louis	90 15	2	37^6	$2 \cdot 37^6$	5131452818	33
ST. LOUIS		5	37^6	$5 \cdot 37^6$	12828632045	35
Lemay	90 17	3	37^6	$3 \cdot 37^6$	7697179227	34

TABLE 2:
Multipliers and their logarithms to the base
of the base multiplier of 23
for the example of Figure 7.

Multiplier	Logarithm, base 23
$1 \cdot 23 = 23$	1
$2 \cdot 23 = 46$	1.2210647
$3 \cdot 23 = 69$	1.3503793
$5 \cdot 23 = 115$	1.5132964
$1 \cdot 23^2 = 529$	2
$2 \cdot 23^2 = 1058$	2.2210647
$3 \cdot 23^2 = 1587$	2.3503793
$5 \cdot 23^2 = 2645$	2.5132964
$8 \cdot 23^2 = 4232$	2.6631942
$1 \cdot 23^3 = 12167$	3
$2 \cdot 23^3 = 24344$	3.2210647
$3 \cdot 23^3 = 36501$	3.3503793
$5 \cdot 23^3 = 60835$	3.5132964
$8 \cdot 23^3 = 97336$	3.6631942
$13 \cdot 23^3 = 158171$	3.8180367

TABLE 3:
Table 1.1 labelled as a reversed logarithmic sum graph

City Suburb METRO	LONG-ITUDE	FIBO-NACCI LABEL	BASE MULTI-PLIER	FACTORED WEIGHT	NODE WEIGHT	LOG BASE 37 NODE
Salem	70 54	21	37	$21 \cdot 37$	777	1.746657
Lynn	70 57	13	37	$13 \cdot 37$	481	1.629043
Quincy	71 00	8	37	$8 \cdot 37$	296	1.509974
Brockton	71 01	5	37	$5 \cdot 37$	185	1.394708
Cambridge	71 07	3	37	$3 \cdot 37$	111	1.269430
Boston	71 07	2	37	$2 \cdot 37$	74	1.169991
BOSTON		1	37	$1 \cdot 37$	37	1
Longueuil	73 30	8	37^2	$8 \cdot 37^2$	10952	2.509974
Verdun	73 34	5	37^2	$5 \cdot 37^2$	6845	2.394708
Montreal	73 35	3	37^2	$3 \cdot 37^2$	4107	2.269430
Laval	73 44	2	37^2	$2 \cdot 37^2$	2738	2.169991
MONTREAL		1	37^2	$1 \cdot 37^2$	1369	2
Camden	75 06	13	37^3	$13 \cdot 37^3$	658489	3.629043
Phil.	75 13	8	37^3	$8 \cdot 37^3$	405224	3.509974
U. Darby	75 16	5	37^3	$5 \cdot 37^3$	253265	3.394708
Norris.	75 21	3	37^3	$3 \cdot 37^3$	151959	3.269430
Chester	75 22	2	37^3	$2 \cdot 37^3$	101306	3.169991
PHILADELPHIA		1	37^3	$1 \cdot 37^3$	50653	3
Scar.	79 12	21	37^4	$21 \cdot 37^4$	39357381	4.746657
Toronto	79 23	13	37^4	$13 \cdot 37^4$	24364093	4.629043
NYork	79 25	8	37^4	$8 \cdot 37^4$	14993288	4.509974
York	79 29	5	37^4	$5 \cdot 37^4$	9370805	4.394708
Etobicoke	79 34	3	37^4	$3 \cdot 37^4$	5622483	4.269430
Missi.	79 37	2	37^4	$2 \cdot 37^4$	3748322	4.169991
TORONTO		1	37^4	$1 \cdot 37^4$	1874161	4
Wind.	83 00	13	37^5	$13 \cdot 37^5$	901471441	5.629043
Warren	83 03	8	37^5	$8 \cdot 37^5$	554751656	5.509974
Detroit	83 10	5	37^5	$5 \cdot 37^5$	346719785	5.394708
Dearb.	83 15	3	37^5	$3 \cdot 37^5$	208031871	5.269430
Livonia	83 23	2	37^5	$2 \cdot 37^5$	138687914	5.169991
DETROIT		1	37^5	$1 \cdot 37^5$	69343957	5
ESLou	90 10	5	37^6	$5 \cdot 37^6$	12828632045	6.394708
SLou	90 15	3	37^6	$3 \cdot 37^6$	7697179227	6.269430
Lemay	90 17	2	37^6	$2 \cdot 37^6$	5131452818	6.169991
ST. LOUIS		1	37^6	$1 \cdot 37^6$	2565726409	6
NOrl	90 05	5	37^7	$5 \cdot 37^7$	474659385665	7.394708
Marr	90 06	3	37^7	$3 \cdot 37^7$	284795631399	7.269430
Meta	90 11	2	37^7	$2 \cdot 37^7$	189863754266	7.169991
NEW ORLEANS		1	37^7	$1 \cdot 37^7$	94931877133	7

