3. ARTICLES

The Paris Metro: Is Its Graph Planar?

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"Over the river and through the woods,
To Grandmother's house we go.
The horse knows the way to carry the sleigh
Through the white and drifting snow."

Song of unknown origin

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The reader should read this article with a map of the Paris Metro in hand.

In the Euclidean plane, crossing lines intersect at a point in the plane; the line segment determined by X, Y and X', Y' intersect at a point Z (Figure 1). The graph that includes the four nodes X, Y, X', Y' and the two edges XY' and $X'Y$ does *not* have a fifth node at any other location (Figure 2). To make this viewpoint consistent with our narrow Euclidean mindset, think of stretching the edge $X'Y$ so that there is no visual hint of "intersection" – the horse knows to go over the river even though an aerial view of the wintry landscape sees the river and road as two "intersecting" dark tracings across the white, snowy backdrop.

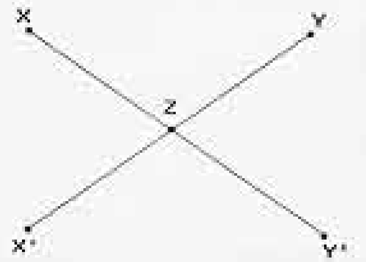

Figure 1. Draw nodes X and Y, left to right, horizontally. Draw nodes X' and Y', left to right, horizontally below the first set. Join X to Y' and join X' to Y using straight segments. Label their intersection as Z.

Planar graphs

To capture this idea more formally, we introduce the concept of embedding; the approach and material in this section follows closely that of Harary (1969, pp. 102-113). A graph is *embedded* in a surface when it is drawn on that surface in such a way that no two edges intersect (geometrically). The graph in Figure 2 has not been embedded in the plane: the edges $X'Y$ and XY' intersect. The graph in Figure 3 has been embedded in the plane. The

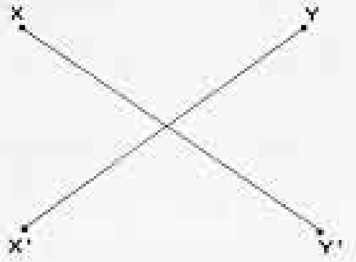

Figure 2. Draw nodes X and Y, left to right, horizontally. Draw nodes X' and Y', left to right, horizontally below the first set. Join X to Y' and join X' to Y using straight segments.

connection pattern of the graphs in Figures 2 and 3 is identical: topologically, they are said to be *homeomorphic*. They are equivalent structural models. Thus, we distinguish between a planar graph and a plane graph. A graph is *planar* if it can be embedded in the plane (as can Figure 2); a graph is *plane* if it has already been embedded in the plane (as has Figure 3). The graph in Figure 2 is planar but not plane; the graph in Figure 3 is both planar and plane.

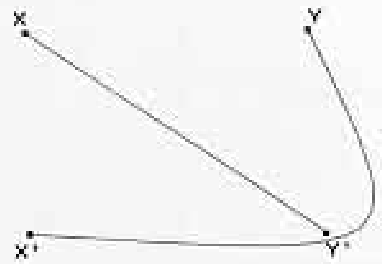

Figure 3. Draw nodes X and Y, left to right, horizontally. Draw nodes X' and Y', left to right, horizontally below the first set. Join X to Y' using a straight segment and join X' to Y using a curved line that does not pass through the segment joining X to Y'.

A graph that cannot be embedded in the plane is called *nonplanar*. There are two nonplanar graphs of particular importance. One is the graph composed of two sets of three nodes: think of one set of three nodes arranged horizontally and of the other set as arranged horizontally below the first set. Edges join each node of the top set to each node of the bottom set: a total of nine edges (Figure 4 shows the detail of labeling). This set is denoted as $K_{3,3}$. The other critical nonplanar graph is denoted as K_5. It is composed of a pentagon and all edges joining the nodes (Figure 5 shows detail).

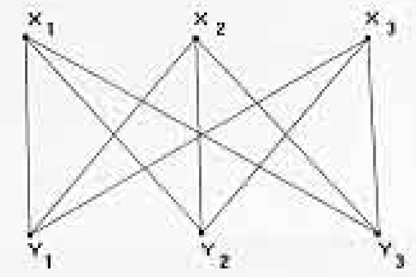

Figure 4. Draw nodes X_1, X_2, X_3 from left to right as one set of nodes arranged horizontally. Draw nodes Y_1, Y_2, Y_3 from left to right as another set of nodes arranged horizontally, below the first set. Draw edges X_1Y_1, X_1Y_2, X_1Y_3; X_2Y_1, X_2Y_2, X_2Y_3; X_3Y_1, X_3Y_2, X_3Y_3 to form the nonplanar $K_{3,3}$ graph.

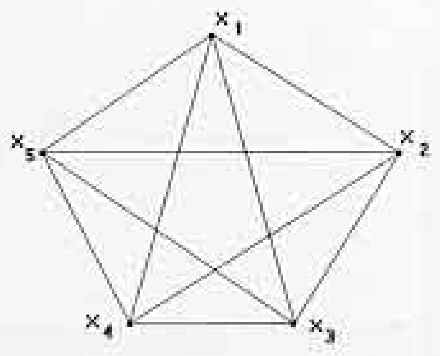

Figure 5. Draw nodes X_1, X_2, X_3, X_4, and X_5 arranged as nodes of a regular pentagon. Join the nodes as a pentagon: along edges X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_5X_1. Join the remaining nodes: along edges X_1X_3, X_1X_4, X_2X_4, X_2X_5, X_3X_1, X_3X_5.

