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ON INDUCING A NON-TRIVIAL, PARSIMONIOUS, HIERARCHICAL GRAMMAR FOR A GIVEN SAMPLE OF SENTENCES

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ABSTRACT

ON INDUCING A NON-TRIVIAL, PARSIMONIOUS, HIERARCHICAL GRAMMAR FOR A GIVEN SAMPLE OF SENTENCES

by
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The thesis presents an algorithm which, for a given sample of sentences, induces a grammar that (1) can generate the given sample of sentences, (2) is non-trivial in the sense that it does not merely enumerate the sentences of the sample, (3) is hierarchical in the sense that the sentences of the sample are derived through relatively long sequences of sentential forms, (4) is, at the same time, parsimonious in the sense that the grammar contains a relatively small number of rules of production, (5) contains recursive rules of production under appropriate specified conditions, and (6) contains disjunctive rules of production (and generalizations) under appropriate specified conditions.

The thesis formally defines the notion of a regularity in a sample of sentences, presents a combinatorial algorithm for searching for regularities, and argues that searching for local regularities is an appropriate basis for a grammar discovery algorithm.

The Grammar Discovery Algorithm presented involves (1) a search of the given sample of sentences for local regularities—this search being an exhaustive process within a limited range, (2) the consideration non-exhaustively of various possible transformations of the given sample via grammatical rules of production that are developed from the local regularities, (3) a selection process among such transformations, and (4) the reapplication of these steps until the sample is fully

resolved and a grammar induced.

The thesis contains and describes a computer program implementing the Grammar Discovery Algorithm.

While the emphasis is on grammar discovery, there is also discussion on the equivalent problem of inducing axioms and rules of inference in a formal system.

The Grammar Discovery Algorithm is justified in terms of its internal consistency, its satisfying of various heuristic criteria and by examples.

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TABLE OF NOTATION

A	Starting set (axioms) of general derivation system.
V_T	Terminal Alphabet of grammar or general derivation system.
V_N	Non-Terminal Alphabet of grammar or general derivation system.
R	Rules of Production of grammar or general derivation system.
$\mathcal{G} = \langle V_T, V_N, R, A \rangle$	General Derivation System.
S	Starting Symbol of grammar.
$\mathcal{G} = \langle V_T, V_N, R, S \rangle$	A Grammar.
C_1	Number of symbols in alphabet of the Sample (V_T).
Y	The sample of sentences. A sequence over V_T or the current alphabet.
T	Indexing parameter for sample.
N	Length of Sample Y.
N_{max}	Length of longest sentence in Y. (If there is no initial punctuation in the sample, N_{max}=N).
V_c	The current Alphabet. V_T is the current alphabet for level one.
C	Number of symbols in current alphabet (that is, C_1 plus non-terminal symbols added). $C=C_1$ for level one.
M	Length of Sequence of symbols.
g,h	Indices typically used to index positions in a regularity, the context part or predicted part of a regularity.
ϵ	The value $1/_C g$.
p, p_{ij}	Conditional Probabilities.
%	Symbol meaning "predicted".
_	Symbol meaning "context".
#	Symbol meaning "Don't Care".

$\mathcal{M}(I)$	Mask—Sequence of Length M over symbols "%", "_", and possibly also "#" (if mask is ternary).
$\gamma(I)$	Context-Sub-Sequence—Sequence of Length M over current alphabet augmented by "%" and possibly also "#".
$\phi(I)$	Predicted-Sub-Sequence—Sequence of Length M over current alphabet, augmented by "_" and possibly also "#".
$\mathcal{R} = \langle\gamma,\phi, P, Y\rangle$	A regularity in sample Y whose context-sub-sequence is γ, whose predicted-sub-sequence is ϕ, and whose conditional probability is p.
Δ	The domain sequence of a regularity.
$\mathcal{T}$	A transformation.
λ	Coefficient of parsimony.
λ_r	Coefficient of Recursive Parsimony.
n	Number of Rules of Production.
n_r	Number of Recursive Rules of Production.
.	The period (Initial punctuation mark).

I. INTRODUCTION

A. STATEMENT OF THE PROBLEM

The processes of deduction and induction are fundamental in mathematics and the axiomatized sciences. These same two processes also appear in the study of generative grammars.

We define a *general derivation system* $\mathcal{G} = \langle V_T, V_N, R, A \rangle$ to be a system in which V_T is the set of *terminal symbols*, where V_N is the set of *non-terminal symbols*, where R is a set of *rules*, and where A is the *starting set*. The starting set contains strings of terminal and non-terminal symbols. The rules are pairs of symbol strings. The first string, called the *antecedent* (or simply the *left side*), is a non-empty string of terminal symbols and/or non-terminal symbols. The second string of the pair, called the *consequent* (or simply the *right side*), is a string of terminal symbols and/or non-terminal symbols, or is empty.

A derivation system generates sentences by starting with any one of the strings in the starting set, and by applying the rules, one after the other. Each application of a rule consists of substituting the consequent sub-string for the antecedent sub-string. Each symbol string so produced is, in general, a string of terminal symbols and non-terminal symbols, and is called a *sentential form*. A sentential form consisting only of terminal symbols is called a *sentence*.

A generative *grammar* is a derivation system in which the starting set A consists only of a single distinguished non-terminal of length one. We customarily call this symbol "S", the *starting symbol*, and we

call the rules of a grammar *rules of production.*

A *formal system* consists of two derivation systems. First, there is a set V_T of *symbols of discourse* about the subject at hand, and a set of *formation rules* R_F for the formation of *well-formed sentences* over V_T. This derivation system is a grammar and has a starting symbol S and whatever non-terminal symbols are needed to express the rules R_F. The second derivation system has a subset A of the well-formed sentences of the first derivation system as its starting set. This subset consists of the accepted truths of the system, called *axioms*. There is also a set of truth-preserving rules R_I, called *rules of inference,* for transforming one true, well-formed sentence into another. By recursively applying the axioms and the rules of inference, one derives the true sentences Y_T of the formal system, called *theorems*. The non-terminal symbol set of this second derivation system is typically the same as its terminal symbol set. Thus, any sentential form of this second derivation system is a sentence and also a theorem.

The process of generating sentences in a derivation system is called *derivation*. The process of generating theorems in a formal system is called *deduction*. In both cases, V_T, V_N, R, and A are given. Typically, these two processes present themselves in the setting of finding the derivation, if any, that can generate a particular given sentence. For grammars, this problem is the *parsing problem*. For formal systems, it is well to remember that all theorem proving is merely a parsing problem. Parsing and theorem-proving algorithms are either *bottom-up* in the sense that they start with the given sentences and consider possible sentential forms from which these sentences can be immediately derived, or *top-down* in the sense that the derivation starts with the starting set A.

Induction is the inverse process for deduction. In induction, one begins with an observed, limited set Y of the true sentences Y_T. The Copernican doing induction tries to discover a parsimonious set of axioms A and a set of rules of inference R_I which together could have generated the given set Y.

Grammar discovery is the inverse process for derivation in a grammar. In the *grammar discovery problem*, one starts with a set V_T of C_1 terminal symbols and a set Y of sentences over this V_T. Y may be presented, for example, as a concatenation of the individual sentences, with or without initial punctuation between the sentences to identify the sentences.

The problem we discuss herein is the problem of finding a non-trivial, parsimonious, hierarchical grammar $\mathcal{G} = \langle V_T, V_N, R, \text{"S"} \rangle$ which could have generated the given sentences Y. This problem requires finding a set of rules R of production and a suitable set of non-terminal symbols V_N.

Grammar discovery and induction are necessarily bottom-up processes.

In what follows, the main emphasis will be on the grammar discovery problem, since this is the underlying problem common with the induction problem for formal systems.

When we say that we are seeking a *non-trivial* grammar, we are referring to any grammar except a grammar which merely catalogs the sentences of the sample Y—an example of which is the grammar in which the starting symbol "S" may be rewritten as the disjunction of the observed sentences of Y. Note that the adjective "non-trivial" applies to this well-defined, definite enumerative situation. When we say that we are seeking a *hierarchical* grammar, we are referring to a

grammar which, as a minimum, is non-trivial, but which has the additional quality of generating its sentences through relatively long sequences of sentential forms. Note that the adjectives "hierarchical" is less precise and more connotative than the adjective "non-trivial". The connotation is that each small phrase and substructure of a sentence is generated by a separate application of a rule of production—in contrast to a situation in which large phrases are generated by one application of one rule. When we say that we are seeking a *parsimonious* grammar, we are referring to a grammar with a relatively small number of rules of production. Note that the adjective "parsimonious" is also a connotative word. The connotation is that a few rules of production are each applied many times in the generation of sentences—in contrast to a situation in which many rules are each used only in a few specific cases. A hierarchical grammar without recursions tends to be unparsimonious. A grammar may be parsimonious without being hierarchical—for example, if it is trivial. A grammar may be hierarchical without being parsimonious, as, for example, a grammar which fails to employ recursions, or employs superfluous non-terminal symbols or superfluous rules of production.

Note that we state the grammar discovery problem as a problem of finding *a* parsimonious and *a* hierarchical grammar, but not necessarily the *most* parsimonious and *most* hierarchical grammar—which are qualities that are not defined here. Similarly, for lack of definition, we do not require the finding of a necessarily interesting grammar, nor a significant grammar, nor a linguist's grammar, nor a grammar conforming to some known or supposed physiological or psychological model, nor the "best" grammar.

The grammar discovery problem, of course, admits of several

variations, and some of these variations have appeared within the very limited literature on grammar discovery and induction. For example, the machine alleging to solve the grammar discovery problem may receive more than a fixed sample of sentences. The machine may, for example, receive, or ask to receive, answers to certain questions in order to assist it in solving the problem. These questions may take the form of asking whether certain new sentences which it constructs are in the language produced by the grammar which is the desired solution to the problem. Or, non-sentences identified as such may be introduced with the sample sentences. Or, the machine may ask for a larger sample of sentences or identified non-sentences if it needs them (Gold, 1967). Then again, the machine may be assumed to be receiving a continuing, infinite input of sample sentences (Goodall, 1962). Another possibility is that the machine can receive, or ask to receive, certain cues, such as the number (or an upper bound on the number) or internal states which an automaton of a certain type would have to have in order to accept the language generated by the grammar which is the solution to the problem (Pao, 1969). Or, the machine may be told that the given sample contains all the sentences of the language (as in Gold's "text" method of presentation) or that the sample contains sentences whose production requires the use of all the transitions in a particular automaton (Pao, 1969). Or, the sample may be assumed to be structured in various ways—for example, the sample may be assumed to contain (or tend to contain) all of the sentences of the language whose length is not longer than the longest sentence of the sample (Feldman, 1969).

We deal here, however, with what seems to be the most basic and general problem—the kind of grammar discovery problem one would

encounter in deciphering an unknown message from outer space, or in decoding an ancient language from a given scroll, or in finding the substructures in a coded message—that is: *given* a fixed, finite sample of sentences and no other information of any kind (that is, no other information about the source of the same; no information about the type of grammar producing the sentences; no information about the statistical properties or completeness of the sample; no non-sentences; no cues; and no information in response to queries or requests) *find* a non-trivial, hierarchical, parsimonious grammar that could have generated the sentences of the sample.

B. REVIEW OF PREVIOUS WORK IN THE FIELD

The literature in the field of induction is very small.

Solomonoff (1964) treats the problem as a special case of sequence extrapolation using enumerative methods which are not readily extendable, and which are highly dependent on such artifacts as order of sentences within the sample.

Gold (1967) proposed a theory of language learning in which the learner receives sequences of correct sentences of an unknown language and successively guesses grammars of specified types. If the guessing converges to one correct grammar, the language has been "identified in the limit." Gold proves several decidability results concerning language identifiability in the limit, and proposes several categorizations of language learning models, and methods of presenting information to such learning models.

Feldman (1969) proposes various concepts of grammatical complexity and proposes approaches to inducing a least complex grammar for a given sample. Feldman (1969) includes a heuristic method of merging symbols of the alphabet to achieve recursive rules from a finite sample.

Pao (1969) deals with the induction of finite state languages and "delimited" languages--a subset of context-free languages. Pao specifies conditions on the sample to guarantee a solution; and then, given certain cues, the algorithm constructs an automaton, evaluates it, and modifies it until a solution is attained.

Caianello (1970) discusses the Procustes Algorithm for feature extraction and form analysis of language, and uses semigroup homomorphisms similar to what are called "transformations" or "recodings" herein.

In a highly suggestive, short paper, Goodall (1962) discusses the problem of induction from the point of view of logical types, and the Godel Incompleteness Theorem. Goodall develops a probabilistic automaton for a given ongoing infinite sequence of input symbols. In this paper, Goodall suggests, without specifying, the idea of recoding the given sample into a new "higher level" sample using "resolving transformations" that optimize the conflicting qualities of "entropy" (information) and "parsimony." This thesis started as an attempt to attribute some consistent and reasonable meaning to these suggestive terms and other metaphorical observations in the Goodall paper, but no claim is made that the definitions and constructions herein of these same terms in any way correspond to that intended by Goodall.

In addition to work in the immediate field of induction, there is, of course, a large literature in the fields of pattern recognition (e.g. Uhr); cryptography (e.g. Smith and Gaines); picture analysis; sequence forecasting and correlation, including continuous functions are numerical sequences.

C. PRINCIPAL CLAIMS OF THIS THESIS

In this thesis, we formalize the notion of a regularity in a sample of sentences, present a combinatorial algorithm for searching for regularities, and argue that searching for local regularities is a suitable beginning for a grammar discovery algorithm— an approach not found in the literature.

We then present the Grammar Discovery Algorithm which, for a given sample of sentences, induces a grammar that (1) can generate the given sample, (2) is non-trivial in the sense that it does not merely enumerate the sentences of the sample, (3) is hierarchical in the sense that the sentences of the sample are derived through relatively long sequences of sentential forms, (4) is, at the same time, parsimonious, in the sense that the grammar contains a relatively small number of rules of production, (5) contains recursive rules of production under appropriate specified conditions, and (6) contains disjunctive rules of production (and generalizations) under appropriate specified conditions. We do not claim that the demonstratably non-trivial, hierarchical, and parsimonious grammar that we seek and find here is an interesting grammar, is a significant grammar (in the sense, perhaps, that it conforms to some known or supposed physiological or psychological model),is a "linguiist's" grammar, or is the "best" grammar. Rather, we only claim that (1) the grammars we seek and find here do possess the above three qualities and that these qualities are reasonable objectives for a grammar discovery algorithm; (2) the grammar discovery algorithm herein satisfies certain heuristic criteria; and (3) the *A*lgorithm handles a variety of interesting examples that indicate it is not inherently limited in its capacity.

D. OVERVIEW OF THIS THESIS

Part II of this thesis contains a description of the Grammar Discovery Algorithm.

In II. A., we discuss the method of presenting the sample of sentences of the Grammar Discovery Algorithm. The Grammar Discovery Algorithm begins with a sample of sentences Y produced by an unknown grammar $\mathcal{G} = \langle V_T, V_N, R, \text{"S"} \rangle$. The sentences are presented as a concatenation of sentences, with or without initial punctuation (for example, periods) between the sentences to identify the sentences. The terminal alphabet V_T is also given--or can be trivially inferred from the sample. The goal is to find a parsimonious, non-trivial, hierarchical grammar for the given sample of sentences—that is, to find a set R of rules of production, and whatever non-terminal alphabet symbols V_N are needed to write the rules R. One of the non-terminal symbols in V_N will be designated as the starting symbol ("S") of the grammar.

The Grammar Discovery Algorithm is divided into three parts: the Search Phase, the Recoding Phase, and the Selection Phase.

In II. B., we describe the first main phase of the Grammar Discovery Algorithm—namely, the Search Phase. In II. B. 1 through II. B. 6, we formally develop and define the idea of a regularity in a sample.

Informally, a "regularity" exists whenever the occurrence of certain symbols somewhere in a sentenc in the sample implies that certain other symbols occur elsewhere in the sentence with a conditional probability greater than mere chance.

In the Search Phase, one is interested in finding *regularities* in the given sample of sentences. The search is for *local*, not

global, regularities. It is not obvious that any grammar discovery algorithm should begin with a search for regularities, but an argument will be given to justify this search (II. B. 8). The regularities will also be categorized by type in a manner suggestive of the various Chomsky types of grammars (e.g. context-sensitive unrestricted rewrite, etc). (II. B. 7) The grammatical types of the rules production that ultimately will be induced will parallel the types assigned to the regularities.

We present the algorithm of the Search Phase in II. B. 9, and give examples (II. B. 10).

The Search Phase is an exhaustive, combinatorial process—within the limits imposed by the principle of searching only for local regularities.

The Search Phase ends with the generation of various tables of statistics about the various regularities discovered.

The second main phase of the Grammar Discovery Algorithm is the Recoding Phase. In the Recoding Phase, the local regularities found in the Search Phase are used to transform (recode) the given sample of sentences into a new set of sentences. This recoding is specified by (1) a set of grammatical rules of production, which specify which substitutions are to be made, and by (2) a Recoding Procedure, which specifies when and where these substitutions are to be made. The grammatical rules of production are developed from the local regularities discovered in the Search Phase. These rules ultimately become the rules of production of the induced grammar. The Recoding Procedure is essentially a punctuating process that partitions the given sample into appropriate phrases and substructures. The Recoding Procedure operates so as to minimize the introduction of rules of pro-

duction into the induced grammar. For example, before a new rule of production is introduced into the induced grammar, the algorithm considers the possibility of instead introducing one recursive rule (to replace one existing rule and the proposed new rules) as well as the possibility of re-using the new rule alone or in conjunction with other existing rules. The Recoding Phase is a non-exhaustive process, in contrast to the Search Phase.

In II. C., we describe this Recoding Phase. We define the notion of a transformation (recoding) in II. C. 2. We show how to develop grammatical rules of production from local regularities in II. C. 3, and show the connection between the grammatical type of the rule and the mask of the local regularity from which the rule was developed. In II. C. 4, we describe the Recoding Procedure—which, with the rules of production, defines the transformation (recoding).

The new set of sentences resulting from a recoding contains most of the information and structure of the given sample of sentences. Because structural regularities found in the original sample are replaced with short markers, what were originally global regularities in the original sample tend to become near-by local regularities in the new set of sentences. These short markers ultimately become the non-terminal symbols of the induced grammar.

The original given sample of sentences is regarded as the set of sentences of the first level in the Grammar Discovery Algorithm. These sentences are composed entirely of terminal symbols. The set of sentences resulting from the recoding then becomes the sentences of the next level of the process. These sentences contain non-terminal symbols as well. These sentences will now be searched for their local regularities, and another recoding performed. The alternate

application of first the Search Phase, and then the Recoding Phase is what produces the non-trivial, hierarchical quality of the grammar that is developed.

The third main phase of the Grammar Discovery Algorithm is the Selection Phase. After the Search Phase is completed, several (but not all) possible recodings of the sentences are considered. The Selection Phase is a non-exhaustive process that oversees these several trial recodings and selects one recoding which is then actually applied to the sample to produce one new set of sentences. This selection is made on the basis of three criteria: the *entropy*, *parsimony*, and *recursive parsimony* of the transformation, the parsimonious quality of the induced grammar derives from this selection from among the various possible recodings. Upon completion of this *actual recoding*, the 3 main phases are repeated, using the new set of sentences, instead of the original sample. The processes continue until the conditions of the Termination Procedure are fulfilled, thus ending the grammar discovery algorithm.

The result is a set of rules of production R, and a non-terminal alphabet V_N with which to express them, and a starting symbol "S". Together with the given terminal alphabet V_T, these elements will constitute a grammar $\mathcal{G} = \langle V_T, V_N, \text{"S"}, R \rangle$ that could have produced the given sample of sentences.

In II. D., we describe this Selection Phase. We define the entropy, parsimony, and recursive parsimony of a transformation in II. D. 2. We define the notion of resolving transformation in II. D. 4. We graph the conditional probabilities of the regularities of the transformation in II. D. 3. And, in II. D. 5, we describe the process of making several trial recodings (each parameterized by the

length of the longest regularity used), and the algorithm for selecting the actual recoding.

In II.E., we describe the process of inducing recursions from a finite sample of sentences. In II.E.1. we first discuss the motivation for inducing recursions—namely, the ability of the induced grammar to generate an infinity of sentences. Without recursions, there can only be a finite number of sentences generated by a grammar. We define recursions in II.E.2, and discuss different possible approaches to inducing recursions in II.E.3. The different approaches can be divided according to whether they operate on the sentences of the sample or on the rules of production of the induced grammar. In II.E.4, we present the sentence oriented method of inducing recursions. This method has limited application. Finally, in II.E.5, we discuss the rule-oriented method of inducing recursions.

In II.F., we discuss the related problems of inducing generalizations and inducing disjunctions. The discussion parallels the discussion of recursions. In II.F.1, we present the motivation for inducing generalized rules when the sample is a formal system. In II.F.2, we present a combinatorial rule-oriented method for inducing generalizations. Then, in II.F.3, we turn to disjunctions. We discuss the motivation for inducing disjunctive (non-deterministic) rules of production in grammar induction—namely, the ability of the induced grammar to generate a variety of essentially different sentence types. Without disjunctions, there can only be essentially one sentence type generated by the grammar. In II.F.4, we present a sentence-oriented method for inducing disjunctions that is based on finding ensembles of substitution instances that have maximal or near-maximal entropy. We conclude this section with a discussion of the similarity of the problem of

inducing generalizations and inducing disjunctions.

In II.G. we discuss the Termination Procedure and Convergence of the Algorithm.

In II.H. we discuss the heuristic considerations for choosing the upper limits on lengths of regularities to be searched for in the Search Phase.

In II.I, we discuss the various parameters of the Recoding Procedure.

In II.J., we elaborate on the idea that the Grammar Discovery Algorithm presented here for linear sequences of symbols is applicable to a broad variety of induction and pattern recognition problems presented in different formats. This section also serves to extend the definitions and concepts of the Grammar Discovery Algorithm by providing illustrations from a simple two-dimensional pattern recognition problem.

In II.K, we present a variety of examples of the Algorithm.

Throughout part II, we point out how the basic features of the Grammar Discovery Algorithm—namely, its Search Phase involving local regularities, the Recoding Phase, and the Selection Phase—work to produce a non-trivial, hierarchical, and parsimonious induced grammar.

In Part III, we discuss the self-punctuating nature of the Grammar Discovery Algorithm. In II.B.2, we define the notion of extending a regularity, of a regularity being preserved under extention, and of a maximal regularity (III.B.2). In III.B.4, we argue that a maximal regularity will usually not be missed or lost by an unfortuitous choice of limits on the length of regularity in the Search Phase.

In III.C., we argue that the choice of parameters in the Recoding

Phase and the combinatorial incompleteness of the Recoding Phase does not usually cause the Algorithm to miss maximal regularities in the sample—unless it finds equally desireable maximal regularities. In III.D., we make the observation (quite contrary to intuition) that the presence or absence of initial punctuation between the individual sentences of a sample does not qualitatively or even qu nti-tatively (combinatorially) complicate grammar induction.

In Part IV, we discuss several additional features of the Grammar Discovery Algorithm, including ternary masks and "don't care" conditions (IV.A); alphabet-enlarging versus the alternative (Huffman) type recodings (IV.V); recodings with noise in the sample and what noise represents (IV.C); and the generation of contest-free rules of production by an algorithm which appears to be inherently context-sensitive (IV.D).

In Part V, we discuss the justification for the Algorithm.

In the appendices, we describe the input to the Fortran IV computer program implementing the Grammar Discovery Algorithm (Appendix A); we exhibit the program in Appendix B; and we present additional examples in Appendic C.

II. DESCRIPTION OF THE GRAMMAR DISCOVERY ALGORITHM

A. PRESENTATION OF THE SAMPLE

Let V_T be a set of C1* symbols called the *terminal alphabet.*

The Grammar Discovery Algorithm is said to operate in one of two modes depending on whether initial punctuation is present in the sample. We say that the Grammar Discovery Algorithm is operating in the *first mode* if there is no initial punctuation present in the sample—that is, the sample Y consists of one long sequence of terminal symbols

$$Y = Y(1), Y(2), \ldots , Y(N)^{**}$$

in which the "sentences" of the sample are concatenated together.

We say that the Grammar Discovery Algorithm is operating in the *second mode**** if *initial punctuation* is present between the sentences of the sample. This initial punctuation, if present, is conventionally the period. In this mode, the sample appears as a string of symbols (a sentence) followed by a period, another string of symbols (another sentence), another period, etc.

Thus, the *sample* is a sequence of symbols over the terminal alphabet, with or without initial punctuation between the individual

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*Mathematical variables which are capitalized herein refer to variables that appear in the computer program implementing the Grammar Discovery Algorithm.

**Mathematical vectors which appear in the computer program implementing the Grammar Discovery Algorithm are written herein in the parenthesized style common in computer languages.

***The variable MODE in the computer program specifies whether the Grammar Discovery Algorithm is operating in the first mode or in the second mode.

sentences. The sample consists of only this symbol sequence. No additional information of any kind is given. No assumptions as to the statistical properties or completeness of any other aspect of this sample are made. Note that this method of information presentation is neither Gold's "text" method (in which the sample is presumed to contain every sentence of the language!) nor Gold's "informant" method (in which non-sentences appear) (Gold, 1967).

The presentation of the sample Y as a sequence of symbols, rather than as a square, rectangle, tree, graph, or other structure or raster containing symbols conforms to the predisposition in the study of grammars of natural and artificial languages and in the study of formal systems to consider linear sequences of symbols. However, virtually nothing in what follows is actually dependent on this linear method of presentation. Thus, pattern recognition problems where the sample consists of other structures or rasters of symbols are equally amenable to the methods described herein.

B. THE SEARCH PHASE--FINDING REGULARITIES IN A SAMPLE OF SENTENCES

1. INTRODUCTION TO THE SEARCH PHASE

The first main phase of the Grammar Discovery Algorithm is the Search Phase.

In the Search Phase, we search for local regularities in the given sample of sentences. Informally, a "regularity" exists whenever the occurence of certain symbols somewhere in a sentence in the sample implies that certain other symbols occur elsewhere in the sentence with a conditional probability greater than mere chance.

In this section, we proceed to define formally the notion of a cylinder set in a sample; to generalize this concept using the idea of a mask; to categorize these masks by grammatical type; to define formally the notion of context-sub-sequence, the predicted-sub-sequence, and the domain sequence; and to formally define and illustrate the idea of a regularity in a sample. We then argue that if a grammar discovery algorithm is to begin with a search for regularities, the length of those regularities must be small—that is, we should only search for local regularities. We present the algorithm of the Search Phase.

The Search Phase is an exhaustive, combinatorial process--within the limited range established by the principle of searching only for local regularities.

2. CYLINDER SETS IN A SAMPLE

Let V be a set of C symbols, $a_1, a_2, \ldots, a_C$. Call V the *alphabet*. A *sentence* over V is a sequence

$$s = y_1 y_2 \cdots y_n$$

where each y_i, $1 \leq i \leq n$, is in V. Our interest here is in finite sentences, although most of the statements we make would apply equally to infinite sentences.

Let Y be a given sample of sentences. In the terminology of automata theory and information theory (Khinchin, 1956), each sentence in the set of all possible sentences over V_T is an *elementary event* of some space of events. Each subset of Y (and Y itself) then is an *event*. We are interested in a particular kind of event—the cylinder set.

An *index sequence*

$$Z = (t_1, t_2, \ldots, t_h)$$

is a sequence of different non-zero positive integers. An *allowable index sequence* for a given set Y of sentences is an index sequence in which no integer is greater than the length of the shortest sentence in Y. Two index sequences are *disjoint* if they contain no integers in common. The *cardinality* of an index sequence X, denoted $|Z|$, is the number of integers in Z.

A *cylinder set* ξ, or a *cylinder*, for a given sample Y of sentences is the subset of all sentences in Y such that

$$y_{t_i} = a_i$$

for all i = 1, ..., h, and where each $a_i \in V$, and where the sequence $(t_i)_{t=1}^{h}$ is an allowable index sequence for the set Y. ξ may, of course, turn out to be empty. Note that the cylinder set ξ is defined in terms of 3 parameters: (1) the sample Y of sentences, (2) the allow-

able index sequence Z, and (3) a sequence of h a_i's from the alphabet V.

Consider the cylinder set ξ defined for a sample Y of sentences, and index sequence $Z = (t_i)_{i=1}^{h}$, and h a's from the alphabet V. Let $W = (w_j)_{j=1}^{g}$ be an index sequence disjoint from Z. Let σ be the cylinder set defined over the cylinder set ξ (which is a subset of the sample Y of sentences), an index sequence W, and g a's from V. A *positive regularity* exists in a sample Y of sentences if, for some *non-empty* cylinder set ξ defined as above, and for some cylinder set σ defined as above, then the conditional probability

$$\frac{|\sigma|}{|\xi|} > \frac{1}{C^g}$$

We denote the quantity $1/C^g$ by ϵ. Note that this quantity is the probability associated with a "pure chance" event—that is, it is the probability of occurence of particular string over C symbols of length g if each of the strings occurs equiprobably.

It will be customary herein to number the cylinder sets, such as ξ and σ, so that we will then be able to index the conditional probabilities (and thereby the positive regularities) with two numerical indices and refer to them as P_{ij}, where i is the index number assigned to cylinder set ξ, and j is the index number assigned to σ.

The h positions of the index sequence Z constitute the *context part* of the positive regularity; and the g positions of the index sequence W constitute the *predicted part* of the positive regularity.

Note that a positive regularity in a given sample is specified by 4 parameters: (1) the sample, (2) a context part, (3) a predicted part, and (4) the conditional probability $|\sigma|/|\xi|$

In what follows, we will assume that the union of the h+g integers in the disjoint index sequences Z and W together constitute h+g consecutive integers—or, equivalently, that the context part

and the predicted part cover h+g contiguous positions of the sentences of the sample. We will discuss the occasions when this is not the case. separately later in the section of TERNARY MASKS.

3. MASKS

It will prove advantageous to generalize the above approach to cylinder sets and to detach the cylinders from specific positions $t_1, \ldots, t_h$. That is, if a "1" in position 1 and a "1' in position 2 reliably predict a "0" in position 3, and a "1" in position 4 and a "1" in position 5 reliably predict a "0" in position 6, we should be able to generalize these two observations about the sample and identify them as being a regularity applying to sequences of length 3. Indeed, often the same phrase will appear many times in a sample—but it will usually appear in many different relative positions within the sentences. We should therefore be able to generalize these regularities and identify them as one regularity.

Khinchin accomplishes this generalization by introducing a time shift operation. This approach is equivalent to the various "window" schemes in pattern recognition approaches (Uhr, 1963).

However, even with the time shift, all the observations are specific to the symbols that occur in the various positions. That is, if an "a" in the first position (note: not position 1) and a "c" in a third position reliably predict a "b" in the second position, and a "d" in a first position and an "f" in a third position reliably predict an "e" in the fifth position, then we should also be able to generalize these two observed regularities and identify them as one regularity applying to certain sequences of length 3.

Thus, it is desirable to make a further generalization and detach the specific context symbols and the specific predicted context symbols from the regularity, and express the fact that certain positions may constitute a context that reliably predicts certain predicted positions. This leads to the following definition:

A *binary mask*, $\mathcal{M}$, or a *mask*, is a binary sequence over the symbols "_" and "%"—provided the sequence contains at least one "_" and at least one "%". The symbol"_" denotes "the context." The symbol "%" denotes "the predicted part." An *M-mask* is a mask of length M. For example, "_ % _" is a 3-mask in which the context is the first and third position, while the predicted part is the second position.

The *universe of binary masks* is all 2^M-2 binary sequences over "_" and "%" provided the sequence contains at least one "_" and at least one "%". By implication, the shortest masks are of length 2.

For example, the universe of binary masks of length 3 consists of the 6 masks "_ _%", "_%%", "%%_", "%_ _", "_ % _", and "%_%". Note that "_ _ _" (all context) is not considered a mask and has no meaning. Also "% % %" (all predicted part) is not considered a mask at this point in this paper. It can, however, be given a reasonable meaning by interpreting it as the mask which catalogs unconditional probability of occurrence of strings of length 3 in the sample.*

*The control parameter LAWL controls whether this extra mask is used in the computer program implementing the *G*rammar *D*iscovery *A*lgorithm.

4. TYPES OF MASKS

The universe of M-masks is partitioned by *grammatical type* as follows:

All masks of the form "$\_^{M-k}\%^{k}$" $k\neq 0$ are called *strictly left-sensitive*. All masks of the form "$\%^{k}\_^{M-k}$" $h\neq 0$ are called *strictly right-sensitive*. Both of these two types of masks are considered to be of the same level of grammatical complexity--that is, context-sensitive. "_%" is an example of a strictly left-sensitive 3-mask, and "%_" is an example of a strictly right-sensitive 2-mask.

All masks of the form "$\_^{k_1}\%^{k_2}\_^{M-k_1-k_2}$", where $k_1\neq 0$ and $k_2\neq 0$ and $k_1+k_2\neq M$, are called *strictly context-sensitive*. M must be at least 3 to have a strictly context-sensitive mask. "_%_" is an example of a strictly context-sensitive mask.

All other masks are called *strictly unrestricted rewrite*. These masks are characterized by more than two switches from "_" to "%" or from "%" to "_" (if they start with a "_"), or by two or more such switches (if they start with a "%"). "%_%" is the shortest strictly unrestricted rewrite mask.

The justification for the choice of these names for these subsets of masks will be seen later, in the section wherein we use the regularities derived from the masks to develop the rules of production of the induced grammar.

Masks can have a *radix* of either 2 or 3. If the radix of a mask is 3, the mask is called a *ternary mask*. A *ternary mask* is a sequence over the symbols "_", "%", and "#". The symbols "_" and "%" are defined as for binary masks. The symbol "#" denotes "don't care."

There are several important differences and effects when one uses ternary masks, instead of binary masks. Ternary masks (1) allow

a separation in positions between the symbols of a regularity; (2) allow a kind of generalization operation (opposite in effect to the substitution rule of inference in logic) over whatever may be in a certain position; and (3) may allow certain cryptographic regularities to be more readily discovered. Ternary masks will be discussed in full later.

For example, of the 6 masks of length 3, "_ %%" and "_ _ %" are strictly left-sensitive; "%% _" and "%_ _" are strictly right-sensitive; "_ % _" is strictly context sensitive; and "% _ %" is strictly unrestricted rewrite.

5. CONTEXT SUB-SEQUENCES, PREDICTED SUB-SEQUENCES, AND THE DOMAIN SEQUENCE

The *context-sub-sequence* γ is a sequence of length M over the C symbols of the current alphabet Vc and the predicted symbol "%" defined, for I = 1, 2, ..., M as

$$\gamma(I) = \begin{cases} Y(T-M+I) & \text{if } \mathcal{M}(I) = \text{"\_"} \\ \text{"\%"} & \text{if } \mathcal{M}(I) = \text{"\%"} \end{cases}$$

The symbol "%" is merely a filler symbol in γ, and its appearance here does not carry any of the meaning that the symbol otherwise has. Because of this convention, note that the context-sub-sequence γ contains in itself all the information of the binary mask $\mathcal{M}$ from which it was produced.

The *predicted-sub-sequence* ϕ is a sequence of length M over the C symbols of the current alphabet Vc and the context symbol "_" defined, for I = 1, 2, ..., M as

$$\phi(I) = \begin{cases} Y(T-M+1) & \text{if } \mathcal{M}(I) = \text{"\%"} \\ \text{"\_"} & \text{if } \mathcal{M}(I) = \text{"\_"} \end{cases}$$

Here again "_" is a filler symbol in ϕ. Note that the predicted-sub-sequence contains in itself all the information of the binary mask $\mathcal{M}$ from which it was produced.

If the mask is a ternary mask, then $\gamma(I)$ and $\phi(I)$ are "#" whenever $\mathcal{M}(I)$ is "#".

Finally, we define the *domain sequence* as the sequence of length M over the current alphabet V_c obtained as follows:

$$\Delta(I) = \begin{cases} \gamma(I) & \text{if } \phi(I) = \text{"\%"} \\ \phi(I) & \text{if } \gamma(I) = \text{"\_"} \end{cases}$$

for I = 1, ..., M. Note that Δ consists of the symbols from the current alphabet Vc (and not "%" or "_") occurring position-by-position in either γ or ϕ, and represents the symbols actually appearing in Y.

6. DEFINITION OF A REGULARITY

A *regularity* $\mathcal{R} = < \gamma, \phi, P, Y >$ is defined as a quadruple consisting of a context-sub-sequence γ (a sequence of length M over the symbols of the current alphabet Vc and the symbol "%"), a predicted-sub-sequence ϕ (a sequence of length M over the symbols of the current alphabet Vc and the symbol "_"), a real number P representing the conditional probability that the occurrence of γ in the sample Y predicts the occurrence of the predicted-sub-sequence γ in Y. When the sample Y intended is clear, we may write $\mathcal{R}$ as $<\gamma, \phi, P>$. The *length of a regularity* is M, which is the common length of γ and ϕ and Δ.

In English text, for example, one regularity is $\mathcal{R}$ = <"Q%", "_U", 1.0>. This regularity is of length 2. This regularity expresses the fact that the letter "Q" is always followed by the letter "U". The binary mask $\mathcal{M}$ from which this regularity is derived is "_%", and is of length 2. This mask is a strictly left-sensitive mask. The context-sub-sequence γ of this regularity is "Q%", and the predicted-sub-sequence ϕ is "_U". Both, of course, are also of length 2. The conditional probability P that the context-sub-sequence predicts the occurrence of the predicted-sub-sequence in the sample is 1.0. The domain sequence Δ for any occurrence of this regularity is "QU"—that is, "QU" are the symbols that actually occur in the sample. Δ is also of length 2.

Note that the frequency of occurrence of a given context-sub-sequence (the "Q" in the above case) does not enter into the definition of the regularity since conditional probabilities are used. It should be noted, however, that the notion of a mask can be extended in order to accomodate the measurement of probabilities of occurrence of single symbols, 2-grams, 3-grams, etc. in the sample. This extention is

accomplished by allowing a mask to consist entirely of the predicted symbol (%). With a vacuous context, the conditional probability P_{ij} now becomes the probability of occurrence of the predicted-subsequence ϕ. The probability of occurrence is not, however, used anywhere in the Grammar Discovery Algorithm.

7. TYPES OF REGULARITIES

We classify regularities and P_{ij}'s by the magnitude of the conditional probability P_{ij}. If the P_{ij} is exactly 1.0, we say that the regularity (or, the P_{ij} itself) is of *type 1*, and we call it *structural*. The regularity that "U" follows "Q" in English is a structural regularity. If P_{ij} is near 1.0; that is, if P_{ij} is less than 1.0 but greater than or equal to $1.0 - 1/c^g$, then we say that the regularity (and the P_{ij}) is of *type 2*. If the P_{ij} is less than $1.0 - 1/c^g$ but greater than or equal to 1/2, then we say that the regularity (and the P_{ij}) is of *type 3*. If the P_{ij} is less than 1/2 but greater than or equal to $\varepsilon = 1/c^g$ then we say that the regularity (and the P_{ij}) is of *type 4*. The regularities of these 4 types are the positive regularities. If the P_{ij} is less than ε, but greater than 0.0, we say that the P_{ij} is of *type 5*, and we call it *noise*. If the P_{ij} is exactly 0.0, we say that the P_{ij} is of *type 6*. We are interested, of course, in the P_{ij} that are 1.0 or near 1.0.

8. RATIONALE FOR SEARCHING FOR LOCAL REGULARITIES

It is not obvious that searching for local regularities in a sample of sentences is the way to begin a grammar discovery algorithm. Indeed, both practical and theoretical considerations seem to advise against this kind of approach.

First of all, it would appear that, in searching for regularities in a sample of sentences produced by an unknown grammar, one would be obliged to consider dependencies between symbols widely separated from one another in the sentence. This point is made in Chomsky (1956, p. 286) in discussing the distance in the "relations of dependencies" that may occur as a result, for example, of self-embedding in natural languages such as English. In this regard, consider the sentence

++p+ppp

which is a well-formed sentence of the propositional calculus (in Polish Notation), and which is generated from the single recursive rule of production "p → +pp". When one parses this sentence, the first symbol (i.e. the "+") is in fact generated with the same application of the rule of production as the last symbol (i.e. the final "p"). Now the number of possible regularities observable in M positions ($M \geqslant 2$) of sentences over an alphabet of c symbols is

$$\sum_{h=1}^{m-h} C^h \cdot \binom{m}{h} \cdot C^{m-h}$$

(where h is, as before the number of positions in the context part of the regularity). This is equal to

$$(2^m - 2)\ C^m \stackrel{d.}{=} F_m$$

which is very large for anything but small m. Indeed, in the simplest case, that of the binary alphabet (C=2), we have

m	2	3	4	5	6	7
F_m	8	48	224	960	3968	16,128

Moreover, we are naturally interested in the possibility of sentences of indefinite length; that is, in sentences that are produced by automata with cyclic state transition diagrams, and in sentences produced by grammars with at least one recursive rule of production. A finite sample of such sentences—which is, of course, the only situation we would encounter—contains a longest sentence; but, still, that sentence could be very long. Thus, there would seem to be practical combinatorial obstacles to any algorithm which is based on searching for regularities over m symbols, for even modestly large m, much less the m one would encounter in practice.

Even without the practical combinatorial obstacles that seem to surround analysis of sentences of non-small length m, there is a theoretical problem in searching for regularities among distant symbols. Consider a finite sample of sentences over an alphabet V with C symbols, wherein the longest sentence is of length m. Unless the sample contains at least one instance of each of the C^m possible sequences of length m, (which would make the grammar very uninteresting, indeed) there will be at least one, and in general very many, conditional probabilities that are greater than $1/c^g$, for each h = 1, 2, 3, ..., m-1. Thus, there will be a least one positive regularity in any such sample. The fact that positive regularities do appear will be used later. But for the moment, it will be used to make another argument.

Consider again all the well-formed sentences of the propositional calculus generated by the single recursive rule of production "P → +PP ". The sentences produced by this rule are all of length m = 3, 5, 7, 9, ... There are $2^{\frac{m-3}{2}}$ sentences of length m. And, in particular, there are

8 sentences of length 9; namely,

+P+P++PPP
++P++PPPP
+P+++PPPP
++++PPPPP
+P+P+P+PP
++P+P+PPP
+P+++P+PP
++++P+PPP

Suppose further, we are given a sample of sentences in which no sentence is of length greater than 9, and that only 7 of the 8 sentences of length 9 are included in the sample. Suppose it is the last sentence which is not present. Then the conditional probability that there is a "P" in position 6, given that there is a "++++P" in positions 1 through 5, respectively, and given that there is a "PPP" in positions 7 through 9, respectively is 1.0. Since c=2 and g=1 here, this is greater than $1/c^g = 1/2$, and indeed therefore indicates a "positive" regularity. This discovered structural regularity is, of course, an artifact of the sample. If the sample were understood to be *complete*, in the sense of containing all of the sentences of the grammar, then this regularity would be faithful to the grammar. However, this sample of sentences is typical in that it is admittedly an *incomplete* sample arising from a grammar which is capable of producing other sentences. In a larger sample, it is possible that the missing sentence would occur as a sentence. Or, it may very well appear as a sub-sentence (which we can call *phrase* or *clause*) of a longer sentence. Thus, when we search for regularities of length equal to the length of the longest sentences of the sample, the regularities we find are likely to be artifacts of the sample size, and therefore highly dependent and variable on the sample size. Any grammar discovery algorithm based on finding regularities of this quality would be unstable. Our goal, of course, is that the grammar

produced not depend on sample size, after a certain point. A grammar discovery algorithm should converge, and thereafter be stable.

A final objection to searching for regularities of long length is that their discovery may indeed tell something about each itemized long sentence present in the sample, but tell us little about the sub-structure of the sentence. Thus, even in the absence of other objections, the regularities produced would tend to lead to trivial information (i.e. cataloging the long sentences)--rather than information about the sub-structure of the sentences.

Thus, if a grammar discovery algorithm is to begin with a search for regularities, the length of regularities considered must be small--both absolutely, to avoid practical combinatorial problems--and relatively, to avoid instability and triviality problems. For these, and other reasons that will be discussed later, it is necessary (and also sufficient) to consider only small M in searching for regularities in a grammar discovery algorithm. The way to find global properties is to look for local properties.

9. ALGORITHM OF THE SEARCH PHASE

In the grammar discovery algorithm, we consider masks of length M = 2, 3, 4, ..., up to some pre-determined, small upper limit on M.* The selection of this upper limit M2 will be discussed later.

For each such M, we consider all possible masks $\mathcal{M}$ of length M which are of grammatical type up to some pre-determined limit**—perhaps, for example—all masks no more complex than strictly context sensitive. The masks are considered in order of increasing grammatical complexity—up to the pre-determined limit. That is, context sensitive masks are considered, for example, before unrestricted rewrite masks. This pre-determined limit is selected on the basis of what type of grammatical rules one is seeking. It does not matter whether one is too generous in admitting rules that are more complex, because the rules are not only generated in order of increasing grammatical complexity, but also are adopted into the grammar in that order. Thus, all the M-1 strictly right sensitive masks of length M, and all M-1 strictly left sensitive masks of length M are generated and considered first. They, in turn, are followed by the M(M+2)/2 strictly context sensitive masks. These masks are then followed by the strictly unrestricted rewrite masks of length M, of which there are

$$2^M - 2 - 2M - 2 - M(M+2)/2$$

Suppose we denote, for the moment, the symbols of a sentence in the sample as

$$Y(1), Y(2), \ldots, Y(d),$$

where d is the length of the sentence. Then, given the current M we are considering, and the current mask $\mathcal{M}$, we examine the sentence,

*The variable M2 is this upper limit in the computer program.

**The variable GRAM controls this in the computer program.

symbol by symbol. We consider the sub-strings of length M terminating in position T=M, M+1, ..., d ($d \geq M$), and match each with the current mask—that is, we match the symbols

$$Y(T-M+1), \ldots, Y(T=M)$$

with mask positions 1, 2, ..., M. A context-sub-sequence γ and a predicted-sub-sequence ϕ are then developed from each such matching, as indicated by the definition of γ and ϕ.

Each distinct context-sub-sequence γ that is produced is given a distinct index number. Each distinct predicted-sub-sequence ϕ that is produced is also given a distinct index number. This allows us to refer to the conditional probability P_{ij} that predicted-sub-sequence j occurs, given that context-sub-sequence i occurred, by using the two indices i and j. In the computer program implementing the Grammar Discovery Algorithm, the index number assigned is the natural number equivalent to the M symbols of γ and ϕ modulo (C+2)+2.* These two natural numbers together thus abbreviate all the information needed to identify the sub-sequence as either a predicted sub-sequence or a context-sub-sequence--because a context-sub-sequence contains symbols from the alphabet and the "%", but never the "_", while a predicted-sub-sequence contains symbols of the alphabet and the "_", but never the "%".