TABLE 4.0: Distances between all metro areas

	BOS	MONT	PHIL	TOR	DET	SL	NO
BOSTON	0	255	263	429	615	1034	1349
MONTREAL		0	388	312	523	974	1394
PHIL			0	331	444	808	1086
TORONTO				0	211	662	1112
DETROIT					0	452	936
ST LOUIS						0	596
NEW ORLEANS							0

TABLE 4.1: Boston-area cities

	Salem	Lynn	Quincy	Brock.	Cambr.	Boston
Salem	0	4.29	19.1	31.6	15.3	21.4
Lynn		0	15.1	27.8	10.2	17.2
Quincy			0	12.6	10.9	5.96
Brock.				0	22.4	13.6
Cambr.					0	9.21
Boston						0

TABLE 4.2: Montreal-area cities

	Longue.	Verdun	Laval	Mont.
Longueuil	0	6.6	11.3	4.64
Verdun		0	9.29	3.54
Laval			0	7.35
Montreal				0

TABLE 4.3: Philadelphia-area cities

	Camden	Chester	U Darby	Norris.	Phila.
Camden	0	15.2	9.12	18.3	7.7
Chester		0	9.64	18.4	13.0
Upper Darby			0	11.2	3.5
Norristown				0	10.7
Philadelphia					0

TABLE 4.4: Toronto-area cities

	Scar.	Miss.	N. York	York	Etob.	Tor.
Scarborough	0	24.3	11.0	14.8	19.5	10.8
Mississauga		0	17.9	10.4	6.27	13.5
North York			0	7.66	11.8	8.23
York				0	4.75	5.12
Etobicoke					0	9.23
Toronto						0

TABLE 4.5: Detroit-area cities

	Windsor	Warren	Dear.	Livonia	Detroit
Windsor	0	16.3	12.8	20.7	9.18
Warren		0	20.0	19.3	13.9
Dearborn			0	10.5	6.27
Livonia				0	11.5
Detroit					0

TABLE 4.6: St. Louis-area cities

	E St. L.	Lemay	St. Louis
E. St. Louis	0	6.29	4.49
Lemay		0	1.79
St. Louis			0

TABLE 4.7: New Orleans-area cities

	Met.	Mar.	New O.
Metairie	0	7.61	5.98
Marrero		0	5.84
New Orleans			0

TABLE 5:
New data added — Ann Arbor

City Suburb METRO AREA	LONG-ITUDE	FIBO-NACCI LABEL	BASE MULTI-PLIER	FACTORED WEIGHT	NODE WEIGHT	LOG BASE 37 NODE
Wind.	83 00	21	37^5	$21 \cdot 37^5$	1456223097	5.746657
Warren	83 03	13	37^5	$13 \cdot 37^5$	901471441	5.629043
Detroit	83 10	8	37^5	$8 \cdot 37^5$	554751656	5.509974
Dearb.	83 15	5	37^5	$5 \cdot 37^5$	346719785	5.394708
Livonia	83 23	3	37^5	$3 \cdot 37^5$	208031871	5.269430
Ann Arbor	83 45	2	37^5	$2 \cdot 37^5$	138687914	5.169991
DETROIT		1	37^5	$1 \cdot 37^5$	69343957	5
ESLou	90 10	5	37^6	$5 \cdot 37^6$	12828632045	6.394708
SLou	90 15	3	37^6	$3 \cdot 37^6$	7697179227	6.269430
Lemay	90 17	2	37^6	$2 \cdot 37^6$	5131452818	6.169991
ST. LOUIS		1	37^6	$1 \cdot 37^6$	2565726409	6
NOrl	90 05	5	37^7	$5 \cdot 37^7$	474659385665	7.394708
Marr	90 06	3	37^7	$3 \cdot 37^7$	284795631399	7.269430
Meta	90 11	2	37^7	$2 \cdot 37^7$	189863754266	7.169991
NEW ORLEANS		1	37^7	$1 \cdot 37^7$	94931877133	7

TABLE 6:
Specializations of sum graphs

Type of graph	Characteristics
Sum graph (Figure 7)	Variable resolution at local and global scales, only. Shape, size, and connection pattern of parts to whole suggested by global label.
Sum graph with base multiplier (Figure 9)	Variable resolution at intermediate and global scales. Relative shape, size, and connection pattern of parts to whole suggested by multiple labels associated with split regions.
Logarithmic sum graph (Figure 10)	Confines sum graph labels to a single unit for each subgraph. Deals well with split regions; is not itself a sum graph. Label on cataloging node suggests relative shape, size, and connection pattern of parts to the whole
Reversed sum graph (Figure 11)	Not itself a sum graph. Sole function is to assign an integral value to the cataloging node of each subgraph.
Augmented reversed logarithmic sum graph (Figure 13)	Combines characteristics of logarithmic and reversed sum graphs. Added edges join cataloging nodes. Linkage patterns are suggested at local, intermediate, and global levels of resolution.