ADAPTING AND USING U.S. MEASURES TO STUDY IRISH TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

by

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To my parents John and Martha Delaney and to the memory of their parents John and Catherine Delaney and Jim and Mary Ryan.

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Go raibh míle maith agaibh go léir.[1]

[1] This Irish expression is pronounced "gu – rev – mee-la – mah – ha – gut – gu - layer" and it means "Thank you all very much." The literal translation is "May a thousand good things happen to each of you."

Preface

The subject of the dissertation is a study of Irish teachers' mathematical knowledge for teaching, using a construct and instruments designed in the United States. Although the most widely used language of education in both countries is English, I have identified several differences of terms and spellings, mostly minor, between the two countries. Early in the dissertation writing process I appreciated the need to be as consistent as possible about the language I used throughout. When I refer to schools in Ireland I use class levels used in Ireland i.e. junior infants, senior infants, first class to sixth class. When I refer to materials used in Irish schools I use the terms used in Ireland. When it is clear that I am referring to education in both countries or generally (not specifically to Ireland), I use terms widely used in the United States. Such distinctions were not always obvious and despite my best efforts there may be some inconsistencies. Where the word is the same but spelling differences exist (e.g. color and colour, liter and litre, analyze and analyse, program and programme), I have generally used the U.S. spelling, unless it is a transcription of direct speech from an Irish participant in the study. I include a glossary below to support readers of this dissertation from different countries.

Term/Spelling Used in the United States	Term/Spelling Used in Ireland
Math	Maths
Popsicle sticks	Lollipop sticks
Elementary school	Primary school
Middle school and High school or Secondary school	Post-primary school
Grade level	Class level
Partitive model of division	Equal sharing model of division
Measurement model of division	Repeated subtraction model of division
Pre-kindergarten	Junior Infants (4 – 5-year-olds)
Kindergarten	Senior infants (5 – 6-year-olds)
First Grade	First class (6 – 7-year-olds)
Second Grade	Second Class (7 – 8-year-olds)

Third Grade	Third Class (8 – 9-year-olds)
Fourth Grade	Fourth Class (9 – 10-year-olds)
Fifth Grade	Fifth Class (10 – 11-year-olds)
Sixth Grade	Sixth Class (11 – 12-year-olds)
Notebook	Copy or copybook
Rectangular prism	Cuboid
“Sum” refers to addition	“Sum” also refers to addition but traditionally teachers and students refer to any calculation (including subtraction, multiplication, division) generically as a sum.
Mathematical practices	Mathematics skills
Teacher certification	Teacher qualification
A school where all subjects (except English) are taught through the medium of Irish	Gaelscoil
Irish speaking region	Gaeltacht
Student teaching or practice teaching	Teaching practice
Prime Minister	Taoiseach
Professional development institutes	Courses for teachers

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List of Abbreviations

2-D	Two-dimensional (shapes)
3-D	Three-dimensional (shapes)
CCK	Common content knowledge
ERIC	Education Resources Information Center (Database of educational articles)
IRT	Item response theory
KCS	Knowledge of content and students
KCT	Knowledge of content and teaching
N&O	Number and operations
OECD	Organisation for Economic Co-operation and Development
PISA	Program for International Student Assessment
SCK	Specialized content knowledge
TEDS-M	Teacher Education and Development Study in Mathematics
TIMSS	Trends in International Mathematics and Science Study
U.S.	United States (when used as an adjective)
RMSEA	Root Mean Square Error of Approximation

Abstract

Around the world, in many countries, teacher educators, researchers and policymakers are interested in the mathematical knowledge needed to teach effectively. This dissertation used a nationally representative sample to investigate Irish primary teachers' knowledge of mathematics using an instrument based on the construct of mathematical knowledge for teaching (MKT). Because MKT was based on studies of teaching practice in the United States, the study included an examination of the equivalence of the construct and the validity of the instrument for use in Ireland.

To establish the usability of the instrument, 501 teachers from a representative sample of Irish schools completed a teacher knowledge survey; ten additional teachers who completed the survey were videotaped teaching four lessons each. Ten Irish lessons were analyzed to examine construct equivalence between the mathematical demands of teaching in this sample of Irish practice and the mathematical demands of the teaching studied by U.S. researchers to develop the construct of MKT. Multiple-group factor analysis complemented the lesson analysis. Validity was examined by coding forty Irish lessons for the mathematical quality of the instruction; these codes were correlated with teachers' MKT scores. Factor analyses of teachers' responses, and comparison of tasks identified in Irish lessons with tasks that formed the basis of MKT, suggested that the constructs were sufficiently similar to use the measures in Ireland.

Results showed that Irish teachers' scores on MKT measures were moderately correlated with the mathematical quality of their instruction, suggesting that items were measuring knowledge used in instruction. Although MKT varied among teachers, performance on algebra items was strong and teachers were skillful at identifying and classifying student errors. Teachers demonstrated good knowledge of fractional representations. Applying properties and definitions of shapes, numbers and operations, and attending to student explanations and evaluating student understanding were more difficult.

This study suggests that measures based on the construct of MKT as conceptualized in the United States are valid for use in at least one setting outside the United States. Methods of establishing conceptual equivalence are identified; future research should refine further cross-cultural measurement of teachers' mathematical knowledge for teaching.

Chapter 1

Adapting a Practice-Based Construct to Study Irish Teachers' Mathematical Knowledge

> "Success in the future will be strongly dependent on growing the skills of our population and ensuring that levels of scientific and mathematical literacy increase. This places new demands on our education system, from primary level upwards."[2]

Like former Irish Prime Minister Bertie Ahern, many political leaders aspire to raise literacy levels in mathematics and science as a way to build a productive and prosperous future. Increasing a country's mathematical literacy, however, may place additional demands on teachers' mathematical knowledge. This dissertation studies the mathematics Irish primary teachers know and need to know. But the techniques used and outcomes reported have relevance beyond any single country.

"Mathematical knowledge for teaching" is a construct about the mathematics teachers need to know. Deborah Ball and her colleagues at the University of Michigan developed the construct in the United States by studying mathematics teaching and its knowledge demands. I apply the U.S. construct to study Irish teachers' mathematical knowledge. Educators in both countries stand to benefit from such a study: Ireland gains a readymade construct of mathematical knowledge, and data from a new setting tests aspects of the construct for the U.S. research program. With reference to this specific case, I discuss theoretical and practical issues that arise when a practice-based[3] construct is moved across countries. When governments want students to become more mathematically literate, teachers too may need to learn more mathematics. The study will be of interest to researchers who want to use the construct of mathematical knowledge for teaching in their countries.

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[2] From Taoiseach (Prime Minister) Bertie Ahern's foreword to *Ireland's Strategy for Science, Technology and Innovation 2006-2013*, (p. 2). Bertie Ahern resigned from the position of Taoiseach on May 6, 2008.

[3] By practice-based construct, I mean a construct that has been developed substantially from studying the practice of teaching. In this study it refers to the construct of mathematical knowledge for teaching which was developed in the United States by studying the practice of teaching and teacher knowledge literature.

I became aware of mathematical knowledge for teaching and the work of the Learning Mathematics for Teaching Project at the University of Michigan when I began to research the topic of teacher knowledge of mathematics. My interest in the topic was aroused when, as a teacher educator in Ireland, I encountered several prospective primary teachers who themselves found primary school mathematics difficult. I enrolled as a graduate student at the University of Michigan and for over 4 years I participated as a project member in research on mathematical knowledge for teaching. My contribution has primarily focused on developing and applying instruments to measure teachers' mathematical knowledge for teaching. This work has led me to the research question at the center of this dissertation: To what extent and how can measures of mathematical knowledge for teaching developed in the United States be used to study mathematical knowledge for teaching held by Irish primary teachers? I will return to this question shortly and explain both the question and mathematical knowledge for teaching in more detail. First I step back to provide some context for the question by describing aspects of the mathematics education environment in Ireland.

Most of what is known about the mathematical knowledge of practicing teachers in Ireland is anecdotal. The remarks below were overheard in relation to knowledge held by teachers and prospective teachers:

1. Fourth class student to parent: "No Ma, ordinary rectangles are not parallelograms. Only rectangles pushed out of shape are. Our teacher said so and it's in the book as well."
2. Teacher to principal regarding class allocation for next year: "Please don't give me fifth class. I'd never be able for the maths."
3. Mathematics teacher educator to colleague: "You know I gave a sixth class exam to my student teachers and not one of them got full marks on it and a fifth of them scored 50% or less on the exam."

Irish Teachers' Mathematical Knowledge

Teachers are typically expected to increase students' mathematical literacy. Less attention has been paid, however, to how teachers themselves become mathematically proficient. Mathematical proficiency among Irish teachers has been assumed rather than actively developed – possibly because prospective teachers have completed 5 years of secondary school mathematics after completing their primary school mathematics program. Prospective teachers, however, do not necessarily develop their knowledge of primary school mathematics by studying mathematics in secondary school. But even if they entered teacher education programs with strong knowledge of primary mathematics, such knowledge may be insufficient for the kind of work teachers need to do. A teacher needs to know definitions of shapes which are mathematically accurate and comprehensible to students of different grades, for example. A teacher needs to be able to interpret a textbook, which may contain incorrect or incomplete information. A teacher who perceives that her mathematical knowledge is inadequate may be limited professionally by opting to teach junior class levels, not by choice but out of concern about not knowing enough mathematics to teach senior classes. Comments such as those above suggest that primary school teaching requires strong mathematical knowledge, knowledge unevenly held among teachers.

Various responses to the comments are possible. Some readers may wring their hands and say that educational standards are dropping and recall that teachers in the past knew more, akin to the teacher portrayed in Goldsmith's (1783) poem:

> The village all declar'd how much he knew;
> 'Twas certain he could write, and cypher too;
> Lands he could measure, terms and tides presage,
> And even the story ran—that he could gauge.
> In arguing, too, the parson own'd his skill,
> For ev'n though vanquish'd, he could argue still;
> While words of learned length and thundering sound
> Amaz'd the gazing rustics rang'd around;
> And still they gaz'd, and still the wonder grew
> That one small head could carry all he knew.

Other readers may dismiss the comments as atypical and unrepresentative of most Irish teachers. Neither the nostalgic response nor the dismissive response takes seriously the issue of teachers' mathematical knowledge as a teaching resource. A more productive response might be to consider the context in which these teachers work, by reviewing some of what is known about the outcomes of mathematics education in Ireland.

Outcomes of Mathematics Education in Ireland

Many indicators are positive. In 2005, the per capita number of Irish graduates in mathematics, science and technology was double the European average.[4] Irish 15-year-olds performed at the Organisation for Economic Co-operation and Development (OECD) mean in mathematics in the 2006 Program for International Student Assessment (PISA) study (Eivers, Shiel, & Cunningham, 2007). In October 2006, at the launch of a report on fourth class students' mathematics achievement, the Minister for Education

[4] http://www.education.ie/robots/view.jsp?pcategory=10861&language=EN&ecategory=40272&link=link001&doc=28970. Accessed on March 25, 2008

and Science described as heartening "an improvement in aspects of the pupils' achievement such as their ability to reason, to undertake algebra, in their understanding of shape and space and their ability to manage data."[5] Learning support provision in mathematics increased while demand for it fell between 1999 and 2004.[3] Although encouraging signs exist, other facts indicate shortcomings in Irish mathematics education.

Shortcomings are evident in uneven achievement among students and an overemphasis on routine tasks. Primary school students attending schools designated as disadvantaged are disproportionately represented at lower bands of achievement on standardized tests compared to their counterparts in other schools (Department of Education and Science, 2005b). In the national study of fourth class students mentioned above, performance was low on the higher order practice of problem solving (Shiel, Surgeoner, Close, & Millar, 2006). At secondary school level 10% of Leaving Certificate students failed their mathematics examination in 2007.[6] In the 2006 PISA study relative student performance in mathematics was weaker than performance in science and reading (Eivers et al., 2007). Reforms of secondary school mathematics are to be piloted from September 2008 because procedures are overemphasized in current practice.[7] Clearly, many students are being failed by the Irish education system and procedural performance appears to be more widespread than conceptual understanding.

Primary Teaching is Mathematically Demanding Work

Policymakers aspire to increase mathematical literacy and to develop in students practices such as problem solving. Such aspirations are expressed in strategies like *Ireland's Strategy for Science, Technology and Innovation 2006-2013* mentioned earlier and in the national mathematics curriculum. A new primary school mathematics curriculum was introduced in 1999 and teachers are expected to deliver a program which

> should be sufficiently flexible to accommodate children of differing levels of ability and should meet their needs. These will include the need for interesting and meaningful mathematical experiences, the need to apply mathematics in other areas of learning, the need to continue studying mathematics at post-primary level, and the need to become mathematically literate members of society (Government of Ireland, 1999a, p. 3).

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[5] http://www.education.ie/robots/view.jsp?pcategory=10861&language=EN&ecategory=40280&link=link001&doc=32771 . Accessed on March 25, 2008

[6] http://www.ireland.com/newspaper/frontpage/2007/0815/1187036480218.html. Accessed on March 25, 2008

[7] http://www.independent.ie/national-news/chalk-and-talk-maths-teaching--to-go-in-shakeup-1326586.html. Accessed on March 25, 2008

These inclusive and ambitious aspirations can only be achieved when the curriculum is enacted with mathematically competent instruction. When the Minister for Education and Science launched the 1999 curriculum, he highlighted a "greater emphasis" in the mathematics curriculum on the "use of concrete materials" as a resource.[8] No mention was made, however, of developing another resource in teaching, teachers' mathematical knowledge. Substantial professional development was provided to support teachers in implementing the curriculum in all subjects, but in mathematics the emphasis was placed on equipping teachers with new teaching methodologies (Delaney, 2005). Little or no provision was made to ensure that teachers acquired the mathematical knowledge needed to implement the curriculum.

It is not surprising that developing teachers' mathematical knowledge was overlooked when implementing the curriculum. Teachers have typically had few requirements or opportunities to develop their mathematical knowledge prior to or during their professional career. The lack of attention paid to developing teachers' mathematical knowledge is consistent with a belief that teaching primary mathematics is mathematically trivial work. To enter a primary teaching programme in Ireland, the mathematics requirement is the lowest passing Leaving Certificate grade. Prospective teachers are required to study no mathematics when in college, although some do. Professional development rarely, if ever, is focused on developing teachers' mathematical knowledge. The implicit message is clear: minimal mathematical knowledge is needed to teach mathematics to primary school students. International mathematics education researchers disagree.

Having provided some context about Ireland I now discuss the theoretical orientation of the dissertation. It is rooted in work done by Lee Shulman and his colleagues.

Research on Teachers' Subject Matter Knowledge

Two decades ago Shulman drew attention to subject matter knowledge – the "missing paradigm" in most research on teaching (Shulman, 1986). Mathematics education research suggests that teachers don't need just any kind of subject matter knowledge. Evidence indicates, for example, that beyond a certain level, university mathematics courses have little impact on mathematics instruction or on student achievement (Begle, 1979; Borko et al., 1992). Influenced by Shulman and other

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[8] http://www.education.ie/home/home.jsp?maincat=&pcategory=10861&ecategory=40216&sectionpage=12251&language=EN&link=link001&page=2&doc=15005. Accessed on March 25, 2008.

researchers, Ball and Bass (2003b) have been to the fore in highlighting the kind of mathematical knowledge needed to teach mathematics. Ball, a teacher educator, educational researcher and former elementary school teacher, and Bass, a research mathematician, have worked with colleagues to mathematically analyze the work of teaching. Building on traditions of studying teaching and studying teachers' mathematical knowledge, Ball and Bass study the mathematical work of teaching, primarily by looking at records of practice, including videotapes of lessons. They have used their own research findings and work of other scholars to identify a particular type of mathematical knowledge specific to the work of teaching, called mathematical knowledge for teaching, often referred to by its acronym, MKT.

The Construct of Mathematical Knowledge for Teaching

The construct of MKT conceptualizes the specialized knowledge teachers need. It includes knowing the mathematics students are expected to learn but it includes much more. Consider the following example from Ball and Bass.[9] Imagine three students who have multiplied the numbers 35 x 25 in three different ways (see Figure 1.1). For most people being able to calculate the answer is sufficient. A teacher needs to do more. Ball and Bass put it as follows:

> Suppose, for example, a teacher knew the method used in (B). If a student produced this solution, the teacher would have little difficulty recognizing it, and could feel confident that the student was using a reliable and generalizable method. This knowledge would not, however, help that same teacher uncover what is going on in (A) or (C).
>
> Take solution (A) for instance. Where do the numbers 125 and 75 come from? And how does $125 + 75 = 875$? Sorting this out requires insight into place value (that 75 represents 750, for example) and commutativity (that 25×35 is equivalent to 35×25), just as solution (C) makes use of distributivity (that $35 \times 25 = (30 \times 20) + (5 \times 20) + (30 \times 5) + (5 \times 5)$. Even once the solution methods are clarified, establishing whether or not each of these generalizes still requires justification.
>
> Significant to this example is that a teacher's own ability to solve a mathematical problem of multiplication (35×25) is not sufficient to solve the mathematical problem of teaching – to inspect alternative methods, examine their mathematical structure and principles and to judge whether or not they can be generalized (2003b, p. 7)

In this extract Ball and Bass document how mathematical demands specific to the work of teaching differ from those faced by other adults who use mathematics in the course of

[9] Although I have been a member of this research group for over four years, in this section I treat the construct of MKT as part of the existing mathematics education literature.

their work. This is just one example of one task in one topic area. Many, many more tasks of teaching requiring specialist mathematical knowledge have been identified by Ball and Bass and by other scholars of teachers' mathematical knowledge. Completing the mathematical tasks of teaching requires significant mathematical knowledge. Teachers need MKT in order to provide accurate and understandable explanations and definitions, respond to students' questions and mathematical ideas, pose good questions, assess learning and plan future work, represent mathematical ideas and link representations to one another, evaluate textbooks, and choose materials (Ball & Bass, 2003b).

Student A	Student B	Student C
35	35	35
x25	x25	x25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Figure 1.1 Three students' attempts to multiply 35 x 25 (Ball & Bass, 2003b).

Why Mathematical Knowledge for Teaching Might Be Specific to U.S. Teaching

The construct of MKT emerged from studying teaching and mathematics education literature in the United States. Researchers, led by Ball and Bass, used their knowledge of teaching and of mathematics to scrutinize teachers' work and to identify its mathematical demands. Ball and Bass argue that teaching is "a form of mathematical work" in which mathematical knowledge influences how questions are posed and answered, how tasks are chosen and modified, how discussions are managed, and how classroom materials are interpreted (2003b, p. 6). Because the construct is closely linked to practice, [10] however, and because U.S. practice and U.S. literature were studied when elaborating the construct, it might not apply to teaching mathematics in Ireland. Studies of teaching across countries suggest that country-specific differences exist in how teachers teach (Schmidt et al., 1996; Stigler & Hiebert, 1999). If the practice of mathematics teaching differs in Ireland and the United States, and if the knowledge needed is determined by the work of teaching, teachers in each country may need and use different mathematical knowledge. Given such differences, the construct of MKT as elaborated by studying teaching in the United States would not be appropriate for investigating the mathematical knowledge Irish teachers need.

But if the practice of teaching is similar in Ireland and the United States, the construct of MKT could be useful for Irish teachers and teacher educators. U.S. research could be used to inform policy and practice in Ireland with regard to the mathematical knowledge teachers need. Such research would be credible because empirical evidence supports the importance of MKT in U.S. teaching. U.S. researchers have demonstrated an association between strong MKT and better quality mathematics instruction (Hill et al., in press). Furthermore, higher MKT has been associated with higher student achievement (Hill, Rowan, & Ball, 2005). If the construct of MKT were similar in Ireland, instruments used to study MKT in the United States, including multiple-choice measures, could be applied to the study of Irish teachers' mathematical knowledge and many potential benefits, such as those listed below, would follow.

Potential Benefits of Using Mathematical Knowledge for Teaching Measures in Ireland

First, the measures could be used at national level to generate a baseline of Irish teachers' MKT. Little is currently known about Irish teachers' mathematical knowledge and consequently, it is difficult to specify what might be an improvement. Second, what Irish teachers currently know could be used to illustrate the expert knowledge teachers

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[10] I define practice in this context as the "habitual doing of" (Simpson, 2004) the tasks of teaching.

possess and need in order to teach mathematics. Third, better understanding of Irish teachers' MKT could inform teacher education at pre-service and in-service levels. Related to this, the measures could be used to evaluate teacher learning following professional development initiatives, and to evaluate teacher education programs. Fifth, the measures could be used to study the relationship between teachers' MKT and student knowledge. If Irish teachers' MKT predicts student learning, developing teachers' MKT would be one concrete way to increase mathematical literacy. Finally, by associating demographic details with teachers' performance on the measures, inferences could be made about how teachers develop MKT, whether it is associated with teaching experience or with Leaving Certificate grades, for example. All of the benefits listed here relate to administering the measures to *groups* of teachers. The U.S. measures are not validated for, and may not be used for, evaluating MKT held by individual teachers. Notwithstanding this restriction, the benefits of learning more about Irish teachers' MKT make studying the possible application of the construct in Ireland worthwhile.

Benefits of the Study

The study is worthwhile because it can potentially contribute to increasing mathematical literacy in Ireland. By learning more about what knowledge Irish teachers need when teaching, and about what knowledge teachers possess, teachers themselves, policymakers and teacher educators can attempt to bridge the gap between what is known and what needs to be known. Because interest in the topic of teachers' subject matter knowledge has only become widespread in the last two decades it is not surprising that little is currently known about Irish teachers' mathematical knowledge. This study addresses the gap in what is known. In doing so it shines a light on the mathematical work of teaching in Ireland and on the mathematics Irish teachers know.

This study is not just of interest to Irish educators who stand to learn more about the mathematical knowledge needed by Irish teachers. An important aspect of the study centers on how a construct developed by studying practice in one setting is evaluated for its relevance in another setting. Researchers have been criticized in the past for lifting a theory or an instrument successfully developed and used in one country and applying it in a second country without establishing if such an application is appropriate (Flaherty et al., 1988; Johnson, 1998). In this study I explicitly describe the process of establishing the similarities and differences of the construct of MKT in United States and Ireland. Describing this process is an important contribution because many mathematics education researchers currently devote considerable attention to cross-national studies. Apart from studies of student knowledge across countries such as TIMSS and PISA, one study is currently investigating primary and secondary teacher education to determine how different countries prepare teachers to teach mathematics. It studies the policy and context, the processes, and the outcomes of teacher education in 18 countries (Tatto et al., 2008). Such a study must make assumptions that the content knowledge of mathematics, the pedagogy knowledge of mathematics and the knowledge of teaching needed by teachers is the same in all countries. We do not know if that is true and it is possible that if the tasks of teaching vary, the knowledge required may vary. I did not assume that MKT is equivalent in both countries but instead I incorporated testing equivalence into the research question.

Research Question

In order to examine the suitability of the construct of MKT for Ireland, I posed the question that has guided this study: To what extent and how can measures of MKT developed in the United States be used to study MKT held by Irish primary teachers? MKT is the mathematical knowledge needed by teachers to do the work of teaching. What is currently known about MKT is based almost entirely on U.S. mathematics teaching. Although Irish teachers use mathematical knowledge when they teach, little is known about it. To respond to the first part of the research question I investigated whether the construct of MKT in Ireland resembles the construct of MKT in the United States and identified many similarities and a few differences. The research question refers to MKT measures. These are multiple-choice – or selected-response (Yen & Fitzpatrick, 2006) – questions based on mathematics teaching scenarios that can be administered to teachers. Unlike interviews or lesson observations, which have been used to evaluate teacher knowledge, the MKT measures can be administered to large groups of teachers. In asking the research question I hoped to use the measures to study Irish teachers' MKT, but only if the measures were appropriate for use in Ireland.

How the Research Question was Addressed

The scope of this study is quite broad because it combines studying MKT in Ireland and evaluating equivalence of a practice-based construct in two settings. Therefore, I addressed the overarching research question by decomposing it into three sub-questions:

(1) How well does the construct of MKT, developed in the United States, describe MKT held by Irish teachers?

(2) How do the multiple-choice instruments developed to measure MKT in the United States measure Irish teachers' MKT, when adapted for use in the new setting?

(3) What MKT do Irish teachers possess?

The approach I used to respond to the questions was grounded in studying the practice of teaching from a mathematical perspective. In order to compare the construct of MKT in Ireland with the construct as currently elaborated in the United States, I first studied a sample of ten videotaped lessons taught by Irish teachers and used open coding to identify the mathematical tasks in which the Irish teachers engaged. I compared these tasks to teaching tasks which undergirded the construct of MKT in the United States. By demonstrating that mathematical tasks observed in Ireland were similar to tasks that had informed the construct in the United States, I concluded that the construct of MKT in the United States adequately describes MKT needed by Irish teachers. This qualitative analysis was complemented by comparing factor analyses of U.S. and Irish teachers' responses to the multiple-choice items.

Having established equivalence of the construct in the two settings, I tested the validity of using the adapted U.S. measures in Ireland. Guidelines for adapting the measures were described in an earlier study (Delaney, Ball, Hill, Schilling, & Zopf, in press) and these guidelines were applied to adapting items in the present study. Validity of the adapted measures was evaluated by investigating if teachers' scores on the measures could predict the mathematical quality of the teachers' mathematics instruction. If the measures were suitable for use in Ireland, videotaped lessons taught by teachers with high MKT scores would display higher quality mathematical instruction than lessons taught by teachers with lower levels of MKT. Possessing more MKT is not an end in itself but it is an invisible resource which can support teachers' practice, and contribute to the mathematical quality of instruction in classrooms (Hill et al., in press). I found that instruction coordinated by teachers with higher MKT scores generally exhibited a higher mathematical quality.

When I had established construct equivalence of MKT in both countries, and validated how the items would be interpreted, the third part of the question could be addressed: what MKT do Irish teachers possess? Multiple-choice items based on various mathematics teaching scenarios were used to answer the question. The Learning Mathematics for Teaching Project developed the items using scenarios inspired by both direct observation of teaching and mathematics education literature. The scenarios, in multiple-choice format, were presented to teachers in a national representative sample of Irish primary schools. The results, therefore, should generalize well to Irish primary teachers. The findings pointed both to mathematical topics and practices where Irish teachers' MKT is strong, and to areas where future professional development might be targeted. In order to report Irish teachers' scores on the MKT measures, a scale was created using item response theory (IRT). The scale is different to a raw score because it takes into account the difficulty of the items.

Answering all three sub-questions allowed me to address the overarching question. The U.S. multiple-choice measures are based on a construct that is largely similar in Ireland and in the United States. Furthermore, use of the measures to make claims about knowledge related to mathematics instruction in Ireland was validated by noting that MKT scores were a moderate to good predictor of the mathematical quality of instruction. Although MKT varies substantially among Irish teachers, some patterns emerge. Irish teachers have strong knowledge of graphical representations for fractions and are competent in identifying and classifying types of computation mistakes made by students. They had difficulties attending to student explanations and evaluating student understanding. Details of these findings will be presented in the chapters which follow. These results matter not only for Irish educators but for mathematics educators in other countries outside the United States. I have demonstrated that the MKT items can be moved from their U.S. context and used successfully in another country. This allows researchers in other countries to benefit from resources invested in developing MKT instruments in the United States, and subsequently to contribute to the ongoing development of the construct, by documenting results and possibly studying specific aspects of the construct in new contexts. In the following chapters I will elaborate more on the methods used and on the outline of findings presented here in Chapter 1.

Organization of the Dissertation

Chapter 2 sets in context and describes research on teachers' mathematical knowledge. It provides an overview of how MKT has been studied in the United States and reviews some of the literature that informed the construct. It describes attempts to measure teacher knowledge on a large scale. It outlines previous investigations of Irish teachers' mathematical knowledge and indicates the specific contribution made by this study.

Chapter 3 describes in more detail the background to the techniques used to study Irish teachers' MKT. These include grounded theory, factor analysis, video-coding, and IRT analyses of responses to multiple-choice questions. The chapter describes the data collected and how the various data were analyzed.

Each of the chapters 4, 5 and 6 corresponds to one sub-question of the main dissertation question. In some ways these chapters read like stand-alone chapters because each one describes a mini-study in itself. All three contribute to part of the overarching question. Chapter 4 describes in detail the process used to establish equivalence of MKT in both countries using both factor analyses, and open-coding of ten Irish lessons to identify mathematical tasks of teaching in Ireland. The work of teaching mathematics in ten Irish lessons is compared to tasks of teaching that informed the construct of MKT. Chapter 5 addresses the validity of using the multiple-choice measures to make inferences about the relationship between MKT and the quality of mathematics instruction. Chapter 6 describes the mathematical knowledge held and needed by a representative sample of Irish teachers. It describes the variation that exists among teachers and the particular strengths and difficulties identified in teachers' responses to the multiple-choice measures.

Chapter 7 takes the form of a hypothetical discussion between parties interested in the results of this dissertation.[11] I use this format to summarize the dissertation findings and to respond to possible questions and criticisms. I am joined in the conversation by an educational researcher, a comparative psychologist, a primary school teacher, and an educational policymaker. Together we explore issues that arise in this first national application in Ireland of a U.S. practice-based construct, to study Irish teachers' mathematical knowledge.

According to the opening quotation, the Irish Government recognizes the key role primary schools can play in increasing mathematical literacy. This study shows that teachers' mathematical knowledge is a resource distributed unevenly among teachers. Consequently, primary school students may participate in mathematical instruction that varies in quality. Internationally, policy makers and educators paid little heed to teachers' mathematical knowledge until recently. This was understandable because research offered conflicting evidence about the level and type of mathematical knowledge teachers need. Consensus is now growing that a special type of mathematical knowledge is needed to do the work of teaching. This study provides a portrait of MKT held by Irish teachers and provides baseline information about knowledge held and

[11] I first came across this format in Ball (1988). Ball's format was inspired by both Phillips (1988) and Lakatos (1976).

knowledge needed which can inform future planning and teacher education in mathematics. When teachers' mathematical knowledge is systematically developed in a country, teachers will be better placed to increase the mathematical literacy of the population.

Chapter 2

A Background to Studying Mathematical Knowledge for Teaching on a Large Scale

Establishing baseline data on the mathematical knowledge held by large numbers of Irish teachers is not an end in itself; its importance lies in its potential impact on student achievement. Higher achievement in mathematics tests is desirable because many Irish students achieve poorly in secondary school mathematics examinations (Hourigan & O'Donoghue, 2007b), which excludes them from several university programs and limits their career options. Even better would be improved examination results accompanied by stronger conceptual understanding of mathematics so that students can communicate and reason about ideas, and solve problems in flexible ways. Better instruction is needed to raise student achievement. Pre-service and in-service teachers can be better equipped to coordinate higher quality mathematical instruction if teacher educators and policymakers know more about teachers' mathematical knowledge as a resource in teaching. Improving teacher knowledge alone will not raise student achievement but it is an important component in enhancing the quality of instruction in schools.

When I read the literature about teachers' mathematical knowledge the metaphor that comes to mind is of a large pile of tangled yarn (see figure 2.1) hidden inside a dark box which must be sorted, using only touch. Multiple colors, fibers, lengths and thicknesses characterize the yarn and it has the potential to be woven into a tapestry that could be both beautiful and functional. Several weavers, individually and collectively, have proposed ways in which the yarn may be sorted. Some strands are long, and seem promising for weaving but such strands take time to sort. Other strands are easy to sort but too short to be of any real use. Some pieces have been disentangled from the pile and measured. Many strands are independently viable and others are inter-woven. The pile includes strands that look like yarn but closer inspection reveals them to be made of paper and plastic. Although, some weavers have described and conceptualized what is in the darkened box and have provided supporting evidence for their claims, further sorting will likely be required.

Figure 2.1. Tangled yarn of teachers' mathematical knowledge. Image taken from http://thetextilefiles.blogspot.com/2007_05_01_archive.html on March 15, 2008.

The strands of yarn correspond to different types of knowledge that have been identified as important for teaching mathematics. Researchers have used different methods to identify both the content and the form of teachers' knowledge. Many hypotheses have been offered as to the composition of the subject matter knowledge teachers need, including knowledge of the discipline of mathematics, knowledge of school mathematics (e.g. curriculum knowledge, knowledge of students), general school knowledge (e.g. pedagogical knowledge), different forms and epistemologies of knowledge, and relationships among the different types. In addition to identifying and elaborating the various strands of teachers' knowledge, researchers have attempted to measure it, study how it is acquired and learn about how it is related to instruction and to student achievement. Describing previous attempts to study and measure the strands of teachers' mathematical knowledge is central this chapter.

In order to study Irish teachers' mathematical knowledge I begin to disentangle some of the strands by describing in detail the construct of teacher knowledge at the center of this study, MKT, and how it was developed. My goal is to describe the kind of research that informed MKT as a construct. I begin by defining what is meant by knowledge of mathematics for teaching and by instruction. This is followed by an overview of the construct of MKT and how the construct has been developed in the United States. Shulman's professional knowledge categories are described next. An overview of literature on mathematical knowledge and on the study of teaching provides a context for the tradition in which Ball and Bass worked and the kind of literature (including cross-national studies) on which they could draw when developing the construct. I next discuss the measurement of teachers' mathematical knowledge with particular reference to measuring teachers' mathematical knowledge on a large scale (i.e. with 100 teachers or more). Because of my specific interest in *Irish* teachers' knowledge, findings of previous studies of Irish teachers' mathematical knowledge are summarized. I conclude by outlining limitations in how the construct of MKT can be used outside the United States and how this dissertation addresses some of the limitations.

The books and articles cited in this chapter were selected in order to present representative studies of teachers' mathematical knowledge. Reviews of past research have been conducted by Begle (1979), Fennema and Franke (1992) and, Ball, Lubienski and Mewborn (2001) and I consulted several references cited in each of these reviews. Additional articles on the topic were identified by conducting searches on ERIC using several descriptors and terms including "mathematics instruction," "pedagogical content knowledge," and "teacher characteristics." I had a particular interest in articles describing studies that measured the MKT of large numbers of teachers. Articles in a special edition of *Measurement: Interdisciplinary Research and Perspectives*, (Volume 5, issues 2 & 3) were consulted. The ERIC database was used to identify relevant articles from Ireland and I supplemented this with articles published in proceedings of two "Mathematics Education in Ireland" research conferences in 2005 and 2007, and with work of which I was already aware.

Defining Knowledge of Mathematics for Teaching

Little consensus exists as to what is included in a definition of mathematical knowledge. In her early research Ball (1988) distinguished between knowledge *of* mathematics and knowledge *about* mathematics. The former referred to the "understanding of substance – topics, concepts, procedures" and the latter to ideas about "where [mathematics] comes from, what it is good for, and how right answers are established" (p. 39). The impact of beliefs on instruction and planning has been identified by other researchers. Thompson (1984), for example, contrasted the teaching of a teacher who believed that mathematics develops the ability to reason logically, with a teacher who believed that methods and procedures guarantee right answers. Other researchers explicitly claim that beliefs should be part of any model of teacher knowledge (Cooney, 1999). I accept the importance of teachers' knowledge about mathematics in instruction, and even of its importance in interaction with teachers' knowledge of mathematics. Nevertheless, the focus in this study is specifically on teachers' knowledge *of* mathematics, of aspects such as language and practices. A vast amount is known about teachers' beliefs generally (Richardson, 2003) and in relation to mathematics specifically (e.g. Beswick, 2007); Irish teachers' beliefs about mathematics would merit a separate study in itself.

A second distinction made in teacher knowledge is between form and content (Sherin, Sherin, & Madanes, 2000). The form of the knowledge refers to how knowledge is "organized and represented in a teacher's mind" (p. 364). Examples include knowledge packages (Ma, 1999), agendas, scripts and routines (Leinhardt, Putnam, Stein, & Baxter, 1991), and Schoenfeld's (2000) use of Morine-Dershimer's idea of lesson image. Such metaphors provide ways of thinking about *how* teachers hold their mathematical knowledge. The content of teachers' knowledge is "what the knowledge is for, or what it is about" (Sherin et al., 2000, p. 364). Examples include pedagogical content knowledge and Shulman's other categories of professional knowledge (1986) or mathematics-for-teaching (B. Davis & Simmt, 2006). My interest is to establish a national picture of Irish teachers' mathematics and for this reason I am focusing on the content of what they know rather than its form. In summary, the knowledge that will be studied in this dissertation is the *content* of Irish teachers' knowledge *of* mathematics.

Defining Instruction

Instruction is another term used which may be open to more than one interpretation. I use the term as defined by Cohen, Raudenbush and Ball (2003) which is broader than simply teaching; it stresses the interactive nature of the relationship among teacher, students and content. Instruction is neither "something done to learners by teachers" nor an event, but a "stream … that flows in and draws on environments – including other teachers and students, school leaders, parents, … state agencies and test and textbook publishers" (p. 122). This conception of instruction is both ambitious and realistic: ambitious because, although this conception has been around for two centuries (Cohen et al., 2003), relatively few models for such dynamic instruction exist and therefore many teachers have experienced education that was more passive than interactive; realistic because only such a conception of instruction reflects the competing interests a teacher struggles to coordinate from day to day and the complex relationship between teacher knowledge and other aspects of instruction. When I refer to instruction in this study I consider teacher's knowledge as one resource in managing the interaction with students and subject matter in the context of the wider environments in which the interaction occurs.

In the next section I will describe the theory and construct of MKT. I explain how the construct has been developed to date in the United States and outline the domains of the construct that have been identified in that setting.

The Theory of Mathematical Knowledge for Teaching

Central to this study is the theory of MKT, a theory framed by studying practice from a disciplinary mathematics perspective. The theory of MKT is primarily concerned with what teachers need to know, but its starting point is the knowledge demands of effective teaching (Ball, Thames, & Phelps, in press). Ball, Bass and their colleagues assume that what elementary teachers need to know is not the mathematical content typically taught in advanced university-level mathematics courses but a particular type of knowledge needed to do the work of teaching which they label MKT. The construct of MKT has been conceptualized by both studying the mathematical work of teaching and drawing on mathematics education literature.

The first part of this work involves studying the practice of teaching from a mathematical perspective. Ball described this way of working as follows:

> We seek to analyze how mathematical and pedagogical issues meet in teaching – at times intertwining, at times mutually supporting, and at times creating conflicts. Through analyses of mathematics in play in the context of teaching, the project extends and challenges existing assumptions of what it is about mathematics that elementary teachers need to know and appreciate, and where and how in teaching such understandings and appreciation are needed (Ball, 1999, p. 28).

Studying the relationship between knowledge of mathematics and knowledge of teaching is at the heart of this approach. It is assumed that a mathematical perspective and a teaching perspective highlight different aspects of mathematics teaching and that the combined perspective will yield insights into the mathematical nature of the work of teaching.

In addition to studying actual practice when conceptualizing the construct of MKT, Ball and her colleagues studied the work of teaching documented in mathematics education literature. Ball wrote about the process of developing MKT:

> The work builds on two recent lines of research – one on teachers' subject matter knowledge and its role in teaching; and the second on the interplay of mathematics and pedagogy in teaching and teachers' learning (Ball, 1999, p. 22).

The first line of research to which Ball refers was inspired by Shulman (1986) and it includes investigation of teacher knowledge needed to teach several subjects (e.g. Sam Wineburg in History; Pam Grossman in English). The second line of research was also inspired by Shulman's work and it relates more specifically to mathematics education. It includes literature from Simon (1993), Borko and Eisenhart (1992) and Thompson (1984). Research on secondary teachers' knowledge by Ball (1990) and others, and Liping Ma's (1999) study of Chinese teachers' mathematical knowledge also informed the work.

Therefore, two sources of data informed the construct of MKT as it is currently understood. The first is studying practice to identify tasks of teaching and to analyze the mathematical components of these tasks. Much of the teaching analyzed by Ball and Bass and their colleagues came from a complete set of records of practice – including teacher notes, video tapes, audiotapes, transcripts, and student work – from Deborah Ball's third grade class in the school year 1989-1990 (Ball, 1999; Ball & Bass, 2003b). Mathematics education literature complemented the analyses of practice. The second data source is teacher knowledge literature. Tasks of teaching identified in the analyses of practice and in the teacher knowledge literature formed the basis of the construct of MKT as elaborated in the United States. Before describing the work in more detail I define what I mean by a theory and a construct because the distinction is important for the study.

A theory is “a scheme or a system of ideas or statements held as an explanation or account of a group of facts or phenomena” (Simpson, 2004). With regard to MKT the theory relates to the idea that the subject matter teachers need to know is the subject matter knowledge demanded by the work of teaching. In this sense the theory of MKT can be regarded as etic (Pike, 1954) in that it describes a generalized approach to and belief about knowledge that can be related to all countries. In this sense educators in Ghana, Brazil, South Africa or any country could study the work of teaching and identify the knowledge demands of the work in that setting. In contrast, a construct is emic in that it is valid for one country or setting at a time. It “attempts to describe the pattern” (Pike, 1954, p. 8) of MKT in one country or setting. In this sense the specific details we know about MKT are emic; they relate to the construct as it applies to teaching studied in the United States. Therefore, when I refer to the theory of MKT, I refer to the general principle about teachers’ knowledge being determined by the nature of their work; when I refer to the U.S. construct, as I do more frequently, I refer to what has been written and specified about MKT as it has been elaborated in the United States based on existing literature and a limited sample of U.S. teaching.[12] I now describe an example of a task of teaching analyzed from a mathematical perspective.

The example below illustrates the kind of mathematical analysis of practice in which Ball and Bass engaged when developing the construct of MKT (2003b). It relates to a teacher who wishes to test if her students can order decimal numbers from smallest to largest. The teacher wanted to choose decimals that would indicate whether students understood decimals. The teacher considers the lists below:

(a)	.5	7	.01	11.4
(b)	.60	2.53	3.14	.45
(c)	.6	4.25	.565	2.5

The lists contain whole numbers and numbers with one or more decimal places. If you look at the solutions, however, something becomes apparent:

[12] Because the term “construct” is being used to refer to MKT that applies in a given setting, it will sometimes be used to refer to Ireland. When I refer to Ireland I will qualify the term by using the adjective “Irish.” Generally when I refer to “the construct of MKT” or “the construct” it refers to the U.S. construct of MKT.

(a)	.01	.5	7	11.4
(b)	.45	.60	2.53	3.14
(c)	.565	.6	2.5	4.25

Lists (a) and (b) could be answered correctly if students treated all numbers in each list as whole numbers. In other words, by ignoring the decimal points – a mistake students new to decimals frequently make – students would still order the lists of decimals correctly. If students solved (a) and (b) correctly the teacher could not be sure the students understood decimals. List (c) is different. If the decimal points are ignored on this list .6 would be written first which is not the smallest number in the list. Therefore, the most suitable list for the teacher to use to assess understanding of decimals is list (c). Ball and Bass point out that the teacher's work here involves more than being able to order the decimals. A mathematical perspective is required to conduct

> an analysis of what there is to understand about order, a central mathematical notion, when it is applied to decimals. And it also requires thinking about how ordering decimals is different from ordering whole numbers. For example, when ordering whole numbers, the number of digits is always associated with the size of the number: Numbers with more digits are larger than numbers with fewer. Not so with decimals. 135 is larger than 9 but .135 is not larger than 9. (p. 9)

In line with the disciplinary perspective they adopt, Ball and Bass conclude by arguing that teaching be recognized as "mathematically intensive work, involving significant and challenging mathematical reasoning and problem solving" (p. 13). Ball and Bass claim that (a) testing if students can order decimal numbers by size is part of the work of teaching, and (b) doing that work effectively involves selecting appropriate numbers to test student understanding. They do not claim that teachers in the United States or elsewhere either possess or apply such knowledge when they teach. But the example illustrates how insights into knowledge needed for teaching can be acquired by analyzing the practice of teaching from a mathematical perspective.

Domains of Mathematical Knowledge for Teaching

The Learning Mathematics for Teaching Project has hypothesized four specific domains of MKT to reflect the kind of knowledge teachers are expected to hold. The domains are common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. Common content knowledge (CCK) refers to mathematical knowledge "used in settings other than teaching" (Ball et al., in press). One example of CCK includes being able to recognize two-dimensional shapes such as a square, a rectangle or a triangle. Specialized content knowledge (SCK) is “mathematical knowledge and skill uniquely needed by teachers in the conduct of their work” (Ball et al., in press). It includes knowing a range of definitions of shapes that are both comprehensible to students of different age levels, and mathematically accurate and complete. Such knowledge would help teachers resolve disagreements when students argue about whether a rectangle can be classified as a parallelogram (it can) or a square as a rectangle (it can). Knowledge of content and students (KCS) “combines knowing about students and knowing about mathematics” (Ball et al., in press). For example, teachers need to know that many students who can recognize the square in Figure 2.2a will think that the shape is different when rotated 45 degrees as in Figure 2.2b. When teachers know about this misconception they can plan their teaching so that students’ likely misconceptions are challenged. The fourth domain of MKT, knowledge of content and teaching (KCT), refers to knowledge of mathematics in combination with knowledge of teaching. Choosing instructional materials and knowing how to sequence a topic are part of this knowledge domain. For example, a teacher may need to design or select a poster to illustrate shapes for students that includes non-examples and non-stereotypical examples of shapes in order to strengthen students’ understanding of the shapes (Clements & Sarama, 2000). Ball, Thames and Phelps (in press) have represented the sub-domains of MKT visually as shown in Figure 2.3.

