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A NETWORK APPROACH TO VIBRATION ANALYSIS

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF FIGURES
(Concluded)

ABSTRACT

This report discusses the application of impedance-network theory to the analysis of mechanical vibrating systems. A description of the relevant mechanical impedance parameters of a machine-mount-foundation system, a translation from mechanical to electrical network concepts, and a review of pertinent four-terminal-network theory led to a method of computing the one-dimensional vibration-response characteristics of distributed as well as lumped parameter systems. The usefulness of the network approach is demonstrated by the analysis of mechanical filters, and by the computation of the response of a sample machine-mount-foundation system. The capabilities and limitations of the method are discussed.

SECTION I. NETWORK THEORY

INTRODUCTION

In the continuing search for better engineering methods directed toward achieving a higher degree of vibration isolation in mechanical systems, the "mechanical filter" concept has been proposed[1] as a possibly fruitful source of further investigation. This paper is the result of carrying out the suggested investigation. In the course of the research, it became apparent that the results obtained with respect to the filter evaluation were equivalent to conclusions reached by authors of several earlier articles.[6,7] However, by arriving at these conclusions independently and in a somewhat more comprehensive context, a method of solution was evolved which is likely to be of general interest inasmuch as: (1) the procedure involves a conceptually simple method of analyzing, in one dimension, the behavior of complex vibrating systems, and (2) the criteria for evaluating the degree of isolation achieved in the system emphasize those design parameters which are of current interest.

Thus it is primarily the purpose of this paper to report this method in a sufficiently elementary and complete form to permit its use without previous familiarity on the part of the user, and secondarily to summarize the body of well-known results which constitute a reply to the memorandum of Ref. 1.

OBJECTIVES OF THE "NETWORK" APPROACH

The problem with which we shall concern ourselves arises from the basic question: how can the isolation of a vibration source from its surrounding environment be best achieved? This question leads naturally to the need for parameters which best describe the degree of isolation. The selection of these parameters, in turn, stimulates the search into the mechanics of the system to determine design criteria which permit optimization of the isolation parameters.

A present, the art of vibration isolation of heavy machinery supported by complex structures has not been developed sufficiently to provide a satisfactory solution to the fundamental problem. Many questions remain unsettled with regard to which parameters best measure the degree of isolation of the system, and consequently which physical variables ought to be controlled to optimize the degree of isolation.

In the past, a clarification of these difficulties was attempted by resorting to the theoretical analysis of one- or two-degree-of-freedom, lumped parameter idealizations of the real physical system. This "harmonic oscillator" idealization provided a fairly suitable means of mount analysis for low-frequency vibration sources, i.e., frequencies corresponding to wavelengths which are large compared to the dimensions of the mount. However, it has not provided any insight into the higher frequency phenomena termed wave effects nor has it proved adequate in the analysis of typical systems in which the machine and foundation do not behave as a simple combination of masses, springs, and dash-pots. Thus it represents an oversimplification of the problem since in many cases information of primary interest is suppressed.

An extension of this method is suggested by the fact that sound possesses, in addition to an oscillatory property, a wave property or propagation characteristic, as it is called. Thus one may be led to an attempt to formulate the problem in terms of the wave equation subject to appropriate boundary conditions. This also has been successfully attempted with regard to the analysis of mount behavior, but it has not been applied, insofar as the author is aware, to the analysis of an entire vibrating complex. This may be due in part to the algebraic difficulty encountered in determining and applying the correct boundary conditions to the solution of the wave equation for this type of problem.

To avoid this difficulty, a modified approach is suggested when one recalls the electrical engineer's preference for solving steady-state circuit problems. Rather than solve simultaneous differential equations subject to boundary conditions, he prefers to introduce the concept of impedance and reduce the differential equations to algebraic equations. This "network" approach has its analog in treating mechanical systems also, and is particularly promising in its application to vibration-isolation problems in view of the current work being done to improve techniques for measuring mechanical impedance.

Hence the method is intended to perform the following functions:

1. To provide a better approximation than lumped parameter theory allows in the theoretical prediction of an entire vibrating system's behavior.

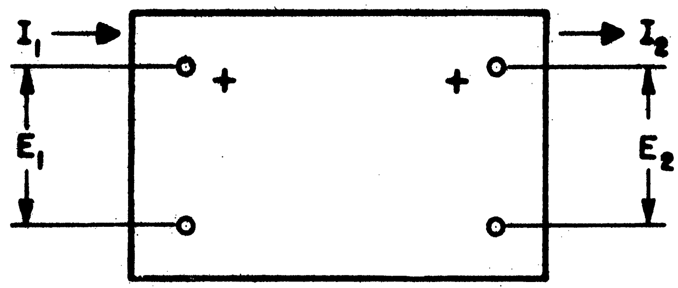
2. To emphasize the specific types of impedance measurements needed to compute, on the basis of this theory, the system response to oscillatory forces.

3. To provide a clearer understanding of the relation of isolation parameters currently used in evaluating the performance of system components to impedance quantities.

4. To demonstrate a conceptually simple approach to the analysis of entire systems.

Since this method is similar to the procedures employed in the analysis of certain types of electrical circuits, and relies heavily on the concepts and terminology of electrical engineering, it seems appropriate to begin with a review of those electrical concepts which will be used most frequently. Specifically, the electrical results needed will be those of four-terminal-network theory and the related transmission-line theory. It might be well at the outset to keep in mind that the force-voltage, current-velocity analog will be employed later in the development.

FOUR-TERMINAL NETWORKS



GENERAL FOUR-TERMINAL NETWORK

FIGURE 1

Figure 1 represents the general four-terminal network. The network within the "blackbox" is arbitrary, subject to the restrictions that the elements be linear and bilateral and that no sources are contained therein. Reference conditions for voltage and current are indicated on the diagram. General loop-analysis methods yield the equations (see, e.g., Ref. 3)

$$E_1 = z_{11}I_1 - z_{12}I_2$$

$$E_2 = z_{21}I_1 - z_{22}I_2 \quad .$$

Interpreting these equations, we see that if the output end is open, $I_2 = 0$, and

$$E_1 = z_{11}I_1 \ , \qquad E_2 = z_{21}I_1 \quad .$$

Consequently z_{11} may be interpreted as the input impedance of end 1 with end 2 open-circuited. Similarly, z_{21} represents the transfer impedance under the same conditions. If the input end is open, $I_1 = 0$, and

$$E_2 = -z_{22}I_2 \ , \qquad E_1 = - z_{12}I_2 \quad .$$

Thus z_{22} and z_{12} are, respectively, the input and transfer impedances of end 2 with end 1 open. The relation between input voltage and current and output voltage and current may be put in a convenient form by defining the following parameters:

$$A = \frac{z_{11}}{z_{21}}$$

$$B = \frac{|z|}{z_{21}} \qquad \text{where } |z| = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} = z_{11}z_{22} - z_{21}z_{12}$$

$$C = \frac{1}{z_{21}}$$

$$D = \frac{z_{22}}{z_{21}}$$

In terms of these parameters, the following identities are readily verified:

$$E_1 = AE_2 + BI_2 \;; \tag{1}$$

$$I_1 = CE_2 + DI_2 \;. \tag{2}$$

Solving for E_2 and I_2 gives

$$E_2 = DE_1 - BI_1 \;; \tag{3}$$

$$-I_2 = CE_1 - AI_1 \;. \tag{4}$$

The characteristic impedance, Z_0, of the network is defined to be that terminating impedance which causes the input impedance of the network to have the value of the terminating impedance. Z_0 may be found from the expression for input impedance in the following way. By definition, the input impedance looking into the left of the network is

$$Z_1 = \frac{E_1}{I_1} = \frac{AE_2 + BI_2}{CE_2 + DI_2} \;.$$

Since the receiving (output) end impedance is

$$Z_r = \frac{E_2}{I_2} \;,$$

then

$$Z_1 = \frac{A\,Z_r + B}{C\,Z_r + D} \;.$$

For a symmetrical network, $z_{11} = z_{22}$, and consequently $A = D$. Then, by definition of characteristic impedance, $Z_1 = Z_r = Z_0$, and it is found immediately that

$$Z_0 = \sqrt{\frac{B}{C}} \quad . \tag{5}$$

The propagation constant, γ, may be defined by

$$\frac{E_1}{E_2} = \frac{I_1}{I_2} = A + \sqrt{A^2 - 1} = e^{\gamma} \quad , \tag{6}$$

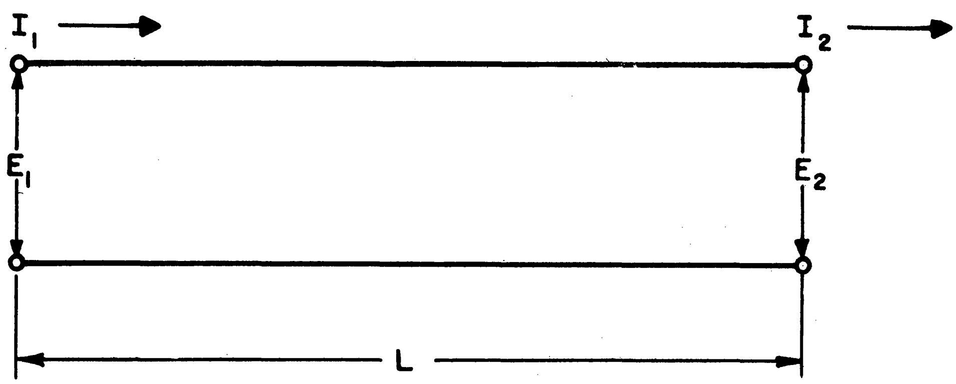
where the ratios are taken for a network terminated in its characteristic impedance. It is then easily shown (see Ref. 3) that the following relations hold for a symmetrical network:

$$A = D = \cosh \gamma \quad ; \tag{7a}$$

$$B = Z_0 \sinh \gamma \quad ; \tag{7b}$$

$$C = \frac{\sinh \gamma}{Z_0} \quad . \tag{7c}$$

The equations describing the properties of four-terminal networks are closely related to those describing the properties of "transmission lines." For if the section of transmission line depicted in Figure 2 were treated as a four-terminal network, precisely the same conclusions would be reached as above except that each γ that appears would be replaced by γL. Thus the four-terminal network treated above represents a unit length of transmission line. Therefore all the relations, especially Eqs. (1) through (7), may be applied directly to a transmission line simply by substituting γL for γ.



TRANSMISSION LINE

FIGURE 2

This brief review of the electrical theory will be sufficient for the present, pending a somewhat detailed account of the analysis of the components of a typical mechanical system. This will consist essentially of two parts. In the first, the machine and foundation will be treated immediately in terms of their electrical analogs. In the second part, a conventional mechanical analysis of the mount will be presented to compare results with the summary of the four-terminal-network theory given above.

PROPERTIES OF THE MECHANICAL ELEMENTS

To fix our ideas in the following considerations, we shall address our attention to the problem of analyzing, as completely as possible in one dimension, the characteristics of a vibrating mechanical complex composed essentially of three basic elements:

1. sources of vibratory energy;
2. elastic elements which serve to transmit the vibratory energy throughout the system (transmission paths); and
3. terminations of the transmission paths.

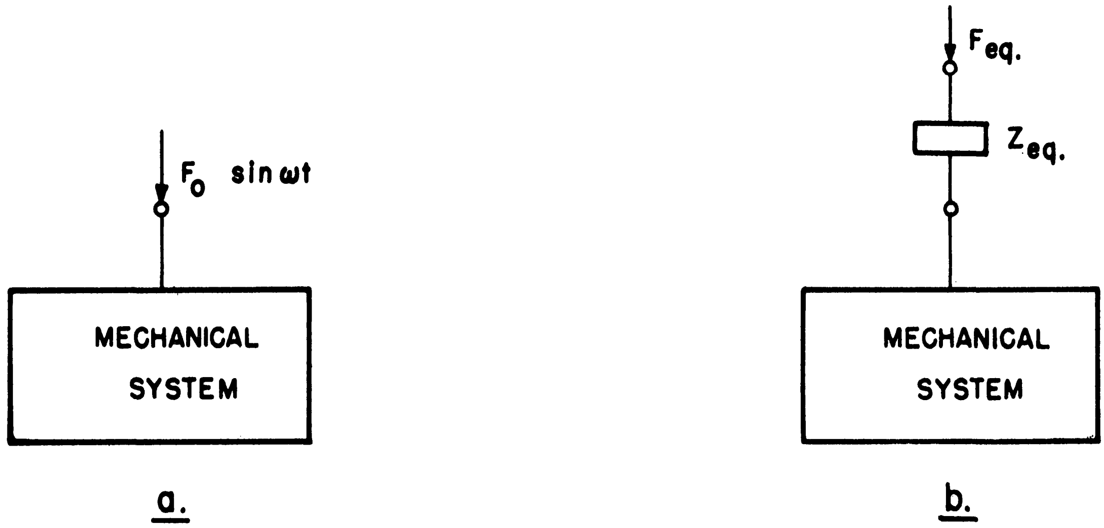
Since such a model represents a highly idealized mechanical system which might apply equally well to an endless variety of physical realizations, we may choose without loss of generality the labels machine, mount, and foundation, to refer respectively to the elements of this system with the understanding that these terms are used in a symbolic rather than real sense.

The Machine.—A machine may be defined, in the sense mentioned above, as any active component of a vibrating system. The term "active" has a meaning analogous to the electrical concept, implying here that a machine contains sources in its make-up as contrasted with the entirely passive nature of all other elements in the system. The idealizations of a machine may assume two forms: (a) a simple vibratory source, e.g., a constant amplitude force or velocity generator; or (b) a mechanical network of simple sources connected together by mechanical elements displaying some combination of the properties of inertia, compliance, and dissipation.

The first of these machine models may be symbolized simply by a vector labeled to indicate the amplitude and frequency of the causal mechanism creating the vibrations, as in Figure 3a. The model representing the second possibility, Figure 3b, may be deduced from the use of the mechanical equivalent of Thevenin's theorem. To apply this theorem, we must first digress briefly to consider the concept of mechanical impedance.

Mechanical impedance is the complex quotient of the alternating force applied to the system by the resulting linear velocity in the direction of the force at its point of application. With symbols, one writes

$$Z_m = F/v \text{ (mechanical ohms)} \quad , \tag{8}$$



SIMPLE MACHINE IDEALIZATION

FIGURE 3

where Z_m is the mechanical impedance, F is the root-mean-square value of the applied sinusoidal force, and v is the root-mean-square value of the velocity. Because Z_m is complex the phase angle between F and v must be known. The similarity between mechanical impedance and the corresponding electrical quantity is immediately apparent by comparing Eq. (8) with the electrical definition

$$Z_e = \frac{E}{I} .$$

Since the statement of the definition will be sufficient for the present, further comment on this concept will be reserved until it is again needed in connection with mount analysis.

The form of Thevenin's theorem applicable to the mechanical network idealization of a machine states that any linear mechanical network containing any number of force and velocity sources of the same frequency can be viewed as an equivalent constant force source, F_{eq}, acting through an equivalent (series) impedance, Z_{eq}, as depicted in Figure 3b. The impedance of the machine must in general be experimentally measured and expressed in the complex form

$$Z_m = R_m + iX_m ,$$

where R_m is the mechanical resistance and X_m is the mechanical reactance.

It may be noted that the measurement of this particular type of impedance entails certain nontrivial difficulties which arise from the physical dissimilarity between mechanical and electrical networks. To satisfy the requirements of Thevenin's theorem, one can, in some types of electrical networks, replace each voltage generator by its internal impedance and then measure the total impedance as seen looking into the two output terminals of the network.

The interpretation of this procedure in mechanical terms becomes immediately complicated since the force generators in a machine cannot be easily isolated to measure their respective internal impedances. Indeed, even if this could be done, there yet would remain the serious difficulty of physically replacing these sources by their impedances to make the total impedance measurement of the machine.

Thus it appears that any direct measurement of machine internal impedance analogous to the type made in some electrical circuits is at best an unlikely possibility. Certainly a procedure involving measuring the velocity occurring at the feet of a machine in response to a force applied to the feet does not provide the impedance value to be used in connection with an application of Thevinin's theorem.

One possible method of circumventing these difficulties is suggested by the electrical technique employed when the previously mentioned method is not applicable. This is done by terminating the electrical network with a variable known impedance and determining the load impedance for which maximum power transfer is obtained. Because the condition of maximum power transfer requires that the load impedance be the complex conjugate of the source impedance, obtaining the condition of maximum power transfer establishes the source impedance in terms of a readily measured load impedance. The problem of the feasibility of this procedure for mechanical systems or of the determination of other possible procedures will be left to the enterprise of the experimentalists for solution. The purpose of the discussion at this point is simply to emphasize that a very specific type of machine internal impedance measurement is needed in connection with the general method of system analysis to be presented.

With this brief look at the properties of a machine, we next consider the foundation, leaving the more extended analysis of the mount to the last.