*It should be noted that many of the operations of the Search Phase, as well as later phases such as the Recoding Phase, are amenable to being performed by parallel computers—either by general purpose parallel computers which will represent the next generation of computers, or by a special purpose parallel computer. The combinatorial obstacles to performing grammar induction on large samples of unknown text will thus be greatly reduced in practice.

**The natural number corresponding to the context-sub-sequence is called SQCTXT in the computer program. The natural number corresponding to the predicted-sub-sequence is called SQPRED. The reason for the first "+2" is the inclusion of the symbols "_" and "%". The reason for the second "+2" is the inclusion of the symbols "#" and the initial punctuation mask.

The pair of 2 natural numbers together are sufficient to recreate the context-sub-sequence γ, the predicted-sub-sequence ϕ, and the mask $\mathcal{M}$.

The procedure for doing the above depends somewhat on the mode in which the Grammar Discovery Algorithm is operating. If initial punctuation is present in the sample (the second mode), the examination above proceeds through each separate sentence. Naturally, if T+M-1>d, then masks of that length M are not considered for that sentence. (We can, if we like, allow the initial punctuation mask appearing at both ends of a given sentence to be considered as part of that sentence; however, we never allow inter-sentence examinations). If there is no initial punctuation in the sample, the only restriction is that T+M-1>d.

In either mode, the entire sample Y is examined under the current mask $\mathcal{M}$.

Upon completion of this examination, it is now possible to compute the conditional probability P that a given predicted-sub-sequence ϕ arising from a particular mask $\mathcal{M}$ of length M occurred, given the occurrence of a particular context-sub-sequence γ arising from that same mask.

This process is then repeated for each mask being considered.

It is convenient to sort these conditional probabilities into descending numerical order, from 1.0 on down. The P_{ij}'s are then classified into types (I, II, III, IV, and V) as previously described.

Note that for a given context-sub-sequence with h context positions, one would expect that most of the $(M-h)^C = g^C$ possible predicted-sub-sequences will not actually occur in the sample. In a non-pathological sample, there are relatively few context-sub-sequence

and predicted-sub-sequence pairs that will actually occur.*

The result of the *S*earch *P*hase is the production of tables of regularities that occur in the sample.

*In the computer program implementing the Grammar Discovery Algorithm, the possibility of an excessive number of small, insignificant P_{ij}'s is dealt with by providing for the early culling of the tables of regularities. This early elimination of P_{ij}'s is controlled by the parameter EARLY.

10. EXAMPLES

Example A: To illustrate the ideas developed above, consider the unpunctuated (first mode) symbol string Y as follows: THE¬BIG¬CAT RAN¬THE¬BAD¬CAT¬RAN¬THE¬BIG¬RAT¬RAN¬THE¬BAD¬RAT¬RAN¬

The terminal alphabet for this sample Y of length N = 64 over C1 = 12 symbols is A ={A,B,C,D,E,G,H,I,N,R,T,¬}, where "¬" is an undistinguished, but visible symbol to denote the blank.

The table below illustrates the kind of masks, context sequences, predicted sequences, and conditional probabilities that one obtains in analyzing the sample.

SOME REGULARITIES FOR LEVEL 1

LENGTH M	CONDITIONAL PROBABILITY p_{ij}	TYPE	MASK $\mathcal{M}$	CONTEXT Sub-Sequence γ	PREDICTED Sub-Sequence δ
3	1,00000	1	_%%	C%%°	_AT
3	1.00000	1	__%	TH%	__E
3	1.00000	1	%%_	%%G	BI_
3	1.00000	1	%__	%AN	R__
3	1.00000	1	%_ _	R%T	_A_
3	0.50000	3	_%%	B%%	_IG
3	0.50000	3	_%%	B%%	_AD

The first line of this table, for example, indicates that the letter C is always followed by the letters AT in the sample. This forms the "word" CAT. The second line of this table indicates that the letter pair TH is sufficient to determine the following letter which is E. These 2 lines illustrate left-sensitive rules.

Line 3 of this table indicates that the letter G is sufficient context on the right to predict, with 100% reliability, the preceding two letters, namely B and I. Line 4 indicates that the context of R on the left and the context of T on the right is sufficient to determine

the one intervening letter A, making the word RAT. This is an example of a strictly context sensitive rule.

The only mask not illustrated for length 3 above is the unrestricted rewrite mask %-%. This is the only unrestricted rewrite mask of length 3.

The last two lines of this table indicate that the letter B is followed by IG and AD equally often(i.e. with probability .50).

Using the terminology we have developed, the information on line 1 represents the type I regularity $\mathscr{R}$ = <"C%%", "_AT", 1.0,Y> of length M = 3.

Example B: Now let us consider an example based on the following sequence of 272 bits found in the memory of an IBM 360 computer:

(1) "1110 0110 1100 1000 1100 0101 1101 0101 0100 0000 1100 1001
1101 0101 0100 0000 1110 0011 1100 1000. 1100 0101 0100 0000
1100 0011 1101 0110 1100 0100 1100 0010 1100 0101 0100 0000
1101 0110 1100 0110. 0100 0000 1100 1000 1110 0100 1101 0100
1100 0001 1101 0101 0100 0000 1100 0101 1110 0101 1100 0101.
1101 0101 1110 0011 1110 0010 0100 0000"

Here the alphabet A = {0,1} and N = 272 and C1 = 2.

When the memory of the computer is dumped, the sequence of 272 bits would conventionally be presented as 68 half-bytes (4 bit sequences). The name conventionally attached to each 4 bit sequence is the hexadecimal equivalent of the 4 bits in binary. Using these names, the dump might read:

(2) "E6C8C5D5 40 C9D5 40 E3C8C5 40 C3D6E4E2C5 40 D6C6 40 C8E4D4C1D5
40 C5 E5 C5D5E3 E2 40"

When the 68 half bytes are then combined into 34 full bytes and presented as EBCDIC characters (as is the case in the EBCDIC dump), we see

(3) "WHEN IN THE COURSE OF HUMAN EVENTS"

If we now search the sample (1) for regularities, we find for a regularity length M of 2 the following regularities:

LEVEL= 1 M= 2

NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)	0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)	0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)	4
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)	4
NUMBER OF TYPE 5 SEQUENCES (NOISE)	4
NUMBER OF P(I)	12
NUMBER OF POSSIBLE P(I)	8
SEPARATION VALUE BETWEEN TYPES 3 AND 4	C.50000
EPSILON FOR DEFINING TYPES 2 AND 3	0.34000

#	LEVEL	LENGTH	PROB	TYPE	MASK	CONTEXT	PREDICTED
6	1	2	0.57	3	_%	1%	_0
2	1	2	0.57	3	%_	%1	0_
7	1	2	0.55	3	_%	0%	_0
4	1	2	0.55	3	%_	%0	0_
3	1	2	0.45	4	%_	%0	1_
8	1	2	0.45	4	_%	0%	_1
1	1	2	0.43	4	%_	%1	1_
5	1	2	0.42	4	_%	1%	_1
11	1	2	0.31	5	%%	%%	00
10	1	2	0.25	5	%%	%%	10
12	1	2	0.25	5	%%	%%	01
9	1	2	0.19	5	%%	%%	11

Note that the mask "%%" is included in the above table and that it catalogs the probabilities of occurrence of the various sequences of symbols appearing in the sample.

Similarly, if we search the sample for regularities of length no greater than 3, we obtain the following table of regularities:

LEVEL= 1 M= 3

NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)	0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)	0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)	8
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)	8
NUMBER OF TYPE 5 SEQUENCES (NOISE)	24
NUMBER OF P(I)	40
NUMBER OF POSSIBLE P(I)	32
SEPARATION VALUE BETWEEN TYPES 3 AND 4	0.50000
EPSILON FOR DEFINING TYPES 2 AND 3	0.33333

#	LEVEL	LENGTH	PROB	TYPE	MASK	CONTEXT	PREDICTED
22	1	3	0.61	3	__%	11%	__0
38	1	3	0.60	3	%__	%11	0__
6	1	2	0.57	3	_%	1%	_0
26	1	3	0.57	3	__%	00%	__0
2	1	2	0.57	3	%_	%1	0_
42	1	3	0.57	3	%__	%00	0__
28	1	3	0.56	3	__%	01%	__0
40	1	3	0.55	3	%__	%10	0__
7	1	2	0.55	3	_%	0%	_0
4	1	2	0.55	3	%_	%0	0_
23	1	3	0.52	3	__%	10%	__0
43	1	3	0.51	3	%__	%01	0__
44	1	3	0.49	4	%__	%01	1__
24	1	3	0.48	4	__%	10%	__1
3	1	2	0.45	4	%_	%0	1_
8	1	2	0.45	4	_%	0%	_1
39	1	3	0.45	4	%__	%10	1__
27	1	3	0.44	4	__%	01%	__1
41	1	3	0.43	4	%__	%00	1__
1	1	2	0.43	4	%_	%1	1_
25	1	3	0.43	4	__%	00%	__1
5	1	2	0.42	4	_%	1%	_1
37	1	3	0.40	4	%__	%11	1__
21	1	3	0.39	4	__%	11%	__1
20	1	3	0.31	5	_%%	0%%	_00
36	1	3	0.31	5	%%_	%%0	00_
11	1	2	0.31	5	%%	%%	00
15	1	3	0.30	5	_%%	1%%	_00
30	1	3	0.30	5	%%_	%%1	00_
32	1	3	0.28	5	%%_	%%1	10_
16	1	3	0.27	5	_%%	1%%	_01
14	1	3	0.26	5	_%%	1%%	_10
10	1	2	0.25	5	%%	%%	10
31	1	3	0.25	5	%%_	%%1	01_
19	1	3	0.25	5	_%%	0%%	_10
12	1	2	0.25	5	%%	%%	01
35	1	3	0.25	5	%%_	%%0	01_
34	1	3	0.24	5	%%_	%%0	10_
17	1	3	0.23	5	_%%	0%%	_01
33	1	3	0.20	5	%%_	%%0	11_

18	1	3	0.20	5	_%%	0%%	_11
9	1	2	0.19	5	%%	%%	11
52	1	3	0.17	5	%%%	%%%	000
29	1	3	0.17	5	%%_	%%1	11_
13	1	3	0.17	5	_%%	1%%	_11
51	1	3	0.14	5	%%%	%%%	010
47	1	3	0.13	5	%%%	%%%	100
48	1	3	0.13	5	%%%	%%%	001
50	1	3	0.12	5	%%%	%%%	101
46	1	3	0.11	5	%%%	%%%	110
49	1	3	0.11	5	%%%	%%%	011
45	1	3	0.07	5	%%%	%%%	111

Some of the regularities we obtain with an M of 4 are

LEVEL= 1 M= 4

NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)	0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)	0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)	16
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)	32
NUMBER OF TYPE 5 SEQUENCES (NOISE)	64
NUMBER OF P(I)	112
NUMBER OF POSSIBLE P(I)	96
SEPARATION VALUE BETWEEN TYPES 3 AND 4	0.50000
EPSILON FOR DEFINING TYPES 2 AND 3	0.25000

#	LEVEL	LENGTH	PROB	TYPE	MASK	CONTEXT	PREDICTED
142	1	4	0.70	3	%___	%101	0___
87	1	4	0.68	3	___%	110%	___0
96	1	4	0.67	3	___%	101%	___0
85	1	4	0.65	3	___%	111%	___0
139	1	4	0.63	3	%___	%011	0___
147	1	4	0.63	3	%___	%111	0___
90	1	4	0.61	3	___%	100%	___0
22	1	3	0.61	3	__%	11%	__0
98	1	4	0.61	3	___%	010%	___1
38	1	3	0.60	3	%__	%11	0__
93	1	4	0.60	3	___%	011%	___0
138	1	4	0.60	3	%___	%001	0___
135	1	4	0.58	3	%___	%100	1___
134	1	4	0.58	3	%___	%110	0___
144	1	4	0.58	3	%___	%010	1___
6	1	2	0.57	3	_%	1%	_0
26	1	3	0.57	3	__%	00%	__0
2	1	2	0.57	3	%_	%1	0_
42	1	3	0.57	3	%__	%00	0__
28	1	3	0.56	3	__%	01%	__0
40	1	3	0.55	3	%__	%10	0__
7	1	2	0.55	3	_%	0%	_0
4	1	2	0.55	3	%_	%0	0_

And, some of the regularities we obtain with an M of 5 are shown below:

LEVEL= 1 M= 5

NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)	C
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)	2
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)	33
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)	79
NUMBER OF TYPE 5 SEQUENCES (NOISE)	174
NUMBER OF P(I)	288
NUMBER OF POSSIBLE P(I)	256
SEPARATION VALUE BETWEEN TYPES 3 AND 4	0.50000
EPSILON FOR DEFINING TYPES 2 AND 3	0.20000

#	LEVEL	LENGTH	PROB	TYPE	MASK	CONTEXT	PREDICTED
291	1	5	0.86	2	____%	1111%	____0
419	1	5	0.86	2	%____	%1111	0____
394	1	5	0.74	3	%____	%0011	0____
287	1	5	0.73	3	____%	1010%	____1
400	1	5	0.73	3	%____	%1011	0____
270	1	5	0.72	3	____%	0110%	____0
272	1	5	0.70	3	____%	1101%	____0
142	1	4	0.70	3	%___	%101	0___
410	1	5	0.70	3	%____	%0101	1____

In later sections, we will refer to these tables as part of other examples.

C. THE RECODING PHASE--DEVELOPING RULES OF PRODUCTION FROM REGULARITIES

1. INTRODUCTION TO THE RECODING PHASE

The Recoding Phase is the second main phase of the Grammar Discovery Algorithm.

In the Recoding Phase, the local regularities found during the Search Phase are used to develop grammatical rules of production which, in turn, are used to transform the given set of sentences into a new set of sentences. The new set of sentences contains most of the information and structure of the original set of sentences. This transformation (recoding) process works by replacing contiguous predicted symbols of local regularities occurring in the sample with single symbols. Thus, what were originally global regularities in the given sample tend to become local regularities in the new set of sentences. The recoding that is done is defined, in part, in terms of grammatical rules of production; and these rules of production ultimately become the rules of production of the grammar being induced. The resulting new sentences can later be treated anew as a sample and this new sample can then be searched for regularities, reusing the methods of the Search Phase. And, this new sample can then be recoded at this higher level, yielding additional rules of production and a newer set of sentences. The alternate application of first the Search Phase, and then the Recoding Phase is what produces the hierarchical (and non-trivial) quality in the grammar being induced.

2. TRANSFORMATIONS (RECODINGS)

We are concerned here with mappings from a sequence Y of length N over C1 symbols to a new sequence Y' of length N' over C1' symbols. A *transformation* (or *recoding*) $\mathcal{T}$ as used here will refer to such a mapping. If C1' > C1, we say that the mapping is *alphabet-enlarging*.

A transformation is *presented* in terms of a set of (1) grammatical rules of production, and (2) a Recoding Procedure. Both are necessary to define the transformation. Each grammatical *rule of production* consists of an *antecedent* (*left*) side, which is a symbol string over the alphabet of Y'; an arrow; and a *consequent* (*right*) side, which is a symbol string over the alphabet of Y. An example of a rule of production is

$$0A0 \rightarrow 0111O \quad .$$

We read this as "replace A by 111 whenever A is found in context with a zero on both sides of it."

Note that the set of rules of production is not sufficient, in general, to uniquely define the mapping from Y to Y'. There will often be many overlapping occurrences of the context-sub-sequences of the local regularities. Hence there are many different overlapping opportunities for applying the rules of production. It is therefore necessary to establish an order of precedence for developing the rules of production. The *Recoding Procedure* is an algorithm for scanning the sequence Y for occurrences of the context-sub-sequences of the various local regularities and for specifying in what order the rule of production should be developed from these regularities.

The sample Y, as initially presented, is said to be the sample of the *first level* . Each transformation, when applied to a sample, maps that sample from its current level to a sample of one higher *level*. It should be noted that each level above the first represents

one sentential form in the derivation of the terminal string (which is the sample at the first level). Thus, each level above the first will contain some non-terminal symbols.

We adopt the convention here that a transformation of a given sample Y will always be over the complete domain Y. This will in general necessitate the application of the identity transformation on certain substrings of Y. This convention guarantees that the entire domain Y is "covered" by each transformation, and this completeness of covering facilitates later comparisons between transformations.

3. DEVELOPMENT OF RULES OF PRODUCTION FROM LOCAL REGULARITIES

In the Search Phase, certain local regularities were discovered. Recall that a regularity $\mathscr{R} = \langle \gamma, \phi, P, Y \rangle$ consists of a context-sub-sequence γ, a predicted-sub-sequence ϕ, and an associated conditional probability P. The local regularities are now the basis for developing the rules of production of the grammar.

Suppose we are given a context-sub-sequence γ from some regularity $\mathscr{R}$ of length M. Suppose further that for some integral $T > 0$,

$$Y(T-M+I) = \gamma(I)$$

for each I = 1, 2, ..., M for which $\gamma(I)$ is not "%" or "#" *—that is, suppose the context specified by the context-sub-sequence ϕ is present in the sample Y; and, that its terminal position is position T of Y. Suppose the Recoding Procedures indicates it is appropriate that a rule of production be developed using this occurrence of the context part of this regularity, and suppose further that we are generating context-dependent rules of production and that we are using only alphabet-enlarging recodings. Then the rule of production would be produced as follows:

1. The consequent (right) side of the rule consists of the domain sequence

$$\Delta = \Delta(I), \ldots, \Delta(M)$$

—which is of length M.

2. The antecedent (left) side consists of a sequence of symbols constructed as follows: Begin with the context-sub-sequence γ and consider each sub-string of contiguous predicted symbols "%" in γ. Enlarge the current alphabet V_C by a single new symbol, say "N".

---

* "#" is the "don't care" symbol and is discussed more fully later.

This new symbol will be a non-terminal symbol in the induced grammar. Now, insert this single new symbol N into the context-sub-sequence γ in place of the entire substring of contiguous predicted symbols. If there is more than one sub-sequence of contiguous predicted symbols in γ, introduce another new non-terminal symbol into V_C for each such sub-sequence. Note that there can be more than one such sub-sequence only when the mask from which the regularity was derived was of strictly unrestricted rewrite type. Note that the antecedent (left) side thus produced is of length M or less. Note that this decrease in length is the result of replacing the entire contiguous predicted substring by a single new symbol. Note also that the symbols of the current alphabet V_C occurring in context or don't care positions in the sample are left unchanged. Finally, note that the grammatical type of the rule of production developed in this way is exactly the grammatical type of the mark from which the regularity $\mathscr{R}$ was derived.

Suppose, for example, that $\mathscr{R}$ = <"11%%", "__00", 1.0, Y> is a regularity in a given sample Y—that is, in Y, the symbols 11 are invariably followed by 00. The context-sub-sequence γ for this structural regularity is "11%%". The predicted-sub-sequence ϕ is "␣␣00" The conditional probability P is 1.0. The mask "__%%" gave rise to this regularity, and this mask is a left-sensitive mask. The domain sequence Δ is "1100". The sequence Y begins 0100110001..., so that the context part of the context-sub-sequence "11%%" occurs in Y so that its final symbol (i.e. the second "%") is at position T = 8 of Y. The rule of production developed from this regularity of length 4 in Y will be the left-sensitive rule

$$11N \rightarrow 1100,$$

where N is understood to be a non-terminal. In this example, "N"

replaces the two contiguous predicted symbols, which in this case, are the two 0's.

The alternative to generating context-dependent rules of production is to generate context-free rules. If context-free rules are to be generated, the antecedent (left) side will consist solely of a single new non-terminal symbol. In this event, the antecedent (left) side will, of course, be of length 1. If we are considering only $M \geqslant 2$, then it will be shorter than the consequent side. The subject of generating context-free rules will be discussed in detail later.

An alternative to an alphabet-enlarging code is a recoding using a Huffman code, or a similar scheme for optimal recoding. In this approach, the alphabet is not enlarged. Instead, a distinctive symbol string in the original alphabet is inserted in place of the symbols in Y. The details of possible non-alphabet-enlarging recodings and the advantages and disadvantages of this kind of recoding will be discussed later.

Thus, as we have seen, the length of the regularities searched for in the Search Phase limits the length of the consequent (right) side of the rules of production developed for the induced grammar. Thus, if M is the length of the longest regularity searched for in the Search Phase, M is also the length of the longest consequent side of a rule of production in the induced grammar. This in turn, means that if the sample Y is of length N, then regardless of how the rules of production are applied to accomplish the recoding of Y, the recoded version of Y cannot be of length shorter than N/M. Since M is generally fairly small, the image of the sample Y under a Recoding is

of a length relatively close to the length of Y itself. This means that Y is not recoded by any one Recoding into an extremely short completely-resolved image. In other words, if we look at the way the sample Y is derived from the rules of production using the ultimate induced grammar, we see that the derivation is indeed a hierarchical derivation—that is, a derivation requiring relatively many intermediate sentential forms (levels). Note that it is the smallness of M which results in this hierarchical characteristic. Note also that the hierachical grammar thus induced is also not trivial. Thus, both the hierarchical and non-trivial character of the induced grammar follow from the decision to search only for *local* regularities in the Search Phase.

Finally, note that while certain Unrestricted Rewrite *masks* can be accomodated in the above procedure (for example, the mask "%%_%%"), there is no provision in general for the development of length-increasing rules of production (that is, rules in which the consequent (right) side is longer than the antecedent (left) side). (See Ginsburg and Greibach [1966]). However, since every rudimentary event (Smullyan [1959]) can be accepted by some linear bounded automaton (Myhill [1960]), and hence context-sensitive language, the formal systems of Smullyan are within the scope of this Algorithm.

4. THE RECODING PROCEDURE

We noted earlier that a transformation is specified in terms of both (1) a set of grammatical rules of production, and (2) a Recoding Procedure. The *Recoding Procedure* is an algorithm for scanning the sequence Y for occurrences of the context-sub-sequences of various local regularities, and for specifying in what order the rules of production should be developed from those regularities.

The Recoding Procedure implemented in the computer program for the Grammar Discovery Algorithm admits of numerous variations. These variations will be described in detail as the reason for each possible variation arises. For now, we will consider merely the basic features of Recoding Procedure.

In the Search Phase, masks of various possible lengths, from a length of M1 to a length of M2, were used to find local regularities. We take these same M1 and M2 from the Search Phase for use in the Recoding Procedure. If M2 is the mask of longest length considered in the Search Phase, then the first occurrence of a context-sub-sequence associated with that mask cannot be earlier than position T = M2 in the sequence Y. Therefore, the Recoding Procedure starts scanning the sequence Y at position T = M2. The sequence Y is examined for the presence of a sub-sequence of Y of length M which terminates at position T such that

$$Y(T-M+I) = \gamma(I)$$

for each I = 1, 2, ..., M for which $\gamma(I)$ is not "%" or "#". In general, context-sub-sequences may be identified with various sub-sequences of Y. The purpose of the Recoding Procedure is to select one context-sub-sequence from among the several possibilities. The sub-sequence of Y that is selected is the one which meets the following conditions:

(1) The sub-sequence selected is of longest length, of the lengths between M1 and a parameter MBE which is less than or equal to M2. This range of lengths is further constrained by 2 considerations. First, there must indeed be a sub-sequence in Y of the length L being considered. This means that $T \geqslant L$, for each L considered. Also, if the sequence Y has initial punctuation, then the sub-sequence in Y of length L must not include an initial punctuation mark. Note that an initial punctuation mark is always assumed to precede any seuqence Y, and the first case is merely a special case of the second.

The second constraint is that none of the positions occupied by the sub-sequence of length L have been recoded as yet on this level. That is, our aim is to find regularities which "cover" the sequence Y no more than once at any position. If multiple covering were allowed at any one level, then (i) the resulting grammar would be ambiguous, and (ii) it might be necessary to establish a precedence ordering among the rules of production of the grammar induced.

(2) The regularity selected is the one whose conditional probability is of the highest type, among those types which are allowed in the Recoding. The lowest allowable type* is specified in advance, and is typically type I, II, or occasionally III.

(3) The regularity selected has highest conditional probability, given condition (2). This is accomplished by the sorting of the conditional probabilities P(I) associated with the regularity.

(4) The regularity selected is the regularity whose mask is of simplest grammatical type. That is, a regularity arising from

---

*The variable AllOW in the computer program specifies the lowest allowable type of regularity to be used in recoding.

a left-sensitive mask will be preferred to a regularity arising from a strictly-context-sensitive mask, etc. Naturally, the type of masks that can be selected are limited by the types of masks used in the Search Phase.

After this local regularity is selected, a rule of production may be developed from it, in the manner described in the preceding section.

The sample Y is then recoded, as specified by that rule of production. Before a new rule of production is introduced however, the possibility of instead introducing a recursive rule of production is considered. The procedure for inducing recursions works with rules of production developed at previous levels of recoding. Thus, if the grammar discovery algorithm is operating on its first level, no test for recursion is performed, and the rule of production developed from the regularity is automatically added to the induced grammar. If the algorithm is already on the second level, or on a higher level, the test for recursion will be performed. This test is fully described in a later section. If a recursion is then induced, one recursive rule of production is added to the induced grammar. This recursive rule replaces one non-recursive rule of production which is already in the induced grammar.

The Recoding Procedure operates so as to minimize the introduction of new rules of production. Thus, when a new non-recursive rule is to be introduced, the entire sequence Y is immediately searched for additional possibilities of applying that rule. These recodings are then performed in preference to any other possible recoding. Similarly, whenever a new recursive rule of production is introduced, a similar back-tracking search is made through all previous levels to see if the recursive rule can be applied. Moreover, with recursive rules,

the Recoding Procedure considers the possibility of immediately re-applying recursive rules (even of the same positions of Y which have just been recoded), so that as much of Y as possible is recoded using the newly induced recursive rule.

Also, the Recoding Procedure operates in such a way that it first scans Y looking only for the possibility of inducing recursive rules of production. After this scan is done, the sequence Y is rescanned for the possibility of inducing non-recursive rules. This double scanning approach given preference to inducing a recursive rule. Because a recursive rule replaces an existing rule already in the grammar, this approach helps minimize the number of rules in the induced grammar.

The Recoding Procedure operates in such a way that the degree of "covering " of the sample is maximized—that is, as many as possible of the symbols of Y are either in the context part or predicted part of some regularity and some rule of production.

Once a non-recursive rule is introduced into the induced grammar, opportunities to reapply that rule are always considered before any further rules are added to the grammar.

5. EXAMPLE

In this section, we illustrate the development of rules of productions from regularities, and illustrate the application of the rules using a recoding procedure.

We begin with an illustration of a context-sensitive rule of production.

Suppose the given sample of sentences is as follows:

```
------------------------- LEVEL 1 ---------------------------------
SYMBOL STRING OF LENGTH 137 AND USING ALPHABET OF SIZE  14

.THE¬BIG¬CAT¬RAN¬.THE¬BAD¬CAT¬RAN¬.THE¬BIG¬COG¬RAN¬.THE¬BAD¬
DOG¬RAN¬.THE¬BIG¬CAT¬SAT¬.THE¬BAD¬CAT¬SAT¬.THE¬BIG¬DOG¬SAT¬.
THE¬BAD¬COG¬SAT¬.
```

Note that the terminal alphabet for the sentences of this sample include 14 Roman alphabet symbols and the period (as initial punctuation).

Suppose further that the following regularity has been identified in the sample:

$$\langle \text{"\%\%E¬"},\ \text{"TH\_\_"},\ p \rangle$$

This regularity is a left-sensitive context-sensitive regularity because the mask from which the regularity was derived is a left sensitive mask (i.e. "%%__").

The development of a rule of production from this regularity proceeds as follows: The two contiguous predicted positions (namely positions 1 and 2) are consolidated and replaced by the new non-terminal symbol ("0" in this case). Together with the two context positions, these three symbols become the antecedent (left) side of the rule of production. The consequent (right) side of the rule consists of the four terminal symbols from the regularity. The rule of production is thus

$$0E\neg \longrightarrow THE\neg$$

In a similar manner, the following 7 rules of production might be developed:

```
TENTATIVE RULES OF PRODUCTION FOR M OF   4
0E¬   -----> THE¬
1G¬   -----> BIG¬
2AT¬  -----> CAT¬
3AN¬  -----> RAN¬
4D¬   -----> BAD¬
5G¬   -----> DOG¬
6AT¬  -----> SAT¬
```

Applying these rules of production to the sample, one recodes the sample and obtains the following:

```
NEW STRING

•0E¬1G¬2AT¬3AN¬•0E¬4D¬2AT¬3AN¬•0E¬1G¬5G¬3AN¬•0E¬4D¬5G¬3AN¬•0
E¬1G¬2AT¬6AT¬•0E¬4D¬2AT¬6AT¬•0E¬1G¬5G¬6AT¬•0E¬4D¬5G¬6AT¬•
```

Note that the rules of production in this example were context-sensitive because the mask of the regularity from which the rules were derived were context-sensitive masks.

It would seem that only context-sensitive rules of production can be derived. Such is not the case. Later, after motivating and defining the concept of maximal regularity (in section III.B.3 and IV.D), we will develop context-free rules of production. However, it should be noted now that if there is a procedure for developing context-free rules of production, then their application to the given sample, using the recoding procedure, is exactly the same as with context-sensitive rules of production.

To illustrate this statement,recall example B from the previous section. The sample in Example B was the following binary sequence:

```
--------------------------- LEVEL 1 -------------------------------
SYMBOL STRING OF LENGTH 272 AND USING ALPHABET OF SIZE  2

1110011011001000110001011101010101C000C011C01001111010101010O
0000111000111100100011C001010100000C11CCCC011110101101100010O
11CC0010110001C1C1CCCCC011010110110001100100000110010001110
0100110101001100000111010101010000000110001011110010111100101
1101010111100011111CC0100100C0000
```

Suppose there is a justification for developing context-free rules of production in the Grammar Discovery Algorithm. Suppose that the following four context-free rules are developed:

```
TENTATIVE RULES OF PRODUCTION FOR M OF   2
A   ----->   11
B   ----->   10
C   ----->   01
D   ----->   00
```

Then the recoding of the given sample, using these four context-free rules of production and the recoding procedure, would yield the following new sample of sentences:

```
NEW STRING


ABCBADBDADCCACCCCDDDACBCACCCCDDDABDAACBDADCCCDDDADDAACCBADCD
ADDBADCCCDDDACCBACCBCDDCACBCABCDACCDADDCACCCCDDDADCCABCCABCC
ACCCABCAABCBCDDD
```

In the examples that accompany several of the following sections of this paper, we will use context-free rules of production in many cases, although the rationale for developing these rules of production will not be fully developed until sections III.B.3 and IV.D.

D. THE SELECTION PHASE—SELECTING RECODINGS

1. INTRODUCTION

The third main phase of the Grammar Discovery Algorithm is the Selection Phase. The Selection Phase oversees several possible trial recodings and selects one which is then actually applied to the sample of sentences to produce one new set of sentences. This selection is made on the basis of three criteria: the entropy, the parsimony, and the recursive parsimony of the transformation.

In this section, we define the concepts of entropy of a transformation, parsimony of a transformation, and recursive parsimony of a transformation. We define a resolving transformation, and we present the algorithm for trying recodings, evaluating them, and then selecting one recoding as the actual recoding to be performed on the sample. The idea of the P_{ij}-graph is developed.

2. ENTROPY, PARSIMONY, RECURSIVE PARSIMONY OF A TRANSFORMATION

Typically, there is more than one rule of production in each transformation. Each rule of production is derived from a local regularity. Recall that each regularity $\mathcal{R} = \langle \gamma, \phi, P_{ij} \rangle$ consists of a context-sub-sequence γ, a predicted-sub-sequence ϕ, and an associated conditional probability P_{ij}. We define the *entropy* of a transformation to be

$$- \sum P_{ij} \log_2 P_{ij}$$

(using the convention that $0 \log_2 0 = 0$), where the P_{ij} are the conditional probabilities associated with the regularities from which the rules of production of the transformation were derived. Note that the sum here is taken over the (reduced) set of rules of production. Note that this measure depends only on knowing the *set* of rules of production of the transformation.

Note, for example, that if each local regularity from which the rules of production of the transformation was developed is a structural regularity (i.e. the context-sub-sequence predicts the predicted-sub-sequence with 100% reliability), then the entropy of the transformation will be zero. As an illustration of this, consider the string Y

... abcdefghefghabcdefgh...

This string might be recoded using the two strictly-context-sensitive rewrite rules

aNd → abcd

eMh → efgh ,

which each have 100% reliability. The resulting string Y' would then be

...aNdeMheMhaNdeMh....

Note that no information is lost by this transformation for the positions actually recoded.

On the other hand, if the local regularities used to develop the rules of production of a transformation all had conditional probabilities of $\varepsilon = 1/c^g$ (i.e. where the occurrence of the context tells us "nothing" about the "predicted" part), then the entropy of such a transformation would be maximal. Information is lost by such a transformation; the original sequence Y cannot be recovered by applying any kind of inverse of production of this transformation to the encoded string Y'. The information lost would be maximal for the positions actually encoded. For example, suppose 001 is followed with 00 or with 11 with probability 1/2, and that the rule of production is

$$001A \rightarrow 00100.$$

Y may be the sequence ...00100,00111,00100,00111... (the commas are added here only as a visual aid) and Y' may be ...001A,00111,001A, 001A... Here Y cannot be recovered from Y'.

The *weighted entropy* of a transformation is the same sum as the entropy of a transformation, except that the sum is taken over each application of the rules of production. Thus, rules that are used more often in the actual recoding count more in this sum. Note that this measure is dependent on both the set of rules of production and also the Recoding Procedure—which determines how many times each rule is applied.

The fewer rules of production in a grammar, the more parsimonious it is.

Accordingly, we define the *parsimony* of a transformation as $\lambda \cdot n$, where n is the number of (reduced) rules of production, and

where λ is a positive real number called the *coefficient of parsimony*.* Note that n is a measure of the complexity of the transformation, and indirectly of the induced grammar (Goodall).

To illustrate the idea of entropy and parsimony, consider the following trivial grammar which can be inferred from any given sample of sentences $\alpha_1, \alpha_2, \ldots, \alpha_w$ over an alphabet V_T. The grammar is $\mathcal{G} = \langle V_T, S, S, P \rangle$ where the rules of production P are the w rules:

$$S \rightarrow \alpha_1$$
$$S \rightarrow \alpha_2$$
$$\ldots$$
$$S \rightarrow \alpha_w$$

In this case, the entropy of the transformation here is zero, because all transformations are 100% reliable. However, the parsimony here is high --- namely λw.

We define the *combined entropy-parsimony measure* of a transformation to be

$$-\sum P_{ij} \log_2 P_{ij} + \lambda \cdot n$$

Note that there is a trade-off between the entropy and the parsimony of a transformation. The combined entropy-parsimony measure for 2 different transformations can be the same if the one with more rules of production has correspondingly less entropy (i.e. if the regularities used to develop the rules of production of the transformation have greater reliability).

Note also that while probability of occurrence of the different contexts is not explicitly considered in the Grammar Discovery Algorithm (because conditional probabilities are used throughout), probabilities of occurrence are reflected implicitly in the following

*The variable LAMBDA in the computer program is the coefficient of parsimony.

way: A rule of production developed from a regularity whose context part that occurs rarely in the sample will not be applied often in transforming the sample, and therefore other rules of production will be needed to transform the remainder of the sample, and therefore parsimony will be less. The ideas of "entropy" and "parsimony" were suggested by a paper of Goodall (1962). However, it is likely that Goodall's entropy was a measure of information (and therefore was dependent on probabilities of occurrence and not conditional probabilities). The idea of a combined measure and a trade-off between these two quantities appears in Goodall (1962).

One recursive rule of production generally replaces many (indeed, an infinity) of non-recursive rules of production.

The *recursive parsimony* of a transformation is defined as

$$\lambda_r \cdot n_r$$

where n_r is the number of recursive (reduced) rules of production, and where λ_r is a positive real number, called the *coefficient of recursive parsimony.**

The *combined entropy-parsimony-recusive-parsimony measure* of a transformation is defined to be

$$-\sum P_{ij} \log_2 P_{ij} + \lambda \cdot n + \lambda_r n_r$$

Typically, recursive rules of production are more desireable than non-recursive rules. Thus, λ_r would be chosen so that

$$\lambda_r << \lambda$$

so as to give relatively greater weight in the measure to the less desirable non-recursive rules. One might even assign a value of zero to recursive parsimony.

*The variable LAMR in the computer program is the coefficient of recursive parsimony.

Note that in all of the foregoing cases, the identity transformation would have no entropy at all, since the application of an identity transformation is a completely reliable recoding. The combined measure of entropy, parsimony, and recursive parsimony actually used in the computer program implementing the Grammar Discovery Algorithm is an *ad hoc* combination of the above features. We have not attempted to derive the particular function chosen from a series of desired axiomatized properties, or to show that the function (or its class) is the only function (or only class of functions) that satisfies the axioms. The properties that the function does, however, satisfy are as follows:

(1) A rule of production developed from a structural (Type I) regularity whould have zero entropy. Thus, rules developed from either type I regularities or identify transformations will contribute zero to entropy.

(2) Each rule of production should count equally against parsimony. Certainly, measuring grammatical complexity can be done in other ways (and better ways) than counting rules of production; however, we will use this simple approach suggested by Goodall (1962) here. One important implication of this requirement is that each different identity transformation should count as a rule in determining parsimony. Thus, we do not treat the identity transformation as one transformation any more that we treat any other rule of production having different predicted parts as being the same. Note that if identity transformation did not count against parsimony, they would not contribute to the H measure at all (since their entropy is zero); and, if that were the case, the transformation with minimal H would always be the identity transformation on the whole sample, and the initial sample would then be perpetuated from

level to level.

(3) Each recursive rule of production should count equally against recursive parsimony.

(4) The entropy for each application of each rule should be normalized to reflect the number of symbols recoded with it. If this normalization were not done, longer rules would tend to be favored over shorter rules. Carried to the extreme, the best recoding would be accomplished with very long rules--rules so long that they are 100% reliable solely by virtue of their capturing artifacts of the sample.

(5) The relative weights to be given to the 3 attributes of entropy, parsimony, and recursive parsimony should be determined by a weighted sum. The trade-off should be between entropy on the one hand and the two kinds of parsimony on the other.

The weighted, normalized, combined entropy-parsimony-recursive-parsimony used in the computer program is therefore

$$- M_{ij} \sum P_{ij} \log P_{ij} + \lambda \cdot n + \lambda_r n_r$$

where this sum is taken over each application of each rule of production, and where M_{ij} is the length of the regularity associated with P_{ij}. Note that

$$\sum M_{ij} = N,$$

where N is the length of Y.

The idea of reliably recoding, or "bunching together," of portions of a sample of data is found in a variety of settings. It is well known in psychology and physiology [Miller, 1956] that human beings are well able to classify gradations in pitch, hue, loudness, count, smell, taste, and a variety of sensory information into about 7 categories. If faced with much more than 7 categories, humans lose the ability to distinguish reliably and must "bunch" the data together so

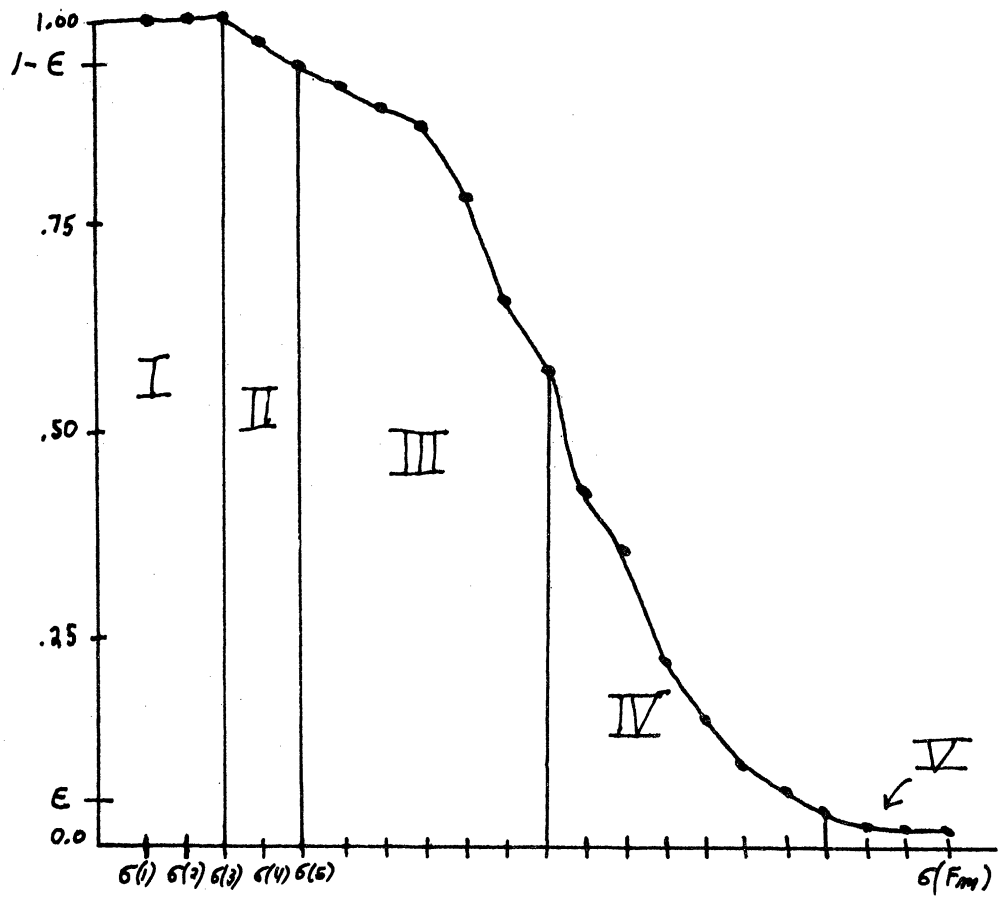
that items within the bunch may first be distinguished and gradated. Then, at the next higher level, representatives of the bunches may then be distinguished and gradated.

The pervasiveness of this number 7 (plus or minus 2) seems also to extend into the social and political behavior of humans. As C. Northcote Parkinson (1957) has noted in an examination of the English Privy Council and other cabinet bodies, human decision-making bodies tend to function best with about 8 members. When more than this number are added to a committee, an inner committee of smaller size tends to form to do the actual decision-making and to integrate the views of factional representatives from the larger body.

Although not within the scope of this study, it seems possible that, for humans, the tradeoff between parsimony and entropy comes when more than 7 items must be integrated. This level (of parsimony) in turn implicitly defines the degree of faithfulness of representation (entropy) which necessarily must be accepted by humans in a variety of sensory and social and political settings.

3. THE p_{ij}-GRAPH OF A TRANSFORMATION

Whenever we have a collection of probabilities from a set of regularities abstracted from a given sample, we may not only classify them as to type (i.e. type I, II, III, IV, and V), but we may sort them into descending order by magnitude. Such a graph, which we call the *p_{ij}-graph* has the general appearance seen below:



In the figure, σ denotes the permutation of the F_m p_{ij}'s which sort them into descending order. $\sigma(i)$, where i is an integer from 1 to F_m, is the ordered pair (i,j) that identifies the p_{ij}. Goodall (1962) presented such a graph, although apparently erroneously using the actual probabilities of occurrence. These probabilities, of course, are rarely near 1.0. However, if we instead consider conditional probabilities (as we do herein) or even relative probabilities (perhaps normalizing all probabilities of occurrence of an M-gram by dividing by the largest), there are several provocative and valid points in Goodall. As always, the Goodall terminology is highly metaphorical and suggestive. The high probabilities at the left end of the graph represent the "structure" in the sample. The *structure* is that portion of the sample that is reliably present. Structure is associated with the type I or II regularities in the sample. Structure may be such facts as Q's are always followed by U's in English; that periods are reliably followed by spaces; that most birthdays in the social security files have a "19" in them; or that certain self-synchronizing codes have a certain punctuating symbol at the beginning of each phrase (e.g. the peak of a sawtooth wave at the beginning of a television line image). Since the structure is reliable, all of the structure may be abbreviated and replaced by a shorter marker (or even removed altogether!). What is structure at one level is structure at the next level--regardless of how it is recoded, abbreviated, or deleted.

The *message* portion of a sample is the part that is unpredictably variable at this level of analysis. The entropy of this portion of the sample is high because this is the part of the sample which contains the information. A key point is that what is "unresolvable message"

at one level is not necessarily unresolvable at a higher level. Unlike structure, message is not always message. Of course, message cannot be recoded (except by an identity transformation) because it has no as-yet-identified internal structure.* Message corresponds to the type III and IV regularities in the sample.

The *noise* portion of the sample is that part of the sample which occurs with very low probability. Noise corresponds to the type V regularities--indeed, the "irregularities." Noise represents all the exceptions, contradictions, error, and randomness of the sample--things which occur so rarely that their non-occurrence in the sample is so predictable that they cannot be considered message (which is unpredictable variability). Whether noise represents "error" (as, for example, would be the case in an incoming encoded message) or "exceptions" (that is, information which indeed is correct, but rare) depends on the nature of the sample. Accordingly, one would want to preserve exceptions for analysis at a higher level and discard error as information that is incompatible with the sample. This is an external decision.

On the p_{ij}-graph, structure, message, and noise appear as one proceeds from left to right.

The p_{ij}-graph described earlier is of greatest interest when we plot only the p_{ij}'s used to develop rules of production that are actually used in a transformation. We call this graph the *p_{ij}- graph of a transformation.*

---

*There is one possible exception to this statement which is discussed in the sentence-oriented method for inducing disjunctions. The exception occurs when a message sequence is one of an ensemble of possible substitution instances for a predicted sub-sequence that has maximal or near maximal entropy. See II.E.4.

