THE UNIVERSITY OF MICHIGAN
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THE EVALUATION OF THE REFRACTION ANGLES IN GEODETIC MEASUREMENTS FROM TWO OBSERVATION STATIONS

By

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TABLE OF CONTENTS

TABLE OF CONTENTS (CONT'D)

LIST OF TABLES

LIST OF ILLUSTRATIONS

SYMBOLS AND ABBREVIATIONS

C = the constant of the perfect Gas Law

g = the acceleration due to the force of gravity

K = a constant which varies slightly with the moisture content of the air

R = the mean radius of curvature of an arc joining two points on the earth's surface

l_a = the distance between stations A and C

l_b = the distance between stations B and C

μ = the index of refraction of white light in the air

α = a measured vertical angle

θ = an angle subtended by an arc at the center of the earth

Ω = the refraction angle

T = the temperature of the air measured from absolute zero (-273° C.)

P = the pressure of the air

σ = the radius of curvature of a ray of light

k = the coefficient of refraction

h_a = the height of Station A above mean sea level

h_b = the height of Station B above mean sea level

h_c = the height of Point C above mean sea level

r_a = the probable error of the refraction angle at Station A

r_b = the probable error of the refraction angle at Station B

r_{h_a} = the probable error of the height of Point C computed from Station A

r_{h_b} = the probable error of the height of Point C computed from Station B

SYMBOLS AND ABBREVIATIONS (CONT'D)

$d\alpha_a$ = the probable error of the measured vertical angle at Station A

$d\alpha_b$ = the probable error of the measured vertical angle at Station B

$\frac{dT}{dh}$ = the lapse rate of temperature

EST = Eastern Standard Time

INTRODUCTION

Terrestrial refraction is the bending of a ray of light when it travels between any two points within the atmosphere. This phenomenon causes all objects to appear to be at higher elevations than they actually are, except when the air near the earth's surface is strongly heated, causing the density of air to increase with height so that the objects appear to be at lower elevations.

Terrestrial refraction is a matter of great importance in the subject of trigonometrical leveling. The exact knowledge of it is important in correcting the measured vertical angles resulting from the effect of refraction and reducing them to their actual values.

Previous investigations have dealt with the problem of refraction and some formulas have been developed. These formulas are based on meteorological factors, i.e. pressure, temperature, and lapse-rate of temperature along the path of light; but because of the difficulties of determining the exact values of the pressure, temperature and lapse rate of temperature, the computations arrived at by these formulas are uncertain in most cases.

The present investigation deals with a method of determining the refraction angles in which the meteorological factors are eliminated from the computation.

The method involves observing one point from two observation stations. These observation stations are located in such a way that the paths of light rays from the point to the two stations will be close to each other; also, the difference between the lengths of the two paths is such as to provide an appreciable discrepancy between the apparent heights of the observed point.

On the basis of this method, new formulas have been developed. A practical application of the new method was evolved and tested by running many experiments in different weather conditions. The results show that an adequate degree of accuracy was achieved.

CHAPTER I

GENERAL STATEMENTS

Two methods exist for determining the difference in elevation between points on the earth's surface. The first is by running lines of spirit level. The principle of this method is well-known and can be found in any surveying text. Moreover, the accuracy of this method is attested by any surveying text. The second method is trigonometrical leveling, by which the differences in elevations are determined from the horizontal or geodetic distances between points and the measured vertical angles after correcting them for the effect of curvature and refraction.

The second method is faster and easier than the first, but it is not as accurate as the first method, mainly because of the uncertainty in determining the refraction angles.

Refraction of light in air has been recognized for a long time, and some coefficients have been introduced; but none of these coefficients can be used with confidence unless the meteorological factors along the path of a ray of light are the same as those under which the coefficients were introduced. Also, previous investigations have produced some formulas for computing the refraction angles, but the accuracy of the results computed from these formulas depends upon exact knowledge of the pressure, temperature, and lapse rate of temperature along the paths of light.

Bomford's Geodesy[1] gives the following formula for computing

1 Brigadier G. Bomford, Geodesy, Great Britian: Oxford at the Clarendon Press, 1952, p. 153, formula 4.15

the refracting angles: (Figure 1)

$$\Omega = \frac{1}{L}\int_{0}^{L} \frac{L-\ell}{\sigma}\, d\ell \,, \qquad (1\text{-}1)$$

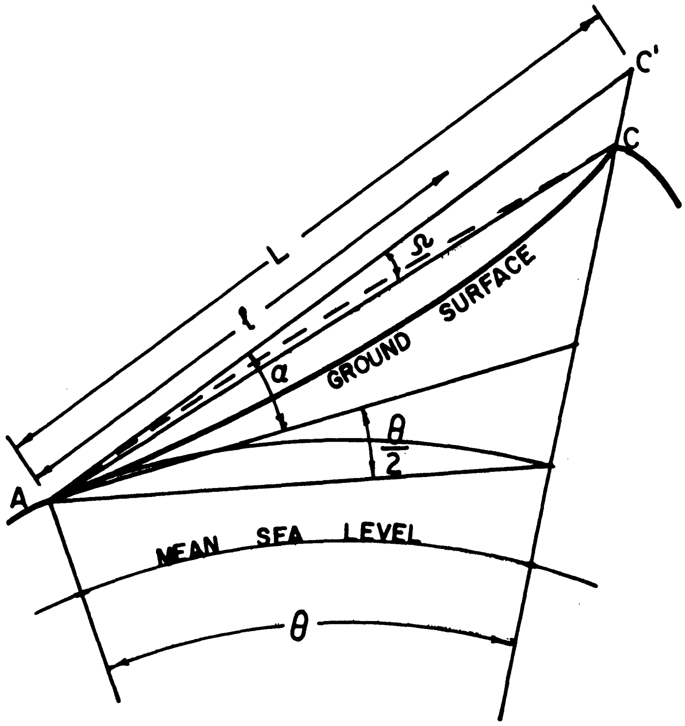

Fig. 1 EFFECT OF REFRACTION

where

Ω = the refraction angle,

L = the length of the path of a ray of light,

ℓ = a length along the path of light measured from the observation point,

σ = the radius of curvature of the points along a path of a ray of light.

$\frac{1}{\sigma}$ is given by the following formula:[1]

$$\frac{1}{\sigma} = \frac{K\cos\alpha}{C\mu} \frac{P}{T^2}\left(\frac{g}{C} + \frac{dT}{dh}\right), \tag{1-2}$$

where

K = a constant which varies slightly with the moisture of the air,

α = the measured vertical angle,

P = the pressure,

C = the constant of the perfect gas law,

μ = the index of refraction of white light in air,

T = the temperature measured from absolute zero (-273° C.)

g = the acceleration due to the force of gravity,

$\frac{dT}{dh}$ = the lapse rate of temperature.

Substituting equation (1-2) in equation (1-1), one gets

$$\Omega = \frac{1}{L}\int_0^L \left[\frac{K\cos\alpha}{C\mu} \frac{P}{T^2}\left(\frac{g}{C} + \frac{dT}{dh}\right)\right]\left(L - l\right) dl. \tag{1-3}$$

Equation (1-3) is the fundamental formula for computing the refraction angle Ω, and in order to determine Ω with adequate accuracy, pressure, temperature, and lapse rate of temperature must be known at all points along the path of a ray of light. Since it is impossible to measure the meteorological factors at so many points, this is an impracticable method of determining the refraction angles, and the method usually followed is to assume that the lapse rate of temperature is constant along the path of a ray of light and that the average temperatures and pressures along the path are those computed or measured at one-third the distance along

1 Ibid, formula 4.12

the path measured from the observation station. This method of substitution probably gives good results if the lapse rate of temperature is determined exactly, and if the assumed values of the pressure and temperature are equal to the actual values. Practically, the pressure and the temperature may be determined within one percent or better,[1] but the doubtful item is the lapse rate of temperature. For example, attempts were made in connection with the present investigation to measure the lapse rate of temperature, and the following results were obtained: On March 21, 1957, five attempts were made which gave results that varied from 0.00221 degrees Centigrade per foot to 0.00725 degrees Centigrade per foot. On March 28, 1957, seven attempts were made which gave results that varied from 0.00242 to 0.00672 degrees Centigrade per foot. Such variations make it difficult to determine the exact value of the lapse rate of temperature; and an uncertainty of 0.001 in the lapse rate of temperature causes an uncertainty of 10 per cent in the computed angle of refraction[2]. Therefore, refraction angles computed from formulas based on the knowledge of the meteorological factors are uncertain.

Up to the present time there has been no method of determining the refraction angles with acceptable accuracy and without including the pressure, the temperature, and the lapse rate of temperature in the computation. No doubt the discovery of such a method will be a great benefit in the determination of terrestrial refraction.

1 Ibid, line 20.

2 Ibid, line 28.

CHAPTER II

METHOD AND FORMULAS

The density of air is not constant, but varies with height; therefore a ray of light passing between two points within the atmosphere suffers a continuous change in direction. Thus, the path of a ray of light is a curve in its shape and not a straight line. This phenomenon causes displacement of the apparent positions of the observed objects.

Relative Refraction.

In Figure 2 assume A P C is the path of a ray of light from point A at height h_a to point C at height h_c. Assume α_a is the observed angle of elevation C at point A above A D, which is the tangent to the earth's level surface. Assume A C_a is the tangent to the path A P C at point A. Then the angle C_a A C (Ω_a) will be the angle of refraction and C C_a the appeared displacement of point C.

$$\Omega_a = k_a \, \theta_a \, , \qquad (2\text{-}1)$$

where

k_a = the coefficient of refraction,

θ_a = the angle subtended by the arc A F at the earth's center.

Bomford's Geodesy gives the following equation for determining the coefficient of refraction[1]:

$$k_a = 28000 \frac{P}{T^2} \left(0.0104 + \frac{dT}{dh} \right) , \qquad (2\text{-}2)$$

1 Ibid., p. 159, formula 4.17a

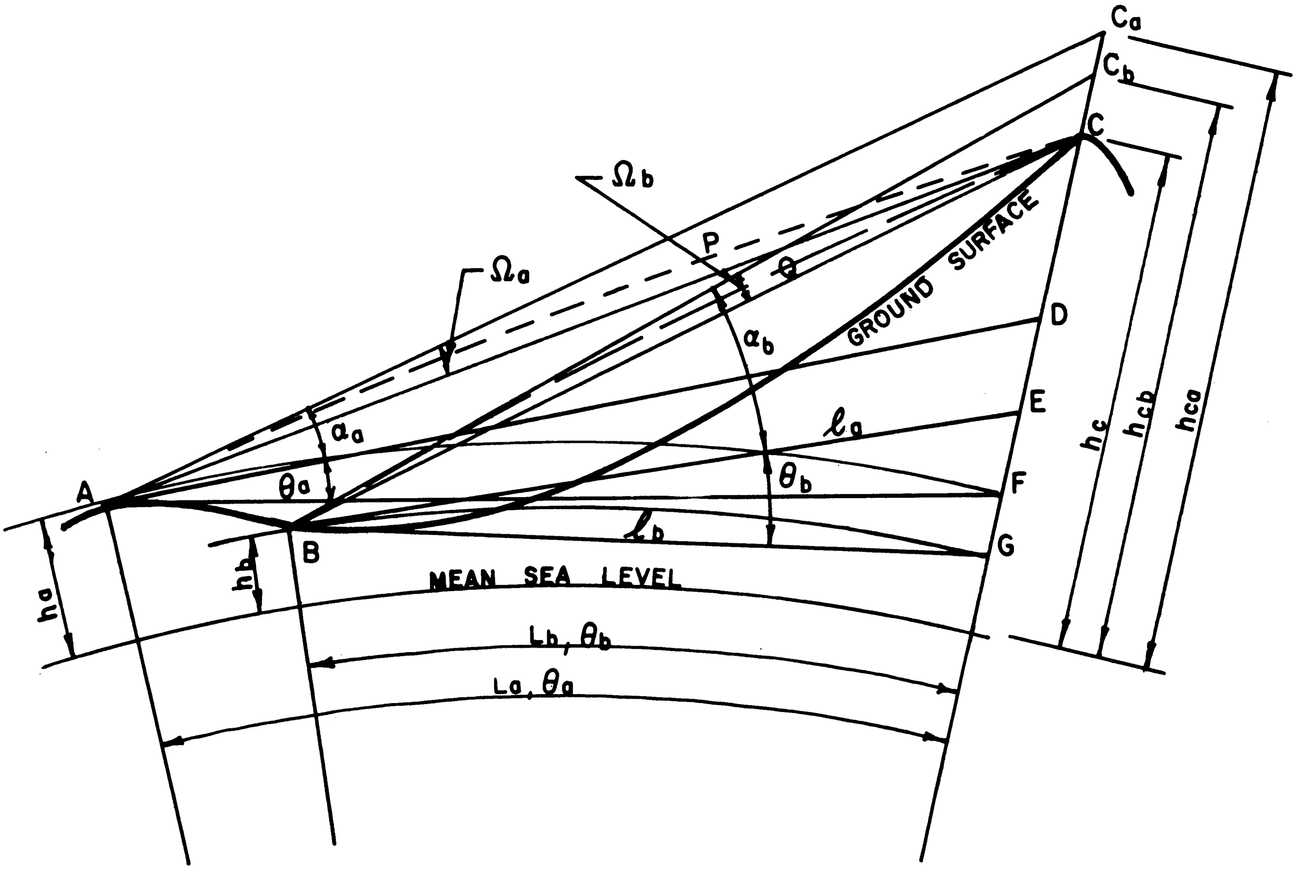

Fig. 2 RELATIVE POSITIONS OF THE OBSERVATION STATIONS A AND B

where

k = the coefficient of refraction,

P = the pressure measured in inches of mercury,

T = the temperature measured from absolute zero (-273° C.),

$\frac{dT}{dh}$ = the lapse rate of temperature in Centigrade per foot.

