4. ARTICLES

Equal-Area Venn Diagrams of Two Circles: Their Use with Real-World Data.

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General Problem

We are concerned with populations whose members have two discrete characteristics; that is, characteristics which are either present or absent in each member. In populations having two characteristics, the populations can be described by the proportion of the population that has both characteristics present, one or the other (but not both) characteristics present, or both characteristics absent. One well-known way to present this data is with a two circle Venn Diagram in which each circle represents one of the characteristics, the intersection of the circles represents the members with both characteristics, and the region outside both circles but within the universe of discourse (depicted as a bounded figure surrounding the circles — often a rectangle) represents the members with neither characteristic. Appropriate regions might then be labelled with suitable percentages, whether or not the geometric intersection pattern is suggestive of the numeric partition of the sample. At this point, it may be useful to the reader to draw a two-circle Venn diagram.

For example, consider a population of countries with national child vaccination programs. Some of the countries in the population use a campaign strategy, some a clinic-based strategy, some use both strategies, and a few use neither strategy. Construct a Venn diagram to represent this grouping of mixed strategies. Draw Circle α on the left and draw an intersecting Circle β on the right. Draw a rectangle that is large enough to easily contain all of the intersecting circle configuration. In this Venn diagram, Circle α could represent countries using a campaign strategy and Circle β could represent countries with a clinic-based strategy. Partition each of these circles according to their intersection pattern using the following notation.

Notation

Let the symbol A denote the area of Circle α that does NOT also lie within the Circle β.
Let the symbol B denote the area of Circle β that does NOT also lie within the Circle α.
Let the symbol AB denote the area of the intersection of Circles α and β.

In the vaccination program interpretation of the two circle Venn diagram, Area A is proportional to the countries using only a campaign strategy, Area B to the countries using only a clinic-based strategy, and Area AB to the countries using both a campaign and a clinic-based strategy. Countries using neither strategy are not represented; indeed, only participant populations will be considered for the remainder of this analysis, although we do note the existence of the logical category of the "neither" class.

Data such as this is sometimes illustrated with bar or pie charts. Such illustrations are inadequate because they do not allow easy portrayal on a single diagram of both the intersecting areas and the total percentage of each characteristic. The advantage of using a diagram of intersecting circles is that it portrays all the data clearly in a single diagram.

The objective here is to draw the intersecting circles so that the different areas are exactly proportional to the data, in much the way that an equal area map is drawn so that different areas are exactly proportional to the size of the landmass. Equal area graphic displays, be they maps or diagrams, are critical in making accurate visual comparisons of mapped or plotted data.

The remainder of this paper is devoted to displaying the detail of the calculations required to construct equal-area Venn diagrams from real-world data. Fundamentally, these calculations rest on the problem of finding the radii of the circles and the location of their centers, given the various common (intersecting) and non-common areas.

Definition of the two-circle problem

Given two intersecting circles, α and β, and given their common and non-common areas, AB, A and B, respectively. Find the radii of both circles, r_A and r_B, and the distance between the centers of the two circles d_{AB} such that the centers can be located and the circles drawn. It is clear that if the center of Circle α is at the origin, the Cartesian coordinates of the center of Circle β are $(d_{AB}, 0)$.

We can transform the three areas into percentages of the total area covered by the two intersecting circles by noting that the total area covered is $A + B + AB$.

$$A\% = 100 \times (A/(A + B + AB)) \tag{1}$$

gives the A-only area percent;

$$B\% = 100 \times (B/(A + B + AB)) \tag{2}$$

gives the B-only area percent;

$$AB\% = 100 \times (AB/(A + B + AB)) \tag{3}$$

gives the AB area percent;

No generality is lost by requiring Circle α to be the larger circle and by standardizing the size of Circle α by setting its radius equal to 1: $r_A = 1$. (Naturally, this assumes that $\alpha > \beta$ and that the bigger real-world characteristic is assigned to Circle α.) As a result, the area $A + AB$ of Circle α is π. The problem can now be restated, more simply, as follows.

Given: $A\%$, $B\%$, and $AB\%$, where $A\% + B\% + AB\% = 100$.

Find: r_B, the radius of Circle β, and d_{AB}, the distance between the centers.