The Foundation.—It might be well to stress at the outset that the elements of the system which are termed "foundations" may appear physically in a form quite different from that intuitively suggested by the term. For example, suppose the problem consisted of isolating a vibration-sensitive, floor-mounted piece of equipment from a strongly vibrating machine attached to the same floor. In this case, the floor, whose transmission characteristics are to be modified, plays the role of a transmission path or "mount." Hence the vibration-sensitive item must be considered the termination to the transmission path, and therefore is termed the "foundation." Analyzing such a system further, it might develop that mounting the vibrating source machine on an isolation device is the only change permissible in the system. The problem then reassumes the conventional configuration of a machine-mount-foundation by treating the floor as the foundation. This apparent difficulty in deciding what is to be called the termination of a transmission path may be resolved in a given situation by applying the following two rules:

1. Starting from the machine, one selects the number of connected transmission paths whose characteristics are expected to be modified or controlled according to certain specifications. Whatever then exists in physical connection with the terminal end of the last connected transmission path will be termed the foundation.
2. The impedance of all the (passive) elements thus composing the foundation may be expressed in terms of an equivalent foundation impedance

$$Z_f = R_f + iX_f$$

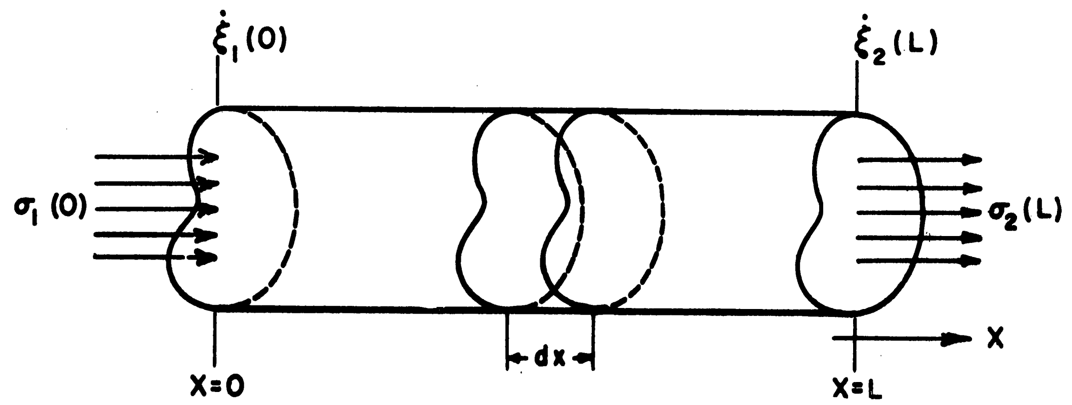
by means of Thevenin's theorem for passive, linear elements.

In general it will be necessary to measure experimentally the mechanical impedance of a foundation in the force direction at the mount's point of connection. Utilization of this concept permits the possibility of considering to some degree the interacting effects of a complex foundation on a machine-mount combination.

The Mount.—Since the problem is being considered in one dimension only, the mount may be treated as a series of connected, individually homogeneous, elastic cylinders of constant cross section, i.e., we consider only the propagation of plane waves in the mount. The model chosen to represent a homogeneous section of the mount may be described, due to its homogeneity, by the following quantities.

ρ = density
c = velocity of sound in the mount material
E_1 = Young's modulus of elasticity
S = cross-sectional area
L = length of the resilient element
m = ρSL = mass of the element

Consider the element shown in Figure 4. Let $\sigma(x)$ be the positive (tensile) stress acting on a cross-sectional element of thickness dx, and $\xi(x)$ denote the displacement of this cross-sectional element from its equilibrium position in the positive x direction. Hence, $\dot{\xi}(x)$ will symbolize particle velocity at a point, x, along the rod. At the end $x = 0$, the element is being acted upon by a sinusoidal force creating a stress and velocity, denoted by σ_1 and $\dot{\xi}_1$, respectively. This activity sets up a stress and velocity distribution along the rod. The respective quantities thus transmitted to the end $x = L$ are labeled σ_2 and $\dot{\xi}_2$. These longitudinal vibrations are described by a second-order partial differential equation which may easily be derived by considering the forces acting



IDEALIZATION OF THE MOUNT

FIGURE 4

upon a cross-sectional element of thickness dx. Since the forces are due to the elasticity and viscosity of the material, we consider them individually. Let μ be the coefficient of viscosity, defined so that it represents the factor of proportionality between the viscous force and the product of area times the time rate of change of strain in the element. Thus the component of the force due to viscosity is defined by

$$F_v = \mu S \frac{\partial}{\partial t}\left(\frac{\partial \xi}{\partial x}\right) = \mu S \frac{\partial^2 \xi}{\partial t \, \partial x} .$$

A similar role is assumed by Young's modulus of elasticity which relates the elastic force and the product of area times strain in the element. Hence, the force component due to elasticity is defined by

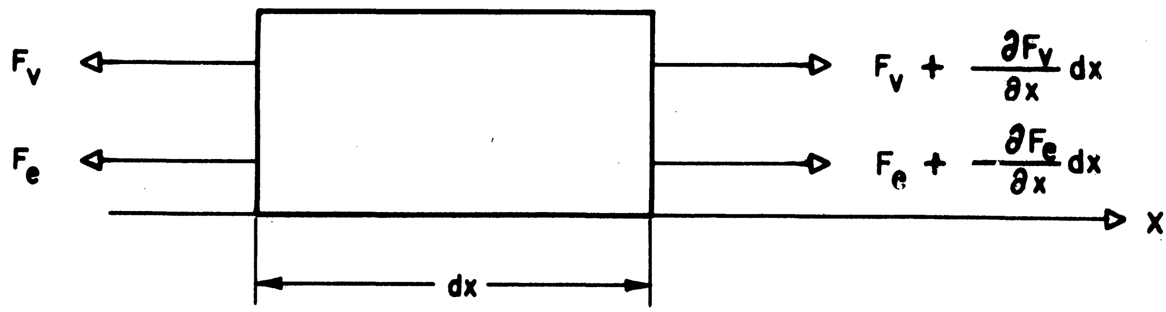
$$F_e = E_1 \cdot S \frac{\partial \xi}{\partial x} .$$

The total force acting on the element dx is depicted in Figure 5. Writing Newton's second law for the element shown, we have

$$\frac{\partial F_e}{\partial x} dx + \frac{\partial F_v}{\partial x} dx = \underbrace{\rho S \, dx}_{m} \cdot \underbrace{\frac{\partial^2 \xi}{\partial t^2}}_{a} .$$

Then by performing the indicated differentiation on F_v and F_e, the equation of motion becomes

$$E_1 \frac{\partial^2 \xi}{\partial x^2} + \mu \frac{\partial}{\partial t}\left(\frac{\partial^2 \xi}{\partial x^2}\right) = \rho \frac{\partial^2 \xi}{\partial t^2} . \tag{9}$$



FORCES ACTING ON A CROSS-SECTIONAL ELEMENT OF THE MOUNT

FIGURE 5

This may be put into the form of the familiar wave equation by assuming the steady-state solution to Eq. (9) is of the form

$$\xi = \xi(x)\, e^{i\omega t} \;, \quad \omega = 2\pi f \;, \quad f = \text{excitation frequency} \;, \tag{10}$$

and defining a complex modulus E so that

$$E = E_1 + i\, E_2 \;, \tag{11}$$

where $E_2 = \mu\omega$. Then differentiating $\partial^2\xi/\partial x^2$ with respect to time, Eq. (9) becomes

$$(E_1 + i\mu\omega)\, \frac{\partial^2 \xi}{\partial x^2} = \rho\, \frac{\partial^2 \xi}{\partial t^2} \;,$$

and substituting the modulus E defined in Eq. (11), we finally obtain

$$\rho\, \frac{\partial^2 \xi}{\partial t^2} - E\, \frac{\partial^2 \xi}{\partial x^2} = 0 \;, \quad 0 \leq x \leq L \;, \quad t > 0 \;. \tag{12}$$

Equation (12) completely describes the propagation of plane longitudinal waves in a linear homogeneous, elastic, viscous element of constant cross section. The solution to this equation, which may be verified by substitution, is a linear combination of expressions describing waves traveling in negative and positive x directions, respectively:

$$\xi(x,t) = (C_1 e^{\gamma_m x} + C_2 e^{-\gamma_m x})\, e^{i\omega t} \;, \tag{13}$$

where γ_m, the mechanical propagation function is defined by

$$\gamma_m = \alpha_m + i\beta_m \tag{14a}$$

$$\alpha_m = \text{attenuation constant} \tag{14b}$$

$$\beta_m = \text{phase constant} \quad . \tag{14c}$$

(Note: The subscript m is maintained to differentiate between the mechanical and electrical propagation functions only until equivalence is established. It will then be dropped in the interest of notational simplicity.)

We now proceed to formulate the steady-state solution to Eq. (12). Specifically, we seek an expression relating the stress and velocity at any point x along the mount section to the generating stress and velocity, σ_1 and $\mathring{\xi}$. The total stress at a point x is defined in terms of the complex modulus E as follows:

$$\sigma(x) = E \frac{\partial \xi(x)}{\partial x} \quad .$$

The expression for velocity at any point x may be computed from Eq. (13) by performing the differentiation with respect to time:

$$\mathring{\xi}(x) = i\omega \, (C_1 \, e^{\gamma_m x} + C_2 \, e^{-\gamma_m x}) \, e^{i\omega t} \quad . \tag{15}$$

It now remains only to evaluate C_1 and C_2 in terms of σ_1 and $\mathring{\xi}_1$ (the boundary values at x = 0).

$$\frac{\partial \xi}{\partial x} = \gamma_m \, (C_1 \, e^{\gamma_m x} - C_2 \, e^{-\gamma_m x}) \, e^{i\omega t} \quad . \tag{16}$$

Thus

$$\sigma(x) = E \frac{\partial \xi}{\partial x} = E \cdot \gamma_m \, (C_1 \, e^{\gamma_m x} - C_2 \, e^{-\gamma_m x}) \, e^{i\omega t} \quad . \tag{17}$$

Applying the boundary conditions at x = 0,

$$\sigma(0) = \sigma_1 \; , \qquad \mathring{\xi}(0) = \mathring{\xi}_1 \quad ,$$

to Eqs. (15) and (17) yields

$$\mathring{\xi}(0) = \mathring{\xi}_1 = i\omega \, (C_1 + C_2) \, e^{i\omega t} \quad ,$$

and

$$\sigma(0) = \sigma_1 = E\,\gamma_m\,(C_1 - C_2)\,e^{i\omega t} \quad .$$

Hence,

$$C_1 + C_2 = \frac{\mathring{\xi}_1 \cdot e^{-i\omega t}}{i\omega} \quad ,$$

and

$$C_1 - C_2 = \frac{\sigma_1\,e^{-i\omega t}}{E\,\gamma_m} \quad .$$

Adding these expression, we obtain.

$$C_1 = \frac{1}{2}\left(\frac{\mathring{\xi}_1}{i\omega} + \frac{\sigma_1}{E\,\gamma_m}\right) e^{-i\omega t} \quad .$$

Similarly, subtraction yields

$$C_2 = \frac{1}{2}\left(\frac{\mathring{\xi}_1}{i\omega} - \frac{\sigma_1}{E\,\gamma_m}\right) e^{-i\omega t} \quad .$$

Substituting these values of C_1 and C_2 into Eq. (17), we have

$$\sigma(x) = E\,\gamma_m \left[\left(\frac{\mathring{\xi}_1}{i\omega} + \frac{\sigma_1}{E\,\gamma_m}\right)\frac{e}{2}^{\gamma_m x} - \left(\frac{\mathring{\xi}_1}{i\omega} - \frac{\sigma_1}{E\,\gamma_m}\right)\frac{e}{2}^{-\gamma_m x}\right] \quad .$$

Then using the definitions

$$\cosh \gamma_m x = \frac{e^{\gamma_m x} + e^{-\gamma_m x}}{2} \quad ,$$

and

$$\sinh \gamma_m x = \frac{e^{\gamma_m x} - e^{-\gamma_m x}}{2} \quad ,$$

we may combine terms and write:

$$\sigma(x) = \frac{E\,\gamma_m}{i\omega}\,\mathring{\xi}_1 \sinh \gamma_m x + \sigma_1 \cosh \gamma_m x \quad . \tag{18}$$

An in the same manner, Eq. (15) becomes

$$\mathring{\xi}(x) = \mathring{\xi}_1 \cosh \gamma_m x + \frac{i\omega}{E\,\gamma_m} \sigma_1 \sinh \gamma_m x \quad . \tag{19}$$

The relation between σ_2, $\mathring{\xi}_2$, and σ_1, $\mathring{\xi}_1$, may be obtained from Eqs. (18) and (19) immediately by substituting the second set of boundary conditions: for x = L, $\sigma(L) = \sigma_2$, and $\mathring{\xi}(L) = \mathring{\xi}_2$. Thus

$$S\,\sigma_2 = S\,\sigma_1 \cosh \gamma_m L + \mathring{\xi}_1 Z_0 \sinh \gamma_m L \tag{20}$$

$$\mathring{\xi}_2 = \frac{S\,\sigma_1}{Z_0} \sinh \gamma_m L + \mathring{\xi}_1 \cosh \gamma_m L \quad , \tag{21}$$

where $Z_0 = E\gamma S/i\omega$. It will be demonstrated later that Z_0 represents the "characteristic impedance" of the resilient element. Inspection of Eqs. (20) and (21) shows that the transmission characteristics of the mount are determined by the parameters Z_0 and γ. These equations may now be compared directly with those listed in the summary of four-terminal-network theory. From this comparison, a table can then be drawn up to facilitate the translation from mechanical to electrical terms which will be needed in sections on application of the method.

TRANSLATION FROM MECHANICAL TO ELECTRICAL QUANTITIES

The essential point to note at the outset is that the homogeneous mount section may be regarded as a mechanical transmission line which has as its electric analog the electrical transmission line. To support this assertion, the equations of motion of the parent mechanical transmission line will be shown to have the same form as the analogous electrical transmission line under certain specified conditions.

It may be shown by elemental analysis similar to the above mechanical analysis that the equations describing the voltage and current distributions along the electrical transmission line are given by

$$\frac{\partial^2 E}{\partial x^2} = \gamma_e^2\, E \quad , \tag{22a}$$

$$\frac{\partial^2 I}{\partial x^2} = \gamma_e^2\, I \quad , \tag{22b}$$

where $\gamma_e = \alpha_e + i\beta_e$ is the electrical propagation function. We now determine the conditions under which Eq. (12) describing velocity distribution along a mechanical transmission line has the same form as Eq. (22b). Substituting Eq. (10) into Eq. (12), and performing the differentiation with respect to time, we may write Eq. (12) in the form

$$\frac{\partial^2 \dot{\xi}}{\partial x^2} = - \frac{\rho\omega^2}{E} \dot{\xi} \quad . \tag{23}$$

This form then becomes identical with Eq. (22b) by equating

$$- \frac{\rho\omega^2}{E} = \gamma_m^{\ 2} \quad . \tag{24}$$

Assuming the validity of Eq. (24) for the moment, we may use this relation as a method of evaluating the attenuation and phase functions in terms of mechanical parameters as follows.

Recalling that $E = E_1 + i\mu\omega$, the left side of Eq. (24) may be rationalized and rearranged to read:

$$\frac{-\rho\omega^2}{E_1 + i\mu\omega} = \frac{-\rho\omega^2 \ (E_1 - i\mu\omega)}{E_1^{\ 2} + \mu^2 \ \omega^2} = \frac{\frac{-\rho\omega^2}{E_1} + \frac{i\mu\rho\omega^3}{E_1^{\ 2}}}{1 + \left(\frac{\mu\omega}{E_1}\right)^2} \quad .$$

For most cases of interest $\mu\omega/E_1 \ll 1$. Using this approximation, Eq. (24) reduces to

$$- \frac{\rho\omega^2}{E} = - \frac{\rho\omega^2}{E_1} + i \ \frac{\mu\rho\omega^3}{E_1^{\ 2}} = \gamma_m^{\ 2} = (\alpha_m^{\ 2} - \beta_m^{\ 2}) + i(2 \ \alpha_m\beta_m) \quad .$$

Equating the corresponding real and imaginary parts yields the simultaneous equations

$$\alpha_m^{\ 2} - \beta_m^{\ 2} = - \frac{\rho\omega^2}{E_1} \quad ,$$

$$2 \ \alpha_m\beta_m = \frac{\mu\rho\omega^3}{E_1^{\ 2}} \quad .$$

From the first of these two equations, $\alpha_m^{\ 2}$ may be neglected for small damping coefficients in comparison with $\beta_m^{\ 2}$; thus

$$\beta_m^{\ 2} \simeq \frac{\rho\omega^2}{E_1} = \frac{\omega^2}{E_1/\rho} = \frac{\omega^2}{c^2} = \left(\frac{2\pi}{\lambda}\right)^2 \tag{25}$$

where λ = wavelength, and where c, the velocity of sound, is conventionally computed from the relation

$$c = \sqrt{\frac{E_1}{\rho}} \quad . \tag{26}$$

Substitution of β_m into the second equation results in

$$\alpha_m = \frac{\mu\, \omega^2}{2c\, E_1} \quad . \qquad (27)$$

It is sufficient to note that these values of α_m and β_m are also obtainable in a similar fashion by substitution of Eq. (13) into Eq. (12). This fact justifies the assumption of the equality expressed in Eq. (24).

Since the electrical transmission line is the analog of the mechanical mount, one would expect to find some relation between the mount characteristics and a four-terminal network also. Indeed, this is shown to be the case upon comparison of the sets of Eqs. (20) and (3), and (21) and (4):

$$- S\, \sigma_2 = - S\, \sigma_1 \cdot \cosh \gamma L - \frac{E\gamma S}{i\omega} \mathring{\xi}_1 \sinh \gamma L \qquad (20)$$

$$E_2 = DE_1 - BI_1 \qquad (3)$$

$$- \dot{\xi}_2 = - \frac{i\omega}{E\gamma S} \cdot S\, \sigma_1 \sinh \gamma L - \dot{\xi}_1 \cosh \gamma L \qquad (21)$$

$$- I_2 = C \cdot E_1 - AI_1 \qquad (4)$$

As mentioned earlier, we shall use the force-voltage analog, which provides then the following correspondence:

$$F \leftrightarrow E \,, \qquad \mathring{\xi} \leftrightarrow I \quad .$$

Due to the commonly accepted convention which regards tensile stresses as positive, the relation between a positive force and a positive stress is given by

$$F = - S\, \sigma \quad .$$

Therefore, in terms of stress, we may identify the following two analogous quantities

$$- S\sigma \leftrightarrow E \quad .$$

Now comparing coefficients of corresponding quantities in the above two sets of equations, we find that the mechanical coefficients corresponding to the ABCD parameters are

$$A_m = D_m = \cosh \gamma L \quad , \qquad (7'a)$$

$$B_m = \frac{E\gamma S}{i\omega} \sinh \gamma L \quad , \qquad (7'b)$$

$$C_m = \frac{i\omega}{E\gamma S} \sinh \gamma L \quad . \qquad (7'c)$$

Comparing these with the ABCD parameters derived from considering an electrical transmission line as a four-terminal network, as obtained from (7a), (7b), and (7c) by replacing γ with γL, namely,

$$A_t = D_t = \cosh \gamma L \quad , \tag{7"a}$$

$$B_t = Z_0 \quad \sinh \gamma L \quad , \tag{7"b}$$

$$C_t = \frac{1}{Z_0} \quad \sinh \gamma L \quad , \tag{7"c}$$

we note the following two facts.