The constant 28000 is given as 27000 while actually it should be 28000.

Substituting equation (2-2) in equation (2-1), one obtains

$$\Omega_a = 28000 \frac{P_a}{(T_a)^2} \left[0.0104 + \left(\frac{dT}{dh}\right)_a \right] \theta_a \, , \qquad (2\text{-}3)$$

where

P_a = the average pressure along the path A P C measured in inches of mercury,

T_a = the average temperature along the path A P C measured from absolute zero (-273° C.),

$\left(\frac{dT}{dh}\right)_a$ = the lapse rate of temperature along the path A P C measured in Centigrade per foot.

The lengths of sight in geodetic measurements are relatively very short compared with the length of the radius of the earth, and it is safe to assume, therefore, that

$$\theta_a = \frac{l_a}{R_a + h_a} \, , \qquad (2\text{-}4)$$

where

l_a = the length of the arc AF,

R_a = the radius of curvature of the earth's surface taken at the middle distance between points A and F.

Substituting equation (2-4) in equation (2-3), we have

$$\Omega_a = 28000 \frac{P_a}{(T_a)^2} \left| 0.0104 + \left(\frac{dT}{dh}\right)_a \right| \frac{l_a}{R_a + h_a} \qquad (2\text{-}5)$$

Also, in figure 1, assume point B is another observation station, and let B Q C be the path of a ray of light from point B at height h_b to point C. Assume α_b is the observed angle of elevation of point C at point B above B E, which is the tangent to the earth's surface. Assume B C_b is the tangent to the path B Q C at point B. Then the angle of refraction will be C B C_b(Ω_b), and C C_b is the apparent displacement of point C.

Following the same procedure in deriving equation (2-5), one obtains the following equation for the refraction angle at station B,

$$\Omega_b = 28000 \frac{P_b}{(T_b)^2} \left[0.0104 + \left(\frac{dT}{dh}\right)_b \right] \frac{l_b}{R_b + h_b}, \qquad (2\text{-}6)$$

where

Ω_b = the refraction angle at station B,

P_b = the average pressure along the path B Q C measured in inches of mercury,

T_b = the average temperature along the path B Q C measured from absolute zero (-273° C.),

$\left(\frac{dT}{dh}\right)_b$ = the lapse rate of temperature along the path B Q C in Centigrade per foot,

l_b = the length of the arc BG,

R_a = the radius of curvature of the earth's surface taken at the middle distance between points B and G.

Dividing equations (2-5) and (2-6), one gets

$$\frac{\Omega_a}{\Omega_b} = \frac{28000 \frac{P_a}{(T_a)^2}\left[0.0104 + \left(\frac{dT}{dh}\right)_a\right]\frac{\rho_a}{R_a + h_a}}{28000 \frac{P_b}{(T_b)^2}\left[0.0104 + \left(\frac{dT}{dh}\right)_b\right]\frac{\rho_b}{R_b + h_b}}$$

or

$$\frac{\Omega_a}{\Omega_b} = \frac{\rho_a (T_b)^2 P_a (R_b + h_b)\left[0.0104 + \left(\frac{dT}{dh}\right)_a\right]}{\rho_b (T_a)^2 P_b (R_a + h_a)\left[0.0104 + \left(\frac{dT}{dh}\right)_b\right]} \tag{2-7}$$

Positions of the Observation Stations.

In Figure 2, assume A and B are two observation stations, located so that the paths of sight from the stations A and B to point C have nearly the same azimuth. The distance between the stations A and B is about 2 miles, and the elevations of A and B can differ in accordance with the requirements of the terrain (a difference of a few hundred feet does not affect the results). Then the paths A P C and B Q C will be close to each other; hence the lapse rates of temperatures $\left(\frac{dT}{dh}\right)_a$ and $\left(\frac{dT}{dh}\right)_b$, the average pressures P_a, P_b, and the average temperatures T_a, T_b can be assumed equal.

Determination of the Refraction Angles

When the lapse rate of temperatures, the average pressures, and the average temperatures are taken equal, and if these equalities are substituted in equation (2-7), then the equation becomes

$$\frac{\Omega_a}{\Omega_b} = \frac{\rho_a (R_b + h_b)}{\rho_b (R_a + h_a)}. \tag{2-8}$$

The terms $(R_a + h_a)$ and $(R_b + h_b)$ can be assumed equal, so that equation (2-8) becomes

$$\frac{\Omega_a}{\Omega_b} = \frac{l_a}{l_b} \; . \tag{2-9}$$

In Figure 1, we have

$$CC_a = \Omega_o \, l_a' \, , \tag{2-10}$$

where

Ω_a = the refraction angle at point A,

l_a' = the length of the path A P C;

and

$$CC_b = \Omega_b \, l_b' \, , \tag{2-11}$$

where

Ω_b = the refraction angle at point B,

l_b' = the length of the path B Q C.

Subtracting equation (2-11) from equation (2-10), one obtains

$$CC_a - CC_b = \Omega_a \, l_a' - \Omega_b \, l_b' \, ,$$

or

$$C_bC_a = \Omega_a \, l_a' - \Omega_b \, l_b' \; . \tag{2-12}$$

In Figure 3, we have

$$\frac{FC_a}{\sin\left(\alpha_a + \frac{\theta_a}{2}\right)} = \frac{AF}{\sin\left[90 - \left(\alpha_a + \theta_a\right)\right]} \, ,$$

or

$$FC_a = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) \, , \tag{2-13}$$

and

$$h_{ca} = FC_a + h_a \, . \tag{2-14}$$

Substituting equation (2-13) in equation (2-14), one gets

$$h_{ca} = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) + h_a \, . \tag{2-15}$$

In Figure 4, we have

$$\frac{GC_b}{\sin\left(\alpha_b + \frac{\theta_b}{2}\right)} = \frac{BG}{\sin\left[90 - \left(\alpha_b + \theta_b\right)\right]} \, ,$$

or

$$GC_b = l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) \, ; \tag{2-16}$$

and

$$h_{cb} = GC_b + h_b \, . \tag{2-17}$$

Substituting equation (2-16) in equation (2-17), one gets

$$h_{cb} = l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) + h_b \, . \tag{2-18}$$

Subtracting equation (2-18) from (2-15), one obtains

$$h_{ca} - h_{cb} = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) + h_a - \left[l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) + h_b\right]$$

or

$$C_aC_b = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) - l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) + \left(h_a - h_b\right) \, . \tag{2-19}$$

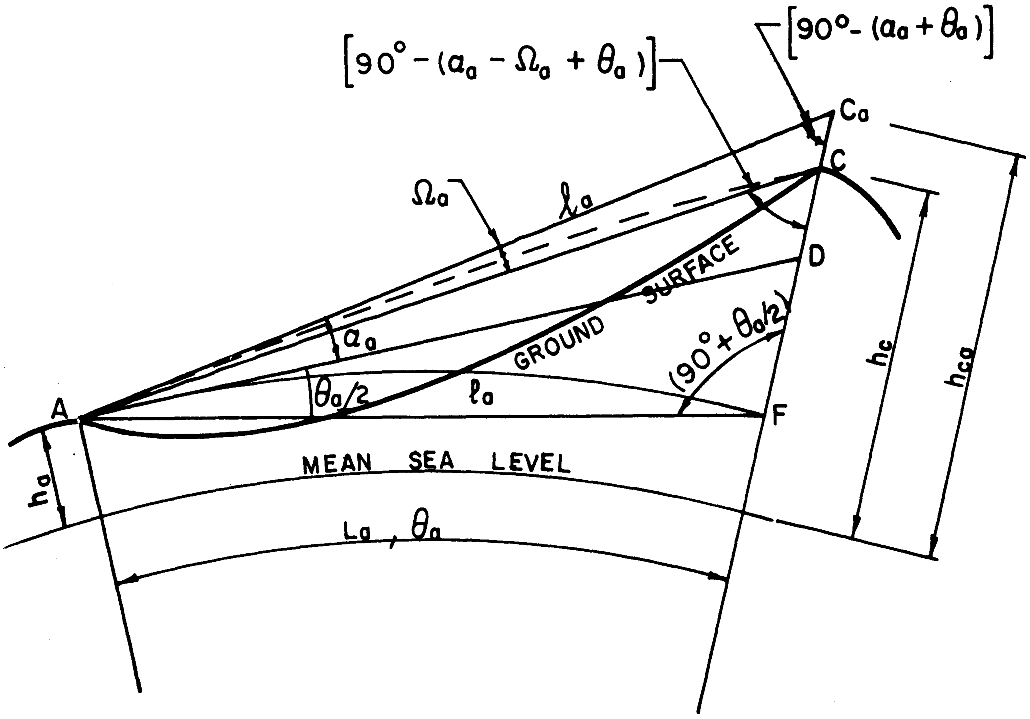

Fig. 3 OBSERVATION FROM STATION A

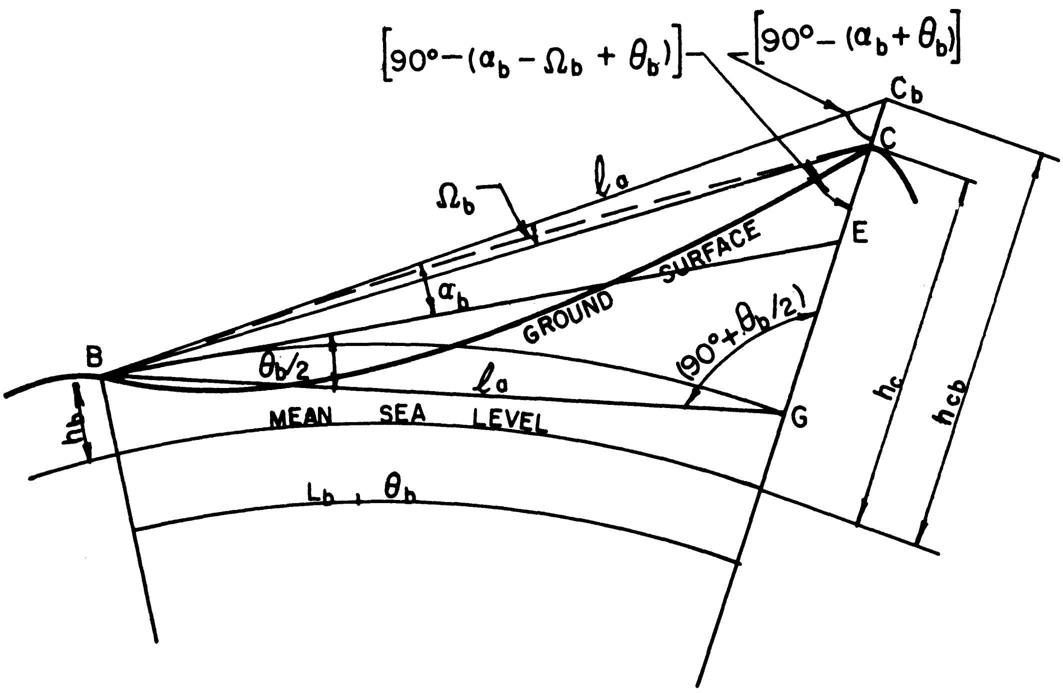

Fig. 4 OBSERVATION FROM STATION B

Substituting equation (2-19) in equation (2-12) we obtain

$$\Omega_a \ell_a' - \Omega_b \ell_b' = \ell_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) - \ell_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) + \left(h_a - h_b\right). \quad (2\text{-}20)$$