For example, if our initial sample is

```
--------------------------  LEVEL 1  ----------------------

SYMBOL STRING OF LENGTH 272 AND USING ALPHABET OF SIZE  2

11100110110010001100010111010101000C0011C010011110101010100
00001110001111001000110010101C00CCC11CCC011110101101100010C
11CC0010110001C1C1COCOCC1101011011000110C1000000110010001110
01001101010011C00001110101010100000011000101111001011110010 1
11010101111100111110C01C01CCOOCO
```

and we use the following 4 rules of production using an M of 2

```
TENTATIVE RULES OF PRODUCTION FOR M OF   2

A   ----->   11

B   ----->   10

C   ----->   01

D   ----->   00
```

we might find the following values for entropy and parsimony:

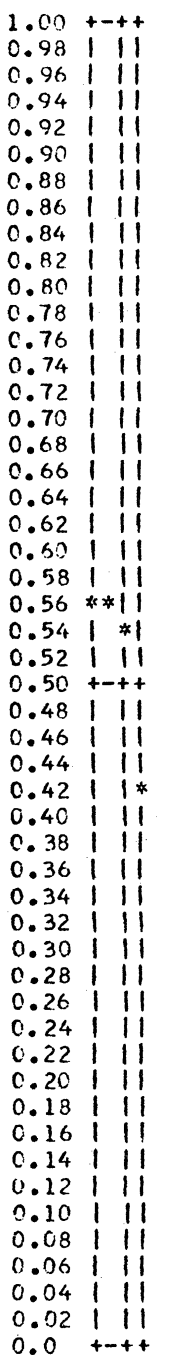
```
ENTROPY TERM                                  65.18735
PARSIMONY TERM                                 4.00000
RECURSIVE PARSIMONY TERM                       0.0
                                             +---------
VALUE OF H FOR THIS RECODING.....            69.18735

NUMBER OF RULES OF PRODUCTION                     4
NUMBER OF RECURSIVE RULES                         0
NUMBER OF TIMES RULES ARE APPLIED               136
```

If we then apply the rules of production, using a standard recoding procedure, we would obtain a p_{ij}-graph such as is found on the next page. This recoding results in the following new string:

```
ABCBADBDADCCACCCCDDDADBCACCCCDDDABDAADBDADCCCDDDADDAACCBADCD
ADDBADCCCDDDACCBACCBCDDCADBCABCDACCDADDCACCCCDDDADCCABCCABCC
ACCCABCAABCBCDDD
```

GRAPH OF P(I) USED IN RECCDING FOR LEVEL 1 AND M OF 2



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)	0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)	0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)	3
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)	1
NUMBER OF TYPE 5 SEQUENCES (NOISE)	0
NUMBER OF P(I)	4

4. RESOLVING TRANSFORMATIONS

If the entropy is not large for a transformation, it must be that p_{ij}'s are mostly near 1.0 or near 0.0 .

Let g be the largest number of predicted positions for any mask in a transformation. This number is always less than or equal to M-1 for the largest M tried in the Search Phase. The *criterion* for a transformation is the value

$$-\frac{1}{C^g} \log_2 \frac{1}{C^g}$$

A *resolving transformation* is a transformation for which the entropy is less than the criterion.

This suggestive term is found in Goodall (1962). The idea there that when a recoding satisfies the criterion, the underlying structure that generated the sample has been identified; and, therefore, the sample is "resolved."

Note that if one of our measures which combine entropy and parsimony is used, some p_{ij}'s may be smaller than 1 - ε; or, if a transformation which is not a resolving transformation is considered, some p_{ij}'s may be smaller than 1 - ε. In both these cases, however, most p_{ij}'s will tend to be near 1, and exceptions will occur only because the introduction of that p_{ij} leads to such a substantial reduction in the number of rules or production that the increase in entropy is justified. In all cases, however, most of the p_{ij}'s will be bunched near 1.0.

In the p_{ij}-graph of a transformation actually used in recoding there will be no p_{ij}'s that are less than ε because we never use regularities so unreliable in a transformation.

In the case of a resolving transformation, there can be no p_{ij}'s equal to ε because one such p_{ij} would alone make the entropy of the transformation larger than the criterion.

These last 2 observations define the features of any p_{ij}-graph of a resolving transformation.

5. TRIAL RECODINGS AND THE SELECTION OF THE ACTUAL RECODING

M1 and M2 are the lower and upper limits, respectively, on the length of context-sub-sequences and predicted-sub-sequences considered in the Search Phase in the search for local regularities. For each M between M1 and M2, a different recoding results, in general, if we allow only regularities of length up to M to be used in developing rules of production in the recoding. Thus, M2-M1+1 recodings are defined. These recodings are parameterized by the index MBE.

In the Selection Phase, these M2-M1=1 different *trial recodings* are each attempted, and the entropy, parsimony, and recursive parsimony of each computed. The recoding which is selected to be the *actual recoding* is the one which is associated with the first M (considering the direction of considering the M's*) which is a resolving transformation**, *or* the M which has the best combined entropy-parsimony-recursive-parsimony measure (if it is not actually a resolving transformation). The *best* combined measure is the least combined measure.

The strings resulting from the application of the actual recoding constitute the strings of the next level of the process. The Search Phase, the Recoding Phase, and the Selection Phase of the Grammar Discovery Algorithm are then applied to the strings of this level, until the Algorithm terminates.

---

*The variable VDIR specifies this order in the computer program.

**The variable WDF in the computer program determines whether the first resolving transformation discovered is used for the actual recoding, or whether (if there is more than one resolving transformation) the best resolving transformation is used.

For example, if we again refer to the example started on page 68, we find that for an M of 3, we develop the following 8 rules of production:

```
TENTATIVE RULES OF PRODUCTION FOR M OF   3

A   ----->   111

B   ----->   001

C   ----->   101

D   ----->   100

E   ----->   011

F   ----->   000

G   ----->   110

H   ----->   010
```

The application of these 8 rules to the sample yields the following new string:

```
ABCDDEFCGCHDFEBBGCHDFEDEGHBDHCFFGFACEEFDGFCDFCFFGCCDEBFFGHBG
HEHDGFECHCFFGBEGHADCGCEGBADHHFOO
```

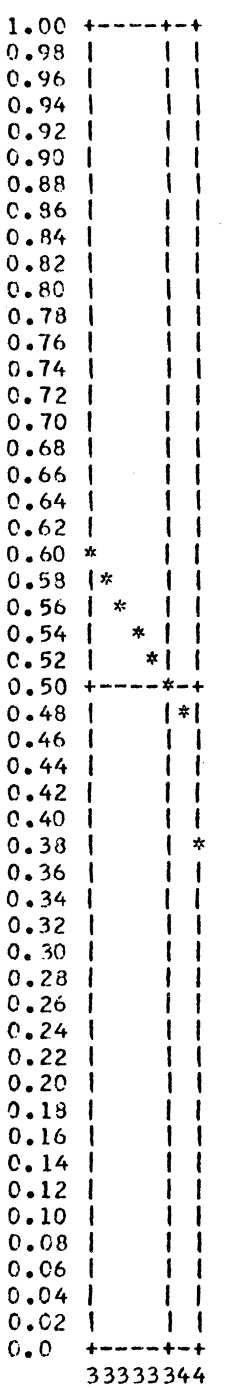
The entropy and parsimony for this transformation are as follows:

```
ENTROPY TERM                                  42.55614
PARSIMONY TERM                                 8.00000
RECURSIVE PARSIMONY TERM                       0.0
                                              +---------
VALUE OF H FOR THIS RECODING.....            50.55614

NUMBER OF RULES OF PRODUCTION                      8
NUMBER OF RECURSIVE RULES                          0
NUMBER OF TIMES RULES ARE APPLIED                 90
```

And, the p_{ij}=graph for the transformation is shown on the next page.

GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF 3



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL) 0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE) 0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE) 6
NUMBER OF TYPE 4 SEQUENCES (MESSAGE) 2
NUMBER OF TYPE 5 SEQUENCES (NOISE) 0
NUMBER OF P(I) 8

Similarly, for an M of 4, we find the following rules:

```
TENTATIVE RULES OF PRODUCTION FOR M OF   4

A   ----->   1110

B   ----->   0110

C   ----->   1100

D   ----->   1000

E   ----->   0101

F   ----->   1101

G   ----->   0100

H   ----->   0000

I   ----->   1001

J   ----->   0011

K   ----->   0010

L   ----->   0001
```

and the following evaluation:

```
ENTROPY TERM                              30.02759
PARSIMONY TERM                            12.00000
RECURSIVE PARSIMONY TERM                   0.0
                                         +--------
VALUE OF H FOR THIS RECODING.........     42.02759

NUMBER OF RULES OF PRODUCTION               12
NUMBER OF RECURSIVE RULES                    0
NUMBER OF TIMES RULES ARE APPLIED           68
```

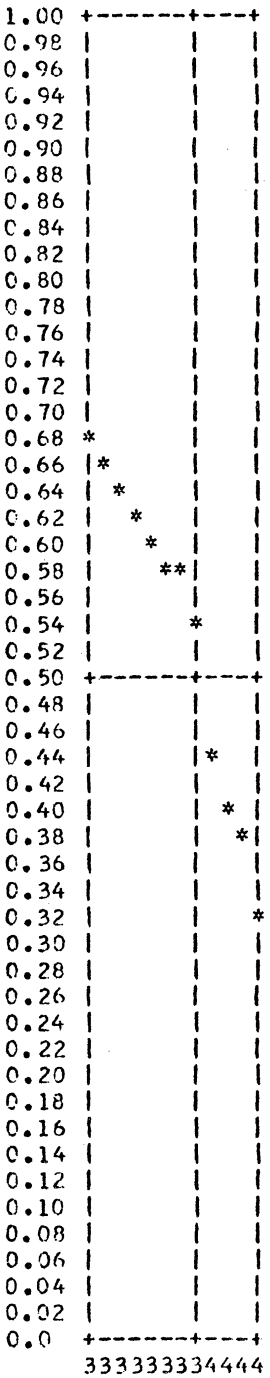
and the following new string:

```
ABCDCEFEGHCIFEGHAJCDCEGHCJFBCGCKCEGHFBCBGHCCAGFGCLFEGHCEAEAE

FEAJAKGH
```

The p_{ij}-graph is found on the next page.

GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF 4



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)	0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)	0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)	8
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)	4
NUMBER OF TYPE 5 SEQUENCES (NOISE)	0
NUMBER OF P(I)	12

If it seems that the combined entropy-parsimony measure (H) is merely a monotonic function of M, consider what happens with an M of 5. Here we get an extraorindary number of rules---27 in all---as shown below:

TENTATIVE RULES OF PRODUCTION FOR M OF 5

A -----> 11100
B -----> 11011
C -----> 00100
D -----> 01100
E -----> 01011
F -----> 10101
G -----> 01010
H -----> 00000
I -----> 11001
J -----> 00111
K -----> 10100
L -----> 00001
M -----> 11000
N -----> 11110
O -----> 01000
P -----> 10110
Q -----> 00010
R -----> 00110
S -----> 00011
T -----> 01110
U -----> 01001
V -----> 11010
W -----> 01111
X -----> 00101
Y -----> 10111
Z -----> 10001
<AA> -----> 10000

For M of 5, the parsimony is therefore quite large. However, the entropy is quite small. With an M as large as 5, there is sufficient context to guarantee a more faithful (and hence reliable, and hence lower entropy) transformation. However, this greater faithfulness is the result of the proliferation of rules of production. There are more rules, each used less often, and each used more reliably. The following is the calculation of H:

ENTROPY TERM	22.21033
PARSIMONY TERM	27.00000
RECURSIVE PARSIMONY TERM	0.0
VALUE OF H FOR THIS RECODING.....	49.21033
NUMBER OF RULES OF PRODUCTION	27
NUMBER OF RECURSIVE RULES	0
NUMBER OF TIMES RULES ARE APPLIED	54

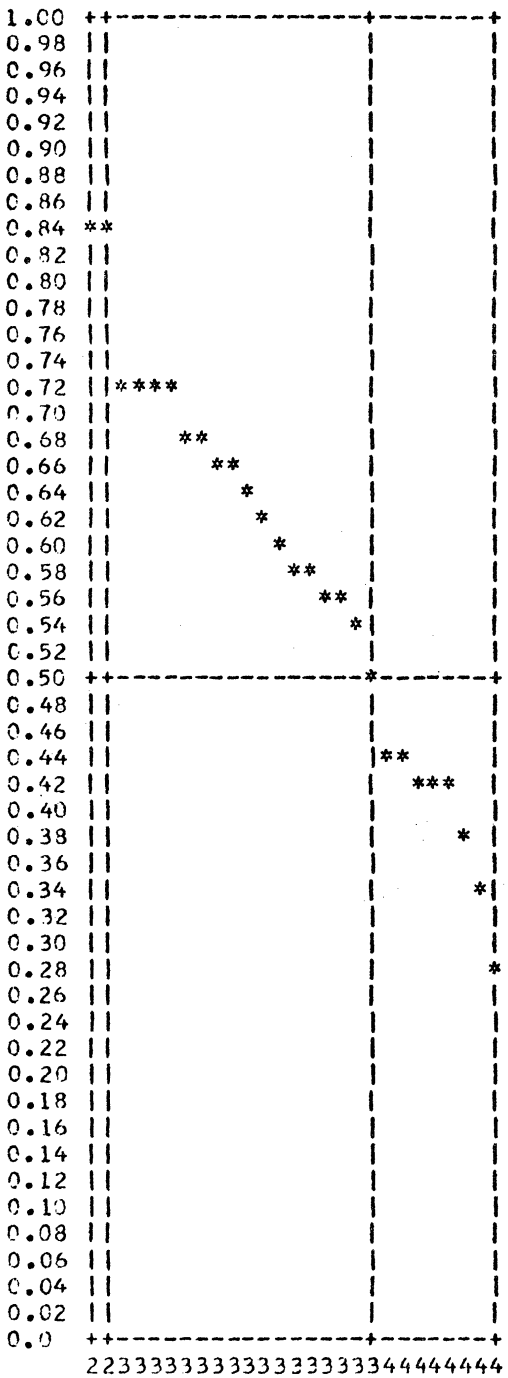
And, the following is the new string that results:

ABCDEFGHIJGKLMNOMFHDJFPCMEQKLFPROSCTUFRLVFHDEIWXVYZNC<AA>OO

(Note that when we exhaust the non-terminal alphabet provided, we use brackets as in Backus-Naur Form to indicate non-terminals).

The p_{ij}-graph for an M of 5 is found on the following page.

GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF 5



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)	0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)	2
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)	17
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)	8
NUMBER OF TYPE 5 SEQUENCES (NOISE)	0
NUMBER OF P(I)	27

Note that for M of 5, we have some type II regularities in the transformation.

Reviewing this example, note that the smallest H is attained with an M of 4. Thus, after considering trial recodings for M of 2,3,4, and 5, we would select the recoding based on an M of 4 for the actual recoding. We then would use the new string obtained by using the recoding based on an M of 4 as the string for input to level 2 of the Algorithm.

E. INDUCING RECURSIONS FROM A FINITE SAMPLE OF SENTENCES

1. INTRODUCTION AND MOTIVATION

Every sample of sentences we encounter is finite; and every finite sample of sentences can be generated by a finite state grammar. In fact, a finite state grammar can be trivially found which generates the sentences in the sample and only those sentences—thus producing the best possible fit between the sample and the grammar induced. A grammar which generates only a finite number of sentences is a *finite cardinality grammar*, and a language with only a finite number of sentences is a *finite cardinality language*. A finite cardinality grammar need not be a finite state (regular) grammar, and a finite state (regular) grammar need not be a finite cardinality grammar.

Of course, for reasons of economy and to satisfy our own intuitive requirements, we insist that the grammar induced by a grammar discovery algorithm not always be either a finite state grammar or a finite cardinality grammar. Thus, we allow the induction of non-finite-state-grammars—that is of left-sensitive, right-sensitive, strictly context-sensitive, and strictly unrestricted rewrite grammars—all from a finite sample (which is, of course, a finite cardinality language itself). Similarly, we allow the induction of non-finite-cardinality-grammars from a finite sample. The only way for a grammar (by which we mean, of course, a finite grammar—i.e. one with a finite number of rules) to generate more than a finite number of sentences is for the grammar to have a recursion. Allowing the induction of recursions from a finite sample is as important as allowing the induction of non-finite-state rules of production from a finite sample.

Indeed, without recursion, many simple relations must be expressed

by numerous different rules of production. Moreover, if the sample of sentences is extended—say by application of a recursion—and we are not allowed to induce grammars with recursions, then new rules of production must then be added to explain the extension. Each extension leads to a proliferation of additional rules of production. In the case of this kind of extension of the sample, a grammar which did not contain a recursion would be neither economical, nor stable (under the extension).

2. DEFINITION OF A RECURSION

Given a grammar $G = \langle V_N, V_T, P, S \rangle$. Let $V = V_N \cup V_T$. Let w and y be strings over V*. We say that w *immediately derives* y, which we write as

$$w \Rightarrow y ,$$

if there exists a z_1, z_2, u, and v, all in V*, such that $w = z_1 u z_2$ and $y = z_1 v z_2$, and such that $u \rightarrow v$ is a rule of production in P. We say that w *derives* y, which we write as $w \overset{*}{\Rightarrow} y$, if there exists strings $w_0, w_1, \ldots, w_n$ such that $w = w_0$, and such that $y = w_n$, and such that $w_i \Rightarrow w_{i+1}$ for each i from 0 to n-1. A *sentential form* is a string $w \in V^*$ such that $S \Rightarrow w$. A *sentence* is a string $w \in V_T^*$ such that $S \overset{*}{\Rightarrow} w$.

A grammar is said to contain a *recursion* if there is a non-terminal $N \in V_N$ such that

$$\alpha N \beta \overset{*}{\Rightarrow} \gamma N s$$

where $\alpha, \beta, \gamma, s \in V^*$, provided that it is not the case that $\alpha = \gamma$ and $\beta =$,

Note that rules such as $N \rightarrow N$ and $N \rightarrow N$ and $NM \rightarrow MN$ do not count as recursions in the above definition. We use only the noun "recursion" in referring to the above idea. An individual rule of production $\alpha N \beta \rightarrow \gamma N s$ is called a *recursive rule* if the same non-terminal N appears in both the antecedent (left) side and consequent (right) side of the rule, and provided that it is not the case that $\alpha = \gamma$ and $\beta = s$, and provided that $|\alpha N \beta| \neq |\gamma N s|$.

3. APPROACHES TO INDUCING RECURSIONS

There are several possible approaches to inducing recursions from a finite sample.

Feldman (1966,1967) proposed an algorithm for inferring an unambiguous, recursive regular grammar from a given sample. This algorithm begins with the construction of a non-recursive, regular, grammar that generates exactly the given sentences, then merges non-terminals in this grammar to get a recursive regular grammar which seems to be a "reasonable generalization" of the sample. The algorithm ends with a simplication process.

To generate the intermediate grammar which exactly generates the given sentences, Feldman processes the strings of the sample in order of decreasing length. To the extent that the sample does not have different sentences of equal length, this procedure eliminates the effect of order of the sentences within the sample. Rules of production are developed one-by-one, as they are needed to generate each sentence in the sample in turn. Specifically, starting with a (the) longest sentence $a_1a_2...a_n$ of length n, Feldman would generate n-1 rules as follows: The first rule is $S \rightarrow a_1A_1$. The next n-3 rules are $A_i \rightarrow a_{i+1}A_{i+1}$, for i = 2, ..., n-2. The n-1 -th rule is $A_{n-3} \rightarrow a_{n-1}a_{n-2}$ --which is a rule which is not in the form of a rule of a regular grammar, and which he calls a "residue" rule. "Residue" rules in general are rules of the form $A \rightarrow w$, $w \in V_T$, $/w/ \geq 2$. New rules are added as necessary as additional sentences from the sample are considered.

As each sentence is considered, new rules of production are added only to the extent required to guarantee the generation of that sentence. These new rules may even be terminating rules—which are like

"residue" rules except that their consequent (right) side is of length 1. Terminating rules come about when all but the last symbol of the sentence under consideration can be generated with the previously developed rules. The rules so developed may be combined and written non-deterministically.

To obtain recursion, rules of production are now merged. Each residue rule is merged with a non-residue rules, thus eliminating the "residue" rules. The general principle is that after such merging, the resulting grammar must still generate all the sample, plus as few new short sentences as possible. This merging is accomplished as follows: Whenever the non-terminal on the antecedent (left) side of a residue rule occurs on the consequent (right) side of a non-residue rule, the non-terminal of the left side of this non-residue rule is substituted for the non-terminal on its consequent (right) side which was in common with the residue rule. This eliminates the residue rule, and makes the non-residue rule involved into a recursive non-residue rule. Note that the resulting grammar is now entirely in the form of a regular grammar. Also, note that this procedure guarantees that the longest sentence of the sample is generated by a recursive rule. Shorter sentences of the sample may be generated either with the aid of a recursive rule, or only by non-recursive rules. The tendency is that the shorter the sentence, the more non-recursive rules have been constructed; hence, it is more likely that it can be generated by a sequence of non-recursive rules, followed by one terminating rule.

The non-residue recursive grammar thus produced is now simplified to remove equivalent productions. This simplication results in no change in the language generated.

A second algorithm by Feldman (1969) infers a "pivot" grammar for a given set of sentences. A pivot grammar is an operator grammar in which a terminal symbol which separates non-terminals in a production appears in no other way. Linear grammars are a special case of pivot grammars, but the class of pivot grammars is much broader than the linear grammars.

The algorithm begins with the sample of sentences and the knowledge of which terminal symbol(s) are the pivot terminal symbols. The algorithm produces a pivot grammar.

The main strategy of the algorithm is to find self-embeddings. Each sentence is examined to see if it has a proper substring which is also a sentence of the sample. If it does, a "loop non-terminal" is substituted for the longest such substring. This results in a new sentence. This new sentence begins part of the sample under consideration. If no sentences have such substrings, the sample is scanned to see if all sentences have the same first symbol, or the same last symbol. If this is the case, the common symbol is trimmed off, and the process described above is then applied to the trimmed sentences.

4. SENTENCE-ORIENTED METHOD OF INDUCING RECURSION

The methods for inducing recursions can be divided into two categories. First, there are methods that operate on the sentences of the sample—that is, the methods that look for some indication in the sample that a recursive rule of production is justified. Second, there are methods that operate on the rules of production developed at each level of the grammar discovery algorithm.*

We examine the sentence-oriented approaches first.

A theorem of Bar-Hillel *et al.* (1962) states that for every context-free language L, there exist constants p and q, depending only on L, such that *if* there is a sentence z in L of length greater than p, then z may be written as a concatenation uvwxy, where $|vwx| \leq q$, with $|vw|>0$, and further that each sentence

$$u\, v^i\, w\, x^i\, y$$

is in L for $i \geq 0$. Thus, if L contains one sufficiently long sentence, an infinity of other long sentences are stated to also be in L. A proof of this theorem appears in Hopcroft and Ullman (1969) under the name "uvwxy theorem."

The special case of the uvwxy theorem for regular languages is a well-known result, usually presented in discussions of whether a given regular language is finite or infinite. This result is the theorem stating that if a regular language contains a sentence of length greater than the number of non-terminal symbols in the grammar (which is the same as the number of states in the finite automaton associated with the grammar), then the language is infinite, and in fact contains an infinity of sentences of the form

*The variable WCV specifies which approach is to be used in the computer program, only the rule-oriented approach. The variable KRECUR specifies whether any attempt to induce recursions should be made.

$$uv^i w y \quad , \quad i \geq 0.$$

The uvwxy theorem, and its proof, suggest a sentence-oriented approach for inducing the presence of a recursion in a given sample of sentences. If the sample of sentences is sufficiently rich and varied, there should be some instances in the sample of substrings of the form $v^i w y^i$. We should attempt to recognize these instances. In a sentence, a substring such as v^i is the product of recursion. This sub-string can be generated by the application of some recursive rule i times.

The detection of v^i can be accomplished with masks of even length M, wherein the first M/2 positions of the mask are context positions, and the last M/2 positions are predicted positions--that is,

$$\underbrace{\quad}_{\frac{M}{2}} \; \% \; \underbrace{\quad}_{\frac{M}{2}}$$

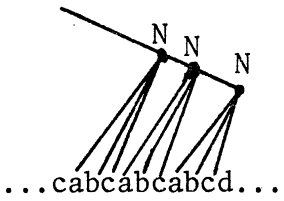
(A mask with the context and predicted symbols reversed works equally well). The given sample of sentences at the current level is searched for regularities associated with this series of masks of even length. The only occurrence of regularities associated with this mask can be over those strings where there is a repeated sub-string. The conditional probability associated with this regularity should be near 1.0 for a recursion to exist. That is, most occurrences of the sub-string should be followed by another occurrence of the same sub-string--the only exception being the last occurrence within the sentence. Note that the conditional probability cannot be exactly 1, unless the entire sample consists of repetitions of the sub-string.

If the recursive rule is a non-self-embedding rule, then we would find only v^i in a sentence. If the recursive rule is a self-embedding one then there would also be an x^i sub-string.

Whenever a repeated substring α is discovered, a new non-terminal can be introduced into the non-terminal vocabulary of the induced grammar, and the recursive rule

$$N \rightarrow \alpha\, N$$

can be added to the induced grammar. Each occurrence of α in the sample is then replaced with the non-terminal "N". The derivation tree for this is as follows:



The recoded sample is then

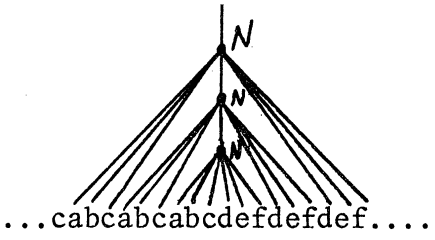
...cNNNd....

There is no good procedure for deciding whether to write the rule as N → abcN or N → cabN, when a terminal such as "c" appears at the beginning of the repeated substrings. There is no good criterion for deciding whether to make the recursive rule left recursive or right recursive. There is no good way to decide whether the sequence immediately following the repeated substring should be written as a disjunctive alternative to the recursive symbol—that is,

N → abcN | d

or whether another formulation is more desirable.

Moreover, an obvious self-embedding such as N → abcNdef with the derivation tree



may yield the pair of recursive rules

$$N \rightarrow abcN$$

$$M \rightarrow defM$$

and a recoded sample of

...cNNNMMM...

Moreover, this method has no obvious extension to context-sensitive or unrestricted rewrite rules; and the method also requires a rather large sample size in general.

5. RULE-ORIENTED METHOD OF INDUCING RECURSION

In the rule-oriented method of inducing recursion, we examine the rules of production generated at each successive level of the grammar discovery process, and try to induce the presence of recursion.

Consider, for example, a language generated by the grammar with "p" and "+" as the terminals and with the single strictly-context-free recursive rule

$$p \rightarrow +pp$$

A sample of sentences (with initial punctuation) from this language might be

(1) ++pp+pp.++ppp.+pp.+p+pp.++pp+pp.+++pppp.++pp+p+pp.++pp++ppp.

In practice, the Grammar Discovery Algorithm would discover the single non-recursive rule

$$A \rightarrow +pp$$

at level 1. The original sample would then be recoded using this rule of production. The result would be a set of sentences containing the original terminal symbols "p" and "+", and also the non-terminal "A". The substring "+pp" will not, of course, appear in the encoded strings. The result would be

(2) +AA.+Ap.A.+pA.+AA.++App.+A+pA.+A+Ap.

Note that substrings such as "+pA", "+Ap", as well as "+AA" do appear in the image.

The rules of production developed at level 2 include

$$B \rightarrow +AA,$$

which is the obvious non-recursive rule resulting from the recursive structure of the language. However, rules such as

$$C \rightarrow +pA$$

and

$$D \rightarrow +Ap$$

would also be developed since "+pA" and "+Ap" are cohesive sub-strings at this level. Note the resulting exponential growth of the number of rules of production, and of the non-terminals. This growth is typical when the more economical recursive rule (in this case $p \rightarrow +pp$) is not developed.

We now introduce some terminology.

Recursive rules of production for context-free grammars takes the following forms, where A is a non-terminal, and where α and β are Non-null strings:

first, $A \rightarrow A\alpha$, which is called a *left recursion;*

second, $A \rightarrow \alpha A$, which is called a right *right recursion;*

and finally $A \rightarrow \alpha A \beta$, which is called a *self-embedding* ("middle" recursion).

The first two rules are *left regular* and *right regular* rules, respectively, when the length of α is only 1.

If M is large enough (that is, the length of the consequent (right) side of the rule of production is large enough), then recursions will manifest themeslves between adjacent levels of the grammar discovery process. Suppose the rule of production

$$A \rightarrow \alpha B \phi$$

appears in the induced grammar at level i; and suppose the rule

$$B \rightarrow \delta C \varepsilon$$

appears at level i+1; and suppose A and C are ultimately going to be identified with one another to make the recursive rule

$$A \rightarrow \alpha \delta A \phi \varepsilon$$

then if M had been large enough to encompass the length of $\alpha\delta A\phi\varepsilon$,

the rule

$$A \rightarrow \alpha\delta B\phi\varepsilon$$

could have appeared at level i, so that the rule

$$A \rightarrow \alpha\delta A\phi\varepsilon$$

would have been induced at that level.

Thus, it is sufficient to examine rules of production occurring at adjacent levels of the induced grammar, if M is generous enough. (Alternatively, if M is not generous,it will be necessary to expand our examination of rules to non-adjacent levels).*

We now develop a procedure for identifying recursions.

Let R_1 be a rule of production induced at level i-1 of the grammar discovery process, and let R_2 be a proposed new rule of production that is about to be added to the induced grammar at level i. Before any proposed new rule is added at level i ($i \geq 2$), that rule is considered in relation to each rule already in the grammar from level i-1 to see if one recursive rule could replace both. Note that it is here that we limit the detection of recursion to adjacent levels.

The following are the conditions for inducing a recursion:

(1) Rules R_1 and R_2 must be isomorphic in the following sense:

(a) The precondition for isomorphism is that the consequent (right) side of rule R_1 and the consequent (right) side of rule R_2 be of the same length.

(b) Secondly, there must exist a one-to-one mapping ξ from the set of symbols appearing in rule R_1 to the set

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*This latter feature is not now implemented in the computer program for the Grammar Discovery Algorithm.

of symbols appearing in rule R_2, such that if symbol f appears in the consequent (right) side of rule R_1 at position t, then symbol $\xi(f)$ must appear in the consequent side of rule R_2 at position t, for all t between 1 and the common length of the consequent sides of the two rules.

(2) There must exist a symbol σ such that the following two conditions hold:

(a) The symbol σ is a predicted symbol on the antecedent (left) side of rule R_1. (To determine this, the mask from which rule R_1 was developed must be considered).

(b) The symbol σ appears on the consequent (right) side of rule R_2.

The symbol σ is the *recursive symbol*—the symbol linking the two rules. If no recursive rule were identified, there would instead be a chain of rules at successive levels—each linked between adjacent levels by one such symbol.

If the above conditions hold, we have a recursion. The symbol σ is now an extraneous symbol in the induced grammar. Let T be the non-terminal symbol appearing in the same position of rule R_1 at level i-1 as T occurs in rule R_2 at level i. The rule R_2 is extraneous, and should not be added to the induced garmmar. The symbol σ is extraneous and should now be deleted from the current non-terminal vocabulary of the induced grammar. Rule R_1 should be made into a recursive rule R_1' by replacing the symbol σ on its antecedent(left) side by the symbol T , so that T occurs now on both the antecedent(left) side and the consequent(right) side of this modified rule R_1'. Thus, this modified rule R_1' is now a recursive rule of production.

Moreover, T should replace every occurrence of σ in all existing rules as well. Rules which are then identical should then be deleted from the induced grammar. This Rewriting process simplifies the induced grammar.

Finally, the recursive rule just induced should be recursively applied wherever possible to the existing set of sentences. There will, in general, be many opportunities to apply the recursive rule which did not exist before. This back-tracking process has the effect of applying each rule maximally before going on.

Because of these processes of Rewriting all existing rules, and the back-tracking process of recursively applying any new recursive rule wherever possible, it is simpler from the point of view of writing the computer program for this algorithm to induce only one recursive rule at each level. The addition of additional levels to the grammar discovery algorithm in no way complicates the induced grammar.

Thus, in the example, the rule

$$A \rightarrow +pp$$

was a rule at level 1; and (if no search for recursion is made) the rules

$$B \rightarrow +AA$$

$$C \rightarrow +pA$$

$$D \rightarrow +Ap$$

were rules induced at level 2. Level 2 is the first level at which the inter-level search for recursion can be applied. Before accepting

a rule such as

$$B \rightarrow +AA$$

we would discover that it is isomorphic to the rule $A \rightarrow +pp$. That is, the consequent (right) side are both of length 3, and there exists a mapping ξ taking "+" into "+" , and "p" into "A" and this mapping is one-one. The symbol "A" is the recursive symbol σ. T is "p". Replacing "A" with "p", we get the recursive rule

$$p \rightarrow +pp$$

If we now apply this rule whevever possible to the original sample (1), we get a new sample which has a "p" wherever there is an "A" in (2). However, by applying this rule in every possible way, the sample (1) in fact is completely reduced to

p.p.p.p.p.p.p.p.

It should be noted that every recursive rule of production is inherently a disjunctive rule as well. In particular, any recursive rule can be written in the form

$$\alpha NB \rightarrow \phi N\delta \,|\, \Phi$$

where ϕ represents the empty string, where α, B, ϕ and δ are strings, and where "N" is a non-terminal (recursive) symbol. The recursive rules induced above are all of this form. Thus, the rule induced in the above example is, if written completely,

$$p \rightarrow +pp \,|\, \Phi.$$

F. INDUCING DISJUNCTIONS AND GENERALIZATIONS

1. INTRODUCTION AND MOTIVATION FOR INDUCING GENERALIZATIONS

One specific application of the Grammar Discovery Algorithm is the induction problem for formal systems. In general, the rules and meta-rules of formal systems are written in terms of quantified variables. For example, a given property, say commutativity of some binary operation "$\cdot$", may hold "for all" variables taken from a certain set $Z = \{Z_1,\ldots,Z_k\}$. The commutative rule would not be stated as k^2 separate commutative rules (i.e. $Z_1 \cdot Z_2 = Z_2 \cdot Z_1$, $Z_1 \cdot Z_3 = Z_3 \cdot Z_1$, $Z_2 \cdot Z_3 = Z_3 \cdot Z_2$, etc.), but rather as one rule stated in terms of a meta-variable which ranges over the set Z (i.e. $\forall\, x_1,x_2 \epsilon Z \quad x_1 x_2 = x_2 x_1$). Of course, whenever the set Z is not finite, the set of meta-variables is not merely a convenience and economy, but a necessity.

The process of inducing the required meta-rule (which is stated in terms of a quantified meta-variables) from specific instances of the rule is called *generalization*. Generalization in induction is the counterpart of substitution in derivation.

In doing induction in formal systems, it is desireable to have a facility for generalization.

2. INDUCING GENERALIZATIONS COMBINATORIALLY (RULE-ORIENTED METHOD)

The induction of generalized rules can be done combinatorially in a manner similar in concept to the induction of recursions. We begin by allowing the Grammar Discovery Algorithm to proceed in its development of rules level by level. But before a new rule of production R is admitted into the induced grammar (at a level greater than level one), the possibility of instead introducing a generalized rule is considered. The generalized rule, as it exists, would replace the rule R about to be admitted as well as certain rule(s) already in the induced grammar.

The following are the conditions required to induce a generalization:

(1) First, there must be a non-empty set of rules $R_1,\ldots,R_k$ in the induced grammar which are all isomorphic to the rule R. (Isomorphism of rules is defined during the discussion of inducing recursion).

(2) Second, for the h context positions of the consequent (right) side of rule R, suppose only a particular subset of symbols (say, $Z_1,Z_2,\ldots Z_d$) appear in those h positions in both R and $R_1,\ldots,R_k$. Then all d^h combinations of these d symbols in those h positions must occur. (Note, then, that a minimum requirement is $k \geq d^h$).

(3) Third, for all but the h positions, the rules $R_1,\ldots,R_k$ and R must be identical.

Now define Z as the set of d symbols above. $Z = \{Z_1,Z_2,\ldots Z_d\}$. Let $x_1,\ldots,x_d$ be d meta-variables. Let R* be the rule obtained from R by substituting x_i for each occurrence of Z_i in rule R ($1 \leq i \leq d$). Now the rule R* replaces $R_1,\ldots,R_k$ and R in the induced grammar. Rule

R* is the generalized rule; it is written in terms of the d meta-variables $x_1,\ldots,x_d$. For example, suppose a sample from a formal system contains rules of inference such as

$$Z_1Z_2 \rightarrow Z_2Z_1$$
$$Z_1Z_3 \rightarrow Z_3Z_1$$
$$Z_2Z_3 \rightarrow Z_3Z_2$$
$$Z_2Z_1 \rightarrow Z_1Z_2$$
$$Z_3Z_1 \rightarrow Z_1Z_3$$
$$Z_3Z_2 \rightarrow Z_2Z_3$$
$$Z_1Z_1 \rightarrow Z_1Z_1 \qquad \text{(identity rules)}$$
$$Z_2Z_2 \rightarrow Z_2Z_2 \qquad (\ "\ \ \ "\ )$$
$$Z_3Z_3 \rightarrow Z_3Z_3 \qquad (\ "\ \ \ "\ )$$

where

$$Z = \{Z_1,Z_2,Z_3\}$$
$$d = 3$$
$$h = 2$$

the $d^h = 9$ rules of inference are isomorphic. The d^h rules of inference can be consolidated into one meta-rule expressed in terms of the meta-variables x_1 and x_2 universally quantified over Z —namely, $\forall\, x_1,x_2 \varepsilon Z$ $x_1x_2 \rightarrow x_2x_1$.

The above induction of generalized meta-rules is a combinatorial process. Of course, the requirement for the appearance of all d^k variations can be relaxed in practice—with the attendant risk of over-generalizing.*

Finally, it should be noted that the Generalization Process includes a process which might be called the process of finding "negative" regu-

*This combinatorial generalization process is not implemented in the computer program.

larities. Suppose, for example, in a non-binary alphabet that a given symbol of the alphabet does *not* appear in a particular context in the sample. Then a "negative" regularity can be envisioned which states that fact about the sample. In fact, one can envision the masks in the Search Phase as containing "negative" predicted positions—that is, which record the absence of particular symbols in particular contexts in the sample. However, it will be seen that the Generalization Process described above subsumes this process—and indeed, is more general in the sense that pairs, triplets, etc. of absent symbols in particular contexts can be expressed easily.

Let us now digress briefly and discuss the process of inducing disjunctive rules.

3. INTRODUCTION AND MOTIVATION FOR INDUCING DISJUNCTIONS

An *explicit disjunctive rule* is a rule of production in which the consequent (right) side consists of two or more different strings, any one of which can be substituted for the antecedent (left) side. A typical explicit disjunctive rule might be

$$A \rightarrow aB \mid bB \mid c$$

where "A" and "B" are non-terminal symbols, and "a", "b", and "c" are terminal symbols. An *implicit disjunctive rule* is said to exist in a grammar whenever the antecedent (left) sides of two or more rules of production are identical, but the respective consequent (right) sides are not. Obviously implicit disjunctive rules can be collected and written as one explicit disjunctive rules.

Disjunctions already arise in the Grammar Discovery Algorithm—but only in the course of inducing recursions. Each recursive rule of production inherently is a disjunctive rule. In particular, any recursive rule can be written in the form

$$\alpha N \beta \rightarrow \gamma N \delta \mid \phi$$

where ϕ represents the empty string, where α, β, γ, and δ are strings, and where "N" is a non-terminal symbol. The recursive rules induced by the Grammar Discovery Algorithm are all in this form.

But so far, there is no other facility in the Grammar Discovery Algorithm for inducing disjunctions. Indeed, the bottom-up character of the Algorithm guarantees that no implicit disjunctive rule (and therefore no explicit disjunctive rule) can result because different substrings in the sample at any level will never be encoded in the same way. A facility for inducing disjunctions is particularly necessary when the sample has initial punctuation. In that event, the encoding can never proceed beyond a string such as

$$S_1.S_2.S_3. \quad \text{(and so forth)},$$

where the S_i are non-terminal symbols. A facility for inducing disjunctions in any level of the process is needed in order to avoid always writing final rules such as

$$S \rightarrow S_1 \mid S_2 \mid S_3 \mid \ldots$$

where "S" is the starting (non-terminal) symbol of the grammar. This same unattractive situation (i.e. of having to write one final trivial disjunctive rule having a large set of disjunctive choices) is possible, of course, even in the absence of initial punctuation. In any case, this situation can be avoided if the Grammar Discovery Algorithm has a facility to induce disjunctive (non-deterministic) rules of production at any level (that is, not merely at the final level).

Finally, in the absence of a facility for inducing disjunctions (i.e. if all rules were deterministic), only one possible structure (except for variations caused by the disjunctions inherent in recursions) can be represented by the grammar. Thus, while recursions provide the means for generating infinities of sentences, disjunctions provide the means for generating different varieties of sentence.

4. INDUCING DISJUNCTIONS USING ENTROPY (SENTENCE-ORIENTED METHOD)

Up to now, only non-disjunctive (deterministic) rules of production have been induced. These rules of production have been induced as the result of the existence of highly reliable regularities in the sample. For example, the fact that the symbol "a" appears in a position 1 of the sample, and a "e" appears in position 4 of the sample may imply, with 99% reliability, that symbols "bb" appear in positions 2 and 3. Perhaps "cc" appears in positions 2 and 3 with conditional probability .5%. In this situation, a rule of production

$$aNe \rightarrow abbe$$

may be induced, where N is a non-terminal symbol. This rule faithfully represents the sequence of symbols in the sample 99% of the time. Actually, three "regularities" are involved in the above recoding. First, there is the highly reliable Type II regularity

$$\mathscr{R}_1 = \langle \text{"a\%\%e"}, \text{"\_bb\_"}, .99, Y \rangle.$$

This regularity is the basis for the recoding. Then, there are the two Type V regularities (or "non-regularities", if you prefer)

$$\mathscr{R}_2 = \langle \text{"a\%\%e"}, \text{"\_cc\_"}, .005, Y \rangle$$

and

$$\mathscr{R}_3 = \langle \text{"a\%\%e"}, \text{"\_dd\_"}, .005, Y \rangle.$$

These two regularities are in effect ignored because they represent a relationship among the symbols of the sample which occurs only rarely. Note that the entropy of the ensemble of probabilities <.99, .005, .005> is very nearly zero. This near-zero entropy is associated with deterministic rules of production that are induced by the Algorithm.

Now suppose that, in a given sample Y, we have the following three regularities:

$$\mathscr{R}_1 = \langle \text{"a\%\%e"}, \text{"\_bb\_"}, .33, Y \rangle,$$

$$\mathcal{R}_2 = \langle \text{"a\%\%e"}, \text{"\_cc\_"}, .33, Y \rangle$$

and $$\mathcal{R}_3 = \langle \text{"a\%\%e"}, \text{"\_dd\_"}, .33, Y \rangle$$

Now, the three possible *substitution instances* (namely, "bb", "cc", or "dd" in positions 2 and 3) occur with equal probability. Certainly, one could not reasonably induce a rule of production that favored any one of the three possibilities (e.g. aNe → abbe). The reason is that the context here (i.e. an "a" in a first position, and a "e" in the fourth position) does not reliably predict the intervening 2 symbols. Indeed, of the three possibilities that occur, the three occur equally often. On the other hand, there are only three substitution instances--that is, the symbols "ae", "ab", etc. are not possibilities. Note that, in this situation the ensemble of probabilities is <.33, .33, .33> here, and that the entropy of this ensemble is large (and indeed maximal, if only the three subsitution instances that actually occur are considered).

The above suggests a criterion for inducing disjunctive rules, starting at the very first level of the Grammar Discovery Algorithm.

Let W be a set of w regularities $\left\{\mathcal{R}_1 = \langle \gamma, \phi_i, P_i, Y \rangle\right\}_{i=1}^{w}$ having the same context-sub-sequence γ and having non-zero conditional probabilities $\left\{P_i\right\}_{i=1}^{w}$. (In general, $w << C^h$, where h is the number of predicted positions in the mask associated with the regularities). Suppose the entropy of the ensemble of these w non-zero probabilities is maximal, or nearly maximal--that is,

$$\log_2 w - P_i \sum_{i=1}^{w} \log_2 P_i$$

is zero, or small. In this situation, we write the disjunctive (non-deterministic) rules of production

$$\alpha \rightarrow \Delta_1 | \Delta_2 | \ldots | \Delta_w$$

where α is the (common) antecedent (left) side as may be obtained from any of the $\mathcal{R}_i$, and where Δ_i is the Δ-sequence for $\mathcal{R}_i$, $1 \leqslant i \leqslant w$.

Note that we induce deterministic rules of production when the reliability of a single regularity is high (the entropy over the possible substitution instances is zero, or small), and that we induce disjunctive (non-deterministic) rules of production when there are several equally-likely (or nearly equally-likely) substitution instances (i.e. the entropy over the possible substitution instances is maximal or nearly maximal). With intermediate entropy, we do no encoding, and we defer resolution to a higher level. Note also that with either approach, the goal of minimizing the number of rules of production and maximizing the parsimony of the induced grammar is furthered, as is the goal of maximizing the non-trivial and hierarchical quality of the grammar.

In certain samples from formal systems, there is an equivalence between generalized rules and disjunctions. Suppose string α, B, and γ can be disjunctively substituted for a non-terminal N appearing in some string δ (using the rule $N \rightarrow \alpha | B | \gamma$). The string δ may be written as $\delta = \delta_1 N \delta_2$, where δ_1 and δ_2 are strings. Define Z as the set containing α, B, and γ, and then define z as a metavariable ranging over Z. If it is appropriate to interpret the sample as containing theorems of a formal system, we can then rewrite the disjunctive rule

$$N \rightarrow \alpha | B | \gamma$$

as the universally quantified generalization

$$\forall \; z \varepsilon Z \quad \delta_1 z \delta_2$$

for that formal system

Conversely, if we have a universally quantified generalization

$$\forall\, z \in Z \qquad F(z)$$

where F(z) is a rule in which z appears, then we can write

$$N \rightarrow z_1 | z_2 | \ldots | z_n,$$

where $Z = \{z_1, z_2, \ldots, z_n\}$.

As mentioned earlier, implicit disjunctive rules in a grammar can be gathered together and written as one explicit disjunctive rule. We can therefore define the multiplicity of an explicit disjunction as the number of different disjunctive alternatives stated in the rule.

Rules of production are then of three types: (1) explicit disjunctive rules, (2) recursions, and (3) non-disjunctive, non-recursive rules. If a grammar consists only of the third type of rules, at most one sentence can be generated. It is necessary to have disjunctions to attain a variety of sentence structures. If the grammar consists only of rules of the first and third type, there are no more sentences than the product of the multiplicities of the explicit disjunctions. This number is, of course, finite. This product is an upper limit on the number of essentially different structural types for sentences generated by the grammar (of course, through ambiguity and other reasons, this upper limit may not be attained). Indeed, variety in structural types arises only from disjunctions.

If there are any recursions in the grammar, the number of sentences that may be generated by the grammar becomes infinite (assuming the rule can be used).