1. The characteristic mechanical impedance of a homogeneous elastic element treated either as a mechanical four-terminal network or as a mechanical transmission line is defined by

$$Z_0 = \frac{E\gamma S}{i\omega}$$

as indicated earlier.

2. Since $A_m = D_m$ and $A_t = D_t$, the homogeneous resilient element and its electrical analog display the property of symmetry, i.e., $z_{11} = z_{22}$ in both cases. With the correspondence between an electrical four-terminal network thus specified, the minimum information sufficient to characterize the mechanical element may be deduced. Six parameters associated with the element are involved, namely A,B,C,D,Z_0, and γ. However, since $A = D$, this list is immediately reduced to five. There are three conditions imposed on the remaining variables as stated by Eqs. (5), (6), and the following identity:

$$A \cdot D - B \cdot C = 1 \quad ,$$

which is readily verifiable by substitution. We are thus left with the choice of any two of the five as independent variables. Usually the most convenient pair will be Z_0 and γ. Relations (5) through (7) provide a method of computing these functions once the necessary input and transfer impedances of the mount are measured. Since these quantities are generally complicated functions of frequency, it will be necessary to perform a point-wise analysis of the system.

Once again it must be emphasized that very specific conditions must be met in the measurement of these impedance values. With reference to the voltage-current relationships in a four-terminal network, the input and transfer impedances on each side of the network must be obtained with the opposite side open-circuited. Since the current on this side will then be zero, we may interpret the "open circuit" condition mechanically by saying that the input and transfer impedances on each side of the mount must be obtained with the opposite side rigidly constrained so that the velocity on this side will be zero.

It might also be well to clarify at this point the units of mechanical impedance used in this report inasmuch as some variation is found among different authors regarding this matter. To begin with, we note from the expression for Z_0:

$$Z_0 = \frac{E\gamma S}{i\omega} = \left(\frac{E_1 + i\mu\omega}{i\omega}\right)\gamma S = \gamma S \left(\mu - i \cdot \frac{E_1}{\omega}\right)$$

that if damping is neglected, then as μ goes to zero, γ goes to $i\beta$. Hence Z_0 reduces to

$$Z_0 = i\beta S \left(- i \frac{E_1}{\omega}\right) = \frac{SE_1}{c} = \rho c S$$

which is the well-known acoustic resistance for a plane wave of wave front area S. The medium in which the wave is being propagated may be solid or fluid. Now it is conceivable that the entire transmission path to be considered in the problem may have sections which are solid and others which are fluid, and thus it will be necessary to consider the impedances of each case, which are customarily separated into two categories termed mechanical and acoustic impedance, respectively. It is imperative that, if one is to be able to work with these quantities mathematically, they must have the same physical dimensions. Now it is generally agreed that the fundamental definition is

$$Z = \frac{F}{v} ,$$

in accordance with its electrical analog. The divergence of views arises when the impedance is defined in terms of other quantities, such as stress. This seems to be caused by the two facts: (1) the convention regarding positive (tensile) stress is violated on occasion, and (2) the requirement that mechanical impedance have the dimensions

$$D\left[\frac{F}{v}\right] = \frac{MLT^{-2}}{LT^{-1}} = \frac{M}{T}$$

is not met but is compensated for in a variety of ways. A consistent approach with regard to these two considerations dictates that the definition of impedance in terms of stress be written

$$Z = - \frac{S\sigma}{v} \left(\frac{\text{lb (force) - sec}}{\text{in.}}\right)$$

in English units.

So far we have discussed the similarity between electrical and mechanical systems only on the basis of the comparable network equations. However, in regard to the method of constructing the electrical circuit corresponding to a mechanical system, some difficulty may seem to be pending due to our departure from the customary analogy which usually treats the equivalent of mechanical elements as two-terminal networks. However, an excellent system of circuit construction which is highly suited to the consideration of mechanical elements as four-terminal networks has been developed by B. B. Bauer and will be adopted throughout the remaining analysis (see Ref. 4). Table 1 shows the equivalent four-terminal-network form of the various mechanical components. The method of constructing the general circuit corresponding to a mechanical complex consists, briefly, of (1) placing the appropriate four-terminal networks (as obtained from Table 1) in the same relative geometrical position as displayed by their corresponding mechanical elements; (2) providing the coupling between elements by means of ideal transformers of 1:1 turns ratio. The transformers can often be removed afterward by considering the voltage-current relations in such a circuit and placing jumpers across their terminals when possible to do so without upsetting these relationships. Examples of the use of this method will be included later. For further information on the method of this analog and the current-force analog, see Ref. 4.

This essentially completes the groundwork necessary for understanding the application of network methods to specific vibration-isolation problems. We shall at this point include, however, a summary of electrical equations which appear to be applicable to the analysis of a mechanical system.

EXPRESSIONS FOR THE ISOLATION PARAMETERS

General information which shall be of interest later may be derived by considering the circuit in Figure 6. This circuit may be interpreted mechanically as a vibrating machine directly attached to a resilient foundation. The generator and the source impedance, $Z_s = Z_m$, represent the resolution of a general mechanical or electrical network into an equivalent constant voltage generator and series impedance by means of Thevenin's theorem. The impedance of the foundation, measured at the point of connection with the machine, is represented by the complex quantity $Z_r = Z_f$. Now, suppose the general four-terminal network were inserted in this electrical circuit as shown in Figure 7. Inspection of the circuit indicates the following relations hold:

$$E_1 = E_g - I_1 Z_s \quad \text{and} \quad E_2 = I_2 Z_r .$$

Using these results and Eqs. (7") in Eqs. (1) and (2) yields the important ratio

$$\frac{E_g}{E_2} = \frac{(Z_s + Z_0)\ (Z_r + Z_0)e^{\gamma L} - (Z_s - Z_0)\ (Z_r - Z_0)e^{-\gamma L}}{2\ Z_0\ Z_r} . \qquad (28)$$

This may be rearranged to the form

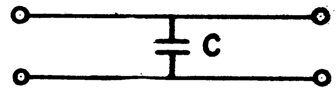
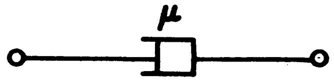
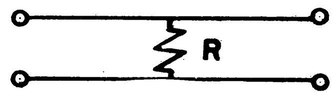
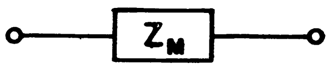
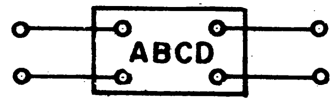
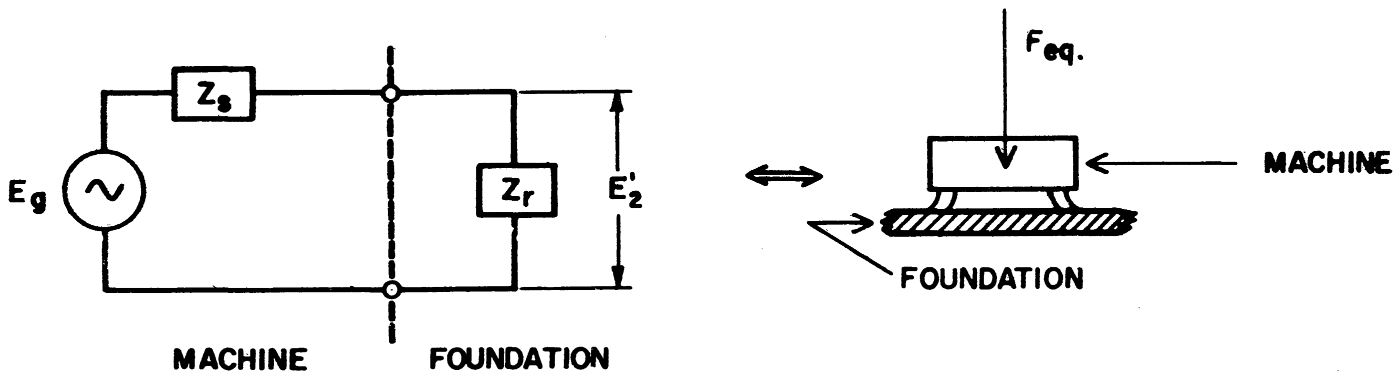
MECHANICAL ELEMENT		ELECTRICAL ELEMENT	
MASS		INDUCTANCE	
SPRING		CAPACITANCE	
DASHPOT		RESISTANCE	
ARBITRARY MECHANICAL COMPONENT		GENERAL FOUR TERMINAL NETWORK	

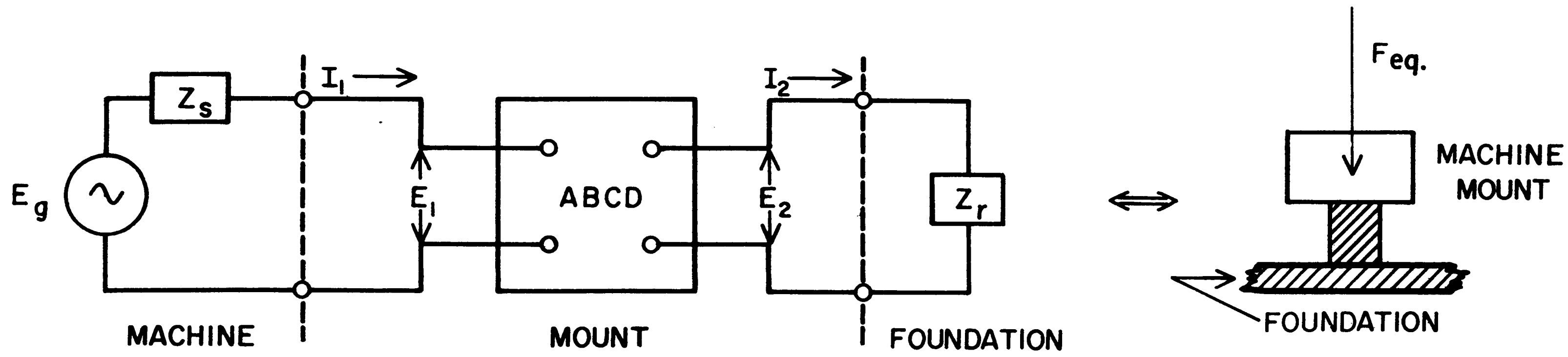
TABLE OF ANALOGOUS COMPONENTS USING FORCE-VOLTAGE ANALOGUE

TABLE I



ELECTRICAL CIRCUIT FOR MACHINE DIRECTLY ATTACHED TO FOUNDATION

FIGURE 6



ELECTRICAL CIRCUIT FOR PEDESTAL MOUNTED MACHINE

FIGURE 7

$$\frac{E_g}{E_2} = \left\{2\sqrt{\frac{Z_s}{Z_r}}\right\} \cdot \left\{e^{\gamma L}\right\} \cdot \left\{\frac{1}{2}\left(\sqrt{\frac{Z_s}{Z_0}} + \sqrt{\frac{Z_0}{Z_s}}\right)\right\} \cdot \left\{\frac{1}{2}\sqrt{\frac{Z_r}{Z_0}} + \sqrt{\frac{Z_0}{Z_r}}\right\} \cdot \left(1 - \Gamma_s \Gamma_r e^{-2\gamma L}\right) , \tag{29}$$

where the reflection coefficients for the sending and receiving end (or input and output end) are given by

$$\Gamma_s = \frac{Z_s - Z_0}{Z_s + Z_0} , \tag{30a}$$

$$\Gamma_r = \frac{Z_r - Z_0}{Z_r + Z_0} . \tag{30b}$$

Equation (29) places in evidence the various correction factors created by insertion of the four-terminal network. The first term, $2\sqrt{Z_s/Z_r}$, on the right accounts for the modification of the voltage (or force) ratio due to the difference between the source and terminal impedance. The factor $e^{\gamma L}$ is seen to represent, by definition, the voltage ratio that would prevail if $Z_0 = Z_s = Z_r$. Since each of the three remaining factors reduces to unity when $Z_0 = Z_s = Z_r$, they represent corrections due to impedance mismatch between the four-terminal network and the terminations. The third and fourth factors partially compensate for mismatch at the input and output ends, respectively, while the last term includes the remainder of mismatch effects. Since the fifth term involves the effects at one end due to mismatch at the other, it is called the interaction term. This term is near unity for small mismatch and high attenuation, but becomes large for the opposite conditions of large mismatch and small attenuation. Now by writing the term $2\sqrt{Z_s/Z_r}$ in the form

$$2\sqrt{\frac{Z_s}{Z_r}} = \frac{(Z_s + Z_r)/Z_r}{\frac{1}{2}\left(\sqrt{\frac{Z_s}{Z_r}} + \sqrt{\frac{Z_r}{Z_s}}\right)} ,$$

Equation (29) becomes

$$\frac{E_g}{E_2} = \frac{\left\{(Z_s + Z_r)/Z_r\right\} e^{\gamma L} \left\{\frac{1}{2}\left(\sqrt{\frac{Z_s}{Z_0}} + \sqrt{\frac{Z_0}{Z_s}}\right)\right\}\left\{\frac{1}{2}\left(\sqrt{\frac{Z_r}{Z_0}} + \sqrt{\frac{Z_0}{Z_r}}\right)\right\} \left(1 - \Gamma_s \Gamma_r e^{-2\gamma L}\right)}{\frac{1}{2}\left(\sqrt{\frac{Z_s}{Z_r}} + \sqrt{\frac{Z_r}{Z_s}}\right)} . \tag{31}$$

Then noting from Figure 6 that

$$\frac{E_g}{E_2'} = \frac{Z_s + Z_r}{Z_r} ,$$

the insertion properties of the network can be described by

$$\frac{E_2'}{E_2} = \frac{e^{\gamma} \circ \frac{1}{2}\left(\sqrt{\frac{Z_s}{Z_0}} + \sqrt{\frac{Z_0}{Z_s}}\right) \circ \frac{1}{2}\left(\sqrt{\frac{Z_r}{Z_0}} + \sqrt{\frac{Z_0}{Z_r}}\right)(1 - \Gamma_s \Gamma_r e^{-2\gamma L})}{\frac{1}{2}\left(\sqrt{\frac{Z_s}{Z_r}} + \sqrt{\frac{Z_r}{Z_s}}\right)} = Ne^{i\eta} . \tag{32a}$$

This equation gives the ratio of what the load (or foundation) potential would be, if the source and load were directly connected, to what it is when the network is inserted. The magnitude of the ratio, N, is termed the insertion ratio, and η is called the insertion angle. For computational purposes, the following form is often more desirable:

$$\frac{E_2'}{E_2} = \frac{(Z_s + Z_0)\ (Z_r + Z_0)\ e^{\gamma L} - (Z_s - Z_0)\ (Z_r - Z_0)\ e^{-\gamma L}}{2\ Z_0\ (Z_s + Z_r)} . \tag{32b}$$

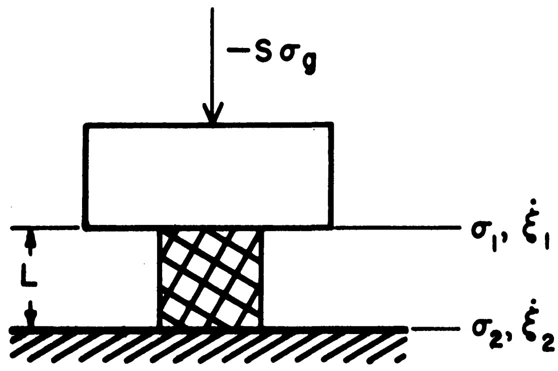
Either Eq. (28) or (32) constitutes a criterion for the evaluation of a given mechanical system. Equation (28) will lead to the conventional notion of force transmissibility, while Eq. (32) is the analog of a recently introduced mechanical concept termed "mount effectiveness." The former provides the overall attenuation achieved in a given mechanical configuration, while the latter determines precisely what is gained (or lost) in the way of isolation by inserting a mount between the machine and its foundation. An extensive treatment of the insertion properties of a mount at low frequencies may be found in Ref. 5.

The formulas collected in this section are sufficient to permit a relatively complete analysis of a mechanical vibration problem in one dimension. The mechanical ingredients needed to feed into this machinery are the particular foundation and machine impedances, discussed earlier, and the two independent parameters (Z_0, γ) necessary to characterize the mount. With these remarks, the development of the fundamentals will be concluded.