Analytic Strategy

In order to solve the two-circle problem, define the *chord* of the intersection to be the straight line joining the two points where the perimeters of the two circles intersect –assuming here, and throughout the remainder of the text, that one of the two circles is not fully contained within the other. There are two situations that arise: one in which the chord lies between the two centers and a second in which the chord lies to one side of both centers. To visualize this relationship, draw one pair of circles with a relatively small area of intersection; in this case the chord lies between the centers. In what follows, this configuration will be referred to as one of type Case I. Alternatively, draw two circles with a relatively large area of overlap; in this case the chord lies on one side of both centers. In what follows, this configuration will be referred to as one of type Case II.

Starting with $A\%$, $B\%$, and $AB\%$, it is straightforward to derive r_B, but not to derive d_{AB}. Therefore, we reverse the situation and seek the function that yields $A\%$, $B\%$, and $AB\%$ given r_A, r_B, and d_{AB}. Several derivations are possible. The simplest one (not using integral calculus) is presented below to maximize accessibility of content.

Functions for $A\%$, $B\%$, and $AB\%$ in terms of r_A, r_B, and d_{AB} were obtained for Case I and for Case II. These functions are sufficiently complex to obstruct the derivation of an inverse function that would yield d_{AB} in terms of $A\%$, $B\%$, and $AB\%$. Consequently, a numerical approach was used in which d_{AB} was calculated for a grid of values of $B\%$ and $AB\%$. The results are presented in Tables 1 and 2. The value r_A is assumed equal to one and r_B is readily calculated from $A\%$, $B\%$, and $AB\%$.

With these results, several options are available to estimate d_{AB} from $A\%$, $B\%$, and $AB\%$. The preferred option depends on the accuracy desired. Option 1 entails interpolating from the data in Tables 1 and 2. Option 2 entails using a polynomial in $B\%$ and $AB\%$ (obtained via regression) to estimate d_{AB}. Options 1 and 2 are the least accurate, both giving answers within one percent accuracy relative to the radius of the largest circle (Circle α). Option 3, which will yield d_{AB} to any desired accuracy, entails searching by trial and error using the functions $B\% = f_1(d_{AB}, r_B)$ and $AB\% = f_2(d_{AB}, r_B)$. The trial and error search is greatly simplified by the fact that r_B can be calculated directly from $B\%$ and $AB\%$.

Derivation of $B\%$ and $AB\%$ as a function of r_B and d_{AB}

General formulæ

Heron's formula for the area of a triangle is based on the lengths, a, b, and c, of its sides. Let $S = (1/2) \times (a + b + c)$. Then the area of the triangle is:

(4) $$(S \times (S-a) \times (S-b) \times (S-c))^{1/2}$$

A sector of a circle is the pie-shaped wedge cut from the center of the circle out to the edge. The region of overlap of two intersecting circles is called a "lune." A sector can be decomposed into a triangle and a lune split longitudinally. We refer to the triangular portion as the "triangle" of a sector and to the lunar portion as the "segment" of a sector. The formula for the area of a sector of a circle with central angle Q (measured in radians) and radius r is:

(5) $$(1/2) \times (Q/\pi) \times (\pi \times r^2) = (1/2) \times (Q \times r^2).$$

The formula for the area of the corresponding triangle of a sector is:

(6) $$(1/2) \times r^2 \times \sin Q.$$

The formula for the area of the corresponding segment of a sector is:

$$(1/2) \times (Q \times r^2) - (1/2) \times r^2 \times \sin Q = (1/2) \times r^2 \times (Q - \sin Q). \tag{7}$$

Case I: Chord lies between the two centers

In Case I, the chord of the lune separates the centers of circles α and β. The distance d_{AB} is the distance between the two centers, measured along the line of centers. Form a triangle using the line of centers as one side of length d_{AB}. The second side is formed by joining the center of circle α to the top intersection point of the lune; the acute angle enclosed by the line of centers and this side has measure Q_A which is 1/2 of the central angle subtending the chord of the lune from the center of circle α. In a similar fashion, join the center of circle β to the same third vertex to complete the triangle. The acute angle enclosed between the line of centers and this side has measure Q_B which is 1/2 of the central angle subtending the chord of the lune from the center of circle β. Let h denote the altitude of this triangle from the vertex of the lune to the line of centers. Let X denote the horizontal distance from the center of Circle α to the intersection with the chord. Let Z denote the area of the triangle with sides of lengths $r_A = 1$, r_B, and d_{AB}. Let K_A and K_B denote areas of the sectors in circles α and β subtended by the chord. Let L_A and L_B denote the areas of the triangles of these two sectors. Finally, let M_A and M_B denote the areas of the segments of the two sectors. Then given r_A, r_B, and d_{AB}, find h, $B\%$, $AB\%$, and $A\%$ as follows.