Figure 2.3 shows that the construct of MKT is made up of both subject matter knowledge and pedagogical content knowledge. Subject matter knowledge consists of CCK, SCK, and a third kind of knowledge (knowledge at the mathematical horizon), described by Ball, Thames and Phelps (in press) as a provisional category recognizing connections among topics throughout the curriculum. Pedagogical content knowledge includes KCS, KCT, and another provisional category, identified by Shulman and his colleagues, knowledge of curriculum. Ball and her colleagues have done much to elaborate these categories in their writing but the framework is such that it can be used by other researchers who can further elaborate aspects of knowledge present in different parts of the model such as the knowledge demands entailed in teaching particular mathematical topics and practices (e.g. Izsák, 2008; Stylianides & Ball, in press). I next take a step backwards to describe how the construct of MKT has been developed in the United States.

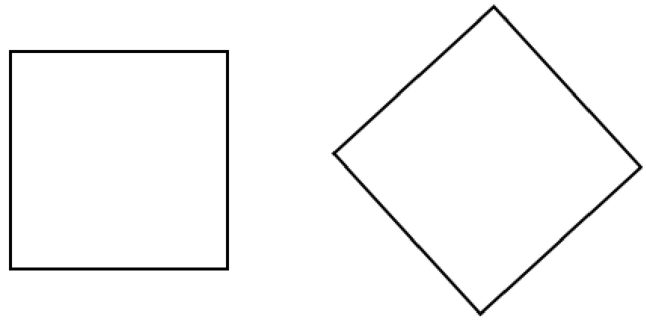

Figure 2.2a and Figure 2.2b. Two different orientations of a square

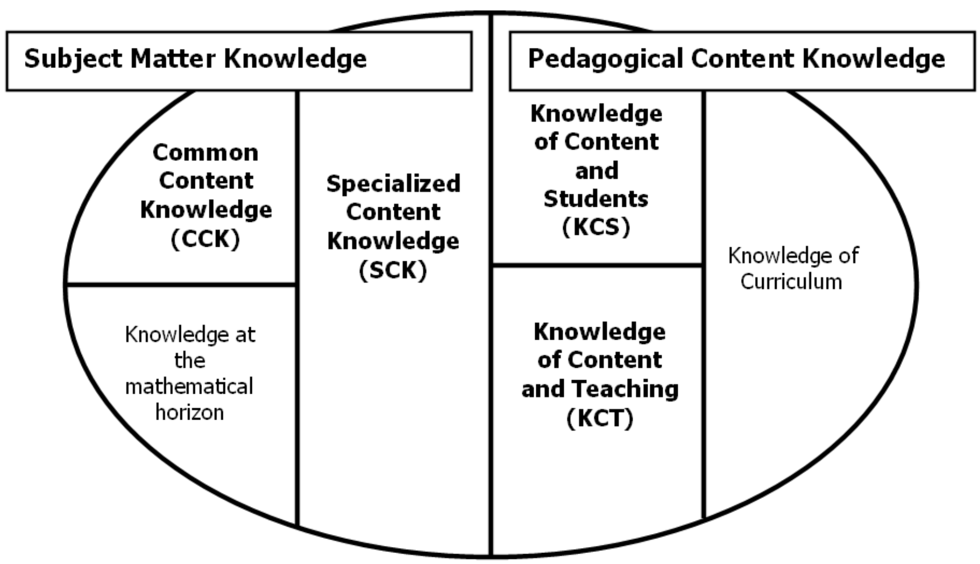

Figure 2.3. Domains of MKT

How the Construct of Mathematical Knowledge for Teaching has been Developed

Two research projects at the University of Michigan, the Mathematics Teaching and Learning to Teach Project[13] and the Learning Mathematics for Teaching Project,[14] have used multiple strategies to develop the construct of MKT. The strategies include observing and studying practice, developing measures, conducting measure validation studies, and drawing on existing literature about teachers' mathematical knowledge and on student thinking. Later in this chapter I survey some of the literature which undergirded the development of the MKT construct. First, I describe in more detail how the practice of teaching was studied.

Studying the practice of teaching from a mathematical perspective is at the heart of the approach used by Ball and her colleagues to identify the mathematical knowledge teachers need. Initially the observation was relatively unstructured with the research team seeking to "annotate and index mathematical issues that shape[d] an account of what [was] happening in the class" and preparing "commentaries on segments of classroom activity supported with evidence from those 'texts'" (Ball, 1999, p. 33). The year-long records of practice from Ball's classroom provided a rich resource for the researchers' annotations and commentaries.

Researchers on the projects bring a variety of perspectives, notably mathematical and teaching perspectives, to bear on their study of practice on the videotapes. Studying the practice of teaching from these perspectives has led Ball and Bass (2003b) to argue that thinking about teaching as "mathematically intensive work" can inform how pre-service and in-service teachers are prepared to meet the mathematical demands of the work (p. 13). The demands identified by Ball and Bass include solving mathematical/teaching problems (e.g. designing explanations that are accurate and comprehensible to students, choosing age-appropriate and mathematically precise definitions, responding to students' questions); unpacking complex mathematical ideas; connecting mathematical ideas across topics (e.g. using area to teach ideas about multiplication); knowing how mathematical ideas develop throughout students' time in school; and making mathematical practices[15] explicit when teaching. Over time additional lesson data have been collected and more structured ways of studying practice have

[13] Principal Investigators: Deborah Loewenberg Ball and Hyman Bass.
[14] Principal Investigators: Deborah Loewenberg Ball, Hyman Bass and Heather C. Hill.
[15] In Ireland the term used would be skills (including reasoning, applying and problem solving, communicating and expressing) whereas in the United States the term "skills" has the more restricted connotation of referring to knowledge of basic number operation facts.

been devised. In particular, a video-coding instrument was developed to study the mathematical quality of instruction evident in lessons and curriculum materials (Blunk & Hill, 2007) and this study of practice contributed to the development of the construct.

Moreover, the design and administration of measures of teachers' MKT have been used to further conceptualize the construct. The multiple-choice measures were developed by members of the Learning Mathematics for Teaching Project and by invited experts including teacher educators and mathematicians (Bass & Lewis, 2005). Data collected from teachers who responded to the measures have revealed underlying factors in the knowledge (Hill, Schilling, & Ball, 2004). Further, the items have been found to measure growth in teacher learning as a result of attendance at professional development institutes (Hill & Ball, 2004). In short, the measures have contributed to the development of the construct (a) by providing specific examples of knowledge that are part of MKT, (b) by providing data as to how the knowledge might be structured, and (c) by showing that they are related to what professional developers want teachers to know.

Mathematics education literature – including literature on elementary teachers' mathematical knowledge, research on secondary teachers' mathematical knowledge, and cross-cultural studies of teachers' knowledge – has informed the development of MKT (Ball, 1999, p. 27). I review some of these studies below. Literature on student thinking also informed the construct (Ball et al., in press). This literature has been particularly useful in developing the domain of KCS and some items designed to measure teachers' KCS can be traced to these articles (e.g. Burger & Shaughnessy, 1986; Falkner, Levi, & Carpenter, 1999).

Despite drawing on work of other scholars and the use of the measures, some readers might be concerned that a construct, which began by studying the work of a relatively small number of teachers, may have limited application, even in the United States. But validation studies indicate that items based on the construct measure knowledge needed by teachers more generally. One study has shown that the MKT measures are a good predictor of the mathematical quality of a teacher's instruction (Hill et al., in press). For example, teachers who score highly on the measures make fewer errors in their lessons and are generally better at responding to students' errors, comments and questions than teachers who get lower scores. Another study found that students whose teachers have higher MKT scores make more progress as measured by a standardized mathematics test than students whose teachers have lower scores (Hill et al., 2005). It is possible that the higher gain scores may be attributable to factors such as general knowledge or aptitude for teaching (Hill et al., 2005). Nevertheless, when considered in light of the connection between higher scores on MKT and higher quality mathematical instruction, the measures based on the construct appear to be measuring mathematical knowledge important for the work of teaching.

Up to now I have described the construct of MKT and how it was conceptualized and developed by Ball, Bass, Hill and their colleagues. I summarized the cyclical process of construct development: studying practice, drawing on literature, developing measures of MKT, administering measures to study teachers' knowledge, and validation work on the measures. Although Ball and Bass used a new approach to study the mathematical knowledge teachers need, namely identifying the mathematical demands of the work of teaching, their work builds on the work of other scholars who studied teachers' mathematical knowledge. I now look in more detail at some of the literature and research programs on which Ball and Bass drew. My survey of the literature includes studies of teachers' and prospective teachers' knowledge of mathematics generally, as well as one prospective teacher's knowledge use in practice. Additional studies of practice offer glimpses of the work of experienced teachers, both teachers with "expert" knowledge and teachers whose inadequate mathematical knowledge constrained their work. I begin with an overview of the work of Lee Shulman, whose work inspired much research into teachers' mathematical knowledge in many curriculum subject areas, including mathematics.

Shulman's Professional Knowledge Domains

Shulman and his colleagues studied how knowledge grows in novice secondary teachers during a teacher education program and during student teaching (Shulman, 1986). Using methods such as regular interviews, observations of teaching and intellectual biographies of novice teachers, the researchers found that teachers' subject matter knowledge influenced "what teachers teach and how they teach it" (Grossman, Wilson, & Shulman, 1989, p. 26). The more influential impact of the work by Shulman and his team, however, was to propose a framework representing the "domains and categories of content knowledge in the mind of teachers" and to "think about the knowledge that grows in the minds of teachers" (Shulman, 1986, p. 9). Shulman proposed three categories of content knowledge: subject matter content knowledge, curricular knowledge and pedagogical content knowledge.

Although Shulman's ideas were developed with secondary teachers using various school subjects, the ideas appealed to researchers of all subjects at all grade levels and the interest was such that by the mid 1990s his model merited a category of its own in one author's review of teacher knowledge literature. "What knowledge is essential for teaching?" was the question associated with the category in which the work of Shulman and like-minded scholars was grouped (Fenstermacher, 1994). The component of Shulman's framework that attracted most attention was pedagogical content knowledge, a particular blend of content knowledge and pedagogical knowledge which includes both knowledge of ways to represent a subject so that others can understand it, and knowledge of what makes learning a topic easy or difficult (Shulman, 1986). This expression of the relationship between content knowledge and pedagogical knowledge influenced and continues to influence both methods of studying teachers' mathematical knowledge and conceptualizations of teacher knowledge. Shulman's work is particularly visible in the specification of the domains of MKT where CCK and SCK represent a further elaboration of subject matter knowledge, and pedagogical content knowledge is further refined by the domains of KCS and KCT (Ball et al., in press). I now present a survey of literature on "the interplay of mathematics and pedagogy in teaching and teachers' learning" on which the construct of MKT builds (Ball, 1999, p. 22). The first two categories studied teachers' and prospective teachers' mathematical knowledge outside the classroom context.

Process-Product Studies and Educational Production Function Studies

Throughout much of the last century substantial resources were invested in studying various teacher characteristics that could predict efficiency in teaching (Barr, 1948). Characteristics investigated included teachers' emotional stability and intelligence, skill in instruction, and interest in teaching and school work (pp. 207-210). Subject matter knowledge was considered, including knowledge of mathematics. Many early studies of teaching sought a correlation between teacher characteristics and outcomes such as student achievement. A teacher's mathematical knowledge was typically determined by considering the number of mathematics courses passed, the teacher's grade point average, or scores on a mathematics test. These characteristics were correlated with students' scores on standardized tests (Grossman et al., 1989). The studies, often referred to as process-product studies (Doyle, 1977), had the advantage of being able to analyze data on hundreds of teachers and students . Most studies, however, found little or no correlation between teacher characteristics and student achievement (Grossman et al., 1989).

Other scholars tried to link more explicitly mathematical features of teaching to student achievement. These studies belong to a category of studies known as educational production function studies (Monk, 1989) and they aimed to use "resources possessed by students, teachers, schools and others" to "predict student achievement on standardized tests" (Hill et al., 2005, p. 374). Begle conducted an early educational production function study in 1972 (Begle, 1972; Eisenberg, 1977). He subsequently reviewed other literature on the relative effectiveness of mathematics teachers based on how they influenced student learning. Begle used his own research and studies by others to conclude that beyond a certain level, mathematical knowledge matters little for student achievement (1979, p. 51). In the studies referred to by Begle mathematical knowledge was measured by tests of general mathematical knowledge administered to teachers (e.g. Begle, 1972); sample items are contained in Appendix 2.1. Begle wrapped up his consideration of the effects of teacher variables on mathematics achievement by stating that "attempts to improve mathematics education would not profit from further studies of teachers and their characteristics" (Begle, 1979, p. 55). Begle's studies and the studies he reviewed are of interest in the study of mathematical knowledge, including the MKT research program, for at least two reasons. First, they provided evidence that proxy measures of teacher knowledge (e.g. math courses studied) and performance on generic mathematics test items are not good predictors of student learning, suggesting that more sophisticated means of studying teachers' mathematical knowledge were needed. Second, they attempted to link teacher knowledge with student achievement, and MKT validation studies still study this link (Hill et al., 2005).

Prospective Teachers' Difficulties with Mathematics

Another set of studies has drawn attention to difficulties prospective teachers have with mathematics, especially with regard to conceptual understanding. In her early work Ball developed interviews to study how prospective teachers responded to scenarios that could arise in their teaching (1988). The scenarios were built around tasks that teachers typically have to engage in such as "responding to unanticipated student questions or novel ideas, examining students' written work, evaluating curriculum materials, and planning approaches to teaching" (p. 30). Sample scenarios included responding to a student who asked about dividing a whole number by zero, and generating representations for a division of fractions problem. Ball presented such tasks to 19 prospective elementary and secondary teachers. Almost three-quarters were unable to provide a representation for the calculation $1\frac{3}{4} \div \frac{1}{2}$ and a similar number failed to give a reason why you cannot divide by zero (Ball, 1990). Simon's (1993) findings in a study of prospective elementary teachers' knowledge of division were consistent with Ball's. He found that prospective teachers had difficulties connecting concrete situations, symbolic representations, computational procedures and abstract ideas. The prospective teachers found it difficult to connect partitive (equal sharing) and quotitive (measurement/repeated subtraction) division. These knowledge deficits seem relevant to teaching because the tasks required knowing meanings of various numbers in the long division algorithm, and relating a quotient and remainder to an answer to a division problem on a calculator. Tasks presented to prospective teachers in these studies – mapping calculations to story problems, analyzing procedures in algorithms – informed subsequent item development by the Learning Mathematics for Teaching Project.

In another study Simon and Blume (1994) observed how mathematical ideas, specifically multiplicative relationships, developed over eight classes of a mathematics course for students majoring in elementary education. Class work was videotaped and small group work was either videotaped or audiotaped and the researchers had access to student journals. The prospective teachers had few problems with procedural understanding but making conceptual connections between multiplication and area were more difficult. Simon and Blume suggested that difficulties experienced by the prospective teachers in acquiring conceptual understanding may be similar to difficulties encountered by elementary school students.

In a report of two studies of prospective teachers' knowledge – one of functions and one of undefined operations – Even and Tirosh (1995) looked at prospective teachers' knowledge of mathematics and their knowledge of students. Like the other studies mentioned above, they concluded that teachers need both procedural knowledge, which they refer to as "knowing that," and conceptual knowledge, "knowing why," in relation to curriculum topics. They make a similar distinction between knowing that and knowing why in relation to teachers' knowledge of students. "Knowing that" refers to "research-based and experienced-based knowledge about students' common conceptions and ways of thinking in the subject matter" (p. 17) and "knowing why" refers to general and specific sources of students' conceptions. Such studies provide insights into areas of difficulty for prospective teachers, especially in relation to conceptual knowledge of curriculum topics and knowledge of students' conceptions. Although some of the scenarios used to identify prospective teachers' difficulties with mathematics seem similar to the types of work teachers need to do in classrooms – explaining why something is true, connecting representations of ideas, and knowing meanings of operations, for example – these studies of prospective teachers were a step removed from actual practice. Borko and her colleagues looked inside a classroom at what happened when one prospective teacher began her practice teaching.

One Prospective Teacher's Difficulties in Practice

The prospective teacher, Ms. Daniels, found it difficult to apply her mathematical knowledge in practice (Borko et al., 1992). Not only did the researchers observe Ms. Daniels teaching, but as part of a larger study they used interviews to assess her and other prospective teachers' subject matter knowledge and pedagogical content knowledge; they observed university mathematics courses for prospective teachers and examined how mathematical knowledge was treated in the math methods course; questionnaires were completed by the prospective teachers and the researchers collected lesson plans, worksheets and course assignments. On paper Ms. Daniels was mathematically knowledgeable. She had successfully completed several university mathematics courses, albeit with only a grade C average, and although she had attended just one of three required courses in elementary school mathematics, she had studied the content of the other two and passed them by examination. Ms. Daniels believed in making mathematics meaningful and relevant for students. During a lesson on division of fractions, however, when asked by a sixth grade student why you invert the divisor and multiply when dividing by a fraction, Ms. Daniels's attempts to explain the procedure were unsuccessful despite her substantial mathematics course work and beliefs about meaningful mathematics. Although Ms. Daniels had studied more mathematics than most elementary teachers, her studies did not help in answering a student's question that could have been (and had been) anticipated. The study suggested that mathematical knowledge in itself may not be sufficient for instruction; instead teachers need a specific type of professional mathematical knowledge to help them explain mathematical procedures and respond conceptually to students' questions.

Borko and her colleagues concluded that university mathematics courses do not provide prospective elementary teachers with the kind of mathematical knowledge they need, and Ms. Daniels's misplaced confidence in her own mathematical competence may have lessened the impact of the methods course. The authors recommend that mathematics courses for teachers need to emphasize conceptual development of elementary mathematics topics. In methods courses prospective teachers need to develop pedagogical content knowledge through practice and reflection. Borko and her colleagues recommend challenging prospective teachers' beliefs about learning, teaching, and learning to teach. Borko's study of mathematics use in teaching supports the research findings above that many prospective teachers do not possess conceptual knowledge of mathematics. In addition, the work of Borko and her colleagues offered insights into the difficulties of teaching mathematics and the benefits of studying practice – principles which influenced work on MKT. Borko's work, however, focused on prospective teachers. Perhaps the study of experienced teachers would reveal more about mathematical knowledge that is useful for teaching. Leinhardt and her colleagues studied practice and the knowledge held by expert and novice teachers.

Expert Teachers' Mathematical Knowledge in Practice

In a study by Leinhardt and Smith (1985) an expert teacher was defined as one whose students had shown consistent, high growth scores in mathematics over 5 years and novices were defined as the best students in the final year of a teacher education program. The data collected were new to the study of teachers' mathematical knowledge: observation of teaching, videotapes of teaching, interviews with teachers and card-sort tasks. [16] As expected, they found a difference between the knowledge held by expert and novice teachers, with expert teachers using fewer categories than novices when sorting mathematics topic cards. More surprisingly, teacher interviews and observations of lessons taught by experts showed differences in knowledge held among experts. Moreover, instruction coordinated by expert teachers differed in the use of representations, in the emphases given when presenting information, and in the amount of conceptual (as opposed to procedural) information presented to students. The authors do not comment on the fact that despite finding differences in the knowledge held by experts, all students of expert teachers had performed well. The similar performances may be due to the small sample of expert teachers (4), or to the fact that student growth scores reported may have been based on procedural rather than conceptual knowledge and higher teacher knowledge may result in more conceptual rather than procedural knowledge (Begle, 1972 and Hunkler, 1968, cited in Grossman et al., 1989).

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[16] That qualitative methods were relatively new to education in the mid 1980s is evident from the justification offered by Leinhardt that "although this type of non-statistical but formal analysis of qualitative data for a small number of cases is new to educational research, it has become a confirmable methodology for psychology (Ericsson & Simon, 1980, 1984)" (Leinhardt & Smith, 1985, p. 251)

Leinhardt's data supports other conclusions that teachers need "a specialized kind of mathematical knowledge – knowledge that is specifically tied to teaching (Leinhardt and Greeno 1986; Shulman 1986)" (Leinhardt & Putnam, 1986), an idea that was developed significantly in developing the construct of MKT. Because Leinhardt studied experts she provides images of what expertise affords in the classroom. For example, Leinhardt, Putnam Stein and Baxter (1991) describe agendas and curriculum scripts used by teachers in which knowledge held by experts differs from novices' knowledge. Agendas are mental plans for lessons which enable the teacher to lay out "the logical sequence for an entire lesson and how it [builds] on previous lessons" (p. 93). Actions in lessons were ordered by a "specific overarching goal" (p. 93). A curriculum script provides an underlying structure "of ideas to be presented and actions to be taken to help students construct the desired knowledge structure" (p. 94-95). A teacher can draw on elements from the script as required in the course of a lesson. How expert teachers hold their knowledge is evident in the explanations and representations employed in their classrooms. Explanations given by expert teachers use representations known to students and use the same representations for many explanations. Experts' explanations were complete and they incorporated skills students already possessed. Compared to novices' explanations they contained more critical features and were less likely to contain errors (Leinhardt, 1989). Furthermore, teachers' knowledge is revealed in how they choose and use representations – analogies, pictures or manipulatives – to enrich explanations. An expert teacher knows that some representations "will take an instructor farther" in explaining material but the expert teacher knows when a "particular representation system has outlived its usefulness" (Leinhardt et al., 1991, p. 108). Leinhardt and her colleagues maintain that substantial mathematical knowledge is needed to "'back up' the accurate, fluid and effective use of representations" (p. 106).

Lloyd and Wilson (1998) use a case study of a ninth grade teacher to show how Mr. Allen, a teacher with 14 years experience, responded to a new curriculum. The researchers found that Mr. Allen's deeply held subject matter conceptions (encompassing "knowledge, beliefs, understandings, preferences and views," p. 249) enabled him to engage in dialogues with his students that consistently reflected his strong understanding of what a function is. In addition, he established links among representations and types of functions in his teaching.

In a year-long self-study of teaching mathematics, Lampert (2001) describes many problems of teaching she had to solve using her mathematical knowledge. She decided, for example, that typical lists of topics, concepts and procedures over-simplify both the work of teaching and descriptions of teaching and she sought an alternative way to frame the work. The problem with thinking about teaching in terms of topics is that "it is not in the situations but across them that the big idea of multiplicative structures comes to be understood" (p. 225). When Lampert reviewed "topics" she taught, including "division and remainders, fractions and decimals, and rate and ratio" (p. 220), she noted they were united by the big mathematical idea of multiplicative relationships. Lampert summarized what she achieved by working with the big idea to the fore: "connecting ideas coherently across problem contexts," "elaborating ideas" in new contexts, "teaching conventional topics within frames of conceptual fields," and monitoring "students' understanding and mastery of ideas and topics" (p. 261). Throughout Lampert's book many other instances illustrate how mathematical knowledge can enhance teaching, from preparing lessons to leading whole-class discussions, from teaching while students work independently to teaching the nature of accomplishment, and from establishing a classroom culture to teaching closure. Lampert's explication of the mathematical work of teaching contributed to the development of the construct of MKT. Teacher knowledge alone cannot solve the problems of teaching but it plays an important role in solving some of them. Despite the insights into teaching that can be gained from studying teachers with expert knowledge, I found many more studies of teachers in whose teaching such knowledge was absent.

Studies of Teachers who did not Exhibit Expert Knowledge in Teaching

Deborah Schifter offers a possible reason why studies of teachers who did not exhibit expert knowledge are more common. The reason is that "knowledge and skills become visible by their absence" and when they have become visible [one] can "turn back to illustrations of effective teaching to see them in place" (Schifter, 2001). Schifter gives an example to illustrate this of a teacher, Mary Ryan, who had prepared for her class a word problem interpretation of the problem $\frac{1}{5}+\frac{2}{5}$. The word problem she had prepared was "One fifth of the boys in the class are absent; two fifths of the girls are absent. What fraction of the class is absent? The total number in this problem does not add up to 3/5 of the class. When Schifter pointed out the error Ms. Ryan changed the problem to one that contributed to a successful lesson: "One fifth of the girls in the class are wearing long sleeved sweaters; two fifths are wearing short sleeved sweaters. What fraction of the girls is wearing sweaters?" The problem the teacher had solved was to be consistent in the whole unit to which the fractions referred.

Additional consequences of teachers' limited mathematical knowledge were described in a series of cases studying how teachers were responding to the California Mathematics Curriculum Framework. In one classroom Cohen (1990) describes how the teacher, Mrs. O, wanted to teach mathematics for understanding but "placed nearly the entire weight of this effort on concrete materials and activities" (p. 318). Mrs. O used concrete materials and physical exercises mechanically while explanations and discussion of mathematical ideas were pushed aside. In one lesson Mrs. O set up an activity where students were required to estimate the length of a desk without an adequate view of the desk or the measuring units. Cohen is critical of Mrs. O's acceptance of wild estimates and the separation of estimation from other computational activities. He concludes that the problem with Mrs. O's teaching is linked to her mathematical knowledge: "Her relatively superficial knowledge of this subject insulated her from even a glimpse of many things she might have done to deepen students' understanding" (p. 322). In another case, Peterson(1990) describes how second grade teacher Cathy Swift's lack of knowledge generally about mathematics, and specifically in relation to how to teach problem solving shaped her teaching. Cathy Swift asked lower order questions in class, keeping classroom discourse to a minimum and viewed problem solving as an add-on to her lessons which could be omitted.

Summary of Insights Gained about Teachers' Mathematical Knowledge

I take a moment now to summarize findings of some of the studies available to Ball and Bass in their development of the U.S. construct of MKT. Studies of both mathematical knowledge for teaching, and mathematics used in teaching are included. Most studies agree that the mathematical knowledge needed for teaching is not trivial. Several researchers looked at knowledge held by prospective teachers, and some studied practicing teachers, both expert and less expert cases. Many prospective teachers have weak conceptual knowledge of topics (noticeably division, fractions and functions). Researchers attempted to relate teachers' knowledge to student achievement. In early studies, proxy measures of mathematical knowledge and generic measures were found to poorly predict student learning. In later, mostly small-scale studies, teachers' mathematical knowledge (or lack of it) was found to impact positively (or negatively) on instruction. Studies of practice suggest that a particular type of knowledge rather than general knowledge of mathematics is important for teaching. Teachers use such knowledge to provide clear and complete explanations and to make connections among representations; they teach big ideas and can relate the big ideas to specific curriculum topics. Teachers who do not use such knowledge appear to have difficulties in writing mathematics word problems, in using materials and setting up activities, in teaching problem solving and in posing questions. Finally, although teacher knowledge is important for instruction, other factors are also involved. The influence of much of the research surveyed above can be seen in the current construct of MKT and how it was conceptualized. The influence can be seen in its acknowledgement of the mathematical complexity of teaching, in its description of specialized mathematical knowledge connected to teaching, in its use of classroom-based scenarios to measure teacher knowledge, in its study of practice, and in its investigation of the relationship between teacher knowledge and student achievement. But Ball and Bass brought something new to their development of the construct: they used a disciplinary mathematical perspective to study the mathematical work of teaching.

Almost all of the research summarized above studied U.S. teaching or U.S. teachers or prospective teachers,[17] but other researchers have compared knowledge of teachers across countries. I present two examples of such studies and then look at some studies which studied teachers' knowledge on a large scale.

Studying Mathematical Knowledge for Teaching across Countries

Comparisons of teacher knowledge across countries have typically relied on studying mathematical knowledge held by handfuls of teachers in each country. A well-known example of studying teachers' mathematical knowledge across countries is Ma's work *Knowing and Teaching Elementary Mathematics* (1999). The instruments she used were teacher interviews based around teaching scenarios developed by Ball (1988). Teachers were asked to respond to various mathematics teaching scenarios such as choosing teaching approaches or responding to students' difficulties or responding to students who had used unorthodox approaches to solve problems. Ma's dataset included responses from 72 Chinese teachers and 23 U.S. teachers. Ma's work differed from some of the studies described earlier. She did not study practice by observing classroom teaching in the way Borko and her colleagues and Leinhardt and Smith did. But the measures she used were grounded in the work of teaching. Ma's combination of

[17] An exception is one part of the Even and Tirosh (1995) paper which studied prospective teachers in Israel.

empirical work, conceptual analyses and illustrative metaphors and similes[18] provide vivid portraits of teachers with "profound understanding of fundamental mathematics" and of its potential influence on instruction. A problem is that the U.S. sample of teachers was atypical and all Chinese teachers taught in schools with which Ma was familiar.[19] The low numbers of teachers and how they were selected make it difficult to generalize Ma's findings to all teachers in either country. Even fewer teachers were included in a comparison of Chinese and U.S. teachers conducted by An, Kulm and Wu (2004) using a mathematics teaching questionnaire, classroom observations and interviews. Although like Ma they found Chinese teachers generally had greater conceptual understanding of mathematics than U.S. teachers, they acknowledge the difficulties in generalizing their findings beyond the samples, especially in a country like the United States where schools are controlled locally. The interest in these studies and the limitations to generalizing their results made the need for large scale measures that could be related to student achievement all the more urgent. A German research group conducted such a study at high school level.

Measuring the Mathematical Knowledge for Teaching of Large Numbers of Teachers

The process-product studies and early educational production function studies summarized above investigated characteristics of large groups of teachers and attempted to relate the characteristics, including teacher knowledge, to student learning. Student learning was not found to be related to teacher knowledge mainly because researchers used either proxy variables for teacher knowledge such as mathematics courses taken or they administered tests of general mathematical knowledge. In contrast, some of the more descriptive studies related what teachers did to student achievement (e.g. Leinhardt & Smith, 1985) or to instruction (e.g. Borko et al., 1992) but they studied only a handful of teachers at a time. Missing were large scale studies of teachers that could relate teacher knowledge to student achievement and instruction. Although vivid portraits of individual teachers can be compelling and insightful, they may be dismissed as atypical. Many reasons make the study of mathematical knowledge held by large groups of teachers desirable: to evaluate growth in teachers' knowledge through professional development, to evaluate teacher preparation programs, to better

[18] One example is a comparing a teacher's profound understanding of fundamental mathematics with a taxi driver's mental map of the city in which the driver works (p. 123).
[19] Although approximately 80% of China's population reside in rural areas (Ho et al., 2000), 60% of Ma's schools were selected from Shanghai, a large city.

understand the relationship between teacher knowledge and other variables such as student achievement, and to compare teacher knowledge across countries.

Measuring the mathematical knowledge of German secondary school teachers.

In Germany secondary teachers' knowledge has been measured on a large scale and linked to both instruction and to student achievement as part of the Cognitive Activation in the Classroom (COACTIV) Project.[20] Baumert and his colleagues used 35 open-ended items to measure the content knowledge and the pedagogical content knowledge of just under 200 secondary school teachers. Items were embedded in contexts of teaching (e.g. students not understanding a concept) and related to knowledge teachers use (e.g. list as many different ways as possible of solving a problem) (Blum & Krauss, 2008). The research project conceived instruction as comprising competence in classroom management, personal learning support for students, and cognitive challenge ("cognitive activating elements") (Kunter et al., 2007). Instruction was evaluated using ratings by students, ratings by teachers and analyses of student written work. Using student ratings alone the researchers found a relationship between teachers' pedagogical knowledge and the level of cognitive challenge provided to students. Although some might be dubious of evaluating the effectiveness of instruction solely on the basis of student ratings, the authors found high levels of reliability in these measures in which individual student ratings were aggregated to produce a class mean. To assess teacher knowledge, items were administered to teachers whose 10th grade students participated in the 2004 German PISA study. Because of the link with PISA, researchers were able to study teacher performance on the items relative to student performance on the PISA measures. They found that "when mathematics achievement in grade 9 was kept constant, students taught by teachers with higher pedagogical content knowledge scores performed significantly better in mathematics in grade 10" (Blum & Krauss, 2008, p. 3). The findings of this study show that it is possible to establish a relationship between student achievement and teachers' mathematical knowledge, measured on a "fairly representative" (p. 2) sample of teachers of a particular grade in a country. This was done by using measures of mathematical knowledge situated in tasks teachers do. Although this method could be feasible for use in Ireland, the items were designed for use with secondary school teachers. Researchers

[20] The directors of the project are Jürgen Baumert (Berlin), Werner Blum (Kassel), and Michael Neubrand, (Oldenburg). For more information see http://www.mpib-berlin.mpg.de/coactiv/index.htm?/coactiv/publikationen/Publikationen.htm (accessed on March 19, 2008)

at the University of Michigan, however, have been working since 2001 on developing MKT measures that could be used with large numbers of elementary school teachers.

Measuring elementary school teachers' mathematical knowledge for teaching on a large scale.

Based on the hypothesized domains of MKT – CCK, SCK, KCS and KCT[21] – Ball, Hill and colleagues developed multiple-choice items to measure teachers' MKT. The items are based on topics in number, algebra and geometry. Like the scenarios from Ball's early work the items were set in teaching contexts and reflected the work teachers do. Items were written by multi-disciplinary teams comprised of mathematicians, teacher educators and teachers, and other researchers (Bass & Lewis, 2005). Items were piloted at Californian professional development institutes in 2001. Pretests and posttests using the measures suggested that teachers' MKT had grown in the course of participating in summer institutes (Hill & Ball, 2004). Psychometric analyses were conducted to establish the consistency of scores over multiple items and they were deemed to be "good to excellent" (Hill et al., 2004, p. 25). Factor analyses were conducted on the items to assess the extent to which the items reflected the hypothesized domains (Hill et al., 2004). I will return to the factor analysis results in Chapter 4. First, I look at a one study which used MKT items to investigate teachers' mathematical knowledge on a national level.

Hill used multiple-choice items to conduct the first ever nationally representative study of mathematical knowledge held by U.S. middle school teachers (2007, p. 96). She used measures based on MKT to study knowledge held by middle school teachers. Overall, teachers found the measures easier than had been anticipated by the authors of the measures. Nevertheless, Hill used the measures to discover that U.S. middle school teachers had stronger knowledge of number than of algebra. Furthermore, their CCK – knowledge likely to be held by adults generally, was stronger than their SCK – knowledge specific to the work of teaching. Hill studied the relationship between teacher performance on the items and teacher characteristics such as credential level, classes taught and teaching experience. In general, higher MKT scores were associated with teachers having taken more mathematics courses, holding mathematics-specific and high school credentials and having high school teaching experience. Although my interest is in using the elementary rather than the middle school items, Hill's study

[21] At the time the study of Irish teachers took place no specific items to measure KCT had been developed.

showed that measures of MKT could be used successfully at a national level, but an important question was whether the items were related to classroom instruction or to student achievement.

In another study Hill, Rowan and Ball (2005) addressed this question by studying teachers' mathematical knowledge and growth in elementary school students' mathematics achievement scores as measured by the CTB/McGraw-Hill's Terra Nova Complete Battery standardized test. They found that teachers' MKT was a significant predictor of student gains in both grades studied (first and third). Being taught by a teacher having an additional standard deviation of MKT increased students' mathematics scores by as much as if the students spent an extra two to three weeks in school in that school year. A test of knowledge of teaching reading administered to the same teachers did not significantly predict growth in student mathematics scores, suggesting to Hill, Rowan and Ball that the effect of teachers' knowledge on student achievement is content-specific and not related to "general knowledge of teaching" (p. 398). The finding of an effect on student achievement, however, is dependent on the knowledge, practices and attitudes tested by the Terra Nova test. In addition, the relationship between MKT and the mathematical quality of instruction has been studied and reported by the Learning Mathematics for Teaching Project as part of its validation of the multiple-choice measures of MKT. A summary of this investigation follows.

Multiple-choice measures are difficult to design but are relatively easy to administer and score. Their use in measuring MKT can only be justified if teacher scores on the measures are related to the quality of instruction and to student achievement. In order to investigate the relationship between instructional quality and teacher scores on the measures, a study of ten U.S. teachers, each videotaped teaching nine lessons, was implemented. A coding rubric was developed to study the mathematical quality of instruction. The rubric was used to code each lesson for features of mathematics instruction such as accuracy of language use, connections made among representations, the quality of explanations and the explicitness of talk about mathematical practices.[22] Teachers' performance on the multiple-choice items was correlated with the mathematical quality of instruction in their lessons and found to be a good predictor of both mathematical quality of instruction and student achievement (Blunk & Hill, 2007; Hill et al., in press).

[22] More details about the video-coding process are given in Chapter 5.

Some concerns and criticisms have been noted in relation to the development of the MKT items. Schoenfeld (2007) expressed concern about the lack of clarity about what is and is not included in the construct of MKT. In addition, he questioned the merits of the multiple-choice format compared to using open-ended items. Garner (2007) advised reconsidering whether the construct of MKT is multidimensional or unidimensional and she proposed that a single construct may account for all the measures. Alonzo (2007) called for the construct and the measures to be open to ongoing revision. Finally, it was noted that the substantial resources – expertise and financial – available for developing and validating such measures are rare, and few other research teams could take on a project of this scope (e.g. Lawrenz & Toal, 2007).

Despite these concerns and points of caution, the construct of MKT offers a useful way of thinking about the mathematical knowledge teachers need to do the work of teaching. The construct emerged from the unique twinning of a mathematical perspective and a practice of teaching perspective. The research instruments that emerged from, and in turn inform, the construct have been used to measure MKT held by large numbers of teachers. The research has yielded both practical and research benefits. Practical benefits include the development of an instrument to evaluate growth in teacher knowledge arising from professional development initiatives, and identification of specific examples of knowledge that might be included in a mathematics course for prospective teachers or in professional development for practicing teachers. Benefits to researchers include provoking discussions about what is included in the construct of MKT (Schoenfeld, 2007) and how instruments to measure MKT are validated (Engelhard & Sullivan, 2007). The instruments have furthermore been positively associated with the mathematical quality of instruction and with growth in student achievement. To date all this work has been conducted in the United States, in the context of U.S. teaching. I now turn my attention to work that has been done to study Irish teachers' mathematical knowledge.

Measuring Teachers' Mathematical Knowledge in Ireland

Concerns are frequently expressed in Irish media about student achievement in mathematics. These concerns generally reach a crescendo when the annual Leaving Certificate[23] results are issued. In 2007, 10% of students failed mathematics.[24] Behind

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[23] A post secondary school national examination

[24] E.g. Irish Times article from August 15, 2007. Accessed at http://www.ireland.com/newspaper/frontpage/2007/0815/1187036480218.html on March 19, 2008

the media headlines, several national and international reports have studied the mathematics achievement of Irish students. Most studies considered teacher variables that might explain student scores. For example in the 1995 TIMSS study, Mullis et al (1997) looked at teacher characteristics such as certification, degrees held, age, gender and teaching experience. Cosgrove, Shiel, Oldham and Sofroniou (2004) reported specifically on teacher characteristics collected as part of the 2003 PISA study. They had data on teacher gender, country of birth, years teaching mathematics, qualifications held, participation in in-career development, class time spent on various activities, teachers' views on mathematics, assignment of homework, assessment and emphasis placed on aspects of the curriculum. *The 2004 National Assessment of Mathematics Achievement* (Shiel et al., 2006) reported on classroom environment factors and mathematics achievement. The classroom variables that were documented included gender, teaching experience, qualifications, teachers' attendance at and satisfaction with mathematics professional development, use of resources and time allocated to teaching mathematics. Despite the range of studies of mathematics achievement in Ireland in the last 10 years, like the missing paradigm to which Shulman (1986) referred, few large scale studies, if any, have referred to teachers' mathematical knowledge as a possible variable associated with classroom instruction or student attainment.

Several factors may account for the absence of teacher knowledge as a variable in the studies mentioned. First, international studies of student achievement in mathematics do not include such a component and Ireland has participated in TIMSS in the past and it still participates in PISA. Second, no consensus exists as to the knowledge teachers need to possess and therefore, it would be difficult to choose an instrument to measure teachers' knowledge. Third, on practical grounds, administering a test to teachers when a test is being administered to students would be a difficult organizational feat, although this has been done in Germany. But despite no references to teachers' mathematical knowledge in major studies of mathematics, smaller scale studies have expressed concerns about teachers' mathematical knowledge.

An early concern about Irish teachers' mathematical knowledge dates to the 1920s. A conference was summoned to report to the Minister for Education about the suitability of the National Programme of Primary Instruction. Among the group's recommendations was one that "the present state of mathematical knowledge among women teachers left us no alternative but to suggest that both algebra and geometry be optional for all women teachers" (National Programme Conference, 1926, p. 12). There was an additional recommendation that teachers' notes for mathematics would be "worded in language as un-technical as possible so that teachers, especially the older ones, may be helped and not puzzled and frightened, as many of them appear to be" by the current notes (National Programme Conference, 1926, pp. 16-17). No specific evidence was provided to justify the recommendations made but the entire report was written based on oral evidence from witnesses, written evidence in response to a press advertisement and reports by Department of Education inspectors. The teachers' mathematical difficulties appear to have been basic and related to not knowing the content of the syllabus because the proposed solution was that "the Department issue detailed specimen syllabuses" to ensure that "all teachers may understand easily and exactly the meaning of the programme" (National Programme Conference, 1926, p. 12).

More recently, one study (Greaney, Burke, & McCann, 1999) looked at whether pre-service teachers who studied university level mathematics were perceived by Department of Education inspectors to be better at teaching the subject. Like Begle (1979) had found in the United States, teachers who had studied mathematics to degree level were not perceived to be better at teaching mathematics than those who had studied other subjects to degree level. However, the numbers who had studied mathematics were small (17 in one dataset and 11 in the other). Furthermore, teachers were rated on their "teaching performance relative to other teachers" (p. 27) and it is possible that criteria for rating teaching performance may have differed among inspectors. Although this study was ambitious in its intent, it was limited by the instruments available to the authors to rate teaching. More recently, research on mathematical knowledge has focused on evaluating what is known by prospective teachers, without relating this knowledge to instruction.

Four recent studies (Corcoran, 2005; Hourigan & O'Donoghue, 2007a; Leavy & O'Loughlin, 2006; Wall, 2001) have investigated aspects of mathematical knowledge held by Irish pre-service elementary school teachers. In all four cases several respondents exhibited shortcomings in their mathematical knowledge. One study found that most prospective teachers had superficial understanding of the mean (Leavy & O'Loughlin, 2006) and another found that prospective teachers' knowledge of some key mathematical concepts was mostly procedural (Corcoran, 2005). Hourigan and O'Donoghue (2007a) identified weaknesses in several areas of common subject matter knowledge, including operations with decimals and finding the area of an irregular shape. It is notable that the researchers are pre-service teacher educators and all were sufficiently concerned about the mathematical knowledge held by some prospective teachers to have engaged in formal study of the prospective teachers' mathematical knowledge.

Previous Irish studies have been limited in various ways. First, non-representative samples were used. Second, in several cases knowledge held by prospective teachers and not practicing teachers was studied. A third problem with the Irish studies described here is that they used as instruments items not explicitly related to the practice of teaching. Some items used were taken from national or international tests of elementary or middle school students; others were previously used in studies of elementary or middle school or undergraduate college students; and some items were designed to reflect mathematics curriculum content. In only a few exceptional cases were items specifically designed to measure knowledge specialized to the work of teaching. One attempt to link knowledge and classroom teaching used a potentially subjective method to rate teaching. A study of practicing teachers from a national, random representative sample of schools using MKT measures could address some of the limitations found previously. But such a study raises some practical research problems, including evaluating the suitability of the MKT measures for use in Ireland.

Measures developed in Germany were used to study the knowledge held by German teachers. Measures developed in the United States were used to study U.S. teachers. In Ireland it would be difficult to secure the resources needed to develop measures independently and to validate them for use in Ireland. Instead of starting with a blank page and developing new measures of teachers' mathematical knowledge in Ireland, it made sense to adapt and validate existing measures. The German and U.S. measures were grounded in the practice of teaching, in each country's specific practice of teaching. Based on their study of instructional methods across three countries Stigler and Hiebert (1999) argue that teaching is a cultural activity and that instruction differs from country to country. A study of teaching in six countries prompted Cogan and Schmidt to describe a characteristic pedagogical flow recognizable in instruction in each country studied. They found that differences in instruction among countries were greater than any differences within countries (Cogan & Schmidt, 1999). If a construct and measures are explicitly grounded in instruction in one country and if instruction varies across countries the construct and measures may not travel well. The construct of MKT emerged from studying U.S. practice and it is possible that the items based on the construct are not suitable for use in Ireland. Although the items are related to U.S.

instruction they may not show any relation to instruction in Ireland. Any researcher who wishes to use the U.S. measures in a country outside the United States faces this problem and no previous research exists on how the construct of MKT in the United States might be similar to or different from constructs of MKT in other countries. This is a problem that includes, but goes beyond the relatively straightforward work of translating items (see Delaney et al., in press). It extends to investigating the meaning of the construct and evaluating the validity of the measures for use in a setting outside the United States.