From equation (2-9), we have

$$\Omega_b = \frac{\ell_b}{\ell_a} \Omega_a . \quad (2\text{-}21)$$

Substituting equation (2-21) in equation (2-20), one gets

$$\left(\ell_a' - \frac{\ell_b}{\ell_a} \ell_b'\right) \Omega_a = \ell_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) - \ell_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) + \left(h_a - h_b\right),$$

or

$$\Omega_a = \frac{\ell_a}{\ell_a \ell_a' - \ell_b \ell_b'} \left[\ell_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) - \ell_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) + \left(h_a - h_b\right) \right]. \quad (2\text{-}22)$$

Applying the same procedure above and substituting for Ω_a in equation (2-20), one obtains

$$\Omega_b = \frac{\ell_b}{\ell_a \ell_a' - \ell_b \ell_b'} \left[\ell_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) - \ell_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right) + \left(h_a - h_b\right) \right]. \quad (2\text{-}23)$$

When the lines of sight are nearly horizontal, the lengths ℓ_a and ℓ_a', ℓ_b and ℓ_b' are very nearly equal and may be assumed equal without affecting the results. For the same reason, $\ell_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right)$ and $\ell_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec\left(\alpha_b + \theta_b\right)$ can consequently be assumed equal to $\ell_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right)$ and $\ell_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right)$. Applying these equalities in equations

(2-22) and (2-23) we obtain

$$\Omega_a = \frac{\ell_a}{(\ell_a)^2-(\ell_b)^2}\left[\ell_a \tan\left(\alpha_a+\frac{\theta_a}{2}\right) - \ell_b \tan\left(\alpha_b+\frac{\theta_b}{2}\right) + \left(h_a-h_b\right)\right], \qquad (2\text{-}24)$$

and

$$\Omega_b = \frac{\ell_b}{(\ell_a)^2-(\ell_b)^2}\left[\ell_a \tan\left(\alpha_a+\frac{\theta_a}{2}\right) - \ell_b \tan\left(\alpha_b+\frac{\theta_b}{2}\right) + \left(h_a-h_b\right)\right]. \qquad (2\text{-}25)$$

When the lines of sight are considerably inclined, then ℓ_a', ℓ_b' can be computed as follows (See Figure 3):

$$\frac{ACa}{\sin\left(90+\frac{\theta_a}{2}\right)} = \frac{AF}{\sin\left[90-\left(\alpha_a+\theta_a\right)\right]},$$

or

$$\ell_a' = \ell_a \cos\frac{\theta_a}{2} \sec\left(\alpha_a+\theta_a\right).$$

The term $\cos(\frac{\theta_a}{2})$ is very small and may be taken to equal one, so that

$$\ell_a' = \ell_a \sec\left(\alpha_a+\theta_a\right). \qquad (2\text{-}26)$$

Using Figure 4, we have

$$\frac{BC_b}{\sin\left(90+\frac{\theta_b}{2}\right)} = \frac{BG}{\sin\left[90-\left(\alpha_b+\theta_b\right)\right]}$$

or

$$\ell_b' = \ell_b \cos\left(\frac{\theta_b}{2}\right) \sec\left(\alpha_b+\theta_b\right)$$

The term $\cos(\frac{\theta_b}{2})$ is very small and may be taken to equal one so that

$$l_b' = l_b \sec\left(\alpha_b + \theta_b\right). \tag{2-27}$$

Substituting equations (2-26) and (2-27) in equations (2-22) and (2-23), we obtain

$$\Omega_a = \frac{l_a}{(l_a)^2 \sec(\alpha_a+\theta_a) - (l_b)^2 \sec(\alpha_b+\theta_b)} \Big[l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a+\theta_a) - l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec(\alpha_b+\theta_b) + (h_a - h_b) \Big], \tag{2-28}$$

and

$$\Omega_b = \frac{l_b}{(l_a)^2 \sec(\alpha_a+\theta_a) - (l_b)^2 \sec(\alpha_b+\theta_b)} \Big[l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a+\theta_a) - l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec(\alpha_b+\theta_b) + (h_a - h_b) \Big]. \tag{2-29}$$

<u>Determination of the Height</u>.

In Figure 3, we have

$$h_c = h_{Ca} - C_a C. \tag{2-30}$$

Substituting equations (2-10) and (2-15) in equation (2-30), one obtains

$$h_c = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec\left(\alpha_a + \theta_a\right) - \Omega_a l_a' + h_a. \tag{2-31}$$

Substituting equation (2-26) in equation (2-31), one gets

$$h_c = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a + \theta_a) - \Omega_a l_a \sec(\alpha_a + \theta_a) + h_a \cdot \quad (2\text{-}32)$$

For small angles of α_a and θ_a , one can reduce equation (2-32) to

$$h_c = l_a \tan\left(\alpha_b + \frac{\theta_a}{2}\right) - \Omega_a l_a + h_a \cdot \quad (2\text{-}33)$$

Following the same procedure for the measurements at point B, we obtain

$$h_c = l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec(\alpha_b + \theta_b) - \Omega_b l_b \sec(\alpha_b + \theta_b) + h_b \quad (2\text{-}34)$$

And for small angles of α_b and θ_b , one can reduce equation (2-34) to

$$h_c = l_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right) - \Omega_b l_b + h_b \quad (2\text{-}35)$$

The values of Ω_a and Ω_b are given in equations (2-28) and (2-29).

Air Density Inversion.

When the air is strongly heated because of the radiation of heat from the earth's surface, so that the density of the air increases with the height, then the paths of light rays bend downward convexly. This is an exceptional case, but sometimes it does happen on a hot day over sandy terrain. This phenomenon causes the objects to appear to be at lower elevations.

Figure 5 illustrates the courses of the paths of light, and shows the apparent positions of point C when it is observed under such conditions from points A and B. This case can be detected when the position of point C appears to be at a lower elevation from point A than from point B.

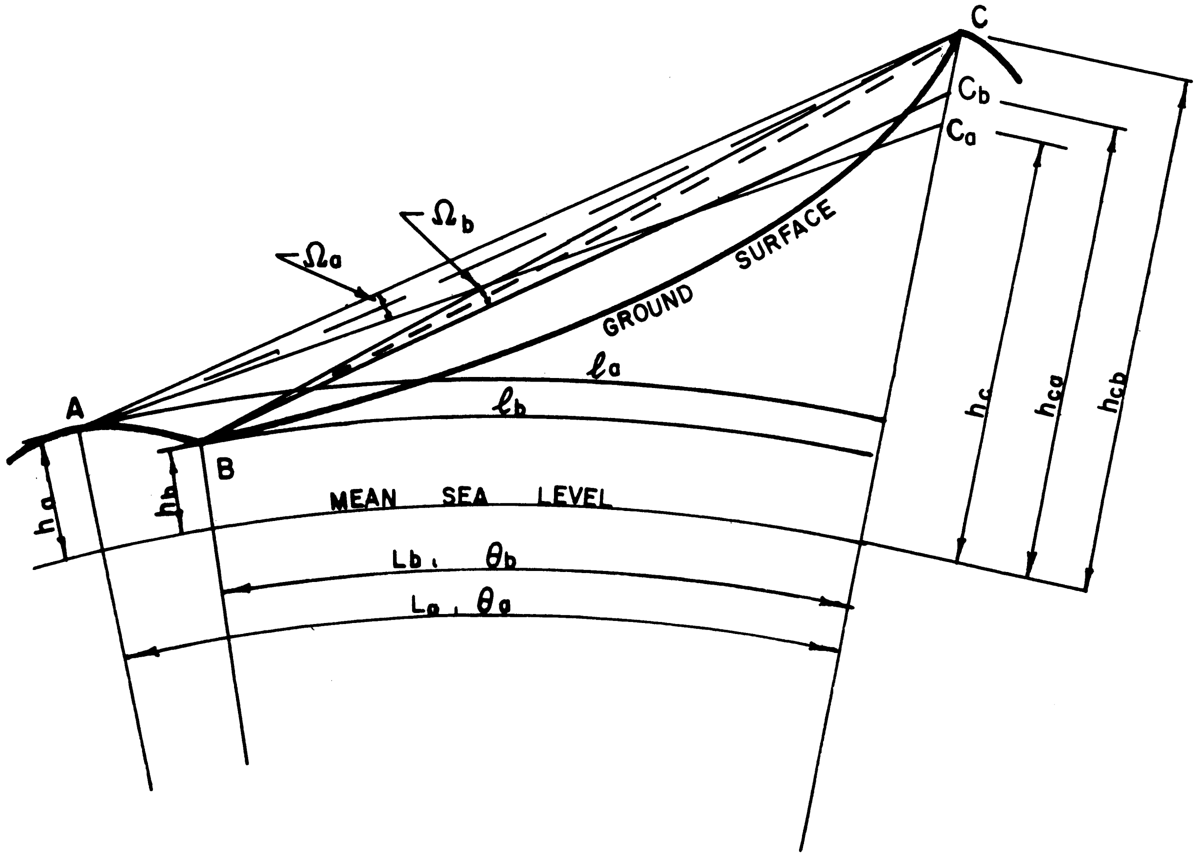

Fig. 5 AIR DENSITY INVERSION

The method of calculation is the same as that given previously in this chapter, except in the determination of height, where the error due to refraction must be added to the apparent height, so that equations (2-32) and (2-34) become

$$h_c = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a + \theta_a) + \Omega_a l_a \sec(\alpha_a + \theta_a) + h_a ; \qquad (2\text{-}36)$$

and

$$h_c = l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec(\alpha_b + \theta_b) + \Omega_b l_b \sec(\alpha_b + \theta_b) + h_b . \qquad (2\text{-}37)$$

CHAPTER III

DISCUSSION OF ERRORS

After presenting the method of determining the refraction angles and the height of a point from two observation stations, it is desirable to discuss the errors and the percentage of these errors that might occur from the assumptions used. Also, in order to complete the discussion of errors, it is necessary to derive formulas for computing the probable errors due to the probable errors of measurements.

Errors of Assuming Equal Meteorological Factors Along the Paths.

The lapse rate of temperature is the most important factor in the determination of the atmospheric refraction. Its value changes rapidly early in the morning and late in the afternoon, and its variability is least at mid-day, that is from about the 11th hour to about the 14th hour. Therefore, for more accurate work, the measurements must be made during these hours.

In Chapter II, the relative positions of the observation stations are given, and it is pointed out that the two paths are traveling in the same direction, over the same terrain, and close to each other; therefore the lapse rates of temperature along the two paths are equal. For the same reasons the average pressures and temperatures along the paths are equal when the paths coincide, or there will be small discrepancies when they do not coincide, but these have a negligible effect on the results. For example, make the following assumptions: that station A and B in Figure 2 are located at distances of 25 and 23 miles respectively from point C; and that the elevation of station B is lower than that of station A by 500 feet; and that the elevation angle at station A is 3 degrees. These assumptions will provide about 170 feet difference in elevations between the two paths at one-third the distance between A and C measured from station A. Then the discrepancy in the pressure is about 0.2 inches of mercury and in the temperature is about 0.5 degrees Centigrade. These discrepancies affect the results by 1 per cent. Therefore it is safe to assume that the average temperatures and pressures are equal along the paths.

Errors in the Difference of the Apparent Elevations.

The difference of the apparent elevations C_a C_b[1] is an important factor in determining the refraction angles; thus it is important to know the percentage of the probable error in the length C_a C_b.

Table 1 has been prepared to show the percentage of the probable errors for different lengths of sight. The table was prepared on the assumption that angular errors are independent of the lengths of sight; further, it is assumed that the probable error in measuring the vertical angles is 0.3 seconds of arc, and the refraction rate is 3.5 seconds per mile. These two assumptions are based on the results which were obtained in connection with the present investigation under good weather conditions.

The results which are given in column 6 of Table 1 are reasonable; however, to secure better results in long lengths of sight, such as 20 miles or more, the distance between stations A and B may be made more than 2 miles.

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1 C_a C_b is given in Figure 2.

TABLE I

PERCENT OF ERROR IN THE LENGTH $C_a C_b$

Lengths of sight in miles	Refraction errors in feet	Difference $C_a C_b$	Probable errors in feet	Total Probable Errors in C_aC_b feet	Percent of Error
5	2.238	2.149	0.0384	0.066	3
7	4.387		0.0537		
10	8.954	3.939	0.0767	0.12	3
12	12.893		0.0921		
20	35.816	7.521	0.153	0.23	3
22	43.337		0.169		
30	80.586	11.104	0.230	0.34	3
32	91.690		0.246		
50	223.850	18.270	0.384	0.55	3
52	242.120		0.399		

Probable Error of the Refraction Angles.