From equation (4),

$$S = (1 + r_B + d_{AB})/2. \tag{8}$$

From equations (4) and (8), we get the area of the triangle as

$$Z = (S \times (S-1) \times (S - r_B) \times (S - d_{AB}))^{1/2}. \tag{9}$$

From equation (9),

$$h = 2 \times Z/d_{AB}, \tag{10}$$

because $Z = (1/2) \times h \times d_{AB}$.

From equation (10),

$$Q_A = \text{Arcsin}\,(h/r_A), \tag{11}$$

because $\text{Sin}\,Q_A = h/r_A$; also from equation (10),

$$Q_B = \text{Arcsin}\,(h/r_B), \tag{12}$$

because $\text{Sin}\,Q_B = h/r_B$.

From equations (5) and (11), the sector A area is

$$K_A = {r_A}^2 \times (2 \times Q_A)/2 = Q_A, \tag{13}$$

and from equations (5) and (12), the sector B area is

(14) $$K_B = {r_B}^2 \times (2 \times Q_B)/2 = Q_B.$$

From equation (9), the sum of the areas of the two sectors is

(15) $$L_A + L_B = 2 \times Z.$$

To find the area AB of the intersecting area (lune), view it as the sum of the two segments of the two sectors. From equation (7): $AB = M_A + M_B = (K_A - L_A) + (K_B - L_B) = (K_A + K_B) - (L_A + L_B)$ so that from equations (9), (13), (14), (15), $AB = Q_A + Q_B \times {r_B}^2 - 2 \times Z$. Using equations (11) and (12), it follows that $AB = \text{Arcsin}(h/r_A) + \text{Arcsin}(h/rB) \times {r_B}^2 - 2 \times Z$ and finally, noting that r_A=1, that

(16) $$AB = \text{Arcsin}(h) + \text{Arcsin}(h/r_B) \times {r_B}^2 - 2 \times Z.$$

The B-only area is found by subtracting the area of the lune from the area of the whole circle as

(17) $$B = (\pi \times {r_B}^2) - (AB).$$

Subtracting out the extra intersection, the total area covered by the circles, denoted TOTAL, is (from equation (16))

(18) $$\text{TOTAL} = (\pi) + (\pi \times {r_B}^2) - AB.$$

[Some may recognize the formula in (18) as one form of the Principle of Inclusion and Exclusion–ed.]

From equations (16) and (18) it follows that

(19) $$AB\% = 100 \times AB/\text{TOTAL};$$

from equations (17) and (18) it follows that

(20) $$B\% = 100 \times B/\text{TOTAL};$$

and from equations (19) and (20) it follows that

(21) $$A\% = 100 - AB\% - B\%.$$

These results hold for d_{AB} greater than or equal to X, the distance from the origin to the chord, but not greater than $r_A + r_B$, that is:

(22) $$X \leq d_{AB} \leq r_A + r_B,$$

where, from equation (11), $X = \text{Cos}(Q_A/2)$.

Case II: Chord to one side of both centers

The same definitions apply as in the previous section, except in relation to the following situation. Draw two intersecting circles and associated lines, labelling them as follows. Draw the larger of the two circles on the left. Insert the center of the large circle as a distinguished dot. Draw a smaller circle intersecting the larger one in such a way that the center of the large circle is contained within the smaller circle. Much of the small circle is therefore necessarily contained within the large circle. Note the center of the small circle as a dot. Draw the chord joining the two intersection points of the small and large circles; half of it has length h. Draw the line segment joining the two circle centers, of length d_{AB} and extend the segment to intersect the chord. The small circle now contains a right triangle which in turn contains a triangle with an obtuse angle. Label the radius of the larger circle as r_A; label the radius of the smaller circle as r_B. Label the constructed central angle in the larger circle as Q_A and the constructed central angle in the smaller circle as Q_B. The area of the obtuse triangle is Z_1 and the area of the difference between the right triangle and the obtuse triangle is Z_2.