Note, however, that the number of essentially different structural types of sentences is still bounded. Recall also that it was noted earlier that every recursive rule is inherently a binary disjunctive rule---one of the disjunctive possibilities being the recursive part of the rule, and the other being the so-called "base" string. Note also that the recursive rule can always be written so that this base string is the empty string.

A *normal-form grammar* here is a grammar such that all of the rules of production are rules of one of the following types:

(1) explicit disjunctive rules,

(2) recursions with an empty base string, and

(3) non-disjunctive, non-recursive rules.

For every grammar, there is a normal-form grammar that generates exactly the same language---thst is, there is an equivalent normal-form grammar.

In general, for a normal-form grammar, the number of essentially different sentence structures is bounded above by the product of multiplicities of the explicit disjunctive rules and 2^{n_r}, where n_r is the number of recursive rules.

Note, from the description of the Grammar Discovery Algorithm, that every induced grammar produced by the Algorithm is a normal-form grammar.

G. TERMINATION PROCEDURE

The Grammar Discovery Algorithm terminates when the sample Y of the current level consists only of one symbol. This symbol becomes the starting symbol of the induced grammar. The Grammar Discovery Algorithm arrives at this situation in one of two ways. If the Grammar Discovery Algorithm is operating in the first mode (i.e. there is no initial punctuation between sentences), then this situation is arrived at naturally by the reduction of Y to a length of one. If the Algorithm is operating in the second mode (i.e. there is initial punctuation between the sentences of the sample), then the sample at the next-to-last level may have consisted of a concatentation of single non-terminal sentences and initial punctuation marks (i.e. $s_1.s_2.$ and so forth). In that case, the reduction of the sample from this level to the final level may be accomplished by (a) removing repetitions of the same non-terminal from the sample, and (b) Inducing Disjunctions over the single symbols. Typically this disjunction occurs at an intermediate level, and process (a) then applies to the repetitions at the final level. Given a finite sample Y, the Grammar Discovery Algorithm converges to some induced grammar after a finite amount of time.

If $M1 \geqslant 2$ (as is always the case) and if context-free rules only are being generated (as described in a later section), then each application of each context-free rule reduced the length of the sample by at least one symbol. Even if each transformation involves only one application of one binary constituent rule (M of 2), the Algorithm would terminate after either N-1 levels (N being the length of the sample, if there is no initial punctuation in the sample) or N_{max}-1 levels (N_{max} being the length of the longest sentence in the sample, if there is initial punctuation).

If context-sentitive rules are being generated, the reduction in the length of the sample Y from level to level depends on their being more than one contiguous predicted symbol in the mask of at least one regularity used at least once in the recoding. One can assure this by considering only masks with this property, or by requiring that there be at least one application of such a rule as part of each transformation (or, of course, by having at least one context-free rule in each transformation).

In practice, of course, one usually "covers" the entire sample Y or nearly all of it) so that these requirements are virtually automatically satisfied by any transformation.

H. CHOICE OF LIMITS ON LENGTH OF REGULARITIES IN THE SEARCH PHASE

In the Search Phase of the Grammar Discovery Algorithm, masks of length M = M1, ..., M2 are considered. The lower limit M1 and the upper limit M2* on the length of mask to be considered are determined from the following considerations:

(1) Clearly one cannot scan the sample Y with an M greater than the length N_{max} of the longest sentence in Y. If the Grammar Discovery Algorithm is operating in the first mode (i.e. with no initial punctuation in the sample), then M2 cannot be larger than N—the sample length. If the Grammar Discovery Algorithm is operating in the second mode (i.e. with initial punctuation), M2 is limited above by N_{max}—the length of the longest individual sentence in the sample Y.

(2) An M of 1 is appropriate only in the case where each symbol in the sequence Y is statistically independent from the others, or in the case where Y consists entirely of 1 symbol repeated endlessly. In the case of one symbol repeated endlessly, an H (information rate) of zero is attained. In the case of statistical independence among the single (different) symbols, an information rate greater than zero is attained; and indeed H is maximal (given the alphabet size) when the symbols occur equally often. These two cases are so uninteresting that we specify $M1 \geq 2$ to avoid them.

(3) If M is large, the number of different sequences of length M over C symbols is large. Thus, the number of appearances of any particular sequence in the sample is relatively small. Certainly, if a sequence only occurs once, twice, or 3 times, the statistics about these sequences will not be meaningful. Thus, an M should be tried which is not too large relative to the total length N of the sample. Also,

*M1 and M2 are parameters in the computer program.

if $N \ll C^M$, it is not even possible for all possible sequences of length M to appear, even if, in these N symbols, no sequences appeared more than once. Thus, M should be chosen so that

$$\log_C N \gg M.$$

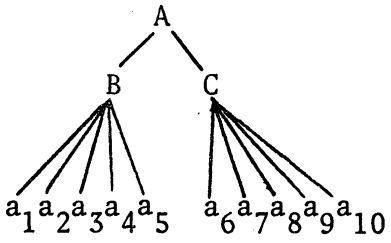
(4) In attempting to induce recursions, the point was made that if there is a recursion, it will manifest itself in the tentative rules induced at consecutive levels, provided M is large enough. Thus, either M should be large enough to encompass any recursion or the search for recursions should extend between non-consecutive levels. This could be done in the same fashion as the search for recursions between consecutive levels.

It should always be remembered that the entire approach to grammar discovery described here is based on the idea of using relatively small M in searching for regularities. If a regularity is missed at one level because M was too small, then we claim the regularity will be detected at a higher level. Thus, if the maximum length of M is limited to say 5, and a Type I (100% reliable) regularity of length 10 exists in the sample, that regularity will be detected at a higher level of the process. In the simplest case the first 5 symbols may be encoded as a second new symbol. If the first 5 symbols at the first level reliably predict the five succeeding symbols, then this regularity involving 10 symbols at the first level manifests itself as a regularity involving two symbols at the second level—namely, the first new symbol reliably predicts the second new symbol as its immediate successor.

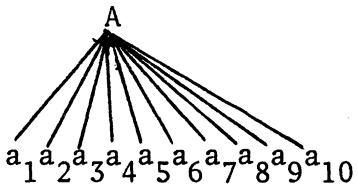
It should be emphasized that finding two regularities of length 5 at the first level of the process, and then finding one regularity of length 2 at the second level is not at all the same as finding the one regularity of length 10 at the first level. This point is made in

Goodall (1962). Combinatorics aside, the important difference is that the first approach discovered a *hierarchical* relation in the sample, and offers a significant insight into the sub-structure of the phrase of length 10, while the other approach merely cataloged the occurrence of a gross event.

The difference is between



and



Indeed, if M is large enough, the subsequences occurring in any finite sample become quite regular and unique--the limiting case of this being a completely trivial, non-hierarchical, and non-parsimonious grammar.

I. CHOICE OF PARAMETERS IN THE RECODING PROCEDURE

The Recoding Procedure described earlier is only one possible recoding procedure. Several variations in this Recoding Procedure are possible. For example, the sample Y may be scanned from right to left, instead of from left to right, in the search for opportunities for developing rules of production. Or, the scan could be started at some intermediate symbol. Or, the positions could be scanned in some order determined by some other considerations—such as the value of the conditional probabilities encountered, etc.

Similarly, the regularities could be considered in ascending, rather than descending, order of length*—thereby giving preference to longer regularities.

Also, the position for starting the scan is variable.** If the length of the regularity varies between M1 and M2, and the scan of Y is started at position T < M2, then ovbiously a regularity of length less than or equal to T may well be found to recode some or all of these first T positions, to the exclusion of a regularity of length M2.

The Recoding Procedure can also operate so that whenever recoding is attempted using phrases no longer than M that this recoding is actually done using *only* phrases of length M. This is called a *strict recoding****, and is appropriate only when the sample is believed to come from a "uniform" code source (Fano). This variation in the Recoding Procedure would be only rarely appropriate.

Another variation in the Recoding Procedure concerns the allowed range in values of the conditional probabilities of the regularities

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*This variation is under the control of the variable VDIR in the computer program implemented in the Grammar Discovery Algorithm.

**This variation is under the control of the variable VSTART.

***This variation is under the control of the variable STRICT.

used in recoding. The conditional probabilities associated with each regularity are categorized by type (i.e. Type I, Type II, etc.), and must specify whether the recoding prodedure can use only absolutely reliable (Type I) regularities, or whether Type II, or Type III regularities can also be used.****

Finally, Recoding Procedures can differ according to the order in which the various criteria for selecting regularities are applied. For example, one can consider all the Type I regularities which are also of length M, and then the other Type I regularities of different length. Alternately, one can consider all the regularities of length M which are also of Type I, and then consider other regularities of length M which are of different type.

The impact of these variations will be discussed in a later section.

Finally, it should be noted that the Recoding Procedure, even with the variations noted above, is only an abbreviation itself of a more combinatorially *complete* recoding procedure—that is, a recoding procedure which is exhaustive in the sense that it includes all possible variations in recoding. This complete recoding procedure would not scan the sample from left to right, or right to left, or from the middle out. In this complete procedure,the procedure would start by considering all possible ways of partitioning the given sample Y of length N into parts, with no part containing more than M symbols. The number of such partitions is the same as the number of partitions of the integer N into parts no greater than M (Riordan, 1958), and this number is very large. Then all the possible regularities that might be applied to each part of each partition must be considered. And for each partition, and each such part, each possible regularity

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****This variation is under the control of the variable ALLOW.

must be substituted into the part, and a recoding attempted, and entropy, parsimony, and recursive parsimony of that encoding computed. And, when a recoding is finally completed at one level, all the possible different recodings at each successive level must be considered before that recoding is finally selected to be the recoding at the present level.

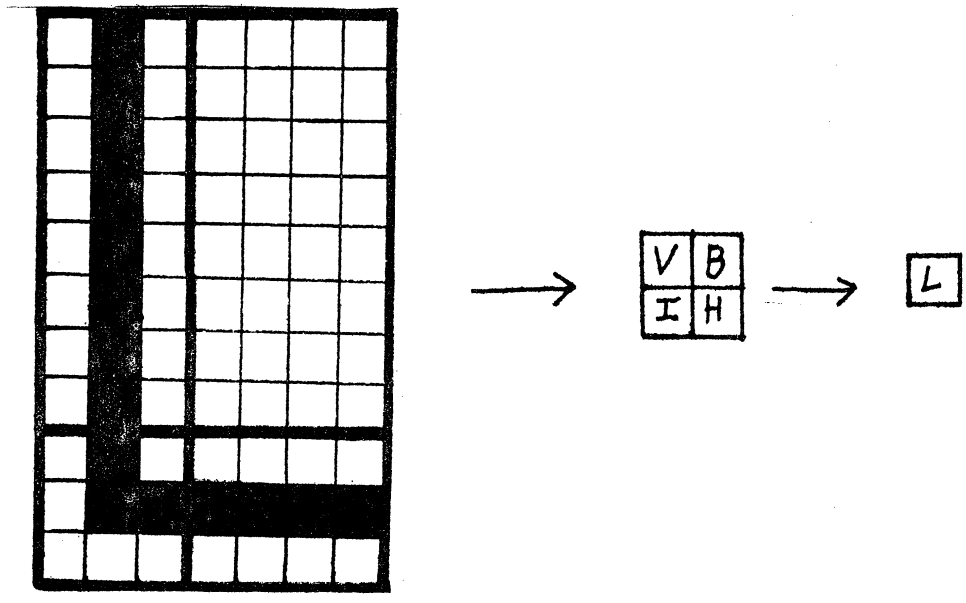
J. EXTENSIONS OF THE GRAMMAR DISCOVERY ALGORITHM TO SAMPLES OTHER THAN LINEAR SEQUENCES OF SYMBOLS

Earlier we made the statement that the methods of induction and grammar discovery that we described herein are not particular to the format for presentation of the sample. We then proceeded to describe a specific Grammar Discovery Algorithm dealing with the problem of inducing a grammar to generate a given sample of sentences which is presented as a linear sequence of symbols. In this section, we return to this point and argue for the general applicability of the methods of induction and Grammar Discovery Algorithm in terms of a two dimensional pattern recognition problem. It will be seen that this rephrasing is not particular either to the Grammar Discovery problem for sentences, or to the two dimensional pattern recognition problem, and that the main features and insights of the Grammar Discovery Algorithm apply to other problems, as well as similar problems presented in other formats.

Let us consider the simplest kind of two dimensional pattern recognition problem. The *sample* will consist of a two dimensional raster (matrix) of digital symbols. The raster may, for example, consist of binary digital symbols representing the digitalized image of certain items to be recognized—perhaps printed letters of the arabic alphabet. The *alphabet* for this sample is the set of whatever symbols appear in the raster--perhaps 0 and 1 in the case of a binary raster. A raster is said to have *initial punctuation* if certain submatrices in the raster are separated in some way—for example, if there were 11 x 7 sub-matrices outlined such that each submatrix is presumed to contain one letter of the arabic alphabet. In contrast, a raster would be unpunctuated if no such initial punctuation of the raster were specified. A *mask* is a submatrix over the context symbol "_", the predicted symbol

"%", and—if the mask is ternary—the don't care symbol '#'. δ and ϕ are submatrices defined in a manner analagous to the subsequences δ and ϕ of the Grammar Discovery problem. A *regularity* is defined exactly as for the Grammar Discovery problem except that δ is the context-sub-matrix and ϕ is the predicted-sub-matrix. In the Search Phase, various masks are considered, and regularities are catalogued and characterized by type according to their conditonal probability.

In the recoding phase, the domain is partitioned ("covered") by sub-matrices. Whenever the context-sub-matrix of a regularity of allowably high type occurs in one of the parts of the domain, this part is recoded. Rules of production are developed form regularities to express this recoding in a way exactly analogous to the 1-dimensional case. For example, the letter "L" occurring as in a 12 x 12 raster amongst a large sample of arabic letters might be recoded as shown below.



Here an 8 x 3 pillar of 1's is recognized as a feature of many letters, and recoded as "V" (vertical line); the 3 x 3 vertical-right intersection sub-matrix is recognized and recoded as "I"; and the 3 x 4 hori-

zontal line might be recognized and recoded as "H". The 9 x 5 blank area might be recoded as "B". Thus, the letter "L" is recoded as a 2 x 2 sub-matrix contining a "V", "I", "H", and "B". Indeed, this description is precisely what an "L" is—namely a vertical line segment intersecting at its end with the end of a horizontal line segment. Note that both the vertical and horizontal segment may be represented by recursive rules. If L's occur often in the sample, this 2 x 2 sub-matrix might be given a name and recoded.

One should note in passing that this entire paper is about induction and pattern recognition of samples of discrete symbols. Suppose instead of doing a basic symbol-by-symbol *matching* operation, one does a *correlation* between time segments of continuous signals. It may be possible to extend the notion of regularity to continuous symbols. The time shift operation inherent in the matching of symbols has an immediate analog for continuous signals. Defining the context part, the predicted part, and the don't care part of a signal merely involves considering the portion of a given signal restricted to a particular time domain with the correlation coefficient playing the role of the conditonal probability, The idea of a regularity amongst a sample of continuous signals may be defined in the obvious way.

The universe of possible masks is infinite, but it is possible to imagine some discrete time intervals being used to make this universe finite.

The concepts of entropy, parsimony, and recoding all have obvious analogs.

It should be noted that the operation of digitalizing a continuous signal is itself a recoding, or punctuating process, in exactly the sense we have been discussing throughout this paper. In digitalizing

data, a *fidelity criterion* (Chomsky & Miller) is used to associate almost identical continuous signal segments and to encode them under a common discrete digital name.

III. THE SELF-PUNCTUATING NATURE OF THE GRAMMAR DISCOVERY ALGORITHM

A. INTRODUCTION

The choices of the upper and lower limits on the length of regularities searched for in the Search Phase and the choice of parameters in the Recoding Procedure affect the partitioning (punctuating) of the sample into the short parts which are in turn recoded. Since the choice of parameters in the Recoding Procedure and in the values of M1 and M2 are specified externally according to heuristic considerations, the grammar that is ultimately induced would seem to be the product, not of the Grammar Discovery Algorithm, but rather of these external choices. These external choices would seem to lead to two principle types of effects: First, there would seem to be the effect of choosing M2 too small in the Search Phase and thereby "missing" longer regularities in the sample. Second, there would seem to be numerous effects of partitioning the sample for purposes of recoding—perhaps of missing regularities because the regularity encompasses symbols that end up in different parts of this partition—or, perhaps, of losing the "synchronization" of the symbols in the sample (as, for example, in a sample produced by a uniform code source).

It will be seen that both the choice of M2 and the choice of parameters in the Recoding Procedure ultimately concern the partitioning, or "punctuating", the sample into parts. In fact, the Grammar Discovery Algorithm is essentially a punctuating process (or a "selective punctuation" as Goodall would call it).

More importantly, the Grammar Discovery Algorithm is "self-punctuating," and recovers from the presumed predetermining effects of the choice of M2 and of synchronization.

B. EFFECT OF CHOICE OF LIMITS ON LENGTH OF REGULARITIES IN THE SEARCH PHASE

1. INITIAL DISCUSSION

Let us first consider the possible limiting and predetermining effects of the choice M2. (M1 is normally 2 and is not an issue). Suppose that the maximum length of regularity searched for in the Search Phase is M2, but that a regularity of length M, with M>M2, exists in the sample Y, say in positions $t_{h+1},\ldots,t_{h+M}$. The regularity that might be missed might for example, be that symbol Y(h+1) in position 1 of the regularity, and the symbol Y(h+M) in position M together reliably determine all the intermediate symbols Y(h+2), ...,Y(h+M-1). If M is, say 8, the regularity would be characterized by a context sub-sequence of "a%%%%%%h" and a predicted sequence of "_bcdefg_" and a mask of "_%%%%%%_" and a conditional probability of 100% (Type I - absolutely reliable). That is, the regularity is <"a%%%%%%h", "_bcdefg_", 1.0>. If M2 is only 6, this regularity would appear to be missed. We claim that if a regularity is missed at one level because M2 was too small, then the regularity will be detected as a regularity at a higher level. We now proceed to make this statement more precise.

2. EXTENTIONS OF A REGULARITY

First, it is necessary to know that the regularity which may be "missed" is the longest regularity encompassing the symbols involved. This does not weaken the statement; it does not sacrifice generality; indeed, it would seem to make the statement harder to verify.

Recall that a regularity $\mathscr{R} = \langle \gamma, \phi, P, Y \rangle$ is characterized by its context-sub-sequence γ (a sequence of length M over the current alphabet V_c augmented by the predicted symbol "%"; the predicted-sub-sequence ϕ (which is a sequence of length M over the current alphabet V_c augmented by the context symbol "_"), and the value P (which expresses the conditional probability that the context-sub-sequence γ predicts the predicted-sub-sequence ϕ in the sample Y).

A regularity $\mathscr{R} = \langle \gamma, \phi, P, Y \rangle$ may be *p-extended* to the left (right) to a new regularity $\mathscr{R}' = \langle \gamma, \phi, P, Y \rangle$ by the addition of a predicted position—that is, by increasing the length of γ and ϕ by one, by annexing a "%" to the context-sub-sequence γ on the left (right) thus forming γ', by annexing a specified symbol from the current alphabet V_c to the corresponding position of the predicted-sub-sequence ϕ thus forming ϕ', and by changing the conditional probability P to P' to express the conditional probability that γ' predicts ϕ' in the sample Y. Note that there are $2 \cdot C$ possible p-extentions to the left (or right) for a given regularity. Note also that $P' \leqslant P$.

Similarly, a regularity $\mathscr{R} = \langle \gamma, \phi, P, Y \rangle$ may be *c-extended* to the left (right) to a new regularity $\mathscr{R}' = \langle \gamma', \phi', P', Y \rangle$ by the addition of a context position—that is, by increasing the length of γ and ϕ by one, by annexing a "_" to the predicted-sub-sequence ϕ

on the left (right) thus forming ϕ', by annexing a specified symbol from the current alphabet V_c to the corresponding position of the context-sub-sequence γ thus forming γ', and by changing the conditional probability P to P' to express the conditional probability that γ' predicts ϕ' in the sample Y. Note that there are $2 \cdot C$ possible c-extentions to the left (or right) for a given regularity. Note also that $P' \geqslant P$.

Of course, the type of a regularity may change under extention—p-extentions being in general of lower type (less reliable) than the regularity from which they were derived, and c-extentions being in general of higher type (more reliable) than the regularity from which they were derived.

In our discussions, if it does not matter whether an extention is a c-extention or a p-extention, it will be called simply an *extention*.

3. DEFINITION OF A MAXIMAL REGULARITY

A regularity $\mathscr{R}=\langle\gamma, \phi, P, Y\rangle$ of types I, II, or III is said to be *preserved (p-preserved) (c-preserved) under extention* provided that $P' \geq P$—or the extention is of the same type as $\mathscr{R}$.

A regularity $\mathscr{R}$ is said to be *maximal (p-maximal) (c-maximal)* provided it is not preserved under extention (p-extentions) (c-extention) for any possible extentions (p-extentions) (c-extentions) of it.

The existence of a maximal regularity means that a particular sequence appears as a unit in the sample Y but that the symbols surrounding this unit vary widely. For example, if $\mathscr{R}$ = <"ab%","_ _c", 1.0, Y> is a regularity, it means that whenever "ab" appears in Y, the symbol "c" always follows. If $\mathscr{R}$ is a maximal regularity, it means that the sequence "abc" appears in the sample Y embedded in all variety of different environments, perhaps as "dabce", "fabcg", etc. If $\mathscr{R}'$=<"a%","_b",1.0,Y> is also a regularity in Y, it is not maximal because $\mathscr{R}'$ may be extended to $\mathscr{R}$. Equivalently, "ab" does not appear in a variety of different contexts in Y, but rather also appears with a "c" to its right.

4. PERSISTENCE OF MAXIMAL REGULARITIES AFTER RECODINGS

We now claim that a maximal regularity of length M will usually not be missed by the Grammar Discovery Algorithm because M2 (where M>M2) was chosen too small. If a maximal regularity exists at one level of the sample, it will usually persist and continue to exist after the recoding at the next higher level. Thus, an unfortuitously small choice of M2 will not cause the maximal regularity to be lost. "usually" is our "almost always" and it means "in all cases except possible for a case whose occurrence requires the coincidence of several independent events each of very small probability "— in effect, "usually" refers to second order (and lower) effects.

Let us assume that a maximal regularity $\mathcal{R} = \langle \gamma, \phi, p, Y \rangle$ of specified type (Type I, II, or III) exists in the sample Y, and that the regularity is of length M>M2. Suppose that the context sequence appears q times in the sample Y. Let Δ be the domain sequence of $\mathcal{R}$ in Y. Say that Δ appears q' times in Y. Note that p=q'/q.

Let Δj denote the j-th occurrence of the sequence Δ in Y (j=1,...,q). Let $\Delta j(i)$ denote the i-th symbol in the j-th occurrence of Δ in Y where i=1,...,M, and j=1,...,q.

Let a_j be the symbol appearing in Y to the left of the left-most symbol of the j-th occurrence of Δ in Y (j=1,...,q), and let b_j be the symbol appearing in Y to the right of the right-most symbol of the j-th occurrence of Δ in Y.

Note that because $\mathcal{R}$ is maximal, no symbols in the sequence $\langle a_1,\ldots,a_q \rangle$ or $\langle b_1,\ldots,b_q \rangle$ or in the sequence of ordered pairs $\langle (a_1 b_1),\ldots,(a_q,b_q) \rangle$ will appear more than $\frac{q}{2}$ times (since $\mathcal{R}$ is of type III or better).

There are apparently three cases to consider:

(1) None of the strings $a_j \Delta_j b_j$ of length q+2 are encoded by the transformation operating at this level of the Grammar Discovery Algorithm.

(2) some of the symbols in Δj are encoded by a transformational rule, but neither the symbol a_j nor the symbol b_j is also encoded by the same application of this rule—that is, only symbols interior to Δj are encoded.

(3) Some symbols of Δj are encoded by a transformational rule, and either the symbol a_j or the symbol b_j is also encoded by the same application of this rule. (Note, that both a_j and b_j cannot be encoded by one application of one rule, since M<M2).

Case (1) poses no problem as to "missing" regularities, since no encoding occurs. At a higher level some encoding may occur, but even then one of the cases below will then apply. It should be noted, however, that the Grammar Discovery Algorithm operates in practice so that most symbols of Y do in fact get "covered" by the rules at each level. When we say that the symbols are "covered", we do not necessarily mean that they are changed, but that they will at least be part of a context-sub-sequence. In summary, this case does not occur often, and is no problem when it does occur.

In case (2), some symbols entirely interior to Δj are encoded.

Suppose that the k symbols $\Delta j(h),\ldots,\Delta j(h+k-1)$, $k \leq M$. are the symbols interior to Δj which are encoded—that is, are the domain of the transformation $\mathscr{T}$ being applied to the sample Y. Note that this transformation $\mathscr{T}$ is made only because there exists some regularity $\mathscr{R}' = \langle \gamma', \phi', P', Y \rangle$ of length k of allowably* high type in the sample. As a matter of notation, note that the Δ of $\mathscr{R}'$

*The variable ALLOW controls this in the computer program.

is $\Delta j(h),\ldots,\Delta j(h+k-1)$, and we call this substring Δ'. Note also that by $\mathcal{T}(\Delta)(I)$, we mean the I-th symbol in the image under $\mathcal{T}$ of Δ.

In a transformation, contiguous symbols of the domain Δ' are encoded as new non-terminal symbols. If the image of Δj, that is $\mathcal{T}(\Delta j)$, is of the same length or longer than Δ (the latter occurring only when Unrestricted Rewrite Rules are being generated), no regularity $\mathcal{R}$ is "missed" because at some higher level the reduction in length finally occurs, and one of the cases here will then apply. If $\mathcal{T}(\Delta j)$ is shorter than Δj, then $\mathcal{T}(\Delta')$ was shorter than Δ'. Say the length of $\mathcal{T}(\Delta')$ is f, where $f<k$. Then the image of Δj is the substring

$$\mathcal{T}(\Delta j)=\Delta j(1),\ldots,\Delta j(h-1),\ \mathcal{T}(\Delta')(1),\ldots,\ \mathcal{T}(\Delta')(f),\Delta j(h+k),$$
$$\ldots,\Delta j(M)$$

which is of length M-(k-f).

There are now two sub-cases. In the first sub-case, M-(k-f) is still greater than M2. In this case, the regularity $\mathcal{R}$ is not "missed," because the second sub-case below will apply at a higher level when the necessary reduction in length finally occurs.

The second sub-case is that $M-(k-f)\leq M2$. The original maximal regularity $\mathcal{R}$ of length M now manifests itself as a regularity $\mathcal{R}'' = \langle \gamma'', \phi'', P'', Y\rangle$. We show that $\mathcal{R}''$ exists by constructing it.

First we extend the transformation $\mathcal{T}$ (which is a semigroup homomorphism) to the domain V_c U ("_" U "%"), by saying that the extention $\mathcal{T}^*$ is

$$\mathcal{T}^*(x)\begin{cases}(x) & \text{if } x \in V_c \\ x & \text{if } x \in \{\text{"\_"},\ \text{"\%"}\}\end{cases}$$

Now we have, for I=1,...,M-(k-f), the context-sub-sequence

$$\gamma''(I)=\begin{cases}\gamma(I) & \text{if } 1\leq I\leq h-1\\ \mathscr{T}^*(\gamma')(I+h+1) & \text{if } h\leq I\leq h+f-1\\ \gamma\left(I+(k-1)\right) & \text{if } h+f\leq I\leq M-(k-f),\end{cases}$$

the predicted sub-sequence

$$\phi''(I)=\begin{cases}\phi(I) & \text{if } 1\leq I\leq h-1\\ \mathscr{T}^*(\phi') & (I+h+1) \quad \text{if } h\leq I\leq h+f-1\\ \phi\left(I+(k-f)\right) & \text{if } h+f\leq I\leq M-(k-f)\end{cases}$$

and the conditional probability

$$P''=P\cdot P' \quad .$$

Note that if the types of regularities allowed for making the transformation $\mathscr{T}$ are only of Type I (100% reliable), then $\mathscr{R}''$ will not only be detected—but its unconditional probability p'' will be the same as that of $\mathscr{R}$ (since p' =1.0). If the allowed types of regularity used in the recoding are of type II, then $\mathscr{R}''$ will either still be of type II or usually at least of Type III (since it would, in general, take a fairly large number k of recodings to make

$$(1-\varepsilon)^k < \frac{1}{2}$$

where ε was defined earlier to be $1/c^k$ which is very small. Note that this k is, in general, only 1, and will be more than 1 only if case (1) applies at some point, or if, in case (2), $\mathscr{T}(\Delta j)$ is the same length or longer than Δj for some recodings.

If type III regularities are used in the recoding (particularly, if the conditional probability of $\mathscr{R}'$ is $(\sqrt{\frac{1}{2}})$, then regularities of type III (particularly, if the conditional probability of $\mathscr{R}$ is $(\sqrt{\frac{1}{2}})$, may be lost. This is of course the reasons why regularities of Type III (particularly, the lower range of Type III are not very suitable for recodings).

It should be noted that this discussion of case (2) assumed *one*

rule of production (derived from *one* regularity $\mathcal{R}'$) was used to encode the symbols Δj. In general, more than one rule of production could be invoked. However, the argument is not altered—although the probability p" would then be the sum of say, two small probabilities. It should be emphasized that because M2 is always small, it would be most unlikely that as many as three rules of production would be applied within one such domain.

Case (3) really never happens. By hypothesis, $\mathcal{R}$ is maximal. Hence there is no regularity of type III or of higher type that encompasses a_j (or b_j) and some of the symbols of Δj—for if there were, there would then have to be a regularity of length M+1 encompassing the sequence $a_j\Delta_j$ (or $\Delta_j b_j$) of length M+1—and this is impossible since the fact that $\mathcal{R}$ is maximal means that no such regularity exists.

We return now to the example cited earlier—the question of whether the regularity $\mathcal{R}$=<"a%%%%%h", "_bcdefg_", 1.0> of length M = 8 would be missed if the maximam length of regularity searched for in the Search Phase was M2 = 6. This regularity states a relationship among 8 symbols—and, in particular, that two widely separated symbols determine the intermediate symbols. We will assume that this regularity $\mathcal{R}$ is maximal—that is, that the string "abcdefgh" appears in a wide variety of different contexts in the sample. With a M2 of 6, the Grammar Discovery Algorithm will be recoding the sample using regularities up to length 6. We assume that some recoding "abcdefgh" does occur at this level. If some recoding does occur, it is the result of the existence of some regularity $\mathcal{R}'$ within the 8 symbols—perhaps that "bc" are always followed by "defg". This recoding would be accomplished by the rule of production

bcN → bcdefg

where N is a non-terminal. Thus, the 8 symbols at the first level would be encoded as "abcNh" in every case. Now, at the next level, it would be noted that an "a" appearing in position 1 and an "h" appearing in position 5 reliably predicts the occurrence of "bcN" as intermediate symbols. This regularity would be noted because its length if 5, which is less than M2, which is 6. Thus, at level 2, this regularity $\mathscr{R}$" =<"a%%%h", "_bcN_", 1.0> can be used to develop the rule

aPh → abcNh

where P is a non-terminal. Note that since "abcdefgh" appears in a variety of different contexts at level one, so will "abcHn" at level two. Thus, no encoding involving the symbols to the left of "a" and no encoding involving symbols to the right of "h" will be developed.

C. EFFECT OF CHOICE OF PARAMETERS IN THE RECODING PROCEDURE

Now let us consider the possible limiting and pre-determining effects of the syncronization, punctuating, or partitioning of the sample as it is determined by the choice of parameters or the Recoding Procedure. Thus punctuating is the product of choices which either are external choices determined according to heuristic considerations, or which are choices that are an integral part of the Recoding Procedure itself and the fact that the Recoding Procedure is not a combinatorially exhaustive procedure for partitioning the sample.

We claim that a maximal regularity $\mathscr{R} = \langle \gamma, \phi, P, Y \rangle$ will usually not be "missed" by the Grammar Discovery Algorithm because of the partitioning of Y by the Recoding Procedure, except when a regularity of equally high type is used instead as the basis for partitioning and recoding.

As before, case (1) (where there is no encoding) presents no problem of losing regularities.

In case (2), we are concerned about the domain (the Δ sequence) of a maximal regularity $\mathscr{R}$ being split between two parts of the partition of Y. (Obviously, if the Δ of $\mathscr{R}$ lies entirely within one part of the partition of Y, there is no problem).

First of all, the partition is not likely to divide a maximal regularity. The partition is not imposed indiscriminately on Y--independent of its regularities. Indeed, the partition is the consequence of the existence of regularities of allowably high type and suitable length. More often than not, the hypothesized maximal regularity will itself determine the partition and therefore be located within 1 part of the partition. Recall, in particular, that the

Recoding Procedure considers the regularities in order of decreasing type (that is, Type I first), so that the regularity chosen as the basis for making the partition of Y will, at worst, be of the same type as $\mathscr{R}$ itself. Also, within a given type, the regularities are considered in order of decreasing reliability, so that the regularity chosen will have a conditional probability at least as high as that of $\mathscr{R}$. Moreover, since the longer regularities are, in practice, considered first (even if the shorter regularities are considered first, it is the longer regularities that are more reliable, so that they tend to appear first in any case) maximal regularity will tend to be used as the basis for making the partition in the first place (in preference to any shorter regularity). Thus, the problem of a maximal regularity being divided is not likely to arise in the first place.

However, suppose this division does occur. Indeed, in a highly structured sample, there may be many reliable regularities that can be used as a basis for partitioning and recoding, and the Δ of these regularities will tend to overlap. But because of the order of considering regularities for recoding purposes, only regularities at least as reliable as $\mathscr{R}$ will be invoked.

D. EFFECT OF PRESENCE OR ABSENCE OF INITIAL PUNCTUATION IN THE SAMPLE

The purpose of this section is to make the assertion (quite contrary to intuition) that the presence or absence of initial punctuation separating sentences in the sample is of little consequence in the grammar discovery process; and that if it is absent, the discovery of sentence boundaries is qualitatively the same punctuation problem as grammar discovery in general, and is moreover not even quantitatively (i.e. combinatorially) much more difficult.

It is important to see that the entire Grammar Discovery Algorithm is a punctuating process—that is, a process of partitioning the sample Y into appropriate parts and then recoding the parts. The problem of inducing appropriate sentence boundaries is no different than the process of inducing appropriate phrase boundaries within sentences.

Naturally, if one wants to induce sentence boundaries, it is necessary that the sample contain repetitions of various sentences (or at least significant parts of them)—just as the induction of phrase boundaries requires repetitions of the phrases. Indeed, it is the repetition of features which establishes them as features. With this precondition in mind, let us use a particular constructed sample Y to argue the main assertion (above).

Suppose that a sample Y consists of 10 copies of 100 different sentences, each sentence being of length 20—the particular numbers and the uniformity of the number of copies and uniformity of sentence length being unimportant to our argument. The sample thus consists of 2,000 sentences. The 2,000 sentences appear in jumbled disorder (if the sentences appeared in a regular order, they would not be individual sentences !). If the Grammar Discovery Algorithm is

operating in the second mode (i.e. with initial punctuation between sentences), the sample appears as a string of 2,000 sentences each separated by an initial punctuation mark (the period). If the Grammar Discovery Algorithm is operating in the first mode (i.e. with no initial punctuation), the sample appears as a string of 20,000 symbols.

In either mode, the Grammar Discovery Algorithm first searches for local regularities and then recodes them. In the second mode, no search for regularities is made across initial punctuation marks. The following happens in the first mode: Since the sample is jumbled, the two sentences on either side of each of the 10 occurrences of a given sentence α will, in general, be different. Indeed, the probability that a particular sentence β appears, say, to the left of the given sentence α is only 1/100; and, even with the Birthday Problem phenomena, the odds are against any duplication of any sentence to the left of any of the sentences, and certainly against any reliable occurrence of such a duplication. Thus, it is most unlikely that any of the symbols at the left end of α are related in any way with the symbols occurring to their left. Since the partitioning and recoding of Y is done because of the occurrence of regularities, the symbols within each sentence may be recoded, but groups of symbols spanning the sentence boundary will not. Indeed, each sentence may be what we called a maximal regularity (but it need not be, of course).

Thus, although we cannot predict how the recoding of Y will proceed, we can predict that no groups of symbols spanning a sentence boundary will ever be recoded together. Thus, the sentence boundary will be preserved, as the recoding proceeds from level to level.

Even if each sentence finally becomes encoded as a single non-terminal at a high level, these single non-terminals will display no regularities amongst each other—indeed,the sequence of 2,000 single non-terminals will have maximal entropy or near maximal entropy. This may, of course, be an appropriate moment for inducing disjunctions—but the point is simply that even at this level, no regularities should appear—either for M of 2, 3, 4, etc. because the sentences (being independent units) were jumbled.

Thus, appropriate sentence boundaries will be induced by the same punctuating process as phrase boundaries.

Note also that the search for regularities is not even appreciably shortened by the presence of initial punctuation. If the sentences have average length of 20, most of the searching for local regularities (with small M's of up to 4, 5, or 6) does not occur over the sentence boundaries anyways. Thus, the presence of initial punctuation (across which the search is not made) does not appreciably reduce the combinatorics of the Search.

IV. ADDITIONAL FEATURES OF THE GRAMMAR DISCOVERY ALGORITHM

A. TERNARY MASKS AND DON'T CARE CONDITIONS

In the description of the Grammar Discovery Algorithm, the masks used to search for regularities in the Search Phase were all *binary masks*--that is, they were sequences of length M over only the symbols "_" ("context") and "%" ("predicted"). The masks were also *compact* in the sense that the mask expresses a relation within a certain substring of the sample, and all symbols appearing between the left-most symbol of that substring and the right-most symbol of that substring are either in the context part of the regularity or the predicted part.

One may wonder about regularities involving relations between two or more substrings that are widely separated by symbols that are not part of the regularity--as, for example, might occur in mechanically encoded messages. For example, an "a" in a certain position in the sample and a "b" occurring 6 positions later may reliably predict the symbol occurring half-way between them (in position 4).

There are two approaches to finding such regularities.

If binary compact masks are being used, this regularity will be detected as a regularity at a higher level--particularly if the distance between the substrings is moderately large or large. Or, this regularity may be detected in the Generalization Process--that is, a sequence of rules of production are found in which one of more of their context positions are found to vary "freely" over the entire current alphabet V_c and these rules are then replaced by one generalized rule having a universally quantified metavariable in the appropriate positions. This approach is particularly appropriate when the distance between the substring of the relation is rather small.

The second approach is more certain and involves using ternary masks.* A *ternary mask* $\mathcal{M}$ is a sequence of length M over the symbols "_", "%", and the symbol "#" (which is called *don't care*). There are 3^M ternary masks of length M as opposed to only 2^M binary masks of length M. To illustrate the use of the "don't care" symbol in ternary masks, consider the following example: The ternary mask is "_#%" (Note that the smallest ternary mask using all 3 symbols is of length 3). This mask refers to the relationship in which a particular symbol predicts a third symbol, regardless of the second symbol. A regularity based on this mask might be <"a#%", "_#b", 1.0, Y>--that is, the symbol "a" appearing in any position of the sample Y reliably predicts the occurrence of symbol "b" two positions later. To express this same regularity using only binary masks would require C (the number of symbols in the current alphabet) separate regularities--namely, <"aa%", "__b", 1.0, Y>, <"ab%", "__b", 1.0, Y>, <"ac%", "__b", 1.0, Y>, *etc*. In general, if there are h don't care positions in a mask, one ternary mask replaces C^h separate binary masks. Thus, there are $2^M C^M$ binary regularities of length M, and there are $3^M C^{M-h}$ ternary regularities.

The smallest non-degenerate situation where both a binary and ternary mask might be used occurs for values of M=3 (since the mask must contain at least one "_", one "%", and one "#"), C=2 (a binary alphabet), and h=1 (at least one don't care position). For this case, the binary mask approach is more efficient than the ternary mask approach. But when

$$h > \log_C \left(\frac{3}{2}\right)^M$$

*The variable FR controls the use of binary or ternary masks in the computer porgram. Ternary masks are not implemented in the computer program at this time.

--which encompasses most all other situations (particularly since C will tend to be moderately large, and M is usually small), the ternary masks will involve fewer combinations. Moreover, it should be remembered that the Generalization Process is itself a combinatorial process, so that any apparent efficiency of binary masks in the Search Phase is lost in the Generalization Process. Also, the Generalization Process requires a rather large and complete sample before it can operate (at least without making conjectures), and this is another factor in favor of the efficiency of ternary masks.

B. ALPHABET-ENLARGING VERSUS HUFFMAN-TYPE ENCODINGS

As mentioned earlier, there are two types of transformations: alphabet-enlarging, and non-alphabet-enlarging.

In the *alphabet-enlarging* transformation (which is the approach used above), there is a non-terminal alphabet. Strings containing these non-terminals from time to time replace strings in the given sample. Thus, at level one the current alphabet V_c consists only of the terminal alphabet V_T; but at higher levels, new non-terminals are added as needed (as rules of production are developed) and the current alphabet V_c becomes the union of the original terminal alphabet V_T and these added new non-terminals. Because these non-terminals are new symbols which are not found in the original sample at level one, a string containing one of these non-terminals can be recognized as being a string from a level higher than level one.*

In the *non-alphabet-enlarging* transformation, non-terminal symbols are not used. The sample at each level (with the possible exception of the last and highest level) consists only of the symbols occurring at level one. The encoding is done by mapping one string over the given alphabet to another string over the same alphabet. In general, not all strings over the given alphabet are possible in the given sample Y, so that some of the "impossible" strings are available to serve the function which the new non-terminals serve in the alphabet-enlarging approach. Generally, these "impossible" strings will be rather long strings. Note that the two sides of the

*The variable EXR in the computer program specifies whether alphabet-enlarging or Huffman-type recoding is to be done. Although Huffman encoding was attempted in early stages of this work (and the subroutine HUFF remains in the computer program), this feature is no longer operative in the computer program.

rules of production developed from this approach are strings over the same alphabet. Note also that the antecedent (left) side is almost always going to be longer than the consequent (right) side of the rule. This is in contrast to the situation when alphabet-enlarging transformations are used. Thus, when alphabet-enlarging transformation is used, the Grammar Discovery Algorithm is inherently a context-sensitive (or simpler) porcess; while when the non-alphabet-enlarging approach is used, unrestricted rewrite rules emerge.

Naturally, one must choose the new long strings with care--so that they cannot be confused with natural strings in the sample. These new longer strings must be treated as an encoded unit, and we are not concerned with the statistics or internal regularities that these strings may exhibit. Since the new strings tend to make more symbol strings possible in the sample, the total entropy increases as a result of the addition of them. An optimal coding procedure, such as the Huffman code procedure (Fano), can be used to produce these new strings in a manner that it is possible to reverse the code and recover the original sample. When a Huffman coding procedure is used on a sample of messages which are themselves strings over the encoding alphabet, it is necessary to encode every symbol in the original sample. As this type of encoding is applied from level to level, all possible strings become possible in the sample, and the information rate of the sample increases to a maximum.

The two approaches can be combined into a *limited-alphabet* transformation. In this approach, the number of non-terminals that can be added is limited to a fixed number h. If more non-terminals are needed, combinations (in the Huffman sense) of the allowed non-terminals must be used. Unrestricted rewrite rules would then result.

C. RECODING WITH NOISE IN THE SAMPLE

When a recoding is based only on Type I regularities,* no information is lost in the recoding. Whenever a recoding is based on a Type II or lower type regularity $\mathcal{R} = \langle \gamma, \phi, P, Y \rangle$, then (1-p)·100% of the time the transformation is not faithful to the sample. There can be two motivations for using regularities of Type II or lower type:

(1) The sample Y is assumed to contain a small amount of noise and is therefore assumed not to be 100% accurate itself. In this situation, if a regularity is found which has a conditional probability of nearly 100% accurate itself. In this situation, if a regularity is found which has a conditional probability of nearly 100%, it is assumed that the parts of the sample that do not conform to the regularity are in error. The use of a regularity with conditional probability near 100% therefore has the effect of correcting the sample and removing the alleged errors in it.

(2) The sample Y is accepted as being 100% accurate, but one desires to develop a very simple grammar for the sample. Parts of the sample which do not conform to regularities whose conditional probabilities are near 100% are assumed to represent "exceptions", and one is not willing to sacrifice the parsimony necessary to account for all the exceptions. Thus, the simplified grammar will be one that represents most of the sample most of the time. This simplified grammar will have fewer rules of production (more parsimony), but will have higher entropy (more information from the sample is lost).

*The variable ALLOW in the computer programs regulates the type of regularity allowed in the recoding.

D. THE CONTEXT-FREE CASE

In the Recoding Phase, certain strings of symbols are recoded. Up to now, this recoding (assuming an alphabet-enlarging encoding) always has involved substituting a new non-terminal symbol for a string of contiguous symbols that are well predicted by a certain context. Thus, the antecedent (left) side of any rule of production that is ultimately developed has less than or the same number of symbols as the consequent (right) side of the rule. Thus, each rule developed will be context-sensitive, and the grammar ultimately induced will also be context-sensitive. Thus, a grammar discovery algorithm based on contexts and regularities is inherently a context-sensitive process.

It might appear that because the whole Recoding Phase is based on context and conditional probabilities that only *strictly*-context-sensitive rules can be developed. Indeed, regularities that have no context part (as, for example, regularities developed from the mask $\%^{M}$) merely record frequencies of occurrences of substrings—rather than any relationship among symbols in the sample. Therefore, it would appear that every rule must have a non-empty context and therefore be strictly-context-sensitive. In fact, however, context-free rules (and regular rules—which are context-free rules with an M of 2) can be developed in one of *several* ways by the Grammar Discovery Algorithm.

First of all, a context-free rule can be induced using the Generalization Process. Whenever a combinatorially complete set of strictly-context-sensitive rules exists (that is, given h context positions in a rule, all C^{h} possible strictly-context-sensitive rules appear in the grammar), the Generalization Process can then induce a

context-free rule to replace the entire set of strictly-context-sensitive rules. Certainly, if a given substitution is made in all possible contexts, then the substitution can be made without regard to that context—that is, the substitition is context-free. However, because the Generalization Process is a combinatorial process, this approach is generally not available—whether for reasons of the large combinatorics involved or because not all C^h strictly-context-sensitive rules appear. The latter condition is indeed most restrictive. One can loosen the Generalization Process by allowing the generalization if no exceptions are found—that is, if among all the contexts that do appear, the substitution is invariably made, then a context-free substitution can be generalized. However, even this approach involves an exhaustive examination of a fair number of cases.