At the outset of this chapter I compared understanding teachers' knowledge to sorting strands of yarn. By taking some strands and studying them in a new location new dimensions may in time become visible. Much remains to be learned about the construct of MKT generally, such as the relationship of beliefs to knowledge and its impact on student learning in ways other than those measured by gain scores on standardized tests. This study addresses one substantial gap in the literature: the application of MKT to investigate the content of teachers' knowledge of mathematics in a new country. By responding to the challenges of adapting measures for another English-speaking country and by taking seriously the challenges of establishing construct equivalence and validity in a new setting, this study contributes to the literature of studying teachers' mathematical knowledge in and beyond the United States. New insights can be gained about the practice of teaching when it is seen in light of practice in another country (Hiebert, Gallimore et al., 2003). If practices and constructs can be communicated clearly, similarities and differences in context and application can be considered. This study endeavors to be careful about applying the U.S. construct of MKT in Ireland. A study like this one, although not a cross-national study of teachers' knowledge, is a pre-requisite to responding to calls for cross-national studies (Alonzo, 2007). It is the first national study of primary teachers' MKT outside the United States.

Chapter 3

Using Multiple Techniques to Study Practice from a Mathematical Perspective

This study is guided by the question: To what extent, and how, can measures of MKT developed in the United States, be used to study the MKT held by Irish teachers? I addressed the question by surveying a national sample of several hundred primary teachers and therefore, the findings can be generalized to all Irish primary teachers. I studied Irish teachers' knowledge of mathematics, not as an end in itself but to better understand it as a resource that teachers can use to enhance instruction and thereby raise student achievement in mathematics. Although much can be learned from in-depth, close-up case studies of individuals or small groups of teachers, this study investigates MKT at a national level. The study can be classified as descriptive (National Research Council, 2002), with a focus on describing the MKT held by Irish teachers. Before describing teachers' MKT it describes the work of primary mathematics teaching observed in a sample of Irish lessons and evaluates how the construct of MKT in the United States is similar to or different from the construct of MKT in Ireland.

The dissertation is grounded in a body of research that studies the practice of teaching from a disciplinary mathematics perspective (Ball, 1999). Two major sets of data were collected and in this chapter I describe how these data were used to respond to the three different aspects of the research question: assessing the construct equivalence of MKT across countries, evaluating the validity of using the U.S. measures in Ireland and estimating the MKT held by Irish teachers. The processes of collecting the lesson video data and the survey data will be described. To conclude the chapter, I summarize the specific techniques used for data analysis.

Design of Study

In order to determine how U.S. measures can be used to study Irish teachers' MKT I have subdivided the question into three parts and I respond to the parts in multiple ways. The different techniques used to address the parts of the question are united by studying the practice of teaching from a disciplinary mathematics perspective. The U.S. construct of MKT has been developed to date by studying conceptions of the work of teaching from a mathematical perspective, and the research instruments used in this study were originally developed by or are informed by the MKT research program (Ball, 1999; Ball & Bass, 2003b). Ball described how she and her research group studied teaching from a mathematical perspective in order to gain insights into the mathematical knowledge needed for teaching:

> One central analytic task is to probe the particulars of the cases we are examining: to uncover mathematical issues that can be seen to figure in particular moments of teaching practice, to seek connections with other moments, and to consider the role such elements of mathematics play in teaching. The other is to identify what is generalizable and what is specific to particular approaches to teaching, or to the specific cases which we are studying (Ball, 1999, p. 33).

In this approach to studying teacher knowledge, analysis begins with a specific case of the work of mathematics teaching which is examined closely and in context, using a mathematical focus to understand the case. Particular instances of practice are connected with other instances and with relevant literature to identify common mathematical issues that arise and to better understand the mathematical work of teaching. The analysis is not, however, confined to studying specific instances of practice but goes beyond them to look for elements that may be generalizable as well as recognizing that some aspects may remain specific to one classroom or teaching situation.

The discipline of mathematics [25] informs the analysis of teaching in Ball's work to the extent that the tools and building blocks of mathematics – mathematical tasks, answers, commencement and conclusion of a mathematical line of work, promising ideas and approaches, explanations, expression of ideas, logical consistency and disciplinary convention, who is involved in contributing to, and shaping the outcome of mathematical activity[26] – are used not to do mathematics, but to recognize and understand the actions of teachers doing the mathematical work of teaching. Using a mathematical perspective enables the researcher to take seriously the mathematical work that transpires in the instructional interactions among teacher, students and mathematics (Bruner, 1960). By taking seriously the mathematical aspects of teaching it is possible to identify the mathematical nature of the work and its knowledge demands.

[25] Although there are many different foundations on which mathematical thoughts rest (P. J. Davis & Hersh, 1981), there are also likely to be "general considerations which are honored by all mathematicians at all times," a kind of "metamathematics" (Kitcher, 1983, pp. 188-189)
[26] This list was given to me by Mark Hoover Thames (personal communication, March 15, 2008)

A researcher studying teaching from a disciplinary perspective is likely to notice features of teaching different from those that might be seen by a researcher using say, what Sherin and her colleagues describe as a cognitive modeling perspective or a knowledge system analysis perspective (2000). A researcher who observes teaching from a mathematical perspective will attend to mathematical features of what is happening in the classroom and in the work that teachers do. Aspects of teaching such as proving, explaining, justifying, and generalizing, emerge as important in a teacher's work, either through their presence or their absence. Preciseness of language use and concept definition becomes salient. Such features have been noticed when studying teachers' knowledge from other perspectives (Leinhardt & Smith, 1985) but identifying the presence and absence of mathematical features becomes more prominent and aspires to be more comprehensive when applying a mathematical perspective.[27]

In analyzing data for this study I drew on literature from cross-national studies and I used a mathematical perspective to study teaching. Literature from cross-national studies was useful for addressing the construct equivalence of MKT in Ireland and the United States, even though the study is not a cross-national one in the sense of comparing knowledge held by teachers in two countries. Studying the practice of teaching using a mathematical perspective is useful because it keeps the focus on how mathematical knowledge is deployed to do the work of teaching. I used techniques from grounded theory, video lesson analysis, classical test theory, and IRT to analyze the data.

Data Sources

The study relies substantially on two major sources of data (see Figure 3.1). One is the responses of over 500 Irish teachers to a survey of multiple-choice measures of MKT, developed from the practice of teaching in the United States. Survey items were adapted[28] for use in Ireland (see Delaney et al., in press) and administered to teachers in a national representative sample of Irish primary schools. The second major source of data is a set of 40 video-taped and transcribed mathematics lessons, taught by ten Irish teachers who completed the same MKT survey as the national sample of teachers. Teachers for the video study were selected by attempting to recruit what Patton (2002) called a "typical case" sample. The sample is, therefore, "illustrative not definitive" (p.

[27] Ideas in this and the previous paragraph have benefited from discussions with Mark Hoover Thames.
[28] I use the terms adapt and translate interchangeably to describe the process of making the items sound familiar to Irish teachers.

236). These rich data sources of Irish MKT will be supplemented by other sources, including responses to survey items by U.S. teachers and by U.S. literature about MKT.[29] I now describe how the data were collected and the techniques used to analyze them.

[29] Other data were collected in Ireland, including a pilot sample with 100 teachers (see Delaney et al., in press); a focus group with 4 Irish teachers about the item translations; follow-up interviews with teachers who completed the pilot study; and interviews with the teachers who were video-recorded teaching the lessons. Although these data sources may be occasionally used, they are not central to the dissertation and, therefore, will not be discussed here.

Sub-Question	Data	Analysis Technique	Mathematical Perspective
Is the construct of MKT similar in both settings?	1. Video lessons taught by 10 Irish teachers and	(i) Grounded theory (open coding) description	Identifying the mathematical tasks of teaching in each lesson
	MKT literature from the United States and	(ii) Mini-literature review	Collating mathematical tasks of teaching identified in U.S. literature
	Items developed in the United States	(iii) Study of text	Collating mathematical tasks of teaching identified in U.S. MKT items
	2. Responses to items from 501 Irish teachers and	(i) Exploratory factor analysis	What underlying structure can be identified in Irish teachers' responses?
	Responses to forms B_01 from U.S. teachers	(ii) Confirmatory factor analysis	Is the factor structure identified in the responses from U.S. teachers similar to the factor structure found in responses from Irish teachers?
Do the instruments developed to measure MKT in the United States measure MKT in Ireland?	1. 40 Video lessons taught by 10 Irish teachers	Video coding of lessons for the mathematical quality of instruction	Using codes developed in the United States to study the mathematical quality of instruction
	2. Survey forms completed by the same 10 Irish teachers	Study of covariance between MKT and mathematical quality of instruction	Investigating how well the mathematical instruction observed in individual lessons in Ireland is related to the items developed from U.S. practice
What MKT do Irish teachers Possess?	1. Responses to items from 501 Irish teachers	(i) Variations in IRT scores on a 2-parameter model (ii) Difficulties of items on a 2 parameter model	Items used are based on the construct of MKT, grounded in a small number of classrooms

Figure 3.1

An overview of the data collected for the study and the techniques used to analyze the data

Data Collection: Video study[30]

Sample.

In order to recruit ten teachers for the video part of the study, I asked teacher educator and school principal acquaintances to recommend teachers who might be willing to be videotaped teaching a series of mathematics lessons. In order to produce "typical case" samples of mathematics teaching in Ireland (Patton, 2002) I explained that I sought typical teachers teaching typical lessons. Although typical cases were sought, their typicality cannot be confirmed. Participating teachers were required to complete a survey of MKT items, to be interviewed and to be videotaped teaching four lessons. Although many teachers might be daunted at such a prospect, no teacher I spoke to about taking part in the study declined to participate. The sample of teachers varied in their interest in mathematics. One teacher had a postgraduate qualification in mathematics education; some other teachers claimed that mathematics was their favorite subject as students and as teachers; some teachers expressed no opinion about mathematics; and one teacher recalled negative experiences of learning mathematics in school.

The lessons were taught between May and December 2007. The eight female and two male teachers had from 3 to 30 years teaching experience, with an average of 14 years. The teachers graduated from five different teacher education programs. Class levels taught varied, and all classes but one were single-grade. One teacher taught senior infants (5 – 6-year-olds), three teachers taught second class (7 – 8-year-olds), two taught third class (8 – 9-year-olds), three taught sixth class (11 – 12-year-olds), and one taught a multi-grade combination of fourth, fifth and sixth class students. Geographically most schools were located in Dublin suburbs with one inner city school, one multi-grade rural school and one single-stream rural school. The schools served students from a wide range of socio-economic backgrounds. A variety of school types was represented: co-educational, all-boys and all-girls; schools designated as disadvantaged and schools not so designated. One teacher taught in a private school that followed the national curriculum; the teacher was a fully recognized and probated primary school teacher and had attended professional development for the revised 1999 curriculum. No all-Irish speaking school was included in this part of the study because I planned to receive

[30] This part of the project was funded in part by the Learning Mathematics for Teaching Project at the University of Michigan.

assistance in video-coding from non-Irish speaking U.S. researchers trained in using the video codes.

Procedure.

Each teacher who participated in the video study was asked to read and sign a consent form (see Appendix 3.1), and permission was sought from the school principal to conduct research in the school (see Appendix 3.2). On my behalf, teachers asked parents to sign consent forms permitting their son or daughter to participate in the lesson (see Appendix 3.3). Students whose parents did not give such consent were asked to sit outside the range of the camera during filming. All teachers who participated in the video study completed the MKT survey (see below) and were interviewed about their mathematics teaching and about their responses to items on the survey form.

Each teacher taught four lessons.[31] Lessons were taught close together in time (generally over a two to three week period for each teacher), with times agreed to suit both the teacher's and my availability. Teachers chose the topics they wanted to teach, although they were asked to include, if possible, two different topics over the four lessons. All but one teacher did this. The guideline given for lesson length was the length of the teacher's regular mathematics lessons.

I videotaped all lessons using a single camera,[32] positioned at the back of the classroom because my interest was in studying the teacher and the quality of mathematics instruction coordinated by the teacher. The camera generally remained focused on the teacher unless a student asked a question. When the teacher was monitoring student work distant from me in the classroom I sometimes focused on the work of students sitting closer to the camera. When the teacher or a student wrote on the board I focused on that material. Each teacher was asked to wear a radio microphone connected to a digital voice recorder. Therefore, two audio records exist for most lessons and one video record. Class materials used in the recorded lessons were requested and scanned. After each lesson the teacher was asked to state the primary focus of the lesson. Following the lessons, the audio file of the lesson was used to transcribe the lesson. Transcribers were hired to produce first drafts of lesson transcripts. Cleaning of transcripts is ongoing.

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[31] Four lessons were selected because in the U.S. study four lessons per teacher was deemed to be the number of lessons needed per teacher to be safe in making inferences about the mathematical quality of teaching (Blunk & Hill, 2007).

[32] The camera used was a Canon XL1s. Details about it are available from http://www.canon.co.uk/For_Home/Product_Finder/Camcorders/Digital/XL1s/ and http://www.calstatela.edu/tvf/equip/equipmentpdfs/xl1s.pdf.

Data Collection: Survey

Instrument.

The instrument used to measure the MKT held by Irish teachers was Form B_01, a set of items which had previously been administered to U.S. teachers (Hill & Ball, 2004; Hill et al., 2004).[33] I decided to use one complete form used in the United States rather than select items from multiple forms because this would indicate how useful the measures in general would be in Ireland. If items were picked from multiple forms it was possible that items selected might be biased towards Irish teaching and that only particular MKT items would work in Ireland. By taking one entire form it is likely that if it worked well in Ireland, other forms could be used in future. Using one form would make it possible to compare factor analysis results on the Irish and U.S. form. Finally, the specific form, B_01, was administered under exam-like conditions in both the United States and Ireland.

The multiple-choice items[34] on form B_01 related to the MKT sub-domains CCK, SCK, and KCS. The strands covered were number and operations, and patterns, functions and algebra. I included additional geometry items which were not part of the B_01 form in the United States. Although these items added to the length of a form that was already relatively long, it was important to learn about Irish teachers' knowledge of the "shape and space" curriculum strand. The geometry items chosen were those used in the pilot study (see Delaney et al., in press). No items related to the measures and data strands of the Irish curriculum had been developed at the time the survey was administered. Table 3.1 gives an overview of items on the form by sub-domain and by curriculum strand.

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[33] The reliability of this form when used in the United States was estimated at 0.83 using a two parameter model (Hill, 2004b). In terms of the content topic of the items Hill noted that "content was represented" more broadly on the 2001 forms than on forms piloted in 2002-2003 (p. 7). No items had negative point biserial correlation estimates and 10 of 56 items had slopes lower than 0.5 (Hill, 2004a, 2004c, 2004d). These data exclude geometry items because they were not included on the original B_01 form.

[34] Sample items are included throughout the dissertation e.g. Figure 4.2, Figure 6.1 and Figure 6.7. Other released items can be seen at http://sitemaker.umich.edu/lmt/home (accessed on March 4, 2008).

Table 3.1

Breakdown of survey items, by curriculum strand and by sub-domain

	Number & operations	Patterns, functions & algebra	Geometry*	Total
SCK	10	5	-	15
CCK	15	8	-	23
KCS	18	-	-	18
Geometry*	-	-	28	28
	43	13	28	84

*Note: Geometry items have not been classified into SCK, CCK and KCS

Although English is the primary language of school instruction in Ireland, items on the U.S. form needed to be adapted for use in Ireland so that teachers would not be distracted by terms or names not familiar to them (Hambleton, 1994). Such distractions could adversely affect how teachers performed on some test items (Yen, 1993). In the pilot study the process of adapting MKT items for use in Ireland was studied carefully and documented. That process, which included a focus group discussion to check the suitability of the items, was documented in Delaney et al. (in press). Guidelines produced for translating the pilot study items were followed in the current study[35] and the guidelines were consistent with those recommended by the International Test Commission (Hambleton & de Jong, 2003). The survey was then ready for administering to the sample of teachers.

Sample design.

When choosing participants I first selected a random representative sample of schools from Ireland's total number of 3293 primary schools. The list of schools published by the Department of Education and Science included the number of students in each school but not how many teachers. I estimated teacher numbers using student enrolment and additional teacher allocation data published by the Department of Education and Science. For the purpose of drawing a sample, each school was treated as a cluster and all teachers in the chosen schools made up the sample of teachers eligible to participate. This is sometimes called a "take all" approach. My goal was to select enough schools to ensure that at least 500 teachers would complete the survey because that number of responses (or more) is desirable for applying a 2-parameter IRT

[35] One item that had been translated for the pilot study was included in its original form for this study and one pilot study item that had not been adapted was adapted for this study. This enabled comparison of how the items performed differently when adapted. See Delaney et al (in press).

model (Hulin, Lissak, & Drasgow, 1982)[36] and for conducting factor analysis (Gorsuch, 1983).[37]

When selecting schools all primary schools listed on the website of the Department of Education and Science[38] were stratified by type and by region. The school type strata included disadvantaged, Gaeltacht, Gaelscoil, ordinary[39] schools, and special schools.[40] Schools designated as disadvantaged are selected for targeted support by the Department of Education and Science to support the education of students affected by social or economic impediments. At present, provision for supporting disadvantaged students in Ireland is in transition because of a 5-year plan, *Delivering Equality of Opportunity in Schools,* launched in 2005 by the Department of Education and Science. I initially used three separate strata to classify schools as disadvantaged when identifying the population to be sampled because the website records of designated schools did not reflect the new arrangements. When drawing the final sample, however, these categories were merged because of the small numbers of schools in each one. The categories of Gaelscoil and Gaeltacht school are of interest because these teachers teach mathematics through the medium of the Irish language and may have a particular perspective on teaching the subject. Furthermore, many (but not all) of the teachers in these schools speak Irish as a first language or they may have learned some or all of their mathematics through the medium of Irish. Teachers who learn mathematics through Irish may find it more difficult to develop their competence in mathematics because fewer Irish medium textbooks and ancillary resources are available and mathematics and mathematics methods courses in some colleges are offered only through the medium of English. A separate list of special schools and special classes exists. Special schools, however, were excluded as clusters from the study because the schools enroll students of both primary and post-primary school age. Teachers of special classes in mainstream primary schools were included in the study.

[36] This is likely to be a conservative estimate because Hulin et al's experiment was based on having 30 items and 500 respondents. I had 84 responses (56 excluding geometry items).
[37] Gorsuch writes that an "absolute minimum ratio is five individuals to every variable, but not less than 100 for any analysis" (p. 332). With 84 items, this would suggest the need for having at least 420 respondents in this study.
[38] List downloaded from www.education.ie on May 12, 2006.
[39] This term is used by the Department of Education and Science to describe schools that do not fit into specific categories such as All-Irish school or Gaeltacht school.
[40] Dedicated schools for students with special educational needs (e.g. Down syndrome, autistic spectrum disorder, attention deficit disorder).

The second set of strata used when designing the sample was school region. It is possible that teachers and schools vary by region. For example, teachers in schools near colleges of education or near universities may be more likely to participate in research projects or in postgraduate studies, or teachers in western counties may be more likely to be taught by teachers from the Gaeltacht.[41] There may be greater concentrations of multi-grade schools in some counties. For these reasons, schools were stratified according to their region in the country: Dublin, Leinster (excluding Dublin), Munster, and Connacht/Ulster.

Based on the list of stratified schools and estimated teacher numbers a sample of schools was selected by a staff member at the Center for Statistical Consultation and Research (CSCAR) at the University of Michigan, using PROC SURVEYSELECT in SAS software.[42] A random sample of schools was drawn from each stratum (having first merged the three categories describing disadvantaged schools). Appendix 3.6 shows the number of schools selected in each category, 87 in total. All teachers in each school were invited to participate. It was estimated that this number of schools would result in a sample of 606 teachers, but this was likely to be an underestimation because some schools (e.g. all-Irish speaking schools) have lower student-teacher ratios and in some schools enrolments may have increased over the previous year. In fact, the sample produced a total of 670 teachers from whom data could be collected.

Survey administration design.

I decided that the survey would be administered in my presence or in the presence of a research assistant representing me. This was done for several reasons. I believed that by setting a specific day and time to complete the questionnaire participants would be less likely to postpone doing it and this would increase the response rate. This administration process helped ensure that respondents did not confer with others and that answers given were based on a teacher's mathematical knowledge on the day the questionnaire was completed. The process ensured that all teachers completed the questionnaire under similar conditions. Finally, administering the survey in this way helped prevent items from being inadvertently released to teachers

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[41] The Gaeltacht refers to an Irish speaking area mainly confined to the western and southern coastal areas. A quota of places exists for teacher education applicants from these areas and consequently some of these teachers may have lower Leaving Certificate points than teachers generally.

[42] SAS Version 9.1.3 was used. SAS and all other SAS Institute Inc. product or service names are registered trademarks or trademarks of SAS Institute Inc., Cary, NC, USA.

other than those participating in the study – a concern because items may subsequently be used to evaluate teacher learning in professional development courses.

Administering the questionnaire in this way created some challenges. The schools were located all over Ireland and consequently, substantial travel was required (at least one visit to each participating school and sometimes more than one visit). This was expensive and time-consuming and meant that more time was required to administer the questionnaire than would have been required for a postal survey, for example.[43] Local circumstances in each school dictated when the survey was administered. In the vast majority of cases teachers completed the questionnaire after school. In other instances it was completed as part of a staff meeting. The specific timing may have prevented some teachers from participating who would otherwise have agreed to be involved. For example, if some teachers in a school agreed to complete the survey on a particular day after school when other teachers had after-school commitments, the teachers with commitments might be discouraged from participating.[44] Although every teacher who expressed interest in participating was accommodated, it is possible that not all potentially interested teachers expressed their interest. All participants received a small gift token in recognition of their time given to the study.

The process of survey administration.

The data were collected from schools between June and December 2006. Because of the number of schools involved, their geographical spread throughout 24 of Ireland's 26 counties, and the need to expedite the process, I recruited seven assistant survey administrators to assist me with survey administration. Those recruited were all either retired school principals or practicing teacher educators. In addition, they all worked as supervisors of students on teaching practice in Coláiste Mhuire Marino. Therefore, all assistant survey administrators knew how schools operate and were familiar with following protocols when conducting school-based assessments where consistency is important. I prepared survey administration guidelines to ensure that surveys were administered consistently (see Appendix 3.7).

I made contact with seventy-nine schools at least twice by phone (see Appendix 3.4) and once in writing in May and June 2006. The initial phone call was to introduce the study and request permission to send the letter (see Appendix Figure A3.5) to the principal, and the second call was a follow-up to the letter to ask if any teachers in the

[43] The funding I received from the Department of Education and Science and from Coláiste Mhuire Marino to conduct the study was essential for addressing this problem.
[44] Notwithstanding this, surveys were administered on more than one occasion in several schools.

school had expressed interest in participating. Eight schools were contacted by the assistant survey administrators directly. This occurred when the administrator was acquainted with a school principal or a teacher on the staff who might boost the response rate.

Almost every principal I spoke to was receptive to the study. Several principals commented that June was a difficult time of the year for teachers to complete the survey because they are busy with tasks such as writing school reports, preparing booklists for the following year, and preparing and correcting tests. Making arrangements to administer the survey was complicated because the survey needed to be completed in the presence of an administrator and because administration required between 60 and 90 minutes. In conjunction with one teacher from each participating school (the principal in the vast majority of cases) suitable times were scheduled to suit circumstances in the school. Although I had hoped to complete data collection in June 2006, only 310 teachers had completed the survey by that time. Many teachers who wanted to take part said it would be more convenient for them to do so in the new school year. Therefore, I decided to continue collecting data until December 2006. Schools who had declined to participate, and some in which the initial participation rate was low, were contacted again by phone or by a personal visit between September and December 2006 and again invited to participate. This increased the number of participants.

Response to Survey Form

The total number of teachers in the sample was 670. Of these teachers 75% (n=503)[45] completed the survey. In 83% (n=72) of the schools, at least one teacher completed the survey. In schools where at least one teacher completed the survey, the average school participation rate was 86% and 42 schools had a 100% response rate. At least six additional teachers agreed to take part but no convenient time could be found to administer the survey.

The response rate of 75% is high considering that teachers were asked to give up between 60 and 90 minutes to do what many teachers considered to be a mathematics test, in the relatively formal setting of having a researcher present. The strong response can be attributed to at least three factors. Many Irish teachers are favorably disposed towards educational research either because they have been involved in it in some way or they believe that it may benefit students. Many principals

[45] Only 501 of these responses were analyzed because two respondents who sat beside one another during the survey administration produced sets of answers which were identical except for one or two items.

said this when I spoke to them and they encouraged staff members to participate. A second factor in the relatively high response rate is that the nature of the research meant that every school was contacted at least twice by phone and once in writing and many schools were contacted more than that. When teachers in a school agreed to participate, a venue and time for completing the questionnaire were agreed and the researcher was present to collect the forms at that time. Moreover, I visited many schools in person to ask the principal and/or the teachers if they would be willing to participate in the study. This direct contact contributed to the high response rate. The third factor is that every teacher who participated in the study received a small token of appreciation.

Demographics of respondents.

Demographic details of respondents were collected. In the final sample 84% of respondents (423) were female and 15% (75) were male. Three did not state whether they were male or female. In the entire population there were 26,282 teachers on 30, 2005 – 4,493 (17%) men and 21,789 (83%) women – so the response has a similar gender composition to the primary teaching population. English was the first language of 470 (94%) respondents and 20 (4%) had Irish as their first language. Two respondents were bilingual and nine did not answer this question. More than half the participants had 11 or more years teaching experience (see Table 3.2). Institutions from which teachers received their teaching qualification are listed in Table 3.3. Noteworthy is the fact that 16% of teachers surveyed received their initial teacher education in institutions other than the traditional Irish providers of teacher education.

Table 3.2

The number and percentage of teachers in the study by years of teaching experience

Experience	Number of Teachers	Percentage of Teachers
1 Year*	46	9
2 to 5 years	112	22
6 to 10 years	77	15
11 to 20 years	70	14
21 or more years	191	38

*191 teachers completed the questionnaire between September and December 2006 and a small number of them would have just begun teaching in September 2006. Because there was no option for "less than one year" these teachers may have ticked the box corresponding to having one year's experience. Four teachers did not state how long they had been teaching and one form was completed by a student currently enrolled in one of the colleges of education but who was working as a substitute teacher in a school on the day the questionnaire was administered.

Table 3.3

Where participants in the study received their teaching qualification

	Number of Teachers*	Percentage of Teachers
Carysfort	63	13
Church of Ireland College of Education	7	1
Coláiste Mhuire Marino	26	5
Froebel College	29	6
Hibernia College	21	4
Mary Immaculate College	147	29
St. Patrick's College	140	28
Other	59	12

*11 teachers did not respond to this question.

Recording data.

When survey forms were completed, I numbered them and recorded teachers' responses in Microsoft Excel. Annotations to items, if present, were recorded by means of "comment" labels on Excel. If no evidence of attempting an item was present a "9" was recorded and if an item appeared to have been attempted but no single response was clearly selected (e.g. by doodling on the page or recording two answers), an "8" was recorded. Subsequently, responses were recoded as correct or incorrect, based on an answer key prepared by the Learning Mathematics for Teaching research project. The choice of "I'm not sure" was marked as incorrect as were items where respondents had chosen two answers.

Missing data

One issue to be addressed in any study is that of missing data. The questionnaire given to teachers in this study was in the format of a test but several respondents left questions unanswered. This is not surprising because it was a low-stakes test. No rewards or favors (e.g. promotion, new job) were available to those who responded to every question. Although a token of appreciation was offered to all participants, it was not conditional on attempting all questions. In fact, the test instructions explicitly advised respondents that they were "under no obligation to complete the questionnaire, or to answer all questions presented in it" (consent letter to teachers, Appendix 3.5). Out of 84 items, the number attempted ranged from 12 on one form to all 84 on 171 forms. On average 78 items were attempted and 90% of teachers answered 70 questions or more. Nevertheless, missing data cannot be ignored when calculating teachers' scores on the test.

Reasons for not completing an item are many. The item may be too difficult or may be perceived as being too difficult; it may have been accidentally skipped if, for example, the respondent turned two pages instead of one; the respondent may have run out of time or may have been interrupted. Unfortunately, the survey form alone rarely reveals why an item is unanswered. But the reason for missing data matters. Some missing answers relate to the central question of participants' mathematical proficiency. For example, a highly proficient participant may know enough to know that she does not know an answer and consequently, chooses to leave the item blank (De Ayala, Plake, & Impara, 2001). Other missing answers do not relate to a teacher's proficiency. An individual who needs to finish early may skip items that require substantial reading and attempt items that seem as if they can be answered quickly. In short, some reasons for missing data relate to a respondent's proficiency level and other factors relate to "personality characteristics and demographic variables" (De Ayala et al., 2001, p. 214). A specific problem that may be relevant for future studies of MKT outside the United States is that willingness to guess answers varies among countries (Ludlow & O'Leary, 1999).

In attempting to deal with the problem of missing responses, psychometricians distinguish between "not-reached" items and "omitted" items. A not-reached item is one a participant did not consider and an omitted answer relates to an item the participant read but did not answer (Lord, 1980). Consecutive unanswered items at the end of a questionnaire are considered to be not-reached, whereas omitted responses are those that occur throughout the form. It is usually not known if these classifications are correct in a given instance but they are considered the best way to deal with the problem of missing responses (Lord, 1980, p. 182). When calculating IRT scores for teachers, all items after a teacher's final attempted question were considered to be not-reached, reducing the total out of which their scores were marked. I decided to code them as not-reached because despite having no time limit for completing the questionnaire, some teachers indicated in advance that they had to finish at a specified time and at least one teacher was interrupted when completing the form and was unable to continue. Only 31 teachers had incomplete items towards the end of the survey. Other omitted answers were marked incorrect because I have no sample-wide evidence about what motivated teachers to leave such items unanswered. By deciding to mark them as incorrect, teachers' MKT proficiency levels may be underestimated.

Data Analysis

The two major sources of data described above – video recordings of lessons and survey responses to items developed from studying teaching – are grounded in the practice of teaching. The data provide the raw material for (a) evaluating how the construct of MKT in the United States is similar to or different from the construct in Ireland, (b) validating the survey instrument as a means of measuring MKT among Irish teachers, and (c) measuring the MKT held by Irish teachers using a two-parameter IRT model. The data will be used to describe the mathematical work of teaching in Ireland and to look for similarities and differences with tasks of teaching that undergird the U.S. construct of MKT. The data will further be used to examine the relationship between the mathematical quality of instruction of Irish lessons and teachers' MKT. Finally, the data will show how Irish teachers performed on the measures and their relative strengths and weaknesses with regard to MKT.

Multiple techniques will be used to analyze the data and address the research questions. I now describe four of the techniques: grounded theory, factor analysis, video coding, and IRT.

Assessing construct equivalence.

The first goal in the study was to investigate whether the construct of MKT as identified in the United States is similar to MKT in Ireland. In other words, I wanted to establish if the construct is equivalent in both settings. I followed steps described by Singh (1995) to study three aspects of construct equivalence: conceptual equivalence, factorial similarity and factorial equivalence. Factor analysis is a popular technique for establishing construct equivalence (e.g. van de Vijver & Leung, 1997). Qualitative techniques are also recommended (Johnson, 1998) but frequently researchers do not have the resources to use them (Ferketich, Phillips, & Verran, 1993). I used a qualitative approach to compare mathematical tasks of teaching identified in Irish lessons to tasks of teaching that informed the development of the U.S. construct of MKT. I did this by first using grounded theory to describe the work of teaching in Ireland.

Grounded theory is a research technique in which interviews or video data are studied and coded into categories, concepts and dimensions, typically in order to generate theory (Corbin & Strauss, 2008). Grounded theory methods can also be used for description and that is how they are used in this study – to create a list of mathematical tasks of teaching identified in Irish lessons. Ten lessons provided the data for this part of the study, one taught by each of the ten teachers recorded on video, as described above. I identified mathematical tasks of teaching in Ireland using open coding, an approach which allows codes to emerge from the data. My previous knowledge of and experience with MKT as a member of the Learning Mathematics for Teaching Project, however, likely acted as a "sensitizing construct" (van den Hoonaard, as cited in Brenner, 2006, p. 360) when I studied Irish teaching. I was more likely to notice teachers' use of language, definitions and explanations than if I had no prior experience with MKT. When I had developed a robust description of several aspects of the work of teaching in Ireland using this technique I compared the tasks identified in Irish lessons with the descriptions of tasks of teaching that had informed the U.S. construct of MKT, as described in literature about MKT. This method was necessary because what is known about the mathematical work of teaching differs between both countries. In the United States researchers have been studying the practice of teaching and describing the mathematical work of teaching for decades (e.g. Leinhardt & Smith,

1985) and several articles describing the construct of MKT have now been published where tasks of teaching that informed the construct are listed and described (e.g. Ball & Bass, 2003b). In Ireland no such literature exists. For this study little purpose would be served by re-analyzing U.S. lessons to identify tasks of teaching that have already been documented. Therefore, the grounded theory descriptions of tasks of teaching in Ireland are compared to descriptions of tasks documented in the United States in order to establish conceptual equivalence. The method will be described in more detail and evaluated in Chapter 4.

As previously mentioned a quantitative approach, exploratory factor analysis, is frequently used to establish construct equivalence (van de Vijver & Leung, 1997) between two groups. Factor analysis offers a way to assess if variables in a survey share a common domain. The construct of MKT is hypothesized to consist of CCK, SCK, KCS, and KCT. The first three domains have been identified in U.S. survey responses (Hill et al., 2004). Factorial similarity and equivalence (Singh, 1995) can be established by identifying the same factors in Irish teachers' responses. Two types of factor analysis are used: exploratory and confirmatory.

Exploratory factor analysis was used to identify covariances among the variables in teachers' responses to survey items (Berends, 2006) on the questionnaire administered to the 501 teachers. One problem with using exploratory factor analysis to establish construct equivalence is that when comparing factors the "factorial similarity" can be underestimated (van de Vijver & Leung, 1997, p. 98).[46] In addition, the approach is purely data driven (Berends, 2006) and does not take hypothesized domains into account. Confirmatory analysis, in contrast, allows the hypothesized domains to be specified in advance and the computer program can measure the extent to which the construct fits the observed data (Berends, 2006). Multiple-group confirmatory factor analysis was used to compare the factor loadings of responses to the items by Irish and U.S. teachers.

I used MPlus (Muthén & Muthén, 1998-2007) software to conduct the factor analyses. The questionnaire used in this study contains items known as testlets (Wainer & Kiely, 1987) or item bundles. These are series of three or more items linked to a

[46] The problem is that if exploratory factor analysis is conducted separately on two populations, a particular type of rotation, called "target rotation" is required to offset a problem where the relationship between the factors may be underestimated. Using target rotation in factor analysis is problematic because very few of the programs for target rotation run on personal computers (van de Vijver & Leung, 1997, pp. 90-93).

common stem (see Figure 6.7, for example). Some common software packages (e.g. SPSS) do not adjust for possible measurement error caused when using testlets as factor indicators. MPlus takes such measurement error into account[47] and therefore, I used this software to do factor analyses on the data.

Assessing validity of measures.

When the extent of construct equivalence of MKT in both settings had been considered (see chapter 4), I assessed whether the multiple-choice measures were valid in the Irish setting. I was interested in determining whether the measures were actually measuring MKT and whether they could predict the mathematical quality of instruction offered by teachers. In order to do this I used a technique used to investigate the validity of items in the United States (Hill, Ball, Blunk, Goffney, & Rowan, 2007). This technique used two data sources: the 40 videotaped lessons taught by Irish teachers and the same teachers' responses to items on the survey form. Central to this part of the study is an instrument developed at the University of Michigan to study the mathematical quality of instruction in lessons.[48] The instrument is a set of video-codes designed to capture the mathematical quality of instruction in lessons and it includes five categories of codes: (a) instructional formats and content, (b) knowledge of the mathematical terrain of the enacted lesson, (c) use of mathematics with students, (d) mathematical features of the curriculum and the teacher's guide, (e) use of mathematics to teach equitably.
Three categories – excluding (a) and (d) – were used to assess the mathematical quality of instruction observed in the Irish video tapes. Mathematics lessons were first partitioned into five-minute clips before coding the instruction. Each lesson was coded by two people who each watched the complete lesson first, and watched it again to independently code it. Finally, the individual codes were reconciled to produce one agreed set of codes for the lesson. Other members of the Learning Mathematics for Teaching Project who were trained to use the codes assisted me with this coding.[49] Codes for teachers' mathematical quality of instruction were correlated with their performance on the multiple-choice measures to study the relationship between

[47] Personal correspondence on MPlus discussion board: http://www.statmodel.com/discussion/messages/8/2863.html?1202347368. Accessed on March 5, 2008

[48] More details about this are available at http://sitemaker.umich.edu/lmt/home. Accessed on March 5, 2008.

[49] Those who helped are Merrie Blunk, Yaa Cole, Amy Jeppsen, Jennifer Lewis, Laurie Sleep, Deborah Zopf.

teachers' MKT and the mathematical quality of their instruction. More detail about this part of the study is provided in Chapter 5.

Reporting Irish teachers' levels of Mathematical Knowledge for Teaching.

The analyses described so far summarize how I established construct equivalence and validity of the measures of MKT for use in Ireland. First, conceptual equivalence was established by comparing the tasks of teaching in ten Irish lessons with tasks that informed the construct of MKT. Subsequently, the validity of measures was established by correlating performance on the measures with instruction. Having completed both stages, I could then ask: What MKT do Irish teachers possess? Scores on the measures could be presented in various ways. Reporting the number of items answered correctly is problematic because items vary in difficulty. Take, for example, two respondents with the same score. One may have correctly answered relatively easy items and the other may have shown greater proficiency by answering items that were more difficult. A raw score or a per cent score conceals such differences. In addition, the MKT items are not criterion-referenced and consequently, there is no expected performance level by which to judge teachers' scores. For that reason reporting raw scores would not be meaningful. Furthermore some items are better at predicting a respondent's overall MKT proficiency than others. Using IRT scores to report performance on the items takes these problems into account (Bock, Thissen, & Zimowski, 1997). The scale used to estimate MKT proficiency and to present the results in this study has an average of 0 and a standard deviation of 1.

IRT has another advantage over raw or per cent scores. It estimates the difficulties of specific items on the same scale as the scale score. An average item has a difficulty of 0. This means that a person of average proficiency has a 50% likelihood of answering the item correctly (Hambleton, Swaminathan, & Rogers, 1991). An easy item would have a difficulty of -3 and a difficult item would have a difficulty of $+3$, indicating that a person with a level of corresponding proficiency has a 50% likelihood of responding correctly. This feature of IRT will be used to identify patterns of items on the survey that Irish teachers found more and less difficult. Bilog-MG version 3 IRT software (Zimowski, Muraki, Mislevy, & Bock, 2003) was used to estimate respondent proficiencies and item difficulties.

Summary

In this chapter I have given an overview of the multiple techniques used to investigate the research question: To what extent and how can the construct and measures of MKT developed in the United States, be used to study the MKT held by Irish teachers? The question was divided into three parts, each addressed in turn using the multiple techniques listed (see Table 3.1). Construct equivalence of MKT in both settings was investigated by comparing grounded theory descriptions of the mathematical work of teaching in Ireland with documented descriptions of mathematical tasks of teaching in the United States. Both exploratory and confirmatory factor analyses, including multiple-group confirmatory factor analysis, of Irish teachers' responses to the survey items complement the descriptive comparisons. The covariance between MKT scores and the mathematical quality of instruction observed in the mathematics teaching of ten teachers was studied to validate the measures. Finally, Irish teachers' MKT proficiency scores are reported using IRT scales and mathematical tasks teachers found relatively difficult and easy are identified. I begin in Chapter 4 by looking at construct equivalence of MKT in both settings.

Chapter 4

Evaluating Construct Equivalence of Mathematical Knowledge for Teaching in Two Settings

In this chapter I consider how measures based on the U.S. construct of MKT can be meaningfully used in Ireland. For over a decade mathematics educators, teachers, mathematicians and other researchers at the University of Michigan have been developing the construct by systematically studying records of mathematics teaching (e.g. videotapes of lessons, copies of student work, teacher's plans and reflections) to identify the mathematical demands of teaching (Ball & Bass, 2003b). Mathematics teaching was analyzed to identify the mathematical knowledge teachers use or might use when doing the work of teaching. The researchers supplemented the analyses of practice with literature on teachers' mathematical knowledge. The researchers have developed an instrument to measure teachers' MKT based on the construct. At first glance it might seem reasonable that such a construct and its associated measures could inform the study of mathematics knowledge and consequently professional development for pre-service and practicing teachers outside the United States. Indeed, educators from several countries have expressed interest in the construct of MKT and in using the items that attempt to measure it. Past experience in using common test instruments among different cultural groups, however, would urge caution (e.g. Straus, 1969). Criticism of researchers who take an instrument used successfully in one setting and apply it in another has been a theme in the literature of cross-cultural and cross-national[50] studies (Johnson, 1998).

The history of applying theories across cultures is not a happy one (Sue, 1999). As often as not researchers consider a construct that works in one country to be universal. Such mistaken assumptions of universality limit constructs and their relevance

[50] In the literature both terms, cross-cultural and cross-national, are used, with the former appearing to be more frequent (e.g. Johnson, 1998). In this dissertation I generally prefer the term cross-national because it more specifically describes the process of attempting to use a U.S. theory and measuring instrument to study MKT in Ireland. Within each of these countries are many cultures (Triandis, 2007). On the grounds that most countries are neither nations nor states as understood in the West, Teune (1990) considers the term "country" to be preferable.

(e.g. van de Vijver & Leung, 2000). MKT was developed entirely in the United States, based on analyses of U.S. teaching and U.S. literature on teacher knowledge. Teaching may well be culturally specific (Stigler & Hiebert, 1999) and if that is true the construct of MKT as currently conceived may be specific to the United States. The work of teaching in Ireland may differ from conceptions of the work of teaching that informed the construct of MKT. Consequently, measures based on the exclusively U.S. construct of MKT would have limited meaning in Ireland. In other words, the instrument may measure knowledge needed to teach mathematics in the United States but not knowledge needed to teach in Ireland. I made no assumption that MKT as currently conceptualized was universal. Instead, testing aspects of the construct equivalence of MKT between the United States and Ireland was built into the study.

The Need for Equivalence in Cross-National Research

My goal is to study Irish teachers' mathematical knowledge using measures developed in the United States to study U.S. teachers' MKT. Before using the measures, however, I wanted to be sure that the construct of MKT on which the measures are based was similar to the knowledge that Irish teachers use or might use when teaching mathematics. So I needed to investigate the extent to which the construct of MKT is equivalent in both settings. To do this I drew on literature about cross-national and cross-cultural comparisons and cross-national and cross-cultural equivalence. If the construct was equivalent, then it was possible that the same instrument could be used to measure it, but if substantial differences were found an alternative instrument may be required.

Many terms have been used in cross-cultural research to describe aspects of equivalence. In enumerating over fifty of them Johnson (1998) claims that not all have been well-defined and considerable overlap exists among them. Some types of equivalence (such as measurement equivalence and scalar equivalence) relate to practical aspects of using instruments to compare characteristics of people in different countries. What is of more interest for this dissertation is equivalence that relates to the *meaning* of what is being studied; I want to know if MKT as elaborated in the United States refers to the MKT used in or needed for teaching in a sample of Irish lessons. To assess construct equivalence of MKT in both countries I followed procedures summarized by Singh (1995). Singh outlines six steps that contribute to construct equivalence and five of them are relevant to this study: functional equivalence, conceptual equivalence, instrument equivalence, factorial similarity and factorial equivalence (see Figure 4.1). The sixth, measurement equivalence, would only be of interest if comparing MKT across countries was a concern. In this chapter I concentrate on conceptual equivalence, factorial similarity and factorial equivalence. Instrument equivalence has been dealt with elsewhere (Delaney et al., in press) and functional equivalence will be addressed using a logical argument.