The following formulæ can then be readily deduced. From equation (4),

(23) $$S_1 = (r_A + r_B + d_{AB})/2 = (1 + r_B + d_{AB})/2;$$

from equations (5) and (23),

(24) $$Z_1 = (S_1 \times (S_1 - 1) \times (S_1 - r_B) \times (S_1 - d_{AB}))^{1/2};$$

from equation (24),

(25) $$h = 2 \times Z_1/d_{AB}$$

because $Z_1 = (1/2) \times d_{AB} \times h$; from equation (25)

(26) $$Q_A = \text{Arcsin}\,(h),$$

because $\text{Sin}\,(Q_A) = h/r_A = h$; from equation (26)

(27) $$Q_B = \text{Arcsin}\,(h/r_B),$$

because $\text{Sin}\,(Q_B) = h/r_B$; from equations (25) and (27)

(28) $$Z_2 = (1/2) \times h \times r_B \times \text{Cos}\,(Q_B));$$

from equations (5) and (26)

(29) $$K_A = \text{Sector } A \text{ area} = (1/2) \times (2 \times Q_A) \times {r_A}^2 = Q_A;$$

from equations (24) and (28)

(30) $$L_A = \text{Triangle Area of Sector } A = 2 \times (Z_1 + Z_2);$$

from equations (7), (29), (30)

(31) $$M_A = \text{Segment Area of Sector } A = K_A - L_A;$$

from equations (5) and (27),

(32) $$K_B = \text{Sector } B \text{ Area} = (1/2) \times (2 \times Q_B) \times {r_B}^2 = Q_B \times {r_B}^2;$$

from equation (28)

(33) $$L_B = \text{Triangle Area of Sector } B = 2 \times Z_2;$$

from equations (32) and (33)

(34) $$M_B = \text{Segment Area of Sector } B = K_B - L_B;$$

from equations (31) and (34)

(35) $$\text{Area } W = M_B - M_A;$$

from equation (35)

(36) $$AB = \text{Area of Circle } B - \text{Area } W = \pi \times {r_B}^2 - W;$$

thus,

(37) $$B = W;$$

from equation (35)

TOTAL, the total area covered by the circles

(38) $$= \text{Area of Circle } A + \text{Area } W = \pi + W;$$

from equations (36) and (38)

(39) $$AB\% = 100 \times AB/\text{TOTAL};$$

from equations (37) and (38)

(40) $$B\% = 100 \times B/\text{TOTAL};$$

and from equations (39) and (40)

(41) $$A\% = 100 - AB\% - B\%.$$

These results hold for d_{AB} greater than or equal to zero, but not greater than X, the distance from the origin to the chord, that is:

(42) $$0 \leq d_{AB} \leq X,$$

where from equation (26) $X = \cos(Q_A)$.

Methods for Computing r_B and d_{AB}.

The computation of r_B given $A\%$, $B\%$, and $AB\%$ is straightforward. However, this is not the case for d_{AB} in light of the fact that we did not obtain a function for d_{AB} in terms of $A\%$, $B\%$, and $AB\%$. We present three numerical methods for estimating d_{AB}, each with a different level of accuracy. First, however, we derive $AB\%$ and $B\%$ as a function of A, B, and AB, and r_B as a function of $AB\%$ and $B\%$. These derivations of $AB\%$, $B\%$ and r_B are the same for all three methods of estimating d_{AB}.

r_B as a Function of $B\%$ and $AB\%$

Let A, B, and AB be the B-circle only area, the A-circle only area, and the area of intersection, respectively, and let $TA = A + B + AB$; then,

(43) $$A\% = 100 \times A/TA, \quad A = A\% \times TA/100;$$

(44) $$B\% = 100 \times B/TA, \quad B = B\% \times TA/100;$$

(45) $$AB\% = 100 \times AB/TA, \quad AB = AB\% \times TA/100;$$

(46) $$\text{Circle } A \text{ area} = A + AB = \pi;$$

(47) $$\text{Circle } B \text{ area} = B + AB = \pi \times {r_B}^2.$$

Substituting equation (46) in equation (47): $B + AB = \pi \times {r_B}^2 = (A + AB) \times {r_B}^2$,

(48) $${r_B}^2 = (B + AB)/(A + AB).$$

Substituting equations (43), (44), and (45) in equation (48):

(49) $${r_B}^2 = \frac{(B\% \times TA)/100 + (AB\% \times TA)/100}{(A\% \times TA)/100 + (AB\% \times TA)/100} = \frac{B\% + AB\%}{A\% + AB\%}.$$

(50) $$A\% = 100 - B\% - AB\%,$$

because $A\% + B\% + AB\% = 100$.

Substituting equation (50) into equation (49):

$${r_B}^2 = \frac{B\% + AB\%}{(100 - B\% - AB\%) + AB\%} = \frac{B\% + AB\%}{100 - B\%}.$$

Thus,

(51) $$r_B = ((B\% + AB\%)/(100 - B\%))^{1/2}.$$

Look-up Table Method for Estimating d_{AB}

Table 1 (at end of article) gives the value of d_{AB} to 6 decimal places for all values of $AB\%$ from 0 to 100 and for $B\%$ from 0 to 50 in 5 percentage point increments for values of d_{AB} from 0 to $r_A + r_B$. Note from Table 1 that some of the values are calculated using the procedure for Case I and some using the Case II procedure.