A second approach is based on the idea of maximal regularity, which was defined earlier. Whenever a rule is developed from a maximal regularity $\mathscr{R} = \langle \gamma, \phi, P, Y \rangle$, a context-free rule can be induced, --namely $N \rightarrow \Delta$,

where N is a new non-terminal, and where Δ is the Δ-sequence for $\mathscr{R}$. The justification for the writing of this context-free rule in this situation is that the regularity $\mathscr{R}$ cannot be extended. This is the case because Δ appears in the sample Y in a wide variety of contexts (the distribution of symbols surrounding Δ have a non-zero and persumably high entropy), and therefore Δ is "free" of its contexts. However, determining that a regularity is maximal (although involving only the checking of $2 \cdot C$ p-extensions and $2 \cdot C$ c-extensions) is itself as small-scale combinatorial process and somewhat unnatural. However, this second approach suggests a third approach.

The third approach is based on the fact that regularities are

considered in the Recoding Phase in decreasing order of their length. Thus, if any regularities discovered in the Search Phase are maximal regularities, they will be considered first in the Recoding Phase. Thus, we can simply write

$$N \rightarrow \Delta$$

whenever we have a regularity $\mathscr{R}$. Moreover, note that if $\mathscr{R}$ is not maximal, nothing is lost by this approach because at a higher level, the maximal regularity will still appear as a regularity encompassing N and various other symbols. Suppose the Δ-sequence of a maximal regularity $\mathscr{R}'$ of length M' is Δ', and that a regularity $\mathscr{R}$ of length M M < M' is used to write the rule

$$N \rightarrow \Delta$$

where Δ is the Δ-sequence of $\mathscr{R}$. Then, with Δ a proper substring of Δ', the maximal regularity $\mathscr{R}'$ manifests itself at a higher level over the string

$$\Delta(1)'\cdots\Delta'(h)\ N\ \Delta'(k)\cdots\Delta'(M')$$

of length M'-M. Using the maximal regularity to write a rule now (assuming that M'-M is now M2), we might get $A \rightarrow \Delta'(1)\cdots\Delta'(h)\ N\ \Delta'(k)\cdots\Delta'(M')$, where A is a non-terminal. Thus, we would have two rules, both context-free, and this third approach works.*

*The variable RIT controls whether context-free rules are written in in this way in the computer program.

V. EXAMPLES

In this section, we consider a number of examples that illustrate the Grammar Discovery Algorithm.

EXAMPLE A: Consider the following sample of sentences:

```
AA.AAAA.AAAAAA.AAAAAAAA.AAAAAAAAAA.
```

This sample presumably is from the language whose sentences are all of the form A^{2n}, where $n \geq 1$. The terminal alphabet consists only of the single symbol "A".

The input to the Grammar Discovery Algorithm might be as follows:

```
MAXIMUM NUMBER OF LEVELS TO BE TRIED                3
SMALLEST M TO BE TRIED                              2
LARGEST M TO BE TRIED                               4
DIRECTION OF CONSIDERING M IN RECODE             DESCEND
LAMBDA -- COEFFICIENT OF PARSIMONY              1.00000
LAMR -- COEFFICIENT OF RECURSIVE PARSIMONY      0.10000
SOURCE OF INITIAL SYMBOL STRING                    READ
MODE -- INITIAL PUNCTUATION MARK                    2ND
TYPE OF GRAMMAR DESIRED (MASKS TO BE TRIED)      LS-RS
BIT - CONTROLS ANTECEDENT SIDE OF RULES             CF
WORST  TYPE OF P(I) ALLOWED TO ENTER CODE           4
USE OF BEST M STRICTLY                            YES
FR -- RADIX OF MASK                                 2
PRINT CONTROL -- PRINT UNSORTED P(I)               NO
PRINT CONTROL -- PRINT SORTED P(I)                 NO
PRINT CONTROL -- PRINT GRAPH OF P(I)-S             ALL
PRINT CONTROL -- PRINT SEQUENCE NUMBERS           YES
SUBROUTINE USED FOR RECODING                   RECODE
OVERRIDE VALUE FOR MBEST (MCH)                      0
EARLY ELIMINATION OF UNALLOWABLE P(I)              NO
INCLUDE ALL-PREDICTED MASK                         NO
START RECODE AT TIME OF MBEST                     YES
HSEL -- METHOD FOR COMPUTING H                      4
CUT-OUT ON FINDING GOOD M                          NO
METHOD OF FINDING RECURSIONS                     RULE
INCLUDE IDENTITY RULES                             NO
TEST FOR RECURSION                                YES

SIZE OF TERMINAL ALPHABET (INITIAL STRING)          1
TERMINAL ALPHABET:                              A
INITIAL PUNCTUATION MARK (MODE=2)               .

SIZE OF BASIC NON-TERMINAL ALPHABET                10
SYMBOLS OF THE BASIC NON-TERMINAL ALPHABET: 0123456789
```

At level 1, for an M of 2, the Grammar Discovery Algorithm will develop exactly one possible rule of production, as follows:

```
--------------------------  LEVEL 1  ----------------------------------

SYMBOL STRING OF LENGTH  35 AND USING ALPHABET OF SIZE   1

AA.AAAA.AAAAAA.AAAAAAAA.AAAAAAAAAA.


LEVEL=   1      M=   2


NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)        2
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)          0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)          0
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)          0
NUMBER OF TYPE 5 SEQUENCES (NOISE)            0
NUMBER OF DIFFERENT MASKS USED                2
NUMBER OF POSSIBLE M-SEQUENCES                1
NUMBER OF POSSIBLE P(I)                       2
NUMBER OF P(I)                                2
NUMBER OF DIFFERENT CONTEXTS                  2
SEPARATION VALUE BETWEEN TYPES 3 AND 4     0.50000
EPSILON FOR DEFINING TYPES 2 AND 3         0.34000

TENTATIVE RULES OF PRODUCTION FOR M OF    2

0   ----->   AA


ENTROPY TERM                               0.0
PARSIMONY TERM                             1.00000
RECURSIVE PARSIMONY TERM                   0.0
                                          +---------
VALUE OF H FOR THIS RECODING.........      1.00000

NUMBER OF RULES OF PRODUCTION                 1
NUMBER OF RECURSIVE RULES                     0
NUMBER OF IDENTITY RULES                      0
NUMBER OF TIMES RULES ARE APPLIED             1

CURRENT STRING Y


AA.AAAA.AAAAAA.AAAAAAAA.AAAAAAAAAA.

NEW STRING


0.00.000.0000.00000.
```

The p_{ij}-graph for the one rule actually used in the recoding is rather sparse:

```
GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF   2

1.00 *
0.98 |
0.96 |
0.94 |
0.92 |
0.90 |
0.88 |
0.86 |
0.84 |
0.82 |
0.80 |
0.78 |
0.76 |
0.74 |
0.72 |
0.70 |
0.68 |
0.66 |
0.64 |
0.62 |
0.60 |
0.58 |
0.56 |
0.54 |
0.52 |
0.50 +
0.48 |
0.46 |
0.44 |
0.42 |
0.40 |
0.38 |
0.36 |
0.34 |
0.32 |
0.30 |
0.28 |
0.26 |
0.24 |
0.22 |
0.20 |
0.18 |
0.16 |
0.14 |
0.12 |
0.10 |
0.08 |
0.06 |
0.04 |
0.02 |
0.0  +
     1
NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)       1
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 5 SEQUENCES (NOISE)           0
NUMBER OF P(I)                               1
```

Similarly, for an M of 3 of level 1, the sample admits of the development of only one rule of production:

```
LEVEL=   1      M=   3



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)          4
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)            0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)            0
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)            0
NUMBER OF TYPE 5 SEQUENCES (NOISE)              0
NUMBER OF DIFFERENT MASKS USED                  4
NUMBER OF POSSIBLE M-SEQUENCES                  1
NUMBER OF POSSIBLE P(I)                         4
NUMBER OF P(I)                                  4
NUMBER OF DIFFERENT CONTEXTS                    6
SEPARATION VALUE BETWEEN TYPES 3 AND 4      0.50000
EPSILON FOR DEFINING TYPES 2 AND 3          0.33333


TENTATIVE RULES OF PRODUCTION FOR M OF   3

0   ----->   AAA


ENTROPY TERM                                0.0
PARSIMONY TERM                              1.00000
RECURSIVE PARSIMONY TERM                    0.0
                                          +---------
VALUE OF H FOR THIS RECODING.........     1.00000

NUMBER OF RULES OF PRODUCTION                  1
NUMBER OF RECURSIVE RULES                      0
NUMBER OF IDENTITY RULES                       0
NUMBER OF TIMES RULES ARE APPLIED              1


CURRENT STRING Y



AA.AAAA.AAAAAA.AAAAAAAA.AAAAAAAAAA.

NEW STRING



AA.0A.00.00AA.00 0A.
```

Note that the new string contains both the original terminal symbol "A" and the induced non-terminal "0".

The p_{ij}-graph for M of 3 is as sparse as for M of 2, and will not be shown here.

For an M of 4, one rule is again induced at level 1, as follows:

```
TENTATIVE RULES OF PRODUCTION FOR M OF   4
0   ----->   AAAA
```

Application of this rule leads to the following new string:

```
AA.O.CAA.CO.COAA.
```

We omit the p_{ij}-graph and other output for M of 4.

Note that we have temporarily suppressed consideration of the effect of identity rules in this example. In each case the total value of H consisted only of the value of parsimony. For M of 2, 3, and 4, the value of parsimony was the same. The best M, therefore, would be the shortest M and an M of 2 would be chosen, as noted by the following:

```
BEST M IS    2 WITH H OF    1.CCCOO
```

More significantly, however, is that 2 is in fact the best M when the effect of the identity rules are considered. For an M of 3 or 4 (but not an M of 2), a recoding of the entire sample requires application of an indentity rule which transforms the terminal symbol "A" into itself. This rules, when given any non-zero weight at all, is sufficient to dictate the choice of M of 2 as the best M.

In Example 2, the role of these identity rules will again be crucial.

To review level 1, the following is a table of the regularities observed in the sample

REGULARITIES FOR LEVEL 1

#	LEVEL	LENGTH	FREQ	TYPE	MASK	CONTEXT	PREDICTED
1	1	2	1.00	1	%_	%A	A_
2	1	2	1.00	1	_%	A%	_A
3	1	3	1.00	1	_%%	A%%	_AA
4	1	3	1.00	1	__%	AA%	__A
5	1	3	1.00	1	%%_	%%A	AA_
6	1	3	1.00	1	%__	%AA	A__
7	1	4	1.00	1	_%%%	A%%%	_AAA
8	1	4	1.00	1	__%%	AA%%	__AA
9	1	4	1.00	1	___%	AAA%	___A
10	1	4	1.00	1	%%%_	%%%A	AAA_
11	1	4	1.00	1	%%__	%%AA	AA__
12	1	4	1.00	1	%___	%AAA	A___

and the following is the one non-identity rule of production induced at level 1:

---------- RULES OF PRODUCTION FOR LEVEL 1----------

0 -----> AA

The new string resulting from application of this one rule of production is as follows:

NEW STRING

0.00.000.0000.00000.

The Algorithm now accepts this new string as its input for level 2:

---------- LEVEL 2 ----------

SYMBOL STRING OF LENGTH 20 AND USING ALPHABET OF SIZE 2

0.00.000.0000.00000.

For M of 2 at level 2, the Grammar Discovery Algorithm proceeds just as it did for level 1. The same regularities that were observed at level 1 are observed in this particular sample, except that they occur in terms of the induced non-terminal "0" instead of in terms of the terminal symbol "A".

The rule of production 1 ⟶ 00 is the rule of production which the Algorithm begins to induce. However, before inducing a new rule of production at a level above level 1, the possibility of instead inducing a recursive rule of production must be considered. Note that the rule 1 ⟶ 00 is isomorphic to the rule 0 ⟶ AA, and that the symbol "0" is common to both the antecedent side of the rule at level 2, and the consequent side of the rule at level 1. The same symbol positions of both rules are predicted and context symbols.

Therefore, instead of inducing the rule 1 ⟶ 00 at level 2 and adding this rule to the induced grammar, we instead induce the recursive rule A ⟶ AA. We delete 0 ⟶ AA as a rule, and we suppress the rule 1 ⟶ 00. We obtain

```
TENTATIVE RULES OF PRODUCTION FOR M OF   2
RECURSION NO.   1
A   ----->   AA
```

The new rule (which is a recursive rule) is applied in all possible ways, yielding:

A.A.A.A.A.

The termination conditions are now satisified, since the sample is now entirely reduced to sentences of length 1. The induced grammar is thus A $\longrightarrow$ AA. No further M's need be considered at level 2; however, if we did continue we would induce A $\longrightarrow$ AAA for M of 3, and A $\longrightarrow$ AAAA for an M of 4. The parsimony again would be identical for M of 2, 3, and 4, except for the fact that identity recodings would be necessary when A $\longrightarrow$ AAA and A $\longrightarrow$ AAAA are used. Thus, again, an M of 2 would be the best M.

EXAMPLE B: Suppose the sample consists of a lightly different set of sentences---namely, sentences of the form A^n, $n \geq 2$, as follows:

```
------------------------- LEVEL 1 ------------------------------
SYMBOL STRING OF LENGTH  63 AND USING ALPHABET OF SIZE   1

AA.AAA.AAAA.AAAAA.AAAAAA.AAAAAAA.AAAAAAAA.AAAAAAAAA.AAAAAAAA
AA.
```

The analysis will proceed in a manner similar to that of Example A, except that now two rules of production will always result. The two rules will be either A $\longrightarrow$ AA and A $\longrightarrow$ AAA, or A $\longrightarrow$ AA and an identity rule mapping A into itself. The additional rule, in both cases, accommodates the sentences of odd length.

Note that, as in Example A, other rules (such as perhaps A $\longrightarrow$ AAAA or A $\longrightarrow$ AAAAAA)will not be developed because A $\longrightarrow$ AA subsumes both and is more parsimonious.

EXAMPLE C: In this example, we consider a sample of well-formed sentences of the propositional calculus in Polish notation, with one term and one operation.

The sample is

P.PP+.PP+PP++.PPPP+++.PPP++P+.PP+P+P+.PPP++.PP+P+.

The input to the Grammar Discovery Algorithm is as follows:

```
MAXIMUM NUMBER OF LEVELS TO BE TRIED                    3
SMALLEST M TO BE TRIED                                  2
LARGEST M TO BE TRIED                                   3
DIRECTION OF CONSIDERING M IN RECODE                    DESCEND
LAMBDA -- COEFFICIENT OF PARSIMONY                      1.00000
LAMR -- COEFFICIENT OF RECURSIVE PARSIMONY              0.10000
SOURCE OF INITIAL SYMBOL STRING                         READ
MODE -- INITIAL PUNCTUATION MARK                        2ND
TYPE OF GRAMMAR DESIRED (MASKS TO BE TRIED)             LS-RS
RIT - CONTROLS ANTECEDENT SIDE OF RULES                 CF
WORST  TYPE OF P(I) ALLOWED TO ENTER CODE               4
USE OF BEST M STRICTLY                                  YES
FR -- PARIK OF MASK                                     2
PRINT CONTROL -- PRINT UNSORTED P(I)                    NO
PRINT CONTROL -- PRINT SORTED P(I)                      NO
PRINT CONTROL -- PRINT GRAPH OF P(I)-S                  ALL
PRINT CONTROL -- PRINT SEQUENCE NUMBERS                 YES
SUBROUTINE USED FOR RECODING                            RECODE
OVERRIDE VALUE FOR WORST (4CH)                          0
EARLY ELIMINATION OF UNUSABLE P(I)                      NO
INCLUDE ALL-PREDICTED MASK                              NO
START RECODE AT TIME OF WORST                           YES
HSEL -- METHOD FOR COMPUTING H                          4
CUT-OUT ON RECODING CODE                                NO
METHOD OF RECODING RECURSIONS                           RULE
INCLUDE IDENTITY RULES                                  NO
TEST FOR RECURSION                                      YES

SIZE OF TERMINAL ALPHABET (INITIAL STRING)              2
TERMINAL ALPHABET:                                      +P
INITIAL PUNCTUATION MARK (MODE=2)                       .

SIZE OF BASIC NON-TERMINAL ALPHABET                     10
SYMBOLS OF THE BASIC NON-TERMINAL ALPHABET: 1234567890
```

For an M of 2, three rules of production are developed, as follows:

```
LEVEL=  1      M=   2



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)        0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)          2
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)          4
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)          0
NUMBER OF TYPE 5 SEQUENCES (NOISE)            2
NUMBER OF DIFFERENT MASKS USED                2
NUMBER OF POSSIBLE M-SEQUENCES                4
NUMBER OF POSSIBLE P(I)                       8
NUMBER OF P(I)                                8
NUMBER OF DIFFERENT CONTEXTS                  4
SEPARATION VALUE BETWEEN TYPES 3 AND 4    0.50000
EPSILON FOR DEFINING TYPES 2 AND 3        0.34000

TENTATIVE RULES OF PRODUCTION FOR M OF    2

1    ----->    PP

2    ----->    ++

3    ----->    P+
```

The value of H obtained upon applying these rules is as follows:

```
ENTROPY TERM                                  1.20941
PARSIMONY TERM                                3.00000
RECURSIVE PARSIMONY TERM                      0.0
                                            +---------
VALUE OF H FOR THIS RECODING.....             4.20941

NUMBER OF RULES OF PRODUCTION                    3
NUMBER OF RECURSIVE RULES                        0
NUMBER OF IDENTITY RULES                         0
NUMBER OF TIMES RULES ARE APPLIED                3
```

The new string obtained is

```
NEW STRING



P.1+.1+12.112+.13+3.1+33.13+.1+3.
```

The p_{ij}-graph for this recoding is as follows:

```
GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF   2

1.00 +++
0.98 |||
0.96 |||
0.94 |||
0.92 |||
0.90 |||
0.88 |||
0.86 |||
0.84 |||
0.82 |||
0.80 |||
0.78 |||
0.76 |||
0.74 |||
0.72 |||
0.70 **|
0.68 |||
0.66 |||
0.64 |||
0.62 |||
0.60 |||
0.58 |||
0.56 |||
0.54 |||
0.52 |||
0.50 +**
0.48 |||
0.46 |||
0.44 |||
0.42 |||
0.40 |||
0.38 |||
0.36 |||
0.34 |||
0.32 |||
0.30 |||
0.28 |||
0.26 |||
0.24 |||
0.22 |||
0.20 |||
0.18 |||
0.16 |||
0.14 |||
0.12 |||
0.10 |||
0.08 |||
0.06 |||
0.04 |||
0.02 |||
0.0  +++
     223
NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)       0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         2
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)         1
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 5 SEQUENCES (NOISE)           0
NUMBER OF P(I)                               3
```

For an M of 3, we obtain the following:

```
LEVEL=   1     M=   3



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)        0
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)          4
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)          7
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)          6
NUMBER OF TYPE 5 SEQUENCES (NOISE)           15
NUMBER OF DIFFERENT MASKS USED                4
NUMBER OF POSSIBLE M-SEQUENCES                8
NUMBER OF POSSIBLE P(I)                      32
NUMBER OF P(I)                               32
NUMBER OF DIFFERENT CONTEXTS                 16
SEPARATION VALUE BETWEEN TYPES 3 AND 4    0.50000
EPSILON FOR DEFINING TYPES 2 AND 3        0.33333


TENTATIVE RULES OF PRODUCTION FOR M OF   3

1   ----->   PP+
```

The value of H is as follows:

```
ENTROPY TERM                              0.64752
PARSIMONY TERM                            2.00000
RECURSIVE PARSIMONY TERM                  0.0
                                        +--------
VALUE OF H FOR THIS RECODING.........     2.64752

NUMBER OF RULES OF PRODUCTION                2
NUMBER OF RECURSIVE RULES                    0
NUMBER OF IDENTITY RULES                     0
NUMBER OF TIMES RULES ARE APPLIED            2
```

The string is then recoded as follows:

```
NEW STRING


P.1.11+.PP1++.P1+P+.1P+P+.P1+.1P+.
```

Note that the rule 1 ⟶ PP+ is reapplied several times in the recoding.

After considering tentative recodings based on an M of 2, 3, and 4, the Algorithm concludes that

```
BEST M IS     3 WITH H OF     2.64752
```

and then uses the rule

to recode the sample. The resulting string is then the input to the Algorithm at level 2.

```
---------------------  RULES OF PRODUCTION FOR LEVEL 1------
1    ----->    PP+
```

At level 2, the Search Phase again searches for regularities in the new sample, developes rules of production, and computes the value of H for the resulting transformations. For an M of 2 at level 2, we obtain the following:

```
LEVEL=    2      M=    2


NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)          1
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)            2
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)            4
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)            1
NUMBER OF TYPE 5 SEQUENCES (NOISE)             10
NUMBER OF DIFFERENT MASKS USED                  2
NUMBER OF POSSIBLE M-SEQUENCES                 16
NUMBER OF POSSIBLE P(I)                        32
NUMBER OF P(I)                                 18
NUMBER OF DIFFERENT CONTEXTS                    7
SEPARATION VALUE BETWEEN TYPES 3 AND 4    0.50000
EPSILON FOR DEFINING TYPES 2 AND 3        0.34000
```

The value of H is as follows:

ENTROPY TERM	2.15596
PARSIMONY TERM	4.00000
RECURSIVE PARSIMONY TERM	0.0
VALUE OF H FOR THIS RECODING.....	6.15596

NUMBER OF RULES OF PRODUCTION	4
NUMBER OF RECURSIVE RULES	0
NUMBER OF IDENTITY RULES	0
NUMBER OF TIMES RULES ARE APPLIED	7

The p_{ij}-graph for an M of 2 at level 2 is as follows:

```
GRAPH OF P(I) USED IN RECODING FOR LEVEL 2 AND M OF   2

1.00 *-++
0.98 | ||
0.96 | ||
0.94 | ||
0.92 | ||
0.90 | ||
0.88 | ||
0.86 | ||
0.84 | ||
0.82 | ||
0.80 | ||
0.78 | ||
0.76 | ||
0.74 |*||
0.72 | ||
0.70 | ||
0.68 | ||
0.66 | *|
0.64 | ||
0.62 | ||
0.60 | ||
0.58 | |*
0.56 | ||
0.54 | ||
0.52 | ||
0.50 +-++
0.48 | ||
0.46 | ||
0.44 | ||
0.42 | ||
0.40 | ||
0.38 | ||
0.36 | ||
0.34 | ||
0.32 | ||
0.30 | ||
0.28 | ||
0.26 | ||
0.24 | ||
0.22 | ||
0.20 | ||
0.18 | ||
0.16 | ||
0.14 | ||
0.12 | ||
0.10 | ||
0.08 | ||
0.06 | ||
0.04 | ||
0.02 | ||
0.0  +-++
     1223
NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)       1
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         2
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)         1
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 5 SEQUENCES (NOISE)           0
NUMBER OF P(I)                               4
```

For an M of 3 at level 2, the Algorithm considers developing the rule 2 ⟶ 11+, but first checks to see whether a recursion is a possibility. In fact, one recursive rule can replace both 2 ⟶ 11+ and the rules 1 ⟶ PP+ developed at level 1. We thus obtain the following recursion instead:

```
TENTATIVE RULES OF PRODUCTION FOR M OF   3
RECURSION NO.    1
P    ----->    PP+
```

Upon applying the recursion once, we obtain the following new string:

P.P.11+.PP1++.P1+P+.1P+P+.P1+.1P+.

Then upon applying the recursion in every possible way to the new string (and taking into account the fact that the non-terminal "1" is now superseded by the recursive symbol "P"), we obtain the following as the recoding:

P.P.P.P.P.P.P.P.

The conditions for terminating the Algorithm now obtain, and not further levels are considered.

EXAMPLE D: In this section, we analyze the behavior of the Grammar Discovery Algorithm for a language---rather than merely a particular finite sample of sentences from the language. We do this to demonstrate that the kinds of results one obtains from the Algorithm for small samples of sentences, from languages with relatively simple grammars, over relatively small alphabets can be expected to obtain also for larger samples from languages with more complex grammars over alphabets with large numbers of symbols. This particular example also serves to illustrate the full range of grammar induction devices discussed, including those which are not implemented in the computer program for the Algorithm, or which require large combinatorial searches---namely, context sensitive regularities, maximal regularities, recursions, disjunctions, as well as context-sensitive and context-free rules of production.

With this in mind, consider the language generated by the following rules:

$V \longrightarrow abV \mid \emptyset$

$W \longrightarrow cdeW \mid \emptyset$

$X \longrightarrow fghiX \mid \emptyset$

$P \longrightarrow VWX$

$Y \longrightarrow lmY \mid \emptyset$

$Z \longrightarrow nopZ \mid \emptyset$

$Q \longrightarrow Y \mid Z$

$S \longrightarrow P\ j\ Q$

Similarly, no regularities involving "cde" will appear of length greater than 3 which will be as reliable as those for length 3, and no regularity involving "fghi" of length greater than 4 will be superior to those of length 4.

Finally, similar observations will apply to the two phrases "lm" and "nop".

The second kind of regularity will involve strings longer than the lengths of the basic 5 phrases. These regularities will catalog the propensity of "ab" to follow an "ab". As mentioned above, these regularities will be less reliable than the corresponding shorter regularities.

The third kind of regularity for this language will involve the interface between occurrences of the 5 basic phrases and the symbol "j". These will be of type II or worse.

In order to recode the sample at level 1, each of the 5 basic phrases will have to be recoded in some way internal to themselves. The reason why there will be no inter-phrase recodings is that a type I regularity internal to the phrase exists in each case, while any inter-phrase regularity will be of type II or worse. Exactly which recoding will be used will depend on several externally-specified parameters. For example, if context-sensitive rules are searched for, the Algorithm will find and use a regularity such as

< "c%e", "_d_", 1.0 >

whereas if only right-sensitive regularities are searched for, something such as

< "%%e", "cd-", 1.0 >

might be used. The Recoding Procedure, and in particular the point in the sample where recoding begins, all affect the exact recoding chosen. However, even within the variability permitted by these external choices, it can be predicted that the phrase"cde" will be recoded as a unit, and that, for example, no phrase involving "cdecde" will be used (of length 6). Thus, the initial sample of sentences will be punctuated into sub-phrases of length 2, 3, and 4 corresponding to the 5 basic phrases.

However, these phrases are recoded, the opportunity to recode them at level 2 will occur, and it is at level 2 that the recursive structure of the sample will be discovered. Again, the exact way in which the recursions will be discovered will depend on some externally=specified choices. If, for example, the "uvwxy" method of inducing recursions is being used, the Algorithm will note that there are recurrences of the 5 basic phrases "ab", "cde", etc. or recurrences of whatever symbols now represent these phrases. If the rule-oriented method of inducing recursions is being used, then at level 2 there may be recurrences of non-terminals such as V, W, X, Y, and Z. These non-terminals may be recoded, in a manner similar to that of Examples A and B into other non-terminals, and the recursion discovered at level 3.

In any event, the recursive structure of the sample will be discovered somewhere between levels 1 and 3 of the recoding regardless of how the recoding actually proceeds.

Also, the symbol "j" will not be recoded at all because it is the exceptional suffix to "fghi" and the exceptional

Note that the sentences of this language have the general form

$$(ab)^r(cde)^s(fghi)^t\ j\ (lm)^u\ (nop)^v$$

where $r,s,t \geq 0$ and either u or v is 0, while the other is > 0.

In the Search Phase, one finds several different kinds of regularities. First, there are regularities within the 5 phrases "ab", "cde", "fghi", "lm" and "nop". Second, there are regularities occurring between each of these 5 phrases and themselves. Third, there are regularities occurring between 2 different ones of these 5 phrases as well as with the symbol "j."

Consider first the first kind of regularity. Among these will be the regularity which catalogs the fact that an "a" always predicts a "b" to the right. This regularity is a type I left-sensitive regularity. There is also the fact that a "b" is almost always--but not always---preceded by an "a". This is a type II right-sensitive regularity. Similarly, a "b" is usually followed by an "a", but occasionally is followed by a "c" or even an "f". No regularity involving "a" and "b" of length greater than 2 will be as reliable as those of length 2 because, as will turn out, the string "ab" is a maximal regularity.

Similarly, within the phrases "cde" and "fghi", there will be regularities. Some of these will be context-sensitive regularities (something that was impossible in a phrase of length 2), such as the type I regularity

$$\langle\ \text{"c\%e"}\ ,\ \text{"\_d\_"}\ ,\ 1.0\ \rangle$$

Again, all regularities will be within the phrases.

prefix to whatever follows.

The "j" will be followed approximately equally often by either "lm" or "nop." After these 2 phrases are recoded internally, the non-terminals into which they have been recoded will appear as the two possible substitution possibilities following the "j". Thus, a situation where the suffix to the "j" presents maximal entropy will exist. If the phrase "lm" is recoded as Y, and the phrase "nop" is recoded as "Z", then the disjunction $Q \longrightarrow Y \mid Z$ can be induced.

With the 3 recursions discovered at the beginning of each sentence, and the 2 recursions and 1 disjunction at the end of each sentence discovered, each sentence will have the form VWX j Q, or perhaps P j Q, depending on the intermediate recoding that occurred. When each sentence of the sample is reduced to a common form, the conditions for terminating the Algorithm have been satisfied. Thus, in this case, we have induced the original grammar, or one virtually the same as the original grammar.

Note that the different recursions do not get "tangled" because the phrases in which they are embedded are all maximal regularities.

EXAMPLE E: Here again we shall consider a language rather than a finite sample of sentences in order to illustrate a situation which cannot be represented by any finite sample.

Consider an ergodic source generated strings over a terminal alphabet A of size C_1. Perhaps the grammar which describes the language is

$$S \longrightarrow (a_1 \quad a_2 \quad a_3 \quad \ldots \quad a_{C_1}) \quad S.$$

Any finite subset of the sentences generated by such a grammar (even a subset that is complete in the sense that is includes all sentences of the language up to a certain specified length) has certain regularities which are artifacts of the finiteness of the sample. The nature of ergodicity is precisely the opposite---namely, that there are essentially no regularities in the sample. Therefore, to discuss the ergodic case, one cannot consider any specific finite sample.

Recall that the Algorithm is given an externally specified range of M's. This range is typically from a lower value of 2 up to a small whole number such as 5. The Algorithm is also given an externally specified direction for considering the M's for recoding purposes (almost always descending---and always descending when context-free rules are being generated). Regardless of these two choices, the entropy associated with any ergodic sample is always the maximal value. Thus, a disjunction expressed in terms of phrases of length M (the first M) is induced. If an M of 1 is consider, the disjunction is in terms of single singles from the alphabet. (Note that the only useable mask for an M of 1 is the "predicted" mask which catalogs probability of occurrence).

Thus, the Grammar Discovery Algorithm produces a reasonable result for an ergodic source generating the sentences of the given sample.

The above discussion suggests, in effect, an alternative definition of ergodicity. In this paper, the notion of the universe of masks is well defined, as is the concept of the universe of possible regularities for a given value of M. A source is ergodic, therefore, if there are no regularities in the sentences produced by the source except for type VI regularities having a conditional probability p_{ij} equal to $\epsilon = 1/C^g$, where C is the size of the alphabet, and g is the number of predicted positions in the regularity. In a more practical vein, if the limiting value of the p_{ij}'s are their respective $1/C^g$, then we can call the source ergodic.

EXAMPLE E: In this example, we will consider a very simple example of sentences from English. The sample is as follows:

```
.THE¬BIG¬CAT¬RAN¬.THE¬BAD¬CAT¬RAN¬.THE¬BIG¬DOG¬RAN¬.THE¬BAD¬
DOG¬RAN¬.THE¬BIG¬CAT¬SAT¬.THE¬BAD¬CAT¬SAT¬.THE¬BIG¬DOG¬SAT¬.
THE¬BAD¬DOG¬SAT¬.
```

The input to the Grammar Discovery Algorithm is as follows:

```
MAXIMUM NUMBER OF LEVELS TO BE TRIED                3
SMALLEST M TO BE TRIED                              2
LARGEST M TO BE TRIED                               4
DIRECTION OF CONSIDERING M IN RECODE                DESCEND
LAMBDA -- COEFFICIENT OF PARSIMONY                  1.00000
LAMR -- COEFFICIENT OF RECURSIVE PARSIMONY          0.10000
SOURCE OF INITIAL SYMBOL STRING                     READ
MODE -- INITIAL PUNCTUATION MARK                    2ND
TYPE OF GRAMMAR DESIRED (MASKS TO BE TRIED)         LS-RS
RIT - CONTROLS ANTECEDENT SIDE OF RULES             CF
WORST  TYPE OF P(I) ALLOWED TO ENTER CODE           4
USE OF BEST M STRICTLY                              YES
FR -- RADIX OF MASK                                 2
PRINT CONTROL -- PRINT UNSORTED P(I)                NO
PRINT CONTROL -- PRINT SORTED P(I)                  NO
PRINT CONTROL -- PRINT GRAPH OF P(I)-S              ALL
PRINT CONTROL -- PRINT SEQUENCE NUMBERS             YES
SUBROUTINE USED FOR RECODING                        RECODE
OVERRIDE VALUE FOR MBEST (MCH)                      0
EARLY ELIMINATION OF UNALLOWABLE P(I)               NO
INCLUDE ALL-PREDICTED MASK                          NO
START RECODE AT TIME OF MBEST                       YES
HSEL -- METHOD FOR COMPUTING H                      4
CUT-OUT ON FINDING GOOD M                           NO
METHOD OF FINDING RECURSIONS                        RULE
INCLUDE IDENTITY RULES                              NO
TEST FOR RECURSION                                  YES

SIZE OF TERMINAL ALPHABET (INITIAL STRING)          14
TERMINAL ALPHABET:                                  ABCDEGHINORST¬
INITIAL PUNCTUATION MARK (MODE=2)                   .

SIZE OF BASIC NON-TERMINAL ALPHABET                 10
SYMBOLS OF THE BASIC NON-TERMINAL ALPHABET:         0123456789
```

The search phase produces the following for an M of 2:

```
LEVEL=   1     M=   2



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)      19
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)        11
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 5 SEQUENCES (NOISE)          16
NUMBER OF DIFFERENT MASKS USED               2
NUMBER OF POSSIBLE M-SEQUENCES             196
NUMBER OF POSSIBLE P(I)                    392
NUMBER OF P(I)                              46
NUMBER OF DIFFERENT CONTEXTS                28
SEPARATION VALUE BETWEEN TYPES 3 AND 4   0.50000
EPSILON FOR DEFINING TYPES 2 AND 3       0.34000


TENTATIVE RULES OF PRODUCTION FOR M OF   2

0    ----->   TH

1    ----->   E¬

2    ----->   BI

3    ----->   G¬

4    ----->   CA

5    ----->   T¬

6    ----->   RA

7    ----->   N¬

8    ----->   BA

9    ----->   D¬

<00>   ----->   DO

<01>   ----->   SA
```

The value of H for M of 2 is as follows:

```
ENTROPY TERM                              1.50000
PARSIMONY TERM                           12.00000
RECURSIVE PARSIMONY TERM                  0.0
                                        +---------
VALUE OF H FOR THIS RECODING.........    13.50000

NUMBER OF RULES OF PRODUCTION               12
NUMBER OF RECURSIVE RULES                    0
NUMBER OF IDENTITY RULES                     0
NUMBER OF TIMES RULES ARE APPLIED           12
```

The sample is then recoded as follows:

```
.01234567.01894567.0123<00>367.0189<00>367.012345<01>5.01894

5<01>5.0123<00>3<01>5.0189<00>3<01>5.
```

The p_{ij}-graph for this recoding based on an M of 2 is as follows:

```
GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF   2

1.00 *********--+
0.98 |         |  |
0.96 |         |  |
0.94 |         |  |
0.92 |         |  |
0.90 |         |  |
0.88 |         |  |
0.86 |         |  |
0.84 |         |  |
0.82 |         |  |
0.80 |         |  |
0.78 |         |  |
0.76 |         |  |
0.74 |         |  |
0.72 |         |  |
0.70 |         |  |
0.68 |         |  |
0.66 |         |  |
0.64 |         |  |
0.62 |         |  |
0.60 |         |  |
0.58 |         |  |
0.56 |         |  |
0.54 |         |  |
0.52 |         |  |
0.50 +--------+***
0.48 |         |  |
0.46 |         |  |
0.44 |         |  |
0.42 |         |  |
0.40 |         |  |
0.38 |         |  |
0.36 |         |  |
0.34 |         |  |
0.32 |         |  |
0.30 |         |  |
0.28 |         |  |
0.26 |         |  |
0.24 |         |  |
0.22 |         |  |
0.20 |         |  |
0.18 |         |  |
0.16 |         |  |
0.14 |         |  |
0.12 |         |  |
0.10 |         |  |
0.08 |         |  |
0.06 |         |  |
0.04 |         |  |
0.02 |         |  |
0.0  +--------+--+
     111111111333
NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)       9
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)         3
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 5 SEQUENCES (NOISE)           0
NUMBER OF P(I)                              12
```

For an M of 3, the following tentative rules of production are developed:

```
LEVEL=   1     M=   3



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)      46
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)        34
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 5 SEQUENCES (NOISE)          32
NUMBER OF DIFFERENT MASKS USED               4
NUMBER OF POSSIBLE M-SEQUENCES            2744
NUMBER OF POSSIBLE P(I)                  10976
NUMBER OF P(I)                             112
NUMBER OF DIFFERENT CONTEXTS                98
SEPARATION VALUE BETWEEN TYPES 3 AND 4   0.50000
EPSILON FOR DEFINING TYPES 2 AND 3       0.33333


TENTATIVE RULES OF PRODUCTION FOR M OF   3

0    ------>   THE

1    ------>   ¬BI

2    ------>   G¬C

3    ------>   AT¬

4    ------>   RAN

5    ------>   ¬BA

6    ------>   D¬C

7    ------>   G¬D

8    ------>   OG¬

9    ------>   D¬D
```

The value of H for the M of 3 is

```
ENTROPY TERM                              2.00000
PARSIMONY TERM                           10.00000
RECURSIVE PARSIMONY TERM                  0.0
                                        +---------
VALUE OF H FOR THIS RECODING.........    12.00000

NUMBER OF RULES OF PRODUCTION               10
NUMBER OF RECURSIVE RULES                    0
NUMBER OF IDENTITY RULES                     0
NUMBER OF TIMES RULES ARE APPLIED           10
```

The sample is then recoded as follows:

```
.01234¬.05634¬.01784¬.05984¬.0123S3.0563S3.0178S3.0598S3.
```

For an M of 3, the p_{ij}-graph is as follows:

```
GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF   3

1.00 ******---+
0.98 |     |   |
0.96 |     |   |
0.94 |     |   |
0.92 |     |   |
0.90 |     |   |
0.88 |     |   |
0.86 |     |   |
0.84 |     |   |
0.82 |     |   |
0.80 |     |   |
0.78 |     |   |
0.76 |     |   |
0.74 |     |   |
0.72 |     |   |
0.70 |     |   |
0.68 |     |   |
0.66 |     |   |
0.64 |     |   |
0.62 |     |   |
0.60 |     |   |
0.58 |     |   |
0.56 |     |   |
0.54 |     |   |
0.52 |     |   |
0.50 +-----+****
0.48 |     |   |
0.46 |     |   |
0.44 |     |   |
0.42 |     |   |
0.40 |     |   |
0.38 |     |   |
0.36 |     |   |
0.34 |     |   |
0.32 |     |   |
0.30 |     |   |
0.28 |     |   |
0.26 |     |   |
0.24 |     |   |
0.22 |     |   |
0.20 |     |   |
0.18 |     |   |
0.16 |     |   |
0.14 |     |   |
0.12 |     |   |
0.10 |     |   |
0.08 |     |   |
0.06 |     |   |
0.04 |     |   |
0.02 |     |   |
0.0  +-----+---+
     1111113333
NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)      6
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)        0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)        4
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)        0
NUMBER OF TYPE 5 SEQUENCES (NOISE)          0
NUMBER OF P(I)                             10
```

Note that there are 6 type I transformations, and 4 type II transformations.

For an M of 4, the search phase produces the following:

```
LEVEL=   1     M=   4



NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)      73
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)        81
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)        18
NUMBER OF TYPE 5 SEQUENCES (NOISE)          20
NUMBER OF DIFFERENT MASKS USED               6
NUMBER OF POSSIBLE M-SEQUENCES           38416
NUMBER OF POSSIBLE P(I)                 230496
NUMBER OF P(I)                             192
NUMBER OF DIFFERENT CONTEXTS               219
SEPARATION VALUE BETWEEN TYPES 3 AND 4   0.50000
EPSILON FOR DEFINING TYPES 2 AND 3       0.25000


TENTATIVE RULES OF PRODUCTION FOR M OF   4

0    ----->   THE¬

1    ----->   BIG¬

2    ----->   CAT¬

3    ----->   RAN¬

4    ----->   BAD¬

5    ----->   DOG¬

6    ----->   SAT¬
```

The value of H is as follows:

```
ENTROPY TERM                              0.0
PARSIMONY TERM                            7.00000
RECURSIVE PARSIMONY TERM                  0.0
                                        +--------
VALUE OF H FOR THIS RECODING.........     7.00000

NUMBER OF RULES OF PRODUCTION               7
NUMBER OF RECURSIVE RULES                   0
NUMBER OF IDENTITY RULES                    0
NUMBER OF TIMES RULES ARE APPLIED           7
```

The sample is then recoded as follows:

```
NEW STRING



.0123.0423.0153.0453.0126.0426.0156.0456.
```

For an M of 4, the p_{ij}-graph shows that there are 7 type I regularities in the recoding:

```
GRAPH OF P(I) USED IN RECODING FOR LEVEL 1 AND M OF   4

1.00 *******
0.98 |      |
0.96 |      |
0.94 |      |
0.92 |      |
0.90 |      |
0.88 |      |
0.86 |      |
0.84 |      |
0.82 |      |
0.80 |      |
0.78 |      |
0.76 |      |
0.74 |      |
0.72 |      |
0.70 |      |
0.68 |      |
0.66 |      |
0.64 |      |
0.62 |      |
0.60 |      |
0.58 |      |
0.56 |      |
0.54 |      |
0.52 |      |
0.50 +------+
0.48 |      |
0.46 |      |
0.44 |      |
0.42 |      |
0.40 |      |
0.38 |      |
0.36 |      |
0.34 |      |
0.32 |      |
0.30 |      |
0.28 |      |
0.26 |      |
0.24 |      |
0.22 |      |
0.20 |      |
0.18 |      |
0.16 |      |
0.14 |      |
0.12 |      |
0.10 |      |
0.08 |      |
0.06 |      |
0.04 |      |
0.02 |      |
0.0  +------+
     1111111
NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)       7
NUMBER OF TYPE 2 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 3 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 4 SEQUENCES (MESSAGE)         0
NUMBER OF TYPE 5 SEQUENCES (NOISE)           0
NUMBER OF P(I)                               7
```

The Algorithm then selects the best M, as follows:

```
BEST M IS     4 WITH H OF     7.00000
```

Using the best M (of 4) as the basis for the recoding, the original sample is then recoded, and this recoded version is then used as input to level 2, as follows:

```
------------------------- LEVEL 2 -------------------------
SYMBOL STRING OF LENGTH  41 AND USING ALPHABET OF SIZE  21

.0123.0423.0153.0453.0126.0426.0156.0456.
```

The resolution of this sample at level 2 requires the induction of a disjunction at level 2 --- a feature not now implemented in the computer program. However, if an M of 1 is used at level 2, this sample will be recognized as being derived from a grammar which places a "0" as the first symbol of every string, and then disjunctively places a "1" or a "4" at position 2; a "2" or a "5" at position 3; and a "3" or a "6" at position 4.

VI. FUTURE DIRECTIONS

This paper raises numerous additional questions and suggests further research about grammar induction and grammar discovery. Among these questions are the following:

First, it would be interesting to explore the two-dimensional pattern recognition problem using the methods of the Grammar Discovery Algorithm. The concepts of context-sub-sequences, predicted-sub-sequences, regularities, and hierarchical rules of production would seem to have application to pattern recognition problems as well as grammar induction problems. The kinds of descriptions that the Grammar Discovery Algorithm builds up in analyzing samples would seem to parallel the kind of descriptions one would want a good pattern recognition algorithm to develop about its samples.

Second, a precondition to seriously analyzing the two-dimensional pattern recognition problem would be to rewrite the computer program for implementing the Grammar Discovery Algorithm. The computer program here was an integral part of the research for this paper. As such, the program was constantly changed as the Algorithm was developed. Thus, the computer program is structured in ways peculiar to its development (e.g. the time-consuming hash coding of sequences) but which make it quite inefficient for analyzing large samples.

Third, there are several aspects of the induction problem that are amenable to more formal analysis than has been presented here. For example, the combined measure of entropy and parsimony and recursive parsimony used in the computer program is essentially an *ad hoc* measure based on intuitive considerations. An axiomatic development of a combined measure would be interesting. Also, some of the questions

of convergence and rates of convergence of the Grammar Discovery Algorithm should be possible.

Fourth, it would be interesting to consider extensions of the grammar discovery process that would lend themselves to samples consisting of symbols that are ordered or partially ordered. This would include samples of integers, or of playing cards, or of musical notes which present interesting questions of induction or of identifying underlying structure.

APPENDIX A. DESCRIPTION OF INPUT TO THE COMPUTER PROGRAM IMPLEMENTING THE GRAMMAR DISCOVERY ALGORITHM

The input to the computer program implementing the Grammar Discovery Algorithm consists of three control cards, followed by additional cards containing the sample of sentences.

The first control card contains a variety of parameters which control the operation of the Grammar Discovery Algorithm. All are integer numbers, unless otherwise specified.