SECTION II. APPLICATIONS TO SIMPLE ISOLATION PROBLEMS

RIGID-MACHINE, SIMPLE-ELASTIC-MOUNT, RIGID-FOUNDATION MODEL

The simplest example which exhibits the characteristic features of the theory is that of a rigid vibrating mass separated from an infinite foundation by a conservative, homogeneous, resilient element of small, constant cross section. The model is shown in Figure 8. Using a general four-terminal network to represent the resilient element, the corresponding electrical circuits is constructed as follows: the machine is reduced by Thevenin's theorem to a constant force



MACHINE MOUNTED ON A RIGID FOUNDATION

FIGURE 8

generator, $-S\sigma_g$, acting through an impedance presented by the rigid mass. The elements of the system are arranged geometrically in series; therefore we so arrange the corresponding elements chosen from Table 1. The coupling is provided by ideal transformers of 1:1 turns ratio, indicated by vertical straight lines in Figure 9. Inspection of the circuit shows that attaching four jumpers at ab, cd, ef, and gh will not disturb the voltage-current relationships. Hence the transformers are easily removed in this case. The final circuit is given in Figure 10.

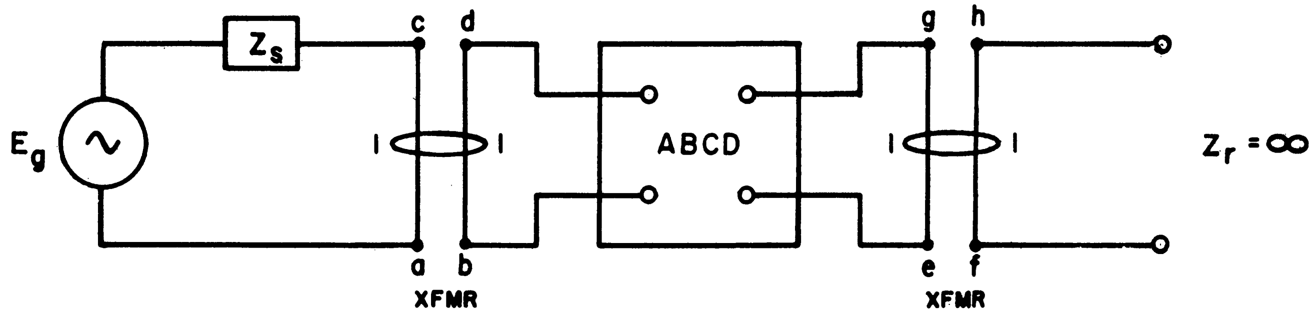
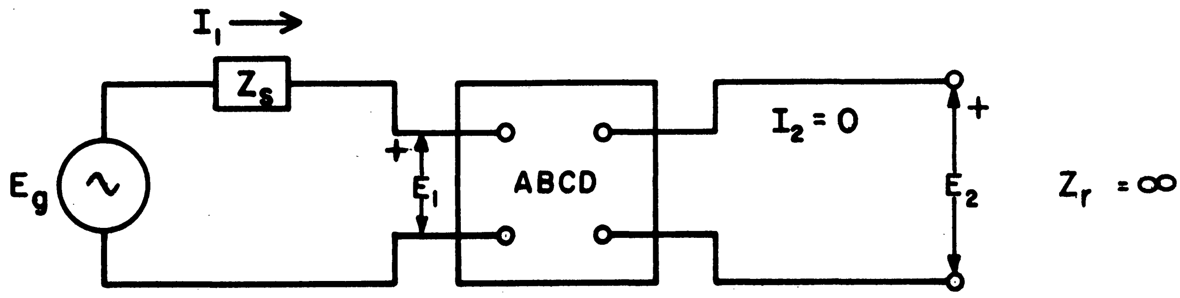


ILLUSTRATION OF CIRCUIT CONSTRUCTION CORRESPONDING TO A GIVEN MECHANICAL SYSTEM

FIGURE 9



ELECTRICAL CIRCUIT AFTER REMOVAL OF TRANFORMERS

FIGURE 10

This circuit is identical to that of Figure 7. Therefore Eq. (28) may be taken over directly. To obtain the transmissibility, the values of Z_0, Z_s, Z_r, and γ are needed to substitute into Eq. (28). Since the foundation has been assumed to be infinite, Eq. (28) may be written in the form

$$\frac{E_g}{E_2} = \left(\frac{Z_s}{Z_0} + 1\right) \frac{e^{\gamma L}}{2} - \left(\frac{Z_s}{Z_0} - 1\right) \frac{e^{-\gamma L}}{2} \quad . \tag{33}$$

Also since the line has no damping, $\mu = 0$, and Z_0 reduces to ρcS. The impedance of the machine is obtained simply from its equation of motion. Thus

$$S\ (\sigma_1 - \sigma_g) = M\ddot{\xi} = i\omega M\dot{\xi}$$

$$i\omega M = -\frac{S\ (\sigma_g - \sigma_1)}{\dot{\xi}} = Z_s \quad . \tag{34}$$

Substituting these values into the above form of Eq. (28), we obtain after some rewriting

$$\frac{E_g}{E_2} = \frac{\sigma_g}{\sigma_2} = \frac{F_g}{F_2} = \cosh \gamma L + \frac{i\omega M}{\rho cS} \sinh \gamma L \quad . \tag{35}$$

Since $\mu = 0$, $\gamma = \alpha + i\beta$ reduces to $\gamma = i\beta$. Using the relations between the hyperbolic functions and the circular trigonometric quantities given by

$$\cosh i\beta L = \cos \beta L \ ,$$

and

$$\sinh i\beta L = i \sin \beta L \ ,$$

the conventional form of force transmissibility in decibels then becomes

$$T = 20 \log_{10} \left| \frac{1}{\cos \beta L - \frac{\omega M}{\rho c S} \sin \beta L} \right| \text{ decibels} . \tag{36}$$

This may be put in computational form by introducing the dimensionless ratios

$$\beta L = \frac{\omega}{\omega_o} \sqrt{\frac{m}{M}} , \tag{37a}$$

$$\frac{\omega M}{\rho c S} = \frac{\omega}{\omega_o} \sqrt{\frac{M}{m}} , \tag{37b}$$

where

$$\omega_o = \sqrt{\frac{K}{M}}, \quad K = \frac{E_1 S}{L}, \quad \beta = \frac{\omega}{c}, \quad c = \sqrt{\frac{E_1}{\rho}} .$$

The transmissibility then takes the form

$$T = 20 \log_{10} \left| \frac{1}{\cos \left(\frac{\omega}{\omega_o} \sqrt{\frac{m}{M}}\right) - \frac{\omega}{\omega_o} \sqrt{\frac{M}{m}} \sin \left(\frac{\omega}{\omega_o} \sqrt{\frac{m}{M}}\right)} \right| . \tag{38}$$

We shall defer computation of other isolation parameters until we consider the effects of resilient foundations. We note in particular that the computation of mount "effectiveness" provides no new information for this case since it has been assumed that the foundation impedance, Z_r, is infinite. From Figure 6, it is seen that this assumption equates E_2 with E_g. Therefore

$$\frac{E_2'}{E_2} = \frac{E_g}{E_2}$$

and the "effectiveness" thus becomes the reciprocal of transmissibility.

The analysis of Eq. (36) has been performed by various authors (see, e.g., Ref. 2). It will be included here for the sake of completeness. The salient features of the system will be shown to be:

1. That at low frequencies, the system response is identical to that predicted by lumped parameter theory.
2. That the transmission characteristics exhibit attenuation bands alternating with pass bands which occur at frequencies for which the length of the mount is an integral number of half wavelengths (wave effects).

3. That a low value of c and a high mass ratio tends to produce attenuation beginning at low frequencies.

To demonstrate the relation between the transmissibility as deduced from this theory and that of lumped parameter theory, it is sufficient to note, from Eq. (36), that since

$$\beta = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

where λ = wavelength, then

$$\beta L = \frac{2\pi}{\lambda} \cdot L \quad .$$

And for $\lambda \gg L$,

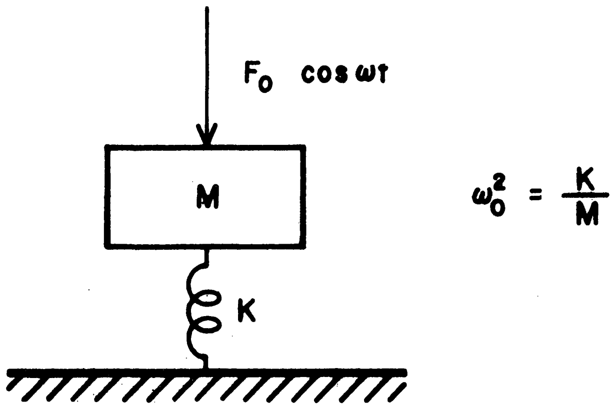
$$\sin \beta L \rightarrow \beta L$$

$$\cos \beta L \rightarrow 1 \quad .$$

Hence the transmissibility at low frequencies becomes, using Eqs. (36), (37a), and (37b),

$$T = 20 \log_{10} \left| \frac{1}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \right| \quad .$$

This is the familiar result obtained from analysis of the lumped parameter model of Figure 11.



LUMPED PARAMETER MODEL OF MACHINE MOUNTED AGAINST A RIGID FOUNDATION

FIGURE 11

The isolation properties of the model as described by Eq. (36) are placed in evidence by solving for those values of ω for which $T = 0$, $T = \infty$, and $T =$ minimum. The points at which $T = 0$ indicate the limits of attenuation and pass bands. Thus

$$T = 0 \quad \text{for} \quad \cos \beta L - \frac{\omega M}{\rho c S} \sin \beta L = \pm 1 \quad .$$

This equation is satisfied under any of the following three conditions:

$$\beta L = n\pi \qquad (n = 0,1,2,\ldots) \quad , \tag{40a}$$

$$\cos \beta L - \frac{\omega M}{\rho c S} \sin \beta L = +1 \quad \text{for} \quad \beta L \neq n\pi \quad , \tag{40b}$$

$$\cos \beta L - \frac{\omega M}{\rho c S} \sin \beta L = -1 \quad \text{for} \quad \beta L \neq n\pi \quad . \tag{40c}$$

Equations (40b) and (40c) above may be simplified as follows. From Eq. (40b),

$$\frac{\omega M}{\rho c S} = \frac{\cos \beta L - 1}{\sin \beta L} \quad \text{for} \quad \beta L \neq n\pi \quad .$$

Recalling that $m = \rho S L$ is the mass of the resilient element, we may rewrite the above equation with the help of the standard trigonometric identities as

$$\frac{\omega M}{\rho c S} = \left(\frac{M}{m}\right) \beta L = -\tan \frac{\beta L}{2} \, , \quad \beta L \neq n\pi \quad . \tag{40'b}$$

Similarly Eq. (40c) becomes

$$\frac{\omega M}{\rho c S} = \left(\frac{M}{m}\right) \beta L = +\cot \frac{\beta L}{2} \, , \quad \beta L \neq n\pi \quad . \tag{40'c}$$

Maximum transmission occurs for $T = \infty$, i.e., when

$$\cos \beta L - \frac{\omega M}{\rho c S} \sin \beta L = 0 \quad , \tag{41}$$

or

$$\left(\frac{M}{m}\right) \beta L = \cot \beta L \, , \; \beta L \neq \frac{n\pi}{2} \quad \text{(resonance points)} \quad .$$

In addition, minimum T is observed when

$$\frac{\partial}{\partial\omega}\left\{\cos\beta L - \frac{\omega M}{\rho c S}\sin\beta L\right\} = 0 \quad .$$

Hence, substituting $\beta = \omega/c$, and differentiating after ω, we have

$$-\frac{L}{c}\sin\frac{\omega L}{c} - \frac{M}{\rho c S}\sin\frac{\omega L}{c} - \frac{\omega ML}{\rho c^2 S}\cos\frac{\omega L}{c} = 0 \quad ,$$

which, after combining terms, becomes

$$\left(\frac{\rho\ SLc + Mc}{\rho c^2 S}\right)\sin\beta L = -\frac{\omega ML}{\rho c^2 S}\cos\beta L \quad .$$

Finally, this can be rearranged to read

$$\left(\frac{\omega\ ML}{Mc + mc}\right) = \left(\frac{M}{M + m}\right)\beta L = -\tan\beta L \quad \text{(minimum transmissibility)} \quad . \qquad (42)$$

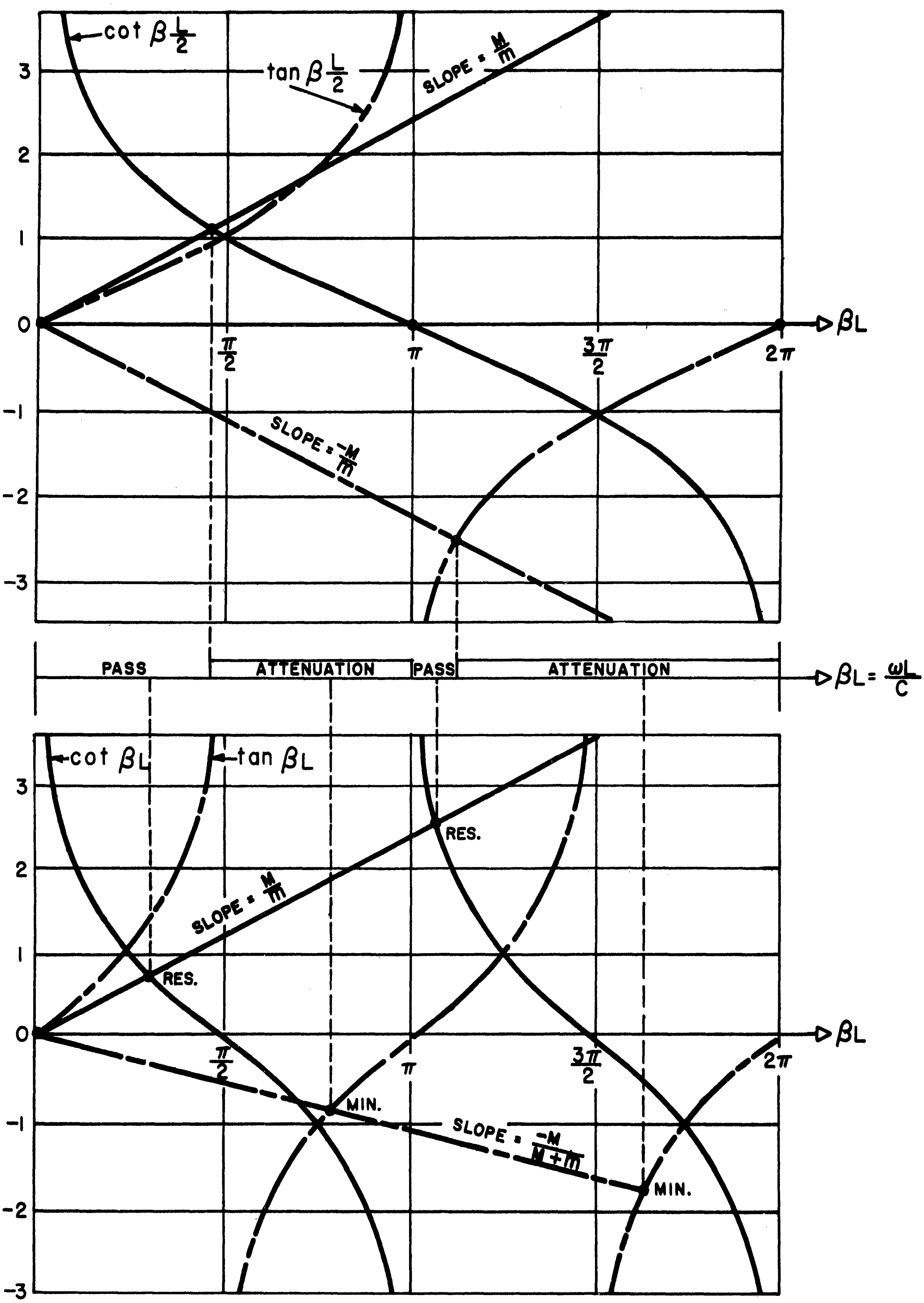
The values of βL (and thus ω) which denote band limits, resonances, and minimum transmissibility are most readily obtained by graphical solution as in Figure 12. In addition to providing the over-all picture of the system's transmission characteristics, consideration of the graphical method of solution is also instructive in regard to the relative influence of the various material parameters M, m, ρ, c, S, and L. From the upper graph, it is apparent that all attenuation bands terminate, and pass bands begin, at the points $\beta L = n\pi$ ($n = 0,1,2,\ldots$). Thus the intersection of the lines

$$y = \pm\left(\frac{M}{m}\right)\beta L$$

with the cot $\beta L/2$ and tan $\beta L/2$ curves corresponding to Eqs. (40'b) and (40'c) determines the inception frequencies of the attenuation bands only.

It is interesting to examine methods of reducing the frequencies at which the attenuation bands begin. From Figure 12 we see that by increasing the slope of the straight lines the intersection points are moved toward the left, the y axis. This corresponds to the lowering of frequency for which attenuation is obtained. Since the slope of the line is proportional to the ratio of machine mass to mount mass, increasing this mass ratio will achieve the desired effect. In addition, it must be recognized that the scale of the abscissa, along which frequency is measured, can be varied. Examining this change of scale more closely, the form $\omega L/c$, where $\omega L/c = \beta L$, shows that fundamentally the abscissa-scale unit is controlled by the length of the mount, L, and the velocity of sound in the mount material, c. Hence, a given point on the abscissa may be made to correspond to a lower frequency by expanding the scale of the abscissa. For example, suppose $\omega_1 \cdot L/c$ = constant. Then for an increase in the L/c ratio, which essentially expands the scale, the value of ω_1 corresponding to the point decreases in inverse proportion.