Equation (2-28) is

$$\Omega_a = \frac{l_a}{(l_a)^2 \sec(\alpha_a+\theta_a) - (l_b)^2 \sec(\alpha_b+\theta_b)} \Big[l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a+\theta_a) - l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec(\alpha_b+\theta_b) + (h_a - h_b) \Big] .$$

In geodetic measurements the angles $(\alpha_a + \theta_a)$ and $(\alpha_b + \theta_b)$ are usually small and in most cases do not exceed 5 degrees of arc; therefore, for the purpose of computing the probable errors, secants of the angles $(\alpha_a + \theta_a)$ and $(\alpha_b + \theta_b)$ can be assumed equal to one. Also $l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a + \theta_a)$ and $l_b \sin\left(\alpha_b + \frac{\theta_b}{2}\right) \sec(\alpha_b + \theta_b)$ can consequently be assumed equal to $l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right)$ and $l_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right)$. Substituting these in equation (2-28), we obtain

$$\Omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2}\left[l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right) - l_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right) + (h_a - h_b)\right] \tag{3-1}$$

Differentiating equation (3-1), one gets

$$d\Omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2}\left[l_a \sec^2\left(\alpha_a + \frac{\theta_a}{2}\right) d\alpha_a - l_b \sec^2\left(\alpha_b + \frac{\theta_b}{2}\right) d\alpha_b\right] \cdot \tag{3-2}$$

For the same reasons which are given above, $\sec^2\left(\alpha_a + \frac{\theta_a}{2}\right)$ and $\sec^2(\alpha_b + \theta_b)$ can be assumed equal to one, therefore equation (3-2) becomes,

$$d\Omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2}\left(l_a\, d\alpha_a - l_b\, d\alpha_b\right). \tag{3-3}$$

Applying the method of propagation of errors[1], we have

$$r_a = \pm \frac{l_a}{(l_a)^2 - (l_b)^2}\sqrt{(l_a\, d\alpha_a)^2 + (l_b\, d\alpha_b)^2}, \tag{3-4}$$

where

r_a = probable error of the refraction angle Ω_a,

l_a = the distance between A and C in miles,

l_b = the distance between B and C in miles,

1 Mansfield Merriman, Method of Least Square, New York: John Willey and Sons, Inc., 1911, p.79, formula (30)

$d\alpha_a$ = the probable errors of the measured vertical angle α_a ,

$d\alpha_b$ = the probable errors of the measured vertical angle α_b ,

By the same procedure the probable errors for the refraction angle Ω_b will be calculated thus:

$$r_b = \pm \frac{l_b}{(l_a)^2 - (l_b)^2} \sqrt{(l_a \, d\alpha_a)^2 + (l_b \, d\alpha_b)^2} \, . \tag{3-5}$$

<u>Probable Error of the Computed Height.</u>

Equation (2-32) is:

$$h_c = l_a \sin\left(\alpha_a + \frac{\theta_a}{2}\right) \sec(\alpha_a + \theta_a) - \Omega_a l_a \sec(\alpha_a + \theta_a) + h_a$$

For the same reasons as those given in line 5, page 20, one can reduce equation (2-32) to

$$h_c = l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right) - l_a \Omega_a + h_a \, . \tag{3-6}$$

Differentiating equation (3-6), we obtain

$$dh_c = l_a \sec^2\left(\alpha_a + \frac{\theta_a}{2}\right) d\alpha_a - l_a \, d\Omega_a \, , \tag{3-7}$$

$\sec^2(\alpha_a + \frac{\theta_a}{2})$ can be assumed equal to one, hence

$$dh_c = l_a \, d\alpha_a - l_a \, d\Omega_a \, . \tag{3-8}$$

Applying the method of propagation of errors[1], we have

$$r_{h_a} = \pm \, l_a \sqrt{(d\alpha_a)^2 + (d\Omega_a)^2} \, , \tag{3-9}$$

or

$$r_{h_a} = \pm \, 0.02558 \, l_a \sqrt{(d\alpha_a)^2 + (r_a)^2} \, , \tag{3-10}$$

1 Ibid

where

r_{h_a} = the probable error of the computed height from point A in feet,

$d\alpha_a$ = the probable error of the measured vertical angle at station A in seconds of arc,

r_a = the probable error of the refraction angle at station A, and its value as given in equation (3-4),

l_a = the distance between stations A and point C in miles.

By the same procedure, the probable error of the height computed from point B is calculated as

$$r_{h_b} = 0.02558\, l_b \sqrt{(d\alpha_b)^2 + (r_b)^2}\,, \tag{3-11}$$

where

r_{h_b} = the probable error of the computed height from point B in feet,

$d\alpha_b$ = the probable error of the measured vertical angle at station B in seconds of arc,

r_b = the probable error of the refraction angle at station A, and its value as given in equation (3-5),

l_b = the distance between station B and point C in miles.

CHAPTER IV

EXPERIMENTS

The experiments, which were carried out during the months of October, 1956 to May 29, 1957, provided a good verification of the method given in Chapter 2.

Preparations.

The experiments were done in Ann Arbor, Michigan. The area is relatively flat and it is covered with trees which make it difficult to find two observation stations on the ground that fulfill the requirement stated in Chapter II. For this reason the two observation stations selected were located on the roofs of two buildings.

The spot selected as Station A was situated over a 4 inch pipe on the northeast corner of the roof of the Water Treatment Plant in Ann Arbor, Michigan. Station B was located at distances of 6.53 feet north and 31.58 feet west of the flagpole of the hospital of the University of Michigan. And the observed point C was taken to be the middle height of the cap at the top of the water stand pipe in Ypsilanti, Michigan.

The instrument used in measuring the horizontal and vertical angles was the Wild Theodolite Type T_2 No. 12137, by which the second of an angle can be read and the tenth of a second can be estimated by the observer. The theodolite was equipped with a striding level in which one division corresponds to 5 seconds of arc.

The plane coordinates position of Station A was determined from the three known Michigan State coordinate position of Barton Hills Golf

Club water tank in Ann Arbor Township, the east water tank of the Ford Bomber Plant in Ypsilanti, and the State Hospital water tank near Ypsilanti, and they are equal to

$$X = 474031.38$$

$$Y = 290334.53$$

The plane coordinates position of Station B was determined from the known Michigan State coordinate position of the flagpole of the University of Michigan Hospital, and they were found equal to

$$X = 482507.45$$

$$Y = 285288.53$$

The Michigan State coordinates position of point C (water stand pipe in Ypsilanti) was known and equal to

$$X = 511351.04$$

$$Y = 271653.72$$

The distances between the points A and C, and B and C were computed from the plane coordinates positions. It was found that

the distance between A and C = 41734.04 feet

the distance between B and C = 31903.93 feet

These distances are at the mean sea level, and when they were reduced to the average height between the stations above mean sea level, it was found that

the distance between A and C = 41735.93 feet

the distance between B and C = 31905.30 feet

The angles subtended by the arc A F(θ_a) and by the arc B G(θ_b) were computed from the formula $\theta = \frac{L}{R}$,

where

L = the distance between two points on the earth's surface,

R = the mean radius of curvature of the arc joining the two points on the earth's surface.

The values of the radii of curvature were taken from Table XI which is given in Hosmer's Geodesy.[1] The values of the angles are

$$\theta_a = 00° - 06' - 51.048''$$

$$\theta_b = 00° - 05' - 14.206''$$

Bench marks near stations A and B were located. The elevations of these bench marks were determined from geodetic bench marks by spirit level lines and trigonometry. The elevation of point C was determined from geological bench mark, and the difference between the geodetical and geological data was considered. Each elevation was carried at least twice, and when the discrepancy between two independent elevations was within the margin of probable error, the mean was taken and adopted as the elevation of the bench marks at the station.

The bench mark near Station A was located on the north edge of the sill of the east window of the upper room and was found equal to 1039.911 feet above mean sea level. The bench mark near Station B was located on the lower side of the iron belt around the flagpole and its elevation was equal to 1007.857 feet above mean sea level. The height of point C was found equal to 928.552 feet above mean sea level.

Procedure of Measuring the Vertical Angles.

The vertical angles were measured consecutively from one station to another. Each set of measurements was made by taking sixteen readings,

1 Hosmer, Geodesy, 2nd ed., New York: John Wiley & Sons, Inc., 1946, p. 428.

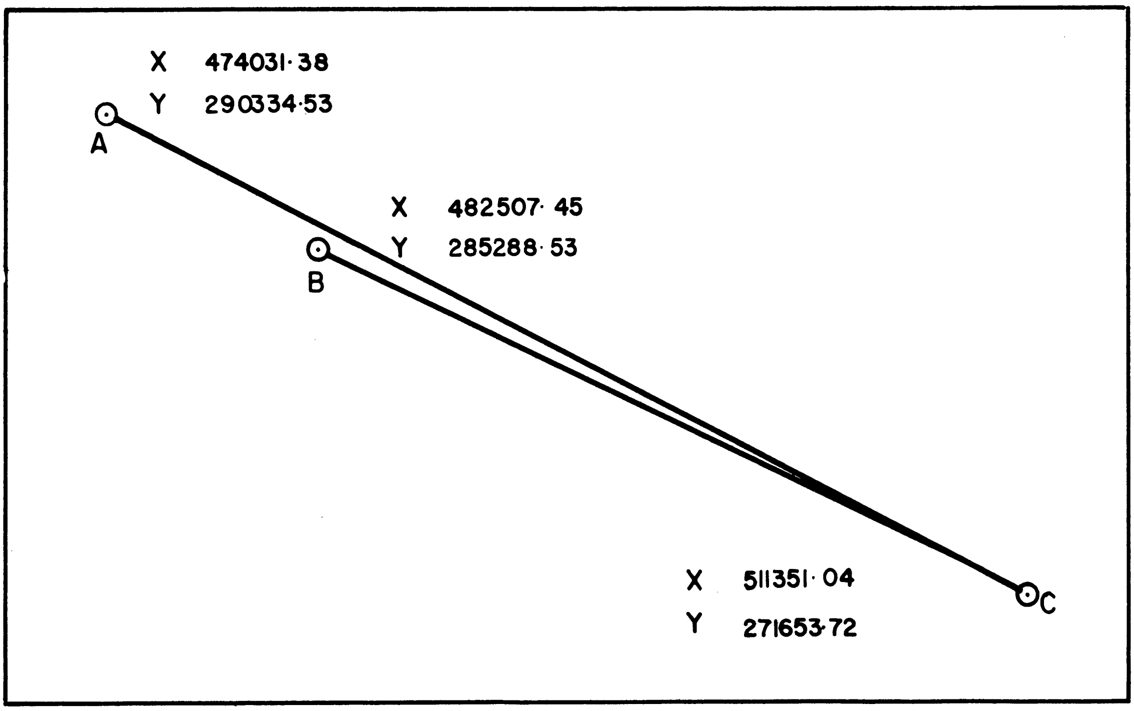

Fig. 6 RELATIVE POSITIONS OF STATIONS A, B, AND POINT C
Scale: 1/8" = 1000'

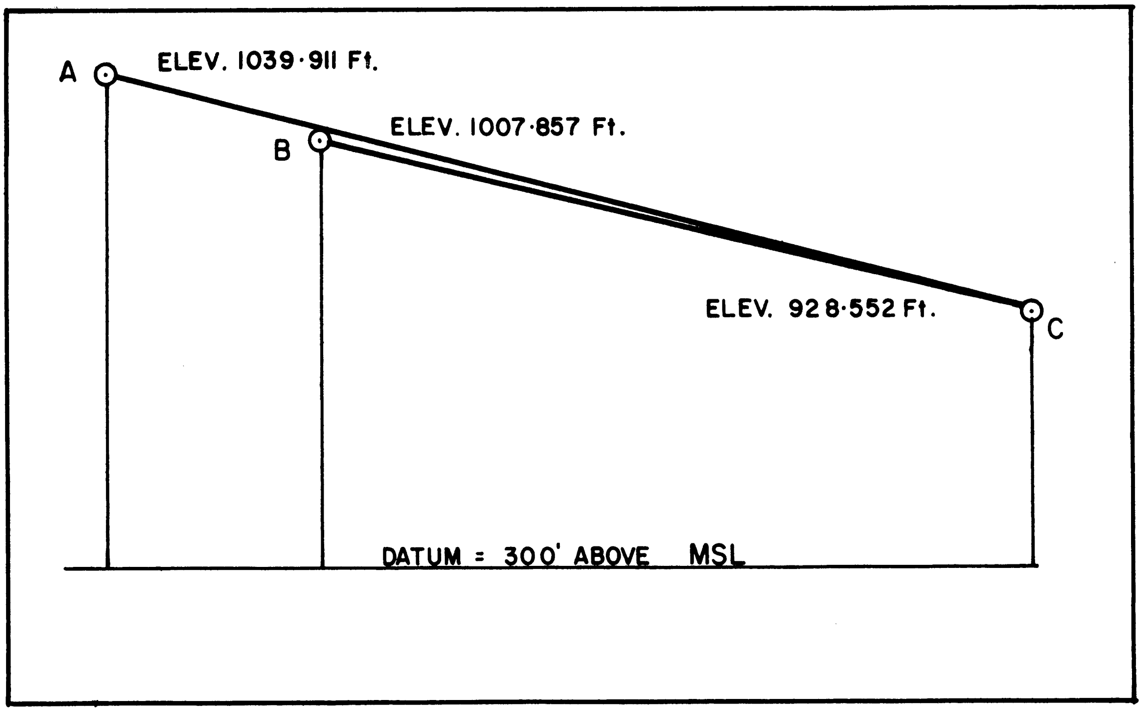

Fig. 7 RELATIVE ELEVATIONS OF THE PATHS AC AND BC
Scale: Vert. 1" = 100'
Horiz. 1/8" = 1000'