Card Columns	Variable Name	Description	Typical Value
1-5	L3	Upper Limit on the number of levels to be tried by the algorithm. This parameter has no theoretical significance, and is used only to conserve computer time in testing. By choosing a large value for L3, its function is effectively negated. (See termination—II.G.)	99
6-10	M1	Lower Limit on length of regularities searched for in the Search Phase. This parameter is almost always set to be 2. (See section II.H. and III.B.)	2
11-15	M2	Upper Limit on length of regularities searched for in the Search Phase. This parameter	8

		is usually a small whole number. (See section II.H. and III.B.)	
16-20	VDIR	Direction of Considering M's in Recoding Procedure. (See section II.C.4.) A value of +1 specifies ascending values, and a value of -1 specifies descending values. The choice of -1 is usual, especially when context-free rules are being generated (IV.D) (See RIT below).	-1
21-25	LAMBDA	Real Number — Coefficient of Parsimony. (See section II.D.2).	.5
26-30	LAMR	Real Number — Coefficient of Recursive Parsimony. Typically LAMBDA << LAMR. (See section II.D.2).	.1
31 32	CONT	Specifies source of the sample. A value of 1 specifies that the sample is to be read in on cards, after the 3 control cards. A value of 2 or 3 specifies that sample is to be generated by experimental subroutines GEN2 or GEN3, respectively. CONT is	1

		usually 1. The subroutine GEN1 reads the cards, if CONT = 1.	
33-34	MODE	Specifies the mode of the initial sample. In the first mode (MODE=1), there is no initial punctuation in the sample, while in the second mode (MODE=2), there is. (See section II.A.)	1 or 2
35-36	GRAM	Specifies grammatical type of masks that are used in the Search Phase to search for regularities. A value of 0 specifies Unrestricted Rewrite type; a value of 1, context-sensitive. If context-free rules are desired, GRAM may be 0 or 1, provided RIT is properly set. If regular rules are desired, GRAM may be 0 or 1, provided RIT is properly set, and provided M2 and M1 are both 2. A value of 0 is most general. (See section II.B.9.)	1
37-38	ALLOW	The worst type of regularity allowed to be used in Recoding. ALLOW may be 1 (only 100% reliable regularities are to be used); or 2 (either Type I or	1, 2, or 3

		Type II may be used); or 3 (Types I, II, or III may be used). It would be unusual for ALLOW to be 4 or 5. (See section II.C.4.)	
39-40	STRICT	A value of 1 specifies the use of only one value of M (namely, MBEST) in Recoding, and is suitable only when the sample is from a uniform code source. A value of 0 allows any value of M from M1 to MBEST (which is limited above by M2) in Recoding. STRICT is almost always 0. (See section II.C.4.)	0
41-42	FR	Radix of Masks. A value of 2 specifies binary masks, and a value of 3 specifies ternary masks (i.e. masks with "don't care" positions). (See section IV.A.)	2 or 3
43-44	PRI	Controls printing of unsorted table of regularities. A value of 1 specifies printing, and a value of 0 specifies no printing.	0
45-46	PR2	Controls printing of sorted	

		table of regularities. A value of 1 specifies printing, and a value of 0 specifies no printing.	
47-48	PRG	Controls Printing of Graph of regularities used in Recod-. ings. A value of 0 specifies the printing of no graphs;aa value of 1 specifies printing of some graphs (namely, when a recoding is actually used); a value of 1 specifies printing of all graphs (i.e. for all trial recodings and the one actual recoding).	1 or 2
49-50	PRS	Controls printing of SEQNO table (table of indices of regularities). A value of 0 specifies no printing, and a value of 1 specifies printing.	0
51-52	EXR	A value of 1 specifies that alphabet-enlarging Recoding as contained in subroutine RECODE is to be used. 2 specifies Huffman type of Recoding is to be used. (See section IV.B.)	1

53-54	MCH	If MCH is not-zero, this value is taken as an override value of MBEST for purposes of Recoding. This parameter is used only to force a value for the "best M" for Recoding, for testing purposes.	0
55-56	EARLY	A value of 1 specifies that early culling of the table of regularities is to be done (to conserve memory space in the computer), and a value of 0 specifies that it is not. This parameter has no theoretical significance. (See section II.B.9.)	0
57-58	LAWL	This variable is no longer used by the program.	
59-60	VSTART	A value of 1 specifies that the scan in the Recoding Phase is to start at position MBEST (rather than M1) of the sample (or if MODE=2, of each individual sentence). (See section II.I. and	0 or 1

III.C. for discussion of effect of VSTART.)

61-62 HSEL — Selects method of computing H (the combined entropy-parsimony-resursive parsimony measure): 1

HSEL	EXPRESSION
1	$\frac{-P_{ij}\sum \log_2 P_{ij}}{NTM} + N_q\lambda + N_r\lambda_r$
2	$N_q \cdot \frac{-P_{ij}\sum \log_2 P_{ij}}{NTM} + N_q\lambda + N_r\lambda_r$
3	$\frac{MBE}{M1} -P_{ij}\sum \log_2 P_{ij} + N_q\lambda + N_r\lambda_r$
4	$-P_{ij}\sum \log_2 P_{ij} + N_q\lambda + N_r\lambda_r$

where here n_q is the number of rules of production, where n_r is the number of recursive rules, where NTM is the number of times the rules are actually applied in the transformation, where MBE is the current M being used (as a maximum M for any rules in this transformation). The usual value of HSEL is 1. The other expres-

		sions were used for testing purposes. (See section II.D.2).	
63-64	WDF	Controls effect of discovering a Resolving Transformation. A value of 1 specifies that if a value of M results in a resolving transformation, then this M is immediately used for the actual recoding at this level. A value of 0 allows the continued examination of M (and thereby allows another equally good or possibly even better M to be used. Note that variable VDIR controls the order of considering the M's (and that this is usually descending order). WDF is usually 1. (See section II.C.4).	1
65-66	WCV	Specifies Approach for inducing recursions: A value of 1 specifies the Rule-Oriented Method (See section II.E.4), and a value of 2 specifies the Sentence-Oriented ("uvwxy") Approach (See Section II.E.5). WCV is usually 1.	1
67-68	WXL	This variable is no longer used by the program.	

69-70	KRECUR	Specifies whether Recursions should be induced. A value of 1 specifies that recursions should be induced (if possible), while a value of 0 specifies that no attempt to induce recursions should be made. (See section II.E.1).	1

The second control card contains the terminal alphabet and the initial punctuation mark (if any). The variable C1 specyfying the size of the terminal alphabet appears in card columns 1-5. The C1 cymbols of the terminal alphabet appear starting in column 6. If MODE = 2, the initial punctuation mark (customarily, the period) appears after the C1 symbols. If the sample contains blanks, it is customary to use the symbol "¬" in place of blanks to improve readability of the analytical tables. This convention in no way identifies the blank as a distinguished symbol to the Algorithm, however.

The third control card contains the non-terminal alphabet. The variable N26 specifies the size of the non-terminal alphabet and appears in card columns 1-5. The N26 symbols starting in column 6 are the non-terminal alphabet symbols. If more than N25 non-terminal symbols are required by the Algorithm, the Algorithm uses bracketed pairs of these N26 symbols in the fashion of backus normal form. It is therefore advisable not to use brackets in either the terminal or non-terminal alphabets. It is customary to use numbers for non-terminals, if letters have been used in the non-terminal alphabet, and vice versa; or, alternately, to use upper case letters for non-terminals and lower-case letters for terminals (if a suitable output printer is available).

If the sample is to be read in from cards (and this is under the control of the variable CONT), then the fourth card is a card containing the value of N, the length of the sample, in card columns 1-5. The fifth and succeeding cards now contain the sample, 80 symbols to a card. The size of the sample includes the initial punctuation in the sample, if any.

The following is an example of input to the computer program implementing the Grammar Discovery Algorithm:

```
    3    2    5   -1  0.5  0.1 1 1 2 1 4 1 2 0 1 2 1 1 0 0 1 1 4 0 1   1
    201
   26ABCDEFGHIJKLMNOPQRSTUVWXYZ
  272
11100110110010001100010111010101C1C1CCCCC011001C011101C1010100000011100011110010000
11000101C1000000011000011111010111C11000100110000101100010101000000011C1011011C00110
010000CC1100010CC111001C011010100110000011101010101000000110001011110010111100101
11010101111000111111000100100000
```

This input would be interpreted as follows:

```
MAXIMUM NUMBER OF LEVELS TO BE TRIED                3
SMALLEST M TO BE TRIED                              2
LARGEST M TO BE TRIED                               5
DIRECTION OF CONSIDERING M IN RECODE -              DESCEND
LAMBDA -- COEFFICIENT OF PARSIMONY                0.50000
LAMR -- COEFFICIENT OF RECURSIVE PARSIMONY        0.10000
SOURCE OF INITIAL SYMBOL STRING                     READ
MODE -- INITIAL PUNCTUATION MARK                    1ST
TYPE OF GRAMMAR DESIRED (MASKS TO BE TRIED)         LS-RS
RIT - CONTROLS ANTECEDENT SIDE OF RULES             CF
WORST  TYPE OF P(I) ALLOWED TO ENTER CODE           4
USE OF BEST M STRICTLY                              YES
FR -- RADIX OF MASK                                 2
PRINT CONTROL -- PRINT UNSORTED P(I)                NO
PRINT CONTROL -- PRINT SORTED P(I)                  YES
PRINT CONTROL -- PRINT GRAPH OF P(I)-S              ALL
PRINT CONTROL -- PRINT SEQUENCE NUMBERS             YES
SUBROUTINE USED FOR RECODING                        RECODE
OVERRIDE VALUE FOR MBEST (MCH)                      0
EARLY ELIMINATION OF UNALLOWABLE P(I)               NO
LAWL                                                YES
START RECODE AT TIME OF MBEST                       YES
HSEL -- METHOD FOR COMPUTING H                      4
CUT-OUT ON FINDING GOOD M                           NO
METHOD OF FINDING RECURSIONS                        RULE
WXL                                                 NO
TEST FOR RECURSION                                  YES

SIZE OF TERMINAL ALPHABET (INITIAL STRING)          2
TERMINAL ALPHABET:                              01

SIZE OF BASIC NON-TERMINAL ALPHABET                26
SYMBOLS OF THE BASIC NON-TERMINAL ALPHABET: ABCDEFGHIJKLMNOPQRSTUVWXYZ
```