GRAPHICAL SOLUTION FOR BAND PASS LIMITS



GRAPHICAL SOLUTION FOR RESONANCE AND MINIMUM TRANSMISSIBILITY POINTS

FIGURE 12

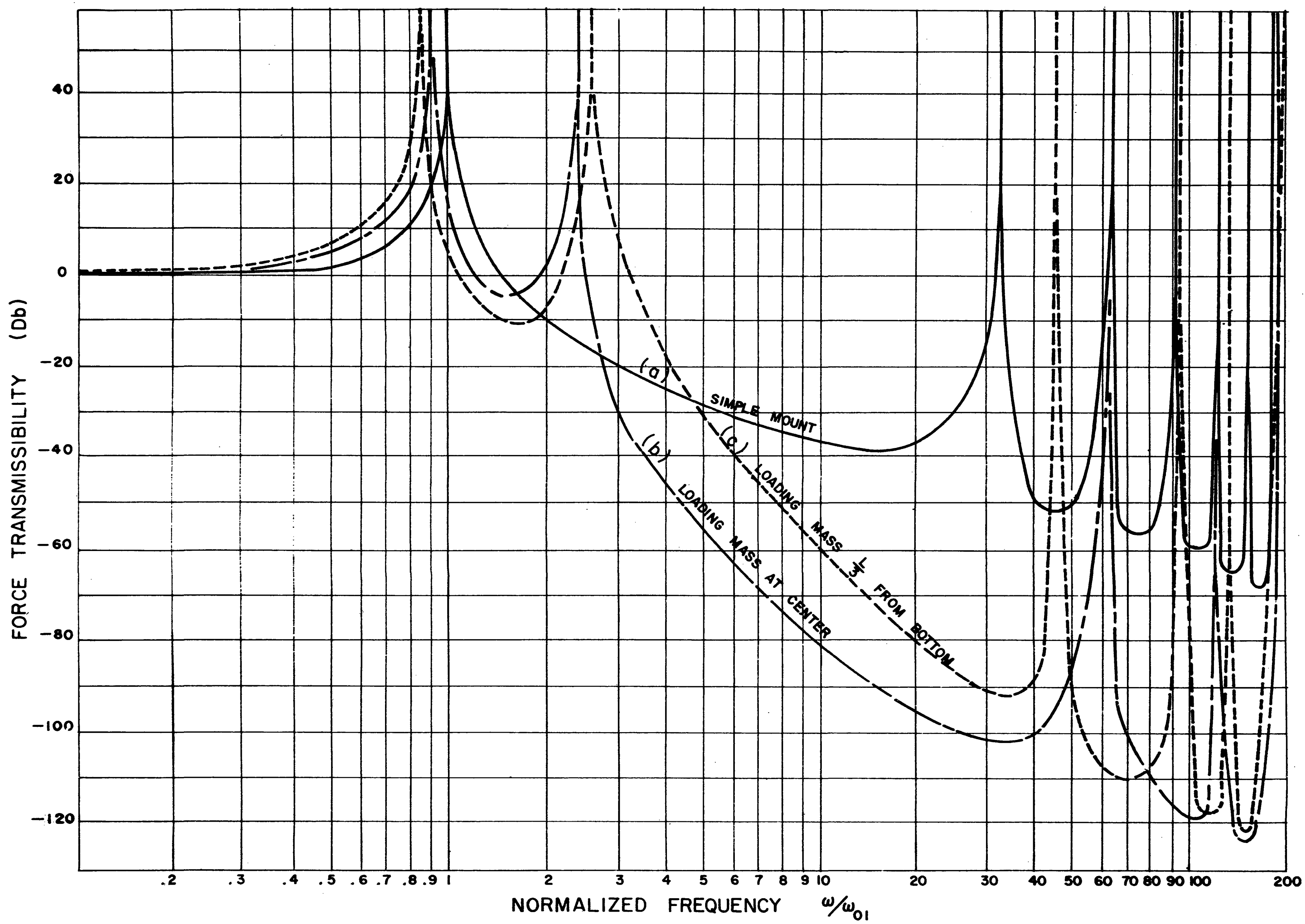
Thus we have determined that the inception points of the attenuation bands may be lowered by increasing the mass ratio or the mount length along with selecting a mount material that propagates sound at a lower velocity. In a particular application, however, it may be more important to control the transmission characteristics in a more elaborate fashion, e.g., to avoid the occurrence of standing waves at certain specified frequencies, than to achieve attenuation at a very low frequency. Therefore one cannot specify on the basis of a general analysis what constitutes the "best" mass ratio, mount length, etc. The theory may be used to indicate the general features of a vibration-isolation problem, from which design information is available to obtain optimum performance.

It is apparent that for this simple case the most important contribution mechanical transmission line theory has offered is its prediction of the pass bands which occur at higher frequencies. The loss of attenuation due to this phenomena, termed "wave effects," had been first observed in practical vibration-isolation devices. Efforts to explain this marked deviation from the predictions of lumped parameter theory encouraged the use of a distributed parameter treatment of the resilient element which had been applied to similar mechanical systems as early as 1930 by R. B. Lindsay.[7] The consideration of distributed parameters associated with a resilient element permits the examination of a resonant condition which does not occur in lumped systems, i.e., the standing waves which are found at frequencies for which the resilient element's length is an integral number of half wavelengths. The elementary treatment of our first example indicated this occurrence of standing wave resonances at regular frequency intervals to indefinitely high frequencies. A plot of the transmissibility, curve (a) in Figure 13, in db attenuation thus reveals, in addition to the "normal mode" peak at low frequencies, the fundamental and higher harmonics of the "wave effect" peaks.

THE MECHANICAL FILTER

In the general formulation of the problem, the possibility is provided for examining connected transmission paths which are individually homogeneous but different from each other in composition. Our next case will involve the simplest example of such a "compound" line. For the time being, the machine and foundation will be relegated, as in the first example, to rather unimportant roles in the system. We may anticipate the results somewhat by considering the fact that constructing a compound line is analogous to the connection of electrical transmission lines having different characteristic impedances. Electrically, this "impedance mismatch" causes a loss in the power transmitted. We therefore expect that some increase in attenuation is to be achieved.

The various sections of a compound line are usually treated as elastic elements. However, one may ignore the elasticity of certain sections as a first approximation, and consider simply the impedance presented by the mass of these sections. We shall now consider an example of this type. In keeping with the desire to maintain conceptual and algebraic simplicity, the machine and founda-



TRANSMISSIBILITY OF SIMPLE MOUNT AND COMPOUND LINES

FIGURE 13

tion will again be considered rigid and the effects of viscosity neglected. The model to be considered is similar to that used in the first example and is given in Figure 14. The machine mass is M_1 and the mass of the interposed mount section is labeled M_2. We shall refer to M_2 as the loading mass. The over-all length of the resilient transmission line will be considered the same as in the first example with the loading mass attached at $x = L/2$. The electrical circuit, shown in Figure 15, is obtained in the same manner as before, using the corresponding elements from Table 1. In this case, the four-terminal networks representing the two halves of the transmission line will naturally have ABCD parameters involving L/2 instead of L as before.

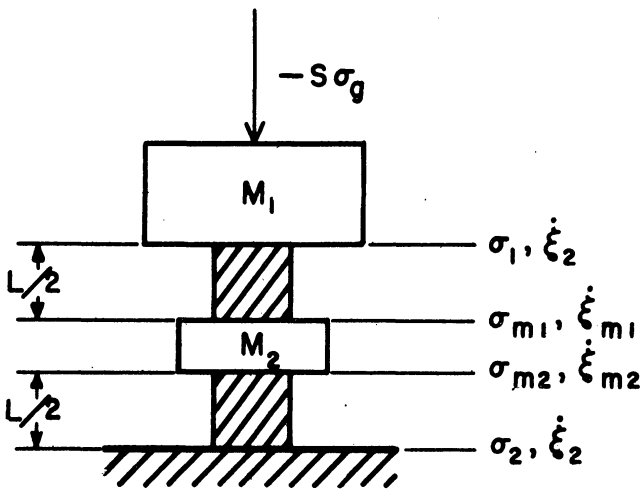
The analysis of this circuit can be conveniently carried out using matrix algebra. Recalling the usual rules of matrix addition and multiplication, consider the E,I relations across the last four-terminal network on the right in Figure 15. We have by Eqs. (1) and (2)

$$E_{m2} = A\,E_2 + B\,I_2 \quad ,$$

$$I_{m2} = C\,E_2 + D\,I_2 \quad .$$

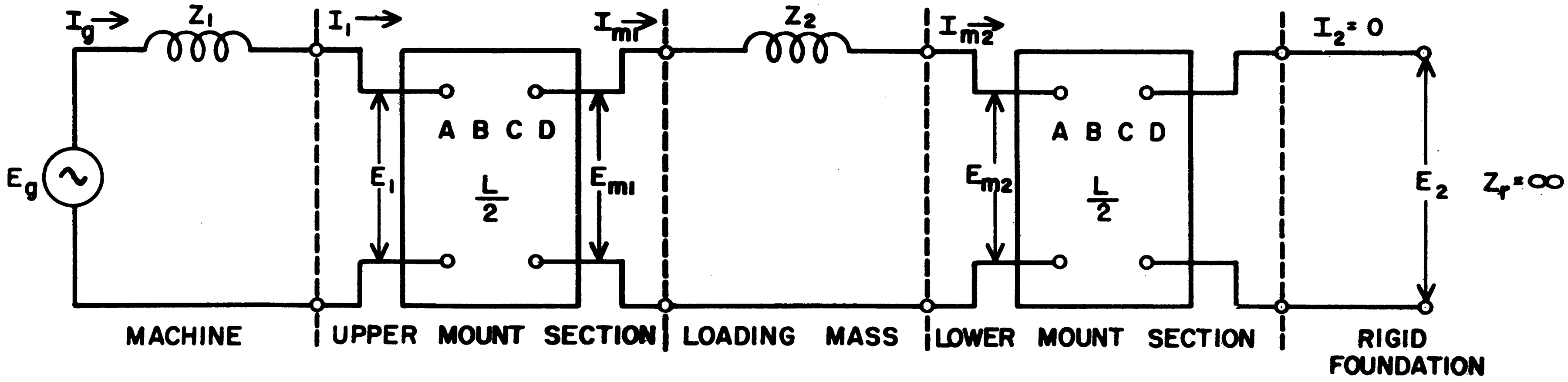
In matrix form these equations may be written

$$\left\| \begin{matrix} E_{m2} \\ I_{m2} \end{matrix} \right\| = \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| \cdot \left\| \begin{matrix} E_2 \\ I_2 \end{matrix} \right\|$$



MACHINE — COMPOUND MOUNT — FOUNDATION MODEL

FIGURE 14



ELECTRICAL CIRCUIT CORRESPONDING TO FIGURE 14

FIGURE 15

Proceeding to the left in Figure 15, similar relations are written for each successive four-terminal network. Hence, for the impedance Z_2,

$$E_{m1} = E_{m2} + Z_2 I_{m2} \;,$$

$$I_{m1} = I_{m2} \;,$$

which, written in matrix form, becomes

$$\begin{Vmatrix} E_{m1} \\ I_{m1} \end{Vmatrix} = \begin{Vmatrix} 1 & Z_2 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} E_{m2} \\ I_{m2} \end{Vmatrix} .$$

Substituting for the E_{m2}, I_{m2} matrix, we have

$$\begin{Vmatrix} E_{m1} \\ I_{m1} \end{Vmatrix} = \begin{Vmatrix} 1 & Z_2 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \cdot \begin{Vmatrix} E_2 \\ I_2 \end{Vmatrix} .$$

Again

$$\begin{Vmatrix} E_1 \\ I_1 \end{Vmatrix} = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \cdot \begin{Vmatrix} E_{m1} \\ I_{m1} \end{Vmatrix} = \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \cdot \begin{Vmatrix} 1 & Z_2 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \cdot \begin{Vmatrix} E_2 \\ I_2 \end{Vmatrix} ,$$

and finally, since

$$\begin{Vmatrix} E_g \\ I_g \end{Vmatrix} = \begin{Vmatrix} 1 & Z_1 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} E_1 \\ I_1 \end{Vmatrix} ,$$

then

$$\begin{Vmatrix} E_g \\ I_g \end{Vmatrix} = \begin{Vmatrix} 1 & Z_1 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \cdot \begin{Vmatrix} 1 & Z_2 \\ 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \cdot \begin{Vmatrix} E_2 \\ I_2 \end{Vmatrix} . \tag{43}$$

Carrying out the indicated matrix multiplication yields

$$\begin{Vmatrix} E_g \\ I_g \end{Vmatrix} = \begin{Vmatrix} [A + Z_1 C]\,[A + Z_2 C] + C\,[B + Z_1 D] & [A + Z_1 C]\,[B + Z_2 D] + D\,[B + Z_1 D] \\ C\,[A + Z_2 C] + CD & C\,[B + Z_2 D] + D^2 \end{Vmatrix} \begin{Vmatrix} E_2 \\ I_2 \end{Vmatrix} .$$

Since $I_2 = 0$, for this case, the above expression reduces to:

$$\left\| \begin{matrix} E_g \\ I_g \end{matrix} \right\| = \left\| \begin{matrix} [A^2 + AC\ (Z_1 + Z_2) + Z_1 Z_2 C^2 + CB + Z_1 CD]\ E_2 \\ [AC + Z_2 C^2 + DC]\ E_2 \end{matrix} \right\| .$$

The voltage and current may immediately be read off as follows:

$$E_g = [A^2 + AC\ (Z_1 + Z_2) + Z_1 Z_2 C^2 + CB + Z_1 CD]\ E_2 \ ,$$

$$I_g = [(CA + Z_2 C^2 + DC)\ E_2] \ .$$

From the voltage equation:

$$E_g/E_2 = A^2 + AC\ (Z_1 + Z_2) + Z_1 Z_2 C^2 + BC + Z_1 CD \ .$$

Utilizing the assumptions that the masses M_1 and M_2 are rigid, the impedances Z_1 and Z_2 are

$$Z_1 = i\omega\ M_1 \ ,$$

$$Z_2 = i\omega\ M_2 \ .$$

The ABCD parameters are given by Eqs. (7"a), (7"b), and (7"c), noting that L/2 must be substituted as the length of each resilient element.

$$D = A = \cosh \frac{\gamma L}{2} = \cos \frac{\beta L}{2} \qquad (\text{since } \mu = 0)$$

$$B = Z_0 \sinh \frac{\gamma L}{2} = Z_0\ i \sin \frac{\beta L}{2}$$

$$C = \frac{\sinh \gamma L/2}{Z_0} = \frac{i}{Z_0} \sin \frac{\beta L}{2}$$

Substitution into the voltage ratio results in

$$\frac{E_g}{E_2} = \cos^2 \frac{\beta L}{2} + \frac{i(Z_1+Z_2)}{Z_0} \sin \frac{\beta L}{2} \cos \frac{\beta L}{2} - \frac{Z_1 Z_2}{Z_0^2} \sin^2 \frac{\beta L}{2} - \sin^2 \frac{\beta L}{2} + \frac{Z_1}{Z_0}\, i \sin \frac{\beta L}{2} \cos \frac{\beta L}{2}$$

$$= \cos \beta L + i \frac{Z_1}{Z_0} \sin \beta L + \frac{i\ Z_2}{2\ Z_0} \sin \beta L - \frac{Z_1 Z_2}{Z_0^2} \sin^2 \frac{\beta L}{2} \ .$$

Finally

$$\frac{E_g}{E_2} = \cos \beta L - \frac{\omega M_2}{\rho c S} \sin \beta L - \frac{\omega M_1}{\rho c S} \left[\sin \beta L - \frac{\omega M_2}{\rho c S} \sin^2 \frac{\beta L}{2} \right] . \qquad (44)$$

The transmissibility in decibels attenuation becomes:

$$T = 20 \log_{10} \left| \frac{E_2}{E_g} \right| = 20 \log_{10} \left| \frac{1}{\cos \beta L - \frac{\omega M_2}{\rho c S} \sin \beta L - \frac{\omega M_1}{\rho c S} \left[\sin \beta L - \frac{\omega M_2}{\rho c S} \sin^2 \frac{\beta L}{2} \right]} \right| . \qquad (45)$$

The dimensionless form of Eq. (45), using the following relations

$$\omega_{o1} = \sqrt{\frac{K}{M_1}} \qquad \omega_{o2} = \sqrt{\frac{K}{M_2}} \qquad m = \rho L S$$

$$\beta L = \frac{\omega}{\omega_{o1}} \sqrt{\frac{m}{M_1}} \qquad K = \frac{ES}{L}$$

$$\frac{\omega M_1}{\rho c S} = \frac{\omega}{\omega_{o1}} \sqrt{\frac{M_1}{m}} , \qquad \frac{\omega M_2}{\rho c S} = \frac{\omega}{\omega_{o2}} \sqrt{\frac{M_2}{m}}$$

becomes

$$T = 20 \log_{10} \left| \frac{1}{\cos \frac{\omega}{\omega_{o1}} \sqrt{\frac{m}{M_1}} - \frac{\omega}{\omega_{o2}} \sqrt{\frac{M_2}{m}} \sin \frac{\omega}{\omega_{o1}} \sqrt{\frac{m}{M_1}} - \frac{\omega}{\omega_{o1}} \sqrt{\frac{M_1}{m}} \left[\sin \frac{\omega}{\omega_{o1}} \sqrt{\frac{m}{M_1}} - \frac{\omega}{\omega_{o2}} \sqrt{\frac{M_2}{m}} \sin^2 \frac{\omega}{2\omega_{o1}} \sqrt{\frac{m}{M_1}} \right]} \right| . \qquad (46)$$

Although the algebraic complexity has already become impressive for this relatively simple problem, the essential characteristics of the system are easily obtained from Eqs. (44) to (46). For instance, it is encouraging to note that Eqs. (45) and (46) reduce to Eqs. (36) and (38), respectively, if M_2 (the loading mass) is chosen negligibly small. The effects of adding M_2 is most easily seen from Eq. (45) by noting that the first of the band-limit equations of the previous example, Eq. (40a), is now changed to

$$T = 0 \quad \text{for} \quad \beta L = 2n\pi \qquad (n = 0,1,2,\ldots) \quad . \qquad (47)$$

The significance of the factor 2 appearing here is that pass bands occur only half as many times as in the previous system.