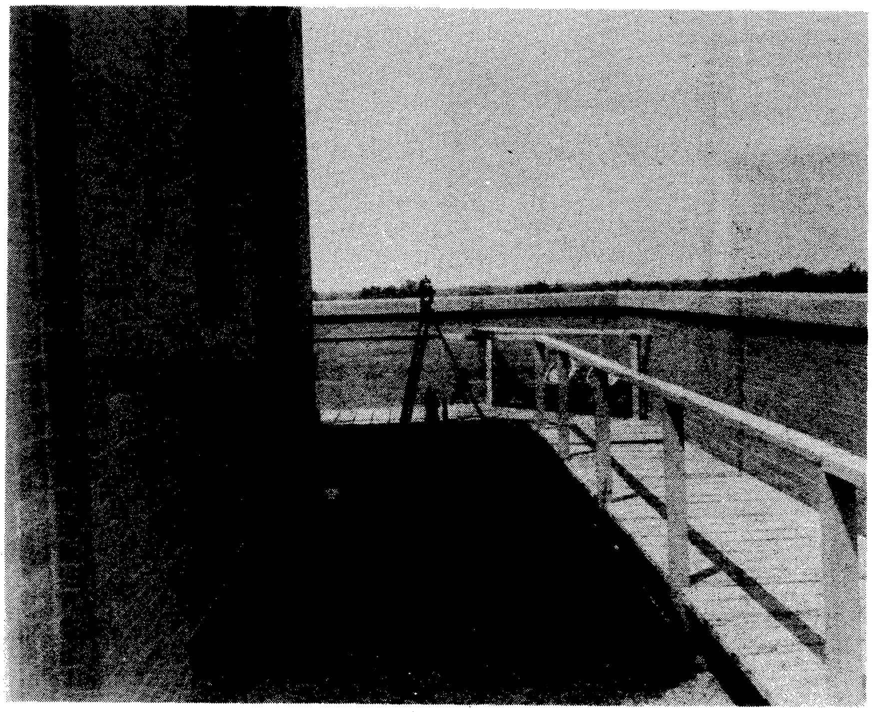

Figure 8. SET-UP AT STATION A

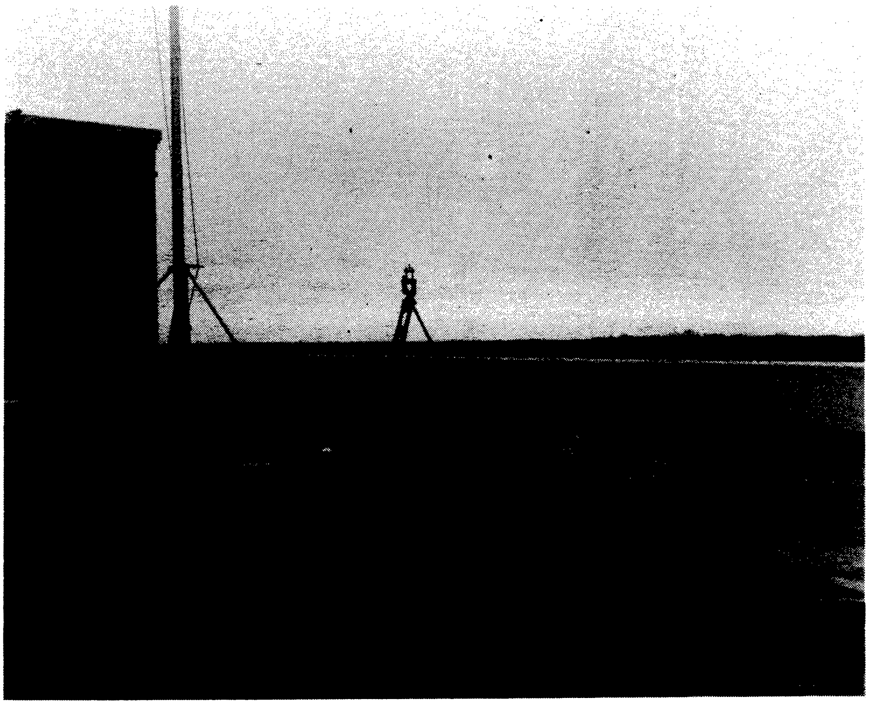

Figure 9. SET-UP AT STATION B

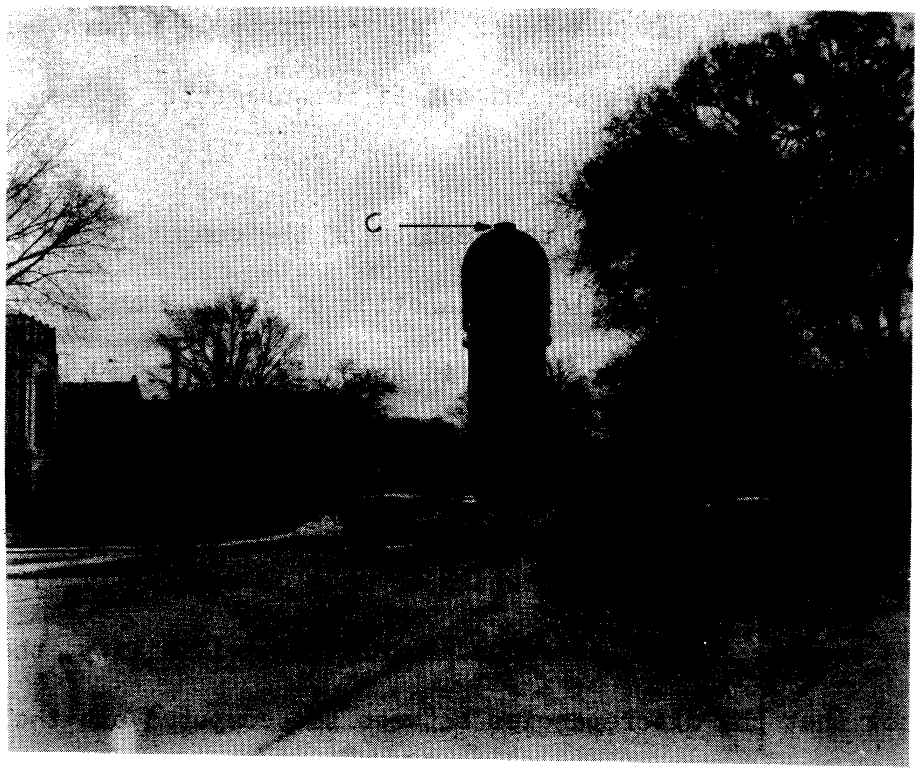

Figure 10. THE OBSERVED POINT C

eight direct and eight inverted.

In the first eleven experiments, one set of measurements was taken, and it was found that the probable errors were high. Then the method of measurement was revised and four sets of measurements were taken at each station. It was found that the probable errors were reduced considerably in this second set of measurements.

Results and Discussion of Results.

The measurements and the results of the computations are all compiled in Table 2. A detailed explanation of Table 2 and a typical example of the computation are given in Appendix A, page 39.

In Table 2, colume 8 gives the values of the refraction angles computed by the new method, and column 10 gives the actual values of the same angles. If the observation which is made on October 11, 1956, is disregarded because it was gained under very hazy visibility, then the results show that the discrepancies between the computed and the actual values are well within the probable errors. Also, if one considers the errors, then the results show that the percentage of errors varies between 0.2 per cent and 5.1 per cent, and the average percentage of the errors is 2.1 per cent.

Examination of the results given in column 12 and column 13 show that the errors in the computed height are well within the margin of the probable error.

Arrangements were made to measure the pressure, the temperature, and the lapse-rate of temperature while three of the observations were in progress, (observations on March 21, March 28, and May 29). Then the refraction angles Ω_a and Ω_b were computed independently by adopting the new and the old method.[1] The results indicate that the accuracy of the new method is greater than that of the older. For example, the average percentage of the errors of the three mentioned measurements computed by using the new method was 2.6 per cent, and that computed by adopting the old method was 20.5 per cent.

1 See Table 3 in Appendix B, page 45.

TABLE 2

MEASUREMENTS AND RESULTS OF COMPUTATIONS

DATE	Est. Time A B	Pressure in inches of mercury A B	Temp. in °F A B	Rod Readings in ft. A B	Measured Vertical Angle A B	Probable Error $d\alpha_a$ $d\alpha_b$	Computed Refraction angles Ω_a Ω_b	Probable Error r_a r_b	Actual Ω_a Ω_b	Computed Height h_c	Probable Error r_{ha} r_{hb}	Actual Error of h_c	Weather
Oct.10 1956	15.15	29.28	51	+0.378	00°-12'-09.3"	±0.61"	29.193"	±2.29"	28.437"	928.399	± 0.479	- 0.153	Partial Clouds Strong Wind Bad Visibility
	16.20	29.36	47	-2.068	00°-10'-35.1"	±0.96"	22.317"	±1.74"	21.328"		± 0.308		
Oct.11	15.40	29.22	55	+0.499	00°-12'-08.60	±0.85"	32.487"	±2.51"	29.737"	927.996	± 0.535	- 0.556	Sunny
	14.45	29.30	58	-2.338	00°-10'-33.45	±0.79"	24.835"	±1.91"	21.237"		± 0.320		Very Hazy
Oct.30	15.40	28.92	67	+0.541	00°-12'-09.56	±0.64"	28.813"	±1.77"	28.986"	928.587	± 0.380	+ 0.035	Cloudy
	14.40	28.98	70	-2.216	00°-10 -33.22	±0.48"	22.026"	±1.35"	22.252"		± 0.174		Moderate Wind
Nov.7	14.00	27.72	53.5	+0.611	00°-12'-10.06	±0.49"	29.300"	±1.56"	28.832"	928.457	± 0.330	- 0.095	Cloudy Strong Wind Very Bad Visibility
	15.10	28.76	54.0	-2.076	00°-10'-34.59	±0.56"	22.399"	±1.19"	21.787"		± 0.204		
Nov.8	12.00	28.84	38	+0.615	00°-12'-10.76	±0.51"	28.599"	±1.95"	28.150"	928.461	± 0.406	- 0.091	Cloudy Strong Wind Bad Visibility
	13.45	28.86	40	-2.138	00°-10'-34.70	±0.83"	21.862"	±1.49"	21.276"		± 0.264		
Nov.9	14.40	28.78	47	+0.765	00°-12'-13.37	±0.46"	26.387"	±1.55"	26.282"	928.531	± 0.327	- 0.021	Sunny
	13.30	28.80	47	-2.663	00°-10'-32.55	±0.59"	20.172"	±1.18"	20.035"		± 0.204		Strong Wind
Nov.12	12.50	28.58	42	+0.503	00°-12 -11.09	±0.66"	27.850"	±2.17"	27.266"	928.434	± 0.458	- 0.118	Partial Clouds Strong Wind Bad Visibility
	10.45	28.59	45	-2.373	00°-10'-33.93	±0.81"	21.290"	±1.66"	20.526"		± 0.286		
Nov.13	11.00	28.79	40	+0.509	00°-12'-07.97	±0.61"	30.109"	±1.77"	30.419"	928.615	± 0.378	+ 0.063	Cloudy
	10.15	28.86	39	-2.346	00°-10'-31.21	±0.54"	23.017"	±1.35"	23.422"		± 0.224		Moderate Wind
Nov.14	14.45	28.61	63.5	+0.427	00°-12'-08.27	±0.73"	28.801"	±2.50"	29.712"	928.736	± 0.525	+ 0.184	Partial Clouds
	13.45	28.68	64.0	-2.625	00°-10'-29.62	±0.97"	22.017"	±1.90"	23.200"		± 0.329		Bad Visibility
Nov.23	14.20	28.90	17.5	+0.458	00°-12'-08.00	±0.88"	29.336"	±2.71"	30.132"	928.713	± 0.575	+ 0.161	Sunny
	13.30	28.94	17.0	-2.429	00°-10'-30.63	±0.92"	22.426"	±2.07"	23.468"		± 0.352		Hazy
Nov 27	12.15	28.77	28.0	+0.527	00°-12'-07.10	±0.29"	31.702"	±0.90"	31.407"	928.492	± 0.191	- 0.060	Partial Clouds
	10.50	28.78	24.5	-2.178	00°-10'-31.87	±0.31"	24.235"	±0.69"	23.849"		± 0 117		Snow on Ground