APPENDIX B: LISTING OF THE COMPUTER PROGRAM IMPLEMENTING THE GRAMMAR DISCOVERY ALGORITHM

```
C
C         GRAMMAR DISCOVERY PROGRAM
C
      IMPLICIT INTEGER ( A-Z)
C
C         REAL SCALARS
      REAL CRIT
      REAL CX
      REAL FPS
      REAL FC
      REAL HBEST
      REAL HP
      REAL LAMBDA
      REAL LAMR
      REAL SEPAR
      REAL TT
C
C         REAL FUNCTIONS
      REAL ALOG
C
C         VECTORS OF LENGTH NS
      DIMENSION Y(500)
      DIMENSION ALPHA(500)
      DIMENSION YNEW(500)
      DIMENSION SECND(500)
      INTEGER DONE(500)
C
C         VECTORS OF LENGTH MTSIZE
      REAL P(500)
      REAL PD(500)
      DIMENSION SQCTXT(500)
      DIMENSION SQLEN(500)
      DIMENSION SQPRED(500)
      DIMENSION TYPE(500)
      DIMENSION INDEX(500)
      DIMENSION INDEP(500)
      DIMENSION TBPHI(500)
      DIMENSION TBNEXT(500)
      DIMENSION TBCNT(500)
C
C         VECTORS OF LENGTH M2MAX
      REAL H(20)
      DIMENSION MASK(20)
      DIMENSION PRE(20)
      DIMENSION POST(20)
      REAL FIRST(20)
      REAL SECOND(20)
      REAL THIRD(20)
      INTEGER BI(20)
      INTEGER BII(20)
C
C         VECTORS OF LENGTH N40
      DIMENSION SCRJ(40)
      DIMENSION SCRJAL(40)
      DIMENSION SCRD(40)
      DIMENSION SCRDAL(40)
```

```
      DIMENSION SCRMAL(40)
C
C
C        VECTORS OF LENGTH  RPMAX
      INTEGER LLIST(50)
      INTEGER RPALEN(50)
      INTEGER RPCLEN(50)
      INTEGER RPLEV(50)
      INTEGER RPIX(50)
      INTEGER RTIX(50)
      REAL POX(50)
      INTEGER DESC(50)
C
C        ARRAYS OF SIZE  RPMAX * M2MAX
      INTEGER RPA(50,20)
      INTEGER RPC(50,20)
      INTEGER RPAM(50,20)
C
C        VECTORS OF LENGTH YRPMAX
      INTEGER YRP(300)
C
C        VECTORS OF SIZE MNDC
      REAL HDJ(500)
      INTEGER ILCLEN(500)
C
C        OTHER VECTORS
      DIMENSION NUMT(5)
      DIMENSION LETTER(75)
      DIMENSION CHAR(78)
      DIMENSION MARKER(3)
      DATA MARKER / '#', '_', '%' /
      DIMENSION BLANKS(5)
      DATA BLANKS / ' ', ' ', ' ', ' ', ' '  /
      DIMENSION ARROW(12)
      DATA ARROW / ' ', ' ', ' ', '-', '-', '-', '-', '-',   '>', ' ',
     1   ' ', ' '   /
      DATA SSS / 'S' /
      REAL*8 NAME(5)
      DATA NAME/'URS','CSG','LS-RS','CFG','RG'/
      INTEGER SOURCE(3)
      DATA SOURCE/'READ','GEN2','GEN3'/
      REAL*8 WAYS(2)
      DATA WAYS/'RECODE','HUFFMAN'/
      REAL*8 TTQ(2)
      DATA TTQ / 'RULE', 'UVWXY' /
      INTEGER TRI(3)
      DATA TRI/'NONE','SOME','ALL'/
      INTEGER ANSW(2)
      DATA ANSW/'NO','YES'/
      INTEGER ORD(2)
      DATA ORD/'1ST','2ND'/
      REAL*8 DIRECT(3)
      DATA DIRECT /'DESCEND', '  ',  'ASCEND'  /
      REAL*8 ANTE(2)
      DATA ANTE/'CONTEXT','CF'/
C
C
C        COMMON STORAGE
C
      COMMON           C1,CONT,L3,M1,M2, PR1, PR2, FR  , LAMBDA,  RIT ,
```

```
      1  MCH, MODE, ALLOW  , STRICT,EARLY,GRAM  ,       NCHAR ,
      2         N26, PRG, EXR , PRS, LAWL, VSTART , VDIR , HSEL, WDF,
      3  WCV, WXL, KRECUR, LAMR
      4 , NS,N,MBE,C,MTSIZE,M2MAX,NQ,TRIAL,CONT,PUNC,LEVEL,N40,FC,NST,
      5  PERC,CTXT,TN,   NTM,RPMAX,NRP,NY,YRPMAX,NRECUR,YRARRW,YRPUNC
      1 , NTP
C
C
C
C       VECTORS AND SCALARS....
C
C         ALLOW=THE WORST TYPE OF P(I) ALLOWED IN THE RECODING
C         ALPHA(T)=ALPHABETIC VERSION OF Y
C         ANSW=ALPHABETIC VECTOR FOR YES-NO PRINTING
C         ANTE=ALPHABETIC VECTOR FOR PRINTING ANTECEDENT TYPES
C         ARROW=ALPHABETIC ARROW
C         BI,BII=VECTORS FOR  TESTING ISOMORPHISM OF RULES
C         BLANKS=ALPHABETIC BLANKS
C         C = # OF DIFFERENT SYMBOLS IN SYMBOL STRING Y AT THIS LEVEL
C           INCLUDES THE SYMBOLS OF MARKER
C           EXCLUDES INITIAL PUNCTUATION MARK (.), IF ANY
C           THIS  INCLUDES THE TERMINALS AND THE NON-TERMINALS
C         C1= # OF DIFFERENT TERMINAL SYMBOLS IN INITIAL STRING Y
C           EXCLUDES THE SYMBOLS OF MARKER
C           EXCLUDES INITIAL PUNCTUATION MARK (.), IF ANY
C         CW=# OF TERMINAL AND NON-TERMINAL SYMBOLS IN CURRENT STRING
C           EXCLUDES THE SYMBOLS OF MARKER
C           EXCLUDES INITIAL PUNCTUATION MARK (.), IF ANY
C         CX= REAL CW
C         CHAR=VECTOR OF NCHAR ALPHABETIC CHARACTERS
C         CONT=CONTROLS METHOD OF PRODUCING INITIAL STRING
C           1=USE GEN1   2=USE GEN2   3=USE GEN3
C         CRIT=THRESHOLD CRITERION FOR H
C         DESC=TYPES OF P(I)'S BEING GRAPHED BY 'GRAPH'
C         DIRECT=ALPHABETIC VECTOR FOR PRINTING DIRECTIONS
C         DONE(I):  0=POSITION I OF Y IS NOT YET RECODED   1=IS DONE
C         EARLY= CONTROLS TIME OF CULLING OUT SMALL P(I)'S
C           1=EARLY CULLING     0=NO EARLY CULLING
C         EPS= EPSILON USED TO SEPARATE TYPE 2 AND TYPE 3 P(I)'S
C         FC=REAL CONSTANT TO CONVERT TO BASE 2 LOGARITHMS
C         FIRST(I)=FIRST TERM OF COMBINED H MEASURE
C         FR=RADIX OF MASKS   2=BINARY   3=TERNARY
C         GRAM=TYPE OF GRAMMAR DESIRED
C           0=UNRESTRICTED REWRITE
C           1=CONTEXT SENSITIVE
C           2=LEFT-SENSITIVE AND RIGHT-SENSITIVE
C           NB: IF CONTEXT-FREE RULES ARE DESIRED, SET RIT=1
C           IN THIS EVENT, GRAM CONTROLS TYPE OF MASKS TO BE TRIED
C         H(M)=COMBINED ENTROPY AND PARSIMONY MEASURE
C         HBEST= BEST VALUE OF H FOR THIS LEVEL
C         HP=REAL CONSTANT FOR CUMULATING H MEASURE
C         HDJ(I)=THE ENTROPY OF THE I-TH DIFFERENT CONTEXT
C           (SUMMED OVER THE DIFFERENT PREDICTED PARTS)
C         ILCLEN(I)=NUMBER OF DIFFERENT PREDICTED PARTS ASSOCIATED
C           WITH CONTEXT I IN ILC ARRAY
C         INDEP=INDEX VECTOR FOR SORTING PD INTO DESCENDING ORDER
C         INDEX=INDEX VECTOR FOR SORTING P INTO DESCENDING ORDER
C         KRECUR:  1=CHECK FOR RECURSION    0=DON'T CHECK
C         L3=NUMBER OF LEVELS TO BE TRIED
C         LAMBDA=COEFFICIENT OF PARSIMONY
```

```
C       LAMR=COEFFICIENT OF RECURSIVE PARSIMONY
C       LAWL=CONTROLS COMPUTATION OF PROBABILITIES OF OCCURRENCE
C         0=DONT  1=DO
C       LETTER=VECTOR OF NON-TERMINAL SYMBOLS
C       LEVEL= TYPE LEVEL. PROCESS BEGINS AT LEVEL=1 (INITIAL Y)
C       LLIST=LIST OF L'S USED TO CATALOG RULES OF PRODUCTION
C       M=TEST SEQUENCE LENGTH
C       M1,M2=RANGE OF VALUES OF M TO BE CONSIDERED
C         M1 >=  2
C         M2<= M2MAX
C         FOR REGULAR GRAMMARS (TYPE 3), SET M1=M2=2
C       M2MAX=DIMENSIONED SIZE OF VECTORS VARYING OVER VALUES OF M
C       MARKER=ALPHABETIC VECTOR = RANGE OF MASK
C         #=DON'T CARE SYMBOL   _=CONTEXT PART     %=PREDICTED PART
C       MASK(M)=VECTOR WHOSE VALUES RANGE OVER THE SET  MARKER
C           MASK:        #     _     %
C           PRE:         Y     Y     %
C           POST:        #     _     Y
C       MCH=OVERRIDE VALUE OF MBEST FOR LEVEL 1    0=NO OVERRIDE
C       MINM=MINIMUM M THAT PRODUCES A RECODING THAT USES AT LEAST
C         ONE RULE OF PRODUCTION
C       MNOC=DIMENSIONED MAXIMUM NUMBER OF DIFFRENT CONTEXTS
C       MODE= INITIAL PUNCTUATION CONTROL
C         1=NO INITIAL PUNCTUATION MARK
C         2=CHAR(C1+4) IS THE INITIAL PUNCTUATION MARK
C       MSC=DIMENSIONED MAXIMUM NUMBER OF PREDICTED PARTS FOR EACH
C         CONTEXT
C       MTSIZE=DIMENSIONED SIZE OF VECTORS CATALOGING SEQUENCES
C         (SUCH AS SQCTXT, SQPRED, SQLEN, TYPE, P, PD, INDEX, LLIST)
C       N=LENGTH OF Y
C       N26=SIZE OF  LETTER
C       N40=DIMENSIONED SIZE OF SYMBOL STRINGS
C       NAME=ALPABETIC VECTOR OF NAMES OF GRAMMAR TYPES
C       NCHAR= SIZE OF VECTOR  CHAR
C         INCLUDES INITIAL PUNCTUATION (IF ANY)
C         INCLUDES INITIAL PUNCTUATION (IF ANY)
C       NOC=NUMBER OF DIFFERENT CONTEXTS
C       NEPS= # OF P(I)'S WITHIN EPS OF 1.0 (TYPE 2)
C       NMESS= # OF MESSAGE P(I)'S (TYPES 2,3,4)
C       NMASK= # OF PERMISSIBLE MASKS
C       NNOISY= # OF P(I)'S BETWEEN EPS AND 0.0 (NOISE) (TYPE 5)
C       NQ=# OF RULES OF PRODUCTION AT THIS LEVEL
C       NRECUR=# OF RECURSIVE RULES OF PRODUCTION AT THIS LEVEL
C       NS=DIMENSIONED SIZE OF Y
C       NSTRU= # OF P(I)'S EXACTLY 1.0 (STRUCTURAL) (TYPE 1)
C       NST=SIZE OF VECTORS CATALOGING SEQUENCES
C         (SUCH AS SQCTXT, SQPRED, SQLEN, TYPE, P, PD, INDEX, LLIST)
C       NT3= # OF P(I)'S BETWEEN 1.0-EPS AND SEPAR (TYPE 3)
C       NT4= # OF P(I)'S BETWEEN SEPAR AND EPS (TYPE 4)
C       NTB=NUMBER OF SEQUENCES CATALOGED FOR CURRENT M
C         (SIZE OF TBCNT, TBPHI, TBNEXT )
C       NTM=# OF TIMES RULES OF PRODUCTION ARE APPLIED IN RECODE
C       NRP=VERSION OF NTP IN SUBROUTINE RECODE
C       NTP=ACTUAL # OF RULES IN RPA, RPC
C       NUMT=VECTOR OF COUNT OF TYPES OF P(I)'S
C       NY=# OF SYMBOLS IN YRP
C       NZ= # OF P(I)'S FOR CURRENT M
C       ORD=ALPHABETIC VECTOR FOR PRINTING MODES
C       P(I)=CONDITIONAL PROBABILITY THAT THE CONTEXT PART OF THE
C         SEQUENCE I PREDICTS THE PREDICTOR PART OF THE SEQUENCE
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C         PD(I)=COPY OF P WHICH IS SORTED INTO DESCENDING ORDER
C           ALSO USED TO SORT P(I)'S THAT ARE USED IN RECODING
C         PDX(I): CONDITIONAL PROBABILITY OF SEQUENCE PAIR THAT
C           PRODUCED THIS RULE OF PRODUCTION
C         PR1=PRINT CONTROL FOR UNSORTED P(I)'S  0=DON'T PRINT  1=PRINT
C         PR2=PRINT CONTROL FOR SORTED P(I)'S    0=DON'T PRINT  1=PRINT
C         PRG: PRINT CONTROL FOR GRAPH OF P(I)'S USED IN RECODING
C             0=DON'T PRINT GRAPH
C             1=PRINT SOME GRAPHS --- ONLY WHEN TRIAL=2
C             2=PRINT ALL GRAPHS
C         PRS: CONTROLS PRINTING OF SEQNO   0=DON'T PRINT  1=PRINT
C         PRE(M)=M-VECTOR  OF CONTEXT PART OF SEQUENCE
C         POST(M)=M-VECTOR OF PREDICTED PART OF SEQUENCE
C         POX(I)=PROBABILITY ASSOCIATED WITH RULE OF PRODUCTION I
C         RIT=CONTROLS FORM OF ANTECEDENT (LEFT) SIDE OF
C           RULE OF PRODUCTION
C           1=(CONTEXT-FREE)= REPLACE ENTIRE PREDICTOR AND PREDICTED
C             SEQUENCES WITH ONE NEW NON-TERMINAL SYMBOL
C           0=REPLACE ALL CONTIGUOUS PREDICTOR SYMBOLS WITH
C             NEW NON-TERMINAL SYMBOL
C         RPA(I,J)=J-TH SYMBOL OF ANTECEDENT SIDE OF I-TH RULE OF PROD
C         RPC(I,J)=J-TH SYMBOL OF CONSEQUENT SIDE OF I-TH RULE OF PROD
C         RPALEN(I)=LENGTH OF ANTECEDENT SIDE OF I-TH RULE OF PRODUCTION
C         RPCLEN(I)=LENGTH OF ACONSEQUENT SIDE OF I-TH RULE OF PROD
C         RPLEV(I)=LEVEL OF RULE I
C         RPAM(I,J)=RECORD OF PREDICTED PARTS FOR RULE I
C         RPMAX=DIMENSIONED MAX # OF RULES OF PRODUCTION IN RPA AND RPC
C         RPIX(I): STATUS OF RULE I IN RPA AND RPC TABLES
C           0=RULE I HAS BEEN MADE INTO A RECURSIVE RULE AND DEACTIVATED
C           -2=NON-RECURSIVE RULE
C           POSITIVE # =RECURSIVE RULE. THE NUMBER POINTS TO THE
C                       ORIGINAL NON-RECURSIVE RULE FROM WHICH IT WAS
C                       DERIVED
C         RTIX(I)=STORAGE FOR RPIX OUTSIDE OF RECODE SUBROUTINE
C         STRUCTURE:    RPA(I,J), RPC(I,J), RPALEN(I), RPCLEN(I),
C                       RPLEV(I), RTIX(I), RPIX(I), POX(I)
C         SCRD,SCRJ=SCRATCH VECTORS
C         SCRDAL,SCRJAL,SCRMAL=ALPHABETIC SCRATCH VECTORS
C         SECOND(I)=SECOND TERM OF COMBINED H MEASURE
C         SEPAR=VALUE SEPARATING TYPE 3 AND TYPE 4 P(I)'S
C         SEQNO(T) = IS INDEX OF ENTRY IN SQCTXT-TABLE CONTAINING PHI OF
C            LONGEST STRUCTURAL SEQUENCE BEGINNING AT TIME T
C         SOURCE=ALPHABETIC VECTOR OF SOURCES OF INPUT SAMPLE
C         SQCTXT(I)= PHI OF THE CONTEXT PART OF THE SEQUENCE
C         SQPRED(I)= PHI OF THE PREDICTED SEQUENCE
C         SQLEN(I)= LENGTH OF STRUCTURAL SEQUENCE I
C         SSS=ALPHABETIC 'S'
C         STRICT:  CONTROLS VARIETY OF M'S ALLOWED IN RECODING
C           0=USE M1...MBEST    1=USE MBEST ONLY
C         T IS TIME
C         TBCNT I)=TEMPORARY COUNT OF OCCURRENCES  OF SUB-SEQUENCES
C         TBPHI(I)=TEMPORARY CONTEXT FOR SUB-SEQUENCES
C         TBNEXT(I)=TEMPORARY PREDICTED PART FOR SEQUENCES
C         THIRD(I)=THIRD TERM OF COMBINED H MEASURE
C         TT= # OF OCCURRENCES OF SEQUENCES PHI (REAL)
C           REPRESENTING PARTICULAR CONTEXT
C         TRI=ALPHABETIC VECTORS FOR PRINTING
C         TTQ=ALPHABETIC VECTOR OF WAYS OF INDUCING RECURSIONS
C         TYPE(I)=THE TYPE OF SEQUENCE I (1,2,3,4,5)
C             TYPE 1 (STRUCTURAL)  P(I) EXACTLY 1.0
```

```
C            TYPE 2 (MESSAGE)      P(I) WITHIN EPS OF 1.0
C            TYPE 3 (MESSAGE)      P(I) BETWEEN 1.0-EPS AND SEPAR
C            TYPE 4 (MESSAGE)      P(I) BETWEEN SEPAR AND EPS
C            TYPE 5 (NOISE)        P(I) BETWEEN EPS AND 0.0
C         VDIR:  CONTROLS ASCENDING OR DESCENDING DIRECTION OF
C           CONSIDERATION OF M'S IN RECODE    -1=DESCENDING  +1=ASCENDING
C         VSTART:  BEGIN SCAN IN RECODE AT TIME OF MBEST FOR EACH SENT
C            (FOR MODE=2, ONLY)   0=NO     1=YES
C         WAYS=ALPABETIC VECTOR OF CODING SUBROUTINES
C         WOV: 1=USE DEFINITIONAL APPROACH TO FIND RECURSIONS
C           2=USE UVWXY APPROACH
C         WDF: EFFECT OF SATISFYING CRITERION OR FINDING COMPLETE
C           RESOLUTION FOR A PARTICULAR M
C           0=NO EFFECT    1=CUT-OUT AND USE THIS  M
C         WXL: INCLUDE IDENTITY RULES (AND FACTOR OF M IN H)
C           0=NO 1=YES
C         Y(T)= LARGE SAMPLE OF CONCATENATED SENTENCES
C         YNEW(T)= NEW Y STRING CREATED BY RECODING
C         YRP(I)=STRING CONTAINING RULES OF PRODUCTION IN CONTIGUOUS
C           FORM. CONTAINS YRARRW BETWEEN ANTECEDENT AND CONSEQUENT
C           SIDES. CONTAINS YRPUNC BETWEEN RULES.
C         YRPMAX=DIMENSIONED MAX # OF SYMBOLS IN YRP
C
C
C      SUBROUTINES....
C
C         BEST  FINDS BEST H
C         CONV  CONVERTS A NATURAL NUMBER INTO A C-ARY SEQUENCE
C         FZERO=ZEROES FLOATING POINT VECTOR
C         GEN1  READS INITIAL SYMBOL STRING FROM CARDS
C         GEN2  IS SPECIAL BINARY SEQUENCE GENERATOR
C         GETMSK  GENERATES MASKS
C         GRAPH  PRINTS GRAPH OF P(I)'S USED IN RECODING
C         GRID  PRINTS OUT MASKS
C         HUFF  IS HUFFMAN CODE SUBROUTINE
C         IEQUAL=SETS VECTORS EQUAL
C         IMATCH=MATCHES INTEGER VALUES
C         ISEEK DOES MATCHING
C         ISUM  FINDS SUM OF VECTOR
C         IXSPRY=SPRAYS INTEGER CONSTINT
C         LOC  LOCATES  SEQUENCES THAT OCCUR IN SYMBOL STRING
C         LOOK   KEEPS LIST OF NEW NON-TERMINAL SYMBOLS AT EACH LEVEL
C         MORE  CREATES MASK FROM TWO SEQUENCES (PRE AND POST)
C         MTEST  TESTS GRAMMATICAL TYPE OF MASK
C         PHI  CONVERTS A C-ARY SEQUENCE INTO THE NATURAL NUMBER
C         RANDU  IS RANDOM INTEGER GENERATOR
C         RECODE  DOES RECODING FROM ONE STRING TO STRING OF HIGHER LEVEL
C         SPRAY  SETS VECTOR TO CONSTANT
C         SYMBOL  PRODUCES SYMBOL STRING FOR PRINTING
C         TLOC  DETERMINES WHICH SEQUENCES APPEAR AT A GIVEN TIME IN Y
C         XFSORT  SORTS BLOCK OF REALS WITHIN ARRAY
C         ZERO  SETS VECTOR TO ZEROES
C
C
C
C
C      INPUT AND OUTPUT...
C           5=INPUT CARDS
C           6=OUTPUT PRINTER
C           7=OUTPUT FILE CONTAINING SUMMARY OF PRINTER OUTPUT
```

```
C
C
C
C        DIMENSIONED SIZES OF VECTORS
C
      NS=500
      M2MAX=20
      MTSIZE=500
      N26=26
      N40=40
      RPMAX=50
      YRPMAX=300
      MNDC=500
      MSC=15
C
C        MAIN LOOP
C
110   CONTINUE
      WRITE(6,790)
790   FORMAT ('1' )
C
C
C
C        READ CONTROL PARAMETERS
C
C
C
C       INPUT CARDS...
C
C         1. CONTROL CARD:
C             4I5:  L3, M1, M2, VDIR
C             2F5.2:  LAMBDA, LAMR
C             25I2: CONT,MODE,GRAM,RIT,ALLOW,STRICT,FR,PR1,PR2,PRG,PRS,
C                   EXR,MCH,EARLY,LAWL,VSTART,HSEL,WDF,WCV,WXL,
C                   KRECUR
C         2. TERMINAL ALPHABET VECTOR ( FORMAT I5, 75A1 )
C             C1=SIZE OF NON-TERMINAL ALPHABET
C             CARD COLUMNS 6...  CONTAIN THE C1 TERMINAL SYMBOLS
C             IF MODE=2, LAST COLUMN AFTER THE TERMINALS CONTAINS
C               THE INITIAL PUNCTUATION MARK (.)
C         3. NON-TERMINAL ALPHABET VECTOR (FORMAT I5, 75A1)
C             N26=SIZE OF VECTOR IN FORMAT I5
C             LETTER=ALPHABET IN FORMAT 75A1
C         4. IF CONT=1, GEN1 READS CARD CONTAINING LENGTH OF Y(FORMAT I5)
C            IF CONT=2, GEN2 READS CARD (FORMAT 16I5)
C         5. IF CONT=1, GEN1 READS IN CARD(S) UNTIL N SYMBOLS HAVE BEEN
C            READ IN UNDER AN 80A1 FORMAT
C
C
      READ (5,4010) L3,M1,M2,VDIR,LAMBDA,LAMR,CONT,MODE,GRAM,RIT,ALLOW,S
     1TRICT,FR,PR1,PR2,PRG,PRS,EXR,MCH,EARLY,LAWL,VSTART,HSEL,WDF,WCV,WX
     2L,KRECUR
4010  FORMAT (4I5,2F5.2,25I2)
C
C
      WRITE (6,4020)
      WRITE (7,4020)
4020  FORMAT ('1')
      WRITE (6,4030) L3,M1,M2,VDIR,LAMBDA,LAMR,CONT,MODE,GRAM,RIT,ALLOW,
     1STRICT,FR,PR1,PR2,PRG,PRS,EXR,MCH,EARLY,LAWL,VSTART,HSEL,WDF,WCV,W
```

```
     2XL,KRECUR
4030  FORMAT (1X,4I5,2F5.2,25I2)
C
C
      WRITE (6,4040) L3,M1,M2,DIRECT(VDIR+2),LAMBDA,LAMR
      WRITE (7,4040) L3,M1,M2,DIRECT(VDIR+2),LAMBDA,LAMR
4040  FORMAT ( '0' /
     8    ' MAXIMUM NUMBER OF LEVELS TO BE TRIED          ', I7 /
     4    ' SMALLEST M TO BE TRIED                        ', I7 /
     5    ' LARGEST M TO BE TRIED                         ', I7     /
     8    ' DIRECTION OF CONSIDERING M IN RECODE          ', 6X, A8 /
     9    ' LAMBDA -- COEFFICIENT OF PARSIMONY            ', F10.5 /
     9    ' LAMR -- COEFFICIENT OF RECURSIVE PARSIMONY    ', F10.5
     7    )
      WRITE (6,4050) SOURCE(CONT),ORD(MODE),NAME(GRAM+1),ANTE(RIT+1),ALL
     1OW,ANSW(STRICT+1),FR,ANSW(PR1+1),ANSW(PR2+1),TRI(PRG+1),ANSW(PRS+1
     2),WAYS(EXR),MCH,ANSW(EARLY+1),ANSW(LAWL+1),ANSW(VSTART+1),HSEL
      WRITE (7,4050) SOURCE(CONT),ORD(MODE),NAME(GRAM+1),ANTE(RIT+1),ALL
     1OW,ANSW(STRICT+1),FR,ANSW(PR1+1),ANSW(PR2+1),TRI(PRG+1),ANSW(PRS+1
     2),WAYS(EXR),MCH,ANSW(EARLY+1),ANSW(LAWL+1),ANSW(VSTART+1),HSEL
4050  FORMAT (
     2    ' SOURCE OF INITIAL SYMBOL STRING               ', 6X, A4 /
     3    ' MODE -- INITIAL PUNCTUATION MARK              ',  6X,A4  /
     2    ' TYPE OF GRAMMAR DESIRED (MASKS TO BE TRIED)   ', 6X,A8 /
     1    ' RIT - CONTROLS ANTECEDENT SIDE OF RULES       ', 6X, A8 /
     1    ' WORST  TYPE OF P(I) ALLOWED TO ENTER CODE     ',I7  /
     8    ' USE OF BEST M STRICTLY                        ',   6X,A4  /
     2    ' FR -- RADIX OF MASK                           ', I7 /
     6    ' PRINT CONTROL -- PRINT UNSORTED P(I)          ', 6X, A4 /
     1    ' PRINT CONTROL -- PRINT SORTED P(I)            ', 6X, A4 /
     1    ' PRINT CONTROL -- PRINT GRAPH OF P(I)-S        ', 6X, A4 /
     5    ' PRINT CONTROL -- PRINT SEQUENCE NUMBERS       ', 6X, A4 /
     2    ' SUBROUTINE USED FOR RECODING                  ', 6X, A8    /
     2    ' OVERRIDE VALUE FOR MBEST (MCH)                ', I7 /
     1    ' EARLY ELIMINATION OF UNALLOWABLE P(I)         ', 6X, A4 /
     7    ' INCLUDE ALL-PREDICTED MASK                    ', 6X, A4 /
     5    ' START RECODE AT TIME OF MBEST                 ', 6X, A4 /
     7    ' HSEL -- METHOD FOR COMPUTING H                ', I7
     8    )
      WRITE (6,4060) ANSW(WDF+1),TTQ(WCV),ANSW(WXL+1),ANSW(KRECUR+1)
      WRITE (7,4060) ANSW(WDF+1),TTQ(WCV),ANSW(WXL+1),ANSW(KRECUR+1)
4060  FORMAT (
     1    ' CUT-OUT ON FINDING GOOD M                     ', 6X, A4 /
     2    ' METHOD OF FINDING RECURSIONS                  ', 6X, A8 /
     3    ' INCLUDE IDENTITY RULES                        ', 6X, A4 /
     4    ' TEST FOR RECURSION                            ', 6X,A4
     5  )
C
C
C        READ IN TERMINAL ALPHABET, INITIAL PUNCTUATION MARK (IF ANY),
C
      READ (5,4070) C1,(CHAR(I),I=4,78)
4070  FORMAT (I5,75A1)
      NCHAR=C1+3+(MODE-1)
C
C        PRINT OUT TERMINAL ALPHABET
      NCX=NCHAR-(MODE-1)
      WRITE (6,4080) C1,(CHAR(I),I=4,NCX)
      WRITE (7,4080) C1,(CHAR(I),I=4,NCX)
4080  FORMAT (   ' ' /
```

```
     1   ' SIZE OF TERMINAL ALPHABET (INITIAL STRING)  ', I7 /
     2   ' TERMINAL ALPHABET:                          ', 75A1
     4   )
      IF (MODE.EQ.2) WRITE (6,4090) CHAR(NCHAR)
      IF (MODE.EQ.2) WRITE (7,4090) CHAR(NCHAR)
4090  FORMAT    (  ' INITIAL PUNCTUATION MARK (MODE=2)            ',A1)
C
C
C        READ IN NON-TERMINAL ALPHABET
C
      READ (5,4070) N26,(LETTER(I),I=1,N26)
C
C        PRINT OUT BASIC NON-TERMINAL ALPHABET
C
      WRITE (6,4100) N26,(LETTER(I),I=1,N26)
      WRITE (7,4100) N26,(LETTER(I),I=1,N26)
4100  FORMAT (  ' ' /
     1   ' SIZE OF BASIC NON-TERMINAL ALPHABET         ', I7 /
     2   ' SYMBOLS OF THE BASIC NON-TERMINAL ALPHABET: ', 75A1    )
C
C        ESTABLISH CONVENTIONS FOR REPRESENTING SYMBOLS
C
C            SYMBOLIC  NUMERIC
C            FORM      FORM
C            (ALPHA)   (Y)
C
C            #         0          DON'T CARE SYMBOL
C            _         1          CONTEXT PART
C            %         2          PREDICTED PART
C            .....     3...C1+2   THE C1 TERMINAL SYMBOLS
C            .         C1+3       THE INITIAL PUNCTUATION MARK, IF ANY
C
C
      DO 115 I=1, 3
      CHAR(I)=MARKER(I)
115   CONTINUE
      C=NCHAR
      DONT=0
      CTXT=1
      PERC=2
      PUNC=NCHAR-1
      YRARRW=DONT
      YRPUNC=PERC
C
C
C
C        OBTAIN INITIAL Y FOR LEVEL 1
C
      GO TO (130,140,150), CONT
C        GEN1: READ SYMBOL STRING FROM CARDS
130   CONTINUE
      CALL GEN1 (Y,N,NS,ALPHA,CHAR,NCHAR)
      GO TO 180
C        GEN2:  SPECIAL BINARY SEQUENCE GENERATOR
140   CONTINUE
C     CALL       GEN2 ( STRING,N,NS,MODE,NCHAR,PUNC)
      GO TO 180
C        GEN3:GENERAL SEQUENCE GENERATOR
150   CONTINUE
      GO TO 180
```

```
180   CONTINUE
C
C        INITIALIZE FOR ALL LEVELS
C
      LEVEL=1
      CATCH=1
      NRP=0
      NTP=0
      NY=0
C
C        CONSIDER VARIOUS VALUES OF  LEVEL=1,...,L3
190   CONTINUE
C
C
C        INITIALIZE FOR THIS LEVEL
C
      MINM=0
      NST=0
      NZ=0
      NDC=0
      DO 192 II=1, MNDC
      HDJ(II)=0.0
192   CONTINUE
C
C        PRINT OUT Y
C
      CW=C-3-(MODE-1)
      WRITE(6,200) LEVEL, N, CW
      WRITE(7,200) LEVEL, N, CW
200   FORMAT(//'1---------------------------  LEVEL', I2,
     9   '  ---------------------------------'   /  /
     2   ' SYMBOL STRING OF LENGTH ', I3,
     3   ' AND USING ALPHABET OF SIZE ', I3  /  )
      CALL SYMBOL (Y,ALPHA,N,   MMA,CHAR,NCHAR,LETTER,N26)
      WRITE (6,220) (ALPHA(III),III=1,MMA)
220   FORMAT ( '0', (/1X,60A1)  )
      WRITE (7,220) (ALPHA(III),III=1,MMA)
C
C        COMPUTE CRITERION FOR THIS LEVEL
C
      FC=ALOG(2.0)
      CX=C-3-(MODE-1)
      CRIT=-1.0   *ALOG(1.0/CX)/FC
      WRITE (6, 225) CRIT
225   FORMAT(//'0CRITERION = ', F10.5  / )
C
C
C        CONSIDER VARIOUS M = M1 ... M2
C
      M=M1
230   CONTINUE
      WRITE (6,240) LEVEL,M
240   FORMAT ('1','LEVEL=',I4,5X,'M=',I4////)
      IF ( PR1 .NE. 0 )   WRITE(6,544)
C
C        INITIALIZE FOR THIS M
C
      NTB=0
      H(M)=9999.
      NSTRU=0
```

```
      NEPS=0
      NT3=0
      NT4=0
      NNOISY=0
      CALL ZERO( TBCNT, NS)
      CALL SPRAY( TBNEXT, NS, -1 )
      CALL SPRAY( TBPHI, NS, -1 )
C
C        EPS AND SEPAR
      SEPAR=0.50
      EPS=1.0/M
      IF(EPS.GE. .34) EPS=.34
C
C
C        CONSIDER VARIOUS MASKS
C
C        CONVENTIONS FOR MASK, PRE, POST  VECTORS
C          #=DON'T CARE SYMBOL   _=CONTEXT PART     %=PREDICTED PART
C
C             MASK:        #    _    %
C             PRE:         Y    Y    %
C             POST:        #    _    Y
C
      NMASK=0
      SIND=0
262   CONTINUE
      CALL       GETMSK( M, FR, GRAM, NMASK, CTXT, PERC, DONT, M2MAX,
     1  MASK, IPART, &421 )
263   CONTINUE
C
C
C        CONSIDER ALL T FROM T=M TO T=N
C          FOR CURRENT VALUE OF M, CONSIDER M-SEQUENCES STARTING AT
C            TIME T=T-M+1 TO TIME T=T
C
      DO 400 T=M,N
C
C        FORM PRE AND POST SEQUENCES
C
      DO 350 L=1,M
      JD=L
      TMJD=T-M+JD
      K=M+1-L
C
C        IF M-SEQUENCE CONTAINS PUNCTUATION MARK, SKIP OVER IT
      IF (MODE.EQ.2.AND.Y(TMJD).EQ.PUNC) GO TO 390
      IF ( MASK(K) .EQ. DONT ) PRE(L)=Y(TMJD)
      IF ( MASK(K) .EQ. DONT ) POST(L) = DONT
      IF ( MASK(K) .EQ. CTXT ) PRE(L)= Y(TMJD)
      IF ( MASK(K) .EQ. CTXT ) POST(L) = CTXT
      IF ( MASK(K) .EQ. PERC )    PRE(L)=PERC
      IF ( MASK(K) .EQ. PERC ) POST(L) = Y(TMJD)
350   CONTINUE
C
C        GET PHI'S FOR EACH SEQUENCE
      CALL PHI (PRE,M,C,XPRE)
      CALL PHI (POST,M,C,XPOST)
C
C        INSERT INTO TABLES
      IF (NTB.EQ.0) GO TO 370
```

```
      DO 360 I=1,NTB
      IF (XPRE.NE.TBPHI(I)) GO TO 360
      IF (XPOST.NE.TBNEXT(I)) GO TO 360
      TBCNT(I)=TBCNT(I)+1
      GO TO 380
360   CONTINUE
370   CONTINUE
      NTB=NTB+1
      TBCNT(NTB)=1
      TBPHI(NTB)=XPRE
      TBNEXT(NTB)=XPOST
380   CONTINUE
390   CONTINUE
C
C
C        END OF LOOP FOR THIS T
C
400   CONTINUE
C
C
C        END OF LOOP FOR THIS MASK
C
      IF(SIND.EQ.1) GO TO 429
      GO TO 262
C
C        ALSO USE THE ALL-PREDICTED MASK
C
421   CONTINUE
      IF(LAWL.NE.1) GO TO 428
      SIND=1
      NMASK=NMASK+1
      DO 422 IZ=1, M
      MASK(IZ)=PERC
422   CONTINUE
      GO TO 263
428   CONTINUE
429   CONTINUE
C
C
C        COMPUTE THE P(I)'S
C
      IF(NTB.EQ.0) GO TO 545
      NZ=0
C
C        MAIN LOOP FOR I=1, NTB
C
      DO 540 I=1,NTB
      IF (TBCNT(I).LE.0) GO TO 540
C
C        NEW AND DIFFERENT CONTEXT DISCOVERED
C        COMPUTE TT=COUNT OF ITS OCCRRENCES
C
      NDC=NDC+1
      TT=0.0
      DO 430 J=I,NTB
C
C        CONSIDER TABLE ENTRIES ONLY ONCE
      IF (TBCNT(J).LE.0) GO TO 430
C
C        MATCH CONTEXTS
```

```
      IF (TBPHI(I).NE.TBPHI(J)) GO TO 430
      TT=TT+TBCNT(J)
C
C        PREVENT FURTHER CONSIDERATION OF PREDICTED PARTS (OTHER THAN
C          FIRST) ASSOCIATED WITH THIS CONTEXT
      TBCNT(J)=-TBCNT(J)
430   CONTINUE
C
C        COMPUTE      THE P,SQCTXT,SQPRED,SQLEN, ETC. VALUES FOR THIS
C          PARTICULAR CONTEXT.  THAT IS, CONSIDER EACH PREDICTED PART
C          ASSOCIATED WITH THIS CONTEXT.
C
      ILCLEN(NDC)=0
      DO 520 J=I,NTB
C
C        CONSIDER TABLE ENTRIES ONLY ONCE
      IF (TBCNT(J).GE.0) GO TO 520
C
C        MATCH CONTEXT
      IF (TBPHI(I).NE.TBPHI(J)) GO TO 520
      NST=NST+1
      NZ=NZ+1
      P(NST)=-TBCNT(J)/TT
      HDJ(NDC)=HDJ(NDC) - P(NST) * ALOG(P(NST))
      TBCNT(J)=0
      ILCLEN(NDC)=ILCLEN(NDC)+1
      SQLEN(NST)=M
      SQPRED(NST)=TBNEXT(J)
      SQCTXT(NST)=TBPHI(J)
C
C
C        CLASSIFY SEQUENCES AS TO TYPE (1,2,3,4, OR 5)
C
      IF (P(NST).GE.1.0) GO TO 440
      IF (P(NST).GE.1.0-EPS) GO TO 450
      IF (P(NST).GE.SEPAR) GO TO 460
      IF (P(NST).GE.EPS) GO TO 470
      IF (P(NST).GE.0.0) GO TO 480
C
C        TYPE 1: P(I) EXACTLY 1.0     (STRUCTURE)
C
440   CONTINUE
      NSTRU=NSTRU+1
      TYPE(NST)=1
      GO TO 490
C
C        TYPE 2: P(I) WITHIN EPS OF 1.0
C
450   CONTINUE
      NEPS=NEPS+1
      TYPE(NST)=2
      GO TO 490
C
C        TYPE 3: P(I) BETWEEN 1.0-EPS AND SEPAR
C
460   CONTINUE
      NT3=NT3+1
      TYPE(NST)=3
      GO TO 490
C
```

```
C         TYPE 4: P(I) BETWEEN SEPAR AND EPS
C
470    CONTINUE
       NT4=NT4+1
       TYPE(NST)=4
       GO TO 490
C
C         TYPE 5: P(I) LESS THAN EPS    (NOISE)
C           NB: NO P(I)=0.0
C
480    CONTINUE
       NNOISY=NNOISY+1
       TYPE(NST)=5
       GO TO 490
490    CONTINUE
C
C
C         DELETE P(I)'S HERE THAT ARE TOO SMALL
C
       IF (.NOT.(EARLY.EQ.1.AND.TYPE(NST).GT.ALLOW)) GO TO 495
       NZ=NZ-1
       NST=NST-1
495    CONTINUE
C
C         PRINT UNSCRTED TABLE OF NON-ZERO AND DEFINED P'S
C
       IF (PR1.EQ.C) GO TO 510
       CALL CONV (SCRJ,M,C,SQCTXT(NST))
       CALL SYMBOL (SCRJ,SCRJAL,M,    MM,CHAR,NCHAR,LETTER,N26)
       CALL CONV (SCRD,M,C,SQPRED(NST))
       CALL SYMBOL (SCRD,SCRDAL,M,    MMQ,CHAR,NCHAR,LETTER,N26)
       CALL GRID ( SCRJ, SCRD, M, DCNT, CTXT, PERC, SCRMAL, MARKER )
       JJ=NST
       MJ=M
       WRITE(6,710) JJ,LEVEL,SQLEN(JJ),P(JJ),TYPE(JJ),
      1 (SCRMAL(JJH),JJH=1,MJ),BLANKS,(SCRJAL(KKJ),KKJ=1,MM),BLANKS,
      2 (SCRDAL(JKL), JKL=1,MMQ )
510    CONTINUE
520    CONTINUE
540    CONTINUE
C
C         PRINT STATISTICS ON TYPES OF SEQUENCES
C
545    CONTINUE
       WRITE(6,550) NSTRU, NEPS, NT3, NT4, NNOISY
550    FORMAT (
      1    'ONUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)', I7 /
      6    ' NUMBER OF TYPE 2 SEQUENCES (MESSAGE)  ', I7       /
      6    ' NUMBER OF TYPE 3 SEQUENCES (MESSAGE)  ', I7       /
      6    ' NUMBER OF TYPE 4 SEQUENCES (MESSAGE)  ', I7       /
      3    ' NUMBER OF TYPE 5 SEQUENCES (NOISE)    ', I7        )
       NPI=(C-3-(MODE-1))**M
       PSIZE=NPI*NMASK
       WRITE(6,570) NMASK, NPI, PSIZE, NZ, NCC, SEPAR, EPS
570    FORMAT (
      1    ' NUMBER OF DIFFERENT MASKS USED        ', I7 /
      2    ' NUMBER OF POSSIBLE M-SEQUENCES        ', I7 /
      2    ' NUMBER OF POSSIBLE P(I)               ',I7  /
      1    ' NUMBER OF P(I)                        ',I7 /
      3    ' NUMBER OF DIFFERENT CONTEXTS          ', I7 /
```

```
     3   ' SEPARATION VALUE BETWEEN TYPES 3 AND 4', F10.5  /
     4   ' EPSILON FOR DEFINING TYPES 2 AND 3    ', F10.5  /
     5   )
C
C
C        SORT THE P(I)'S INTO DESCENDING ORDER
C
      IF (NTB.EQ.0) GO TO 610
      DO 542 JK=1, NST
      PD(JK)=P(JK)
542   CONTINUE
      CALL XFSORT ( PD, MTSIZE, INDEX,  1, NST )
C
C        PRINT SORTED TABLE OF NON-ZERO AND DEFINED P'S
      IF ( PR2 .EQ. 0 ) GO TO 548
      WRITE (6,544)
544   FORMAT ('0      #   LEVEL  LENGTH    PROB    TYPE',7X,
     1  'MASK    CONTEXT    PREDICTED' / '0' )
      DO 546 L=1, NST
      JJ= INDEX(L)
      MJ=SQLEN(JJ)
      CALL CONV ( SCRJ,MJ, C, SQCTXT(JJ)   )
      CALL SYMBOL ( SCRJ, SCRJAL,MJ,     MM, CHAR , NCHAR ,LETTER,N26)
      CALL CONV ( SCRD,MJ, C, SQPRED(JJ)   )
      CALL SYMBOL ( SCRD, SCRDAL, MJ,     MMQ , CHAR, NCHAR ,LETTER,N26)
      CALL GRID ( SCRJ, SCRD, M, DONT, CTXT, PERC, SCRMAL, MARKER )
      WRITE(6,710) JJ,LEVEL,SQLEN(JJ),P(JJ),TYPE(JJ),
     1 (SCRMAL(JJH),JJH=1,MJ),BLANKS,(SCRJAL(KKJ),KKJ=1,MM),BLANKS,
     2 (SCRDAL(JKL), JKL=1,MMQ )
546   CONTINUE
548   CONTINUE
C
C
C        ATTEMPT RECODING, COMPUTE H FOR THIS M
C
      TRIAL=1
      MBE=M
      CALL       RECODE(Y,YNEW,SEQNO,ALPHA,  LLIST,SQCTXT,SQLEN,SQPRED,
     1  TYPE,      P,PD,INDEX,INDEP,       CHAR,  SCRJ,SCRJAL,SCRD,
     2 SCRDAL,  POST,BI,BII,PRE,MASK,H,FIRST,SECOND,THIRD,  LETTER ,
     3 RPALEN, RPCLEN,RPLEV,RFA,RPC,RPAM,YRP,NUMT,ARROW ,DONE   ,DESC
     4 , PCX, RPIX, RTIX )
C
C
C        COMPUTE NSUM, NMESS, MINM
C
      IF (ALLOW.EQ.1) NSUM=NSTRU
      IF (ALLOW.EQ.2) NSUM=NSTRU+NEPS
      IF (ALLOW.EQ.3) NSUM=NSTRU+NEPS+NT3
      IF (ALLOW.EQ.4) NSUM=NSTRU+NEPS+NT3+NT4
      IF (ALLOW.EQ.5) NSUM=NSTRU+NEPS+NT3+NT4+NNOISY
      NMESS=NZ-NNOISY-NSTRU
      IF(NTM.EQ.0) GO TO 600
      IF (MINM.EQ.0) MINM=M
600   CONTINUE
C
C
C        CHECK FOR (1) ALL SEQUENCES STRUCTURAL SEQUENCES,
C          (2) H(M) IS 0.0, OR (3) H(M) IS LESS THAN CRITERION
C
```

```
      IF(WDF.NE.1) GO TO 592
      IF(NUMT(1).NE.0 .AND. NUMT(2).EQ.0 .AND. NUMT(3).EQ.0
     1    .AND. NUMT(4).EQ.0    .AND. NUMT(5).EQ.0    ) GO TO 650
      IF(FIRST(M).EQ. 0.0 ) GO TO 650
      IF(H(M).LE.CRIT) GO TO 630
592   CONTINUE
C
C
C        END OF LOOP FOR M
C
610   CONTINUE
      M=M+1
      IF (M.LE.M2) GO TO 230
C
C
C
C        FIND BEST M --- LOCAL MINIMA AMONG THE H(M)
C
      CALL BEST(H,M1, M2, HBEST, MBEST, MINM, FIRST,SECOND,THIRD,M2MAX)
      WRITE (6,620) MBEST,H(MBEST)
      WRITE (7,620) MBEST,H(MBEST)
620   FORMAT (       '0BEST M IS ', I4, ' WITH H OF ', F10.5  )
      GO TO 670
C
C        CRITERION SATISFIED
C
630   CONTINUE
      MBEST=M
      WRITE (6,640) MBEST,CRIT,H(MBEST)
      WRITE (7,640) MBEST,CRIT,H(MBEST)
640   FORMAT ( '0M OF ', I4,   ' SATISFIES CRITERION OF ', F10.5,
     1  ' WITH  H OF ', F10.5  )
      GO TO 670
C
C        ALL P(I) ARE STRUCTURAL
C
650   CONTINUE
      MBEST=M
      WRITE (6,660) M,LEVEL
      WRITE (7,660) M,LEVEL
660   FORMAT(  '0M OF ', I4, ' GIVES COMPLETE RESOLUTION AT LEVEL',I2//)
C
C        OVERRIDE VALUE OF MBEST (FOR LEVEL 1 ONLY)
C
670   CONTINUE
      IF (.NOT.(MCH.NE.0.AND.LEVEL.EQ.1)) GO TO 690
      MBEST=MCH
      WRITE (6,680) MBEST
      WRITE (7,680) MBEST
680   FORMAT ( '0M OF ', I4, ' IS OVERRIDE VALUE ' )
690   CONTINUE
C
C
C        PRINT THE SQ-TABLES
C
      WRITE (6,700) LEVEL
700   FORMAT ( '1REGULARITIES FOR LEVEL ', I4 )
      WRITE(6,544)
      IF (NST.EQ.0) GO TO 730
      DO 720 L =1, NST
```

```
      I=INDEX(L)
      M=SQLEN(I)
      CALL CONV (SCRJ,M,C,SQCTXT(I))
      CALL SYMBOL (SCRJ,SCRJAL,M,   MM,CHAR,NCHAR,LETTER,N26)
      CALL CONV (SCRD,M,C,SQPRED(I))
      CALL SYMBOL (SCRD,SCRDAL,M,   MMQ,CHAR,NCHAR,LETTER,N26)
      CALL GRID ( SCRJ, SCRD, M, DCNT, CTXT, PERC, SCRMAL, MARKER )
      JJ=I
      MJ=M
      WRITE(6,710) JJ,LEVEL,SQLEN(JJ),P(JJ),TYPE(JJ),
     1 (SCRMAL(JJH),JJH=1,MJ),BLANKS,(SCRJAL(KKJ),KKJ=1,MM),BLANKS,
     2 (SCRDAL(JKL), JKL=1,MMQ )
710   FORMAT ( 3I8, F8.2, I8, 7X, 80A1 )
720   CONTINUE
730   CONTINUE
C
C
C        DO RECODING WITH M=MBE (BEST M)
C
      TRIAL=2
      MBE=MBEST
      CALL       RECODE(Y,YNEW,SEQNO,ALPHA,  LLIST,SQCTXT,SQLEN,SQPRED,
     1  TYPE,       P,PD,INDEX,INDEP,       CHAR,  SCRJ,SCRJAL,SCRD,
     2 SCRDAL,  POST,BI,BII,PRE,MASK,H,FIRST,SECOND,THIRD,  LETTER ,
     3 RPALEN, RPCLEN,RPLEV,RPA,RPC,RPAM,YRP,NUMT,ARROW ,DONE  ,DESC
     4 , PCX, RPIX, RTIX )
C
C
C        END OF LOOP FOR THIS LEVEL
C
      GO TO (740,760), CATCH
740   CONTINUE
      LEVEL=LEVEL+1
      IF (LEVEL.LE.L3) GO TO 190
      WRITE (6,750)
      WRITE (7,750)
750   FORMAT ( 'ONO MORE LEVELS TO BE CONSIDERED' / )
      CALL SYMBOL (Y,ALPHA,N,   MMA,CHAR,NCHAR,LETTER,N26)
      WRITE (6,220) (ALPHA(III),III=1,MMA)
      WRITE (7,220) (ALPHA(III),III=1,MMA)
C
C
C        PRINT FINAL RULE OF PRODUCTION
C
760   CONTINUE
      GO TO ( 762, 772), MODE
762   CONTINUE
      CALL SYMBOL (Y,ALPHA,N,   MMA,CHAR,NCHAR,LETTER,N26)
      WRITE(6,770) SSS, ARROW, (ALPHA(III), III=1, MMA )
      WRITE(7,770) SSS, ARROW, (ALPHA(III), III=1, MMA )
770   FORMAT ( '0' / ' ', 129A1  /  ( ' ', 19X, 110A1 / )   )
      GO TO 778
772   CONTINUE
      L=0
      DO 775 T=1, N
      L=L+1
      YNEW(L)=Y(T)
      IF(Y(T).NE. PUNC) GO TO 775
      CALL SYMBOL( YNEW, ALPHA, L , MMA, CHAR, NCHAR, LETTER, N26)
      WRITE(6,770) SSS, ARROW, (ALPHA(III), III=1, MMA )
```

```
      WRITE(7,770) SSS, ARROW, (ALPHA(III), III=1, MMA )
      L=0
775   CONTINUE
778   CONTINUE
C
C        END OF PROCESSING OF THIS DATA SET
C
780   CONTINUE
      GO TO 110
      END
```

```
$COPY RECODE *SINK*@¬CC
      SUBROUTINE RECODE(Y,YNEW,SEQNO,ALPHA,  LLIST,SQCTXT,SQLEN,SQPRED,
     1  TYPE,      P,PD,INDEX,INDEP,       CHAR,  SCRJ,SCRJAL,SCRD,
     2 SCRDAL,  POST,BI,BII,PRE,MASK,H,FIRST,SECOND,THIRD,  LETTER ,
     3 RPALEN, RPCLEN,RPLEV,RPA,RPC,RPAM,YRF,NUMT,ARROW ,DONE   ,DESC
     4 , POX, RPIX, RTIX )
C
C        SUBROUTINE FOR RECODING SAMPLE Y
C
      IMPLICIT INTEGER (A-Z)
C
C
C        COMMON STORAGE
C
      COMMON           C1,CONT,L3,M1,M2, PR1, PR2, FR  , LAMBDA,  RIT ,
     1  MCH, MODE, ALLOW  , STRICT,EARLY,GRAM  ,       NCHAR ,
     2         N26, PRG, EXR , PRS, LAWL, VSTART , VDIR , HSEL, WDF,
     3  WCV, WXL, KRECUR, LAMR
     4 , NS,N,MBE,C,MTSIZE,M2MAX,NQ,TRIAL,CONT,PUNC,LEVEL,N40,FC,NST,
     5  PERC,CTXT,TN,   NTM,RPMAX,NRP,NY,YRPMAX,NRECUR,YRARRW,YRPUNC
     1 , NTP
C
C
C
C        VECTORS OF SIZE  NS
      INTEGER Y(NS)
      INTEGER YNEW(NS)
      INTEGER SEQNO(NS)
      INTEGER ALPHA(NS)
      INTEGER DONE(NS)
C
C        VECTORS OF SIZE MTSIZE
      INTEGER SQCTXT(MTSIZE)
      INTEGER SQLEN(MTSIZE)
      INTEGER SQPRED(MTSIZE)
      INTEGER TYPE(MTSIZE)
      INTEGER INDEX(MTSIZE)
      INTEGER INDEP(MTSIZE)
      REAL P(MTSIZE)
      REAL PD(MTSIZE)
C
C        VECTORS OF SIZE NCHAR
      INTEGER CHAR(NCHAR)
C
C        VECTORS OF SIZE N40
      INTEGER SCRJ(N40)
      INTEGER SCRJAL(N40)
      INTEGER SCRD(N40)
      INTEGER SCRDAL(N40)
C
C        VECTORS OF SIZE M2MAX
      INTEGER POST(M2MAX)
      INTEGER BI(M2MAX)
      INTEGER BII(M2MAX)
      INTEGER PRE(M2MAX)
      INTEGER MASK(M2MAX)
      REAL H(M2MAX)
      REAL FIRST(M2MAX)
      REAL SECOND(M2MAX)
```

```
      REAL THIRD(M2MAX)
C
C        VECTORS OF SIZE N26
      INTEGER LETTER(N26)
C
C        VECTORS OF SIZE RPMAX
      INTEGER LLIST(RPMAX)
      INTEGER RPALEN(RPMAX)
      INTEGER RPCLEN(RPMAX)
      INTEGER RPLEV(RPMAX)
      REAL POX(RPMAX)
      INTEGER RPIX(RPMAX)
      INTEGER RTIX(RPMAX)
      INTEGER DESC(RPMAX)
C
C        ARRAYS OF SIZE  RPMAX * M2MAX
      INTEGER RPA (RPMAX,M2MAX)
      INTEGER RPC (RPMAX,M2MAX)
      INTEGER RPAM(RPMAX,M2MAX)
C
C        VECTORS OF SIZE YRPMAX
      INTEGER YRP(YRPMAX)
C        OTHER VECTORS
      DIMENSION NUMT(5)
      INTEGER ARROW(12)
C
C        REAL CONSTANTS AND FUNCTIONS
      REAL HP
      REAL ALOG
      REAL FC
      REAL LAMBDA
      REAL LAMR
C
C
C
C        SUBROUTINE RECODES SAMPLE ACCORDING TO PARAMETERS
C          MAIN  PARAMETERS FOR RECODING:
C            MBE=UPPER LIMIT ON LENGTH OF REGULARITY TO BE USED IN
C              THIS RECODING
C              M=M1,...,MBE
C            TRIAL=INDICATION OF WHETHER THIS RECODING IS TRIAL OR
C              ACTUAL RECODING    1=TRIAL  2=ACTUAL
C            RIT=WHETHER TO WRITE CONTEXT-FREE RULES
C            ALLOW=TYPE OF WORST P(I) ALLOWED IN RECODING
C            STRICT=WHETHER TO DO STRICT RECODING
C            KRECUR=WHETHER TO TRY RECURSIONS
C          OTHER PARAMETERS
C            VDIR=DIRECTION OF SCAN FOR M
C            VSTART=STARTING POINT FOR SCAN
C            HSEL=SELECTION OF H MEASURE (FOR TESTING PURPOSES)
C
C
C        INITIALIZATION
C
      YNL=N
      HP=0.0
      NTM=0
      NCT=0
      NQ=0
      NRECUR=0
```

```
      NHER=0
      FILL=DONT
      CALL ZERO (SEQNO,N)
      CALL ZERO (DONE,NS)
      DO 2040 I=1,M1
      YNEW(I)=Y(I)
2040  CONTINUE
      IF (TRIAL.EQ.2) WRITE (6,2010) LEVEL
      IF (TRIAL.EQ.2) WRITE (7,2010) LEVEL
2010  FORMAT ( '0---------------------  RULES OF PRODUCTION ',
     9   'FOR LEVEL', I2,
     1   '----------------------' )
      IF (TRIAL.EQ.1) WRITE (6,2020) MBE
2020  FORMAT ( '0TENTATIVE RULES OF PRODUCTION FOR M OF ', I3 )
C
C        COPY INTO TEMPORARY RPIX FROM THE PERMANENT RTIX TABLE
C          NB:IF TRIAL=1, THE RPA AND OTHER ASSOCIATED TABLES ARE NOT
C          ALTERED, AND NRP IS NOT CHANGED.
      IF(NTP.NE.0) CALL IEQUAL (RPIX, NTP, RTIX)
C
C
C        LOOP FOR T=M1,...,N
C
      LAST=0
      QLAST=0
      TN=M1
      T=M1
2050  CONTINUE
      YNEW(TN)=Y(T)
      IF (NST.EQ.0) GO TO 2460
      IF (VSTART.EQ.1.AND.T-QLAST.LT.MBE) GO TO 2460
C
C
C        LOOP FOR LR=1,2
C        FIRST TIME: TRY TO USE SEQUENCES THAT BECOME RECURSIVE
C          PROVIDED KRECUR IS SET AT 1( ES)
C          PROVIDED LEVEL IS GREATER THAN 1
C        SECOND TIME: USE ANY SEQUENCE
C
      LR=1
      IF (LEVEL.EQ.1) LR=2
      IF (KRECUR.EQ.0) LR=2
2060  CONTINUE
C
C
C        LOOP FOR LTA=1, ALLOW
C
      LTA=1
2070  CONTINUE
C
C
C        LOOP FOR M
C        M=M1,...,MBE  IF VDIR=+1
C        M=M2,...,M1   IF VDIR=-1
C        M=MBE  ONLY IF STRICT=1
C
      IF (VDIR.EQ.1) M=M1
      IF(VDIR.EQ.-1) M=MBE
      IF (STRICT.EQ.1) M=MBE
C
```

```
2080  CONTINUE
      DO 2120 LLL=1,NST
      L=INDEX(LLL)
      IF (SQLEN(L).NE.M) GO TO 2120
      IF (TYPE(L).NE.LTA) GO TO 2120
C
C        MAKE SURE NO PUNCTUATION OCCURS IN M-SEQUENCE OF Y
      IF (MODE.EQ.1) GO TO 2100
      DO 2090 JD=1,M
      TMJD=T-M+JD
      IF (Y(TMJD).NE.PUNC) GO TO 2090
      QLAST=TMJD
      IF (VDIR.EQ.-1) GO TO 2130
      GO TO 2460
2090  CONTINUE
2100  CONTINUE
C
C        RECREATE CONTEXT-SEQUENCE AND PREDICTED-SEQUENCE
      CALL CONV (PRE,M,C,SQCTXT(L))
      CALL CONV (POST,M,C,SQPRED(L))
C
C        MATCH Y AGAINST CONTEXT AND PREDICTED SEQUENCES
      DO 2110 JD=1,M
      TMJD=T-M+JD
      IF (POST(JD).EQ.DONT) GO TO 2110
      IF (PRE(JD).EQ.PERC.AND.POST(JD).NE.Y(TMJD)) GO TO 2120
      IF (POST(JD).EQ.CTXT.AND.PRE(JD).NE.Y(TMJD)) GO TO 2120
2110  CONTINUE
C
C        LOCATED SEQUENCE OF LENGTH M AT TIME T
C
      GO TO 2150
2120  CONTINUE
2130  CONTINUE
C
C
C        END OF LOOP FOR M
C
      M=M+VDIR
      IF (STRICT.EQ.1) GO TO 2140
      IF (VDIR.EQ.-1.AND.M.GE.M1) GO TO 2080
      IF (VDIR.EQ.1.AND.M.LE.MBE) GO TO 2080
C
2140  CONTINUE
      L=0
2150  CONTINUE
      SEQNO(T)=L
C
C
C        THE SEQUENCE THAT IS FOUND IS
C            1. OF LONGEST LENGTH FROM M1 TO MBE OF THOSE SEQUENCES
C              ENDING AT TIME T (IF VDIR = -1 )
C              OF SHORTEST LENGTH (IF VDIR = +1 )
C              OF LENGTH MBE (IF STRICT = 1 )
C            2. OF HIGHEST ALLOWABLE TYPE
C            3.OF HIGHEST P(I) POSSIBLE
C            4.OF SIMPLEST GRAMMATICAL TYPE
C
      IF (L.EQ.0) GO TO 2450
      M=SQLEN(L)
```

```
C
C        IS THERE ROOM (FROM LEFT) FOR AN M-SEQUENCE
      IF (T+1-M.LE.LAST) GO TO 2450
C
C        ARE ALL M POSITIONS AS YET UNRECODED?
      DO 2160 JD=1,M
      TMJD=T-M+JD
      IF (ONE(TMJD).EQ.1) GO TO 2450
2160  CONTINUE
C
C        CONTRIBUTION TO H
      IF (P(L).EQ.0.0) GO TO 2170
      NTM=NTM+1
      IF(WXL.EQ.0) MWQ=1
      IF(WXL.EQ.1) MWQ=M
      HP=HP-P(L)*ALOG(P(L)) * MWQ
2170  CONTINUE
C
C        CHECK CURRENT L AGAINST LLIST
C          QNEW:  1= L IS NEW TO LLIST    0=L IS OLD
C
      CALL LOOK (LLIST,MTSIZE,NQ,L,ILINE,QNEW)
      LAST=T
      TN=TN-M
      KL=0
      T=T-M
C
C        WRITE CONSEQUENT (RIGHT) SIDE OF RULE OF PRODUCTION
C          CONSEQUENT SIDE IS OF LENGTH M
C
      IF (QNEW.NE.1) GO TO 2190
      DO 2180 JP=1,M
      SCRD(JP)=Y(T+JP)
2180  CONTINUE
2190  CONTINUE
C
C
C        WRITE ANTECEDENT SIDE OF RULE OF PRODUCTION
C
C        RIT=CONTROLS FORM OF ANTECEDENT (LEFT) SIDE OF
C          RULE OF PRODUCTION
C          1=(CONTEXT-FREE)= REPLACE ENTIRE PREDICTOR AND PREDICTED
C            SEQUENCES WITH ONE NEW NON-TERMINAL SYMBOL
C          0=REPLACE ALL CONTIGUOUS PREDICTOR SYMBOLS WITH
C            NEW NON-TERMINAL SYMBOL
C
      IF (RIT.EQ.0) GO TO 2210
C
C        ****************************************************************
C        CONTEXT-FREE CASE (RIT=1)
C        ****************************************************************
C        REPLACE ENTIRE ANTECEDENT(LEFT) SIDE WITH 1 NEW NON-TERMINAL
C
C
C        BRANCH TO RECURSIVE TEST, IF LR=1
      KRET=1
      IF (LR.EQ.1) GO TO 2810
2200  CONTINUE
      IF (LR.EQ.1.AND.LRX.EQ.0) GO TO 2450
      TN=TN+1
```

```