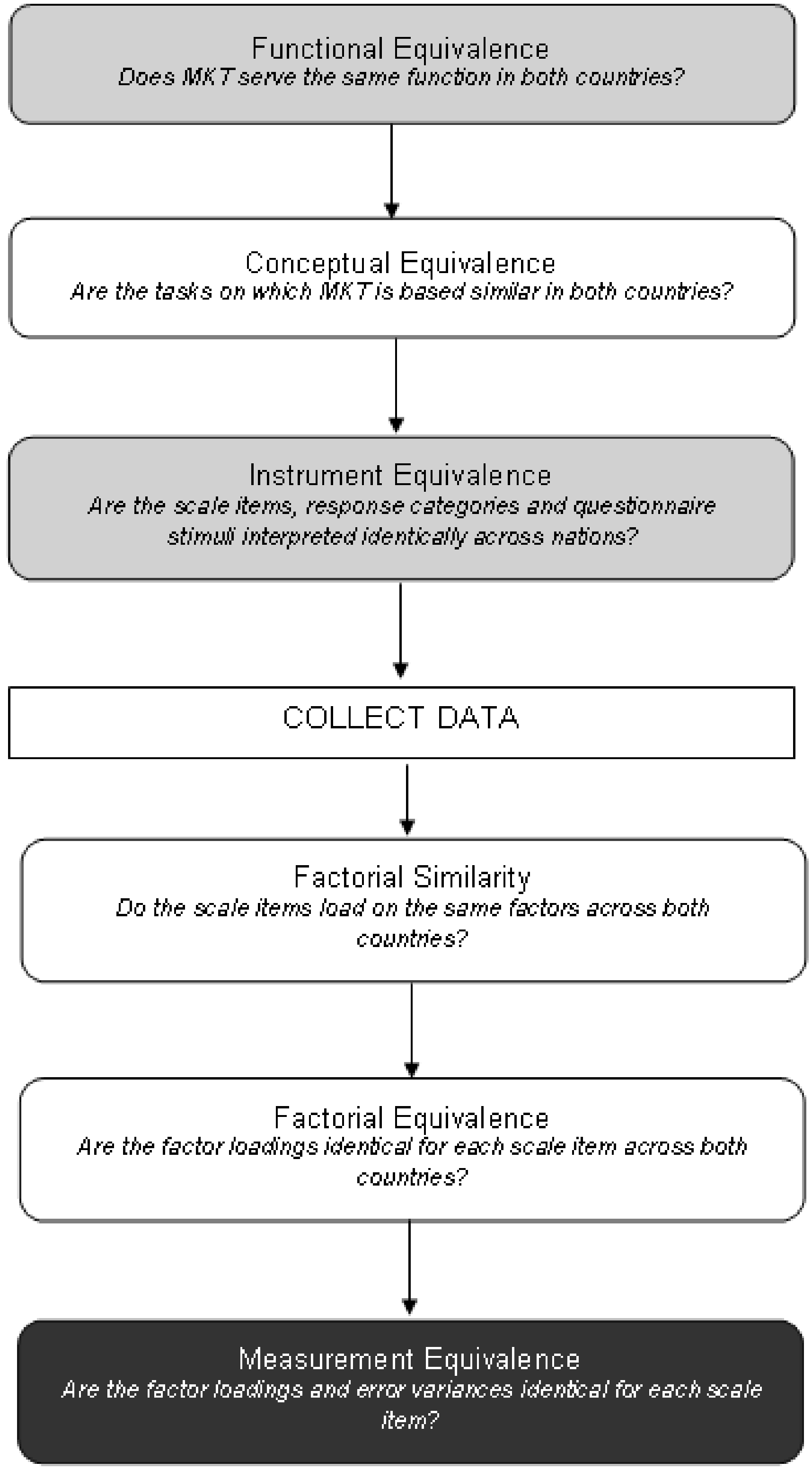


Figure 4.1

Steps in establishing construct equivalence. Figure adapted from Singh (1995)

Rather than addressing Singh's steps in the order presented in Figure 4.1, I begin with instrument equivalence. I briefly consider functional equivalence before devoting much of the chapter to consideration of conceptual equivalence. That is followed by a consideration of factorial similarity and factorial equivalence of the construct of MKT in Ireland and the United States.

The instrument in question refers to multiple-choice items developed to measure teachers' MKT in the United States and these items have been distributed across several test forms. The instrument is equivalent if items are "interpreted identically across nations" (Singh, 1995, p. 605). I adapted U.S. Items for use in an Irish pilot study and my adaptations were discussed over several hours by a focus group of four Irish teachers and separately by one Irish mathematician. They were asked to propose changes where necessary to the items "so that they sounded realistic to Irish teachers" (Delaney et al., in press). Subsequently five respondents to the questionnaire were interviewed about the items and they considered items on that form to be realistic in the context of Irish teaching. Two items were identified that seemed to be interpreted differently by Irish teachers and these were adapted in the final instrument. Although the pilot form and the form used in the study reported here consist of different but overlapping items, data from the pilot study suggests that substantial equivalence exists in how survey items were interpreted by Irish and U.S. teachers.

Functional equivalence relates to whether or not the construct – MKT in this case – serves the same function in all countries (Singh, 1995). Green and White (1976) provide two examples to illustrate functional equivalence: a bicycle can function as a basic means of transport in one country or as a means of recreation in another; shopping can be an integral part of social life in one setting and a chore in another. The question to establish functional equivalence is: has MKT the same role (Teune, 1990) in both Ireland and the United States? MKT is defined as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., in press). Based on that definition, the notion of MKT serves the same function in every country where mathematics is taught. It is the mathematical knowledge needed to teach the subject. Little thought may be given to such knowledge in some countries or the form and content of the knowledge may differ because of different curricula, different teaching traditions or different expectations from education systems. Nevertheless, wherever instruction is concerned with students' acquisition of knowledge, the idea that a teacher needs some mathematical knowledge to teach mathematics seems self-evident and thus the construct of MKT satisfies the requirement of having functional equivalence between the United States and Ireland.[51]

Conceptual equivalence is a different matter. It means that a concept being studied across cultures "should have the same meaning in each culture" (Adler, 1983, p. 37) or it can be "meaningfully discussed in the cultures" being studied (Hui & Triandis, 1985, p. 133). Without this shared meaning any resulting study could be "uninterpretable" because the concept might be understood one way in one setting and differently in another (Green & White, 1976, p. 82). An example is an intelligence test developed in the United States which refers to objects unfamiliar to many children in rural Africa or India. What begins as a test of intelligence in the United States becomes in the new settings a test of "Westernization" rather than of brightness (Straus, 1969). I wish to make claims about mathematical knowledge related to the work of teaching in Ireland, not about mathematical knowledge needed in the United States. Therefore, the question needed to establish conceptual equivalence is: does the construct of MKT mean the same thing in Ireland and in the United States? Or does primary school mathematics teaching in Ireland consist of similar knowledge demands to the knowledge conceptualized in the U.S. construct MKT? To answer this I examined the construct more closely.

Mathematical Knowledge for Teaching: Grounded in U.S. Practice

Studying the practice of mathematics teaching in the United States has been a key element in developing the construct of MKT. This does not mean, however, that MKT as currently theorized describes knowledge held by U.S. teachers. Rather, the construct of MKT refers to knowledge demanded by the work of mathematics teaching in the United States. It is likely, therefore, that wherever teachers engage in similar work, they require the same knowledge. Studying the work of teaching in which a sample of Irish teachers engage and comparing that work to conceptions of the work of teaching that informed the development of MKT will indicate whether the construct is equivalent in each setting. If the work of teaching mathematics in Ireland is similar to conceptions of the work of teaching that informed MKT, substantial overlap likely exists in the construct

[51] It is possible, however, to envisage a hypothetical, extreme case of a teacher-proof curriculum where every line to be spoken by the teacher in every lesson is scripted, where every possible student response (error, question, suggestion, mathematical idea etc.) is anticipated and matched to an appropriate teacher response, where textbooks and materials are chosen by experts other than teachers and where students' understanding is continuously monitored by pre-designed assessments. In such a case the role of a teacher becomes one of manager or facilitator and no mathematical knowledge is needed to do the work of teaching. Functional equivalence would not apply because MKT would not be needed in such a setting.

in both settings. But if the tasks of teaching differ, then the construct is likely to differ. In order to justify this claim, I am going to describe how the construct of MKT emerged from the study of practice.

Studying practice involves looking at the work of teaching, in this case mathematics teaching. But studying practice is not just a matter of observing and describing what happens. Because teaching is complex and much of what happens is invisible, discerning observation is required. Ball and Bass (2003b) put it as follows:

> Casual observation will no more produce insight about teaching and learning than unsophisticated reading of a good mathematics text will produce mathematical insight. Teaching and learning are complex and dynamic phenomena in which … much remains hidden and needing interpretation and analysis (p. 6).

This quotation highlights the active rather than passive nature of studying practice. How one interprets and analyzes practice depends on one's prior experiences and on how those experiences are informed by disciplinary knowledge, and by one's life and work history. Ball and Lampert write about how an observer's background influences what one can observe when studying teaching:

> an experienced elementary teacher will see things on [a] tape that will be invisible to a policymaker – the structures of the pedagogical moves, for instance. A mathematician will see things not likely to be noticed by an educational researcher and vice versa (1999, p. 389).

A unique component of the development of the construct of MKT was to apply both a mathematics perspective and a teaching perspective to the study of lessons taught by U.S. teachers.

Ball and Bass describe what looking at teaching from a mathematical perspective entails:

> As we analyze particular segments of teaching, we seek to identify the mathematical resources used and needed by the teacher….The goal of the analysis is twofold: First, to examine how and where mathematical issues arise in teaching, and how that impacts the course of the students' and teacher's work together; and second, to understand in more detail and, in new ways, what elements of mathematical content and practice are used – or might be used – and in what ways in teaching (Ball & Bass, 2003b, p. 6).

Two phrases here are particularly important. Not only are the mathematical resources the teacher uses identified when studying practice but the resources *needed* by the teacher and the elements of mathematical content and practice that *might be used*. In other words, studying practice begins with examples of practice but it goes beyond practice in order to identify the mathematical knowledge demands of the situation observed in practice. The researchers apply mathematics and teaching perspectives to identify and analyze knowledge demands of the work of teaching.

If the construct of MKT had emerged exclusively from identifying knowledge U.S. teachers currently *use* when teaching, it would possibly be conceptualized differently. Consequently, instruments based on the construct would be different. Some examples will illustrate this. In the 1999 TIMSS video study Hiebert, Stigler and colleagues (2005) wrote that "the United States was the only country in which no lessons contained instances of developing a mathematical justification" (p. 118). If the Learning Mathematics for Teaching Project video codes had grown exclusively from representative U.S. practice, this finding suggests there would be no code for mathematical justifications. But justification is included in the construct of MKT because it is arguably an important mathematical skill for a teacher to develop in students who make mathematical claims and conjectures in class and it is included in the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000). Similarly, Hiebert and his colleagues wrote in the same article about making connections (among ideas, facts, or procedures) that "virtually none of the making connections problems in the United States were discussed in a way that made the mathematical connections or relationships visible for students" (p. 120). Despite the virtual absence of teachers making connections in the practice of U.S. teaching, "making connections" features in the Learning Mathematics for Teaching Project codes developed to study the mathematical quality of instruction in mathematics lessons. This is because teachers need mathematical knowledge to make such connections among number representations, for example, even if little evidence exists that teachers currently use such knowledge.

Although studying conceptions of the work of teaching mathematics is central to developing the construct of MKT, it was informed by other data. The researchers drew on existing literature about teachers' mathematical knowledge and about student thinking. As the construct developed, measures of MKT were developed and both the process of designing the measures and teachers' responses to the measures contributed to the construct. Validation studies among teachers suggested that the construct had applications beyond the practice and literature that initially informed the construct.

In short, the current construct of MKT emerged from practice in the United States and was supplemented by U.S. teacher knowledge literature and other data. Although the construct of MKT describes mathematical knowledge that would be useful for U.S teachers, it does not claim that U.S. teachers currently hold that knowledge. My goal is to study the conceptual equivalence of the constructs of MKT in the United States and Ireland to decide if measures based on the U.S. construct can be used to make claims about knowledge needed for teaching in Ireland. I do this by identifying tasks of teaching in Ireland to determine how different or similar they are to those that informed the development of MKT in the United States. If a U.S. task of teaching was found to make particular mathematical demands on a teacher's knowledge, the task will most likely make the same knowledge demands on Irish teachers.

Studying Tasks of Teaching in Ireland and Tasks that Informed Mathematical Knowledge for Teaching

One challenge in identifying similarities and differences between tasks of teaching in Ireland and tasks of teaching that informed the construct of MKT is that little or no literature exists about the mathematical work of primary teaching in Ireland. In contrast, the construct of MKT has been informed by substantial primary and secondary data on conceptions of the work of teaching, and researchers developing the construct have contributed to these data by writing research articles (e.g. Ball, 1999; Ball & Bass, 2003b) and designing items to measure teachers' mathematical knowledge. Consequently, I first studied primary school mathematics lessons in Ireland to identify tasks of mathematics teaching there. Having identified a sample of mathematics teaching tasks from Ireland I investigated them to identify how similar or different they were to tasks of teaching that formed the basis of the construct of MKT. I compiled a list of tasks of teaching identified in articles about the construct of MKT in the United States and supplemented the list by analyzing test items written to measure MKT in order to identify tasks of teaching embedded in the items.

It may seem strange at first to analyze primary data from one country (Ireland) and to look for similarities and differences reported in secondary data in the second. But a number of reasons make this possible, necessary and desirable. It is possible because of the wealth of relevant data on conceptions of the work of teaching gathered and documented in the United States. The research articles about MKT are based on study and analysis of extensive primary records of practice over several years by researchers at the University of Michigan and elsewhere. In addition, literature about MKT provides specific instances of teaching tasks that informed the construct. Studying tasks of teaching identified in literature about MKT would indicate more thoroughly and precisely the tasks of teaching that informed the construct than any small-scale attempt to study primary records of practice. In addition, the mathematical demands of primary teaching have not been studied before in Ireland and therefore, I analyzed Irish lessons to identify mathematical tasks of teaching in Ireland. As researchers in other countries engage in similar research it would be wasteful of resources for each country to have to study afresh the work of teaching in their own country *and* in the United States, where data already exist. This study can serve as a template for identifying similarities and differences between the work of teaching in one country and conceptions of the work of teaching that informed MKT in the United States. The template used can be developed and improved in light of its effectiveness in this study. Therefore, I am both using a technique to study similarities and differences in tasks of teaching, and evaluating that technique. I now describe in more detail how I studied the work of teaching in videotapes of Irish lessons.

Studying Videotapes of Irish Mathematics Lessons

Videotapes of lessons provide one way to observe classroom interaction and to study the work of teaching. Hiebert, Morris and Glass (2003) describe a lesson as the smallest unit of teaching containing complex interactions. I used video records of ten lessons and studied teaching tasks in the lessons to identify their mathematical knowledge demands. Video allows interactions to be captured and to be watched and re-watched at a pace that allows close scrutiny of the teacher's work, in a way that is not possible, say with participant observation (Erickson, 1986). To supplement the videotapes, copies of student worksheets from lessons were collected and lesson transcripts were prepared so that the audible classroom discourse could be studied as a written record of the lesson.

Choosing tasks within lessons as the unit of teaching for analysis reveals many of the actions that occur in mathematics teaching each day; but documenting tasks in a larger chunk of teaching, say over a week or a year of teaching (as done by Lampert, 2001) would reveal more about the mathematical demands of teaching. For example, by focusing on tasks within a lesson, one does not observe the teacher's planning notes, or decisions made about which topics to emphasize in the course of the school year, or decisions made that led to current class seating arrangements, or conversations with parents about their child's progress in mathematics, or contributions the teacher may make at staff meetings and so on. To address this shortcoming, I will supplement the list of tasks of teaching identified in the lessons with tasks I have experienced in my 20 years as a primary school teacher and teacher educator in Ireland.

My goal in studying Irish lessons was to study the mathematical work of teaching in Ireland to learn about the demands the work makes on teachers' mathematical knowledge. Ball and Bass (e.g. 2003b) claimed that the mathematical knowledge needed for teaching was determined by the work of teaching. According to their theory, if tasks were similar in both settings, the knowledge required would be similar. Describing the mathematical work of teaching, rather than building theory about it was my goal (Corbin & Strauss, 2008). My intention was not to create an exhaustive list of mathematical tasks in which Irish teachers engage. Rather I was looking for instances of practice that occurred in a finite set of Irish mathematics lessons to determine if they were similar enough to tasks that informed MKT to justify using the measures in Ireland. The sample of teachers in the video study is described in Chapter 3. One lesson taught by each teacher – the third of four – was selected for analysis in this part of the study. Lessons were studied relating to four strands of the Irish curriculum (number, shape and space, measures, and data) with no algebra lessons. Number and geometry lessons appeared most with four lessons each. Although varied approaches to teaching mathematics are evident in the lessons, the tasks observed may not be representative of the work of mathematics teaching in Ireland.

Identifying Tasks of Teaching in Ireland

I carefully studied videotapes of ten mathematics lessons to identify tasks that require "mathematical reasoning, insight, understanding, and skill" (Ball & Bass, 2003b, p. 5) or "mathematical sensibilities or sensitivities [or] mathematical appreciation" (Ball, 1999, p. 28). Such instances where a teacher used, or would have benefited from using, mathematical knowledge occur when "mathematical and pedagogical issues meet" (Ball, 1999, p. 28). For example, a teacher may explain to a student why a circle is not a polygon using a definition of a polygon as part of the explanation. Explaining is a key element of teaching and definitions are central to mathematics so both meet in this example. The mathematical work of teaching refers to "mathematical problems teachers confront in their daily work" (Ball & Bass, 2000a, p. 86). A teacher may need to figure out how a student got a wrong answer or to decide how best to represent the idea of dividing a whole number by a fraction. In short, my goal was to "uncover mathematical issues that can be seen to figure in particular moments of teaching practice" (Ball & Bass, 2000b, p. 200) by Irish teachers. "Uncovering" is a suitable word in this case because to my knowledge primary mathematics teaching in Ireland has never before been analyzed with respect to the mathematical work that *teachers* do. The neutral term "issues" is appropriate because what is of interest is not an appraisal of the teacher's handling of a situation but a judgment of the mathematical needs of the work.

When looking at the intersection of teachers' pedagogical and mathematical work, an important issue was to determine an appropriate task size. A task needed to be big enough to simultaneously constitute a recognizable act of teaching and make mathematical demands on the teacher. It is difficult to imagine a task that meets both criteria being too small. Tasks as minor as writing numerals on the board or asking students to open a book on a particular page indicate a teacher has made mathematical decisions about teaching. A greater problem would be choosing too big a grain size. In one instance a teacher was preparing students for a mathematics test, and in another a teacher in a multi-grade setting was preparing independent work with students in one grade level. Both tasks are recognizable as work teachers do and both require mathematical knowledge but they were too big to be classified as mathematical tasks of teaching. Preparation for a test may range from asking students to learn mathematical definitions to discussing generic problem solving strategies. Similarly, preparing students for independent work may involve solving problems collaboratively or discussing possible approaches to problems. Each activity makes different demands on a teacher's mathematical knowledge and therefore, describing as a task either "preparing students for a mathematics test" or "preparing students for independent work in mathematics" reveals little about the task's mathematical demands. This point was observed in an international video study of mathematics by Kawanaka, Stigler and Hiebert (1999) who found that although teachers in all countries review previous lessons, check homework, learn new concepts and procedures and so on, "there were enormous differences in how those activities were done" (p. 93) which would affect the mathematical knowledge required.

In addition, mathematical tasks of teaching can be nested within one another. For example if a teacher is eliciting a mathematical term, say the word "face" on a polyhedron, this is a task of teaching that needs to be identified. While eliciting the correct term, the teacher may have to respond to an incorrect answer (say, if a student suggests the word "side"). Responding to an incorrect answer is another task of teaching but the tasks overlap. Because my concern was to compile a list of tasks of MKT in Ireland, in such instances I listed each task I noticed, even if it was nested within another task. This is because the study is exploratory in nature and the tasks may not always have a symbiotic relationship. If I opted to name only one task, the other may not be observed elsewhere in this dataset. I recognized that identifying tasks individually in this way simplifies the complex work of teaching but at this stage of the work I believed the separation would lead to better understanding of the work of mathematics teaching and the knowledge demands it makes on teachers in Ireland.

In order to systematically identify tasks of teaching in each lesson, I adapted a lesson table used by Kawanaka, Stigler and Hiebert (1999).[52] I used the table to record the task, the clip in which it occurred, the relevant lesson dialogue and my comments about it. Each task was recorded on a new row of the table. A sample of two pages from the first lesson observed is contained in Appendix 4.1. As the list of tasks grew I developed a glossary to list and describe tasks and to specify possible mathematical knowledge required by the task (see Appendix 4.2). As mentioned above, tasks such as a teacher's interactions with colleagues or with parents about mathematics would not be visible on videotapes of lessons. I used my knowledge of Irish education to supplement the list of tasks identified in the video lessons with tasks of teaching that were not, and in most cases could not be, observed in a video study.

Identifying Tasks of Teaching that Informed the U.S. Construct of Mathematical Knowledge for Teaching

In contrast I identified tasks of teaching that informed the construct of MKT in the United States by studying literature about the construct and analyzing items based on the construct. The construct of MKT was developed by Deborah Ball and Hyman Bass and their research colleagues, arising from both their collaborative study of teaching from a disciplinary mathematics perspective beginning in the mid 1990s, and from other studies of teacher knowledge. In order to review the literature on MKT, I surveyed all published articles (journal articles, book chapters and published conference proceedings) written on the topic by Ball alone or by Ball and Bass between 1999 and 2007. By selecting this set of articles, I would identify conceptions of the work of teaching that informed the construct of MKT. I began with publications from 1999 because Ball's "Crossing Boundaries to Examine the Mathematics Entailed in Elementary Teaching" is the first major article written about this "line of original work in educational research" (p. 15). Ball and Bass have worked with other authors since then, notably Hill and Schilling but later articles tend to be more about the measurement of MKT than about the

[52] Another model that could have been adapted is that proposed by Schoenfeld (2000). It is focused on analyzing teacher decisions and actions and in its fully elaborated form it includes goals, beliefs and knowledge and what corresponds most closely to the work of this chapter "action sequences" (p. 251).

construct itself. Therefore, I have excluded these articles from the selection. I made one exception to these general criteria and included in the review an article by Ball, Hill and Bass (2005) because the article discusses the construct more than its measurement. In order to identify articles I searched on the ERIC database using both authors' names and on Ball's personal website for articles that met the criteria for inclusion. The articles included are as follows: Ball (1999), Ball (2000), Ball and Bass (2000a), (Ball, 2002b), Ball and Bass (2003b), and Ball, Hill and Bass (2005).[53]

In Chapter 2 I surveyed literature by other researchers whose work has influenced the U.S. construct of MKT. Research by Borko, Cohen, Even and Tirosh, Lampert, Leinhardt, Lloyd, Peterson, Schifter and Simon and others was included. Many tasks of teaching that require mathematical knowledge are identified in this body of work, including: providing complete explanations; building on students' previous knowledge; choosing and using representations; making links among representations; deploying concrete materials and mathematics activities; responding to a new curriculum; documenting curriculum content taught; monitoring students' understanding of ideas and topics; connecting ideas across contexts; and writing word problems for fraction calculations. But I did not include these articles when identifying tasks that underlay MKT for three reasons: (a) many of the tasks of teaching described by these researchers (and others) are reflected in the Ball and Bass literature and items; (b) My interest in this study is not to comprehensively document mathematical tasks of teaching identified in literature on teachers' mathematical knowledge in general; and (c) my goal was to use measures of MKT developed at the University of Michigan to study Irish teachers' mathematical knowledge and therefore, I wanted to identify tasks that undergirded the specific construct and measures. Explicit references to the work of teaching mathematics in the Ball and Bass articles were complemented by analyzing items designed to measure teachers' knowledge.

Since 2001 researchers at the University of Michigan have been developing multiple-choice items to measure teachers' knowledge (Hill et al., 2004). The items use teaching contexts and are based on the construct of MKT. Therefore, they provide another window into tasks of teaching that informed the construct. The items are embedded in tasks of teaching and identifying those tasks would supplement the list of

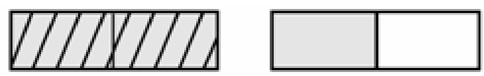
[53] Two articles which appeared to meet the criteria were excluded because they belong to a different body of work (on proof in classrooms) by Ball and Bass (2000b; 2003a). These articles would not provide direct insights into the tasks of teaching that were instrumental in developing the theory of MKT.

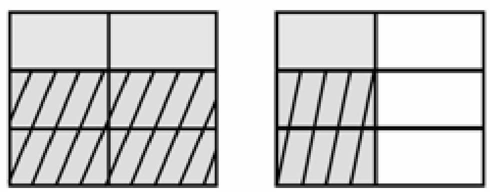
tasks of teaching identified in the literature that informed the construct of MKT. Each item contains two potential sources of mathematics tasks of teaching. One source is the context of the question because questions are usually based around work teachers do such as reviewing student work or discussing a problem with a colleague. The second source is the actual question asked of the teacher which is usually based around work a teacher has to do when teaching or preparing to teach. The item in Figure 4.2 and my analysis which follows it give an idea of the process used to identify tasks of teaching in the items.[54]

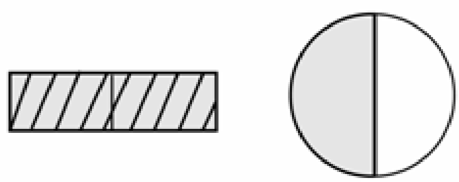
[54] This item is one of a selection of publicly released items and was downloaded from http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf.

17. At a professional development workshop, teachers were learning about different ways to represent multiplication of fractions problems. The leader also helped them to become aware of examples that do <u>not</u> represent multiplication of fractions appropriately.

Which model below <u>cannot</u> be used to show that $1\frac{1}{2} \times \frac{2}{3} = 1$? (Mark ONE answer.)

A) 

B) 

C) 

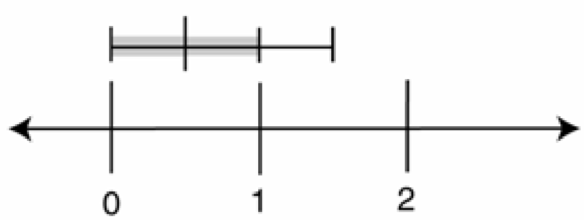
D) 

Figure 4.2. Item 17 on form B_01.

I identified three tasks of teaching implied in this item. First, teachers represent multiplication of fractions problems in multiple ways because they are learning to do that at the professional development workshop. Second, teachers evaluate representations in their work of teaching because according to the item stem, one of the models cannot be used to show the specified operation and a teacher needs to be able to determine which model would not work. Third, teachers connect calculations with representations as part of their work. All three tasks place knowledge demands on the teacher and are consequently part of the U.S. conceptualization of MKT.

Looking for Similarities and Differences among Tasks of Teaching in Ireland and Tasks that Informed Mathematical Knowledge for Teaching

Using four data sources – videotapes of Irish lessons, my experience of Irish teaching, U.S. MKT literature and U.S. MKT items – I set about identifying similarities and differences between tasks of teaching in which Irish teachers engage and tasks used to inform the construct of MKT. My first objective was to make two lists: one of Irish tasks, using tasks from the video lessons and tasks identified from my knowledge of U.S. teaching; and the other of tasks that formed the basis of MKT, using tasks from the MKT literature and the MKT items. In order to do this I needed to scrutinize the articles to identify tasks of teaching that had informed the construct of MKT. But the articles were not intended to be used in this way. They were written more as progress reports on the development of the construct, liberally illustrated with examples of the work of teaching; tasks were integrated into the articles to support the arguments being made and not necessarily recorded in neat, ordered lists that would facilitate my analysis. Some descriptions of tasks were specific and detailed such as selecting definitions, inspecting alternative methods, examining their mathematical structure and principles and judging whether or not they can be generalized (Ball & Bass, 2003b, p. 7); or rescaling a problem for younger or older learners, to make it easier or more challenging (Ball, 2000). The grain size of such tasks fitted well with those I had identified in Irish lessons. Other descriptions of the work of teaching were more general and could be manifested in various ways in the classroom. Examples include figuring out what students know (Ball, 1999) and using representations (Ball, Hill et al., 2005). Still others seemed more like generic tasks or general principles of teaching, e.g. deciding among alternative courses of action (Ball, 1999). Given my decision to use a small grain size of task when studying conceptions of the work of teaching in Ireland, the diverse levels at which tasks that informed MKT were documented made a direct mapping of Irish and tasks that informed MKT difficult.

Issues related to how language was used to describe tasks, and how tasks were demarcated added to difficulties caused by the varying grain size of tasks described above. First, different language can be used to describe similar tasks and conversely similar words can be used to describe tasks that are different. An example of the latter is that in the United States, the word “skills” in mathematics is used to describe mechanical or procedural knowledge of basic computations whereas skills in the Irish curriculum refer to mathematical practices such as applying and problem solving; communicating and expressing, integrating and connecting, and reasoning. Deciding on the boundaries of tasks complicated the analyses of task similarities and differences. Some tasks seemed similar but were not exactly the same. For example, an Irish teacher told students that problems involving division of whole numbers by unit fractions would have whole number answers, whereas multiplication of fractions problems would likely have answers in fraction form. The teacher subsequently encouraged students to check their answers on the basis of this principle. A related task identified by Ball and Bass is where teachers ask students to make sense of solutions different to their own (Ball & Bass, 2003b). In both tasks students are required to examine completed problems (their own in Ireland and those completed by others in the United States) and to evaluate the solutions (using specific criteria in Ireland and generally in the United States). Both teaching tasks could be classed as requesting students to evaluate solutions to problems but depending on how the teachers set up the tasks, different demands may be made on the teacher’s knowledge. I describe this as an issue of how the task is demarcated because although both teachers ask students to evaluate solutions to problems, the terms under which the students do so may be different, depending on the kind of solutions a U.S. teacher presents to her students. I now describe how I looked for similarities and differences with these challenges in mind.

I first compiled a table listing tasks of teaching gleaned from Irish lessons. Next I read each article about MKT and noted references to tasks of teaching identified, regardless of their size (see Appendix 4.3). Notwithstanding the challenges listed above I selected individual tasks observed in the Irish lessons and looked for similar tasks on the list of tasks that had informed MKT. Although tasks may have been expressed differently, I asked if it was reasonable that a given task based on the construct of MKT was similar to or if it incorporated a given Irish task. If a task based on the U.S. construct of MKT was similar to or inclusive of an Irish task, both tasks were part of the work of teaching conceptualized in the construct of MKT. Synthesizing similar U.S. and Irish tasks in this way led to inconsistencies in the grain size of the tasks as will be evident below. Identifying similarities and differences was done iteratively sometimes beginning with the Irish task and looking for a similar task based on the MKT construct and sometimes beginning with the task that had informed the construct of MKT. Instances of similar tasks provided evidence of conceptual equivalence of the construct of MKT in both countries. This is a blunt instrument to use to establish conceptual equivalence but it should indicate if sufficient overlap exists between the constructs of MKT in Ireland and the United States to justify using the U.S. measures in Ireland.[55] Given the challenges of examining similarities and differences in the tasks of teaching, it was important to make the process as transparent as possible. Therefore, in this chapter when I identify similar tasks of teaching I generally present illustrative examples of tasks from Ireland and relate the Irish task to descriptions of tasks taken from the U.S. articles. This enables readers to judge the reasonableness of my assessments of similarities and differences.

Tasks of Teaching in Ireland Similar to Tasks that Informed MKT

Close to 100 tasks were identified in the ten Irish lessons observed (See Appendix 4.2). The challenge I faced in writing this chapter was to describe tasks in sufficient detail to reassure readers that the Irish tasks are similar to tasks that informed MKT and to do so in a reasonable amount of space. Rather than document every task I selected a smaller number of similar tasks in both settings to describe in detail in this section. Remaining tasks are summarized in Appendix 4.4. Even with many common tasks removed, this section of Chapter 4 is quite lengthy. The similar tasks to be described and illustrated are listed in Figure 4.3. The tasks headings are meant to assist readers in following the comparison of tasks. The headings and the sequence in which tasks are presented are not intended to constitute a comprehensive characterization of tasks of mathematics teaching. A more complete task analysis would be required for that. They are merely provided as anchors for reading through the several detailed examples of mathematical tasks of teaching.

[55] I claim this way of comparing similarities and differences of tasks constitutes a "blunt instrument" because of the different ways in which tasks are listed in the literature and because the nature of the lists means that they are not comprehensive. However, the method has potential and part of this study involves evaluating this means of studying conceptual equivalence and considering what it would take to make the instrument sharper.

- Relating representations to number operations
- Eliciting meanings of operations
- Presenting properties of numbers and operations
- Applying mathematical properties
  - Describing and identifying shapes
  - Eliciting properties of shapes from students
- Pressing for mathematical clarification
- Deciding which math ideas to highlight and which to set aside
- Following students' descriptions of their mathematical work
- Eliciting student explanations
- Following and evaluating explanations
- Interpreting student productions:
  - Appreciating a student's unconventionally expressed insight
  - Interpreting and making pedagogical judgments about students questions etc
  - Hearing students flexibly
- Comparing different solution strategies and solutions
- Responding to students
  - Responding productively to students' mathematical questions
  - Helping students who are stuck
- Anticipating student difficulties
- Connecting number patterns and procedures
- Assessing if procedures generalize
- Using concrete materials and visual aids
  - Explaining inadequacies in materials
  - Drawing shapes on the board
- Selecting useful examples
- Presenting estimation strategies
- Using and eliciting mathematical language
  - Using correct and appropriate mathematical terms
  - Being careful in use of general language
  - Eliciting terms
  - Defining and explaining mathematical terms
  - Eliciting meanings of terms
- Attending to concerns for equity
  - Support students in using mathematical language
  - Connecting mathematics to a skill for living
- Connecting ideas to future mathematical work
- Connecting mathematics with the students' environment

Figure 4.3

A summary of tasks of teaching in Ireland similar to tasks that informed MKT. This list includes only those that will be described and illustrated below. For additional similar tasks see Appendix 4.4.

Relating representations to number operations.

One task for teachers is to relate representations to number operations being taught. Representations are necessary for communicating ideas in mathematics (e.g. Zazkis & Liljedahl, 2004). A representation is defined here as “a sign or a configuration of signs, characters, or objects…that…can…symbolize, depict, encode or represent something other than itself” (Goldin & Shteingold, 2001, p. 3). Representations include the use of diagrams, pictures, the number line and so on to communicate mathematical ideas among students and between students and teacher.

In one Irish lesson sixth class students were being taught how to divide a whole number by a unit fraction. The teacher asked a student to draw a picture of “one divided by a quarter” and the student went to the board, drew a square and partitioned it into four equal parts as shown in Figure 4.4. The student pointed out that the square represented one and that you divide it into four. He then hesitated and said he didn’t “see” how to draw it. The teacher addressed the class “Is that one divided by four? Is that one divided by a quarter?” Student responses were mixed so the teacher related it to division with whole numbers. He pointed out that the question is “how many quarters are there in one?” He went on to say “so it is effectively dividing by four, isn’t it? The teacher sensed that the student at the board was uneasy about the diagram and the teacher asked “are you happy with that drawing?” The student replied,

> Yeah, it’s just the answer is all of them, not just one. It’s usually one, because if you’re quartering it, the answer is one of them, but if you’re dividing by a quarter it’s all of them, so that’s what I was drawing, the other way. (SDVS9, C, 2)

The student’s comment captures well the mathematical work of teaching using representations. The teacher must navigate a narrow course between two mathematics problems which are different but easily confused. Finding a quarter of one, and finding how many quarters in one are two different problems, represented by the notation $1 \div 4$ and $1 \div \frac{1}{4}$ respectively. What makes the problem more difficult for the teacher is that the same representation (shown in Figure 4.4) can be used for each problem. Notice that the teacher wavers between asking “is that one divided by four? Is that one divided by a quarter?” as if these problems were the same. Later he equates asking “how many quarters are there in one” with “effectively dividing by four.” The representation if not interpreted carefully might lead one to believe that. The student, however, points out that for the first calculation ($1 \div 4$) the answer is represented by one of the four sections in the square (¼) but for the second calculation ($1 \div \frac{1}{4}$), the answer is represented by all four quarters (4). In this one teaching episode the teacher draws on MKT to understand a student's diagrammatic representation of a fraction calculation, to hear and interpret what the student is saying and to do so when two problems seem similar but are different. If the teacher is not explicit about the differences, student misunderstanding ensues. This is an example from Irish teaching of what is described in the U.S. literature as "representing ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process" (Ball & Bass, 2003b, p. 11).

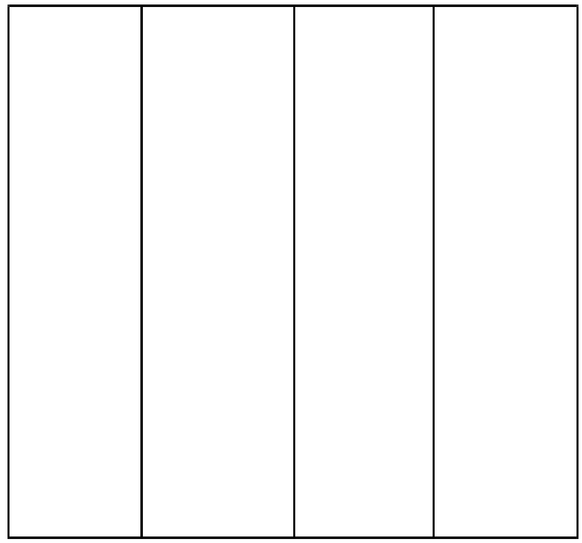


Figure 4.4. Representation drawn by a student to represent 1 ÷ ¼.

Eliciting meanings of operations.

One barrier to good mapping between models can come from different meanings implied in operations, notably subtraction and division. A task of teaching observed in the Irish lessons was eliciting the meaning of an operation. The teacher from the previous example wanted to elicit the meaning of division with whole numbers before introducing the idea of division with fractions. He asked students what came to mind when they saw the expression 72 ÷ 9. The teacher emphasized that he wanted to hear the meaning of the operation rather than the answer when he said “don’t give me an answer. I’m not interested in the answer, OK? What would I be asking you to do if I was asking you to do that sum?”[56] (SDVS9, C, 1). One student responded “you find how many nines there are in seventy-two” and the teacher rephrased this as “I’m dividing 72 into bundles of nine.” What is important about this question is that the teacher intends it to be an introduction to dividing by fractions. But division has two meanings: partitive (equal sharing) and measurement (repeated subtraction). In the partitive model the number of groups is known but the number in each group is not. In the measurement model the size of the groups is known but not how many groups. This distinction matters when it comes to extending the topic to division of whole numbers by fractions because not all interpretations that work well with whole numbers work equally well with fractions. The measurement – repeated subtraction – model works well because sixth grade students can imagine taking half kilograms of butter from a quantity such as three kilograms. The partitive model can work if one thinks about three being one half groups of six. However, the idea of sharing three kilograms of butter equally among one half people makes no sense, and is unlikely to help sixth class students understand division of whole numbers by fractions. Teachers need to think about such issues when eliciting from students the meaning of an operation. The task of working with different meanings of operations is documented in the United States with reference to records from Deborah Ball’s second grade class in the school year 1989-1990. Ball wrote that her 3rd grade students investigated the relationship between the comparison and take away interpretations of subtraction (Ball & Bass, 2000a, p. 93).

Presenting properties of numbers and operations.

Another task of teaching, which was observed in Ireland and which informed the construct of MKT, is making students aware of the properties of numbers. MKT

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[56] In Ireland the term “sum” refers to both the result of combining two addends and to operations generically, depending on the context.

examples include knowing prime numbers (e.g. Form B_01, Item 8) and discussing odd and even numbers (Ball, 1999). In an Irish lesson a teacher of senior infants introduced the property of seven being an odd number. She began by reviewing different ways to make seven (7 + 0 = 7, 6 + 1 = 7 and so on).

> T: How many ways are there of making seven?
> S: Eight, there's eight
> T: Eight ways of making seven. Do we have a double in seven, where there are the same numbers on both sides?
> S: No
> T: Do we have a double? Why don't we have a double?
> S: Because there's (unclear) three
> T: Remember we were sharing out the teddies?
> S: It's a [*sic*] odd number
> T: It's an odd number. When we were sharing out the teddies we couldn't, no matter how we tried, we couldn't share them out so that the two boys had the, both had the...?
> S: Same
> T: Because seven is an odd number. It's not even, like number …?
> S: Six
> T: Six, where we had three plus
> S: Or eight.
> T: Three, or eight. Exactly. It's an odd number so there are no doubles, but there are lots of pairs. (SDVS8, C, 3)

The teacher drew students' attention to a pair absent from the pairs of numbers that make seven; a pair where both numbers are the same. She asked why there was no double and when she reminded the students of an earlier activity where they shared out seven teddies, one student stated that seven is an odd number. In this class of 5-year-old students an odd number is defined as a number where "no matter how we tried, we couldn't share [the teddies] out so that the two boys had the … same." The implied definition of an odd number in this classroom is 2k + 1 where there will always be one teddy left over when the set of teddies is split into two groups of the same size. When the teacher mentioned that an even number is different, students gave examples of even numbers. Describing and eliciting the properties of numbers is another task of teaching.

In teaching primary school mathematics, understanding properties of operations is also important. The long multiplication algorithm, for example, requires students to understand the distributive property (Lampert, 1986). The senior-infants teacher in the previous example was observed performing this task of teaching. Teacher and students had been discussing the "story of seven" and had named pairs of numbers that when added made seven. The teacher would call out a number and students would say what addend made seven. As the following interaction takes place the teacher has just asked the students all pairs that added to seven had been named and most students responded that it was.

T: I think it is; I think it is. Did we do? We did seven. [Teacher turns to write "7 + 0 = 7" on board] What goes with seven to make...seven?
SS: Zero
T: Plus zero equals seven. What was the pair of that?
S: Zero plus seven. [Teacher writes "0 + 7 = 7" across from, and level with "7 + 0 = 7"]
T: Excellent. Are they the same?
SS: No
S: It goes the other way
T: No. It goes the other way, they both have seven and they both have zero. But on this one seven comes first, and on this one zero comes first. So they're pairs. They're friends but they're not the same are they?
S: No.
T: Ok, we did six, what's the total of six? [Teacher writes "6 + 1 = 7" on board]
SS: One, two, one.
T: One, one goes with six to make?
SS: Seven
T: Ok, can you tell me the pair of that?
SS: One and six [Teacher writes "1 + 6 = 7" across from and level with "6 + 1 = 7")
T: Good one plus six equals seven. Well done. SDVS8, C, 2)

The teacher pointed out that 7 + 0 = 7 and asks what the "pair" of that is, eliciting 0 + 7 = 7. The teacher is drawing the students' attention to the commutative property of addition, without using the term. Instead the students are allowed to express the idea in their own way "it goes the other way" which the teacher repeats. She then introduces the terms "pairs" and "friends" to label the concept and asks students to apply what they have learned to produce another commutative pair. One U.S. item (Form B_01, Item 22) requires teachers to recognize whether or not an incorrect answer illustrates lack of understanding of the commutative property.

Applying mathematical properties of shapes

I observed Irish teachers applying mathematical properties of shapes in their teaching. This topic featured in conceptions of the work of teaching that informed MKT where reference is made to defining what a rectangle is (Ball, Hill et al., 2005). Many other teacher tasks related to the properties of shapes are evident in MKT items including: describing properties of shapes, naming shapes based on their definitions, matching specific shapes with their properties, and evaluating the truth of geometry statements (Form GEO_1, p. 7-8; Form GEO_b, p. 6 & 9). Because four of the ten Irish lessons I observed centered on the teaching of two-dimensional and three-dimensional shapes, it is not surprising that I identified tasks of teaching related to the properties of shapes: describing or identifying properties of shapes, eliciting properties of shapes, and comparing shapes or categories of shapes.

I begin with examples of Irish teachers describing and identifying shapes. In one instance a teacher described a shape and asked students to name it. The teacher described one shape as solid "with a circle on what I call the top, a circle on the bottom. There's no sharp edges on it and it's rolling around in my hand" (SDVS4, C, 6). When a student guessed that it might be a circle the teacher reminded the students that it was solid and the correct answer, cylinder, was provided. In another classroom a teacher described the shape she was showing the students as "almost a cylinder" because "there is something missing" (SDVS3, C, 4). The shape in question was a paper towel tube and, according to the teacher, "a top and a bottom" were missing. Another potential problem with the shape, not raised by the teacher, is if a cylinder is a solid shape, how can the paper towel holder be considered a cylinder? Would it be more accurate to think about the paper towel tube as enclosing a cylinder of air? Any discussion of the properties of a cylinder is complicated by the fact that the term "cylinder" is frequently applied to the cylindrical surface itself as well as to the solid bounded by the cylindrical surface, and for mathematicians the term can refer to the lateral sides of the shape, without the "top and bottom caps" (Weisstein, 2003, p. 649). If a teacher wants to describe a shape precisely, as the teacher in the video attempted to do by pointing out what was missing, then describing shapes is another task that makes demands on the teachers' mathematical knowledge.

When a teacher elicits properties of shapes from students, the teacher is doing mathematical work. Sometimes children do not have the language to describe what they observe like the fourth grade student in one class who described a shape that "has a circle at the end and it gets narrow every time." (SDVS4, C, 7). The student was trying to differentiate the cone's pyramid shape from the prism shape of the cylinder. The teacher needs to first recognize what the student is attempting to do and then the teacher can help the student develop more sophisticated ways to describe the shape. Because a teacher is working with learners of mathematics, mistakes can be expected and mistakes place demands on the teacher. For example, one teacher was eliciting examples of circles when a student suggested the globe in the classroom. The teacher asked other students if the globe was a circle and one student said “no, because it’s a three – D shape” Just as the teacher was praising the second student for her answer another student remarked that “it’s called a cylinder” (SDVS10, C, 7). The teacher then elicited the term “sphere” to name the shape of the globe. A teacher needs to be clear about definitions when eliciting properties of shapes. In another lesson a student said that a sphere “has only one face because the face goes round the whole circle” (SDVS3, C, 2). A face, however, is sometimes defined as a polygon shape, and by that definition the surface of a sphere is not a face.

Pressing for mathematical clarification.