TABLE 2

MEASUREMENTS AND RESULTS OF COMPUTATIONS

DATE	Est. Time A B	Pressure in inches of mercury A B	Temp. in °F A B	Rod Readings in ft. A B	Measured Vertical Angle A B	Probable Error $d\alpha_a$ $d\alpha_b$	Computed Refraction angles Ω_a Ω_b	Probable Error r_a r_b	Actual Ω_a Ω_b	Computed Height h_c	Probable Error r_{h_a} r_{h_b}	Actual Error of h_c	Weather
Nov.28 1956	11.45	28.48	30.0	+0.532	00°-12'-03.56	±0.26"	35.306"	±0.81"	34.941"	928.478	± 0.171	- 0.074	Cloudy
	10.30	28.50	28.5	-2.160	00°-10'-29.32	±0.28"	26.990"	±0.61"	26.513"		± 0.105		Snow on Ground
Nov.30	13.15	28.63	26.5	+0.591	00°-12'-07.02	±0.31"	32.000"	±0.91"	31.773"	928.506	± 0.194	- 0.046	Cloudy
	12.00	28.71	26.5	-2.291	00°-10'-30.32	±0.29"	24.462"	±0.70"	24.166"		± 0.117		Snow on Ground
Dec.12	13.55	28.88	27.5	+0.447	00°-12'-05.04	±0.22"	33.236"	±0.76"	33.038"	928.512	± 0.161	- 0.040	Cloudy
	12.30	28.95	29.5	-2.506	00°-10'-28.45	±0.30"	25.408"	±0.58"	25.408"		± 0.101		Hazy
Dec.13	13.40	29.13	27.0	+0.494	00°-12'-06.22	±0.17"	32.154"	±0.61"	32.094"	928.540	± 0.129	- 0.015	Cloudy
	12.25	29.22	29.0	-2.490	00°-10'-29.20	±0.25"	24.580"	±0.47"	24.502"		± 0.082		
Dec.19	14.30	28.97	25.0	+0.480	00°-12'-03.42	±0.82"	35.912"	±2.49"	34.822"	928.332	± 0.529	- 0.220	Partial Clouds
	13.25	29.05	24.0	-2.310	00°-10'-28.84	±0.83"	27.453"	±1.90"	26.028"		± 0.320		Bad Visibility
Feb.27 1957	13.40	28.30	29.0	+0.538	00°-12'-07.50	±0.55"	31.441"	±1.59"	31.032"	928.469	± 0.339	- 0.083	Partial Clouds
	14.40	28.92	30.0	-2.394	00°-10'-30.82	±0.48"	24.035"	±1.21"	23.500"		± 0.201		Hazy
March 8,1957	14.00	28.59	36.5	+0.612	00°-12'-06.22	±0.31"	32.107"	±0.98"	32.677"	928.668	± 0.208	+ 0.116	Partial Clouds
	12.45	28.71	37.5	-2.442	00°-10'-28.72	±0.35"	24.544"	±0.75"	25.291"		± 0.128		
March 21,	13.55	28.76	43.0	+0.529	00°-12'-07.04	±0.35"	31.809"	±1.09"	31.447"	928.478	± 0.231	- 0.073	Sunny
	12.45	28.86	43.5	-2.490	00°-10'-29.86	±0.38"	24.317"	±0.84"	23.842"		± 0.142		Hazy
March 28,	13.50	28.91	43.5	+0.402	00°-12'-10.47	±0.56"	28.183"	±1.77"	27.384"	928.391	± 0.381	- 0.161	Partial Clouds
	12.45	28.99	44.0	-2.238	00°-10'-34.83	±0.42"	21.544"	±1.35"	20.500"		± 0.217		Bad Visibility
May 29	13.20	29.13	79.0	+0.337	00°-12'-10.75	±0.32"	27.196"	±0.98"	26.786"	928.469	± 0.209	- 0.083	Sunny
	12.20	29.17	78.5	-2.171	00°-10'-35.52	±0.34"	20.790"	±0.75"	20.254"		± 0.128		Windy

CONCLUSION

The determination of the refraction angles from two observation stations as presented in this investigation has been conducted by both theoretical and experimental procedures. It has been shown that in the new method extra work is needed in locating a secondary station and making observations at this station, but that this work is simple since the two observation stations are close to each other.

The new method provides a way to determine the refraction angles without using pressure, temperature, and lapse rate of temperature in the computation. It thus eliminates the necessity of measuring these factors, and by doing so, the small uncertainty of lapse rate of temperature and its high effect on the computed refraction angles is avoided.

The results of the experiments that are determined by adopting the new method are satisfactory and of relatively high accuracy; for example, the percentage of the errors of the refraction angles that are determined at station A between March 21, 1957 and May 29, 1957 vary between 1.2 per cent and 2.9 per cent, whereas the percentage of the errors of the refraction angles that are determined by adopting the method of including pressure, temperature, and lapse rate of temperature for the same observations were between 14 per cent and 32 per cent.

Therefore the new method can be used with confidence in determining the heights of high points, and in establishing bench marks in regions that can not be reached, or which are very difficult to reach by lines of spirit leveling.

APPENDIX A

EXPLANATION OF TABLE 2

The measurements and the results of the computations are all compiled in Table 2, and it is felt that an explanation of this table will be helpful to the reader. In order to make the explanation simple, the measurements which were made on March 21, 1957 and the necessary computations are carried out as a typical example

Columns 1 and 2 give the date and the time at which the observations were made.

Columns 3 and 4 give the average pressure and temperature, which were measured at stations A and B.

Column 5 gives the level rod readings. The (+) sign indicates that the line of sight is higher than the bench mark and the (-) sign indicates that it is lower.

Columns 6 and 7 give the mean of the measured vertical angles and the probable errors of the mean. The following example illustrates the method of computation:

<u>At station A</u>

Sets of measurement	Measured zenith angle	Probable error in seconds of arc	Weight	Angle x weight
1	90°-12'-07.89"	± 0.84"	10	78.90"
2	90°-12'-07.13	± 0.82	11	78.43"
3	90°-12'-06.39	± 0.59	20	127.80"
4	90°-12'-07.25	± 0.64	17	123.25
		Total	58	408.38

$$\text{mean} = \frac{408.38}{58} = 7.04''$$

$$\text{probable error of the mean} = 0.59\sqrt{\frac{20}{58}} = \pm 0.35''$$

$$\text{zenith angle} = 90° - 12' - 07.04 \pm 0.35''$$

$$\text{vertical angle} = (-)\ 00° - 12' - 07.04'' \pm 0.35''$$

Column 8 gives the computed values of the refraction angles Ω_a and Ω_b . The following illustrate the computations: equation (2-24) is

$$\Omega_a = \frac{l_a}{(l_a)^2 - (l_b)^2}\left[l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right) - l_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right) + (h_a - h_b)\right].$$

$$\frac{l_a}{(l_a)^2 - (l_b)^2} = \frac{41735.93}{(41735.93)^2 - (31905.30)^2}$$

$$= 0.0000576511$$

$$\alpha_a = -\ 00 - 12 - 07.04$$

$$\frac{\theta_a}{2} = 00 - 03 - 25.52$$

$$\alpha_a + \frac{\theta_a}{2} = -\ 00 - 08 - 41.52$$

$$\text{Log}\tan\left(\alpha_a + \frac{\theta_a}{2}\right) = 7.4028466$$

$$\text{Log}\ l_a = 4.6205101$$

$$\text{Log}\left[l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right)\right] = 2.0233567$$

$$l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right) = -\ 105.525$$

$$\alpha_b = -\ 00 - 10 - 29.86$$

$$\frac{\theta_b}{2} = 00 - 02 - 37.10$$

$$\left(\alpha'_b + \frac{\theta_b}{2}\right) = -\ 00 - 07 - 52.76$$

$$\log \tan\left(\alpha_b + \frac{\theta_b}{2}\right) = \quad 7.3602162$$

$$\log l_b = \quad 4.5038628$$

$$\log\left[l_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right)\right] = \quad 1.8640790$$

$$l_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right) = \quad -73.127$$

ha = 1039.911 + 0.529 = 1040.440[1]

hb = 1007.857 - 2.490 = 1005.367[2]

ha - hb = 35.073

Substituting values in equation (2-24), one gets

$$\Omega_a = 0.0000576511\,(-105.525 + 73.127 + 35.073)$$

Ω_a = 0.0001542167 radians

Ω_a = 31.809 seconds of arc

Applying equation (2-25) and following the same procedure as above, one obtains

Ω_b = 0.0001178919 radians

Ω_b = 24.317 seconds of arc

Column 9 gives the probable error of the computed refraction angles Ω_a and Ω_b. The following illustrate the computation: equation (3-4) is

$$r_a = \frac{l_a}{(l_a)^2 - (l_b)^2}\sqrt{(l_a\, d\alpha_a)^2 + (l_b\, d\alpha_b)^2}\,.$$

Substituting values, l_a and l_b are in miles, we have

1 1039.711 is the height of the bench mark near station A.
2 1006.857 is the height of the bench mark near station B.

$$r_a = \pm \frac{7.904}{(7.904)^2 - (6.042)^2} \sqrt{(7.904 \times 0.35)^2 - (6.042 \times 0.38)^2}$$

(below)

$$r_a = \pm \; 0.304 \times 3.60$$

$$r_a = \pm \; 1.09 \text{ seconds of arc}$$

Applying equation (3-5) and following the same procedure as above, we obtain

$$r_b = \pm \; 0.84 \text{ seconds of arc}$$

Column 10 gives the actual values of the refraction angles Ω_a and Ω_b, they are computed as follows:

h_a =	1040.440
$l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right)$ =	-105.525
observed height =	934.915
actual height =	928.552
effect of refraction =	6.363

$$\Omega_a = \frac{6.363}{41735.93} \times 206264.8$$

$$\Omega_a = 31.447 \text{ seconds of arc}$$

and

h_b =	1005.367
$l_b \tan\left(\alpha_b + \frac{\theta_b}{2}\right)$ =	- 73.127
observed height =	932.240
actual height =	928.552
effect of refraction =	3.688

$$\Omega_b = \frac{3.688}{31905.30} \times 206264.8$$

$$\Omega_b = 23.842 \text{ seconds of arc}$$

Column 11 gives the computed height of point C. The following illustrate the computations: formula (2-33) is

$$h_c = l_a \tan\left(\alpha_a + \frac{\theta_a}{2}\right) - \Omega_a l_a + h_a .$$

Substituting values from page 42, we have

$$h_c = 1040.440 - 105.525 - 41735.93 \times 0.0001542167$$

or

$$h_c = 928.478 \text{ feet}$$

Column 12 gives the probable error of the height h_c. The following illustrate the computations: equation (3-10) is

$$r_{h_a} = \pm 0.02558\, l_a \sqrt{(d\alpha_a)^2 + (r_a)^2}$$

Substituting values, we obtain

$$r_{h_a} = \pm 0.02558 \times 7.904 \sqrt{(0.35)^2 + (1.09)^2}$$

$$r_{h_a} = \pm 0.231 \text{ feet}$$

Applying formula (3-11) and substituting values, we get

$$r_{h_b} = \pm 0.142 \text{ feet}$$

Column 13 gives the actual discrepancy between the computed and the actual values. The actual height of point C is 928.552 feet.

Column 14 gives the weather conditions when the observations were made.

APPENDIX B

DETERMINATION OF THE REFRACTION ANGLES BY INCLUDING THE METEOROLOGICAL FACTORS IN COMPUTATIONS

After presentation of the new method of the present investigation, it is felt desirable to provide some material by which a comparison can be made between the new and the old methods. This material was provided by making arrangements for measuring the temperature at different heights.