Generally, one might look at geometric intersections to suggest whether or not a given graph is planar: simple-looking geometric intersection patterns can often be unscrambled in the plane to eliminate any geometric intersections (as was Figure 2 in Figure 3). More complicated geometric intersection patterns ($K_{3,3}$, K_5) cannot be undone (Harary, 1969). As with the four color problem, and as is often the case, what is a simple problem to consider is in fact a difficult one to solve. It was not until 1930 that Kuratowski finally solved the long-standing problem of characterizing planar graphs. The statement of the theorem is simple; its proof is not (see Harary, 1969, for proof).

Kuratowski's Theorem

A graph is planar if and only if it has no subgraph homeomorphic to K_5 or to $K_{3,3}$.

The K in the notation honors Kuratowski for his achievement. With this elegant theorem in hand, we now turn to consider planarity in the geographic world.

The Paris Metro

The Paris Metro is a subway system that, for the most part, under the streets of Paris, links the classical "Portes" – City "Gates"– to each other as the many routes criss-cross the Seine in association with the various bridges (Figure 6). The Paris Metro map is a graph; there are numerous nodes representing local stations along a single train route as well as larger stations at which one can transfer from one metro route to another. There are directed arcs, forming a cycle, in the south west of the map leading to the Porte d'Auteuil, and in the northwest leading to Pré St. Gervais. All other arcs represent two-way Metro linkages. The map is complicated in appearance; subway lines often follow surface traffic patterns. Pedestrians need access to subway routes from sidewalks. Indeed, the Paris Metro map reflects the surface pattern of the numerous rotary, star-shaped intersections and tortuous "rues" that add much to Parisian charm. The Metro graph is strongly connected; choose any two metro stops – they are mutually reachable within the entire system, although a transfer might be required. Any well-designed mass transit system should clearly have this style of connectedness, lest passengers be stranded. There are a number of nodes with indegree and outdegree in excess of four. Anyone who has traversed the maze of possible transfers at Montparnasse-Bienvenue, for example, will be aware of how complicated a trip from "here" to "there" can be. Because there are quite a few transfer nodes with a number of incident edges, it is natural to consider whether or not a $K_{3,3}$ or a K_5 might be contained as a subgraph of the Metro graph. David Singmaster has shown that the London Underground is non-planar; is the Metro graph planar?

Planarity and the Metro

Indeed, the Metro is not planar, either; when the map is strictly considered as a digraph, it is an easy matter to choose six nodes and a set of edges to form a $K_{3,3}$. If one wishes, however, to eliminate the possibility of a transfer from one train to the other, in order to have direct geo-graphical adjacency as well as graphical adjacency, it is also possible to find a $K_{3,3}$ under these tighter constraints.

The Metro stops of Etoile ("star") and Nation are joined on the north by a single Metro route arching across the northern part of the city; they are joined on the south by a single arch paralleling the southern perimeter of Paris; and, they are joined across a diametral route, through Châtelet as a "center," by a single Metro route passing under the Champs Elysées, Concord, Palais Royal, Hôtel de Ville and the Bastille. When Montparnasse-Bienvenue and Stalingrad are chosen also, as nodes intermediate on these southern and northern arches, along with Gare de l'Est as a final node, this set of nodes can be joined in a $K_{3,3}$ with

Figure 6. Map of the Paris Metro.

only direct geographic linkage (requiring no transfers) between pairs of nodes along distinct edges. Label the nodes as follows (Figure 7):

1. Etoile
2. Montparnasse-Bienvenue
3. Nation
4. Châtelet
5. Gare de l'Est
6. Stalingrad

Each odd-numbered node is joined to each even-numbered node along distinct edges, as required for a $K_{3,3}$. Thus the Paris Metro, viewed as a structural model, is nonplanar; to travel from Montparnasse-Bienvenue to the Gare de l'Est requires, when represented as a

Figure 7. Metro map with labeled nodes and distinguished edges linking the nodes.

map in the plane, that the edge from node 2 to node 5 cross at least one of the other edges of the $K_{3,3}$. The geographical and social implications of this lack of planarity are significant.

Significance of lack of planarity

One might imagine a subway system to exist in a plane parallel to the plane of surface traffic, some number of feet below the surface. Experience with even simple subway systems defeats this notion; trains run on elevated tracks in regions with high water tables or on landfill; their elevation is altered to cross natural barriers such as rivers. There is considerable topographic relief in most subway systems. Natural difficulties can force a subway system out of a planar environment. Thus, collisions between trains on different routes, in different (intersecting) planes, must be considered; the separation of routes into different layers (planes) offers protection from collision–except where the planes intersect.

In the case of Paris, there are Metro lines at different levels; trains enter selected stations at different depth levels. Passengers trying to switch from one Metro route to another at Montparnasse-Bienvenue may recall running up or down stairs and through connecting tunnels to execute a transfer. The Metro map shows the route north from Montparnasse-Bienvenue, toward Odéon, Châtelet, the Gare de l'Est, and the Porte de Clignancourt to "cross" routes 12 (from Mairie d'Issy to the Porte de la Chapelle) and 10 (from the Gare d'Orleans-Austerlitz to the Porte d'Auteuil). If these crossings were "real," rather than over- or under-passes, there could be serious metro collisions at them. Map evidence suggests that it is the Orleans/Clignancourt route that is at a different level as routes 10 and 12 intersect at nearby Sèvres Babylon station. A lack of planarity can be used to advantage by engineers planning new stations or new routes in a tightly-packed transport system.

References

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