On some occasions students’ responses are unclear or incomplete. On such occasions I observed Irish teachers pressing for student mathematical clarification. This is part of attending to, interpreting and handling students’ responses (Ball & Bass, 2003b). Pressing for mathematical clarification was identified in Ireland generally in the context of asking students to either expand on or to clarify a response. In one instance a teacher was checking how many groups of six could be made from nineteen lollipop sticks. One student said the answer was three. The teacher checked the sticks on the student’s desk and probed further:

T: And what did you say you have as well?
S: We also have three groups …
T: Of?
S: Of six
T: And?
S: Six in each group
T: There are six in each and you had something there in your hand?
S: One left over.
T: And one left over good boy. (SDVS5, C, 2)

The student stated that three groups of six could be made with nineteen sticks. The teacher asked the student to expand on this and he gave the size of the three groups. The teacher then asks “And?” to prompt the student to expand further and the student replies that there are six in each group. The teacher required more information so he described each group and pointed to a lollipop stick that had not been part of the response so far. The student added “one left over.” This addition was important for the lesson because the teacher wanted to focus on the remainder and by omitting it from his answer the student was ignoring a key part of what the teacher wanted to teach.

Deciding which mathematical ideas to highlight.

Another task frequently mentioned in the MKT literature is deciding which mathematical ideas to highlight and to explore further in a lesson and which to set aside (e.g. Ball, 2002a; Ball & Bass, 2000a). A teacher needs to use mathematical judgment (in conjunction with pedagogical judgment) to decide whether to allow a particular idea to become part of the agenda in a lesson or to put the idea aside. In one case a student noticed that when he divided 13 lollipop sticks among four people each got three sticks and a third of a stick. But the teacher wanted to focus on the remainder of one, rather than the fractional part so he responded to the student:

> Right grand, relax with that now don't get too carried away with it. That's one lollipop stick isn't that right? Now I can see where you're coming from but don't worry, don't go there for the moment. (SDVS5, C, 1)

The teacher's response indicated that he understood the logic of the student's discovery but the teacher decided to focus on the concept of a remainder in division. Making such decisions requires MKT and decisions taken affect instruction.

Following students' descriptions of their mathematical work.

I observed Irish teachers following students' descriptions of their mathematical work and this task appears in at least one MKT measure (B_01, 26). To describe this task I use an instance of two students describing an idea they had for dealing with the remainder of one when they made four groups of three with 13 lollipop sticks.[57] The teacher wanted the students to notice the remainder but one pair had another idea which they described when the teacher called to their desk:

> S: Yeah Daniel made up this idea
> T: Did he?
> S: Where it's like you divide it into four equal parts and share it out equally.
> T: Ah very good. Interesting. So, you were going to actually split that?
> S: Split it in quarters and share it out. Yeah but like with our pencil…
> T: Yeah but we don't need our pencil though. So, you were going to split it into…?
> S: Quarters and split it between them.
> T: So you were going to split that into four equal parts is that correct? And you were going to put one equal part for each of those. Listen boys you're ahead of the game. You don't need to do that. But well done. Excellent. (SDVS5, C, 2)

The students are describing what they could do with the remainder of one; they could split it up into quarters and allocate one quarter to each of the four groups of three. Following a description can be difficult if information is present that does not seem to fit.

[57] This example is from the same lesson as the previous example but the students are different.

It is not clear from the extract if the teacher knew why the students referred to a pencil. It may have been to mark the quarters on the lollipop stick. The teacher questions the students closely restating what they said to ensure he understands them and possibly to clear up any misunderstanding. In the end he praises the students, affirming that their work made sense but informs them that what they did was not required as part of the activity. In this example the teacher is following a method or solution that may differ from how he would approach the problem or from how he had planned the lesson to proceed. This is part of the mathematical work of teaching in Ireland and it is included in the U.S. conceptualization of MKT.

Eliciting student explanations.

Irish teachers sometimes asked students to explain their answers when students answered a problem or a calculation. The task of eliciting explanations informed the construct of MKT (Ball & Bass, 2000a). One example of this task from the Irish lessons relates to a teacher working on division with a remainder who gave the following problem to his students: “There were twenty-six people going on a tour. Three mini-buses came to collect them. Each mini-bus held eight people. How many people had to wait for an extra mini-bus?” (SDVS5, C, 12). In response to the question one student answered “two people.” The teacher asked “how did you figure that?” to which the student replied “because I counted eight, sixteen, twenty four would be held in three mini-buses. So then I knew there was two left” (SDVS5, C, 12). When the teacher initially heard the answer to the question he knew the student had the right answer but he did not know what thinking led to the right answer or even if the student had worked out the answer himself. By asking the student to explain his answer the teacher learned that the student had linked his thinking to the problem. The student found the number of passengers that could be accommodated on three buses by repeatedly adding eight until he got as close as possible to the number of passengers. At that stage the student probably counted on two to find how many people would have to wait. Encouraging students to explain answers is part of the mathematical work of teaching in Ireland and the United States.

Following and evaluating explanations.

Another mathematical task observed in Irish teaching is that of following and evaluating student explanations. This task also appears in some of the MKT measures (e.g. Form B_01, items 4 and 25). In one sixth class lesson a teacher wrote 72 ÷ 9 on the board and asked the students what came to mind when they looked at that expression.

One student responded that multiplication came to mind. When the teacher asked why the expression brought multiplication to mind, the following exchange took place.

S: You have to see how many times you multiply nine and it still fits into 72.
S: Yeah.
T: Ok yeah. Because multiplication and division …
S: Are the same.
T: they're the same Jack?
S: Almost. Well the basics are.
T: Ok, what do you mean by that?
S: Because really all you're doing is turning the sum around and then swapping …ok so you could have eight times nine equal 72, but in that case you just swap the sum around and 72 divided by nine equals eight. (SDVS9, C, 1)

The student claimed that the basics of multiplication and division are the same, which is true to the extent that division can be defined as missing factors (Parker & Baldridge, 2003). The student explained what he meant by saying multiplication and division are the same when the teacher asked what he meant. The student attempts to describe the inverse relationship that exists between multiplication and division. But this idea is unclear from the language used by the student who talks about "turning the sum around" and "swap[ping] the sum around." In order for other students and the teacher to really understand the student's explanation of his thinking, the inverse nature of the operations needed to be highlighted. The teacher searched for it in the subsequent exchange:

T: Could you add anything else? If you kept going in that plan, going off the track here a little bit but…Yes?
S: There's a word to describe it, equivalent, because like…
T: Mmm, would it be equivalent?
S: No, not really
T: I know what you're thinking, and I can understand where you're coming from, I don't think equivalent is the right word though, because when we talk about equivalence, we're actually talking …
S: It's fractions
T: Well it mightn't necessarily just be fractions, but we're talking about things that are equal, aren't we? You couldn't really say that those two things are equal. They are related certainly. They have something in common. It's related as well, isn't it? And what about…They're four tables aren't they? (SDVS9, C, 1)

The element that would help to clarify the explanation and illuminate a mathematical idea for the class – the inverse nature of multiplication and division – eludes both students and teacher. Yet, for the purposes of this study the example underlines the mathematical work of following and evaluating a student explanation.

Interpreting student productions.

I am using this heading to describe three additional tasks of teaching that were observed in Ireland and which were among the tasks that informed the construct of MKT: appreciating a student's unconventionally expressed insight; interpreting and making pedagogical judgments about students' questions, solutions, problems, and insights; and hearing students flexibly. The first of these tasks is illustrated by the previous example above. The teacher needs to "appreciate a student's unconventionally expressed insight" (Ball, 2000, p. 245) or to "puzzle about the mathematics in a student's idea" (Ball & Bass, 2000a, p. 88) or to respond to an "unexpected student assertion or idea" (Ball, 1999, p. 34). The teacher recognized the need to explore the student's idea because the teacher asked the student what he meant about multiplication and division being almost the same and the teacher was prepared to go "off the track" a little bit to try and make sense of it. The teacher's handling of the exchange could be interpreted as helping the student to "develop, validate and justify" a mathematical claim (Ball, 1999, p. 28). By asking the student what he meant by saying that multiplication and division were almost the same, the teacher was helping the student to develop a mathematical claim. By asking the student if he could "add anything else" the teacher was encouraging the student to either justify or validate his claim but the student was unable to verbalize a justification of his claim in a mathematically precise way. This is a task of teaching I had not identified from my initial viewing of the Irish lessons but it emerged from the iterative process of determining how different or similar tasks of teaching observed in Irish lessons are to tasks that informed the development of MKT. An similar example from tasks that informed the development of the construct of MKT is one where a student in Deborah Ball's class claimed that the number six could be both odd and even (Ball, 1999).

Another task of teaching that informed the construct of MKT is "interpreting and making "pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual)" (Ball & Bass, 2003b, p. 11). In Ireland such a task was observed in a second class lesson where students were discussing three-dimensional shapes, especially properties of a sphere and a cylinder. Note in particular the contribution of the second student.

T: Why could you not stack the spheres on top of each other? What would happen? Why could you not stack spheres on top of each other? Alan?

St: They'll all roll down.

T: They'll all roll and they'll all fall down because they're not, you can't stack them. Excellent

St: If you had a little, like eh, thing, a flat thing…and there's another flat thing you could stack them like that.

T: Yeah. (SDVS3, C, 3)

The teacher asked the students why spheres cannot be stacked on top of spheres. Before waiting for a response she followed up with a second question and repeated the first. One student restated the problem that the spheres would all roll down. The teacher began to explain why but instead repeated part of the question noting that the spheres cannot be stacked. No reference was made to the curved surfaces on the spheres or to the presence of flat faces on a rectangular prism. One student, however, made a statement which used the word “flat.” The student was hesitant in what he said (judging by the irrelevant words “little,” “like” and “eh” and repeated use of the unspecified “thing”) but what he said held the seeds of explaining *why* the spheres cannot stack (because two flat surfaces are needed for stacking) and it had the potential to open a discussion about which shapes *have* flat surfaces because he referred to “another flat thing.” The sentence as uttered by the student was missing mathematical terms that even a student in second class could be expected to know such as “face” or “cuboid” (= rectangular prism) or shape or three-dimensional. Despite these shortcomings, the sentence was an attempt to respond to the teacher’s question and with some work by the teacher it had the potential to elicit rich discussion in the class. The teacher could have explored ideas such as asking the students to name shapes with flat surfaces; asking what category of shapes they are (three-dimensional or polyhedrons); naming the part of the surface that is flat; and discussing the applications of the properties of these shapes in the environment. The work of teaching involves recognizing the potential of such utterances by students, which may be inchoate, and mining them for relevant mathematics to advance students’ mathematical understanding and thinking.

Most of the tasks identified in Irish lessons were identified when doing the open coding of the videos and were refined as more lessons were viewed and analyzed. Additional tasks were included during the iterative process of looking for similarities and differences between Irish tasks and tasks that had informed the construct of MKT. One that was added to the Irish list was that of hearing students flexibly (Ball, 2000, p. 243). I did not identify it originally but when I read about it I could recall instances of it occurring. One was in a lesson where third class students had been asked how many times they could take eight sticks from twenty-nine sticks. The class agreed that the answer is three, remainder five. But one student raised his hand to contribute and when invited to speak, the following exchange occurred.

> S: Eh the way that we done it we counted si…we put them altogether and then we counted six off…and then we put two sticks back on from that five…
> T: To make the…what?
> S: To make the eight …
> T: Good boy
> S: …and then we put it with the other three….
> T: Where did the other three come from?
> S: The three bunches
> T: Oh yes he has it. There were three bundles weren't there? Well done. What was your remainder?
> S: Five.
> T: Boys that's absolutely super. (SDVS5, C, 4)

Despite having watched this clip several times and having read the transcript carefully I am not sure what the students did. It may be that they put all twenty-nine sticks together and then took away three groups of six and then took away another three groups of two and put each group of two with each group of six, resulting in three groups of eight and five loose sticks. If that is what happened, it does not make sense that they would have put two sticks back on "from that five" because they would have had to do this three times. Perhaps they mean to say from that "eleven" because that is how many sticks would have been left after taking away three groups of six. Despite initial skepticism about the approach, however, the teacher seems fully convinced by the end of the exchange that what the students did was correct. What is of interest here is not whether or not the teacher fully grasped what the students were saying, but that when teaching, one task is to be able to hear students flexibly.

Comparing different solution strategies and solutions.

Another task of teaching which informed the development of the U.S. construct of MKT is comparing different solution strategies and solutions. This may involve "sizing up the validity of a child's non-standard procedure" (Ball, 2002a, p. 4), "making sense of methods and solutions different from one's own" (Ball & Bass, 2003b, p. 13) and asking "what, if any, is the method, and will it work for all cases?" (p. 6-7). Above all the teacher needs to be able to "think about things in ways other than their own" (Ball, 2000, p. 243). In my experience with Irish teachers and prospective teachers I find many are skeptical about using alternative or invented algorithms. One teacher in the video study, however, did encourage her students to discuss different ways to solve a problem. The teacher was working with a group of her students to find how much ¼ kg of mushrooms would cost if the price was €0.62 per 100g. One student who was asked to solve it suggested multiplying €0.62 by two and then finding half of €0.62. The teacher commented, "there are a number of ways, why did you choose that?" to which the student replied "'cause … one hundred grams is sixty two cents, so look for two hundred and fifty so you… two and a half, so you want half of that." The teacher asked if the students could think of another way of working it out and one student suggested dividing €0.62 by four and multiplying the answer by ten. This method was based on knowing that one quarter of 100g is 25g and that 25g is one tenth of 250g. The teacher then elicited a third method, which involved finding what a kilo of mushrooms costs by multiplying €0.62 by ten and dividing the answer by four. The teacher concluded that "there's three ways of doing it" (SDVS4, C, 8/9).

Responding to students.

Teachers respond to students in many ways when they teach and the construct of MKT was informed by several such tasks. I include two of them under this heading: responding "productively to students mathematical questions and curiosities" (Ball & Bass, 2003b, p. 11) and helping students who are stuck. I observed a sixth-class teacher responding to a student's mathematical question in a lesson about dividing whole numbers by unit fractions. The student had noticed something about the answers which prompted him to ask the following question:

S: You know like the answer, is it always like a whole number? It's never like twenty-one and a half?

T: Well you answer me? If it's a unit fraction is it always going to be a whole number?

[Interruption to caution two students about behavior]

S: Say if it was three divided by six sevenths or something, it wouldn't be a whole number.

T: It wouldn't be a unit fraction then would it? It wouldn't be a fraction with one on top.

S: It would be…

T: It would change, obviously, if you had a number greater than one on top as a numerator. But that's not what you're doing here. Ok? (SDVS9, C, 6)

The teacher did not answer the student directly but asked the student to look at the pattern. The teacher's response helped refine the student's question by drawing attention to the fact that the fraction divisors were all unit fractions. The teacher pointed out that if the divisors were not unit fractions the answers would not always be whole numbers. But the student asked if the answer is *always* a whole number. The teacher did not seem to resolve this question for the student. This could have been done by referring to the student's prior knowledge about fractions that there is always *x* number of $\frac{1}{x}$ in one and that therefore, when a whole number is divided by a unit fraction the response will always be a whole number multiple of x. Responding productively to student questions is another task of teaching common to both countries.

Sometimes teachers need to “remobilize” students who become stuck (Ball, 2000, p. 243). When helping students who are stuck the teacher might build on something students know how to do, or might remind students what a question is asking. A third-class teacher helped students who became stuck when working on the following problem: “The milkman had 18 bottles of milk in his van. He delivered the same number of bottles to each of four houses. How many were left in the van?” (SDVS5, C, 12). The teacher made the following statement to the class:

> Now remember the question. The boys I’d looked at and checked, they’d forgotten the question. The question is “how many were left in the van?” So you need to find out how many went to each house…and how many were left in the van. Did you do it with the sticks? This boy did it with the sticks. (SDVS5, C, 12)

The teacher helped students who were stuck by emphasizing what they needed to figure out. Students had possibly answered ‘four’, the number of bottles of milk delivered to each house rather than the number left in the van which was the required answer. In addition to clarifying the question the teacher reminds the students that they might use sticks to help them. A sixth-class teacher gave two similar prompts to a student who was working on a word problem and who said he “can’t understand it.” The teacher responds:

> Did you read it again? And? Nothing clicking with you at all? Michael … did you try drawing it? (SDVS9, C)

The student is encouraged to draw the problem because reading it a second time did not help. The teacher seems to think that the student will be able to draw a picture of the problem and that that will lead to understanding.

I take a moment here to recap the purpose of describing tasks of teaching observed in Irish lessons in this chapter. My ultimate goal in the study is to use measures based on the U.S. construct of MKT to study Irish teachers’ mathematical knowledge. First, however, I want to determine if sufficient similarity is present between the work of teaching observed in Ireland and conceptions of the work of teaching that informed the development of the construct in the United States. If so, the items can likely be used to measure knowledge related to the work of teaching in Ireland. In this section I am describing tasks of teaching from Irish classrooms that are similar to those on which MKT is based. I use examples from Irish lessons to illustrate each task of teaching to ensure that what is being described in the Irish lessons concurs with the task description in the U.S. literature or in the U.S. measures of MKT. A final reminder at this point is that in describing specific examples of tasks, my interest is on what the teacher was doing or attempting to do and not in appraising how the teacher was carrying out the task.

Anticipating student difficulties.

Teachers need to be able to anticipate difficulties students are likely to have when doing a particular problem (Ball, 2000). They should be able to order problems according to difficulty for students (Form B_01, item 23). I observed teachers conducting this task in Ireland. One potentially confusing idea for students is calculating how long it takes a train to travel from Destination A to Destination B, if it leaves destination A at 07:35 and arrives in Destination B at 10:23. Students may come up with the following answers:

$$
\begin{array}{rcrr}
 & 9 & & \\
 & \cancel{10} & : & {}^{11}\cancel{2}\;{}^{1}3 \\
- & 7 & : & 3\;5 \\
\hline
 & 2 & : & 8\;8
\end{array}
$$

$$
\begin{array}{rcrr}
 & 9 & & {}^{7}8\;{}^{1}3 \\
 & \cancel{10} & : & \cancel{2}\;\cancel{3} \\
- & 7 & : & 3\;5 \\
\hline
 & 2 & : & 4\;8
\end{array}
$$

In the lower example, the student realizes that it is not possible to take 35 minutes from 23 minutes one hour is renamed as 60 minutes giving a total of 83 minutes from which to take 35. This will give the correct answer. But, in the top example the student has over-generalized from subtracting in the base ten system and renames an hour as 100 minutes, so that 35 is taken from 123 minutes instead of 83. Awareness of the possibility of such a mistake prompted one teacher to ask a student "Working with time, what is the alarm bell Javier if you are adding and subtracting the time, where do you start getting worried?" The teacher later cautions the students to "watch when you are doing your regrouping. Sixty minutes is not like the hundreds, tens" (SDVS6, C, 3).

Connecting number patterns and procedures.

Asking students to search for patterns among numbers is another task of teaching identified in the U.S. MKT items (B_01, item 13). Irish students were encouraged to look for patterns; in the particular instance observed, identifying the pattern was directed towards learning a procedure. A sixth-class teacher asked students to draw diagrams to help them figure out the answers to the following questions:

$$3 \div \frac{1}{4} \qquad 2 \div \frac{1}{4} \qquad 2 \div \frac{1}{3} \qquad 3 \div \frac{1}{2}$$

The answer to each question was recorded on the board. When the students had worked on the four problems the following discussion developed:

T: Well, just what have you noticed, what's the pattern that you've noticed?
S: They're multiplying by the two top ones all the time, except if you put it into twos it'll change. If you put it into like…
T: So, what's the pattern that you've noticed here? Forget about the top one for a second.
S: So, three times four equals twelve. Two times four equals eight, two times three equals six, and three times two equals six.
T: So, if you multiply the whole number by the number …what do we call that?
SS: The denominator
T: By the denominator, of the fraction we're dividing by...
S: It would change because if you put …
T: Grand well, we'll get to that, but just for these ones; ok stick with what's there ok, in front of us. Absolutely dead on. Okay.
S: Pretty easy.
T: Ok… It's what?
S: Pretty easy.
T: Do you get that? Yes? (SDVS9, C, 4)

The teacher first asks the students what pattern they have noticed. The conversation is threaded with one student who wants to point out that if the numerator was a number other than one, the pattern would be different. Although this is true the teacher repeatedly directed students' attention back to division of whole numbers by unit fractions. It is difficult to understand what point the first student who contributed is making when he refers to "the two top ones." Given the context of the rest of the conversation he seems to be referring to how different numerators would affect the pattern but that interpretation is difficult to make from the first statement alone. The teacher directed the students' attention to the pattern and a student identified a pattern by making multiplication statements about each calculation where the whole number was the multiplier and the denominator of the fraction was the multiplicand. The teacher began to formulate the pattern, asking the students to contribute the term "denominator." When one student again raised an exception, the teacher brought the focus back to "what's…in front of us." One student commented that this was "pretty easy."

Assessing if procedures generalize.

Judging whether or not a procedure can be generalized was part of the work of teaching considered when developing the U.S. construct of MKT (Ball & Bass, 2003b). The task was observed in Ireland in conjunction with the algorithm mentioned above where students learned to divide whole numbers by unit fractions. When the students seemed to have understood the procedure the teacher moved onto another task of teaching, enabling students to check if a procedure worked (a) in a specific case and (b) in general. The teacher asked students to check if the "rule" (i.e. multiply the dividend by the denominator in the divisor) worked for 2 ÷ 1/5. The teacher then drew two rectangles on the board and partitioned each one in fifths. He counted the number of fifths in two rectangles and asked the students to calculate the answer using the rule. The teacher then opened another conversation with the students to generalize about the rule:

T: So can we, do you think that's something that we could say is going to happen right across the board? Why not?
S: Because you're going to have to put the numerator as well, into it as well. If it's two then it's going to be a whole different thing.
T: Ok, but if it was one?
S: Oh well yeah it would be the exact same
T: Are you happy that…?
S: If it was two you'd just…
T: But if it was one I'm asking? If the fraction always had a one on top, would you be happy with…?
S: I think I know how to draw…
T: Ok, I'll get back to you on that one, is that ok? (SDVS9, C, 4)

In his opening question the teacher asked the students if the rule would apply in every situation. One student again pointed out that it would be different if the numerator was any number other than one. The teacher attempted to get agreement that the rule applied in circumstances where the numerator was one. One student agreed that "yeah, it would be the exact same." The problem is that this agreement was sought by the teacher and given by the student without a mathematical explanation for why it would always work. It is possible to imagine a general explanation of $y \div \frac{1}{x}$ along the lines that in each unit of y (i.e. $\frac{y}{y}$) there are *x* $\frac{1}{x}$s. Therefore, you can find how many $\frac{1}{x}$s in y by multiplying *x* by *y*. The teacher has shown the students that the algorithm works in a specific case and he set out to show that it works in general. He agreed to return to extending the algorithm to numbers where the divisor's numerator is greater than one.

The missing element of explanation illustrated the mathematical demands that such a task places on a teacher's knowledge.

Using concrete materials and visual aids.

When Ball and Bass and their colleagues were developing the construct of MKT they envisaged that using concrete materials and visual aids was part of the work of teaching. I observed two related tasks in the Irish lessons: explaining inadequacies in concrete materials and visual aids, and drawing shapes on the board. Irish teachers are encouraged to use manipulative materials when teaching mathematics (Government of Ireland, 1999b). One task that informed the thinking about MKT was making "judgments about the mathematical quality of instructional materials and modify[ing] them as necessary" (Ball & Bass, 2003b, p. 11). In several Irish lessons I observed teachers explaining (or needing to explain) inadequacies in materials or drawings used in lessons. In one lesson the teacher used large polygons to teach about the properties of two-dimensional shapes.[58] However the equipment was intended for use in physical education lessons rather than in mathematics class. Each shape had a hole in it (see Figure 4.5) and the teacher commented that "I want you to imagine that there is no hole; that it's covered in, ok. So, if we imagine that this is all red, because it's made from plastic" (SDVS10, C, 1). As the lesson progresses, sometimes it is unclear whether the square being referred to is the internal square (white in Figure 4.5) or the external one (red and white). This matters because the side lengths and location of the angles differ depending on which square students look at.

[58] These were not included as representations because the shapes were used in lessons *as* polygons rather than representations of something else. However, a flat picture of a three-dimensional shape (which is included below) is a representation and could have been included in the earlier section on representations.

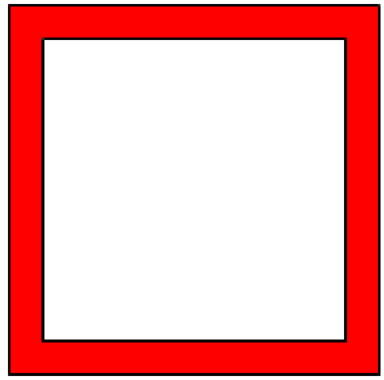

Figure 4.5. Shape used by teacher to discuss properties of a square.

In another lesson a teacher asked students to count how many edges met at a given vertex in a triangular prism. The teacher gave one student a triangular prism made from materials similar to “Frameworks” from Polydron™ to check the answer. The student said that four edges met at each vertex. When the teacher checked this answer she realized that how the triangular prism was constructed using these materials – with two different colors on one edge – made it appear to the student that there was an additional edge meeting at one vertex (SDVS4, C, 5). The teacher quickly identified the cause of the student’s misunderstanding because she was familiar with the teaching materials. In another lesson a teacher used an interactive whiteboard to show examples of spheres. The problem with using a two-dimensional medium to show three-dimensional shapes is that the drawing can never be a sphere. It can be only a representation of a sphere. Using such representations makes it more difficult to examine the properties of shapes and can lead to some misconceptions or difficult-to-check hypotheses such as the student who thought half a lemon was “a sphere cut in half” (SDVS3, C, 3). Shortcomings in materials place additional demands on teachers’ knowledge because they have to judge whether to use inadequate equipment and if they use it to compensate for inadequacies in the materials.

Another task of teaching is to draw mathematical figures and representations on the board. In one lesson a teacher commented about parallel lines that “if I drew them straight they wouldn’t [ever meet]” (SDVS1, C, 2). This teacher’s comment to her students underlines how difficult it is to draw parallel lines, often without appropriate equipment, in the generally demanding environment of a classroom. Drawing shapes on the board requires mathematical knowledge so that they are accurate and suitable for the purposes for which they are intended. For example, a circle drawn freehand may not adequately display a circle’s symmetry or a hexagon intended to be regular may be irregular. This task places demands on a teacher’s knowledge of equipment that can be used to draw mathematical figures, often without advance notice, on the board.

Selecting useful examples.

Choosing useful examples is another part of the mathematical work of teaching that informed the construct of MKT (Ball, Hill et al., 2005). This task is easy to overlook when observing a lesson. It is almost impossible for a teacher to teach a mathematics lesson without choosing some numbers or geometric examples. The examples may be devised by the teacher or taken from a textbook or another source. But examples chosen are so integral to a lesson that they are usually apparent only when an example is poorly chosen or when a particularly apt one is used. In one lesson (not one of the ten at the center of this chapter) a teacher was teaching students about the order of operations where convention dictates that calculations in parentheses precede multiplication and division, which precede addition and subtraction. In Ireland this is often summarized by the mnemonic BOMDAS where B stands for brackets, O for of (as in one quarter of x), M for multiplication, D for division, A for addition and S for subtraction. The mnemonic is somewhat misleading in that by convention multiplication and division are equal in priority to each other as are addition and subtraction. When operations requiring operations of equal precedence appear together, they are conventionally carried out in order from left to right.

In one Irish lesson, a problem from a textbook was 37 – 56 + 28. According to the curriculum statement, students at this class level could be expected to have met negative numbers in context (e.g. golf, sea level or temperature) or to have added "simple positive and negative numbers on the number line", with exemplars ranging from -9 to +5 (Government of Ireland, 1999a, p. 94). When students are asked to apply the BOMDAS rule to this calculation they may begin by attempting to subtracting 56 from 37 but this may be too difficult for them. They may then add 56 and 28 and get 84 but they are still left with the problem of how to subtract 84 from 37. A student may then run into a problem and call on the teacher for help. What does a teacher need to know to help such a student? The teacher needs to know that this number sentence is the same as 37 + (-56) + 28. This means that if a student is not able to add 37 and -56, the student can first add 37 and 28 and then add -56 to (or simply subtract 56 from) the sum to get the correct answer of 9.

This example is problematic in a number of ways. First, because it uses only addition and subtraction, which have equal priority of operation, it is not a good example to show order of operation. Second, given how the calculation is written, students may with some justification interpret the calculation to be 37 – (56 + 28). Third, for students who are not confident in adding negative numbers, the logical way to do this problem is as (37 + 28) – 56 but this means that they must know how to apply the associative property in order to solve the problem. A teacher who unwittingly chose to use such an example in a lesson needs to possess substantial mathematical knowledge to work through it with students.

An example from a geometry lesson where a teacher was careful about the examples of triangles chosen is evident in the following exchange. The teacher asked a student how many sides on an equilateral triangle.

> T: But how many sides are there? Clara?
> S: Three
> T: Three sides exactly, ok, now does a triangle have to be, do all the sides have to be equal?
> S: No
> T: No because we see lots of different shapes of triangles don't we. We often see a lot of different types of triangles. Ok and if you just turn and face the white board for two seconds, I'm just going to draw up some shapes and I want you to tell me if they are triangles or not. (SDVS10, C, 3)

It seems likely that this teacher knew that students can easily develop a stereotypical idea of what a triangle is by seeing examples only of equilateral triangles and she drew some different types of triangles on the board to counter the stereotype. By deliberately choosing examples in this way the teacher is engaging in another task of mathematics teaching, namely connecting what students are currently learning with mathematical work they will do in the future. What the teacher did not do is introduce any counterexamples – shapes that students might think are triangles but which are not. Choosing counterexamples is another aspect of the work of teaching identified in conceptualizing MKT (Ball, 1999).

Presenting estimation strategies.

Recognizing students' estimation strategies is a task of teaching on which MKT is based (Form B_01, item 14) and a related task of teaching, presenting estimation strategies, was observed in Ireland. One teacher asked students to find the average of 47, 43, 44 and 46 "without adding them all up and dividing by the number of numbers you added." A student responded "45," "because 47 is two over, 43 is two under …and then 44 is one under." The teacher suggested that the students could

> Picture them on the number line and if you have your forty-seven where it is and your forty-three they are both a certain distance away, which would match if the middle number was going to be your average. Move the next in, forty-four and forty-six and they are all converging on to your forty-five. So, you could nearly imagine them on a number line and get an average. Would that be enough of a way for getting a definite average or would you have to do the adding all up and the dividing? (SDVS6, C, 2)

Although the teacher asks the students to picture the numbers on a number line she does not relate her explanation to an actual number line in the classroom. For some students a visual representation of a number line may have helped them to follow the teacher's description of how the numbers "converge" onto forty-five. The teacher subsequently points out that although estimating can help, in order to find "definite" answers students need to check them by doing "the adding all up and the dividing." Encouraging students to estimate their answers and presenting estimation strategies is both a task of teaching in Ireland and a task that informed the development of MKT.

Using and eliciting mathematical language.

The use of mathematics terms is an important part of expressing mathematical ideas and I identified some tasks of teaching where teachers needed to attend to mathematical terms. First, teachers need to use correct and appropriate mathematical terms. Several terms are used to convey concepts and to communicate generally in mathematics classes; one teacher asked a student to go to the back of the class and to bring her some spheres (SDVS3, C, 2) and another teacher identified two parallel lines as "the two horizontal ones" (SDVS1, C, 1). Sometimes the specific mathematical meanings of terms are not grasped by students as the following example illustrates. The teacher asked a student to tell her which sides of a rectangle are parallel:

T: Let me see, Sinéad do you spot two others that are parallel?
St: Ah….
T: Do you want to just tell me? So that one and that one are parallel because they go like that.
St: Ah…. the sides?
T: This one and this one [points to adjacent sides]?
St: No, the *sides*
T: The two sides, very good. They're parallel. Will they ever meet?
Sts: No. (SDVS1, C, 2)

The student identifies the "sides" as parallel but when the teacher points to the right vertical side and the lower horizontal side of the rectangle, it becomes clear that the student considers "sides" to refer to only the left and right vertical sides. In this case it seems that the student is using a meaning of side from everyday life when we talk about the side of a building and use expressions such as the "left side" and the "right side." In mathematics, however, all four lines enclosing a rectangle are called sides. By pointing to the adjacent sides the teacher has attempted to challenge the student's limited understanding of sides but the teacher did not pursue the idea further at the time.

Teachers need to be careful in their use of general language to discuss mathematical ideas. In one lesson students used base ten materials (consisting of small cubes, longs, flats and large cubes, each 10 times the size of the previous one) to regroup ten small cubes into one long. In the following extract the teacher discussed how the class began an activity with 12 small cubes but concluded the activity with one long and two small cubes.

T: Who remembers, how many units did we have altogether at the start? Maurice?
S: Twelve.
T: We had twelve units. We had? Twelve units. And how many units did we take away? Tony.
S: Uh. Uh, ten.
T: We took away ten. And we swapped it for a? Long. And how many units were left over?
Ss: Two.
T: And look at your picture on your board. What does your picture say?
S: Twelve.
T: Twelve units. (SDVS7, C)

The central mathematical concept for students to learn by doing this activity is that one ten and two units is an equivalent quantity to twelve individual units. Moreover, the one ten and two units is a more compact way of representing the same quantity. The problem is that the teacher here talks about *taking away* ten small cubes and swapping them for a long. Using the term "swapping" is useful but when the teacher uses the phrase "take away" students may confuse the idea of swapping with the idea of subtraction. Thinking about taking away may interfere with the idea that the two quantities – one long and two small cubes, and twelve small cubes – are equivalent. The previous two examples seem close to a task that informed the development of MKT where "judgments [are made] about how to define terms and whether to permit informal language" (Ball, Hill et al., 2005, p. 21). In the first example a student needed to move from an informal meaning of sides to a mathematical meaning and in the second the teacher needed to be consistent in talking about exchanging materials rather than taking them away.

In addition to the teacher's own use of mathematical terms, Irish teachers elicited terms from students. Examples included a sixth-class teacher eliciting numerator and denominator as the "technical terms" for the numbers above and below the line in fractions (SDVS9, C, 9); and a senior infants teacher eliciting zero as "the other word for none" (SDVS8, C, 1). By relating known terms or ideas to new terms, teachers are scaffolding the use of mathematical language (Ball & Bass, 2003a). Sometimes, when eliciting a term the teacher may elicit alternative terms that make demands on the teacher's mathematical knowledge, as in the following instance. The teacher and students were discussing the net[59] of a shape and the teacher asked for synonyms for a net, in the context of a rectangular prism net and the students responded:

S: a blue print
T: a blue print, another word?
S: a plan
T: a plan, good.
S: a drawing
T: a drawing, anything else? Could you call it a pattern?
S: yeah
T: yeah, of course you could. (SDVS4, C, 7)

The teacher was faced with a difficulty here. Students had several alternatives to the word "net," each of them describing an aspect of a net but none providing an exact synonym. Acceptable alternatives would have been a "development" or a "pattern" (Buekenhout & Parker, 1998). The latter is provided by the teacher. Eliciting mathematical terms from students and providing synonyms is another aspect of the work of teaching in both countries.

In addition to providing synonyms, teachers in the Irish lessons defined or explained mathematical terms. Teachers both sourced definitions from a textbook and devised their own definitions. In some cases a term was explained using words, pictures or representations. One second-class teacher observed used words and a rectangle shape to explain the meaning of parallel:

> What parallel means is that two lines are running beside each other but they will never meet, can you see the way these two lines run straight up, ok, they go straight and they are never going to meet because they will keep going straight, ok? The same with these two sides, see, they are going straight beside each other but they'll never meet. (SDVS10, C, 2)

Students may have difficulty understanding how lines on a rectangle run beside each other or they may wonder about the relevance of the lines "never meet[ing]" if they do not meet now. A mathematician may worry that the teacher did not refer to the equidistance between the parallel lines. The task for the teacher is to make the definition both comprehensible to the students and mathematically precise and complete. The task of giving "mathematically appropriate and comprehensible definitions" was considered in developing the construct of MKT (Ball & Bass, 2003b, p. 11).

[59] A flat surface that when folded encloses a 3-D shape

In the next exchange a teacher referred a student to a mathematical definition to help the student respond to a question. The teacher asked a student why a rectangle is not a regular shape.

T: Why is it not regular Carol?
St: Because the usual rectangle looks like that.
T: No, let's read what a regular and an irregular shape is.
St: [Reads] This is a regular shape. All the sides are the same length. All the angles are the same size. This is an irregular shape. The sides are not all the same length. The angles are not all the same size. (SDVS1, C, 6)

The teacher directs the student to a definition to help her respond to the question. The work of teaching here is knowing where to find precise definitions and knowing how to apply them to concepts that students encounter at elementary school level.

Just as teachers sometimes elicited terms from students rather than providing them for students, I observed Irish teachers eliciting meanings of mathematical terms. In so doing they are helping students develop mathematical definitions (Ball, 1999). Sometimes this was informal and incidental where a teacher remarked to the class "I'm going to give you a pair of dice. Hands up. How many dice am I going to give you?" In this case the teacher was searching for the meaning of the word "pair." In another case the sixth-grade teacher was working one-to-one with a student who was trying to calculate the average age of four students whose ages were given in years and months.

T: So, what would you say the average age there is? You have got an eleven-ten months, a twelve-three months, a ten-nine months and a ten-six months. So they'd all balance off to what? It is only an estimate, only a guess. You can't be wrong really.
S: Around twenty-four or so.
T: That's when you add them all up. I'd be asking you then "what's their total ages?" But I'm asking, "what's their average age?" What does average mean?
S: All together average.
T: That would be total. Average means something different.
S: You add the answer and divide the answer by the number
T: Yeah, that's how you get it but what is average? If the climate people said the average rainfall in the month of June was desperate, what are they really telling you about each day?
S: The rain is [*inaudible*] all the time.
T: All the time. So, it is a roughly – one day you got 10, the next day you got 3. So it all averaged out at what? What would 10 and 3 balance down to? [Teacher places one hand high the other low and brings them together slowly]. [A student from another class enters and gives a message to the teacher]. Go raibh maith agat [= Thank you].
S: [*Inaudible*]

T: [Teacher repeats action with hands] Where would 10 and 3 meet for an average? If you had ten there and three down here, rainfall amounts let's say?
S: Between.
T: Yeah. Whereabouts would they average off at? What number
S: 5.
T: Yeah in around there. So, looking at their ages, where would they average off to?
S: Eleven, because twelve and ten in between.
T: Okay so a twelve and a ten would cancel each other and they would meet at eleven. And then you're left an eleven and a ten. And then you'd probably look at the months and decide. But eleven is a good estimate of an average. So, you can prove it now by adding them all up and dividing. (SDVS6, C, 11)

This exchange begins with the teacher asking the student to estimate the average age of children whose ages range from 10 years and 6 months to 12 years and 3 months. When the student estimates "twenty-four or so" the teacher recognizes the misunderstanding. She says to the student that his answer would be the sum of all four ages. This makes little sense because the combined ages would be closer to double that number. Perhaps the teacher wanted to emphasize that the average estimated by the student is too high because she says that all four ages would be the total ages and she differentiates the total from the average. When she asks the student what the average is he describes the total before describing how to compute the average. The teacher is not satisfied with this and she differentiates "how you get it" (which the student seems to know) with what average is (which the student does not seem to know). The teacher then used an example of rainfall. Initially she referred to the average rainfall in a month being "desperate" which is not a mathematical expression of average. Subsequently she specified two quantities and asks where they would "meet for an average." When the student gave a reasonable estimate, the teacher returned to the problem of the children's ages before asking the student to work out the average. This discussion highlights many aspects of the work teachers do to elicit meanings of terms from students: recognize that a student does not understand a term, figure out what the student thinks the term means, differentiate between understanding the term and knowing a procedure related to the term, choose an example to explain it, choose a representation or words to communicate the meaning, select and use appropriate (mathematical and comprehensible) terms and definitions of terms, scaffold the use of language, and judge when to permit informal language.

Attending to concerns for equity.

A conception of the work of teaching that contributed to the conceptualization of MKT is attending to concerns for equity (Ball, Hill et al., 2005) or helping all students to learn (Ball, 2000). This has long been a task of teaching in Ireland, even if it has not been recognized as such. There have always been disadvantaged students and members of the travelling community in schools who required additional support in learning mathematics (as well as other subjects). This issue has received a good deal of attention from teachers since 2000 because of the arrival of children from several other countries to live and attend school in Ireland. The primary teachers' union, the Irish National Teachers' Organisation published a booklet in 2005 entitled *Newcomer Children in the Primary Education System* in response to this change in the demographics of students in Irish schools. The effect is that supporting all students remains a core task of teaching and it now has greater prominence and greater complexity than it had a decade ago. As described in the U.S. literature the task is quite general and could relate to many of the tasks already mentioned. For example, in one example referred to above a student made the following remark in response to a question about why spheres cannot be stacked: "If you had a little, like eh, thing, a flat thing…and there's another flat thing you could stack them like that " (SDVS3, C, 3). This student attends a school that was designated as disadvantaged and a teacher can attend to equity issues in response to such a comment by supporting the student in using mathematical language more precisely.

Another task observed in Irish lessons which may help attend to equity issues was connecting the mathematics to a skill for living. Mathematical competence is required for many activities in our society from estimating measures when baking to keeping appointments. Mathematical skills are essential when balancing a family budget. Therefore, one task of teaching is to help students connect school work to a skill for living. One teacher had completed working on a problem which asked if it was cheaper to buy potatoes loose by the kilo or to buy a 10kg bag. The teacher then made the following comment:

> Generally children, as a matter of interest when you are shopping, stuff that's bagged for you is generally more expensive than loose. Something to watch out for. I often do that when I'm doing shopping, I look at the label under the loose ones and I look at the price of the bagged ones and almost always it's cheaper to buy the loose ones, unless there's a special offer, in which case … (SDVS4, C, 10)

Connecting the problem in the textbook to this real situation makes demands on a teacher's knowledge. The teacher used language commonly used in the context of shopping: more expensive, cheaper, price, special offer. She told them how to go about comparing prices by looking at the relevant price labels. I include these tasks – responding to a mathematical comment from a student and connecting a mathematics problem to a skill for living – under the heading of equity issues because they offer particular support to students who have been failed by the Irish education system in the past: students who have fewer opportunities than others to develop their mathematical language outside school and those who may leave school early. The task of attending to equity spans many of the other tasks listed in this chapter (Ball, Goffney, & Bass, 2005) but including it as a specific task is a reminder of its importance in any description of the work of mathematics teaching.

Connecting ideas to future mathematical work.

Another task of teaching that informed the development of MKT is anticipating "how mathematical ideas change and grow" (Ball & Bass, 2003b, p. 12) and in the Irish lessons I identified this as connecting the topic currently being taught to material students will work on in the future. If students learn primitive rules of mathematics in primary school they will likely have to revise them in their future study of mathematics. For example, students are often told that they cannot take seven from two or that multiplying makes a number bigger and dividing makes a number smaller. But teachers need to know that seven from two is negative five; multiplying by the identity element, one, does not change the value of a number; multiplying a non-zero whole number by zero or a fraction makes the number smaller. Similarly, dividing any number by one does not result in a smaller answer and dividing two by a quarter yields a quotient larger than both dividend and divisor. In a second-class lesson about triangles a teacher was explicit about the students' future learning. She told her students that

> When you get into older classes in the senior school you'll learn all about triangles, they all have different special names, all the different types of them, so you'll learn all about that when you are in the senior school (SDVS10, C, 4).

In this example the teacher informs her students about something they will learn in the future and by drawing examples of different triangles (without classifying them) on the board, she relates students' future learning to current learning.

Connecting mathematics with the students' environment.

Another mathematical task of teaching observed in Ireland is applying ideas encountered in school to students' environment in and out of school. In some cases a teacher responded to students' attempts to apply mathematics and sometimes the teacher made the application directly for the student. One teacher used such an application when a student claimed that 0.25 litres = 25 millilitres. The teacher commented:

> Now we have a few problems here with this one. Nought point two five, is a quarter, isn't it? What've you written? Twenty-five. There's a huge difference between having twenty-five milliliters and two hundred and fifty milliliters. Isn't there? Two hundred and fifty is the size of that Amigo™ [teacher points to a soft drink container]. All right? Twenty-five would be, you know the, you know Calpol™ [= a brand of children's medicine sold in Ireland]. You know the little spoons you have for medicine. (SDVS2, C, 7)

In order to emphasize the difference between the two measurements, the teacher uses examples with which the student is likely to be familiar. In order to do this the teacher needs to be familiar with measurement benchmarks in the students' environment. A teacher often has to make decisions quickly about whether or not examples of shapes in the environment are legitimate. In one lesson, after hearing there are no corners on a circle a student commented that the shape of the circle is "telling us the answer. It's zero." The teacher responded that "oh yeah it is, yeah it looks like a big zero, yeah very good, excellent" (SDVS10, C, 5). Some might argue, however, that in many instances a "0" is closer to an oval shape than to a circle. Similar mathematical judgment calls had to be made by a teacher when castanets and bongos were suggested as examples of cylinders and an overhead projector was suggested as an example of a cuboid (rectangular prism). This task was described in literature about MKT as connecting "content to contexts effectively" (Ball, 2000).