Physically, this means that the standing waves which previously entertained an antinode at the point of attachment of the loading mass (odd harmonics) have now been suppressed by this mass. Hence, only those standing waves which have the point $x = L/2$ as a node are possible with the mass attached. Thus it is apparent that a single loading mass attached at the center of the rod tends to suppress every other wave resonance, providing in this way relatively wide attenuation bands between the "wave-effect" pass bands as shown by curve (b) in

Figure 13. In fact, attaching the mass at the center may be shown to provide the widest attenuation bands possible for a simple mass-loaded line. Shifting the loading mass off center causes each of the remaining wave-effect peaks to split into two peaks, their separation being controlled by the distance of the mass from the center. As the loading mass moves away from the center, the pairs of transmissibility peaks increase their separation until they finally arrive at the positions of the simple line peaks as the mass is removed from the line. For certain points of attachment, for example, as with the loading mass at L/3 from the bottom as shown by curve (c), Figure 13, a peak moving up in the frequency scale becomes superimposed on the adjacent descending peak, resulting in a transmission characteristic similar to that of center loading. The difference is that the fundamental frequency has been reduced due to lower frequency standing waves occurring in the longer portion of the resilient element. Since the higher peaks occur at regular harmonic intervals, the attenuation bandwidths are reduced for off-center loading.

In addition we note that by introducing impedance mismatch at the center of the transmission line, the amount of attenuation obtained in the attenuation bands has been approximately doubled over that observed in the first example as anticipated. Evidence of this fact is provided in Eq. (45) wherein the coefficient of the predominant term at high frequency, $\sin^2 \beta L/2$, is proportional to the square of the frequency rather than simply proportional as in the first case.

Although more detailed analysis of these equations is necessary when the elasticity of the interposed section is considered, the essential behavior has been sufficiently indicated in the discussion thus far. For further analysis of the suppression of odd harmonic "wave effects" in compound lines and the supporting experimental evidence, see Refs. 6 and 7.

Comparing the foregoing examples a bit further, the occurrence of wave effects as a natural consequence of the elastic element's continuity is again emphasized. It is possible to show, from Eq. (45), that the transmissibility as obtained from this theory reduces at low frequency to that obtained by conventional lumped parameter methods. The equation obtained from Eq. (45) by assuming the length of the resilient element to be small compared to the wavelength is quadratic in ω, yielding the two normal modes of vibration of this system. The two peaks attributable to these modes are clearly evident in curves (b) and (c) in Figure 13.

The entire resilient section, pictured in Figure 14, consisting of a length of elastic element with a loading mass rigidly attached to the center has long been known in acoustics as a "section" of a mechanical filter. It is possible to construct a mechanical filter consisting of an arbitrary number of such sections. The general behavior of such an n-section filter is very much like that of the analogous electrical filter. As additional sections are added to the filter, some attenuation is lost at low frequencies due to the additional normal mode resonances created by the additional "degrees of freedom." However, this is compensated for by the increased attenuation received above the highest normal mode resonance, the attenuation predicted being approximately n times the amount provided by a single filter section. Again when sufficiently high frequencies are reached so that the mass spacing becomes of the order of a half wavelength, a loss of attenuation is observed due to standing waves in the resilient element. Although the pass bands caused by standing waves become increasingly narrow and less detrimental at higher frequencies, their effect is primarily to slow the rate of increase of attenuation with increasing frequency.

This type of mechanical filter corresponds roughly to the electrical low-pass filters. Attempts have been made to try to design a mechanical high-pass filter with very limited success (see Ref. 6). The difficulty encountered in attempting to obtain attenuation starting at zero frequency is essentially that the presence of an infinite mass is needed at some point in the system. Thus the mechanical analog of the electrical high-pass filter is most unlikely to provide any fruitful source of investigation for those interested in obtaining practical results in noise isolation. It appears that the best one can do is to design his isolation system to obtain attenuation of a few frequency bands which are most troublesome. In general, the lower the frequency to be attenuated, the larger the masses involved will have to be.

This, of course, is not a novel result, but is essentially a restatement of the well-known results of lumped parameter theory that the larger the mass and the softer the spring, the lower the resonant frequency will be, and hence the lower the frequency at which attenuation is obtained. Thus it is clear that the "mechanical filter" provides no new mechanism of isolating vibration sources.

SECTION III. NUMERICAL ANALYSIS OF A TYPICAL SYSTEM

Thus far, the examples considered have been elementary to display conveniently the salient features of the theory without unnecessary complications. However, the methods discussed are just as easily applied to entire systems consisting of several vibration sources, transmission paths which have damping, and terminations possessing arbitrary impedances. The difficulties encountered are largely computational rather than conceptual.

As an extension of the theoretical work contained in the previous two sections, this laboratory has attempted an application of network theory to the analysis of a typical heavy-machine noise-isolation problem. The intention was to use representative values of machine, mount, and foundation impedance as the basis for a numerical example illustrating: (1) the applicability of network analysis to a typical system; and (2) the magnitude of the computational problem involved in the analysis of a complex system. However, upon embarking on this investigation it became apparent that the scarcity of experimental impedance data applicable to this method of analysis precluded the possibility of analyzing an actual system. Thus it became necessary to construct an artificial system employing the type of data that are available, to furnish in an indirect manner the basis for the computation. Before reporting the details of this computation, we shall discuss briefly some of the considerations in selecting the characteristics of the artificial system.

BACKGROUND AND DEFINITION OF THE PROBLEM

The best data that were found to be available for computational purposes were the following: (1) impedance curves which were considered to be typical of the internal mechanical impedance characteristics of a heavy machine as measured at four points of support; (2) transmissibility data on several commercial mounts under rated loading; and (3) force-to-velocity ratios which were characteristic of four points on a machine foundation structure.

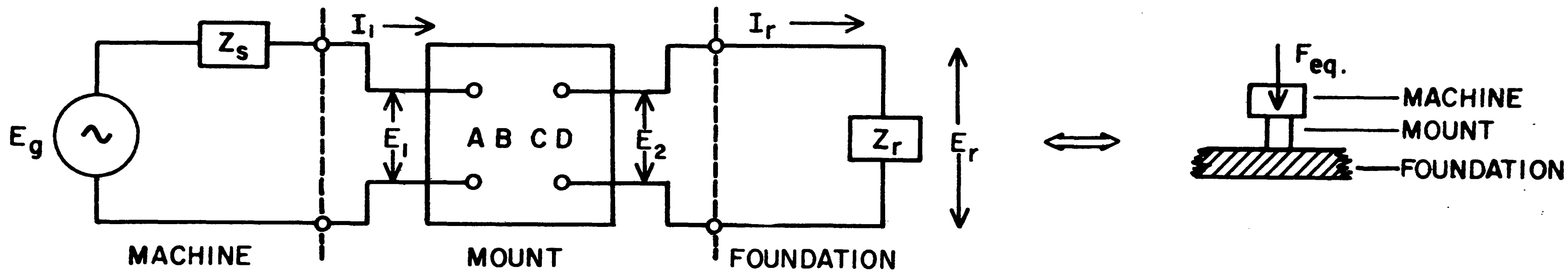
It originally had been hoped that the system analysis could be carried out for the case of a machine having several points of support. However, because values for the transfer impedances appearing between the normal machine-support points were not available, the analysis had to be restricted to a single-pedestal machine mounted through an isolator to a single point on a resilient foundation as shown in Figure 16.

Had the transfer-impedance data been available, the analysis would have been carried out using a circuit configuration similar to that of Figure 17 which shows schematically a general form of the mechanical network for a machine having two points of support. The effects of the multiple mounting and of the transfer impedances can be considered through the use of one circuit of the type shown in Figure 16 for each point of support, with the points of support on both the machine and the foundation side, respectively, interconnected through networks having suitable transfer-impedance characteristics. One could then apply excitation to each of the circuits in succession and compute the response at each foundation terminal. Then by assuming a linear response, the individual responses could be superimposed to obtain a first approximation of the total foundation response due to the vibratory effects of the machine.

ANALYSIS OF THE SYSTEM

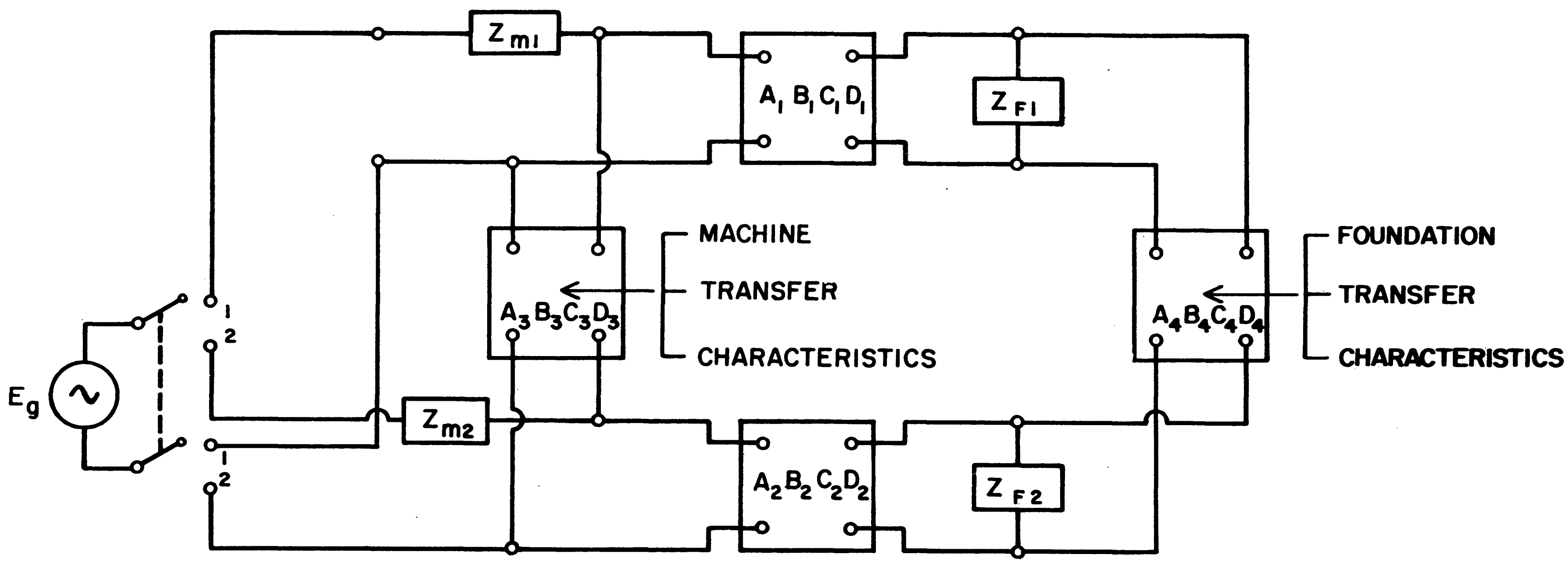
Employing the theory developed in the first two sections, we may draw the circuit analog for a pedestal-mounted machine as in Figure 16. The force-velocity relationships at each end of the mount may immediately be written in matrix form as follows:

$$\begin{Vmatrix} E_g \\ I_g \end{Vmatrix} = \begin{Vmatrix} 1 & Z_s \\ 0 & 1 \end{Vmatrix} \circ \begin{Vmatrix} A & B \\ C & D \end{Vmatrix} \circ \begin{Vmatrix} E_r \\ I_r \end{Vmatrix} \qquad (48)$$



PEDESTAL MOUNTED MACHINE ON A RESILIENT FOUNDATION

FIGURE 16



MACHINE WITH TWO POINT SUSPENSION

FIGURE 17

where

$$E_r = I_r Z_r \quad . \tag{49}$$

Performing the matrix multiplication indicated yields

$$\begin{Vmatrix} E_g \\ I_g \end{Vmatrix} = \begin{Vmatrix} AE_r + BI_r + Z_s\ (CE_r + DI_r) \\ CE_r + DI_r \end{Vmatrix} \tag{50}$$

The ABCD parameters are given by Eqs. (7a), (7b), and (7c), or, more fundamentally, by their definitions in terms of the mount-impedance values, z_{11}, z_{12}, z_{21}, and z_{22}.

Force and velocity transmissibility may be obtained from Eq. (50) by taking 20 log $_{10}$ of the voltage and current ratios. The complex power ratio is obtained by taking the product of these ratios. The real part of this complex number represents the real power ratio, as it is conventionally used. These ratios are given below.

Force ratio:

$$\frac{F_g}{F_r} = \frac{E_g}{E_r} = \left(1 + \frac{Z_s}{Z_r}\right) \cosh \gamma L + \left(\frac{Z_0}{Z_r} + \frac{Z_s}{Z_0}\right) \sinh \gamma L \tag{51}$$

Velocity ratio:

$$\frac{\dot{\xi}_g}{\dot{\xi}_r} = \frac{I_g}{I_r} = \cosh \gamma L + \frac{Z_r}{Z_0} \sinh \gamma L \tag{52}$$

Complex power ratio:

$$\frac{P_g}{P_r} = \frac{E_g I_g}{E_r I_r} = \frac{1}{2} \left[\left(1 + \frac{Z_s}{Z_r}\right) + \frac{Z_r}{Z_0}\left(\frac{Z_s}{Z_0} + \frac{Z_0}{Z_r}\right)\right] \cosh 2\ \gamma L$$

$$+ \frac{1}{2}\left[\frac{Z_r}{Z_0}\left(1 + \frac{Z_s}{Z_r}\right) + \left(\frac{Z_s}{Z_0} + \frac{Z_0}{Z_r}\right)\right] \sinh 2\gamma L + \frac{1}{2}\left[\frac{Z_s}{Z_r} - \frac{Z_r Z_s}{Z_0^2}\right] \tag{53}$$

The mount "effectiveness" parameter may be obtained from (32b):

$$\text{Effectiveness} = \frac{(Z_s + Z_0)\ (Z_r + Z_0)\ e^{\gamma L} - (Z_s - Z_0)\ (Z_r - Z_0)\ e^{-\gamma L}}{2\ Z_0\ (Z_s + Z_r)} \quad .$$

Using the hyperbolic function identities, this reduces to

$$\text{Effectiveness} = \frac{Z_s Z_r + Z_0^2}{Z_0 \, (Z_s + Z_r)} \sinh \gamma L + \cosh \gamma L \quad . \qquad (54)$$

Still another parameter of possible value is the system transfer impedance, Z_T, i.e., the ratio of force generated by the machine to the foundation velocity:

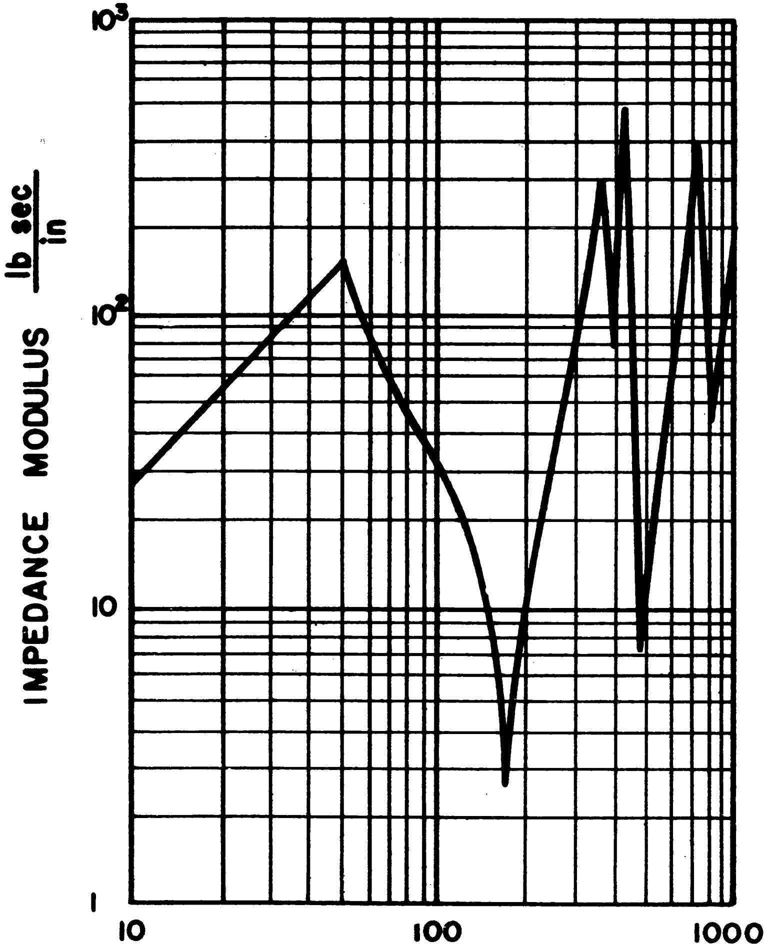
$$\left.\begin{array}{l}\text{System}\\ \text{Transfer}\\ \text{Impedance}\end{array}\right\} = \frac{F_g}{\dot{\xi}_r} = \frac{E_g}{I_2} = (Z_r + Z_s) \cosh \gamma L + \left(\frac{Z_s Z_r + Z_0^2}{Z_0}\right) \sinh \gamma L \quad . \qquad (55)$$

The preceding expressions provide the framework for numerically computing the values of the isolation parameters as a function of frequency once the impedance values which properly characterize the machine, mount and foundation are determined as a function of frequency. We shall now consider the application of these results to the fictitious system which is intended to characterize the degree of complexity found in a typical physical system.