The procedure used to obtain the values in Table 1 is summarized here. For each combination of $B\%$ and $AB\%$ in Table 1, calculate d_{AB} as follows. First calculate r_B using equation (51). Then guess a value for d_{AB} that is approximately correct, and guess whether Case I or Case II applies. (In most areas of the table this is obvious.) Then calculate the values of $B\%$ and $AB\%$ using the guessed value of d_{AB}, the calculated value of r_B and either equations (8) through (20) in Case I, or equations (23) through (40) in Case II. Then adjust the guessed value of d_{AB} up or down and recalculate until the resulting values of $B\%$ and $AB\%$ approximate the desired values as closely as desired (six decimal points) in Table 1. Check the final value of d_{AB} to be sure the correct calculation procedure was used (Case I or II) with the inequalities (22) or (42).

Table 2 contains values of d_{AB} for values of $AB\%$ in the 0 to 10 range. Between 0 and 5, $AB\%$ is in increments of 1. This table was produced because of the large and non-linear increments in d_{AB} in this range of $AB\%$.

If the given values of $B\%$ and $AB\%$ are one of the combinations found in Table 1 or Table 2, then the value of d_{AB} can be obtained directly from the tables to six decimal point accuracy. If the exact values of $B\%$ and $AB\%$ are not in either table, then an interpolation procedure can be used. In Table 1, the procedure would be as follows.

(a) Assume the given values of $B\%$ and $AB\%$ are not along the lower diagonal of the table, so that they are bounded by table values of $B\%$ and $AB\%$ at four corners forming a rectangle within the table. Let $d(i, j)$ be the value of d_{AB} for any values of $AB\%$ (or i) and $B\%$ (or j), within the defined range. Then $d(AB\%, B\%)$ is the value of d_{AB} at the given values of $B\%$ and $AB\%$. If e is the value of $AB\%$ just less than the given $AB\%$, f is the value of $AB\%$ just greater than the given $AB\%$, g is the value of $B\%$ just less than the given $B\%$, and h is the value of $B\%$ just greater than the given $B\%$, then,

$$d_K = \text{estimated value of } d \text{ at point } K$$

$$= d(e, g) + (d(f, g) - d(e, g)) \times (AB\% - e)/(f - e), \tag{52}$$

where K is the intersection point of a horizontal through $d(i, j)$ with the vertical line through g.

$$d_L = \text{estimated value of } d \text{ at point } L$$

$$= d(e, h) + (d(f, h) - d(e, h)) \times (AB\% - e)/(f - e), \tag{53}$$

where L is the intersection point of a horizontal through $d(i, j)$ with the vertical line through h.

$$\text{Estimate of } d(AB\%, B\%) = d_K + (d_L - d_K) \times (B\% - g)/(h - g). \tag{54}$$

(b) Assume the given values of $AB\%$ and $B\%$ are near the lower diagonal of the defined range such that the location is bounded by only three table values (rather than by four table values, as in case (a) above). Use the same labelling as in case (a) above for the bounding table entries; note, however, that d(f,h) is not defined in this case (because of the nearness of the table entry to the lower diagonal). At the boundaries where $B\% = 0$ or $AB\% = 0$, equation (54) holds. Otherwise, we have:

$$d_K = \text{estimated value of } d \text{ at point } K$$

$$= d(e,g) + (d(f,g) - d(e,g)) \times (AB\% - e)/(f - e), \tag{55}$$

where K is the intersection point of a horizontal through $d(i,j)$ with the vertical line through g.

$$d_L = \text{estimated value of } d \text{ at point } L$$

$$= d(e,h) + (d(e,h) - d(e,g)) \times (B\% - g)/(h - g), \tag{56}$$

where L is the intersection point of a vertical through $d(i,j)$ with the horizontal line through e.

$$d(AB\%, B\%) = d(e,g) + (d_K - d(e,g)) + (k_L - d(e,g)) = d_K + d_L - d(e,g)$$

$$= d(e,g) + (d(f,g) - d(e,g)) \times (AB\% - e)/(f - e) + (d(e,h) - d(e,g)) \times (B\% - g)/(h - g). \tag{57}$$

The use of formulas (54) and (57) in conjunction with Table 1 will generally produce answers for d_{AB} within 0.01 of the correct figures, with the exception of the range for $AB\%$ from 0 to 5. In some areas of this range, particularly for $B\%$ greater than 45, the error can be over 0.05. For example, this method produces an estimated value for $d(2.5, 47.5) = 1.7394$, compared to the correct value of 1.7927, an error of 0.053. (This error is 5.3% of the radius of circle A, which is 1, and is $100 * 0.053/1.7927 = 3\%$ of the correct value of d_{AB}.)