Tasks Identified Only in the Irish Sample of Lessons

Although many tasks of teaching identified in the sample of Irish lessons were similar to tasks found in the MKT literature and items, other tasks did not map so easily to MKT tasks using the data available. The tasks observed in the Irish lessons but not recorded in MKT literature or in the MKT items studied are (a) eliciting steps and meanings of procedures, (b) planning and recording teaching, (c) documenting content taught, (d) collecting data from students (for use in lessons about data), (e) comparing ways of representing data, (f) identifying and describing generic problem solving strategies, (g) presenting number facts to students, (h) documenting student progress in mathematics. Although the tasks described here were observed in the Irish lessons and were not identified explicitly in the MKT literature or in the MKT items, it is still possible that they were included in the conceptualization of MKT.

Teachers elicit from students steps and procedures that help the students to complete mathematical tasks. In Irish lessons eliciting steps and meanings of procedures arose frequently in the context of measurement activities. I draw on two examples from Irish classrooms to illustrate this task. In the first example a teacher is eliciting a procedure for reducing fractions and it is done in the context of expressing quantities of milliliters as fractions of liters. The discussion proceeded as follows in the lesson:

T: A thousand millilitres equals one litre. [Teacher writes 1000ml = 1 l on board]. So write them as a fraction of one litre. Now, on the first one they're asking us what? Write five hundred millilitres as a fraction of one litre. What do we do? Carol?
S: Write five hundred over a thousand?
T: Write five hundred over a thousand? Jonathan, would you agree?
S: Yeah.
T: Yeah? Now what do we do then, do we leave it like that?
S: Yeah.
T: Yeah?
SS: No. Break it down.
T: Break down. Okay, _____, break it down please, what do you do?
S: Cross the two zeros off on five hundred and a thousand.
T: Okay, what have you done though?
S: Broke it down to five tenths.
T: Yes, but what did you do? Divided by what?
S: Hundred.
T: Divided by? A hundred, yes. So now five tenths is the same as?
S: A half.
T: A half. So what's five hundred millilitres written as …?
S: A half.
T: a fraction of one litre?
S: A half.
T: A half. So your answer there is a half of a litre. Right? All right so half a litre. (SDVS2, C, 5)

The teacher began the exchange by stating and recording on the board the necessary background information that a liter is equivalent to one thousand milliliters. The teacher then called out the specific problem to be solved. The subsequent exchange alternated between the teacher and various students who together described the stages, and meanings of stages for expressing 500 milliliters as a fraction of a liter. Although most of the information came from the students in this case, the teacher's questions were needed to elicit the information. This is particularly evident as the students move from talking about crossing “the two zeros off on five hundred and a thousand” to saying that they are dividing by one hundred.

In another example a teacher discussed a similar procedure applying it to weight. One teacher worked with her students to find the price of various quantities of fruit and vegetables when the price of a given weight is known. When she asked one student how to find the price of four hundred grams if the price of a kilo was known the following exchange took place:

> T: I’m not looking for the answer Ethna, I’m looking for the process. How will you do it?
> S: Miss, I don’t know. Could you do it as a fraction?
> T: You can tell me what you think you’d do.
> S: Emm, <u>put four hundred over</u> a kilo like it was <u>a thousand</u>
> T: Yeah
> S: And cross out two zeros
> T: Ok so you’d divide by ….., [teacher writes 1000g under the 400g that is already written on the board and puts a line between them to show the fraction $\frac{400g}{1000g}$] putting it like that is it love?
> S: Yeah
> T: Ok. You’re dividing by what?
> S: <u>Cross out the two zeros</u>
> T: By, <u>you’re dividing by what?</u>
> S: T, two
> T: By? You’re dividing by?
> S: Four.
> T: No, you’re dividing by
> S: [Inaudible]
> T: What goes in there four times? You said cross out the two zeros. What are you dividing by?
> S: I don’t know.
> T: No you don’t know, you’re confused. What was she dividing by Damien?
> S: Eh, <u>one hundred</u>
> T: One hundred, Ethna. You were saying a <u>hundred into four hundred goes four time, times. A hundred into a thousand goes ten times</u>. Okay, what will you do now Ethna?
> S: <u>Reduce it</u>
> T: You reduce it to?
> S: <u>Two fifths</u>
> T: Two fifths or you could leave it at four tenths. So, you’re actually looking for four tenths of a kilo. Or you could say, or two fifths, isn’t that right? Four tenths or two fifths
> T: I hope you’re working there Sarah and not just chatting. OK so, how, sorry pet how will you do it? Divide by
> S: <u>Divide by five</u> <u>and multiply by</u>...
> T: Excellent girl, well done.. (SDVS4, C, 9)

When written out in this form this exchange looks disjointed. But if we look at how the teacher supported the student's description of the procedure, several elements are present. Making use of the language spoken in the exchange, a more coherent version of the description would read something like: "Put 400 over 1000. Cross out the two zeros because you are dividing both 400 and 1000 by 100. Now reduce it to two fifths. To find the price of 400 grams, find two fifths of the price of a kilo by dividing it by five and multiplying it by two." I have underlined the lines of dialog above that relate to the description reproduced here. Parts of the description are incomplete. For example, when dividing the numerator and the denominator of 400/1000 four zeros must be crossed out (two in the numerator and two in the denominator) and not just two; it is not clear from the description why or how 4/10 can be reduced to 2/5, although this may be knowledge that is part of the shared knowledge among students in this class and invisible on a videotape of a single lesson. Although the student produces much of the description, as in the previous example the teacher's prompts and questions are pivotal to the student's production, making it another part of the mathematical work of teaching. It is noticeable that although in both cases students refer to crossing out zeros, in each case the teacher draws attention to the fact that this is similar to dividing by 100. But in neither case does the teacher show evidence of knowing the mathematical significance of dividing the numerator and the denominator by 100: i.e. the value of the fraction does not change because it is being divided by a fraction equivalent to one. I found no reference to this task in the literature about MKT or items in the United States.

Primary schools in Ireland are expected to have a written plan stating how each subject is taught throughout the school and the Department of Education and Science expects all teachers to contribute to writing these plans in mathematics as well as in other subjects. The planning should result in a document about aspects of teaching mathematics such as strands and strand units to be taught, teaching methods to be used, assessment and record keeping, provision for students with different needs, equality of participation and access, timetable, homework resources, individual teachers' planning and reporting, staff development, home-school links and community links.[60] In order to engage in planning at school level many demands are made on a teacher's mathematical knowledge. In addition, teachers are expected to prepare long-term (generally for a year or a term) and short-term (generally for a week or a fortnight) plans

[60] Headings taken from part of a planning template at http://www.pcsp.ie/html/ma_plann.php on February 11, 2007.

for teaching mathematics. Reference is made in the MKT literature to planning lessons or mini-lessons (Ball, 1999 and Form B_01, item 24) but no reference is made to longer term or whole-school planning.

Teachers in Ireland prepare reports of topics taught to students. Such reports are distinct from notes kept on individual students' progress. The record summarizes content taught to students over a period of time (often a fortnight or a month). Schools retain such reports for consultation for 2 years and they may be read by teachers and inspectors from the Department of Education and Science. The monthly reports assist teachers when planning and can be a basis for reviewing the whole-school plan. Preparing this record requires teachers to be able to document what they have taught in a way that is meaningful and useful to colleagues. The task was not named in the MKT literature or in items developed to measure MKT by the Learning Mathematics for Teaching research project.

In observing Irish lessons I identified two tasks related to the data strand of the Irish curriculum. The first is to collect data from students. One teacher asked her sixth-class students to open library books and count the number of words in lines on random pages. Students called out various numbers in a haphazard way. How would the teacher ensure that all students contributed and that no student contributed twice? How could the responses be recorded to make them usable in class? If a teacher intended to analyze such data in class the data would need to be systematically analyzed. These activities demand mathematical knowledge.

In addition, teachers need to compare or differentiate between or among different ways of representing data as one teacher did when she asked, "if you looked down at the trend graphs, how are they different to a bar chart?" (SDVS6, C, 4). The teacher listened to specific suggestions from students related to profit and loss and then responded by saying "yeah, they can show a trend, literally the word 'trend.' Whether it is upwards or downwards or whereabouts something is heading" (SDVS6, C, 4). The teacher here was responding to student responses and adding her own explanation of what a trend graph can represent. Much of the focus in developing MKT to date has been on the topics of number, algebra and geometry and the topic of data may be documented more in future. For now data topics are not included in the tasks that informed the construct of MKT.

Another task observed in Irish lessons but not identified in MKT literature is identifying and describing generic problem solving strategies for students. For example, students were told to look for extraneous information in problems and to "pick out information relevant to the question" (SDVS9, C, 11). Another strategy for solving a difficult problem was to try "a slightly simpler version of it" (SDVS9, C, 11). The problem could be kept the same but the numbers in it could be made more manageable. In order to do this work of teaching, teachers need to know the kinds of difficulties primary school students have when solving problems and what problem solving strategies are likely to be helpful. Many references are made to problems throughout the MKT literature and in the items: know what a problem is asking; solve a mathematics problem; anticipate difficulties students may have (Ball, 2000). I identified no reference in the literature or the items, however, to a task similar to the Irish teacher's identification and description of generic problem strategies.

One task done by many Irish teachers is presenting number facts to students. This enables students to respond automatically to problems up to 10 + 10, 20 – 10, 10 x 10 and 100 ÷ 10. Presenting number facts requires a teacher to know properties of numbers and operations that make learning the number facts easier for students. This task is not included among those on which the construct of MKT is based.

Another task of teaching in Ireland involves documenting a student's progress in mathematics in a school report or discussing progress at an individual parent-teacher meeting. In both cases a teacher needs to summarize concisely what a student has learned in mathematics during the year and to recommend strategies for the student to make further progress in mathematics. This is a task in which most Irish teachers engage at least once each year but I found no mention of it among the MKT-related tasks.

That concludes the list of tasks observed in Ireland but not identified in the analysis of the U.S. data. Other tasks of teaching were identified in the MKT items and articles but not in the lessons taught by Irish teachers. Again, it is possible that they occur in Ireland but they were not identified in the ten lessons studied and I have not encountered them in my work as a teacher in Ireland. The tasks are: (a) giving and evaluating mathematical justifications, (b) modifying problems, (c) organizing solutions and creating word problems, (d) writing a word problem to match a division calculation. I now describe each one briefly.

Tasks Identified in MKT Literature and not in Irish Teaching

The first task underlying MKT and not observed in Irish teaching was giving and evaluating mathematical justifications (Ball & Bass, 2003b). In a specific example Ball and Bass use a mathematics problem, “I have pennies, nickels, and dimes in my pocket. Suppose I pull out two coins. How much might I have?” (Ball & Bass, 2003a, p. 37). They suggest that a teacher could build justification into the task by asking questions such as “how do you know that you have all the solutions?” and “suppose someone challenged your solution. How could you prove to them that your answer is right?” (p. 41). Although explanations were observed in Ireland, none of them had the element of deductive reasoning that would be expected in a justification.

The second task not observed in Ireland was modifying a problem or rescaling a problem for younger or older learners to make it easier or more challenging (Ball, 2000). In the lessons observed, Irish teachers tended to rely on problems from textbooks and few, if any, of them were open-ended in the way that the problem described by Ball is:

> Write down a string of 8’s. Insert some plus signs at various places so that the resulting sum is 1,000.

This problem has several solutions but problems used in Irish lessons had single correct answers. As a result no evidence was found of teachers modifying a problem by making it easier or more challenging for students in their classes.

A third task of mathematics teaching identified in the MKT literature but not observed in the Irish lessons is related. If a teacher poses problems with multiple solutions, the teacher needs to figure out how to organize the solutions because the layout can help make visible for students different mathematical aspects of the problem (Ball, 2000). The need to organize solutions in this way did not arise in the Irish lessons studied.

Finally, a fourth task of mathematics teaching that informed the development of MKT in the United States but which was not observed in Ireland was writing a word problem to match a fraction division calculation (Ball & Bass, 2003b). Although one lesson observed in Ireland was taught on the topic of dividing whole numbers by fractions, the teacher did not create a related word problem. This lack of context for fraction problems is consistent with a study that compared Irish primary school textbooks with those in Taiwan and Cyprus. The study found that when worked out examples were used to demonstrate subtracting mixed numbers in both Taiwan and Cyprus the problems were always set in a context such as eating pizzas or comparing meters of fabric. In contrast, neither of the two Irish textbooks presented worked examples of subtracting fractions in the context of a word problem (Delaney, Charalambous, Hsu, &

Mesa, 2007). Based on the evidence of these ten lessons the lack of context for operations with fractions seems to be found in Irish lessons as well as textbooks.

Having studied the mathematical tasks of teaching in Ireland and the United States one cannot help but agree that teaching is "mathematically intensive work" (Ball & Bass, 2003b, p. 13). Moreover, the mathematical tasks identified in the ten Irish lessons are generally similar to those that informed the construct of MKT in the United States. Nevertheless MKT tasks were identified that did not appear in Irish lessons and some tasks of Irish teaching were not listed among the tasks that formed the basis of MKT. This suggests that MKT items will not tap all the knowledge that Irish teachers need to do the work of teaching and it suggests that some knowledge may be sought which is not part of the work of teaching in Ireland. The overall picture, however, is one where substantial overlap exists between MKT tasks and tasks in the Irish lessons. I will return shortly to assess conceptual equivalence in light of these findings. Before that, however, I present additional evidence to address the question of construct equivalence of MKT in both countries.

Assessing Factorial Similarity and Factorial Equivalence

The qualitative analysis above indicates many similarities between the tasks of teaching in the United States and in Ireland. Consequently, the U.S. construct of MKT is likely to be similar to the construct of MKT in Ireland. Two other steps in establishing construct equivalence are to investigate whether factorial similarity and factorial equivalence exist between survey responses in the two countries. In other words, do survey items load on the same factors in both Ireland and the United States? And are the factor loadings identical for each item across countries? (Singh, 1995) If the construct of MKT is equivalent in the United States and Ireland and the survey items are measuring MKT in both countries, the factors and relation among the factors should be the same (Behling & Law, 2000, p. 33). To investigate this I needed to check whether the knowledge teachers used to respond to MKT items was structured in the same way in each country. If the knowledge was structured in the same way, this would be further evidence of the equivalence of the construct of MKT between the countries. The organization of knowledge factors can be assessed using both exploratory factor analysis and confirmatory factor analysis [61] and can be determined only after data have been collected. Exploratory factor analysis identifies common factors among survey

[61] I acknowledge the assistance of Lingling Zhang and Laura Klem from CSCAR at the University of Michigan in conducting the factor analyses. Any errors are my responsibility.

items without prior specification of factors. In studies across countries confirmatory factor analysis has the advantage that hypotheses about factors derived from previous studies can be tested in a new country (van de Vijver & Leung, 1997, p. 99). I conducted exploratory factor analysis on the responses of 501 Irish teachers to survey items and expected to find that survey items were related to the hypothesized sub-domains of MKT. In other words, I anticipated content knowledge items (both SCK and CCK)[62] would load on one factor, KCS items would load on a second factor and algebra items would load on a third factor (See Figure 4.6).[63] The empirical findings, however, provided little evidence to support the conceptualized categories (See Table 4.1).

Although initial analyses cast some doubts on the appropriateness of a three factor solution, I focused on such a solution because three factors were established in previous research (Hill et al., 2004). Contrary to expectations I identified one strong factor on which most content knowledge and algebra items loaded in the three factor exploratory factor analysis solution.[64] Seven KCS items loaded on the same factor. Three algebra items and two KCS items loaded on a second factor. Rather than three underlying factors explaining how Irish teachers responded to the items, this suggested one strong factor, perhaps general mathematical knowledge, could explain teachers' performance on most items. These findings differed from factor analyses conducted on a parallel form (A_01) in the United States[65] and reported by Hill, Schilling and Ball (2004). Correlations among the factors did not appear to be high (see Table 4.2).

Because of the discrepancy between my results and the U.S. results, I re-analyzed responses of 598 U.S. teachers to items on form B_01, using exploratory factor analysis and the results are presented in Table 4.1.[66] In this re-analysis of U.S. data, based on the three factor solution, two factors appeared to explain teachers'

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[62] The current specification of SCK is under review because the CCK and SCK factors did not differentiate themselves in a U.S. validity study but neither can the factors together be considered unidimensional (Schilling, Blunk, & Hill, 2007).

[63] My hypotheses were based on results reported in (Hill et al., 2004). Geometry items were not included in the factor analysis reported here because no geometry items were included in the U.S. form B_01.

[64] By convention, items are considered to load on a factor when the value is 0.4 or higher and 0.3 or higher when n > 175 (Gorsuch, 1983). In this case I used the criterion of > 0.3 to identify factors. In the Hill, Schilling and Ball (2004) study the criterion used was the highest loading on a factor.

[65] I used MPlus software (Muthén & Muthén, 1998-2007), promax rotation and ULS (unweighted least squares) estimation. Hill, Schilling and Ball used ORDFAC software (Schilling, 2002) and promax rotation. No estimation method is specified.

[66] The analyses of the U.S. and Irish data cannot be compared directly with each other because the "spatial orientation of factors in factor analysis is arbitrary" (van de Vijver & Leung, 1997).

performances on items rather than the one factor in the Irish dataset. Most content knowledge items loaded on factor one. Most algebra items loaded on factor three and several KCS items and content knowledge items loaded on the same factor. Although greater evidence of multidimensionality of responses exists among the U.S. respondents, the sub-domains are still not as clearly defined as in the hypothesized model illustrated in Figure 4.6. The correlations among factors did not appear to be high in this model (see Table 4.3). Note that four U.S. items loaded (in excess of 0.3) on more than one factor. I will consider a possible explanation for this below.

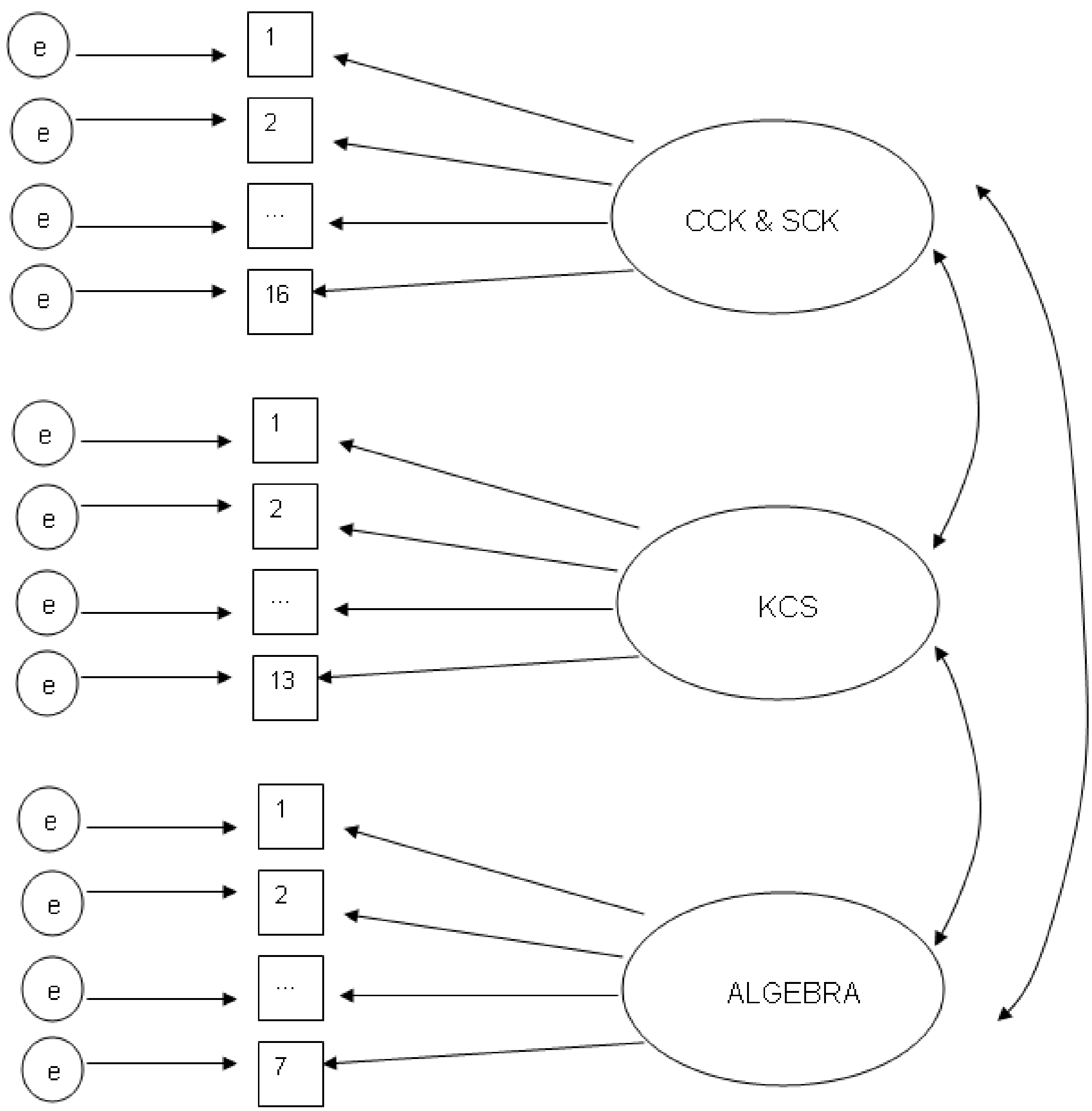

Figure 4.6

The hypothesized common factors (CCK & SCK, KCS and Algebra) that explain the item variances. Measurement error and unique variance explain part of the item variances. Measurement error (e) accounts for the remaining variation not explained by the factors.

Table 4.1

Promax Rotated Factor Loadings with a Three-Factor Solution based on Data from Irish and U.S. Teachers, Form B_01.

	Irish Teachers			U.S. Teachers		
Item	**Factor 1**	**Factor 2**	**Factor 3**	**Factor 1**	**Factor 2**	**Factor 3**
C1(t)	**0.507**	0.041	-0.011	0.293	0.107	0.069
C2	0.281	0.252	0.061	**0.350**	0.083	0.145
C3	**0.634**	0.082	0.149	0.312	0.055	**0.409**
C4	**0.432**	0.117	0.329	**0.717**	-0.015	0.174
C5	**0.418**	0.148	0.159	**0.495**	-0.256	0.424
C6	**0.324**	0.131	0.158	0.273	0.042	0.234
C7	0.186	0.019	-0.015	-0.236	0.042	**0.603**
C8	0.114	-0.165	0.074	0.297	-0.073	0.178
C11	**0.400**	-0.121	0.232	**0.583**	-0.127	0.172
C12	**0.531**	-0.188	0.033	**0.445**	-0.026	0.275
C16	**0.428**	-0.114	0.331	**0.837**	0.009	-0.088
C17	**0.407**	-0.226	-0.079	**0.390**	0.152	0.022
C18(t)	**0.618**	-0.017	-0.055	0.340	-0.030	**0.401**
C19	**0.450**	0.044	-0.077	**0.455**	0.253	-0.059
C20(t)	**0.335**	0.142	-0.122	0.218	0.103	0.265
C21	**0.358**	0.097	0.039	0.137	0.230	0.009
S9	**0.457**	0.139	-0.070	0.250	0.111	0.243
S10	-0.018	0.309	0.122	0.109	-0.241	**0.312**
S13(t)	**0.520**	0.033	0.020	0.186	0.025	**0.438**
S14	0.233	-0.091	0.082	-0.025	0.032	**0.400**
S15	**0.353**	-0.057	-0.082	0.306	0.049	**0.341**
S22	-0.031	0.164	0.064	-0.093	**0.779**	-0.099
S23	0.152	**0.940**	-0.118	-0.174	**0.462**	0.224
S24	-0.051	0.052	**0.435**	-0.212	-0.002	**0.552**
S25	**0.348**	0.001	0.031	0.122	0.024	**0.426**
S26	**0.409**	-0.204	-0.022	0.108	0.217	0.279
S27(t)	**0.382**	0.289	0.056	0.200	0.253	0.131
S28	-0.007	0.137	**0.452**	0.019	-0.145	**0.388**
S29	**0.334**	0.323	0.114	-0.060	0.028	**0.586**
P30	0.193	-0.201	**0.544**	0.246	-0.113	-0.056
P31	0.019	0.047	**0.762**	0.159	-0.025	**0.400**
P32	**0.500**	-0.047	-0.021	0.238	0.088	**0.321**
P33	0.049	-0.245	**0.531**	0.266	0.021	-0.002
P34(t)	**0.578**	0.138	-0.001	0.205	0.121	**0.357**
P35(t)	**0.652**	0.201	-0.019	0.041	0.252	**0.498**
P36	**0.474**	-0.308	0.029	0.236	-0.032	**0.492**

(t)=testlet. C=content knowledge item. S= KCS item. P=algebra item.

Bold print indicates the highest loading above 0.3 in a given row.

Table 4.2
Correlations among Factors in the Three-Factor, Exploratory Factor Analysis Model of the Irish Teachers' Data

	Factor 1	Factor 2	Factor 3
Factor 1			
Factor 2	0.091		
Factor 3	0.447	0.238	

Table 4.3
Correlations among Factors in the Three-Factor, Exploratory Factor Analysis Model of the U.S. Teachers' Data

	Factor 1	Factor 2	Factor 3
Factor 1			
Factor 2	0.434		
Factor 3	0.599	0.415	

Hill, Schilling and Ball (Hill et al., 2004) hypothesized that the algebra items may have been "obscuring relationships among the student thinking items" and they decided to fit a model without the algebra items. I repeated this experiment but this did not yield any clearer factors in either the U.S. or the Irish data. U.S. responses to content knowledge and KCS items were more likely to fall on different factors than Irish items[67] but no clear pattern emerged. Subsequently I applied confirmatory factor analysis to both sets of data.

My goal in applying confirmatory factor analysis was to investigate if specifying the hypothesized factors in advance would provide greater clarity as to the factor loadings. In contrast to the exploratory factor analysis results, the confirmatory factor analysis of Irish data indicated a clear algebra and a clear content knowledge factor (see Table 4.4). Nine KCS items loaded on a KCS factor. Confirmatory factor analysis produced better defined factors than exploratory factor analysis. A similar picture emerged when confirmatory factor analysis was conducted on the U.S. items. Strong content knowledge and algebra factors were identified but one more item (than in the Irish data) loaded strongly on the KCS factor.

One reason for the strong loadings in confirmatory factor analysis is that the factors (CK, KCS and algebra) are strongly correlated among themselves. The correlations among the factors in the Irish data can be seen in Table 4.5. Table 4.6 shows that the correlations among the factors are even higher among U.S. teachers'

[67] A 3 factor model looked most promising for U.S. data with 12 of the 16 content knowledge items loading on factor 1 and 6 of the 13 KCS items loaded on factor 3 and 2 loaded on factor 2.

responses. This suggests that rather than finding separate sub-domains of MKT, there appears to be one higher order factor, possibly MKT itself, which explains most of the variance among responses to items (see Figure 4.7). Statistically there is no difference between Figure 4.6 and Figure 4.7.

By running confirmatory factor analysis separately on the datasets for both countries I established that the factor structure for the Irish and the U.S. form is similar based on adequate model fitting statistics in both settings.[68] I subsequently compared the equivalence of the factor loadings in each country by conducting a multiple-group comparison confirmatory factor analysis in MPlus (Muthén & Muthén, 1998-2007). I set the factor loadings to be equivalent (see Figure 4.8) and showed that I did not have evidence to reject the model with the equivalent constraints.[69] This provides further evidence of both factorial similarity and factorial equivalence in both countries.

The finding is good news for this study: it suggests that the construct of MKT is equivalent in both countries because all items load on the same factors in both countries, i.e. they load on one strong (possibly) MKT factor. Although the evidence of one strong MKT factor and less easily identifiable sub-domains across both datasets supports the factorial equivalence of MKT, it is less good news for the overall MKT project. It suggests that the existence or perhaps the measurement of sub-domains may need to be reconsidered (also suggested by Schilling et al., 2007, in relation to KCS, and CCK and SCK). The difficulty may be explained by items that poorly capture the hypothesized domains. Alternatively, if sub-domains exist, and evidence for this is stronger among the U.S. exploratory factor analysis data than the Irish data, their specification may need to be reconsidered.

[68] The statistic used was RMSEA (Root Mean Square Error of Approximation), which describes the discrepancy between the data fit and a perfect fit. A measure of <0.05 is considered a good fit. The statistics were 0.027 for Ireland and 0.021 for the United States

[69] RMSEA is 0.035 suggesting adequate fit.

Table 4.4. Standardized confirmatory factor analysis for Irish and U.S. teachers.

	Irish Teachers		U.S. Teachers	
	Est.	S.E.	Est.	S.E.
CK				
TC1	**0.489**	0.057	0.376	0.047
C2	0.356	0.058	**0.498**	0.047
C3	**0.717**	0.040	**0.688**	0.037
C4	**0.659**	0.063	**0.803**	0.040
C5	**0.571**	0.060	**0.687**	0.045
C6	**0.439**	0.056	**0.475**	0.049
C7	0.177	0.062	**0.356**	0.052
C8	0.126	0.062	**0.390**	0.052
C11	**0.513**	0.055	**0.607**	0.048
C12	**0.492**	0.054	**0.641**	0.042
C16	**0.588**	0.050	**0.678**	0.040
C17	**0.303**	0.063	**0.457**	0.049
TC18	**0.575**	0.037	**0.656**	0.028
C19	**0.430**	0.060	**0.497**	0.048
TC20	0.293	0.049	**0.507**	0.040
C21	**0.405**	0.057	0.249	0.058
KCS				
S9	**0.463**	0.055	**0.521**	0.049
S10	0.118	0.062	0.238	0.055
TS13	**0.549**	0.041	**0.581**	0.035
S14	**0.259**	0.062	**0.356**	0.053
S15	0.294	0.058	**0.615**	0.045
S22	0.043	0.063	0.262	0.053
S23	**0.317**	0.113	**0.304**	0.088
S24	0.250	0.066	**0.309**	0.061
S25	**0.376**	0.060	**0.513**	0.051
S26	**0.356**	0.064	**0.486**	0.048
TS27	**0.490**	0.045	**0.442**	0.040
S28	**0.315**	0.068	0.279	0.059
S29	**0.482**	0.068	**0.486**	0.064
ALGEBRA				
P30	**0.497**	0.073	0.091	0.070
P31	**0.550**	0.114	**0.498**	0.068
P32	**0.493**	0.063	**0.565**	0.048
P33	**0.341**	0.072	0.253	0.067
TP34	**0.664**	0.037	**0.593**	0.035
TP35	**0.729**	0.039	**0.644**	0.045
P36	**0.435**	0.065	**0.645**	0.044

Table 4.5 Correlations among Confirmatory Factor Analysis Factors in the Irish Teachers' Data

	CK	KCS
CK		
KCS	0.960	
Algebra	0.902	0.859

Table 4.6 Correlations among Confirmatory Factor Analysis Factors in the U.S. Teachers' Data

	CK	KCS
CK		
KCS	0.946	
Algebra	0.936	0.987

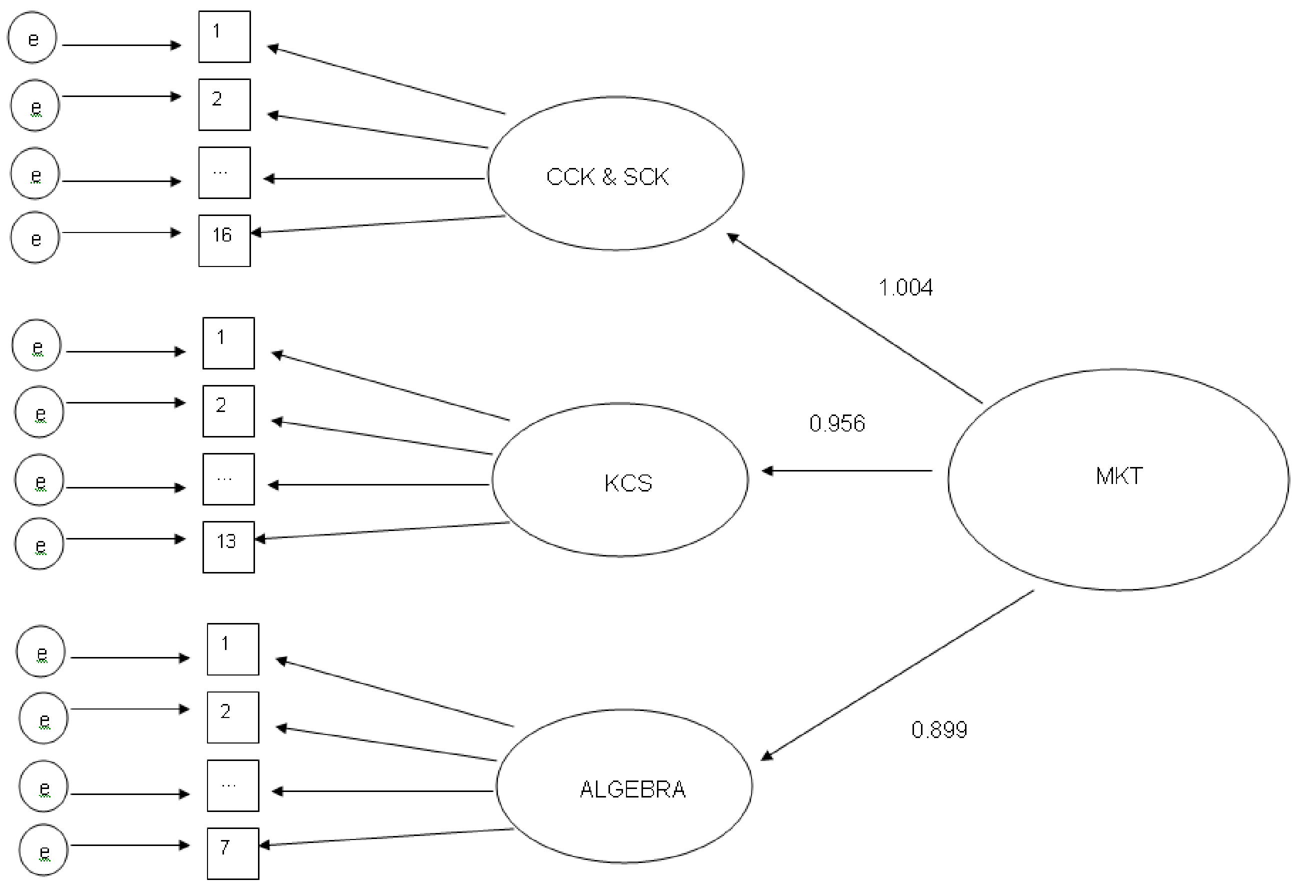

Figure 4.7. An expanded model of the relationship of MKT to the hypothesized factors (CCK & SCK, KCS and Algebra) that explain the item variables. Measurement error (e) explains part of the variable. Numbers on the right refer to loadings of MKT on each of the hypothesized sub-domains.

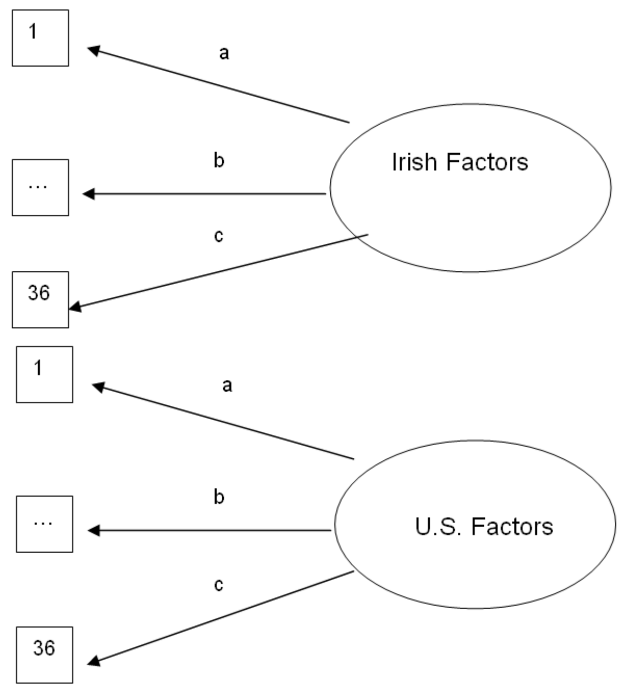

Figure 4.8

A model to illustrate setting equivalent constraints on all factor loadings.

Establishing Construct Equivalence of MKT in Ireland and the United States

I now return to the question at the center of this chapter. That is, how well does the construct of MKT, developed in the United States, reflect MKT in Ireland? The question is important because if the work of teaching in Irish lessons is similar to the work of teaching that informed the construct of MKT, I can use the MKT measures to study Irish teachers' mathematical knowledge. Furthermore, I can claim that the knowledge described by the measures is related to the work of teaching mathematics in Ireland. What evidence exists for conceptual equivalence between MKT in the United States and MKT in Ireland? In order to establish conceptual equivalence, I studied the source of MKT. MKT is a construct of the professional knowledge needed by teachers to do their work. The construct emerged from studying the practice of teaching and conceptions of the work of teaching studied by researchers at the University of Michigan. Ball described this as a process of "combing through records of classroom activity …[looking for]… signs of mathematical activity, places where mathematical issues appear salient" (Ball, 1999, p. 33). Furthermore, the construct was informed by other U.S. research into the mathematical knowledge teachers need. The problem with this is that if teaching is a cultural activity (Stigler & Hiebert, 1999) perhaps the work of teaching observed in Irish lessons differs from the work of teaching which formed the basis of MKT. If the work differs, the knowledge demands of teaching may also differ. Therefore, I combed through video tapes of ten Irish lessons to identify mathematical work engaged in by Irish teachers. In this chapter I described that process and how I compared those tasks to tasks described in MKT literature and in the MKT multiple-choice measures.

The overall picture is one of substantial overlap between Irish tasks of teaching and the tasks that informed the construct of MKT. I illustrated tasks with vignettes from Irish classrooms so that readers can evaluate how tasks were compared. As expected I found some exceptions, areas where tasks found in Irish lessons seemed not to have been considered in developing the construct of MKT and tasks that were considered in developing the construct which did not appear in the Irish lessons studied. Tasks of teaching in the United States not observed in Ireland include giving and evaluating justifications, modifying an open-ended problem to make it easier or more difficult, considering different ways of organizing solutions to a problem, and choosing a word problem for a fraction division problem. Tasks identified in Ireland but not mentioned explicitly in the MKT literature included eliciting steps and meanings of procedures, long term, medium and whole-school planning in mathematics, and keeping a record of mathematics taught to students, collecting data and discussing representations of data, identifying and describing generic problem solving strategies, presenting number facts to students and documenting students' progress in mathematics.

This chapter provides new data about the work of teaching in Ireland. Nevertheless the relatively small sample of lessons and how the sample of teachers was chosen mean that I must be cautious about generalizing any claims about the work of teaching mathematics in Ireland from this study. What I have documented, however, accords well with my experience of the work of teaching in Ireland as a classroom practitioner for 10 years and as a teacher educator for 8 years. Based on the sample of lessons studied, evidence exists of substantial conceptual equivalence between the mathematical tasks of Irish teaching and tasks that informed the construct of MKT. The method used most likely underestimates the extent of conceptual equivalence between the Irish and U.S. constructs of MKT. For example, I know from my research work on MKT that the task "eliciting steps and meanings of procedures" is part of the conceptualization of MKT. Because I found no evidence for it in the relevant articles or in the items, however, it is listed as being a task identified only in Irish teaching. I placed it in this category because part of my interest is to evaluate this means of establishing conceptual equivalence, so that it can be used by researchers who do not have my "inside knowledge" of the research.

My specific interest in this study is to use the MKT measures to study Irish teachers' knowledge. The conceptual equivalence of Irish teaching and tasks that informed the construct of MKT indicates that use of the measures to study Irish teachers is appropriate. Furthermore, the measures are tapping into knowledge that is required to do the work of teaching in Ireland because the work of teaching in Ireland is similar to the conceptions of the work of teaching on which the construct is based. In other words, the measures are tapping into knowledge Irish teachers use when they teach.

The factor analysis findings confirm that both factorial similarity and factorial equivalence exist between responses to MKT measures given by Irish and by U.S. teachers. In the confirmatory factory analysis items loaded on the same factors and in a multiple-group comparison I constrained the factor loadings to be the same and this model could not be rejected. These findings support the construct equivalence of MKT in Ireland and the United States.

One limit applies to what has been established about construct equivalence of MKT between the United States and Ireland and it relates to the purpose of this study.

Singh lists six steps in establishing construct equivalence (1995, pp., and Figure 4.1 above). I claimed that functional equivalence could be established logically and the pilot study (Delaney et al., in press) provided evidence for instrument equivalence. In this chapter I claimed conceptual equivalence, factorial similarity and factorial equivalence. The aspect of construct equivalence not considered is measurement equivalence. This aspect of construct equivalence is relevant only if comparing teachers' knowledge across two countries and the focus in this study is on using MKT measures to study only Irish teachers' mathematical knowledge.

Discussing the Means Used to Establish Conceptual Equivalence

Factor analyses are relatively common in research studying constructs across countries (van de Vijver & Leung, 1997). Less frequent are qualitative processes such as the one used here to compare the tasks of teaching across countries. Therefore, before closing this chapter I reflect on how conceptual equivalence was evaluated. The reflection may be helpful for others who wish to use the measures based on the U.S. construct of MKT in a different country. It is no major surprise that relatively minor differences were found between the work of teaching mathematics observed in a sample of Irish lessons and the tasks which informed the development of MKT. Ireland and the United States share a common language making it easier for ideas, including those about teaching, to move back and forth between the two countries. Much research investment in Irish primary mathematics education is targeted at student achievement, rather than on studies of teaching (e.g. Shiel & Kelly, 2001; Shiel et al., 2006) and other research, although developing, tends to be small-scale and fragmented (e.g. Close, Corcoran, & Dooley, 2007; Close, Dooley, & Corcoran, 2005). Consequently, when revising curricula and preparing teachers, Irish educators draw on ideas and research from other countries, including the United States. It is possible that if U.S. tasks that informed MKT were compared to tasks of teaching in a third country, more differences may emerge. I now make some observations about studying similarities and differences between the work of teaching observed in Ireland and conceptions of the work of teaching which underlies the construct of MKT.

First, I note that the overall process used for comparing the tasks of teaching worked well. I began by doing some open coding of Irish lessons and by identifying tasks of teaching. I believe that open coding of the tasks rather than using tasks identified in the U.S. literature helped to identify tasks specific to Ireland. Adapting the lesson table devised by Kawanaka and his colleagues (1999) helped me to systematically study the mathematical work of teaching as observed in teacher actions. When I had identified tasks of teaching from the ten lessons, I supplemented the list with tasks of teaching in Ireland with which I was familiar and which would generally not appear on videotapes of lessons. After completing the review of MKT literature on tasks of teaching, the iterative process of comparing codes between both settings was helpful. In particular, some tasks emerged that I observed but had not labeled until I saw the task named on the MKT literature list.

Second, comparing tasks from videotapes of Irish lessons with tasks identified in MKT literature was useful for several reasons. The open coding of the Irish video tapes revealed many tasks of teaching in Ireland but I overlooked naming some tasks in the process. Tasks named or described in the MKT literature helped me augment the list of Irish tasks. Because the list of MKT-related tasks was drawn from articles written over a ten-year period, the list is more extensive than any list I could have drawn up by observing a relatively small number of U.S. lessons. Furthermore, the U.S. tasks of teaching had already been identified and discussed by a team of researchers and consequently, the descriptions of the tasks were more refined than they would be if they were created from scratch. The work of primary school mathematics teaching has not been documented in Ireland to date and so comparing literature from each country was not an option. The approach adopted is likely to be feasible for other researchers to use if they wish to learn more about MKT in their country. It would be unreasonable to expect researchers from each new country to start afresh identifying tasks of teaching in the United States when comparing them to tasks in their country. Some difficulties with the approach were also identified.

One difficulty encountered in comparing tasks was that although many tasks of teaching were similar, the language used to talk about them and the way they were presented differed. In particular, I decided to use as small a grain size as possible when describing the Irish tasks. Mostly this worked well but in some cases the matches between the Irish task and what appeared to be an equivalent U.S. task had to be approximate. This is because some of the U.S. tasks were presented in a general way. For example: “listen” (Ball, Hill et al., 2005, p. 17), “hear students flexibly” (Ball & Bass, 2000a, p. 94), “manage discussions” (Ball & Bass, 2003b, p. 6), “attend to mathematical practices as a component of mathematical knowledge” (Ball & Bass, 2003b, p. 12), or “connect content to contexts effectively” (Ball, 2000, p. 243). In other cases a number of tasks were gathered together as one, such as “Interpret and make pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual)" (Ball & Bass, 2003b) That the tasks would be general or concentrated in form is not surprising. Descriptions of the tasks were intended to support a "practice-based portrait" of MKT (Ball, Hill et al., 2005, p. 17) and they were never intended to be used in the way they have been used in this study. Indeed it is remarkable how well the majority of them worked. In order to make my matching of tasks as transparent as possible I liberally employed examples from the Irish lessons to illustrate tasks that I considered similar.