NUMERICAL RESULTS

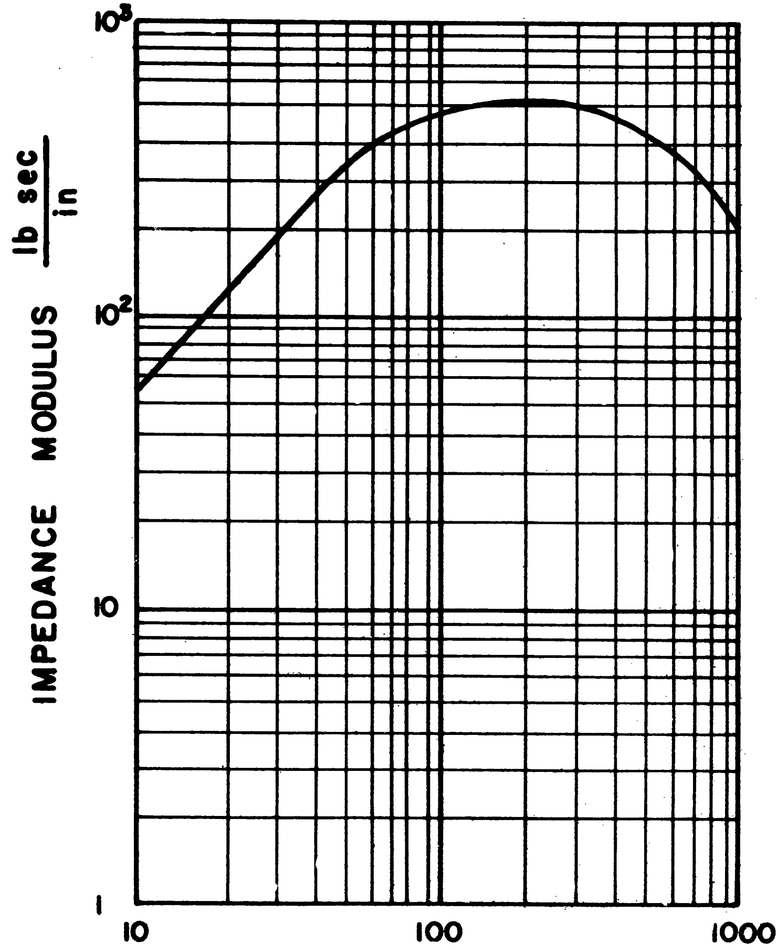
The magnitude of the machine and foundation impedances were taken from the curves shown in Figures 18 and 19 which for the present purpose were considered to be typical. Although only the impedance modulus is plotted in each case, the real and imaginary parts of the machine impedance were obtained by estimating the resistance at resonant points and computing the reactance from this, assuming the resistance to be relatively constant in the vicinity of the peaks. In the case of the foundation, only one resonance existed, at about 1200 cps. The resistance was estimated at this point and the reactance computed for the 10-to-800-cps range, using this value of resistance.

Considerably more effort was necessary to obtain comparable data for the mount. In the past, mount measurements have consisted almost entirely of obtaining "transmissibility" data under specified (usually rated) loading. These data are virtually useless for inclusion in a system analysis using a network-impedance approach. What of course is needed in the way of mount data is the input and transfer impedance of each end with the opposite end rigidly constrained. The procedure followed in this case was essentially to select by trial and error the physical parameters that a simple cylinder of elastic material would have to possess to duplicate the transmissibility curve shown in Figure 20. This curve is the result of experimental measurements made on a conventional commercial mount. The equation for transmissibility was obtained by application of the theory presented in this paper to the physical system on which the measurements were made. Using this theory, one finds the equation for transmissibility to be:



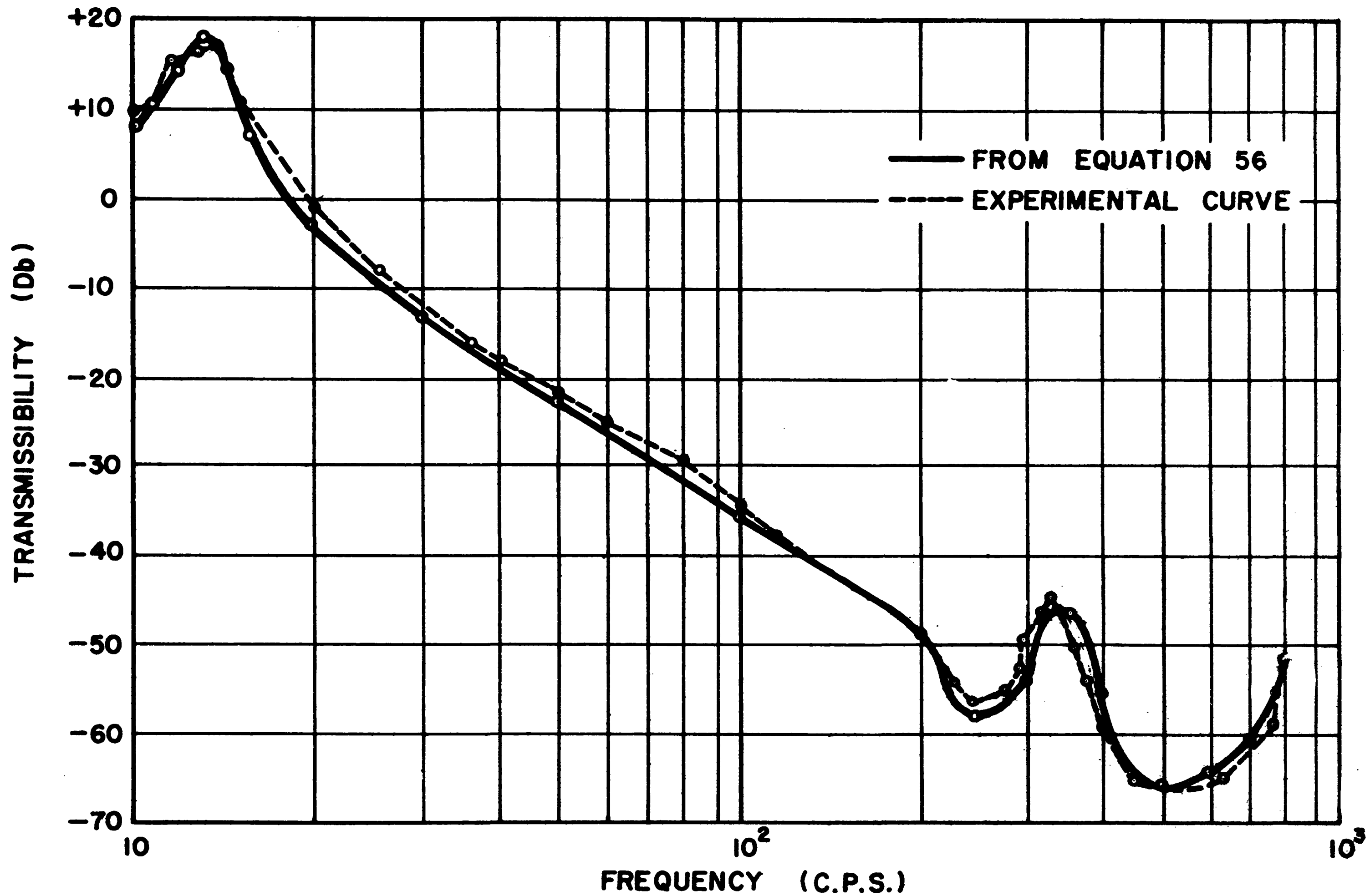
MACHINE IMPEDANCE MEASURED AT POINT OF SUPPORT

FIGURE 18



FOUNDATION IMPEDANCE MEASURED AT POINT OF SUPPORT

FIGURE 19



FORCE TRANSMISSIBILITY OF MOUNT

FIGURE 20

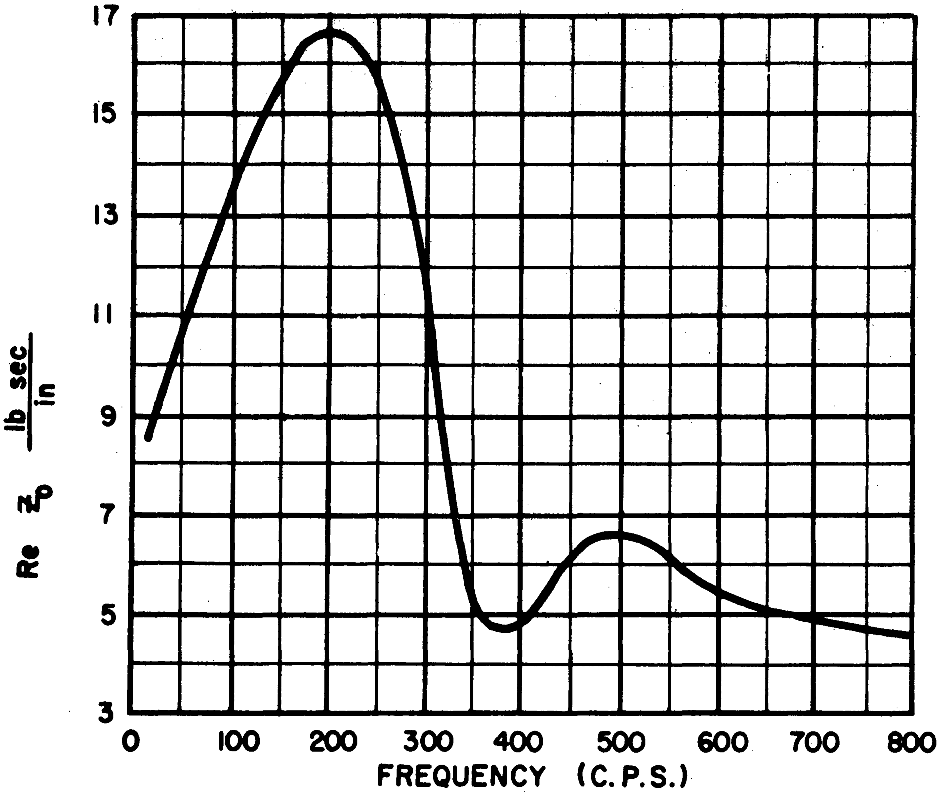
$$T = 20 \log_{10} \left| \frac{1}{\cosh \gamma L + \frac{i\omega M}{Z_0} \sinh \gamma L} \right| . \qquad (56)$$

Thus by considering various trial values of the physical parameters of the material (Young's modulus, velocity of sound, visco-elastic coefficient, cross-sectional area of the cylinder, and length of the cylinder), the proper values were selected so that the number computed from the above equation matched the experimental curve at about twenty points between 10 and 800 cps. These values were then used to compute Z_0, α, and β from the formulas given in Section I. The graphs of these three variables as a function of frequency are given in Figures 21, 22, and 23, respectively.

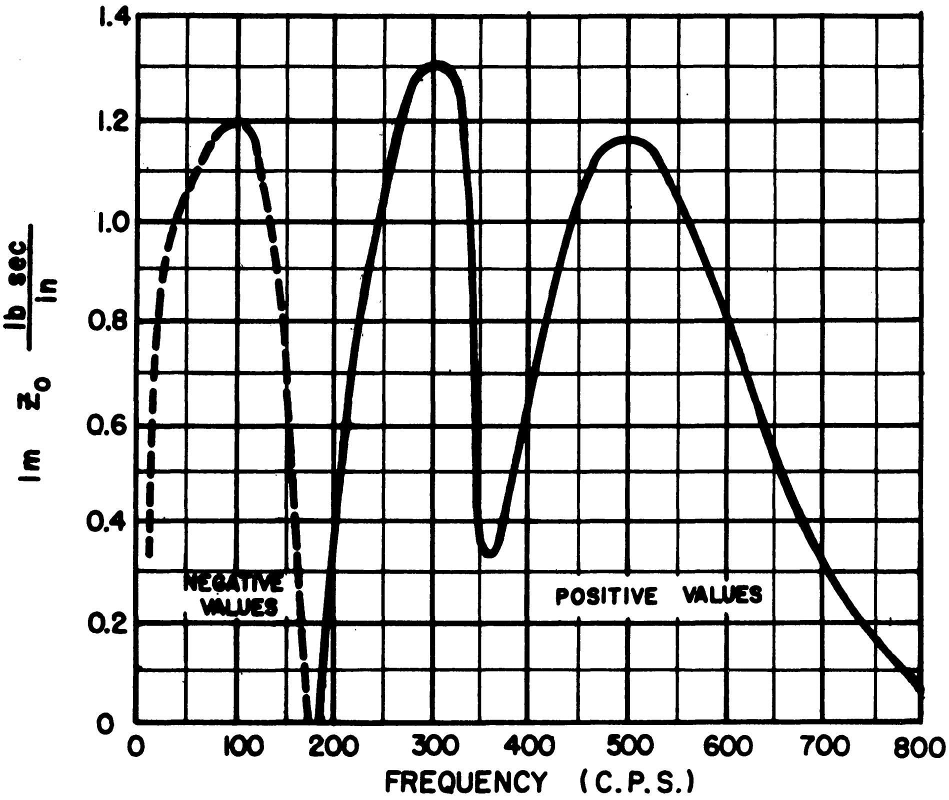
The data used to characterize the mount are drawn entirely from these three figures without further consideration of their possible significance. The graphs of the physical properties of the fictitious mount material are included in Figures 24, 25, and 26 for the sake of completeness. The radius of the cylindrical mount was selected as 2.8 in. and its length as 5 in. It might be mentioned that the length is determined by the frequencies at which the first wave effects occur and by the velocity of sound in the material at these frequencies.

Once these machine-mount-foundation data were compiled, a decision had to be made as to which one of the several isolation parameters listed earlier was to be computed since lack of time prohibited more than one sample computation. Somewhat arbitrarily, the system transfer impedance was selected. Equation (55) was put in the proper form for the computation simply by expanding all complex quantities, rationalizing individual terms, and collecting all real terms and all imaginary terms. The modulus of this complex number was then determined by computation involving some 34 distinct, elementary operations on a standard desk calculator. The time required for the computation of about 25 characteristic points by a moderately experienced laboratory technician was approximately 20 hours. Further practice with the computational procedures might reasonably be expected to reduce the time for a problem of this degree of complexity to about 12 hours. It should be pointed out, however, that the inclusion of effects due to transfer impedance would greatly increase the time of computation. Although the operations still would remain within the scope of a desk calculator's function, the time element would probably encourage the use of automatic computers if there were a considerable number of runs to be made on the same general program of computation. The results of the computation are shown in Figure 27 where the reciprocal of Z_T is shown as a function of frequency. This quantity, $1/Z_T$, is equal to the foundation velocity response to a unit force generated by the machine.

It should be noted that the general behavior of this function bears little similarity to the mount transmissibility or to the reciprocals of the machine and foundation impedances. Rather it represents, as one would expect, a composite

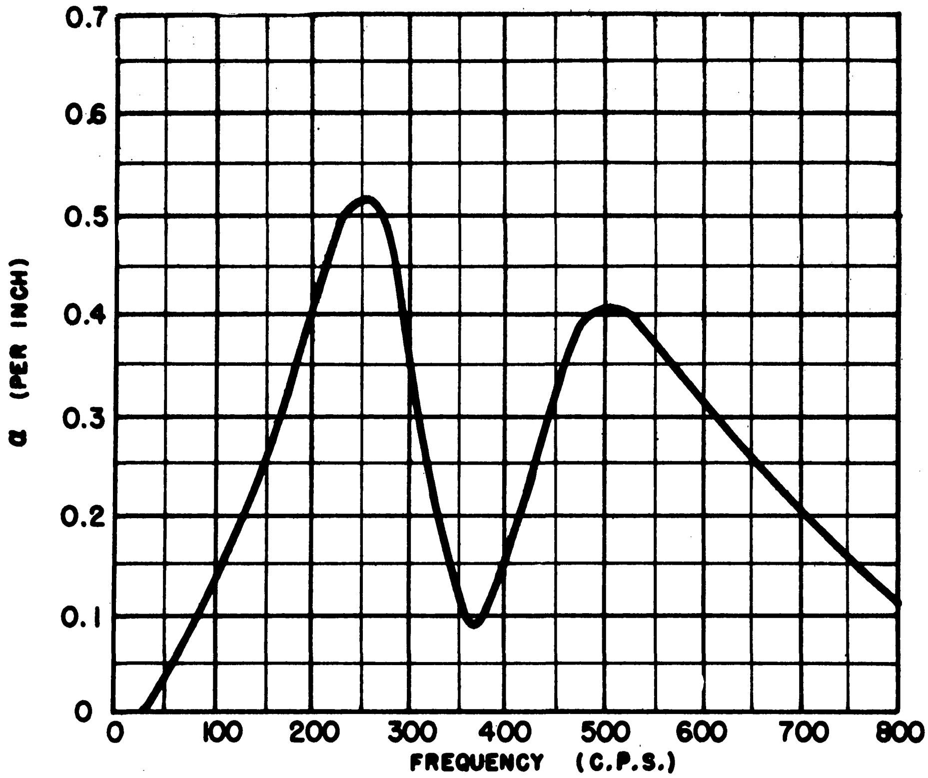


REAL PART OF MOUNT CHARACTERISTIC IMPEDANCE



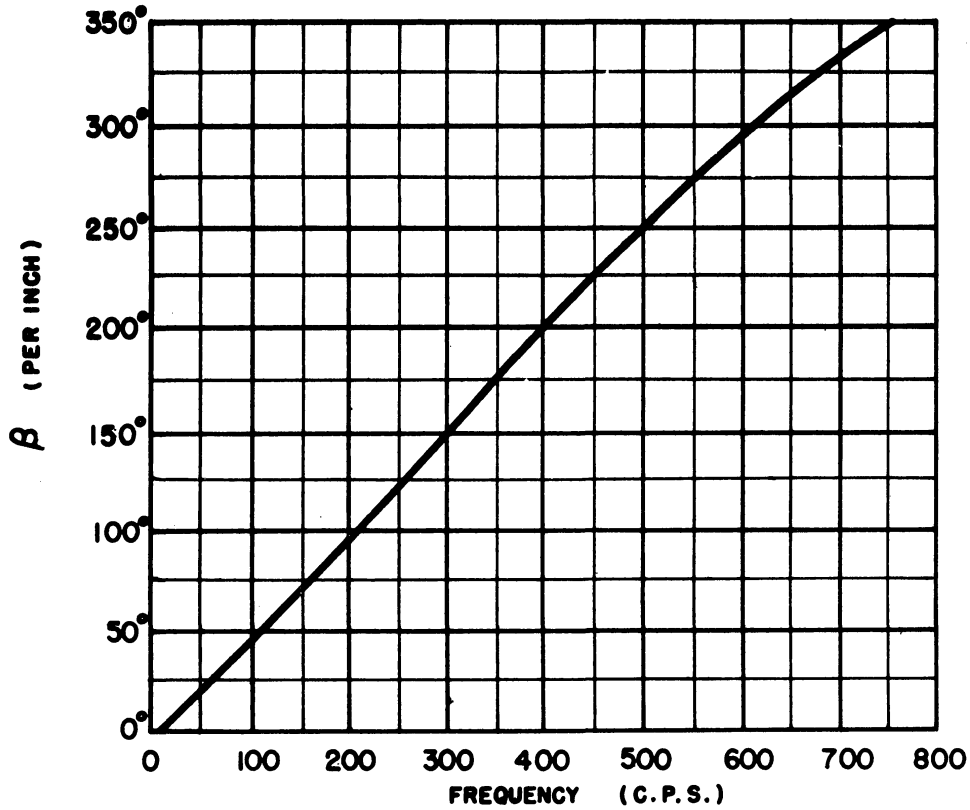
IMAGINARY PART OF MOUNT CHARACTERISTIC IMPEDANCE