If Table 2 is used for values of $AB\%$ between 0 and 5, the error can be reduced to less than 0.02 in the worst cases. For example, the use of Table 2 produces an estimated value for $d(0.5, 49.5) = 1.93016$, compared to the correct value of 1.91299, an error of 0.01717. (This is 1.7% of the radius of circle A and 0.9% of the correct value of d_{AB}.)

Polynomial Estimation of d_{AB}

The regression formulæ were used to obtain polynomials in $AB\%$ and $B\%$ that estimated d_{AB}, in effect interpolating for values of $AB\%$ and $B\%$ between the grid points in Table 1. The range of $AB\%$ and $B\%$ was separated into three subranges and a polynomial was obtained for each subrange. The three subranges are specified below and also denoted graphically, using a variety of typefaces, in Table 3.

$$\text{Subrange 1: } AB\% > 5 \text{ and } B\% \geq 5. \tag{58}$$

$$\text{Subrange 2: } 0 \leq AB\% \leq 5 \text{ and } 0 \leq B\% \leq 50. \tag{59}$$

$$\text{Subrange 3: } 0 \leq AB\% \leq 100 \text{ and } 0 \leq B\% < 5. \tag{60}$$

The polynomials obtained for each subrange are given below.

Polynomial 1 for subrange 1:

$$\text{est}\, d_{AB}{}^1 = c_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 + c_7X_7 + c_8X_8, \tag{61}$$

where

$$\begin{aligned}
c_0 &= 0.994388189 \\
c_1 &= 0.003790799, \ X_1 = AB\% \\
c_2 &= -0.001818030, \ X_2 = B\% \\
c_3 &= -0.129148003, \ X_3 = (AB\%)^{1/2} \\
c_4 &= 0.130891455, \ X_4 = (B\%)^{1/2} \\
c_5 &= -0.000147200, \ X_5 = (AB\%) \times (B\%) \\
c_6 &= -0.000017449, \ X_6 = (AB\%)^2 \\
c_7 &= 0.000081024, \ X_7 = (B\%)^2 \\
c_8 &= -0.004913375, \ X_8 = (AB\% \times B\%)^{1/2}.
\end{aligned}$$

Polynomial 2 for subrange 2:

$$\text{est}\, d_{AB}{}^2 = c_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 + c_7X_7, \tag{62}$$

where

$$\begin{aligned}
c_0 &= 1.003584849 \\
c_1 &= -0.009650203, \ X_1 = AB\% \\
c_2 &= 0.002712922, \ X_2 = B\% \\
c_3 &= -0.089520075, \ X_3 = (AB\%)^{1/2} \\
c_4 &= 0.093223275, \ X_4 = (B\%)^{1/2} \\
c_5 &= -0.000366121, \ X_5 = (AB\%) \times (B\%) \\
c_6 &= -0.000521608, \ X_6 = (AB\%)^2 \\
c_7 &= 0.000075950, \ X_7 = (B\%)^2.
\end{aligned}$$

Polynomial 3 for subrange 3:

$$\text{est}\, d_{AB}{}^3 = c_0 + c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + c_5X_5 + c_6X_6 + c_7X_7 + c_8X_8, \tag{63}$$

where

$$\begin{aligned}
c_0 &= 1.009984781 \\
c_1 &= 0.001043059, \ X_1 = AB\% \\
c_2 &= 0.009094475, \ X_2 = B\% \\
c_3 &= -0.106641306, \ X_3 = (AB\%)^{1/2} \\
c_4 &= 0.088497298, \ X_4 = (B\%)^{1/2} \\
c_5 &= -0.000104701, \ X_5 = (AB\%) \times (B\%) \\
c_6 &= -0.000052845, \ X_6 = (AB\%)^2 \\
c_7 &= -0.000084727, \ X_7 = (B\%)^2
\end{aligned}$$

$c_8 = -0.007209240$; $X_8 = (AB\% \times B\%)^{1/2}$.