Nevertheless, given the potential of this process, I believe that further refinement of how the mathematical work of teaching is described is warranted. The construct of MKT is primarily grounded in the work that teachers do but how that work is described is uneven across the articles. Given the complexity of teaching and the various task sizes and how tasks are nested within one another, describing the work comprehensively would be no easy undertaking. A starting point might be to develop language which describes how the work of teaching is studied in developing the construct of MKT. A glossary of U.S. tasks identified in the course of studying MKT, for example, would help researchers from other countries compare tasks of teaching identified in their country with tasks that informed the construct of MKT. Such a glossary would help reduce any misinterpretation of tasks when identifying similarities and differences. Noting that the same task can be done "with different emphases or arranged" in different ways in different settings (Hiebert, Gallimore et al., 2003), any such glossary would need to consider carefully how tasks are conceived to make investigation of similarities and differences meaningful. The glossary could be accompanied by annotated examples of tasks from the United States or elsewhere. After coding four lessons I prepared a glossary based on the tasks identified in Irish lessons to help classify tasks in subsequent lessons. This glossary evolved as the video analysis continued.

The quantitative technique of confirmatory factor analysis, including multiple-group confirmatory factor analysis, complemented the qualitative data. It provided reassurance that the factor structure and magnitude were not significantly different in the responses to the measures by U.S. and Irish teachers. The qualitative data showed why this is the case.

Finally, my background as a researcher shaped how I described the work of Irish teaching and how I identified similarities and differences between the Irish tasks and tasks described in the MKT data (articles and items). The perspective I bring to the work is that I am a qualified Irish primary teacher and I taught in primary schools from 1987 to 1998. I have been a mathematics teacher educator since 1999. Since January 2004 I have spent a significant amount of time in the United States studying the practice of U.S. teaching as part of the Learning Mathematics for Teaching research team. Therefore, my study of the tasks of teaching across countries has been enhanced by my experience of studying tasks of teaching in the United States and my experience of primary education in Ireland.

Chapter 5

Validating Survey Measures of Mathematical Knowledge for Teaching for Use in Ireland

A Rationale for Validation

Test developers and users are responsible for explicitly stating and justifying how test results can be used and interpreted (Kane, 2006). Test results frequently inform decisions and consequently, test creators and test users must ensure that tests provide accurate and relevant information. Consider a familiar example. A standard driving license is earned by passing a standard driving test. Passing such a test, however, provides little evidence of competence to drive a large truck or bus and consequently, it would be unreasonable to use the test results to claim a commercial driver's license. Few tests link content to purpose and results to their interpretation as transparently as the driving test. Only tests designed and validated for a specific purpose (e.g. to certify competence) can be validly used for that purpose. Validation is a way of formally justifying and appraising the use and interpretation of test results.

I established in Chapter 4 that the construct of MKT is conceptually equivalent in Ireland and the United States. That conclusion relates only to the construct and not to measures developed on the basis of the construct. This chapter addresses the validity of the measures. I first present a general overview of approaches to validity followed by specific reference to validation of MKT measures in the United States. Next I use Kane's validity argument approach to validate the planned interpretations of Irish teachers' scores. Kane's approach to validation involves making an interpretive argument to specify "the proposed interpretations and uses of test results" and a validity argument to evaluate the interpretive argument (Kane, 2006, p. 23). Videotaped lessons from Ireland were used to evaluate the interpretive argument, and in this chapter data from those lessons are presented to support the argument. I conclude by evaluating the proposed interpretation of the measures.

Historical Background to Kane's Argument-Based Approach to Validity

Three categories of test validation have been used: criterion validity, content validity and construct validity. Criterion validity was an early model of validity and is still used for assessing validity of results on college admission tests and employment testing. A test result is compared with a stated criterion (e.g. performance in first year of college) but finding a criterion against which to compare the results can sometimes be difficult (Kane, 2006). A second type of validity, content validity, is established not with reference to a particular criterion but by claiming that performance on a sample of tasks from a domain estimates one's overall performance in a domain, such as academic achievement. This form of validity is important but limited to interpreting scores "in terms of expected performance over some universe of possible performances" (p. 19). For example, playing a complex piece of classical music on piano may indicate general competence in playing classical piano music but may provide little evidence of competence in playing jazz piano. A third type of validity, construct validity, began as a means of assessing the extent to which a test was an "adequate measure" of a particular theory (p. 20). By the 1980s it had become a general method of validation incorporating three principles: validating the interpretation of scores rather than validating a test, specifying a theory grounded in a research program, and considering competing interpretations of test results (p. 22). These historical developments, especially in construct validity, led to Kane's argument-based approach to validity.

Validation in Kane's model requires two steps. One is to propose an interpretive argument stating how results will be interpreted and used "by laying out the network of inferences and assumptions leading from observed performances to conclusions and decisions based on the performances" (p. 23). In the second step the plausibility of the proposed interpretive argument is evaluated. To illustrate how this works in practice, Kane (2004) applied his model to a specific case where test results are used to certify someone to practice in an area such as teaching, law or nursing. The following steps require validation: (a) from participants' observed performance on test items to a specific score; (b) from the specific score to a generalized score over all the test domain; (c) from the test domain to the required knowledge, skills and judgment domain; (d) from the knowledge, skills and judgment domain to the practice domain; (e) from the practice domain to certification.

Many assumptions and inferences are made in moving through these steps from performance on a test to being certified as fit for a field of practice. The inferences and assumptions in each step are different and are validated differently. The specification of the first four steps above is relevant to how scores on MKT items are interpreted and used. Test results are not an end in themselves but a means to inform the design of mathematics instruction. In a series of papers published in *Measurement: Interdisciplinary Research and Perspective,* Hill, Schilling and colleagues have applied Kane's approach to the interpretation and use of MKT measures in the U.S. context.

In Schilling and Hill's (2007) validation of the use and interpretation of MKT measures, steps one, three and four above are renamed and related to three sets of assumptions and related inferences: elemental, structural and ecological. The elemental assumption (step (a) above) relates to individual items in a test and how well the items capture teachers' MKT, and not irrelevant factors such as test-taking strategies. The second assumption tested by Schilling and Hill, structural assumptions and inferences, relates to whether the MKT scales (or subscales) measure no more and no less than the domain of MKT (or its sub-domains CCK, SCK, KCS). Schilling and Hill's third category (step (d) above), and the one of particular interest in this study, relates to ecological assumptions and inferences. This is the step that validates teachers' levels of MKT in light of how MKT affects practice. The assumption is that MKT measures capture teacher knowledge related to effective mathematics instruction.

All stages of the validation process outlined by Kane and by Schilling and Hill matter when validating a test. Despite the conceptualized importance of validation, however, test developers have frequently reported selective results and opted for convenient means of test validation rather than prioritizing appropriate evidence (Schilling & Hill, 2007, p. 70). In this study I focus in depth on one aspect of validation, that of the relationship between teachers' MKT and instruction, or what Schilling and Hill call the ecological assumptions and inferences. I set out to validate the relationship of the adapted U.S. measures to mathematics instruction in Ireland.

Such validation is necessary to show how adapted U.S. multiple-choice questions relate to the mathematical quality of Irish teachers' instruction. Haertel (2004) notes differences between performing on a multiple-choice question and performing in practice. Performing in an actual workplace, he claims, provides a specialized environment, a social context and resources that support carrying out the work and these supports tend not to be present when responding to multiple-choice questions. Although one might question the extent to which a teacher can call on social support from peers as instruction unfolds, most would agree that responding to multiple-choice questions differs from responding to real incidents in context.[70] Although Hill and her colleagues have validated the ecological assumption of MKT for use in the United States, separate validation is required for Ireland in order to investigate if performance on the measures is related to classroom instruction in Ireland. This will be established using Kane's argument-based approach to validity.

Interpretive Argument for Use of Mathematical Knowledge for Teaching Measures with Irish Teachers

The first step in validation is to make an interpretive argument. According to Kane the interpretive argument "specifies the proposed interpretations and uses of test results by laying out the network of inferences and assumptions leading from observed performances to the conclusions and decisions based on the performances" (Kane, 2006, p. 26). The full interpretive argument for using the MKT measures in Ireland is as follows (based partly on Schilling & Hill, 2007):

(1) Assumption: Teachers used their MKT when responding to questions on the form.
Inference: A teacher's chosen response to a particular item was consistent with their mathematical reasoning about the item.

(2) Assumption: Teachers drew on mathematical knowledge used in teaching to respond to the questions.
Inference (a): When responding to the items, teachers used mathematical knowledge used in teaching and their general mathematical knowledge.
Inference (b): Items on the test relate to activities in which teachers regularly engage (or in which they regularly need to engage)

(3) Assumption: The MKT multiple-choice measures captured the mathematical knowledge teachers need to teach mathematics effectively.
Inference: Teachers' scale scores on the measures are related to the quality of the teachers' mathematics instruction. Higher scale scores are related to more effective mathematics instruction and lower scale scores are related to less effective mathematics instruction.

Assumption (3) and its related inference will be evaluated in this chapter. This is not to underestimate the importance of the first two assumptions and their related inferences. In fact, by testing the third assumption I am assuming that the first two hold for Irish

[70] Kane (2004), however, claims that someone who cannot demonstrate knowledge in a test question is unlikely to be able to do so in practice (p. 154). Moreover, the MKT items are embedded in hypothetical instructional contexts.

teachers. If teachers don't use MKT to respond to the question, for example, there is little point in investigating a relationship between MKT and instruction because the score on the measures is providing evidence of something other than MKT. This may happen because of a flawed item or because teachers consistently used incorrect reasoning to choose a correct answer (or correct reasoning to select a wrong answer). I believe it is reasonable to accept assumption (1) because cognitive interviews in the United States found teachers' responses to SCK and the CCK items to be consistent with their reasoning. Although guesswork featured in responses to some KCS items (Schilling et al., 2007), this did not affect the ability of the measures to predict the mathematical quality of U.S. instruction (Hill et al., 2007). Two potentially flawed items were identified in the pilot study and modified to address the flaws before being used in the final survey form (Delaney et al., in press). Assumption (2) will not be addressed in this study because results of U.S. validation studies showed that some refinement of distinctions between SCK and CCK are needed and measurement of KCS may need to be revised (Schilling et al., 2007). Little reason exists to believe that substantially different results would be found between Irish teachers and U.S. teachers in testing assumptions (1) and (2). Assumption (3) is different because it relates to instruction in Ireland. The relationship of MKT measures to the mathematical quality of instruction in Ireland will be investigated because although tasks of teaching in Ireland are similar largely similar to those which undergird MKT, other features of instruction may be different and may interfere with how teachers deploy MKT in instruction.

Mathematical quality of instruction refers to a "composite of several dimensions that characterize the rigor and richness of the mathematics of the lesson" (Hill et al., in press, p. 4) including how teachers represent mathematical ideas and link representations; how they describe, explain and justify ideas and encourage their students to do the same; how accurately they use language; and how explicit they are in talking about mathematical practices. These aspects of instruction are likely to be present in lessons taught by teachers with MKT and missing from lessons taught by teachers lacking MKT. I tested the relationship between Irish teachers' scores on the MKT measures and the mathematical quality of their instruction. Of interest was whether teachers' scores on the multiple-choice measures were associated with instruction that is mathematically rich and free from errors. If such a relationship existed, the multiple-choice measures would be useful for predicting the mathematical quality of instruction among teachers in Ireland and should be of interest to researchers, policy makers and teacher educators.

Data Used to Study the Mathematical Quality of Instruction in Ireland

Samples of mathematics instruction in the form of 40 videotaped lessons from Irish classrooms were collected. Ten teachers were each asked to teach four mathematics lessons and the lessons were videotaped (as described in Chapter 3). The teachers had been teaching for between 3 and 30 years and class levels ranged from senior infants (5-year-olds) to 6^{th} class (12-year-olds) in various school types and locations. Teachers taught the lessons over periods of time ranging from under two weeks to four weeks and they were asked to teach lessons similar to those they regularly taught. Lessons on number operations were the most popular followed closely by number concepts and geometry. Although not a random sample of Irish mathematics instruction, several different classroom environments were represented. Lessons varied in terms of the amount of student discussion, the use of manipulative materials, textbook use, and the balance of whole class work, individual and group work. The focus of this study, however, was not on teaching styles but on the mathematical quality of instruction and a systematic way to evaluate this was required.

The Instrument Used to Assess Mathematical Quality of Instruction

The Learning Mathematics for Teaching research group developed an instrument to assess the mathematical quality of instruction. The instrument consists of 32 features of mathematics instruction known as "codes" grouped in three sections,[71] and an accompanying glossary to explain the codes (Learning Mathematics for Teaching, 2006). The first group of codes considers how the teachers' "knowledge of the mathematical terrain of [the] enacted lesson" is evident in instruction. Sample codes in this section are the teacher's use of technical language (e.g. equation, perimeter, and angle) and general language to describe a mathematical idea (e.g. referring to *exchanging* ten units for one ten); a teacher's selection of representations and links made between or among them; and the presence of explanations. The second category of codes refers to the teacher's "use of mathematics with students" and sample codes include how the teacher *uses* representations; how mathematical work is recorded in class; how the teacher responds to students' errors or expression of ideas; and whether the teacher elicits explanations from the students. The third category of codes considers the teacher's use of

[71] In total there are five sections and around 83 codes. Section 1 relates to instructional formats and content and section 4 relates to the textbook and teachers' guide. Codes from these sections will not be used in my analysis.

"mathematics to teach equitably" in relation to inclusion and participation of students of all races and social classes. Sample codes here refer to the teacher's explicitness about language, mathematical reasoning and practices; the amount of instructional time spent on mathematics; and the teacher's encouragement of a diverse array of mathematical competence. One additional code required coders to estimate teachers' overall level of mathematical knowledge based on the instruction observed. This is referred to below as the "global lesson score." Coders were asked to rate the teacher's knowledge as low, medium or high on the basis of the entire lesson. Given the range of codes to be considered in a given lesson, the process of coding needed to be consistent and explicit.

The Process of Coding the Mathematical Quality of Instruction of Lessons

Lessons were divided into five-minute clips for coding purposes (Learning Mathematics for Teaching, 2006). The coding process involved assigning two randomly paired members of the Learning Mathematics for Teaching team[72] to code a lesson. Each member watched the entire lesson, and then watched it again independently coding features of mathematical instruction in each five-minute segment. When both members had independently coded the lessons they met to reconcile their codes.[73] An accompanying lesson narrative was written for each lesson noting salient points about the mathematical quality of the lesson. The reconciled codes and the lesson narrative became a record of the mathematical quality of instruction in each lesson.

When coding a particular segment of a lesson a number of decisions had to be made. I illustrate the decision process with reference to one code: a teacher's use of conventional notation or mathematical symbols. A coder first decided whether a feature, in this case the use of conventional notation, was "present" or "not present" in a lesson segment. If the teacher wrote the numeral "4" or the word "parallelogram" on the board, a coder may wonder whether they count as mathematical symbols. The glossary clarifies that "by 'conventional notation,' we do not mean use of numerals or mathematical terms"[74] so if no other notation appeared, the relevant category code for the clip would be "not present." The second decision to be made was whether the presence or absence of a feature was appropriate or inappropriate. If, for example, conventional notation was present and mathematically accurate, it was marked as "present and appropriate." On

[72] The research team members involved in the coding consisted of teachers, teacher educators, and others. All have good knowledge of both mathematics and teaching.
[73] This process was followed for 70% of the Irish lessons and the remaining lessons were coded by me alone.
[74] The Video Coding Glossary is available at http://sitemaker.umich.edu/lmt/files/lmt-mqi_glossary_1.pdf. Downloaded on March 25, 2008.

the other hand if a teacher recorded a statement such as the following on the board: $7+6=13+5=18$, it was coded as “present” because it includes the “addition” and “equals” mathematical symbols. But the statement is inaccurate because $7+6 \neq 13+5$ so it would have been coded as “inappropriate.” The overall decision in this case, therefore, is “present – inappropriate.” If the absence of an element seems appropriate, it is coded “not present – appropriate” or if the absence seems problematic it is coded as “not present – inappropriate.” A typical cell to be completed for each code is represented in Figure 5.1.

	a. Conventional notation (mathematical symbols)			
	P		NP	
Clips	A	I	A	I
1				
2				

Figure 5.1. A section of the grid used for video-coding.

The Relationship between Irish Teachers' Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction

I now return to the validity argument to test the inference that teachers' scores on the multiple-choice measures are related to the mathematical quality of the teachers' instruction. I studied the relationship between teachers' knowledge as measured by the MKT measures and how that mathematical knowledge was evident in the mathematical quality of their instruction. The multiple-choice measures and knowledge exhibited in instruction are two different ways of studying the invisible trait of mathematical knowledge for teaching. How teachers use their knowledge to enhance instruction is of primary interest but I wanted to see if the multiple-choice measures were a good way to capture that knowledge.

Data from all ten teachers were used to test the inference. Most teachers were selected based on suggestions from teacher educator colleagues and principals, and two were recruited based on the recommendation of another teacher who had been videotaped. Although typical teachers willing to be videotaped were sought, it is possible that the teacher educators and principals were more likely to suggest teachers they knew to be interested in or competent in teaching mathematics. In addition to being videotaped teaching, the teachers agreed to take the MKT survey taken by the 501 other teachers in the study. Because of how the sample was selected, there was a risk that the teachers would not be representative of the general teaching population. That concern was well founded. In terms of MKT scores the teachers ranged from the 36^{th} to the 97^{th} percentile of Irish teachers (see Table 5.1). In other words all ten teachers are in the top two thirds of Irish teachers, based on MKT scores. Furthermore, six of the ten teachers are in the top quartile of Irish teachers. A wider spread of teachers along the MKT scale would have been good. The relatively narrow range of teachers placed more demands on the MKT measures because they needed to be more sensitive to identify differences among teachers who are relatively close on the MKT scale.

Table 5.1

Irish teachers and their MKT score (range from -3 to +3) and their percentile in the population calculated based on all teachers who participated in the MKT study

Teacher	MKT Score	Percentile
Olive	1.879	97
Nigel	1.309	91
Brendan	1.279	90
Eileen	0.777	83
Clíona	0.677	82
Sheila	0.526	78
Veronica	0.357	57
Hilda	-0.141	46
Caroline	-0.357	42
Linda	-0.431	36

As mentioned above, when video-coders coded each lesson they estimated a global lesson score, to reflect the teacher's overall level of mathematical knowledge. Specifically, coders were asked to state if the teacher's mathematical knowledge was low, medium or high. In several cases coders chose intermediate levels of these bands (i.e. low-medium or medium-high) so in the analysis, I assigned a value to each lesson rating, from 1 (low) to 5 (high) with 2 and 4 representing intermediate levels. Figure 5.2a presents teachers' MKT scores. Teachers' average global lesson scores over the four lessons are presented in Figure 5.2b. Teachers are grouped in bands according to their placement on the scales. Overall the MKT measures were reasonably accurate in predicting the quality of mathematics of teachers' instruction. Three teachers' score bands (Caroline, Hilda and Brendan)[75] were accurately predicted by the MKT measures and all but one of the other teachers were placed in an adjacent band: two in the adjacent higher band (Clíona and Linda) and four in the adjacent lower band (Olive, Nigel, Sheila and Eileen). One teacher's global lesson score (Veronica) placed her two bands below that predicted by her MKT score. Teachers with high MKT scores generally tended to have high global lesson scores and teachers with low MKT scores tend to have low global lesson scores, but there were exceptions. The relationship is a general trend rather than a precise mapping. By looking inside the classrooms of some of the

[75] Pseudonyms are used for all teachers and identifying details have been changed.

teachers, more can be understood about the relationship between teachers' MKT and the mathematical quality of instruction. I first present examples of teachers whose MKT scores were reflected in the mathematical quality of instruction followed by details of teachers who exhibited instruction that was either of higher or lower mathematical quality than predicted by their MKT score.

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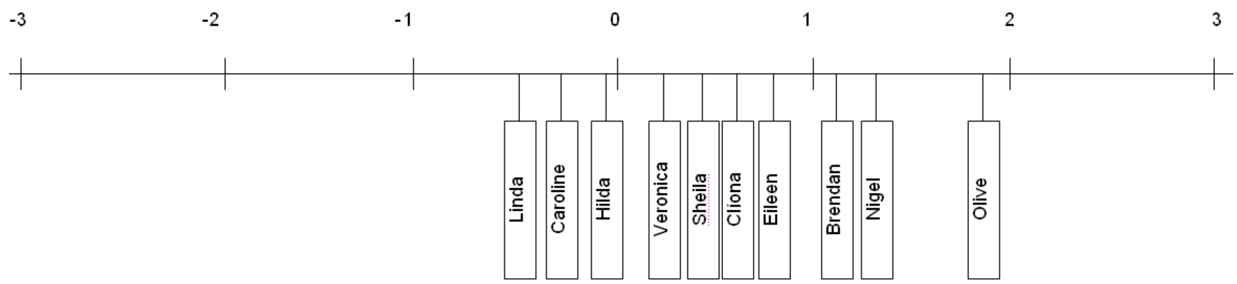

Figure 5.2a
Teachers in the video study ordered according to their IRT scores on the MKT survey (scored from -3 to +3; teachers not placed to precise scale).

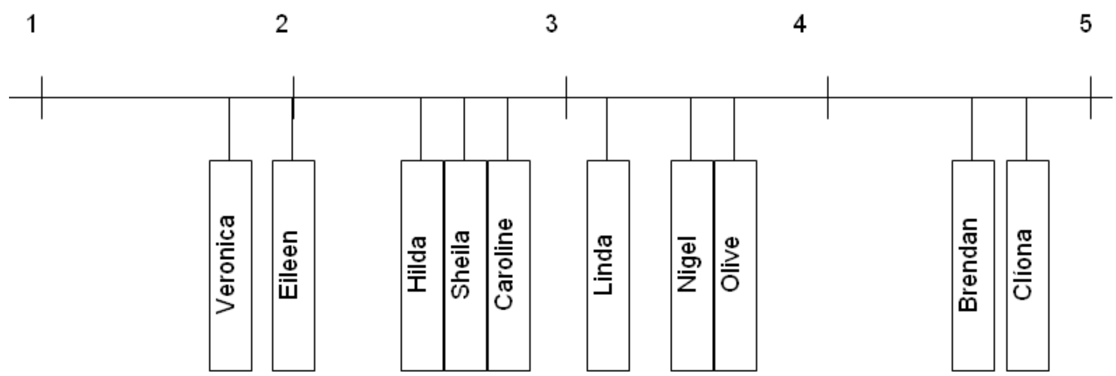

Figure 5.2b
Teachers in the video study ordered according to the overall mathematical knowledge for teaching observed in their instruction, relative to other teachers in the study (scored from 1 to 5; teachers not placed to precise scale).

Irish Teachers with Consistent Mathematical Knowledge for Teaching and Mathematical Quality of Instruction Scores

Both Brendan and Hilda exhibited instruction consistent with their MKT scores. Brendan's MKT score is in the 90^{th} percentile of Irish teachers and his instruction reflected many elements of mathematical quality. An episode from one lesson illustrates this. Brendan and his students were folding paper into halves, thirds or quarters and then folding them again in order to figure out answers to problems such as $\frac{1}{2}$ of $\frac{1}{3}$ and $\frac{1}{4}$ of $\frac{1}{3}$. Aided by Brendan's prompting, the students noticed the pattern whereby the product could be found by multiplying both fractions. The discovery was confusing for some students because in the paper folding activity they had been *dividing* paper but now they could solve the problems using multiplication. One student grappled with the apparent contradiction and asked a question:

> S: Yeah, but it's also division, right?
>
> T: Yeah, it is. Well you are dividing. What you've been doing on the page has been dividing.

Brendan agreed with the student that division is involved in the operation as well. This is correct because in the case of $\frac{1}{2}$ of $\frac{1}{3}$, $\frac{1}{2}$ is an operator that "stretches" $\frac{1}{3}$ one time (i.e. the size of the numerator) and "shrinks" it by dividing it by 2, the size of the denominator (Behr, Lesh, Post, & Silver, 1983). Brendan related his response to the paper folding activity to explain the division component of the calculation. A moment later Brendan's knowledge was tested again when he asked a student to compute $\frac{1}{4}$ of $\frac{1}{3}$. Based on the previous exchange, the student asked if he would do it "as a division or multiplication sum." The following discussion took place as Brendan probed the student:

> T: Well, is it going to work? How would you write it as a division sum?
>
> S: You get a third and divide it by a quarter. You get a twelfth [student writes $\frac{1}{3} \div \frac{1}{4} = \frac{1}{12}$ on the board], so it's the same thing.

The student incorrectly replaced the "of" term with a division symbol and reversed the order of the fractions but he wrote the correct answer, which had been figured out previously using the paper folding activity. Based on this solution, the student claimed that division is the same as multiplying. Brendan, however, knew that the student was incorrect and asked “is it though?” and the student responded as follows:

> Because it’s fractions part of it....Dividing means it gets bigger. When you divide a third by a half it gets bigger, the number. Because if it was over, if it was over one it would be, the number would get smaller…. But if it’s under one it gets bigger.

The student’s statement made further demands on Brendan’s MKT because the statement required deciphering (and meanwhile other students were trying to contribute to the discussion). To decipher the statement Brendan needed to know that when the student referred to dividing making a number bigger, he is referring to division of fractions. When the student referred to the number being “over one” he is referring to division of counting numbers. Brendan also needed to recognize that the specific fraction computation mentioned by the student (dividing a third by a half) was not the question the student was asked to work on but an example chosen by the student to illustrate his point. With little or no time to think, Brendan responded as follows:

> You’re dead right. Maybe the way you’ve written it isn’t exactly accurate. Do you see the third divided by a quarter? Are you dividing it by a quarter or are you dividing it by four?

Brendan’s response signaled that he agreed with the student’s explanation about dividing but the teacher also drew attention to the student’s error by giving a clue to what was wrong: he had written that he was dividing a third by a quarter but it should have been a third divided by four. The student’s reply revealed another misconception as evident in the subsequent exchange:

> S: Same thing basically.
> T: I don’t think so. You’re dividing into quarters, but are you dividing by a quarter?
> S: Oh yeah.

The student had thought that dividing by a quarter was the same as finding a quarter but Brendan used his MKT to distinguish between “dividing into quarters” (i.e. dividing by four) and “dividing by a quarter.” The student’s response of “oh yeah” indicated that he realized his error. Subtle mathematical differences exist between dividing into quarters and dividing by a quarter but teachers need such knowledge. Brendan clarified what needed to be done and posed another question:

> When you're splitting something into four, you're dividing by four, aren't you? You're dividing into four pieces. That's the only thing I'd change in that maths sentence. A third divided by four. How would you write four as a fraction?

One student's response to the question made further demands on Brendan's knowledge: The student responded that four could be written as "sixteen over four" before Brendan elicited another answer, "four over one." Brendan asked why that was correct. One student offered an explanation, which was correct but fell short of an explanation and was difficult to follow:

> Because when you're emm, say if you're multiplying emm four by five but you want to do it in fraction term (sic), you can't emm you can't just put like, say you put five over four you can't do that, so you have to put one over it. So then it would be one eh, over four times one over five or emm… Four over one times five over one...so it'd make it easier

The student took a specific case of multiplying in fraction terms to illustrate how to write whole numbers as fractions. Brendan acknowledged being confused by the response and instead offered his own explanation:

> Well, one over one is one whole, isn't it? So I mean four over one is four whole amounts.

In the episode described above Brendan exhibited knowledge of fractions as operators where the operations of division and multiplication are closely related; he evaluated and responded to a student's incorrect answer; he deciphered a student's inchoate contribution; he distinguished between a student's oral description of a procedure and what the student wrote; he identified student misconceptions and he explained an idea. All these incidents occurred in a period of less than three minutes of a one hour lesson, showing how little time Brendan had to think about his answers. Throughout the four lessons observed, he exhibited similar knowledge making few mathematical errors and using mathematical language appropriately throughout. Both MKT and the mathematical quality of instruction were consistently high.

Like Brendan, Hilda's MKT score was consistent with her mathematical quality of instruction but her scores were lower than his. Hilda's MKT score was in the 46th percentile and her instruction exhibited qualities of both high and low mathematical knowledge. Her use of explanations was characteristic of high MKT and she frequently asked her 2nd class students to explain their work. In one example students had folded a page into quarters and found a quarter of 16 counters by placing an equal number of counters on each quarter of the page. Hilda asked the students what half of sixteen would be and when a student answered eight, Hilda pursued the following explanation:

T: And how did you get that from what you've done here?
S: Because I had four here and I had four here.
T: Yeah?
S: And four and four equals eight.
T: Makes eight. And so what is this part of your page?
S: Half.
T: Good boy, ok. And what did we say about halves and quarters?
S: Halves are bigger than quarters.
T: They are, yeah. And two quarters is the same as a half. Yeah, well done.

In this exchange Hilda wanted the students to see that two quarters equal one half and together with a student she built an explanation of why knowing a quarter of sixteen made it possible to figure out half of sixteen. In addition, Hilda used mathematical terms appropriately in her lessons, including parallel, horizontal and symmetrical. Occasionally students challenged Hilda's knowledge, as they had done to Brendan, such as when a student claimed that a globe was an example of a circle. Hilda corrected the misconception.

On other occasions her instruction exhibited lower mathematical quality such as when she accepted a student's characterization of a rectangle as having "two small sides and two long sides." This definition excludes a square, a special case of a rectangle where all sides are equal in length. In another lesson about a rectangle the following exchange occurred:

T: How many faces would it have? Ailbhe?
S: Two
T: Two faces, front and the back. So because it has two faces, what type of a shape is it? Who can tell me what type of a shape is it? Daniel?
S: 2-D.
T: Good boy, 2-D. And what does 2-D mean? 2-D shape, Joan?
S: It means that it's flat.
T: It's flat exactly, a 2D shape is?
S: Flat.
T: Flat exactly; because it only has two dimensions, it only has two faces, the front and the back. Whereas the 3D shape is?
S: A cube.
T: Bigger like a cube, very good, a cube or a cuboid, because it's got more faces. So that is quite important that we know the difference between 2-D and 3-D shapes, so today were learning all about?
S: 2-D

In this interaction Hilda asked a student how many faces on a rectangle and Hilda agreed with the student's response of two. She named the faces as the front and the back of the rectangle. The error is compounded when three-dimensional shapes were contrasted with two-dimensional shapes as having more faces rather than because they are solid shapes. This lack of knowledge about the dimensions of shapes curtailed the information Hilda gave her students. Earlier in the lesson Hilda defined parallel as follows:

> What parallel means is that two lines are running beside each other but they will never meet. Can you see the way these two lines run straight up? Ok. They go straight and they are never going to meet because they will keep going straight. Ok. The same with these two sides, see, they are going straight beside each other but they'll never meet.

Although Hilda supplements the definition by pointing to the relevant sides of the rectangle, the definition contains terms that could be confusing for second class students such as "running beside each other" and "never going to meet." This is an example of a definition that might be suitable for older students but where some expressions render it unhelpful for younger students. Hilda's instruction exhibited instances associated with high and low MKT. In particular, her responses to students' errors had some evidence of low MKT, whereas she exhibited rich mathematics in her explanations and use of multiple representations, indicators of high MKT.

Irish Teachers with Discrepant Mathematical Knowledge for Teaching and Mathematical Quality of Instruction Scores

In contrast Veronica's instruction was inconsistent with her level of MKT. She scored in the 57^{th} percentile of Irish teachers but her lessons were littered with mathematical errors, especially evident in imprecise use of language for discussing mathematical ideas. In one lesson a student suggested that a sphere was "like a three-d circle" and Veronica repeated this, noting that it was a "very good description." (SDVS3, Lesson D). In the same lesson a student said that a cube had 24 corners and Veronica made no distinction between the four right angles on each face of the cube and the eight vertices. The examples of three-dimensional shapes chosen and accepted by Veronica were often problematic. Any three-dimensional shape with a vaguely rectangular face was considered to be a rectangular prism, such as a stack of paper and an overhead projector. No distinction was made between traffic cones and the mathematical meaning of a cone. In one lesson Veronica was unsure if a two-dimensional visual representation of a rectangular prism was a cube and to test this she measured the edges on the representation of the prism, apparently not realizing that the actual measures are distorted when three-dimensional shapes are drawn in two dimensions. In Veronica's lessons much time was spent on activities such as cutting, and making shapes. These activities, which would not have been out of place in an arts and crafts lesson, contained little mathematical value. On the positive side Veronica asked students to think of examples of three-dimensional shapes in their environment, she encouraged students to explain their thinking and she pushed them to keep trying when tasks were difficult. Overall, however, her relatively low mathematical knowledge seemed to constrain the mathematical quality of instruction in her class.

Of the ten teachers, Veronica's overall mathematical knowledge was most inconsistent with her MKT score. Her global lesson score was two bands below where it might be expected to be based on the MKT score. Why was the mathematical quality of Veronica's instruction different from her MKT score? Several reasons may explain this. Neither Veronica nor her students used a textbook in the observed lessons and this may have deprived the class of a working definition for shapes being discussed. If accurate, comprehensible definitions of shapes had been available, Veronica may have been less accepting of some objects in the environment offered as examples of cones, rectangular prisms and cylinders. In addition, much time in Veronica's lessons was spent making shapes adding little to the mathematics being taught. Such an activity is consistent with the mathematics curriculum which recommends that students construct three-dimensional shapes (Government of Ireland, 1999a). Observing shape construction in practice, however, prompts the paraphrasing of a question asked by Baroody (1989): Can pupils use the activity "in such a way that it connects with their existing knowledge and, hence, is meaningful to *them?* Is the [activity] used in such a way that it requires reflection or thought on the part of students?" (p. 4, italics in original). Evidence from the video lessons suggests that in Veronica's case the answers to both questions were frequently no, and the activities reduced rather than enhanced the mathematical quality of her instruction. Another possible explanation of the inconsistency between Veronica's MKT and the mathematical quality of instruction is her teaching style. She regularly encouraged students to contribute to classroom discussions and she enthusiastically affirmed their contributions. The problem was that in her enthusiasm she sometimes accepted incorrect, inaccurate or incomplete responses and seemed unwilling to challenge students to refine or correct what they said. Furthermore, potentially worthwhile contributions from students were lost in the enthusiastic and lively, but unfocused classroom discussions. In short, Veronica's lessons showed a lower quality of mathematics than expected, possibly because of one of the following factors: the lack of support that the use of a textbook would have provided; her use of activities with little mathematical merit; or her lively discussions combined with an apparent reluctance to challenge the students' responses.

Two other teachers whose global lesson scores were inconsistent with their MKT scores were Clíona and Eileen. Clíona's lesson exhibited a higher mathematical quality of instruction than predicted by her MKT score and Eileen's was lower. Although they were at similar percentiles in terms of their MKT knowledge (82nd and 83rd), the quality of mathematical instruction varied substantially. Eileen's lower than expected mathematical quality of instruction rating may be illustrated with reference to a specific lesson. The lesson centered on a cookery theme, in which she was organizing ingredients needed for the lesson. At the outset of the lesson Eileen asked the students how cooking "ties in with maths." Eileen agreed with several suggestions offered by students: weight, measurement, time, and length but challenged no student to elaborate on how the topics were connected to the cooking theme. She did, however, add ratio to the list but it was explained in an unclear way.

T: Ratio, how does ratio come into it?
St: Five spoons.
St: Five spoonfuls to a cupful of (unclear)
St: It's like fractions and stuff like that.
St: A teaspoonful
T: Exactly. (SDVS6, Lesson B).

Eileen seemed to assume that the students understood potentially complicated ideas and as a result she was frequently less than explicit when explaining terms. Although the seed of the idea of ratio (comparison of quantities) is contained in the exchange above, for a student who had forgotten what ratio is or who had not understood it in the first place this exchange would hardly help. Eileen's own strong mathematical knowledge may have caused her to attribute to students more understanding than was justified by the evidence. She frequently accepted from students and offered to students incomplete explanations.

Using a practical approach (such as cooking) when teaching mathematics is consistent with the Curriculum *Mathematics: Teacher Guidelines* which states that "all number work should be based as much as possible on the children's own experiences and real-life examples used" (Government of Ireland, 1999b, p. 9). The limitations of using real-life examples were evident in this lesson where students were distracted by the context and spent more time engaged in transcribing recipes and deciding who would bring in particular ingredients than on mathematical activities. No doubt, cooking offers multiple opportunities to apply mathematics: doubling or halving ingredients, estimating and weighing quantities, comparing prices of ingredients and so on. One practical example in Eileen's class had great potential for discussing mathematics. A recipe for a custard tart required using 250 ml of egg custard and Eileen wanted the students to make triple the quantity of custard. Students had to figure out the new quantity to be made and the necessary ingredients, based on knowing the ingredients needed to make 1000ml of egg custard. This offered a practical context in which to apply the unitary method but it was lost in the overall excitement of the lesson. There were other examples where Eileen attempted to be ambitious in her teaching (such as calculating probabilities when two dice were thrown) by using interesting contexts but where the mathematics the students were working on was obscured. Eileen chose interesting activities for her students and she regularly encouraged them to look up mathematical ideas in mathematics books. Problems arose when the lesson context overpowered its mathematical context and when Eileen left mathematical ideas vague or incomplete.

Clíona was the other discrepant case. The mathematical quality of her instruction was higher than her MKT score predicted. She had the highest overall lesson score and although her level of MKT is high compared to Irish teachers generally, it was in the middle of the ten teachers discussed here. Clíona's teaching provided opportunities for all students to participate in problem solving and she encouraged them to reason mathematically and to justify their responses. Clíona was careful about her use of language. She conveyed the message to students that they could all do the work required and that effort invested was worthwhile. An extract from one of Clíona's lessons helps explain her style of teaching. In this excerpt she referred to an activity from a previous lesson where the students had used string to measure the circumference of a circle and had made inferences based on the results about the relationship of the circumference to the diameter. Clíona began with a question:

T: What did you learn from that?
S That the diameter, that the circumference is three times bigger than the diameter
T Very good, or approximately. It's not an exact science there. It's approximately three times greater than the diameter.
T So Damien on that information, if I gave you the circumference of a circle how would you establish the diameter or the approximate, the approximate diameter?
S Eh, the

T	If you have your circumference and I'm asking you to give me the approximate diameter how would you do it?
S	Eh fold that in three
T	And?
S	Eh
T	What would you have to do then Damien? You might need another bit of equipment. Can anyone help Damien?
S	Measure it.
T	Yeah, good man. Of course you'd get out your ruler and you'd measure it wouldn't you? So you're folding it in three but come on what else could you do? What would be even easier, as a sum to do that …
S	Divide it by three
T	Good man Robert. Write down your circumference and divide it by three. And what would that give you Robert?
S	Approximately three point seven
T	No, the approximate …
S	The approximate diameter.
T	Good and how would you establish the radius then from that eh Charlotte What's the relationship there between the radius and the diameter?
S	Emm, you ...
T	Radius, diameter, what's the relationship?
S	Divide by two.
T	Thanks Laura. You're very good. (SDVS4, Lesson A)

In this piece of classroom interaction Clíona moved from recalling a previous lesson activity, to posing questions about how to find the length of the radius of a circle. In the course of the discussion she reminded students that describing the relationship of a diameter to the circumference as being a third is approximate. She elicited the operation that could be used to find the length of the diameter if the circumference is known, and she established that the students knew the relationship of the radius to the diameter. She built on students' answers encouraging them to make a link between "folding it in three" and dividing by three. A few hypotheses may help explain why the mathematical quality of Clíona's instruction is higher than suggested by her MKT score. She prepared well for her lessons and frequently referred to her notes and to the textbook. In one case she says "Now children …just give me a moment now. I have it written down here somewhere, what we're going to explore" (SDVS4, Lesson A), indicating that she has planned the lesson material in advance. In another lesson she referred to her notes or to a textbook when explaining the word "vertex." That explanation gives another clue as to her performance when Clíona asked the students for another word for corners:

T:	What other word have we?
Ss:	Vert….vertex…vert-ice
T:	We'll get it right. Vertices. Plural. Vertices. It's a Latin word. Comes from the word "vertex," is a Latin word. So it's one vertex and it's many vertices. So we've faces, we've vertices, and we have?

Clíona responds not just by telling the students the word but by telling them something about the word's Latin origin. Frequently in lessons she looked for synonyms (e.g. for net, and for minus five). Her interest in language generally may help to explain why Clíona was careful and precise in her use of mathematical terms and in her general language when talking about mathematical ideas. A third reason is her teaching situation. The class has three grades and fewer than 20 students in total and Clíona's interaction with the students was like interacting with a large family. Notice in the quotation above Clíona said "*We*'ll get it right." The impression given is of a teacher and students working together to learn. She asked students to describe steps of procedures, to explain and clarify what they meant and she responded to student errors by taking on board the errors and perhaps reframing the question or calling on another student to respond. Sometimes she made mathematical mistakes such as saying that a circle has width and not height, or she confused the mathematical meaning of edge (where two faces meet) with the everyday meaning (edge of a plate). These errors, however, appeared minor compared to the explicitness of her teaching and her encouragement of student effort. Factors such as detailed lesson preparation, attention to precise use of language generally and ways of probing and refining students' answers are unlikely to be measured by the MKT measures but in Clíona's case they enhanced the mathematical quality of instruction.

In summary, nine of the ten teachers' global lesson scores were broadly consistent with their MKT scores. In three discrepant cases other factors served to enhance or detract from the mathematical quality of instruction. In the group as a whole, there is consistency between MKT and mathematical quality of instruction. I now consider another way of testing the inference of the validity argument that the mathematical quality of instruction can be predicted by the MKT measures.

Correlating Mathematical Knowledge for Teaching to Mathematical Quality of Instruction Global Scores and Metacodes

Earlier I described the set of codes used to consider the mathematical quality of instruction. Some of the 32 individual codes were grouped by theme to create a smaller list of codes to describe the mathematical quality of instruction more efficiently. These are sometimes referred to as "metacodes." One metacode, making mathematical connections, refers to whether class time is spent on mathematics or on "busy" activities with little or no mathematical value such as coloring or cutting. A second metacode, responding to students appropriately, relates to how the teacher responds mathematically to student errors or to students' tentative attempts to express mathematical ideas or conjectures. A third metacode, appropriate use of language, refers to the teacher's accurate use of language and notation in instruction. The fourth metacode, total errors, refers to errors made by the teacher either generally or in relation to language specifically (language errors). The fifth metacode, using rich mathematics, describes examples of rich mathematics observed in lessons such as linking representations, explaining and justifying, and being explicit about the use of language and about mathematical practices. The final metacode, equity, refers to the use of mathematics to teach equitably and to include all students by being explicit about language and mathematical practices and by encouraging a diverse range of mathematical competence (Blunk & Hill, 2007).

In the second stage of testing the inference of the validity argument I correlated the metacodes to teachers' MKT scores to study the relationship. I expected to find a good to strong relationship between the ten teachers' performances on the measures and on the metacodes. For example, I expected that teachers with high mathematical knowledge would exhibit instruction rich in mathematical justifications and explanations and that teachers with less MKT would be less likely to do this. The results were more equivocal. An overall positive relationship was found between the MKT scores and the teachers' global lesson score, but the correlation of 0.43 was moderate (see Table 5.2). The moderate correlation between the MKT score and the global lesson score found here contrasts with higher correlations found in similar analyses of U.S. data (Blunk & Hill, 2007). Below I consider a possible explanation for this difference.

A low to moderate correlation (0.358) was found between responding appropriately to students' errors, ideas and questions and MKT scores. As expected, a negative correlation was found between errors made by the teacher and MKT scores. A low correlation between making mathematics connections (i.e. time spent on mathematically productive work) and MKT scores can be explained by most (eight out of ten) teachers scoring equally well on this code. Teachers varied more in their exhibiting rich mathematics in instruction but low correlations between exhibiting rich mathematics (i.e. explaining, justifying, linking representations and being explicit about mathematical practices and reasoning) and MKT scores can be attributed to two teachers with high MKT scores (Olive and Eileen) exhibiting the two lowest scores on this metacode.

Table 5.2 Correlation of teachers' overall MKT scores with metacodes (Spearman's rho).

Scale	Correlation to Total MKT Score
Global score for lesson	0.430
Making Mathematical Connections	0.079
Responding to students appropriately	0.358
Inappropriate responses to students	-0.529
Appropriate use of language	0.188
Total errors	-0.370
Errors of language	-0.103
Rich mathematics	0.055
Equity	-0.261

In attempting to explain the moderate correlation between MKT measures and the mathematical quality of instruction, I noted that validating the KCS items had been problematic in the United States because some teachers had used guesswork or test-taking strategies to answer them (Schilling et al., 2007). Based on this finding I correlated teachers' performances on the MKT items excluding KCS items, with the global lesson score and the metacodes. The findings are contained in Table 5.3.

Table 5.3 Correlation of teachers' MKT scores, excluding KCS items with metacodes (Spearman's rho).