A wire sounding device was set on the Huron Valley Golf Course, which is located at about one-third the distance between Station A and point C measured from Station A, and the temperatures at different heights were measured while the observations from Stations A and B were in progress. This procedure provided a way of determining the temperature and the lapse rate of temperature along the path of light. The figures which are given on pages 47-52 illustrate the results of the measured and the reduced values of the temperature and the lapse rate of temperature.

The pressure was measured at the observation station, and because the paths of light are very nearly horizontal, the pressure was assumed to be constant along the paths. The refraction angles Ω_a and Ω_b and the height h_c are computed and compiled in Table 3 page 45.

TABLE 3

MEASUREMENTS AND RESULTS OF COMPUTATIONS BY ADOPTING METEOROLOGICAL FACTORS

Date	Station	Time Est.	P inches mercury	T °C	$\frac{dT}{dh}$ °C per ft.	$k=28000\frac{P}{T^2}(0.0104+\frac{dT}{dh})$	Refraction angle	Vert. angle Correct. from Curvature & Refraction	Computed h_c ft.	Error in h_c ft.
March 21,1957	A	13.55	28.82	5.25	0.00204	0.0873	35.884"	-00°-09'-17.40	927.665	-0.887
	B	12.45	28.88				27.433"	-00°-08'-20.19	928.203	-0.349
March 28,1957	A	13.50	28.97	4.65	0.00506	0.0556	22.813"	-00°-09'-07.76	929.485	+0.933
	B	12.45	29.01				17.438"	-00°-08'-15.17	929.800	+1.248
May 29, 1957	A	13.20	29.13	20.5	0.0057	.0445	18.290"	-00°-09'-03.52	930.265	+1.713
	B	12.20	29.17				13.982"	-00°-08'-12.20"	929.552	+1.00