Numerical Search on the Inverse Function

The value of d_{AB} can be obtained to any desired accuracy for any combination of $AB\%$ and $B\%$ in the defined range using the same procedure as was used to derive Table 1. We summarize the procedure below.

(1) Given A, B, and AB.

(2) Calculate $B\%$ using equation (44).

(3) Calculate $AB\%$ using equation (45).

(4) Calculate r_B using equation (51).

(5) Estimate an approximate value for d_{AB} using Table 1.

(6) Estimate whether Case I or Case II applies for the calculated values of $B\%$ and $AB\%$ using Tables 1 and 2.

(7) Calculate estimated values of $AB\%$ and $B\%$ using the calculated value of r_B from step 4, the estimated value of d_{AB} from step 5 and equations (19) and (20) for Case I or equations (39) and (40) for Case II.

(8) If the estimated value of $B\%$ obtained in step 7 is too small, increase the estimated value of d_{AB} and recalculate $AB\%$ and $B\%$ by recycling through step 7; if $B\%$ is too big, reduce the estimated value of d_{AB} and recycle through step 7. The size of the adjustment depends on the approximate slope in the region of concern. For example, if we were in a region of Table 1 where d_{AB} increased 0.025 while $B\%$ was increasing by 5 (as is approximately the case for $B\%$ between 10 and 15 and $AB\%$=60), then adjust d_{AB} by 0.005 for each error increment of 1 in $B\%$. In this way, the error in $B\%$ can be made as small as desired by continued recycling. The value of $AB\%$ converges to the desired value along with $B\%$.

(9) Check to be sure that the proper case (I or II) was used by applying the inequalities (22) or (42).

Editor's Note:

In the original submission the author also considered Venn diagrams of three circles, noting that the three-circle Venn diagram contains insufficient degrees of freedom to provide a general solution to a three characteristic situation. The reader interested in generalizations of the two-circle case might wish to examine the literature of Boolean algebra, particularly Karnaugh maps used in the minimization of switching circuits.

More detail is presented in this presentation than would be in traditional publications, suggesting yet another avenue to explore in the dissemination of information across disciplinary boundaries and one way to offer detail that might be required by engineers in the field to implement abstract ideas presented in journals. The increase in cost, to present extra detail that may not be necessary to all, is minuscule in an electronic format.

Author's Note:

The author wishes to thank anonymous referees for suggesting the viewpoint of "equal area" Venn diagrams, and for substantial help in making the context of the problem reflect this viewpoint.

TABLE 1
TWO INTERSECTING CIRCLES PROBLEM
Distance between centers (d), given $AB\%$ and B-ONLY %

	B-Only %										
$AB\%$	00	05	10	15	20	25	30	35	40	45	50
00	1	1.229399	1.33333	1.42006	1.5	1.57735	1.65464	1.7337	1.8163	1.90453	2
05	**0.776393**	0.982242	1.082793	1.164457	1.23779	1.307136	1.374962	1.443	1.5127	1.58547	
10	**0.683772**	**0.868866**	0.961981	1.037539	1.10493	1.168008	1.229156	1.289705	1.350925	1.413929	
15	**0.612702**	**0.782479**	0.86877	0.938611	1.0005	1.05796	1.112964	1.166795	1.220433		
20	**0.552787**	**0.710053**	**0.79011**	0.8548	0.9113	0.96341	1.012665	1.060145	1.106585		
25	**0.5**	**0.646459**	**0.72074**	0.78016	0.831894	0.878849	0.922537	0.963822			
30	**0.452278**	**0.590054**	**0.657608**	0.712452	0.759365	0.801269	0.839453	0.874543			
35	**0.408393**	**0.536264**	**0.599914**	0.649736	0.691905	0.728788	0.761409				
40	**0.367545**	**0.487058**	**0.545676**	**0.590844**	0.628273	0.660049	0.686956				
45	**0.32918**	**0.440711**	**0.494394**	**0.534916**	0.567539	0.594046					
50	**0.292894**	**0.390085**	**0.445404**	**0.48126**	0.508937	0.529864					
55	**0.258381**	**0.354550**	**0.398303**	0.429363	0.451767						
60	**0.225404**	**0.313984**	**0.352752**	0.378617	0.395288						
65	**0.193775**	**0.27465**	**0.308123**	0.328451							
70	**0.16334**	**0.236231**	**0.264046**	0.278096							
75	**0.133975**	**0.198480**	**0.219922**								
80	**0.105573**	**0.160016**	0.174756								
85	**0.078049**	**0.122853**									
90	**0.051817**	0.082696									
95	**0.025321**										
100	**0**										

Case I: Chord joining intersection points lies between the two centers
Case II: Chord lies to one side of both centers.