      IF (LR.EQ.1.AND.LRX.EQ.1) YNEW(TN)=RSYM
      IF (LR.EQ.2) YNEW(TN)=C-1+ILINE+NCT
      KL=KL+1
      SCRJ(KL)=YNEW(TN)
      CALL IXSPRY (DONE,NS,1,T+1,T+M)
      T=T+M
      IF(M.GE.2) CALL IXSPRY(YNEW,NS,FILL, TN+1, TN+M-1 )
      TN=TN+M-1
      GO TO 2360
C
C        RIT=0 CASE
C
2210  CONTINUE
      CALL CONV (PRE,M,C,SQCTXT(L))
      CALL CONV (POST,M,C,SQPRED(L))
      CALL MCRE (PRE,POST,M,MASK,DONT,CTXT,PERC,MODE,PUNC)
C
C        IS MASK STRICTLY CONTEXT-SENSITIVE
C          OR STRICTLY UNRESTRICTED REWRITE
C
      CALL MTEST ( MASK, M,PERC, &2305, &2215 )
C
C        ************************************************************
C        STRICTLY CONTEXT-SENSITIVE CASE
C        ************************************************************
C
C        FIND CONTEXT ON LEFT, IF ANY
C
2215  CONTINUE
      LEFT=0
      RIGHT=0
      DO 2220 LEFT=1,M
      IF (POST(LEFT).EQ.DONT.OR.POST(LEFT).EQ.CTXT) GO TO 2230
2220  CONTINUE
2230  CONTINUE
      LEFT=LEFT-1
C        FIND CONTEXT ON RIGHT, IF ANY
      DO 2240 RL=1,M
      RLQ=M+1-RL
      IF (POST(RLQ).EQ.DONT.OR.POST(RLQ).EQ.CTXT) GO TO 2250
2240  CONTINUE
2250  CONTINUE
      RIGHT=RL-1
C
C        BRANCH TO RECURSIVE TEST, IF LR=1
C
      KRET=2
      IF (LR.EQ.1) GO TO 2810
2260  CONTINUE
      IF (LR.EQ.1.AND.LRX.EQ.0) GO TO 2450
      CALL IXSPRY (DONE,NS,1,T+1,T+M)
C
C        LEFT END (IF ANY)
C
      IF (LEFT.EQ.0) GO TO 2280
      DO 2270 KT=1,LEFT
      T=T+1
      TN=TN+1
      YNEW(TN)=Y(T)
      KL=KL+1
```

```
      SCRJ(KL)=YNEW(TN)
2270  CONTINUE
2280  CONTINUE
C
C
C        REPLACE 'MID' CONTIGUOUS SYMBOLS IN Y WITH 1 NEW NON-TERMINAL
C
      TN=TN+1
      IF (LR.EQ.1.AND.LRX.EQ.1) YNEW(TN)=RSYM
      IF (LR.EQ.2) YNEW(TN)=C-1+ILINE+NCT
      KL=KL+1
      SCRJ(KL)=YNEW(TN)
      MID=M-RIGHT-LEFT
      IF(MID.GE.2) CALL IXSPRY(YNEW,NS,FILL, TN+1, TN+MID-1 )
      TN=TN+MID-1
      T=T+MID
C
C        RIGHT END (IF ANY)
      IF (RIGHT.EQ.0) GO TO 2300
      DO 2290 KP=1,RIGHT
      T=T+1
      TN=TN+1
      YNEW(TN)=Y(T)
      KL=KL+1
      SCRJ(KL)=YNEW(TN)
2290  CONTINUE
2300  CONTINUE
      GO TO 2360
C
C        ************************************************************************
C        STRICTLY UNRESTRICTED-REWRITE CASE
C        ************************************************************************
C
2305  CONTINUE
C        ***RPAM,RECUR,DONE
      NB=-1
      KA=0
2310  CONTINUE
      KA=KA+1
      IF (KA.GT.M) GO TO 2350
      IF (MASK(KA).EQ.NB) GO TO 2330
      NB=MASK(KA)
      KRET=3
      IF (LR.EQ.1) GO TO 2810
2320  CONTINUE
      IF (LR.EQ.1.AND.LRX.EQ.0) GO TO 2450
      TN=TN+1
      IF (LR.EQ.1.AND.LRX.EQ.1) YNEW(TN)=RSYM
      IF (LR.EQ.2) YNEW(TN)=C-1+ILINE+NCT
      NCT=NCT+1
      KL=KL+1
      SCRJ(KL)=YNEW(TN)
      T=T+1
      GO TO 2310
2330  CONTINUE
      IF (MASK(KA).EQ.PERC) GO TO 2310
      T=T+1
      TN=TN+1
      YNEW(TN)=Y(T)
      KL=KL+1
```

```
      SCRJ(KL)=YNEW(TN)
2350  CONTINUE
      GO TO 2360
C
C
C        SAVE ANTECEDENT (LEFT) SIDE OF RULE OF PRODUCTION IN RPA
C
2360  CONTINUE
      IF (QNEW.NE.1) GO TO 2410
      NHER=NHER+1
      NRP=NRP+1
      RPLEV(NRP)=LEVEL
      POX(NRP)=0.0
      IF(LRX.EQ.0) POX(NRP) = P(L)
      IF(LR.EQ.2   .OR.  LR.EQ.1.AND.LRX.EQ.0 ) RPIX(NRP)= -2
C        NB: RPIX CAN ALSO BE SET IN RECURSIVE ROUTINE
      RPALEN(NRP)=KL
      DO 2370 JG=1,KL
      RPA(NRP,JG)=SCRJ(JG)
2370  CONTINUE
C
C        SAVE CONSEQUENT (RIGHT) SIDE OF RULE OF PRODUCTION IN RPC
      RPCLEN(NRP)=M
      DO 2380 JP=1,M
      RPC(NRP,JP)=SCRD(JP)
2380  CONTINUE
      IF (RIT.EQ.0) GO TO 2390
      RPAM(NRP,1)=PERC
      GO TO 2410
2390  CONTINUE
      DO 2400 JB=1,M
      RPAM(NRP,JB)=MASK(JB)
2400  CONTINUE
2410  CONTINUE
C
C        PRINT RULES OF PRODUCTION ONTO 6, AS THEY ARE  PRODUCED
C          SCRJ=ANTECEDENT SIDE OF RULE OF PRODUCTION
C          SCRD=CONSEQUENT SIDE OF RULE OF PRODUCTION
C
      IF (QNEW.NE.1) GO TO 2430
      CALL SYMBOL (SCRD,SCRDAL,M,MMQ,CHAR,NCHAR,LETTER,N26)
      CALL SYMBOL (SCRJ,SCRJAL,KL,MM,CHAR,NCHAR,LETTER,N26)
      WRITE (6,2420) (SCRJAL(J),J=1,MM),ARROW,(SCRDAL(JJ),JJ=1,MMQ)
2420  FORMAT ('0',129A1/(1X,19X,110A1/))
2430  CONTINUE
C
C        IF NEW RULE WAS FOUND, OR NEW RECURSIVE RULE WAS FORMED,
C          TRY TO APPLY IT THRCUGHOUT STRING Y
C
      I=NRP
      MG=RPCLEN(I)
      IF(STRICT.EQ.1 .AND. MG.NE. MBE) GO TO 2450
3538  CONTINUE
      NGL=0
C
C
C        LOOP FOR TX=M1,...
C
      LASTX=0
      QLASTX=0
```

```
      TNX=M1
      TX=M1
3550  CONTINUE
      IF (VSTART.EQ.1.AND.TX-QLASTX.LT. MBE ) GO TO 3630
C
C        IS THERE ROOM (FROM LEFT) FOR AN M-SEQUENCE
      IF(TX+1-MG  .LT. LASTX) GO TO 3630
C
C        ARE ALL M POSITIONS AS YET UNRECODED?
      DO 3570 JD=1,MG
      TMJD=TX-MG+JD
      IF (DONE(TMJD).EQ.1) GO TO 3630
3570  CONTINUE
C
C        MAKE SURE NO PUNCTUATION OCCURS IN M-SEQUENCE OF Y
      IF (MODE.EQ.1) GO TO 3590
      DO 3580 JD=1, MG
      TMJD=TX-MG+JD
      IF (Y(TMJD).NE.PUNC) GO TO 3580
      QLASTX=TMJD
      GO TO 3630
3580  CONTINUE
3590  CONTINUE
C
C        MATCH Y WITH CONSEQUENT (RIGHT) SIDE OF RULE I
      DO 3600 JD=1,MG
      IF (RPAM(I,JD).EQ.DONT) GO TO 3600
      TMJD=TX-MG+JD
      IF (RPC(I,JD).EQ.Y(TMJD)) GO TO 3600
      IF(LRX.EQ.1 .AND. Y(TMJD).EQ.SMID .AND. RPC(I,JD).EQ.RSYM)
     1  GO TO 3600
      GO TO 3630
3600  CONTINUE
C
C        LOCATED SEQUENCE OF LENGTH M AT TIME T
      SEQNC(TX)=L
      NGL=NGL+1
      LASTX=TX
C
C        CONTRIBUTION TO H
      IF (POX(I).EQ.0.0) GO TO 3610
      NTM=NTM+1
      IF(WXL.EQ.0) MWQ=1
      IF(WXL.EQ.1) MWQ=M
      HP=HP-POX(I)*ALOG(POX(I)) * MWQ
3610  CONTINUE
C
C        RECODE
      TNX=TNX-MG
      TX=TX-MG
      MAA=RPALEN(I)
      DO 3620 JD=1,MAA
      TNX=TNX+1
      YNEW(TNX)=RPA(I,JD)
3620  CONTINUE
      IF(MG.GT.MAA) CALL IXSPRY(YNEW,NS,FILL,TNX+1, TNX+MG-MAA)
      TNX=TNX+MG-MAA
      CALL IXSPRY(DONE,NS,1, TX+1, TX+MG)
      TX=TX+MG
C
```

```
      YRP(NY)=SCRD(JP)
2490  CONTINUE
      NY=NY+1
      YRP(NY)=YRPUNC
      CALL SYMBOL (SCRJ,SCRJAL,MA,MM,CHAR,NCHAR,LETTER,N26)
      CALL SYMBOL (SCRD,SCRDAL,MC,MW,CHAR,NCHAR,LETTER,N26)
      WRITE (7,2420) (SCRJAL(J),J=1,MM),ARRCW,(SCRDAL(JJ),JJ=1,MW)
2500  CONTINUE
C
C        COPY INTO PERMANENT RTIX  THE TEMPORARY VECTOR RPIX
C
      IF (NRP.EQ.0) GO TO 2510
      NTP=NRP
      CALL IEQUAL (RTIX,NRP,RPIX)
2510  CONTINUE
2520  CONTINUE
C
C        PRINT OUT ORIGINAL STRING Y
C
      WRITE (6,2720)
      CALL SYMBOL (Y,ALPHA,N,MMA,CHAR,NCHAR,LETTER,N26)
      WRITE (6,2730) (ALPHA(III),III=1,MMA)
C
C        REMOVE 'FILL' SYMBOLS FROM YNEW
C
      CALL REMOVE( YNEW, YNEW, YNL,  YNL, FILL )
C
C        PRINT OUT NEW STRING YNEW
C
      WRITE(6, 2732)
      CALL SYMBOL (YNEW,ALPHA,YNL ,MMA,CHAR,NCHAR,LETTER,N26)
      WRITE (6,2730) (ALPHA(III),III=1,MMA)
C
C
C        TRY TO RE-USE ALL RULES DEVELOPED IN ALL POSSIBLE WAYS
C
      IF(NRP.EQ.0) GO TO 2652
2538  CONTINUE
      NGL=0
C
C        LM=1,2
C        FIRST TIME: TRY TO USE SEQUENCES THAT ARE RECURSIVE
C        SECOND TIME: TRY ANY OTHER SEQUENCE
C
      LM=1
      IF(KRECUR.EQ.0) LM=2
2540  CONTINUE
C
C        LOOP FOR T=M1,...
C
      LAST=0
      QLAST=0
      TN=M1
      T=M1
2550  CONTINUE
      IF (VSTART.EQ.1.AND.T-QLAST.LT.MBE) GO TO 2631
      IF(YNEW(TN).EQ. FILL ) GO TO 2631
      DO 2630 I=1, NRP
      M=RPCLEN(I)
      IF(STRICT.EQ.1 .AND. M.NE. MBE) GO TO 2630
```

```
C        WHEN LM=1, PROCESS ONLY RECURSIVE RULE (RPIX > 0 )
      IF(LM.EQ.1  .AND. RPIX(I) .LE. 0 ) GO TO 2630
C        WHEN LM=2, PROCESS NON-RECURSIVE RULE (RPIX=-2)
      IF(LM.EQ.2 .AND. RPIX(I) .NE. -2 ) GO TO 2630
C        DEACTIVATED RULES ARE SKIPPED
C
C        IS THERE ROOM (FROM LEFT) FOR AN M-SEQUENCE
      IF (T+1-M.LT.LAST) GO TO 2630
C
C        MAKE SURE NO PUNCTUATION OCCURS IN M-SEQUENCE OF Y
      IF (MODE.EQ.1) GO TO 2590
      DO 2580 JD=1,M
      TMJD=T-M+JD
      IF(YNEW(TMJD).NE. PUNC ) GO TO 2580
      QLAST=TMJD
      GO TO 2630
2580  CONTINUE
2590  CONTINUE
C
C        MATCH Y WITH CONSEQUENT (RIGHT) SIDE OF RUL
      DO 2600 JD=1,M
      IF (RPAM(I,JD).EQ.DONT) GO TO 2600
      TMJD=T-M+JD
      IF( RPC(I,JD) .NE. YNEW(TMJD)  ) GO TO 2630
2600  CONTINUE
C
C        LOCATED SEQUENCE OF LENGTH M AT TIME T
C
      NGL=NGL+1
      LAST=T
C
C        CONTRIBUTION TO H
C
      IF (POX(I).EQ.0.0) GO TO 2610
      NTM=NTM+1
      IF(WXL.EQ.0) MWQ=1
      IF(WXL.EQ.1) MWQ=M
      HP=HP-POX(I)*ALOG(POX(I)) * MWQ
2610  CONTINUE
C
C        RECODE
C
      TN=TN-M
      T=T-M
      MA=RPALEN(I)
      DO 2620 JD=1,MA
      TN=TN+1
      YNEW(TN)=RPA(I,JD)
2620  CONTINUE
      IF(M.GT.MA) CALL IXSPRY ( YNEW, NS, FILL, TN+1, TN+M-MA)
      TN=TN+M-MA
      CALL IXSPRY (DONE,NS,1,T+1,T+M)
      T=T+M
2630  CONTINUE
C
C        END OF LOOP FOR T AND TN
C
2631  CONTINUE
      T=T+1
      TN=TN+1
```

```
      IF(T.LE. YNL ) GO TO 2550
C
C        END OF LOOP FOR LM=1,2
C
      LM=LM+1
      IF(LM.LE.2) GO TO 2540
2650  CONTINUE
C
C
C        GO THROUGH LIST OF RULE OF PRODUCTION UNTIL NO APPLICATIONS ARE
C          POSSIBLE
C
      IF (NGL.NE.0) GO TO 2538
C
C        IF TRIAL=1, DELETE  NHER  RULES FROM TABLES
C
2652  CONTINUE
      IF (TRIAL.EQ.1) NRP=NRP-NHER
C
C        IDENTITY RULES
C          (APPROXIMATE COUNT)
C
      NIR=0
      IF(WXL.EQ.0)  GO TO 2659
      CZ=0
      XG=1
      DO 2658 IJ=1, N
      IF(Y(IJ).EQ.PUNC) GO TO 2653
      IF(DONE(IJ).EQ.1) GO TO 2657
      XG=0
      CZ=CZ+1
      IF(CZ.LT. MBE ) GO TO 2658
2653  CONTINUE
      NIR=NIR+1
      NTM=NTM+1
      XG=1
      CZ=0
      GO TO 2658
2657  CONTINUE
      IF(XG.EQ.0) GO TO 2653
      XG=1
      CZ=0
2658  CONTINUE
2659  CONTINUE
C
C
C        COMPUTE H
C
      FIRST(MBE)=2499.
      GO TO (2660,2670,2680,2690), HSEL
2660  CONTINUE
      IF (NTM.NE.0) FIRST(MBE)=HP/(FC*NTM)
      GO TO 2700
2670  CONTINUE
      IF (NTM.NE.0) FIRST(MBE)=(HP*NQ)/(FC*NTM)
      GO TO 2700
2680  CONTINUE
      FIRST(MBE)=(HP*MBE)/(FC*M1)
      GO TO 2700
2690  CONTINUE
```

```
      FIRST(MBE)=HP/FC
      GO TO 2700
2700  CONTINUE
      SECOND(MBE)=NQ*LAMBDA
      SECOND(MBE)= (NQ+NIR) * LAMBDA
      THIRD(MBE)=LAMR*NRECUR
      H(MBE)=FIRST(MBE)+SECOND(MBE)+THIRD(MBE)
      WRITE (6,2710) FIRST(MBE),SECOND(MBE),THIRD(MBE),H(MBE),NQ,NRECUR,
     1 NIR, NTM
2710  FORMAT ( '0' /
     2    ' ENTROPY TERM                          ',  F10.5 /
     3    ' PARSIMONY TERM                        ',  F10.5      /
     4    ' RECURSIVE PARSIMONY TERM              ', F10.5 /
     5    '                                       ',  '+_________' /
     1    ' VALUE OF H FOR THIS RECODING..........',  F10.5 /
     8    /
     1    ' NUMBER OF RULES OF PRODUCTION         ', I7   /
     7    ' NUMBER OF RECURSIVE RULES             ', I7 /
     7    ' NUMBER OF IDENTITY RULES              ', I7 /
     2    ' NUMBER OF TIMES RULES ARE APPLIED     ', I7 /
     4    )
C
C        PRINT OUT ORIGINAL STRING Y
C
      WRITE (6,2720)
2720  FORMAT ( 'OCURRENT STRING Y ' / '0' )
      CALL SYMBOL (Y,ALPHA,N,MMA,CHAR,NCHAR,LETTER,N26)
      WRITE (6,2730) (ALPHA(III),III=1,MMA)
2730  FORMAT ( '0', (/1X, 60A1) )
C
C        AGAIN  REMOVE ALL 'FILL' SYMBOLS FROM YNEW
C
      CALL REMOVE( YNEW, YNEW, YNL,  YNL, FILL )
C
C        PRINT OUT NEW STRING YNEW
C
      WRITE(6, 2732)
2732  FORMAT ('ONEW STRING ' / '0' )
      CALL SYMBOL (YNEW,ALPHA,YNL ,MMA,CHAR,NCHAR,LETTER,N26)
      WRITE (6,2730) (ALPHA(III),III=1,MMA)
C
C        PRINT OUT STRING YRP
C
C
C        PRINT GRAPH OF P(I) ACTUALLY USED IN RECODING
C
      IF (.NOT.(PRG.EQ.2.OR.PRG.EQ.1.AND.TRIAL.EQ.2)) GO TO 2790
      IF (NQ.EQ.0) GO TO 2790
      CALL ZERO (NUMT,5)
      DO 2740 I=1,NQ
      L=LLIST(I)
      PD(I)=P(L)
      K=TYPE(L)
      NUMT(K)=NUMT(K)+1
2740  CONTINUE
      IF (PRG.EQ.C) GO TO 2760
      CALL XFSORT (PD,MTSIZE,INDEP,1,NQ)
      DO 2750 I=1,NQ
      DESC(I)=TYPE(LLIST(INDEP(I)))
2750  CONTINUE
```

```
      CALL GRAPH (PD,NQ,NUMT,LEVEL,MBE,DESC)
2760  CONTINUE
C
C
C        PRINT STATISTICS ON P(I)'S ACTUALLY USED IN RECODING
C
      WRITE (6,2770) (NUMT(IIM),IIM=1,5)
2770  FORMAT (
     1    ' NUMBER OF TYPE 1 SEQUENCES(STRUCTURAL)', I7 /
     6    ' NUMBER OF TYPE 2 SEQUENCES (MESSAGE)  ', I7       /
     6    ' NUMBER OF TYPE 3 SEQUENCES (MESSAGE)  ', I7       /
     6    ' NUMBER OF TYPE 4 SEQUENCES (MESSAGE)  ', I7       /
     3    ' NUMBER OF TYPE 5 SEQUENCES (NOISE)    ', I7       )
      NZZ=ISUM(NUMT,5)
      WRITE (6,2780) NZZ
2780  FORMAT (
     1    ' NUMBER OF P(I)                        ', I7    )
2790  CONTINUE
C
C
C        IF TRIAL=2, SUBSTITUTE RECODED  YNEW  INTO  Y
C        NEW SIZE OF ALPHABET
C        NEW LENGTH OF Y
C        SUBSTITUTE NEW STRING (YNEW) INTO Y VECTOR
C
      IF (TRIAL.NE.2) GO TO 2800
      C=C+NQ+NCT
      N=YNL
      CALL IEQUAL (Y,N,YNEW)
2800  CONTINUE
C
      RETURN
C
C        ------------------------------------------------------------
C
C        CHECK FOR RECURSION
C        ------------------------------------------------------------
C
C
2810  CONTINUE
      LRX=0
      IF (KRECUR.EQ.0) GO TO 2910
      IF (QNEW.EQ.0) GO TO 2910
      IF (LEVEL.EQ.1) GO TO 2910
      IF (NRP.EQ.0) GO TO 2910
      DO 2900 I=1,NRP
C
C        RULE I IS OF PREVIOUS LEVEL
C
      IF (RPLEV(I).NE.LEVEL-1) GO TO 2900
C
C        CONSEQUENT (RIGHT) SIDE OF RULE I FROM PREVIOUS LEVEL, AND
C          CONSEQUENT SIDE OF CURRENT RULE ARE OF SAME LENGTH
C          NB:THIS IS PRE-CONDITION FOR ISOMORPHISM (SEE BELOW)
      LCC=M
      IF (RPCLEN(I).NE.LCC) GO TO 2900
      MA=RPALEN(I)
      DO 2890 J=1,MA
C
C        SYMBOL 'SMID' IS A 'PREDICTED' SYMBOL ON THE ANTECEDENT(LEFT)
```

```
C          SIDE OF RULE I, AT PREVIOUS LEVEL
      IF (RPAM(I,J).NE.PERC) GO TO 2890
      SMID=RPA(I,J)
      DO 2880 JJ=1,LCC
C
C        SYMBOL 'SMID' APPEARS ON CONSEQUENT (RIGHT) SIDE OF RULE AT
C          THIS LEVEL
      IF (SCRD(JJ).NE.SMID) GO TO 2880
C
C        ISOMORPHISM OF CONSEQUENT (RIGHT) SIDE OF RULE I FROM PREVIOUS
C          LEVEL, AND CONSEQUENT (RIGHT) SIDE OF CURRENT RULE
      BILEN=0
      BIILEN=0
      DO 2830 KK=1,LCC
      CALL LOOK (BI,M2MAX,BILEN,RPC(I,KK),BILEV,BINEW)
      CALL LOOK (BII,M2MAX,BIILEN,SCRD(KK),BIILEV,BIINEW)
      IF (BILEV.NE.BIILEV) GO TO 2880
2830  CONTINUE
C
C        HAVE RECURSION
C
      LRX=1
      NRECUR=NRECUR+1
      WRITE (6,2840) NRECUR
2840  FORMAT ( '0RECURSION NO. ', I3  )
C
C        IDENTIFY RECURSIVE SYMBOL
      RSYM=RPC(I,JJ)
C
C        ABANDON GENERATION OF CURRENT RULE
      IF (NQ.NE.0) NQ=NQ-1
C
C        INSERT RSYM WHEREEVER  SMID  OCCURRED IN RULES
      DO 2870 IA=1,NRP
      LZA=RPALEN(IA)
      DO 2850 JA=1,LZA
      IF (RPA(IA,JA).EQ.SMID) RPA(IA,JA)=RSYM
2850  CONTINUE
      LZC=RPCLEN(IA)
      DO 2860 JA=1,LZC
      IF (RPC(IA,JA).EQ.SMID) RPC(IA,JA)=RSYM
2860  CONTINUE
2870  CONTINUE
C
C        GENERATE A RECURSIVE RULE AND DEACTIVATE RULE I
      RPIX(I)=0
      RPIX(NRP+1)=I
      DO 2872 JP=1, M
      SCRD(JP)=RPC(I,JP)
2872  CONTINUE
      DO 2874 JG=1, MA
      SCRJ(JG)=RPA(I,JG)
2874  CONTINUE
      GO TO 2910
C
C        END OF LOOP FOR JJ
2880  CONTINUE
C
C        END OF LOOP FOR J
2890  CONTINUE
```

```
C
C         END OF LOOP FOR I
2900  CONTINUE
C
2910  CONTINUE
      GO TO (2200,2260,2320), KRET
C
C
      END
```

```
      SUBROUTINE BEST ( H, M1, M2, HBEST,MBEST,MINM,FIRST,SECOND,
     1  THIRD, M2MAX )
      IMPLICIT INTEGER (A-Z)
      REAL H(M2MAX)
      REAL FIRST(M2MAX)
      REAL SECOND(M2MAX)
      REAL THIRD(M2MAX)
      REAL HBEST
C
C        SUBROUTINE TO FIND BEST H
C       OUT: HBEST, MBEST
C
      M0=M1
      IF ( M0 .LT. MINM ) M0 = MINM
      HBEST=H(M0)
      MBEST= M0
      DO 10 I=M0, M2
      IF( HBEST .LE. H(I)  ) GO TO 10
      HBEST=H(I)
      MBEST=I
10    CONTINUE
      RETURN
      END
      SUBROUTINE GETMSK (M,FR,GRAM,NMASK,CTXT,PERC,DONT,M2MAX,MASK,IPART
     1,*)
      IMPLICIT INTEGER(A-Z)
      INTEGER MASK(M2MAX)
      IF (NMASK.NE.0) GO TO 2010
      JJZ=2
      KKZ=1
      LATCH=1
      MLIM=M
      IIII=0
      IREG=0
      NTOT=FR**M-2
      IPART=1
2010  CONTINUE
C
C        LOOP FOR IPART
C
2020  CONTINUE
C
      DO 2030 IG=1,M
      MASK(IG)=CTXT
2030  CONTINUE
      GO TO (2040,2050,2110,2140), IPART
C
C        TYPE 3 MASK (REGULAR)
2040  CONTINUE
      IF (M.GT.2) GO TO 2190
      IREG=IREG+1
      IF (IREG.GT.2) GO TO 2190
C        LEFT END
      IF (IREG.EQ.1) MASK(M)=PERC
C        RIGHT END
      IF (IREG.EQ.2) MASK(1)=PERC
```

```
      GO TO 2180
C
C        RIGHT-SENSITIVE AND LEFT-SENSITIVE
2050  CONTINUE
      IF (M.LE.2) GO TO 2190
      GO TO (2060,2090), LATCH
C
C        RIGHT-SENSITIVE
2060  CONTINUE
      MLIM=MLIM-1
      IF (MLIM.GT.0) GO TO 2070
      LATCH=2
      MLIM=M
      GO TO 2050
2070  CONTINUE
      DO 2080 JM=1,MLIM
      MASK(JM)=PERC
2080  CONTINUE
      GO TO 2180
C
C        LEFT-SENSITIVE
2090  CONTINUE
      MLIM=MLIM-1
      IF (MLIM.LE.0) GO TO 2190
      DO 2100 JL=1,MLIM
      JW=M+1-JL
      MASK(JW)=PERC
2100  CONTINUE
      GO TO 2180
C
C        TYPE 1 MASK (CONTEXT-SENSITIVE)
2110  CONTINUE
      IF (M.LE.2) GO TO 2190
      KKZ=KKZ+1
      IF (KKZ.LE.M-1) GO TO 2120
      JJZ=JJZ+1
      IF (JJZ.GT.M-1) GO TO 2190
      KKZ=JJZ
2120  CONTINUE
      DO 2130 LK=JJZ,KKZ
      MASK(LK)=PERC
2130  CONTINUE
      GO TO 2180
C
C        TYPE 0 MASK (UNRESTRICTED REWRITE)
2140  CONTINUE
      IF (M.LE.2) GO TO 2190
      IIII=IIII+1
      IF (IIII.GT.NTOT) GO TO 2190
      CALL CONV (MASK,M,FR,IIII)
      DO 2150 IJK=1,M
      IF (MASK(IJK).EQ.0) MASK(IJK)=CTXT
      IF (MASK(IJK).EQ.1) MASK(IJK)=PERC
2150  CONTINUE
C
      CALL MTEST(MASK,M, PERC, &2180, 2140 )
2180  CONTINUE
      NMASK=NMASK+1
      RETURN
2190  CONTINUE
```

```
      IPART=IPART+1
      IF(IPART.LE. 4-GRAM ) GO TO 2020
      RETURN 1
      END
      SUBROUTINE MTEST(MASK,M,PERC,*,* )
      IMPLICIT INTEGER(A-Z)
      INTEGER MASK(M)
C        TEST MASK FOR GRAMMATICAL TYPE
C          RETURN 1: UNRESTIRCTED RE-WRITE
C          RETURN 2: CONTEXT-SENSITIVE
      NB=-1
      CCC=-1
      DO 2170 IK=1,M
      IF (MASK(IK).EQ.NB) GO TO 2170
      NB=MASK(IK)
      CCC=CCC+1
2170  CONTINUE
      IF( CCC.GE.3  .OR.  (CCC.EQ.2  .AND. MASK(1).EQ.PERC)  ) RETURN 1
      RETURN 2
      END
      SUBROUTINE GRID ( PRE, POST, M, DCNT,CTXT, PERC,  SCRG, ZLM)
      IMPLICIT INTEGER (A-Z)
      INTEGER PRE(M)
      INTEGER POST(M)
      INTEGER SCRG(M)
      INTEGER ZLM(3)
C          #=DON'T CARE SYMBOL   _=CONTEXT PART     %=PREDICTED PART
      DO 100 I=1, M
      K=I
      IF (POST(I) .EQ. DCNT ) SCRG(K)=ZLM(1)
      IF ( POST(I) .EQ. CTXT ) SCRG(K)=ZLM(2)
      IF ( PRE(I) .EQ. PERC ) SCRG(K)=ZLM(3)
100   CONTINUE
      RETURN
      END
      SUBROUTINE GRAPH ( X,N, NUMT , LEVEL, MBE , TY )
      IMPLICIT INTEGER (A-Z)
      REAL X(N)
      INTEGER NUMT(5)
      INTEGER SPOT(5)
      INTEGER IMAGE(51,101)
      INTEGER TY(N)
      REAL COMP
      REAL ORDIN
      DATA BLANK / ' ' /
      DATA VERT / '|' /
      DATA HORZ / '-' /
      DATA PLUS / '+' /
      DATA STAR /'*' /
C
C        GRAPH  PRINTS GRAPH OF P(I)'S USED IN RECODING
C
      WRITE(6,1800) LEVEL, MBE
1800  FORMAT ( '1GRAPH OF P(I) USED IN RECODING FOR LEVEL',I2,
     1    ' AND M OF', I4   /  ' '  )
C
      IF ( N .LE. 101 ) GO TO 150
      COMP=N/100.
      NWIDE=101
      GO TO 170
```

```
150   CONTINUE
      NWIDE=N
      COMP=1.0
170   CONTINUE
C
      DO 200 J=1, NWIDE
      DO 200 I=1, 51
      IMAGE(I,J)=BLANK
200   CONTINUE
C
C        VERTICAL LINES AT RIGHT AND LEFT EDGE
      DO 235 I=2, 50
      IMAGE(I,1)=VERT
      IMAGE(I,NWIDE)=VERT
235   CONTINUE
C
C        FIND LOCATIONS OF VERTICAL LINES DIVIDING GRAPH
      IS=0
      DO 220 K=1,5
      IS=IS+NUMT(K)
      SPOT(K)=IS/COMP
      IF(SPOT(K).EQ.0) SPOT(K)=1
220   CONTINUE
C
C        INSERT VERTICAL LINES DIVIDING GRAPH
      DO 230 K=1,5
      DO 230 I=2, 50
      IF(I.EQ.26) GO TO 230
      IMAGE(I,SPOT(K))=VERT
230   CONTINUE
C
C        HORIZONTAL LINES
      DO 240 II=1,3
      I=1+(II-1)*25
      DO 240 J=1, NWIDE
      IMAGE(I,J)=HORZ
      IF( J.EQ.1  .OR.  J.EQ.NWIDE ) IMAGE(I,J)=PLUS
      DO 238 L=1, 5
      IF(J.EQ. SPOT(L) ) IMAGE(I,J) = PLUS
238   CONTINUE
240   CONTINUE
C
C        INSERT POINTS TO BE PLOTTED INTO GRAPH
      DO 300 J=1, N
      JJ=COMP*J
      IF(JJ.EQ.0) JJ=1
      II=1+ 50.*X(JJ)
      IMAGE(II,JJ)=STAR
300   CONTINUE
C
C
C        PRINT GRAPH
C
      DO 500 I=1, 51
      II=52-I
      ORDIN=.02*(II-1)
      WRITE(6,430) ORDIN, (IMAGE(II,J), J=1,NWIDE )
430   FORMAT ( ' ',  F4.2,  1X,  101A1  )
500   CONTINUE
C
```

```
C        PRINT TYPES OF THE P(I)'S ACROSS BOTTOM OF GRAPH
C
      WRITE(6, 600) ( TY(I), I=1, NWIDE )
600   FORMAT ( 6X, 101I1 )
C
      RETURN
      END
      SUBROUTINE SYMBOL ( Y, ALPHA, N,     K, CHAR,   NCHAR ,LETTER,N26)
      IMPLICIT INTEGER (A-Z)
      INTEGER Y(N)
      INTEGER ALPHA(N)
      INTEGER CHAR(NCHAR )
      INTEGER LETTER(N26)
C
C        CONVERTS NUMERICAL STRING INTO ALPHABETIC SYMBOL STRING
C
C       IN:
C         Y=STRING OF NUMBERS
C           0,1,2,...
C         N=LENGTH OF Y
C         NCHAR=NUMBER OF SYMBOLS IN VECTOR CHAR
C         CHAR=VECTOR OF TERMINAL SYMBOLS
C           INCLUDING INITIAL PUNCTUATION, IF ANY
C           ALSO INCLUDING THE SYMBOLS OF ZLM
C         LETTER= VECTOR OF NON-TERMINAL SYMBOLS
C         N26=SIZE OF VECTOR  LETTER
C       OUT:
C         K=LENGTH OF ALPHA STRING PRODUCED
C         ALPHA=RESULTING STRING OF CHARACTERS
C         ALPHA MUST BE DIMENSIONED TO AT LEAST 4
C           NB: ALPHA AND Y MUST BE DISTINCT
C             IF MORE THAN N26 NON-TERMINALS ARE NEEDED
C
      DATA LP / '<' /
      DATA RP / '>' /
      DATA NBLANK / ' ' /
C
C
      ALPHA(1)=NBLANK
      ALPHA(2)=NBLANK
      ALPHA(3)=NBLANK
      ALPHA(4)=NBLANK
      K=0
      IF(N.NE.0) GO TO 5
      K=4
      GO TO 999
5     CONTINUE
      DO 15 I=1, N
C
C        TERMINAL SYMBOLS (INCLUDING INITIAL PUNCTUATION, IF ANY)
C            INCLUDING ZLM
      IF ( .NOT. (Y(I).LT.NCHAR)) GO TO 6
      K=K+1
      ALPHA(K) = CHAR( Y(I) + 1     )
      GO TO 15
C
C        FIRST N26 NON-TERMINAL SYMBOLS (SINGLE SYMBOL)
6     CONTINUE
      IF( .NOT. ( Y(I).LT. NCHAR+N26)    ) GO TO 7
      K=K+1
```

```
      ALPHA(K) = LETTER ( Y(I) +1-NCHAR)
      GO TO 15
C
C        LATER NON-TERMINALS (PARENTHESES AND 2 NON-TERMINALS)
7     CONTINUE
      L1=(Y(I)-NCHAR)/ N26
      L2= Y(I) - NCHAR - L1*N26  +1
      K=K+1
      ALPHA(K)= LP
      K=K+1
      ALPHA(K)=LETTER(L1)
      K=K+1
      ALPHA(K)=LETTER(L2)
      K=K+1
      ALPHA(K)=RP
15    CONTINUE
999   CONTINUE
      RETURN
      END
      SUBROUTINE MCRE(PRE,POST,M,MASK, DONT,CTXT,PERC,MODE,PUNC )
      IMPLICIT INTEGER(A-Z)
      INTEGER PRE(M)
      INTEGER POST(M)
      INTEGER MASK(M)
C        CREATES MASK FROM  PRE  AND POST  SEQUENCES
      DO 860 JD=1, M
      K=M+1-JD
      IF(PRE(JD) .EQ. PERC ) MASK(K) = PERC
      IF(POST(JD) .EQ. CTXT ) MASK(K) = CTXT
      IF( POST(JD) .EQ. DONT ) MASK(K) = DONT
860   CONTINUE
      RETURN
      END
      SUBROUTINE GEN1 ( Y, N, NS, ALPHA , CHAR, NCHAR )
      IMPLICIT INTEGER (A-Z)
      INTEGER Y(NS)
      INTEGER ALPHA(NS)
      INTEGER CHAR(NCHAR)
      READ ( 5, 12) N
12    FORMAT ( I5)
      READ ( 5, 20)  (ALPHA(I), I=1, N )
20    FORMAT ( 80A1 )
      DO 30 I=1, N
      CALL IMATCH(CHAR, NCHAR, ALPHA(I), Y(I)  )
      Y(I)=Y(I)-1
30    CONTINUE
      RETURN
      END
      SUBROUTINE LOC (NS,N,M1,STRICT,MBE,SQLEN,TYPE,ALLOW,PRE,C,Y,MTSIZE
     1,M2MAX,POST,SQPRED,MODE,DONT,PUNC,INDEX,SEQNO,NST,SQCTXT,PERC,CTXT
     2,VDIR,USED,CLAST,VSTART,M2)
      IMPLICIT INTEGER (A-Z)
      INTEGER Y(NS)
      INTEGER SEQNO(NS)
      INTEGER SQCTXT(MTSIZE)
      INTEGER SQLEN(MTSIZE)
      INTEGER SQPRED(MTSIZE)
      INTEGER TYPE(MTSIZE)
      INTEGER INDEX(MTSIZE)
      INTEGER USED(MTSIZE)
```

```
      INTEGER POST(M2MAX)
      INTEGER PRE(M2MAX)
C
C
C        SEARCH Y FOR ALLOWABLE SEQUENCES
C
C       IN:
C         MBE=VALUE OF M TO BE USED IN THIS RECODING
C       OUT:
C
C
      IF (NST.EQ.0) GO TO 2141
C
C        VARIOUS M = M1 ...  MBEST
      IF (VDIR.EQ.1) M=M1
      IF (VDIR.EQ.-1) M=M2
      IF (STRICT.EQ.1) M=MBE
2123  CONTINUE
C
C        CONSIDER T= M, ... N
      T=M
2125  CONTINUE
      IF (VSTART.EQ.1.AND.T-QLAST.LT.MBE) GO TO 2137
C
      DO 2135 LJ=1,2
      DO 2134 LTA=1, ALLOW
      DO 2133 LLL=1,NST
      L=INDEX(LLL)
      IF (LJ.EQ.1.AND.USED(L).EQ.0) GO TO 2133
      IF (LJ.EQ.2.AND.USED(L).EQ.1) GO TO 2133
      IF (SQLEN(L).NE.M) GO TO 2133
      IF(TYPE(L).NE. LTA ) GO TO 2133
      IF (MODE.EQ.1) GO TO 2129
      DO 2127 JD=1,M
      TMJD=T-M+JD
      IF (Y(TMJD).NE.PUNC) GO TO 2127
      QLAST=TMJD
      GO TO 2137
2127  CONTINUE
2129  CONTINUE
      CALL CONV (PRE,M,C,SQCTXT(L))
      CALL CONV (POST,M,C,SQPRED(L))
      DO 2131 JD=1,M
      TMJD=T-M+JD
      IF (MODE.EQ.2.AND.Y(TMJD).EQ.PUNC) GO TO 2137
      IF (POST(JD).EQ.DONT) GO TO 2131
      IF (PRE(JD).EQ.PERC.AND.POST(JD).NE.Y(TMJD)) GO TO 2133
      IF (POST(JD).EQ.CTXT.AND.PRE(JD).NE.Y(TMJD)) GO TO 2133
2131  CONTINUE
C
C        LOCATED SEQUENCE OF LENGTH M AT TIME T
C
C        SEQNO(T)=THE ROW (L) IN STPHI,STNEXT,STLEN TABLES OF THE
C          LONGEST SEQUENCE OF ALLOWABLE TYPE ENDING AT TIME T
C
      USED(L)=1
      SEQNO(T)=L
      GO TO 2137
C
2133  CONTINUE
```

```
2134  CONTINUE
2135  CONTINUE
2137  CONTINUE
C
      T=T+1
      IF (T.LE.N) GO TO 2125
C
      M=M+VDIR
      IF (STRICT.EQ.1) GO TO 2139
      IF (VDIR.EQ.-1.AND.M.GE.M1) GO TO 2123
      IF (VDIR.EQ.1.AND.M.LE.MBE) GO TO 2123
2139  CONTINUE
C
2141  CONTINUE
C
      RETURN
      END
      SUBROUTINE TLOC (NS,M1,STRICT,MBE,SQLEN,TYPE,ALLOW,PRE,C,Y,MTSIZE,
     1M2MAX,POST,SQPRED,MODE,DONT,PUNC,INDEX,NST,SQCTXT,PERC,CTXT,VSPEC,
     2SPEC,USED,T,L,VSTART,VDIR,QLAST,LAWL,M2)
      IMPLICIT INTEGER (A-Z)
      INTEGER Y(NS)
      INTEGER USED(MTSIZE)
      INTEGER SQCTXT(MTSIZE)
      INTEGER SQLEN(MTSIZE)
      INTEGER SQPRED(MTSIZE)
      INTEGER TYPE(MTSIZE)
      INTEGER INDEX(MTSIZE)
      INTEGER POST(M2MAX)
      INTEGER PRE(M2MAX)
C
C        SEARCH Y FOR ALLOWABLE SEQUENCES
C          GIVEN T
C
C
C       IN:
C         MBE=VALUE OF M TO BE USED IN THIS RECODING
C       OUT:
C
C
      VSPEC=0
      SPEC=0
      IF (NST.EQ.0) GO TO 1639
      IF (VSTART.EQ.1.AND.T-QLAST.LT.MBE) GO TO 1639
C
C        VARIOUS M = M1 ...  MBEST
      IF (VDIR.EQ.1) M=M1
      IF (VDIR.EQ.-1) M=M2
      IF (STRICT.EQ.1) M=MBE
C
1623  CONTINUE
      DO 1633 LJ=1,2
      DO 1632 LTA=1, ALLOW
      DO 1631 LLL=1,NST
      L=INDEX(LLL)
      IF (LJ.EQ.1.AND.USED(L).EQ.0) GO TO 1631
      IF (LJ.EQ.2.AND.USED(L).EQ.1) GO TO 1631
      IF (SQLEN(L).NE.M) GO TO 1631
      IF(TYPE(L) .NE. LTA ) GO TO 1631
      IF (MODE.EQ.1) GO TO 1627
```

```
      DO 1625 JD=1,M
      TMJD=T-M+JD
      IF (Y(TMJD).NE.PUNC) GO TO 1625
      QLAST=TMJD
      GO TO 1639
1625  CONTINUE
1627  CONTINUE
      CALL CONV (PRE,M,C,SQCTXT(L))
      CALL CONV (POST,M,C,SQPRED(L))
      DO 1629 JD=1,M
      TMJD=T-M+JD
      IF (MODE.EQ.2.AND.Y(TMJD).EQ.PUNC) GO TO 1635
      IF (POST(JD).EQ.DONT) GO TO 1629
      IF (PRE(JD).EQ.PERC.AND.POST(JD).NE.Y(TMJD)) GO TO 1631
      IF (POST(JD).EQ.CTXT.AND.PRE(JD).NE.Y(TMJD)) GO TO 1631
1629  CONTINUE
C        LOCATED SEQUENCE OF LENGTH M AT TIME T
      IF (LJ.EQ.1) VSPEC=L
      IF (LJ.EQ.2) SPEC=L
1631  CONTINUE
1632  CONTINUE
1633  CONTINUE
1635  CONTINUE
      M=M+VDIR
      IF (STRICT.EQ.1) GO TO 1637
      IF (VDIR.EQ.-1.AND.M.GE.M1) GO TO 1623
      IF (VDIR.EQ.1.AND.M.LE.MBE) GO TO 1623
1637  CONTINUE
C
1639  CONTINUE
      L=0
      IF (SPEC.NE.0) L=SPEC
      IF (VSPEC.NE.0) L=VSPEC
      IF (L.NE.0) USED(L)=1
C
      RETURN
      END
      SUBROUTINE GEN2 ( STRING,N,NS,MODE,NCHAR,PUNC)
      IMPLICIT INTEGER (A-Z)
      INTEGER STRING(NS)
      REAL PROB(20)
      REAL YFL
C
C        SPECIAL BINARY SEQUENCE GENERATOR FOR SENTENCES OF UNIFORM
C          LENGTH
C
C        TERMINAL SYMBOLS: 0,1  (S0, S1)
C
C
C        RULES OF PRODUCTION:
C
C          S --> XA
C
C          A --> YA
C          A --> ZA
C          A --> WA
C
C          X --> 11
C          Y --> 01
C          Z --> 10
```

```
C           W --> 00
C
C
C       IN:
C         NS=DIMENSICNED SIZE OF STRING
C         NCHAR=NUMBER OF TERMINAL SYMBOLS (INCLUDING PUNCTUATION,IF ANY)
C         MODE=DETERMINES METHOD OF INITIAL PUNCTUATION
C       OUT:
C           N=SIZE OF STRING PRODUCED
C           STRING=THE STRING PRODUCED
C       READ IN FROM CARD:
C           NEAR=APPROXIMATE SIZE OF STRING DESIRED
C         BEGIN=SEED FOR RAND
C         N8=SENTENCE LENGTH
C         DIV: 2=USE NON-TERMINALS Y AND Z   3=USE Y,Z, AND W
C
      SO=3
      S1=4
C
C        READ ONE CONTROL CARD
C
C
      READ (5, 13) NEAR, BEGIN, N8, DIV
13    FORMAT ( 16I5 )
C
      DO 8 L=1, DIV.
 8    PROB(L)=0
C
      N=0
C
C        NEW SENTENCE
C
 4    CONTINUE
      IF ( N .GT. NEAR ) GO TO 30
C
C        LETTER X=11
C          SENTENCE ALWAYS BEGINS WITH AN  X
      N=N+1
      STRING(N)=S1
      N=N+1
      STRING(N)=S1
      J=2
C
C        INSERT Y, Z, OR W RANDOMLY
C
 5    CONTINUE
      CALL RANDU(BEGIN, IY, YFL)
      BEGIN=IY
      IND= YFL*DIV+1.0
      PROB(IND) = PROB(IND) +1.0
      IF ( J .GE. N8)  GO TO 28
      GO TO ( 10, 20, 25), IND
C
C        LETTER Y=01
10    CONTINUE
      N=N+1
      STRING(N)=SC
      N=N+1
      STRING(N)=S1
      J=J+2
```

```
      GO TO 5
C
C        LETTER Z=10
20    CONTINUE
      N=N+1
      STRING(N)=S1
      N=N+1
      STRING(N)=S0
      J=J+2
      GO TO 5
C
C        LETTER W=00
25    CONTINUE
      N=N+1
      STRING(N)=S0
      N=N+1
      STRING(N)=S0
      J=J+2
      GO TO 5
C
C        END OF SENTENCE
C        INCLUDE INITIAL PUNCTUATION MARK, IF MODE=2
C
28    CONTINUE
      IF( MODE .NE. 2 ) GO TO 4
      N=N+1
      STRING(N)=PUNC
      GO TO 4
C
C        PRINT OUT PROBS AS CHECK
C
30    CONTINUE
      DO 38 L=1, DIV
      PROB(L) = PROB(L) / N * 2.0
38    CONTINUE
      WRITE (6, 99) ( PROB(I), I=1, DIV )
99    FORMAT ( 10F10.8 )
C
      RETURN
      END
      SUBROUTINE HUFF( M, C, M2,  P, LEN, KODE, PC, ACT, TAB ,LMIN,MKD,
     1  LIST )
      IMPLICIT INTEGER ( A-Z )
      REAL P(M)
      REAL PC(M2)
      INTEGER LEN(M)
       INTEGER KODE(M, MKD)
      INTEGER ACT(M2)
      INTEGER TAB(M,C )
      INTEGER LMIN(C)
      INTEGER LIST(M)
      REAL SUM
      REAL MIN
C
C        HUFFMAN CODING SCHEME
C          STARTS WITH M MESSAGES OF PROB P(I)
C          CODES THEM INTO SEQUENCES OF C SYMBOLS
C
C       IN:
C         M= # OF MESSAGES
```

```
C         M2= 2*M
C         C=# OF SYMBOLS IN ALPHABET FOR MESSAGE
C         P(I) = PROB OF MESSAGE I
C        MKD=MAX LENGTH OF ANY ENCODED MESSAGE (2ND SUB OF KODE)
C       OUT:
C         KODE(I,J) = CODE FOR MESSAGE I ( USES LEN(I) SYMBOLS)
C         LEN(I)= LENGTH OF CODE FOR ORIGINAL SEQ I
C       SCRATCH VECTORS:
C         PC(I) = ORIGINAL VECTOR P PLUS ADDITIONAL LINES
C         ACT(I) = 0 IMPLIES LINE IS INACTIVE ,   1 IMPLIES ACTIVE
C         TAB(I,J)=THE LINES USED TO CREATE LINE M+I IN TABLE
C           LINE M+1 HAS ONLY MO ENTRIES. OTHER LINES HAVE DEL=C
C         LMIN(I)= LOCATION OF  ONE OF THE I SMALLEST PC'S
C         LIST= C-ARY NUMBER USED IN TREE SEARCH
C
C*************************************************************************
C         CALLING SEQUENCE
C*************************************************************************
C  NEED  NME,CC,PB
C     REAL PC(400)
C     DIMENSION KODE(200,20)
C     DIMENSION TAB(200,2)
C     DIMENSION LEN(200)
C     DIMENSION ACT(400)
C     DIMENSION LIST(200)
C     DIMENSION LMIN(10)
C     MKD=20
C         RECODING
C       IN TO HUFF:
C         NME=NUMBER OF MESSAGES TO BE RECODED
C         CC=NUMBER OF SYMBOLS
C         PB=PROBABILITY VECTOR
C         MKD=DIMENSIONED SIZE OF 2ND DIM OF KODE
C         MKK=TWICE NME
C       PRODUCED  BY  SUBRCUTINE: KODE, LEN
C         KODE(I,J) = CODE FOR MESSAGE I ( USES LEN(I) SYMBOLS)
C         LEN(I)= LENGTH OF CODE FOR ORIGINAL SEQ I
C         SCRATCH FOR HUFF: TAB, ACT, LIST, LMIN
C     IF(EXR.NE.2) GO TO 980
C     IF(NME.EQ.0) GO TO 980
C     MKK= 2* NME
C     CALL HUFF(NME, CC, MKK,PB, LEN, KODE, PC, ACT, TAB ,LMIN,MKD,LIST)
C 980 CONTINUE
C*************************************************************************
C
C
C
C
C
C         INITIALIZE
      DO 7 I=1, M
      ACT(I) =1
      PC(I) = P(I)
      LEN(I) =0
7     CONTINUE
      DO 9 L=1, C
      DO 9 I=1, M
      TAB(I,L) =0
9     CONTINUE
C
```

```
C         FIND MO
      IF ( C .EQ. 2 ) GO TO 20
      CL=C-1
      DO 10 MO=2,C
      NU= M - MO
      Q= NU / CL
      IF ( Q * CL    .EQ. NU ) GO TO 30
10    CONTINUE
20    CONTINUE
      MO=2
30    CONTINUE
C
C         MAIN SECTION
C         FIRST TIME DEL=MO, THEN DEL=C
C
      DEPTH=M
       TDEPTH=0
      NACT=M
      DEL=MO
C
C         FIND  DEL  SMALLEST PROBS
C         GET SUM OF DEL SMALLEST PROBS
C
22    CONTINUE
       SUM=0.0
      DO 70 L=1, DEL
      MIN=999.
      DO 60 I=1, DEPTH
      IF ( MIN .LE. PC(I) .OR. ACT(I) .NE. 1 ) GO TO 60
      MIN = PC(I )
       LMIN(L) = I
60    CONTINUE
      II=LMIN(L)
      ACT( II ) =0
      SUM=SUM + PC( II )
70    CONTINUE
      NACT=NACT - DEL
C
C         CREATE NEW LINE IN PC-TABLE EQUAL TO SUM OF THE DEL SMALLEST
C
      DEPTH=DEPTH +1
       TDEPTH=TDEPTH+1
      ACT(DEPTH) =1
      PC(DEPTH) = SUM
       DO 90 J=1, C
      TAB( TDEPTH, J ) = LMIN(J)
90     CONTINUE
C
      IF ( NACT .LT. 1 ) GO TO 100
      NACT=NACT+1
      DEL = C
      GO TO 22
100   CONTINUE
C
C         CREATE CODE SEQUENCES
C
      DO 120 I=1, M
      LIST(I) =1
120   CONTINUE
C
```

```
C        TREE SEARCH
125   CONTINUE
      II=TDEPTH
      K=0
130   CONTINUE
      K=K+1
      II = TAB( II, LIST(K)    )
      IF ( II .LE. M ) GO TO 140
      II=II - M
      GO TO 130
C
C        REACHED ENDPOINT OF TREE
140   CONTINUE
      IF ( II .EQ. 0 ) GO TO 190
      DO 150 LL =1, K
      KODE(II,LL) = LIST(LL )  -1
150   CONTINUE
      LEN(II) = K
190   CONTINUE
      LIST(K) = LIST(K) +1
      IF ( LIST(K) .LE. C ) GO TO 125
      LIST(K) =1
      K=K-1
       IF ( K .GT. 0 ) GO TO 190
C
C        PRINT CODES
C
      PRH=0
      IF ( PRH .EQ. 0 ) GO TO 361
      DO 360 I=1, M
      JJ= LEN (I)
      WRITE (6, 341 ) I,  ( KODE(I,J), J=1, JJ )
341   FORMAT ( 1X, I4, 5X, 120I1 )
360   CONTINUE
361   CONTINUE
      RETURN
      END
      SUBROUTINE REMOVE ( IN, OUT, NIN, NOUT, FILL )
      IMPLICIT INTEGER(A-Z)
      INTEGER IN(NIN)
      INTEGER OUT(NIN)
      TG=0
      DO 10 T=1, NIN
      IF( IN(T) .EQ. FILL ) GO TO 10
      TG=TG+1
      OUT(TG)=IN(T)
10    CONTINUE
      NOUT=TG
      RETURN
      END
      SUBROUTINE ZERO(NVECT,N)
      ENTRY IZERO(NVECT,N)
      INTEGER NVECT(N)
C        SUBROUTINE TO ZERO VECTOR
      DO 10 I=1,N
      NVECT(I)=0
10    CONTINUE
      RETURN
      END
      SUBROUTINE FZERO(NVECT,N)
```

```
      REAL NVECT(N)
C        SUBROUTINE TO ZERO VECTOR
      DO 10 I=1,N
      NVECT(I)=0.0
10    CONTINUE
      RETURN
      END
      SUBROUTINE SPRAY(NVECT,N,NVAL)
      ENTRY ISPRAY(NVECT,N,NVAL)
      INTEGER NVECT(N)
C        SUBROUTINE TO SET VECTOR TO A CONSTANT
      DO 10 I=1, N
      NVECT(I)=NVAL
10    CONTINUE
      RETURN
      END
      FUNCTION ISUM(NVECT,N)
      INTEGER NVECT(N)
C        (SUM  FINDS SUM OF VECTOR
      ISUM=0
      DO 10 I=1, N
      ISUM=ISUM+NVECT(I)
10    CONTINUE
      RETURN
      END
      SUBROUTINE PHI ( DIGIT, M, C, SUM )
      IMPLICIT INTEGER (A-Z)
      INTEGER DIGIT(M)
C
C        PHI  ASSIGNS AN INDEX NUMBER TO A GIVEN M-SEQUENCE
C          PHI=NATURAL NUMBER CORRESPONDING TO M DIGITS MODULO C
C
C        IN: DIGIT, M, C
C          DIGIT=VECTOR OF M DIGITS MOD C
C        OUT: PHI
C          PHI=NATURAL NUMBER PRODUCED
C
      SUM=0
      E=1
       DO 120 L=1,M
      SUM=SUM+E*DIGIT(M+1-L)
      E=E*C
120   CONTINUE
       RETURN
      END
      SUBROUTINE CONV ( DIGIT, M, C, NUM )
      IMPLICIT INTEGER ( A-Z)
      INTEGER DIGIT(M)
C
C        CONVERTS NUMBER 'NUM' TO M DIGITS MOD C
C
C        IN: NUM, M, C
C          NUM=NATURAL NUMBER
C        OUT: DIGIT
C          DIGITS= M DIGITS MOD C
C
      N=NUM
      DO 10 L=1, M
      Q=N/C
      DIGIT ( M-L+1 ) = N-Q*C
```

```
      N=Q
10    CONTINUE
      RETURN
      END
      SUBROUTINE XFSORT ( V, N, INDEX, N1, N2 )
      ENTRY      FXSORT ( V, N, INDEX, N1, N2 )
      REAL V(N)
      REAL T
      INTEGER INDEX(N)
C
C        INDEXED SORTING RCUTINE.
C          SORTS BLOCK WITHIN VECTOR V INTO HIGH...LOW ORDER
C
C       IN:
C           V=VECTOR OF LENGTH N OF VALUES TO BE SORTED
C         N=DIMENSICNED SIZE OF V
C         N1=INDEX OF BEGINNING OF BLOCK WITNIN V TO BE SORTED
C         N2=INDEX OF END OF BLOCK WITHIN V TO BE SORTED
C       OUT:
C         INDEX(I) = SUBSCRIPT OF I-TH LARGEST ELEMENT OF V
C       CHANGED:
C         THAT PART CF VECTOR V WHICH IS SORTED
C
C         NB: IN MAIN, INDEX VECTOR V BY 2, AND ALL OTHER ASSOCIATED
C           VECTORS BY   INDEX(I)
C
      DO 324 KJ= N1, N2
324     INDEX(KJ) = KJ
      IF ( N1 .GE. N2 ) GC TO 999
      NL= N2-1
      DO 300 L= N1, NL
       IF ( V(L) .GE. V(L+1 )       )  GO TO 300
C        V(L+1) GREATER
      LLL= L+1 - N1
      DO 100 J=1, LLL
        K=L+1 -J
         IF( V(K) .GE. V(K+1)  ) GO TO 200
      T=V(K)
      V(K)       = V(K+1)
      V(K+1) = T
      NT=INDEX(K)
      INDEX(K) = INDEX(K+1)
        INDEX(K+1) = NT
100     CONTINUE
200      CONTINUE
300     CONTINUE
999   CONTINUE
      RETURN
      END
      SUBROUTINE LOOK ( LLIST,       LDIM,LEN,ITEM,LEVEL,NEW)
      IMPLICIT INTEGER (A-Z)
      INTEGER LLIST(LDIM)
C
C         ITEM=CURRENT ELEMENT
C         LDIM=DIMENSIONED SIZE OF LLIST
C         LEN=ACTUAL LENGTH OF LLIST
C         LEVEL=INDEX OF PLACE WHERE ITEM IS FCUND IN LLIST
C         LLIST(I)=LIST OF ITEMS
C         NEW:   0=ITEM WAS IN LLIST ALREADY
C                1=ITEM IS NEW, AND WAS ADDED TO LLIST
```

```
C
C       INITIALIZE: LEN  BEFORE FIRST ENTRY
C
C
      IND=1
      GO TO 50
      ENTRY      LOCKN( LLIST,       LDIM,LEN,ITEM,LEVEL,NEW)
      IND=0
50    CONTINUE
      IF(LEN.EQ.0) GO TO 150
      DO 100 I=1, LEN
      IF(LLIST(I).EQ. ITEM ) GO TO 200
100   CONTINUE
C        NO MATCH
150   CONTINUE
      NEW=1
      IF(IND.EQ.0  .OR.  LEN.LT. LDIM ) GO TO 168
      PRINT 166, LDIM
166   FORMAT ( '0TOO MANY ELEMENTS IN LIST OF LENGTH ', I6 )
C        NB: LEVEL SET TO LAST POSSIBLE LEVEL IN THIS CASE
      LEVEL=LDIM
      GO TO 250
168   CONTINUE
      LEN=LEN+1
      LLIST(LEN)=ITEM
      LEVEL=LEN
      GO TO 250
C        CURRENT  'ITEM'  MATCHES WITH EXISTING ELEMENT OF LLIST
200   CONTINUE
      NEW=0
      LEVEL=I
250   CONTINUE
      RETURN
      END
      SUBROUTINE ISEEK ( NVECT, N, NOW, KODE )
      INTEGER NVECT(N)
      DO 10 I=1, N
      IF ( NVECT(I) .EQ. NOW ) GO TO 20
10    CONTINUE
      KODE=0
      RETURN
20    CONTINUE
      KODE=I
      RETURN
      END
      SUBROUTINE IXSPRY(NVECT,NDIM,NVAL, I1, I2 )
      INTEGER NVECT(NDIM)
      DO 10 I=I1, I2
      NVECT(I)=NVAL
10    CONTINUE
      RETURN
      END
      SUBROUTINE IEQUAL ( JVECT, N, KVECT )
      INTEGER JVECT(N)
      INTEGER KVECT(N)
      DO 10 I=1, N
      JVECT(I)=KVECT(I)
10    CONTINUE
      RETURN
      END
```

```
      SUBROUTINE MATCH ( NVECT, N, NOW, KODE )
      ENTRY IMATCH(NVECT,N,NOW,KODE)
      INTEGER NVECT(N)
      DO 10 I=1, N
      IF ( NVECT(I) .EQ. NOW ) GO TO 20
10    CONTINUE
       PRINT 15, NOW
15    FORMAT ( 'OUNABLE TO MATCH:', A4)
      KODE=0
      RETURN
20    CONTINUE
      KODE=I
      RETURN
      END
      SUBROUTINE RANDU(IX,IY,YFL)
C
C           COMPUTES UNIFORMLY DISTRIBUTED RANDOM REAL NUMBERS BETWEEN
C           0 AND 1.0 AND RANDOM INTEGERS BETWEEN ZERO AND
C           2**31. EACH ENTRY USES AS INPUT AN INTEGER RANDOM NUMBER
C           AND PRODUCES A NEW INTEGER AND REAL RANDOM NUMBER.
C
C        DESCRIPTION OF PARAMETERS
C           IX - FOR THE FIRST ENTRY THIS MUST CONTAIN ANY ODD INTEGER
C                NUMBER WITH NINE OR LESS DIGITS. AFTER THE FIRST ENTRY,
C                IX SHOULD BE THE PREVIOUS VALUE OF IY COMPUTED BY THIS
C                SUBROUTINE.
C           IY - A RESULTANT INTEGER RANDOM NUMBER REQUIRED FOR THE NEXT
C                ENTRY TO THIS SUBROUTINE. THE RANGE OF THIS NUMBER IS
C                BETWEEN ZERO AND 2**31
C           YFL- THE RESULTANT UNIFORMLY DISTRIBUTED, FLOATING POINT,
C                RANDOM NUMBER IN THE RANGE 0 TO 1.0
C
C     REAL YFL
C     IX=65549
C     DO 1 K=1, 100
C     CALL RANDU ( IX, IY, YFL )
C  1  CONTINUE
C
C        REMARKS
C           THIS SUBROUTINE IS SPECIFIC TO SYSTEM/360 AND WILL PRODUCE
C           2**29 TERMS BEFORE REPEATING.  THE REFERENCE BELOW DISCUSSES
C           SEEDS (65539 HERE), RUN PROBLEMS, AND PROBLEMS CONCERNING
C           RANDOM DIGITS USING THIS GENERATION SCHEME.  MACLAREN AND
C           MARSAGLIA, JACM 12, P. 83-89  DISCUSS CONGRUENTIAL
C           THE RANDU TYPE, ONE FILLING A TABLE AND ONE PICKING FROM THE
C           TABLE, IS OF BENEFIT IN SOME CASES.  65549 HAS BEEN
C           SUGGESTED AS A SEED WHICH HAS BETTER STATISTICAL PROPERTIES
C           FOR HIGH ORDER BITS OF THE GENERATED DEVIATE.
C           SEEDS SHOULD BE CHOSEN IN ACCORDANCE WITH THE DISCUSSION
C           GIVEN IN THE REFERENCE BELOW.  ALSO, IT SHOULD BE NOTED THAT
C           IF FLOATING POINT RANDOM NUMBERS ARE DESIRED,AS ARE
C           AVAILABLE FROM RANDU, THE RANDOM CHARACTERISTICS OF THE
C           FLOATING POINT DEVIATES ARE MODIFIED AND IN FACT THESE
C           DEVIATES HAVE HIGH PROBABILITY OF HAVING A TRAILING LOW
C           ORDER ZERO BIT IN THEIR FRACTIONAL PART.
C           POWER RESIDUE METHOD DISCUSSED IN IBM MANUAL C20-8011,
C           RANDOM NUMBER GENERATION AND TESTING
C        INTEGER NUMBERS ARE AL ODD
C
      IY=IX*65539
      IF(IY)5,6,6
    5 IY=IY+2147483647+1
    6 YFL=IY
      YFL=YFL*.4656613E-9
      IX=IY
      RETURN
      END
```

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