Figure 11. WIRE SOUNDING

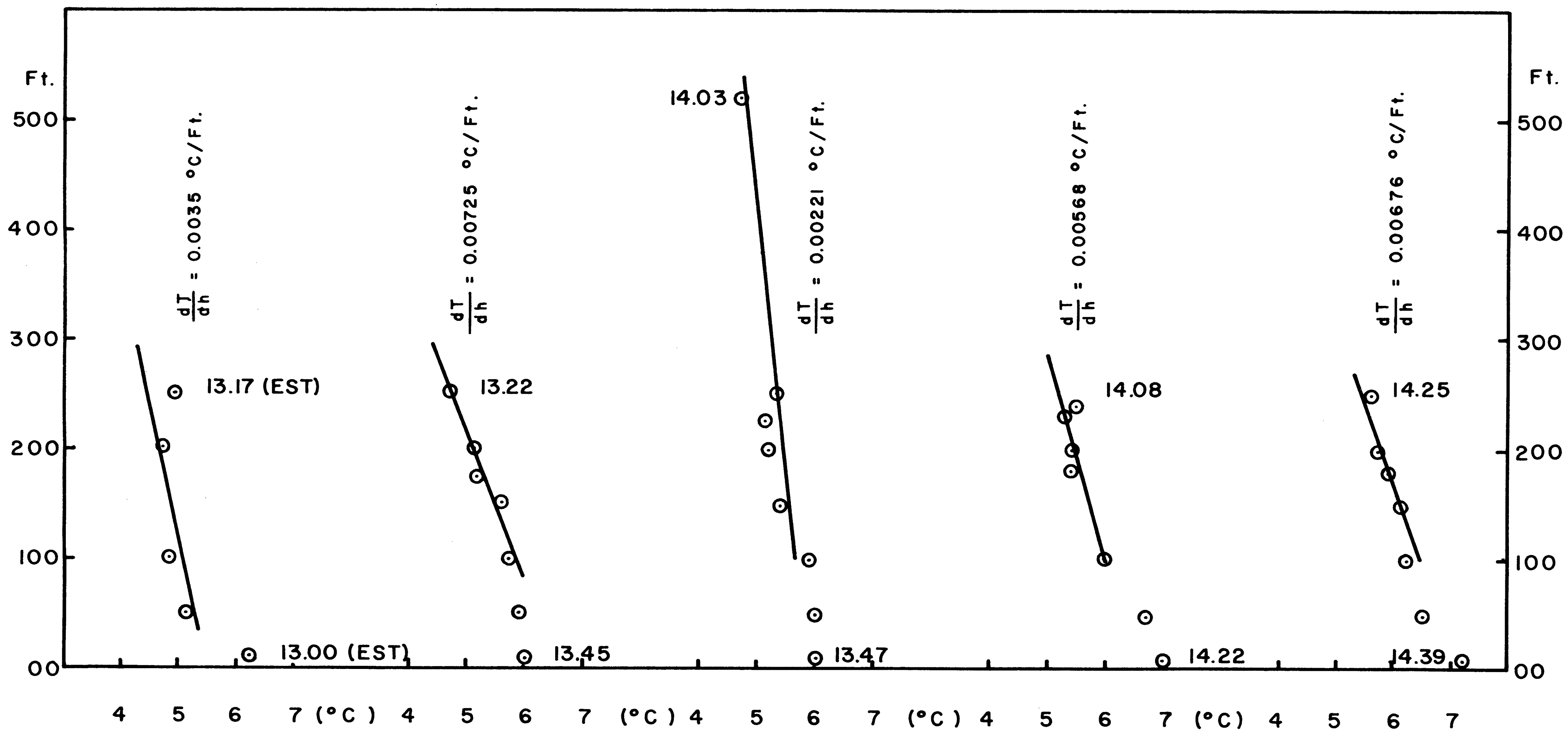

FIGURE 12

TEMPERATURE VS. HEIGHT
HURON VALLEY GOLF COURSE
MARCH 21, 1957

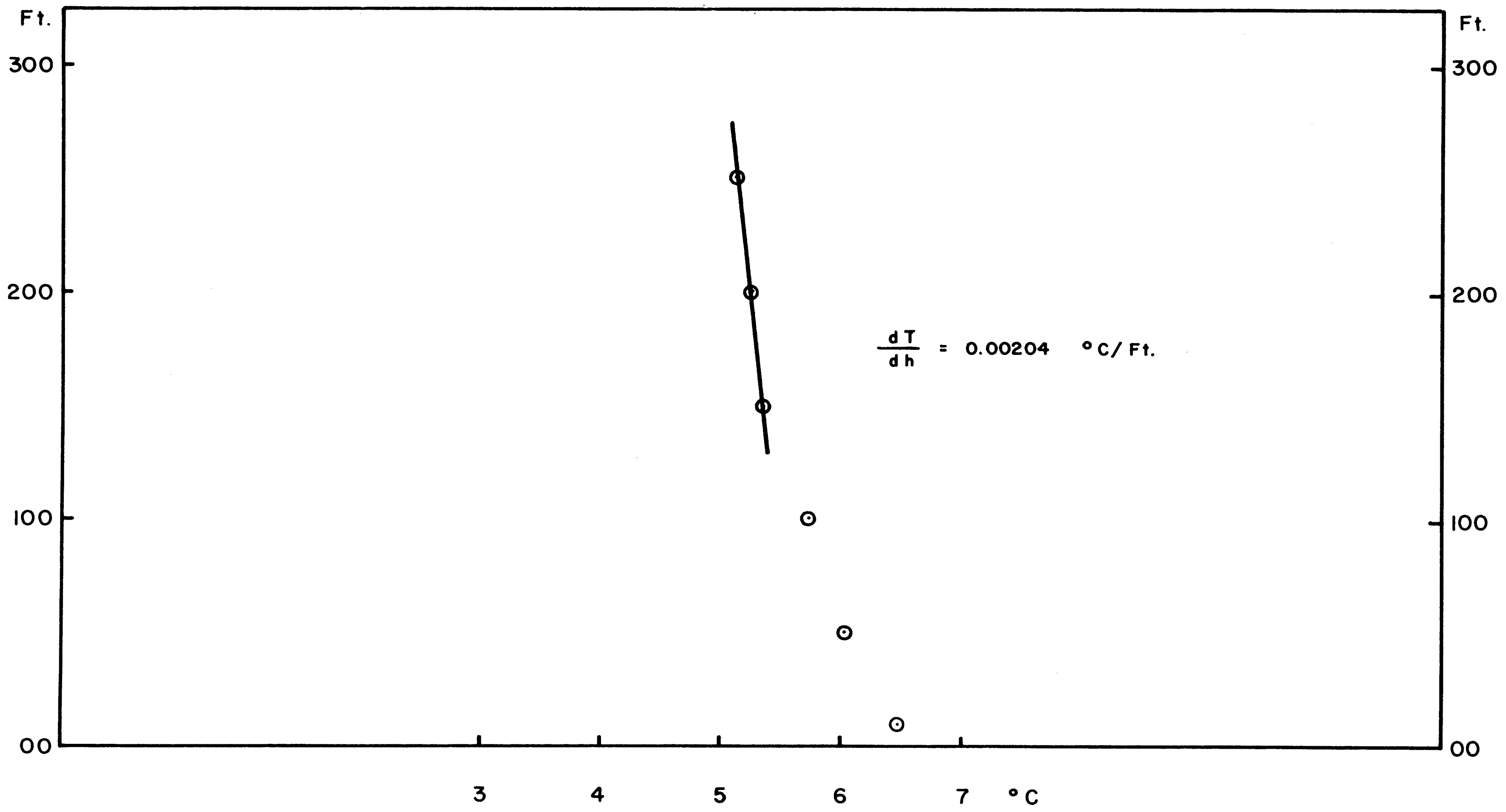

FIGURE 13

AVERAGE VALUES

TEMPERATURE VS. HEIGHT

HURON VALLEY GOLF COURSE

MARCH 21, 1957

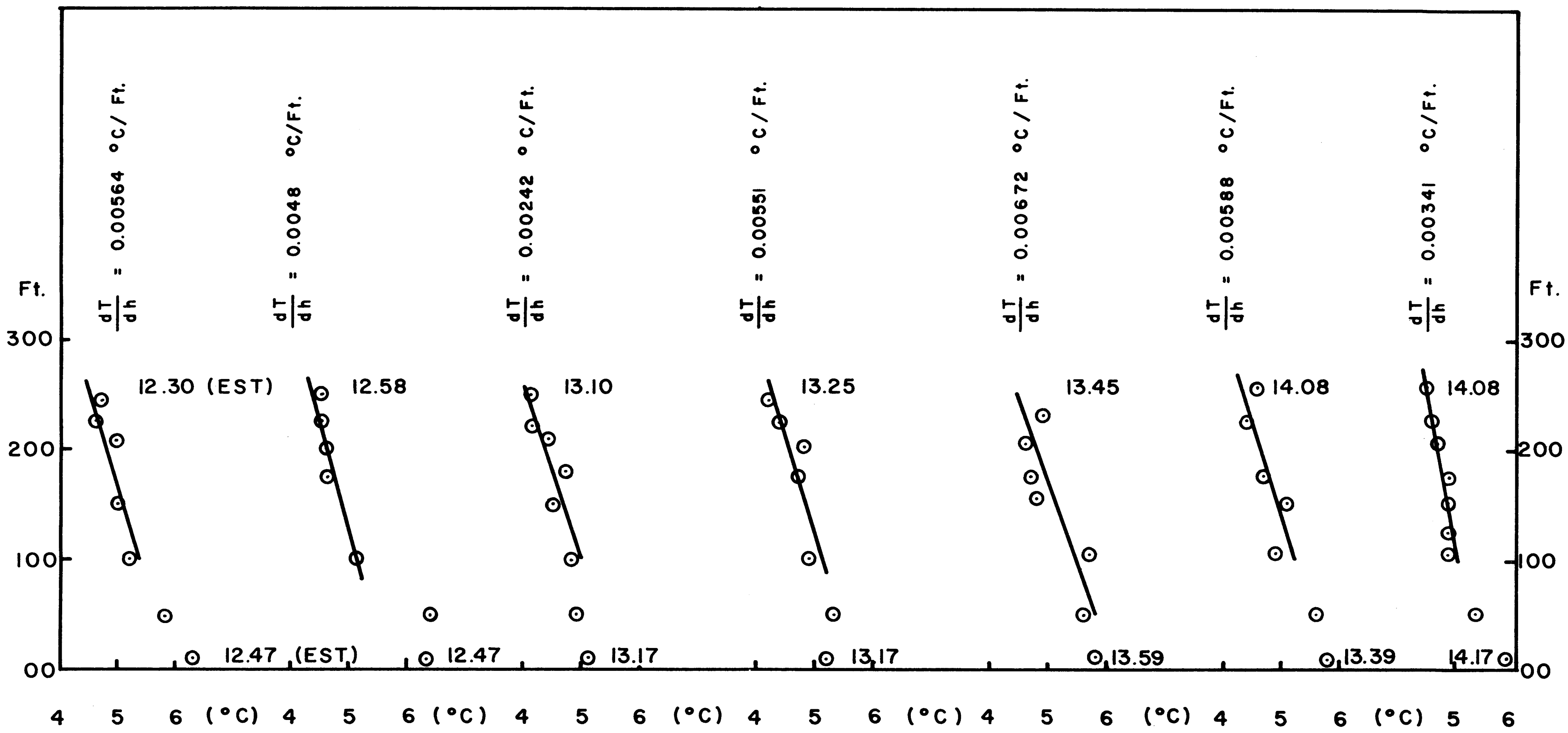

FIGURE 14

TEMPERATURE VS. HEIGHT

HURON VALLEY GOLF COURSE

MARCH 28, 1957

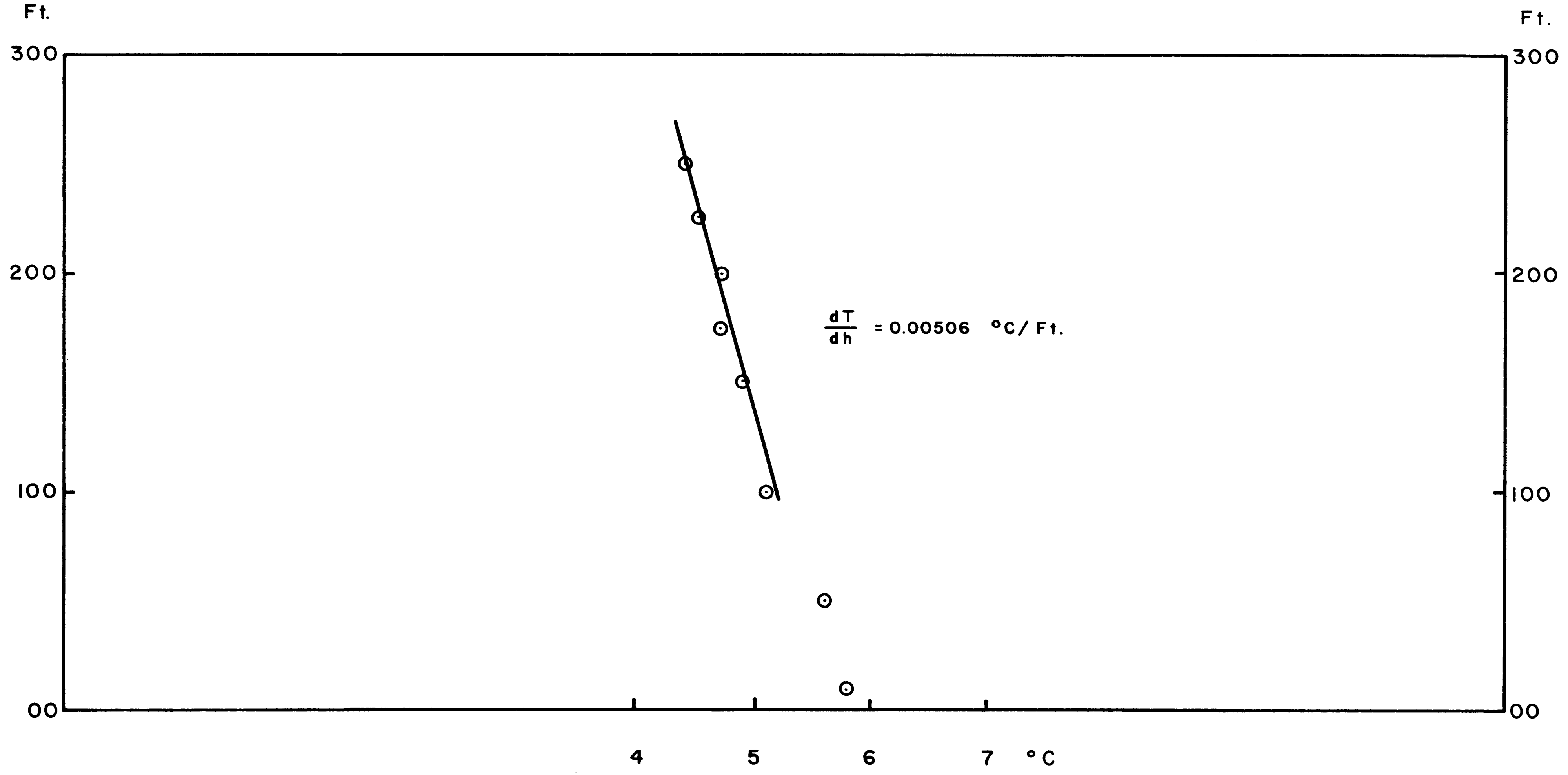

FIGURE 15

AVERAGE VALUES

TEMPERATURE VS. HEIGHT

HURON VALLEY GOLF COURSE

MARCH 28, 1957

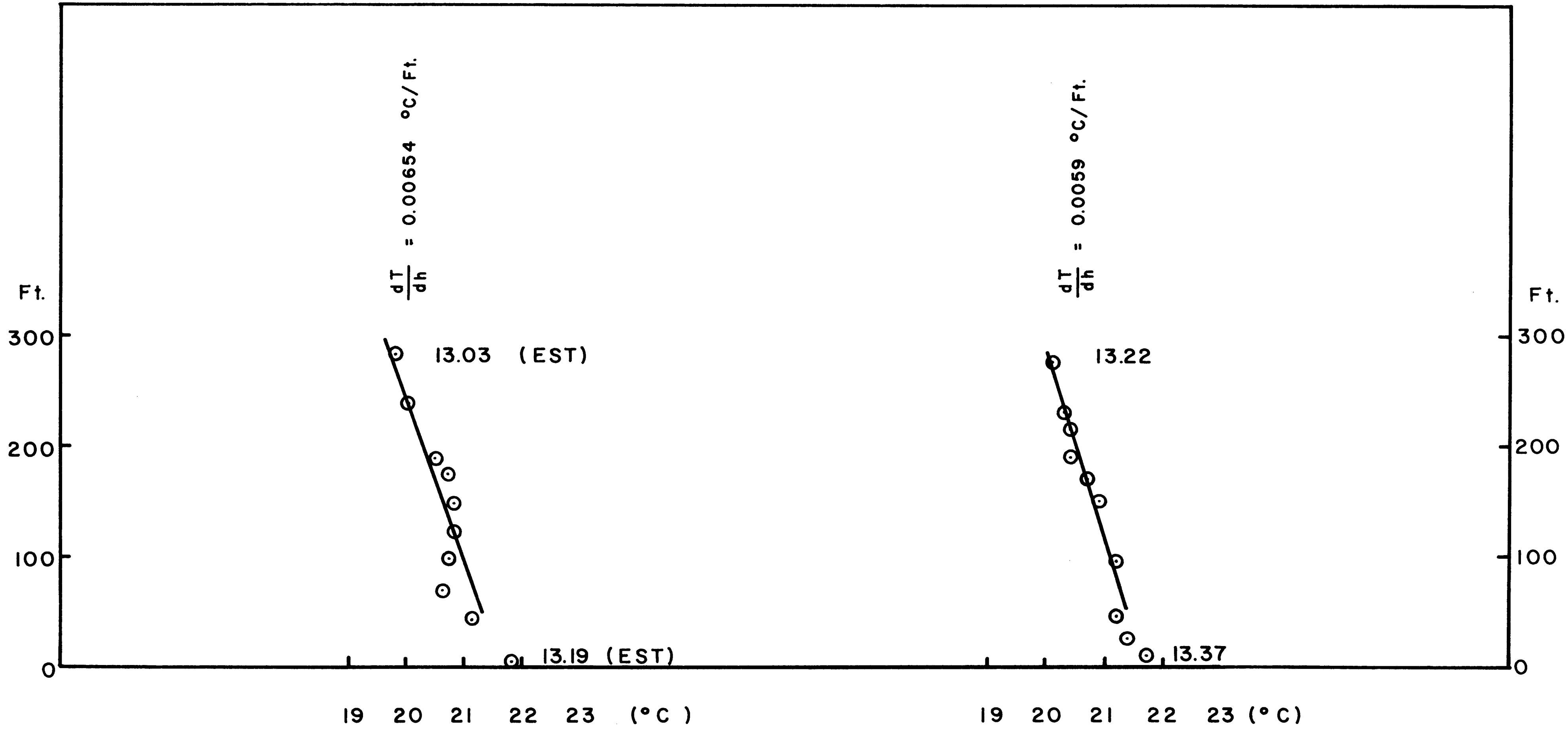

FIGURE 16

TEMPERATURE VS. HEIGHT

HURON VALLEY RIVER

MAY 29, 1957

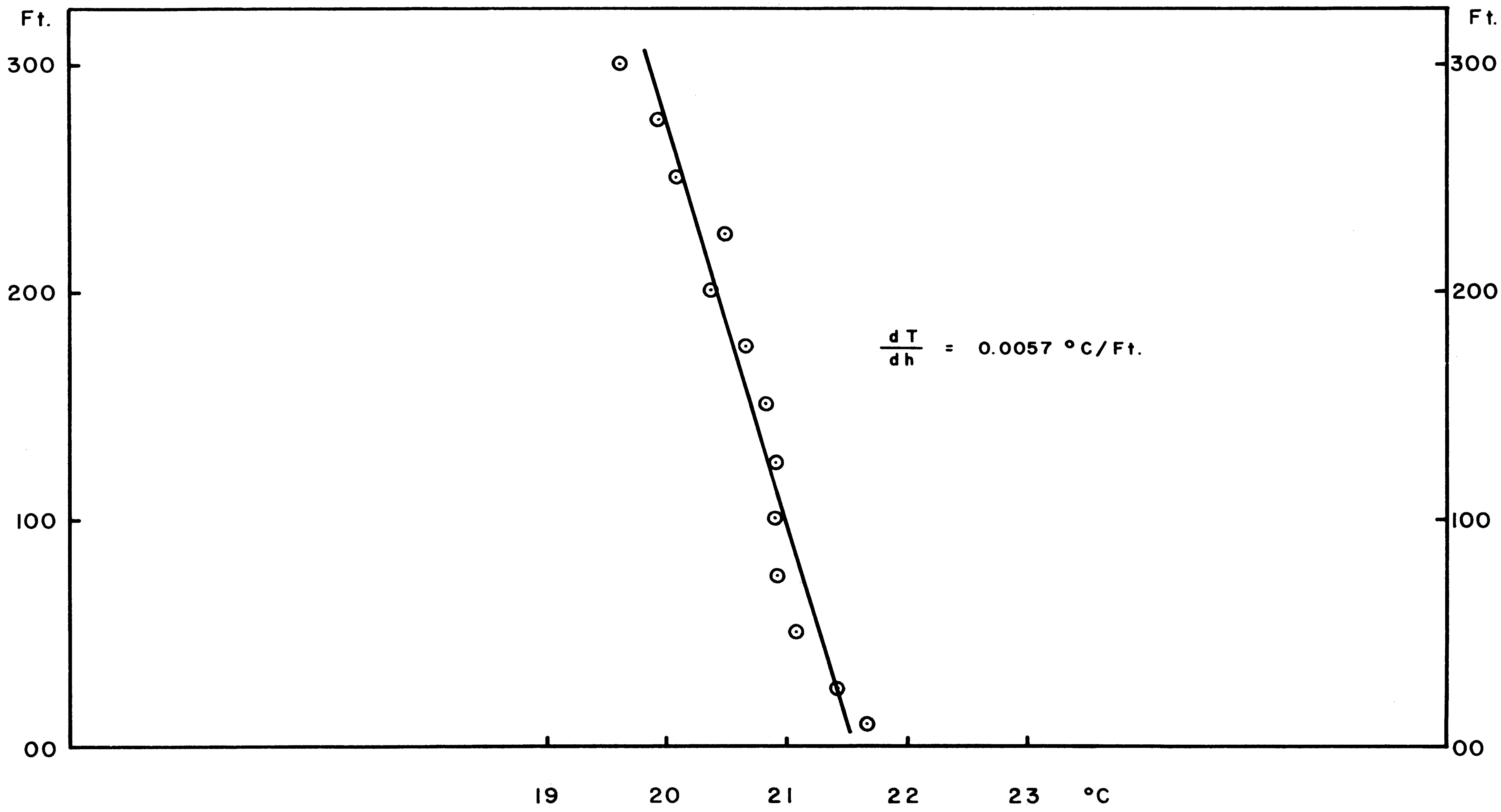

FIGURE 17
AVERAGE VALUES
TEMPERATURE VS. HEIGHT
HURON VALLEY GOLF COURSE
MAY 29, 1957

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