TABLE 2
DATA TABLE FOR TWO-CIRCLE PROBLEM
Distance between centers (d), given $AB\%$ and B-ONLY %
for $AB\% = 00 - 05$

		B-Only %				
$AB\%$	00	01	02	03	04	05
00	1	1.100503	1.142857	1.175863	1.204124	1.229399
01	**0.9**	0.996625	1.041666	1.076413	1.105877	1.132029
02	**0.858578**	**0.949095**	0.993166	1.027487	1.056694	1.082652
03	**0.826794**	**0.91298**	**0.95593**	0.989618	1.018366	1.044
04	**0.8**	**0.882799**	**0.924676**	**0.957696**	0.985978	1.011199
05	**0.776397**	**0.856395**	**0.897278**	**0.929644**	0.957428	0.982242

			B-Only %				
$AB\%$	10	15	20	25	30	35	40
00	1.33333	1.42008	1.5	1.57735	1.65464	1.7337	1.8163
01	1.237653	1.324053	1.402561	1.477746	1.552221	1.627878	1.706373
02	1.187417	1.272749	1.349919	1.423492	1.496065	1.569499	1.645393
03	1.147486	1.231629	1.307493	1.379587	1.450471	1.521965	1.595619
04	1.113248	1.196158	1.270744	1.341437	1.410751	1.480464	1.552075
05	1.082793	1.164457	1.23779	1.307136	1.374962	1.443	1.5127

		B-Only %				
$AB\%$	45	46	47	48	49	50
00	1.90453	1.922958	1.941696	1.960768	1.980196	2
01	1.789387	1.806694	1.824274	1.842146	1.860327	
02	1.725354	1.741988	1.758872	1.776022	1.793457	
03	1.67297	1.689029	1.705319	1.721855		
04	1.627057	1.642596	1.658348	1.674328		
05	1.58547	1.600524	1.615776			

Case I: Chord joining intersection points lies between the two centers
Case II: Chord lies to one side of both centers.

TABLE 3
TWO INTERSECTING CIRCLES PROBLEM
Error in the Estimated Distance between the two centers: (dest - d)

	B-Only %										
AB%	00	05	10	15	20	25	30	35	40	45	50
00	**0.003585**	**-0.00180**	**-0.00022**	**0.002330**	**0.005130**	**0.007043**	**0.009202**	**0.009302**	**0.008917**	**0.000295**	**-0.01170**
05	**-0.00796**	**0.000948**	**-0.00314**	**-0.00465**	**-0.00442**	**-0.00306**	**-0.00110**	**0.000865**	**0.002138**	**0.001824**	
10	*-0.00111*	-0.00320	0.000151	0.000254	-0.00029	-0.00075	-0.00096	-0.00100	-0.00117	-0.00193	
15	*-0.00126*	-0.00331	-0.00005	-0.00039	-0.00121	-0.00152	-0.00099	0.000469	0.002778		
20	*-0.00096*	-0.00263	0.000458	-0.00030	-0.00137	-0.00157	-0.00037	0.002464	0.007088		
25	*-0.00044*	-0.00166	0.001121	-0.00009	-0.00143	-0.00152	0.000319	0.004538			
30	*0.000144*	-0.00065	0.001711	0.000014	-0.00158	-0.00154	0.001003	0.006707			
35	*0.000727*	0.000235	0.002122	-0.00005	-0.00188	-0.00163	0.001762				
40	*0.001248*	0.00930	0.002310	-0.00031	-0.00229	-0.00168	0.002798				
45	*0.001670*	0.001372	0.002251	-0.00074	-0.00272	-0.00151					
50	*0.001965*	0.001534	0.001995	-0.00124	-0.00300	-0.00084					
55	*0.002114*	0.001407	0.001563	-0.00172	-0.00292						
60	*0.002102*	0.001011	0.001052	-0.00195	-0.00207						
65	*0.001914*	0.000389	0.000606	-0.00165							
70	*0.001542*	-0.00037	0.000606	-0.00165							
75	*0.000976*	-0.00113	0.001043								
80	*0.000210*	-0.00160	0.003186								
85	*-0.00076*	-0.00124									
90	*-0.00194*	0.001460									
95	*-0.00334*										
100	*-0.00496*										

Actual Distance from Table 1; Estimated Distance from Polynomials 1, 2, and 3.
Polynomial 1
Polynomial 2
Polynomial 3