FIGURE 21



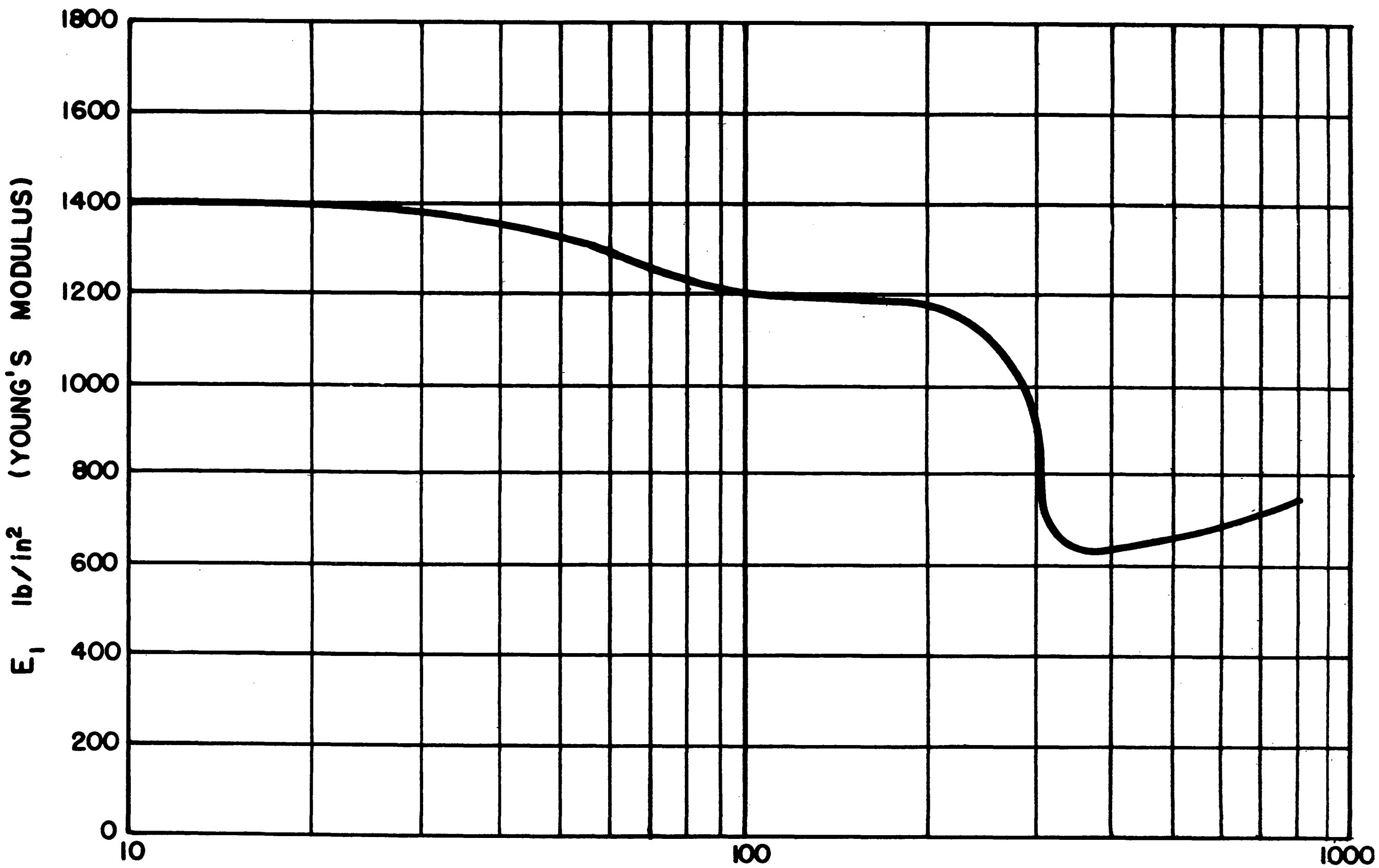
MOUNT ATTENUATION FUNCTION, α

FIGURE 22



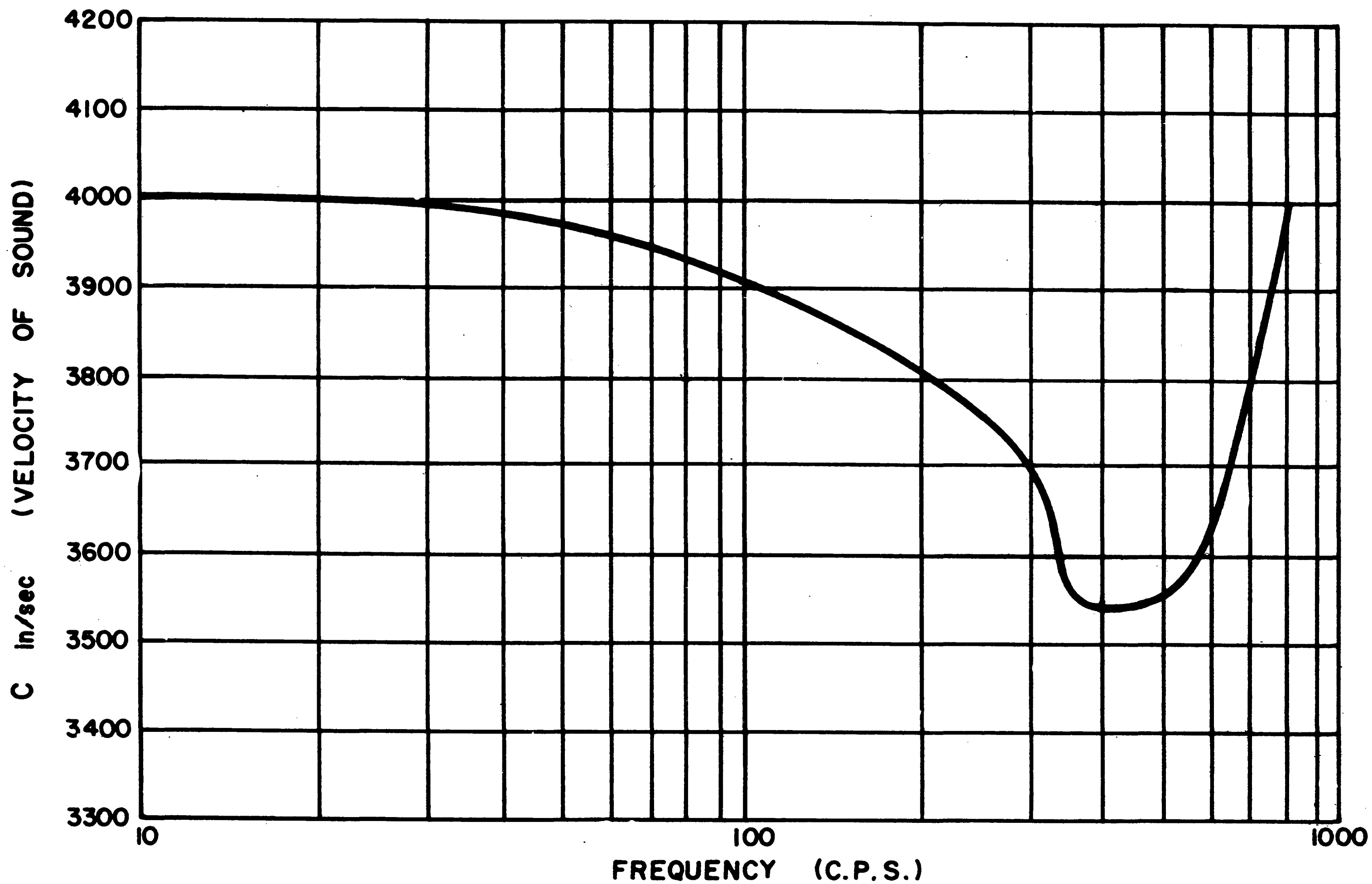
MOUNT PHASE FUNCTION, β

FIGURE 23



YOUNG'S MODULUS OF MOUNT MATERIAL, E_1

FIGURE 24

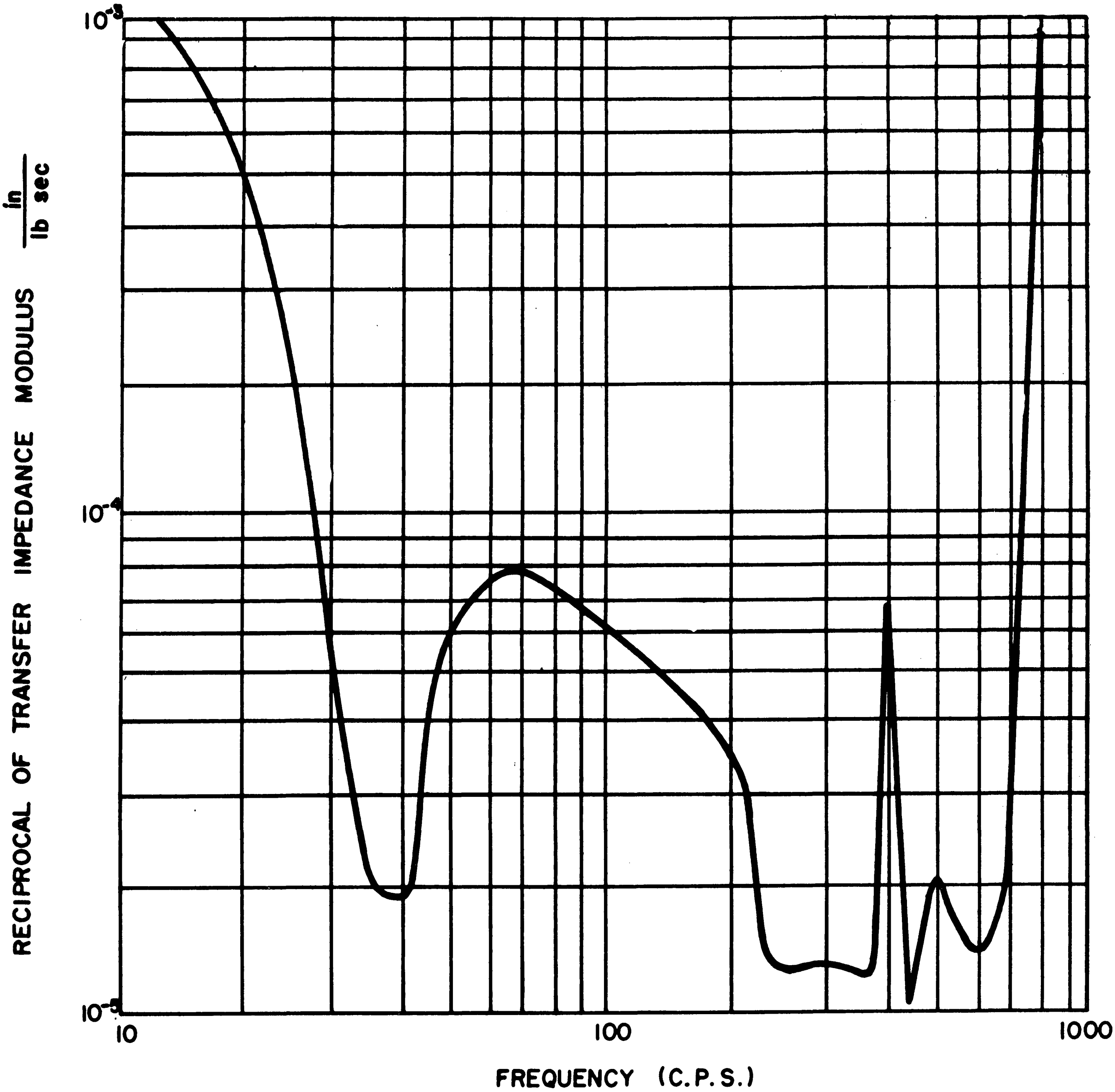


VELOCITY OF SOUND IN MOUNT MATERIAL, C

FIGURE 25

VISCO-ELASTIC COEFFICIENT OF MOUNT MATERIAL, μ

FIGURE 26



FOUNDATION VELOCITY RESPONSE TO UNIT FORCE GENERATED BY MACHINE

FIGURE 27

of the properties of all three elements in the system. A brief look at the behavior of the transfer-impedance function shows that the foundation velocity response in the vicinity of 10 cps is high due to the normal mode resonance of the system. In the frequency range immediately above this, the foundation velocity decreases until a minimum is reached at 40 cps. The curve then shows an increase in foundation velocity which may be attributed largely to a strong machine resonance just above 40 cps. After a relative maximum is reached at 70 cps, the response decreases until a broad minimum is reached in the 250-to-400-cps range. The fact that little increase in foundation velocity appears at the first wave resonance of the mount is apparently due to the high machine and foundation impedances obtained at 360 cps. The irregular behavior found in the 400-to-700-cps range seems to follow the machine impedance characteristic most closely, while at 800 cps an extreme increase in foundation velocity is noted due to a combination of resonance characteristics in all three elements of the system.

The fact that the properties of no one element of the system correspond to the properties of the system over a wide frequency range indicates the need for considering the system as a whole when attempting to evaluate analytically the importance of component properties such as mount or foundation damping, wave effects, and the like.

SECTION IV. SUMMARY AND RECOMMENDATIONS

EVALUATION OF THE NETWORK APPROACH

The analysis of complex vibratory systems has no formulation that is at once simple and complete. In the approach described by this paper, we have only a better approximation to the explanation of vibratory phenomena than the conventional and simpler lumped parameter theory provides. Many obvious limitations remain imposed on the distributed parameter analysis, and these must be kept in mind when evaluating the results of the theory. Specifically, it should be constantly remembered that this is still a one-dimensional theory which considers only the propagation of plane longitudinal waves in structures. In general, there is little reason to assume that this mode of vibration is the most prominent causative factor of detrimental effects in the system. It is simply the easiest mode to treat analytically.

However, the fact that this one-dimensional theory cannot provide all the answers should not give rise to pessimism. For we have here the framework on which to build a better understanding of difficult problems that demand solution. Extensions of the capability of the theory are always possible as, for example, by the addition of empirical corrections which experience suggests or by implicitly taking into account other effects known to be important through better methods of applying the techniques which the present theory affords.

Such better methods in turn help to shorten or improve the experimental labors required to achieve useful practical results.

The functions that this network approach might be expected to serve are two-fold. First of all, the fact that it employs the various impedance quantities outlined in Section I serves to clarify the objectives sought in applying the impedance concept to the analysis of vibrating mechanical systems. For one must have some knowledge of the relation such a concept bears toward the parameters which measure the quality of a system's design, and this is most readily obtained via the mathematical analysis of the system. Since the relation between the isolation parameters and the various impedance quantities is given by network theory, the way is opened to new design approaches in which certain preferred impedance characteristics may be sought in the construction of the system components.

Secondly, the network approach imposes specific conditions which must be met in the experimental measurement of the appropriate impedance quantities. Hence it serves to define the type and form of impedance measurement that must be obtained for fruitful analysis of a vibrating system. The theory also provides the mechanical interpretation of the particular impedances employed which forms the basis of the experimental technique used to measure their values. Thus it gives direction to the experimentalist in making impedance measurements by making clear the specific use that will be made of his data.

Before any real benefits can be derived from the prediction of system behavior provided by this network approach, better impedance measurements must be devised and be made available, especially with regard to the Thevenin equivalent series impedance of the machine and the input and transfer impedances of each end of the mount with the opposite end rigidly constrained. Once these measurements are made available, a better evaluation of the significance of the network approach can be made by comparing its predictions with experimental results.

In regard to the recommendations made on behalf of the "mechanical filter" concept in BuShips Memorandum, Ser 371-M91, it may be stated that the one-dimensional theory of mechanical filters has been worked out in considerable detail by several investigators and is duplicated in a large part in Section II of this report. To summarize the result of this work, mechanical filters do not provide any new mechanism for achieving vibration isolation. Nevertheless the principles involved may find useful application in certain problems since they provide essentially one technique of altering the impedance characteristics of the mount structure. The value of this technique can be determined only after such time as impedance considerations become a significant factor in the design of isolation systems.

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