Scale	Correlation to Score on MKT (exc. KCS) Items
Global Score for Lesson	0.576
Making Mathematical Connections	0.370
Responding to students appropriately	0.455
Inappropriate responses to students	-0.638*
Appropriate use of language	0.321
Total Errors	-0.503
Language Errors	-0.236
Rich mathematics	0.212
Equity	0.042

*Significant at the 0.05 level

A stronger correlation (0.576) was evident between teachers' global lesson scores and their MKT scores when the KCS items were removed from the MKT score. Moderate to good correlations were found for other codes, including between MKT scores and both errors made and inappropriate responses to students (negative correlations as expected). Moderate correlations were found between MKT scores and appropriate responses to students' errors, questions and ideas. Low correlations were found between teachers' appropriate use of language and making mathematical connections. The higher correlations generally, with the KCS items removed, are consistent with problems in the KCS items.

Based on how removing the KCS scores affected the correlations, I studied the relationship between other subscales, and the global scores and metacodes. The algebra results were noteworthy (See Table 5.4) for being the best predictor of teachers' global lesson scores. In other words, teachers' performances on algebra MKT items were the best indicator of the mathematical quality of their instruction. Scores on the algebra subset significantly predicted teachers' making mathematical connections and there was a significant, negative correlation between MKT score and responding inappropriately to students.

Table 5.4 Correlation of teachers' algebra scores with metacodes (Spearman's rho).

Scale	Correlation to Score on Algebra Items
Global score for lesson	0.709*
Making mathematical connections	0.370
Responding to students appropriately	0.515
Inappropriate responses to students	-0.802**
Appropriate use of language	0.624
Total errors	-0.370
Language errors	-0.127
Using Rich Mathematics	0.370
Equity	-0.115

**Significant at the 0.01 level
*Significant at the 0.05 level

Although problems with the KCS items may partly explain why scores on subsets of items differently predict teachers' mathematical quality of instruction, I was surprised generally by the finding because in Chapter 4 the algebra, content knowledge and KCS factors were found to be highly correlated among one another. The relatively strong predictive ability of the algebra items suggested that some items better predict mathematical quality of instruction than others and that the algebra subscale contains a higher concentration of such items than the MKT scale as a whole. This hypothesis could be tested using IRT data. One indicator of how precisely items are measuring a domain is the reliability of the set of measures (see Table 5.5). The high number of items on the form as a whole gives the full set of items the highest reliability but both the geometry and the algebra items are more reliable than the KCS and the number content knowledge items. This is despite the small number of algebra items. Two other IRT measures, the average slope and biserial correlation are also worth considering.

A slope indicates how well items discriminate among teachers with similar knowledge, and a point biserial correlation indicates how well an item relates to the underlying construct. I calculated these for each domain of the test and the results are presented in Table 5.6. These data show that the algebra items discriminate best among teachers who are close together on the MKT scale and because six of the ten teachers are in the top quartile of Irish teachers the more sensitive items were better at predicting teachers' mathematical quality of instruction. Furthermore, the set of algebra items was better related to the underlying MKT construct than the items as a whole. Therefore, the low correlations found between the MKT scores generally and the mathematical quality of instruction are likely due to the measures being more effective at measuring large rather than small differences among scores. One way to think about this is that a classroom math balance would be a good instrument for comparing the weights of different bundles of feathers but would be less efficient at distinguishing between the weights of individual feathers. The lack of sensitivity of the measures is not a problem when measuring MKT of a large number of teachers but can be problematic when a small number is involved, as in the video part of the study.

Another explanation for the low to moderate correlations between the teachers' MKT scores, their global lesson scores and the metacodes relates to the uneven distribution of the video study teachers on the MKT scale. Six teachers were in the top quartile of the population and no teacher was in the lower tercile of teachers. When teachers are located so close together on the scale and when the items are poorly discriminating among them, the sample size is effectively reduced. Therefore, MKT scores and global lesson scores may be inconsistent due to measurement error. Because of the high number of high scoring teachers in the video sample, the lower performing teachers contribute most of the variance to the sample. But two of the lower performing teachers in the sample (Linda and Veronica) are outliers, in that one exhibited higher mathematical quality of instruction than her MKT score predicted and one exhibited a lower mathematical quality of instruction than expected. The mismatch between their MKT scores and the mathematical quality of their instruction reduced further the likelihood of achieving high correlations. Repeating this analysis with a set of randomly selected teachers would be worth considering in the hope of raising the correlation between MKT and the mathematical quality of instruction.

Table 5.5

Reliability details for each domain of items (including only the 501 teachers in the representative sample)

Domain	Number of Items	Reliability	Maximum Information
Total	84	0.929	-0.875
Number content knowledge	25	0.785	-1.375
Knowledge of content and students	18	0.674	-0.875
Algebra	13	0.867	-0.750
Geometry	28	0.870	-0.500

Table 5.6

Average slope and point biserial correlation estimates for each domain of items (including only the 501 teachers in the representative sample)

	Total	Number Content Knowledge	Knowledge of Content and Students	Algebra	Geometry
Average Slope	0.583	0.502	0.408	0.788	0.672
Average Point Biserial Correlation	0.429	0.381	0.315	0.545	0.492

Evaluating the Interpretive Argument

I now return to evaluation of the interpretive argument. The specified inference was that teachers' scale scores on MKT measures were related to the quality of the teachers' mathematics instruction. A higher scale score is related to higher quality mathematics instruction and a lower scale score is related to lower quality mathematics instruction. Based on the correlation between MKT scores and the global lesson scores, teachers' scale scores on the measures are related to the mathematical quality of instruction. However, based on data from the ten teachers examined here, the relationship between MKT as a whole and mathematical quality of instruction is only fair to good. It holds for groups of teachers (i.e. for the group of ten teachers as a whole a relationship exists between their MKT and the quality of instruction). Nine of the ten teachers' relative positions on the MKT scale were broadly similar to their positions on the global lesson score assigned by video-coders. A moderate correlation exists between overall teachers' MKT scores and their global lesson score. It cannot be claimed, however, that the relationship between MKT and mathematical quality of instruction is true on an individual level because discrepant cases were identified. Five teachers exhibited a lower mathematical quality of instruction than predicted. In the case of two of them it may have been because much mathematics class time was spent on non-mathematical activities, explanations were vague, and students' ideas were unchallenged. Two teachers demonstrated a higher level of mathematical quality of instruction than expected. In one case this may have been achieved through detailed lesson preparation, an interest in language generally and by encouraging and challenging students.

For the purposes of this study, the MKT measures can be used to make inferences about the quality of Irish teachers' mathematics instruction generally, but in any specific teacher's case the inference may not hold. Further research might look at characteristics of items that better predict mathematical quality of instruction than others. In addition, further validity studies are needed to confirm the validity of the elemental and structural aspects of teachers' responses to the MKT items. The latter might be done in conjunction with U.S. research, where researchers are considering revising the measurement of KCS and refining the specification for SCK (Schilling et al., 2007).

Results of Irish teachers' performances on the items will be presented in Chapter 6. I have shown that the construct of MKT is similar in both Ireland and the United States. In addition, teachers' MKT results are valid for use at a large group level in that teachers' scores on the items are generally predictive of the mathematical quality of their instruction.

CHAPTER 6

Irish Teachers' Mathematical Knowledge for Teaching

Mathematics education and student achievement in Ireland could benefit if more were known about Irish teachers' mathematical knowledge. But little has been written on the subject to date. A 2002 report, *Preparing Teachers for the 21st Century* (Department of Education and Science), recommended that prospective teachers' competence in mathematics needed to be improved but no specifics were given. Three years later another report found that over a quarter of newly qualified teachers felt poorly prepared to teach mathematics (Department of Education and Science, 2005a). Although content knowledge is not specified, it is likely a factor because other research among prospective teachers identified shortcomings in their mathematical knowledge (e.g. Corcoran, 2005; Leavy & O'Loughlin, 2006). At primary school level a study of student performance in mathematics considered the potential influence of several teacher variables on student achievement, but no reference was made to the possible impact of teacher knowledge (Shiel et al., 2006). A nationally representative study of Irish primary teachers' mathematical knowledge has not taken place before now, most likely because of the absence of suitable measures and practical difficulties of investigating teachers' subject matter knowledge on a large scale (Hill, 2007). In this chapter I present the results of the first investigation of Irish teachers' MKT.

MKT is a resource teachers can tap into when doing the work of teaching mathematics. Compared to other resources for teaching, such as smaller class sizes or classroom materials for example, it has been under-acknowledged and underdeveloped by Irish educators for many years. In this chapter I show that levels of MKT vary widely and I identify strengths and weaknesses of Irish teachers' MKT. By raising awareness among teachers, teacher educators and policymakers of MKT, I hope to improve mathematics instruction and boost student achievement.

Characteristics of Mathematical Knowledge for Teaching

Unlike manipulative materials, textbooks, classroom management skills or other resources for teaching mathematics, MKT cannot be seen or directly observed. As such it is often described as a latent trait. In order to measure a latent trait, a theory is needed to describe it and to indicate what behaviors are associated with having high, moderate or low levels of it (Ludlow, Enterline, & Cochran-Smith, 2008). The construct of MKT and its hypothesized domains of CCK, SCK, KCS and KCT have been described earlier (see also Ball et al., in press). Because MKT supports the tasks of teaching, a teacher who scores well on the measures is expected to be more proficient in performing the tasks than a teacher with a lower score. In other words a high-scoring teacher possesses the knowledge to provide instruction of a higher mathematical quality than a lower scoring teacher. In Chapter 5 evidence was provided to support this hypothesis in the Irish context. Previously a relationship between MKT and mathematical quality of instruction was found in the U.S. context, and documented by Hill and her colleagues (in press). The Hill et al. article draws on research by the Learning Mathematics for Teaching research project and on other literature to identify characteristics of mathematics instruction associated with high and low levels of MKT. Evidence gathered to date suggests that MKT influences teachers' contribution to instruction in several ways. I used the article by Hill et al. to summarize (in Table 6.1) characteristics of instruction managed by teachers with high and low MKT. Although other factors appear to mediate the impact of MKT on instruction – beliefs about mathematics learning, beliefs about equity, beliefs about textbooks, the availability of curriculum materials and the teacher's tendency to replace textbook materials with supplementary materials (Hill et al., in press) – MKT may constrain or enhance instruction in relation to the features listed in Table 6.1.

Table 6.1

Ways in which MKT can enhance instruction and lack of MKT can constrain instruction. All features and citations of literature in table are taken from Hill et al. (in press)

How Possessing MKT Can Enhance Instruction	How Lack of MKT Can Constrain Instruction
• Definitions are mathematically accurate and intelligible for students • Prior discussions with students are reviewed • Mathematical ideas are built sequentially • Examples carefully chosen from a mathematical perspective • Mathematical tasks skillfully chosen and sequenced • Few mathematical errors made • Mathematical explanations are plentiful • Technical mathematical language used carefully • Explicit mathematical language used • Fluent transitions between ‘general’ language used in everyday life and the more specialized language of elementary mathematics • Multiple representations are used • Representations are linked to one another • Teacher is explicit about mathematical practices • Teacher uses student error in the course of instruction • Teacher hears and interprets students’ mathematical statements • A setting is constructed that supports rich mathematical thinking • Classroom activities are connected to mathematical ideas or procedures and not masquerading as mathematics • Students have constant opportunities to think mathematically • Students have constant opportunities to report on their mathematical thinking • Students have opportunities to agree or disagree with one another according to classroom customs • Mathematical thinking and reasoning	• Key parts of definitions are omitted • Parts of lessons seem disconnected from one another in terms of mathematical content • Lessons lack directionality • No mathematical connections are evident across lessons or topics • Numeric examples are not selected strategically • Teacher makes frequent mathematical mistakes • Teacher introduces mathematical missteps and errors to instruction • Mathematics of lesson is poorly recorded • Few explanations made by teacher or students • Teacher makes errors when explaining • No mathematical justification or proof evident in lesson • Lack of precision and care around mathematical language • Language leaves open the possibility of misunderstanding, especially for some students • Teacher makes errors in technical and general language • Important mathematical ideas and problems are taught as procedures, focusing on the mechanics without corresponding explanations • Multiple models are not used to demonstrate mathematical ideas • Teacher is rarely explicit about mathematical reasoning and practices • Teacher is poor at responding to student productions and errors • Rich mathematics absent from lessons • Teacher focuses on activities per se rather than on the goals the activities could serve

are explicit • A commitment to equitable outcomes among students is evident • Culturally appropriate, child-accessible, and sensitive contexts are used • Teacher responds flexibly to students • Multiple solution methods are encouraged • Teacher interprets and responds to students’ thinking (Fennema & Franke, 1992; Fennema et al., 1993) • Attention is placed on *how* to solve problems (Fennema & Franke, 1992; Fennema et al., 1993) • Many types of problems are available to students (Fennema & Franke, 1992; Fennema et al., 1993) • Instructional goals include conceptual understanding as well as skills development (Sowder et al, 1998) • Teacher probes for student understanding (Sowder et al, 1998) • Open-ended questioning and student discussion are used (Swafford et al, 1997) • Teacher emphasizes the conceptual nature of topics (Lloyd & Wilson, 1998)	• Students spend substantial time in mathematics lessons engaged in activities that involve no mathematics • In some lessons mathematics is barely evident • Superficial connections made to mathematical content • Teacher poses no probes to guide students’ exploration • Teacher fails to help students synthesize their exploration • Mathematical quality of instruction drops when teacher departs from the textbook • Inappropriate metaphors used for mathematical procedures (Heaton, 1992) • Teacher accepts wildly inaccurate guesses in a lesson on estimation (Cohen, 1990) • Opportunities to develop student understanding missed (Cohen, 1990) • Teacher fails to push students for explanations and discussions that would lead to mathematical insight (Cohen, 1990) • Shows lack of mathematical sense-making (Heaton, 1992) • Teacher presents material in a way that does not provide a foundation for future development of the topic (Stein et al, 1990) • Teacher has significant difficulties explaining a topic in response to a student question (Borko et al., 1992) • Teacher has trouble talking conceptually about a topic (Thompson and Thompson, 1994)

Measuring Mathematical Knowledge for Teaching

One way to estimate teachers' MKT would be to systematically observe teachers teaching, looking for evidence of high or low MKT[76] qualities in instruction. But many observers would need a lot of time to observe enough teachers to build up a national profile of teacher knowledge in a systematic and rigorous way. With this in mind the Learning Mathematics for Teaching research team at the University of Michigan developed multiple-choice measures based on tasks of teaching. Learning Mathematics for Teaching Project members hypothesize that teachers' MKT determines how they perform on the measures (Lord, 1980, p. 12) and this is the basis for using multiple-choice items to measure MKT. I will illustrate how this might work with reference to one item.

[76] Remember that a teacher's knowledge cannot be measured *directly* by observing the teacher's instruction. Furthermore, Hill and her colleagues (in press) have identified other factors that affect how a teacher's mathematical knowledge impacts on the mathematical quality of instruction.

28. Mrs. Jackson is getting ready for the state assessment, and is planning mini-lessons for students around particular difficulties that they are having with subtracting from large whole numbers. To target her instruction more effectively, she wants to work with groups of students who are making the same kind of error, so she looks at a recent quiz to see what they tend to do. She sees the following three student mistakes:

I	II	III
$\begin{array}{r} {}^{4}\ {}^{12} \\ \cancel{5}0\cancel{2} \\ -\ \ 6 \\ \hline 406 \end{array}$	$\begin{array}{r} {}^{4}\quad {}^{15} \\ 3\cancel{5}00\cancel{5} \\ -\quad 6 \\ \hline 34009 \end{array}$	$\begin{array}{r} {}^{6\,9\,8\,15} \\ \cancel{7}\cancel{0}\cancel{0}\cancel{5} \\ -\quad 7 \\ \hline 6988 \end{array}$

Which have the same kind of error? (Mark ONE answer.)

a) I and II

b) I and III

c) II and III

d) I, II, and III

Figure 6.1 Item No. 28 taken from Form B_01.[77]

[77] Downloaded from http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

The item presents three students' incorrect attempts to do multi-digit subtraction problems. The task is to figure out which students are making the same type of error. Take two hypothetical teachers, one with strong MKT, one with low MKT and imagine how both teachers' knowledge might shape their responses to item 28 in Figure 6.1 (from U.S. Form B_01). The teacher with strong MKT can draw on considerable knowledge to help answer the question correctly. That same knowledge is available to enhance instruction: anticipate, recognize, use and classify student errors; understand the algorithm conceptually; and use place value principles to write equivalent representations of numbers. In contrast, a teacher who approaches the item with low MKT may not recognize the sources of students' mistakes and may respond to the item incorrectly. That teacher may not understand the subtraction algorithm conceptually and may have difficulties applying it when teaching. The teacher may not know what mistakes students are likely to make when applying this algorithm. By having many Irish teachers complete several of these items it was hoped to gain insights into their MKT.

The composition of items on the survey, how the survey was administered and the scoring of items are described in Chapter 3. I briefly summarize some details here. Eighty-seven schools were chosen to represent all schools in Ireland and every teacher in every school was invited to participate in the study. Between June and December 2006 a questionnaire containing 84 MKT items was administered to 501 primary teachers. The items were based on CCK, SCK and KCS and the mathematics topics related mainly to the number, algebra, and shape and space (geometry) strands of the Irish mathematics curriculum. The original items were developed in the United States and adapted for use in Ireland using guidelines developed for the purpose (Delaney et al., in press).

The questions, like the one in Figure 6.1, were based on conceptions of the mathematical work teachers do when teaching mathematics. I briefly summarize the sub-domains of MKT represented in the survey and describe sample item tasks associated with each sub-domain. Some items were designed to tap into teachers' CCK, such as solving mathematics word problems, determining if a number is prime (see Figure 6.2), calculating fractions, and considering the relationship between a rectangle's area and perimeter. Although the tasks mentioned are done by teachers, the knowledge required to do the task is held in common with people who use mathematics in other settings. The item in Figure 6.2, for example, is set in a teaching context but it draws on standard knowledge about prime numbers; respondents don't need to know about students or teaching. SCK tasks are different because they draw on knowledge uniquely needed by teachers in their work. Such items in the survey required teachers to identify fractions of non-standard wholes, to identify non-standard approaches to calculations by students, and to decide how to use base ten materials to represent numbers. Figure 4.2 is an example of such an item. KCS items draw on teachers' knowledge of mathematics *and* their knowledge of primary school students. KCS items in the survey required teachers to solve problems such as identifying mathematical errors students are likely to make and the reasons for such errors, and ordering mathematics word problems by difficulty. Figure 6.1 is an example of a KCS item.[78] In short, the tasks selected for the form were designed to tap into a wide range of mathematical knowledge used by teachers.

[78] A teacher who knows typical student conceptions and misconceptions may draw on such knowledge in addition to mathematical knowledge to respond to the item, which is why it may be classified as KCS. A case can also be made for this to be an SCK item because the item can be solved using only a mathematical analysis.

8. Ms. Chambreaux's students are working on the following problem:

Is 371 a prime number?

As she walks around the room looking at their papers, she sees many different ways to solve this problem. Which solution method is correct? (Mark ONE answer.)

a) Check to see whether 371 is divisible by 2, 3, 4, 5, 6, 7, 8, or 9.

b) Break 371 into 3 and 71; they are both prime, so 371 must also be prime.

c) Check to see whether 371 is divisible by any prime number less than 20.

d) Break 371 into 37 and 1; they are both prime, so 371 must also be prime.

Figure 6.2. Item 8 on Form B_01.

Presentation of Results

Results are presented in two ways. I first give an overview of teachers' performances on the survey items. The items were originally designed in the United States not with specific criteria in mind but to maximize their measurement qualities such as reliability and validity (Hill, 2007).[79] Unlike reporting on a criterion-referenced test therefore, I make no claims about whether teachers have sufficient levels of MKT. Instead IRT[80] was used to create an MKT proficiency scale with an average of zero and a range from -4 to $+4$, where -4 represents a teacher with exceptionally low MKT and $+4$ represents a teacher with exceptionally high MKT. The closer a teacher is to $+4$ the more characteristics of instruction associated with possessing MKT (see Table 6.1) the teacher is expected to exhibit, and the closer a teacher is to -4 the more characteristics of not having MKT instruction are expected. The proficiency scale was developed using responses to items from Irish teachers only. In order to discuss items that are difficult and easy, each item was placed on the same scale from -4 to $+4$. This means that a teacher at -4 on the proficiency scale (i.e. low MKT) would be expected to have a 50% chance of answering correctly an item at -4 on the item scale, so this would be a particularly easy item. In contrast a teacher at $+4$ on the proficiency scale (i.e. high MKT) would be expected to have only a 50% chance of answering an item at $+4$ on the item scale, indicating a very difficult item. The general goal was for the average teacher to answer the average item correctly 50% of the time (Hill, 2007). A teacher with an average MKT score (i.e. 0 on this scale) might be expected to exhibit roughly equal amounts of instruction associated with having and not having MKT.

Variability of Mathematical Knowledge for Teaching Levels

Many Irish teachers performed well on the measures and 15% of them were placed one standard deviation or higher above the mean (See Figure 6.3). Satisfaction with finding strong levels of MKT among some Irish teachers, however, must be tempered by the fact that substantial variation exists among teachers in terms of MKT. The MKT proficiency scale does not map neatly to teachers' raw scores because item difficulties are considered when creating the proficiency scale. Nevertheless, some indication can be given. Each point on the scale roughly corresponds to a teacher answering 12 to 14 per cent more items correctly than the previous point on the scale. In

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[79] Items that everyone answers correctly or incorrectly are not usable. The aim was that the average item would be answered correctly by the average teacher 50% of the time and that there would be a range of items from very easy to very difficult (Hill, 2007).
[80] For an introduction to IRT see *Fundamentals of Item Response Theory* (Hambleton et al., 1991)

other words, a teacher at $+2$ on the scale responded correctly to around 40% more survey items than a teacher at -1 on the scale. To take a more extreme example, a teacher at $+3$ on the scale responded correctly to around 60% more items than a teacher at -2 on the scale. This is a substantial difference in how teachers responded to the items on the questionnaire.

-4	-3	-2	-1	0	1	2	3
Number of teachers	2	12	67	165	182	62	11

Figure 6.3

Numbers of Irish teachers placed on levels of the MKT proficiency scale. Mean = 0.

Another way to consider these findings is that many Irish teachers have the knowledge resources to use accurate definitions that are comprehensible to students, to hear and interpret students' mathematical ideas, to link multiple representations of number concepts, and to skillfully choose and sequence tasks. These teachers are well equipped to manage rich mathematical instruction as envisaged by the 1999 primary mathematics curriculum. Other teachers, however, have only a smattering of such knowledge. Their lessons are likely to be sidetracked into mathematically unproductive work, to be peppered with errors and omissions, and to miss opportunities to develop student understanding. Such teachers are unlikely to have the mathematical knowledge needed to model and encourage mathematical practices such as reasoning, integrating and connecting, and applying and problem solving (Government of Ireland, 1999a). Most teachers are between these extremes but teachers are spread along the scale. Although factors other than teacher knowledge influence instruction, without the mathematical knowledge measured by the items it would be difficult for teachers to coordinate the rich mathematical instruction associated with high MKT.

The spread of scores among Irish teachers cannot be attributed to a disproportionate number of difficult items on the form. In fact, well over two-thirds of the items on the questionnaire were predicted by the two-parameter IRT model to be relatively easy for Irish teachers in the sense that a teacher with a mean level of MKT proficiency was expected to have a greater than 50% chance of answering them correctly. In the absence of a criterion for teachers' performance on this study, it is not possible to say whether or not teachers' MKT levels are sufficient, but the variation in MKT levels among teachers merits attention.

Rather than being a type of knowledge held in more or less similar amounts by every teacher in order to do the work of teaching, the variability of teachers' levels of MKT suggests that among Irish teachers, possessing such knowledge is a matter of chance rather than a given. Because the teachers were selected from a nationally representative sample of Irish schools, Ireland's structures of pre-service and in-service teacher education appear not to be systematically equipping teachers with broadly similar levels of MKT. There seems to be no expectation that all teachers should have this knowledge. Of course some variation among teachers will always exist but the extent of variation found among the teachers in the entire sample – over 60% difference in the number of items answered correctly – seems remarkable,[81] raising the question of how some teachers managed to acquire MKT and others did not. Teachers with high levels of MKT may have acquired it through reading, by reflecting on their teaching, or by applying other mathematical knowledge to the work of teaching or in some other way. No matter how they acquire it, this study suggests that Irish primary teachers possess very different levels of MKT as a resource to enhance their mathematics instruction.

Interestingly, the variation among teachers' MKT differed across topics. Teachers' proficiency levels on number topics varied less than on the survey as a whole (See Figure 6.4). Fewer teachers are in the part of the scale from -1 to -4 (59 compared to 81) and more teachers are placed at either 0 or $+1$ (381 compared to 357). This scenario is not without its problems however; fewer teachers are placed at $+2$ and $+3$, substantial variation remains, and whether or not the mean represents an adequate level of MKT cannot be determined. Nevertheless the table indicates how the distribution of MKT might look if levels varied less among teachers. A policy initiative to improve teachers' MKT levels would aspire to both increase the mean and to reduce the variation so that all students are taught by teachers with similar, strong levels of MKT.

[81] I claim this is remarkable because entry to teaching has always been competitive (Greaney et al., 1999) and entrants to teaching in Ireland have traditionally been in the top quartile of their age cohort in terms of Leaving Certificate results (e.g. Greaney, Burke, & McCann, 1987).

-4	-3	-2	-1	0	1	2	3
Number of teachers	1	5	53	207	174	58	3

Figure 6.4

Numbers of Irish teachers placed on levels of the MKT proficiency scale (Number topics: SCK, CCK and KCS only). Mean = 0.

Some might respond by saying that the level of variation in teachers' MKT is to be expected and possibly even accepted, claiming that there will always be teachers who bring different areas and levels of talent to enhance their teaching. Nevertheless possessing MKT is an important component of students' opportunities to learn mathematics. On reflection, it should come as no surprise that the level of MKT held by Irish teachers varies. Internal and external factors help explain it. An external reason is that for several years prior to the late 1980s researchers in education gave relatively little attention to the topic of teachers' subject matter knowledge. This began to change after Shulman and his colleagues inspired its return to the research agenda (Shulman, 1986; Wilson, Shulman, & Richert, 1987). From the early 1990's there has been a lively interest internationally in studying teachers' subject matter knowledge, especially but not exclusively in mathematics, (e.g. Ball, 1990; Borko et al., 1992; Grossman et al., 1989) and this research is now bearing fruit by linking what teachers need to know with the work they do and describing the knowledge teachers need (Ball & Bass, 2003b). In this sense the lack of attention historically paid to teacher knowledge in Ireland is not exceptional and it contributes to explaining variability in teachers' MKT.

Factors internal to Ireland help explain the variation as well. Ireland's teachers have become more diverse in the last 10 years with teachers certified in other countries[82] and graduates from a new provider of teacher education joining the profession. Furthermore, prospective teachers are not required to study mathematics as part of their teacher preparation program and most prospective teachers study no mathematics after completing secondary school. Moreover, recent in-service education for teachers has focused on conveying teaching methods rather than subject matter knowledge to teachers (Delaney, 2005). As a result, teachers are left to acquire what MKT they can, wherever they can. Research at the University of Michigan has contributed to an awareness of the complexity of the work of teaching mathematics and the benefits of taking seriously teachers' MKT. It seems timely that the type of knowledge teachers need and how they can acquire it be considered in Ireland.

Until now I have focused on the big picture with regard to Irish teachers' MKT. I have shown how MKT is spread among Irish teachers. Many teachers have high levels

[82] I am basing this claim on data provided by respondents to my questionnaire and on the large numbers of teachers who sat the Scrúdú le hAghaidh Cáilíochta sa Ghaeilge in recent years (e.g. 533 in April 2007). This is an Irish language exam for teachers certified outside the state who wish to achieve recognition to teach in Ireland. Source: http://www.scgweb.ie (accessed on February 24, 2008).

of MKT but substantial variability exists. I have described MKT as a resource that is necessary, but not sufficient, for influencing the mathematical quality of instruction. I now turn to the mathematical knowledge teachers displayed when completing the survey. In particular, I identify tasks teachers found more and less difficult.

Performance in Specific Areas of Mathematical Knowledge for Teaching

Irish teachers found more survey items easy than difficult. As previously mentioned, each item is placed on a scale that corresponds to the teacher proficiency scale based on how the Irish teachers responded to the survey.[83] Items with a difficulty level of -4 are extremely easy because a teacher with very little MKT has a 50% chance of answering them correctly. In contrast an item at $+4$ is extremely difficult because even a teacher with much MKT has only a 50% chance of responding correctly. An item of average difficulty will be placed at 0 on the scale. Almost three quarters of the items (61 out of 84) had a difficulty level lower than zero, indicating that on average Irish teachers generally found more items easy than difficult. Figure 6.5 shows how items were distributed among different topics on the form according to difficulty. The average item difficulty level was -0.73 which means that a teacher with a proficiency level of approximately -1 on the scale had a 50% probability of answering the average item on the survey correctly.

[83] Responses to all items were analyzed in a 2-parameter IRT model and the item difficulties are presented by category in Figure 6.5.

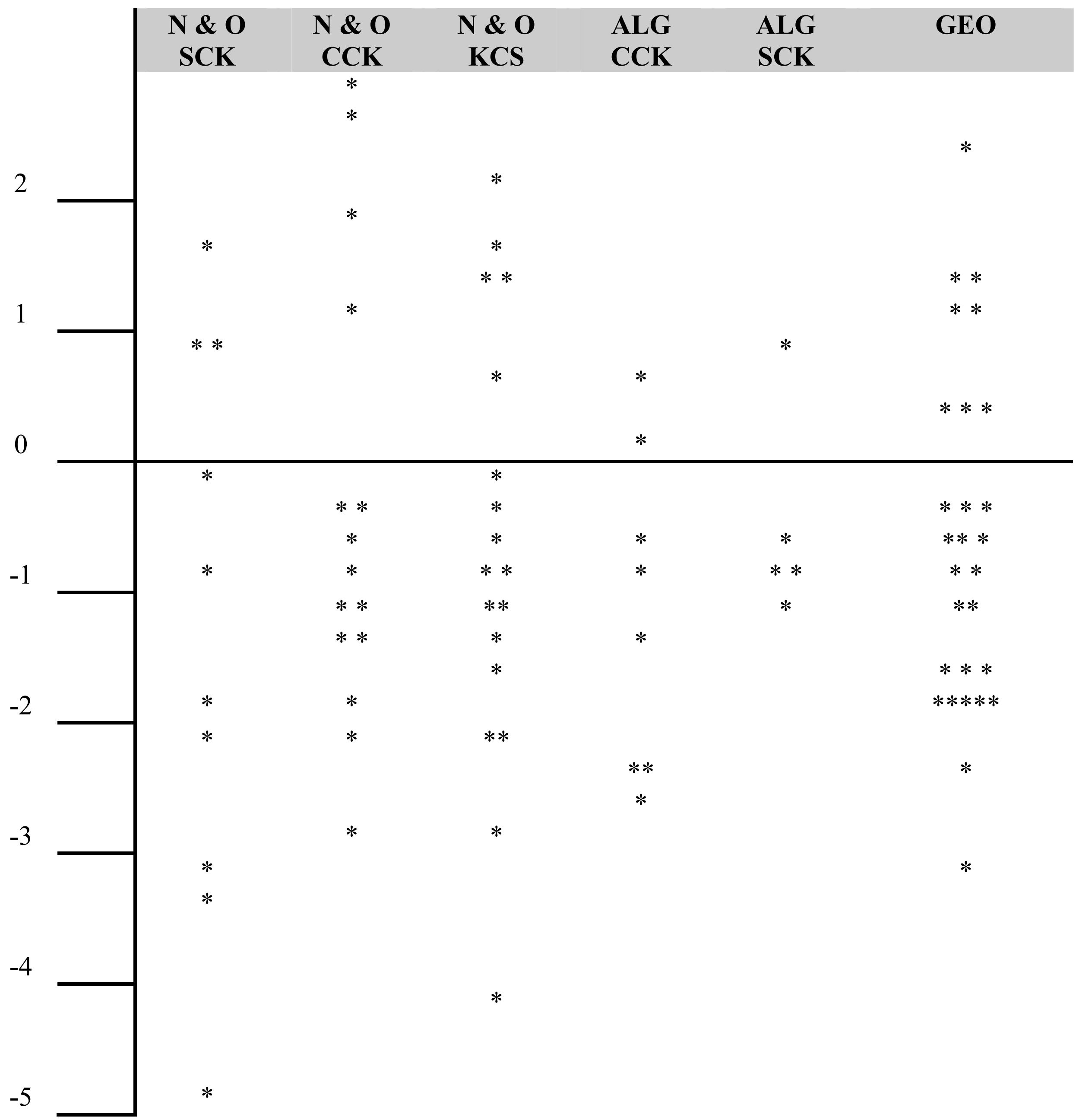

Figure 6.5

The distribution of items by type (number and operations – N & O; algebra – ALG; geometry – GEO; SCK; CCK) and difficulty (mean = 0). * = item.

Looking across the three strands of number, algebra and geometry (shape and space), Irish teachers performed best on algebra items[84] and found geometry items most difficult (see Table 6.2). Performance on the number items was in between, slightly closer to performance on the geometry items. This suggests that Irish teachers' knowledge of algebra is stronger than their knowledge of both number and geometry. Another way to look at the overall performance of Irish teachers is to look at the sub-domains that make up MKT (excluding geometry). When viewed this way, teachers performed best on items requiring SCK and next best on items drawing on KCS. The most difficult items were those requiring CCK. This means that Irish teachers' knowledge, specialized to the work of teaching is stronger than their CCK.[85] This finding, which is particularly noticeable if one looks at number items only, contrasts with that of middle school teachers in the United States among whom CCK is stronger than SCK (Hill, 2007). The difference may be partly explained by one particularly easy SCK item on the survey form. Another hypothesis is that many who enter teaching in Ireland do not have particularly strong CCK[86] but they acquire specialized knowledge needed to perform in the classroom through experience or by taking methods courses. Students who enter the two largest colleges of education study mathematics methods but are required to take no additional content courses after their Leaving Certificate Exam.[87] Evidence from the SCK items provides some support for this hypothesis. The six SCK items on which teachers performed best related to the use of graphical representations of fractions or Dienes materials. Although survey items draw only on mathematical knowledge (and not on knowledge of how to teach), it is likely that prospective teachers may learn some mathematics by working with materials and representations in mathematics methods courses and they may acquire MKT from colleagues when they start to teach.

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[84] This is consistent with findings in the United States and may be due to items not tapping the most difficult knowledge demands of teaching algebra at primary school level.

[85] Caution must be exercised in interpreting these results because performance can be influenced by the particular combination of items on the form. The goal generally in creating such tests is to have items where the average teacher has a 50% of responding correctly to the average item but given that the measures are relatively new it is possible that more difficult CCK items or easier SCK items may have been included in this form.

[86] The mathematics entry requirement to teacher education is the most basic possible, a D3 on ordinary or higher level Leaving Certificate (http://www.education.ie/home/home.jsp?maincat=&pcategory=10900&ecategory=19312&sectionpage=12251&language=EN&link=link001&page=1&doc=16908)

[87] Some students in these colleges, however, will elect to study academic mathematics for one year or to degree level.

Table 6.2

Average difficulty levels for sets of items on the survey form. Higher values indicate more difficult items. N&O = number and operations; ALG = Algebra; GEO = Geometry.

N & O SCK	N & O CCK	N & O KCS	All N & O	ALG CCK	ALG SCK	All ALG	ALL CCK	ALL SCK	GEO
-1.30	-0.36	-0.70	-0.72	-1.16	-0.57	-0.93	-0.64	-1.06	-0.64

I have discussed the variability of MKT among Irish teachers and their overall performance on specific aspects of the questionnaire. I now move on to discuss their performance on specific elements of the questionnaire. The relative difficulty levels of items reveal what Irish teachers know more and less well. Almost three quarters of items on the survey are located below the mean in terms of difficulty. This means that most items were relatively easy for Irish teachers and that Irish teachers performed well in several areas. In Tables 6.3a and 6.3b I identify areas of strengths and areas of difficulty for Irish teachers giving details of the MKT domain to which they relate and the number of items that informed the analysis. I now discuss these areas in more detail beginning with an overview of the findings.

Less difficult areas for Irish teachers, indicating strength in performance in those areas were identifying and classifying mistakes made by students (with one exception), understanding of graphical representations of fractions, and algebra generally. The numbers of items related to the areas of strength are listed in Table 6.3a. Areas that Irish teachers found more difficult were applying definitions and properties of shapes, identifying and applying properties of numbers and operations, attending to and evaluating explanations and linking number and word problems. I considered items with a difficulty level of 1.0 or higher (on the -3 to +3 scale) to be difficult. The numbers of items related to each category are listed in Table 6.3b.

Table 6.3a

Areas of strength in Irish teachers' MKT

Area of Strength	Domain of MKT	Number of Items
Identifying and classifying student mistakes	KCS	3 (+1 exception)
Graphical representations of fractions	SCK	5
	CCK	1
Algebra	Algebra	4

Table 6.3b
Areas of potential development in Irish teachers' MKT

Area for Potential Development	Domain of MKT	Number of Items
Applying definitions and properties of shapes	Geometry	5
Identifying and applying properties of numbers and operations	CCK	3
Attending to and evaluating explanations	KCS	3
	SCK	1
Linking number and word problems	CCK	1

Areas of Strength in Irish Teachers' Mathematical Knowledge for Teaching

Identifying and classifying student mistakes.

Irish teachers generally know how to identify and classify student mistakes. The item shown in Figure 6.1 is a typical example. When applying a conventional subtraction algorithm students in the item made three mistakes. For most adults solving the subtraction problem suffices. A teacher has to do more: check if the student has answered correctly or not; identify any mistake; determine what may have caused the mistake; and in this particular teaching task decide which two errors are similar so that specific students can be supported to rectify the type of error made. A teacher who possesses the knowledge to identify errors is confident enough to allow students to make mistakes and students have no reason to be afraid of getting a wrong answer (Schleppenbach, Flevares, Sims, & Perry, 2007). Helping a student to de-bug or repair errors (Brown & VanLehn, 1980) such as those observed in Figure 6.1 is a productive way to respond to student errors. Teachers who are competent at identifying and classifying errors, as Irish teachers are, have the MKT to use student errors to promote mathematical thinking in their classrooms and to plan further teaching keeping the likelihood of such errors in mind (Schleppenbach et al., 2007).

One exception to this overall strength in identifying and classifying errors was an item where teachers were required to diagnose the cause of an error. Specifically, teachers found it difficult to explain why a student might respond incorrectly to a math problem of the form $a+b=\_\_\_+d$. Researchers have found that primary school students frequently respond to questions in this form by computing either or both of the following sums $a+b+d$ or $a+b$ (Falkner et al., 1999). If teachers know that students frequently interpret the equals sign as an order to compute rather than as an indicator of equality, teachers can plan their teaching to challenge the misconception. This area of teacher knowledge draws on knowledge of students and knowledge of mathematics

(KCS) and is related to identifying and classifying errors because it is knowledge teachers use when they respond to student errors.

Graphical representations of fractions.

Teachers in the Irish study performed well on problems where they were required to work with graphical representations of fractions. The representations included what Ni (2001) classifies as regional area models, a set model, a line segment and number lines. Students' learning of several fraction concepts, including that of equivalence, adding, and subtracting, can be enhanced when teachers use their knowledge of representations and translate between them (Bright, Behr, Post, & Wachsmuth, 1988). Irish teachers need to use their knowledge to make these translations because area models of fractions are the dominant form of representing fractions in Irish textbooks (Delaney et al., 2007) and few problems require students to work across representations. This study shows that teachers have the knowledge necessary to compensate for this shortcoming in textbooks. In another graphical representation of fractions context Irish teachers had little difficulty solving what Saxe (e.g. 2005) and his colleagues call an unequal area problem, which required respondents to identify a fractional part of a square partitioned in unequal parts.

Algebra.

Another positive finding was that Irish teachers performed well on algebra. Among the different groups of items on the survey form only the SCK number items were easier than the algebra items for Irish teachers. This is good because primary students generally find it difficult to make the transition from arithmetical thinking to the "relational thinking" required in algebra; thinking where students notice "number relations among and within" number equations and expressions (Jacobs, Franke, Carpenter, Levi, & Battey, 2007, p. 260). Relational thinking represents a more mathematically sophisticated way for students to understand arithmetic. If teachers can use their knowledge to help students make that transition in their thinking, students' understanding of arithmetic improves and a strong foundation is laid for their subsequent understanding of algebra (Jacobs et al., 2007). From the evidence of this study Irish teachers have the knowledge resources to do this.

When studying algebraic reasoning in a third grade classroom Blanton and Kaput (2005) used several categories to characterize the types of algebraic reasoning they observed. I use some of their categories below in describing Irish teachers' responses to algebra items on the MKT survey. Irish teachers had few difficulties using algebra to generalize mathematical processes in geometry. Most teachers successfully combined knowledge of perimeter and algebra to evaluate perimeter formulae expressed in terms of w (width) and l (length). In addition, they drew on knowledge of area to determine how doubling the width and halving the length of a rectangle affects its area. Few teachers had problems finding functional relationships. The idea here is to develop a rule that describes the relationship between different quantities. Teachers did this in the context of a function (input-output) machine with an additive relationship between the input and output numbers. Another functional relationship arose in the context of students developing a rule to predict the number of exposed faces on a train of cubes. Most teachers successfully identified formulae that would and would not predict the number of faces on a train of any length.

Although the evidence from the teacher responses to this study show that Irish teachers are well placed to improve the teaching of algebra, a priority identified by Department of Education and Science Inspectors in the most recent National Assessment of Mathematics Achievement (Shiel et al., 2006), a possible damper must be mentioned. One survey question involved studying a pattern of 4 shapes repeated once, and required respondents to state what the 83rd shape would be. One way to do this algebraically would be to recognize that every whole number can be written in one of the following forms: $\frac{n+1}{4}$, $\frac{n+2}{4}$, $\frac{n+3}{4}$ or $\frac{n+0}{4}$. When one identifies the form of a given number, it is possible to tell if the shape in that position of the sequence will be the first, second, third or fourth shape in the opening pattern. Solving the problem this way works for all numbers. It is possible, however, to answer the question without using algebraic thinking, and judging by the annotations on nine returned survey forms, some teachers worked this out by counting up to 83 in some form, such as writing 8, 12, 16, 20, 24, etc. below the shapes. The problem is that this will work for finding the 83rd term but for numbers over a few hundred it would be a cumbersome way to find the answer and it does not involve the relational thinking mentioned earlier. In a different context and using very different pencil and paper measures, Jacobs and her colleagues (2007) found that although two groups of teachers performed similarly well on written algebra problems, follow-up interviews revealed that some teachers were thinking algebraically and others were not. It is difficult to know how widespread the arithmetic approach to the algebra item was among Irish teachers but it is an instance where the responses may not tell the full story about teachers' knowledge. Nevertheless, the survey responses indicate that over several items, Irish teachers performed well on algebra.

Areas for Potential Development in Irish Teachers' Mathematical Knowledge for Teaching

Applying definitions and properties of shapes.

The set of geometry (shape and space) items was more difficult for Irish teachers. Item difficulties ranged from -3 to $+3$ but the average difficulty was -0.64. I relate this finding to mixed findings about geometry in the 2004 National Assessment of Mathematics Achievement. The achievement of fourth grade students was significantly better than it had been 5 years earlier and Department of Education and Science inspectors were more satisfied with how geometry was taught. Teachers, however, singled out geometry as an area in which they were less satisfied with the in-career development compared to their satisfaction with the treatment of number (Shiel et al., 2006). Perhaps the spread of responses to geometry items in this study (-3 to $+3$) sheds some light on that finding. Irish teachers have strong knowledge in some areas of geometry, possibly contributing to good teaching and higher student achievement in these topics. They seem to have less MKT in other areas and perhaps these topics were not addressed in professional development, contributing to some teacher dissatisfaction.

Teachers found it easy to identify one parallelogram in a series of two-dimensional figures, some of which were and some were not parallelograms. The easiest to recognize parallelogram, making it the easiest geometry item of all, was the one shown in Figure 6.6. It is not surprising that most Irish teachers recognized this figure because it is the example of a parallelogram typically given in Irish text books (e.g. Barry, Manning, O'Neill, & Roche, 2002; Gaynor, 2002). But recognizing this shape does not indicate if the teacher has the knowledge resource to compensate for inadequate definitions of parallelograms presented in textbooks which frequently refer to rectangles pushed out of shape (Barry et al., 2002; Gaynor, 2002). Such definitions are inadequate because they do not help a student or a teacher to recognize that squares, rectangles and rhombuses, all being quadrilaterals with both pairs of opposite sides parallel, are all special cases of parallelograms. As indicated in Table 6.1, one instructional behavior associated with high MKT is careful use of definitions and in some cases MKT is needed to compensate for inadequate or inaccurate textbook definitions.

Figure 6.6
Irish teachers found this image of a parallelogram easy to identify.

Evidence in this study suggests that Irish teachers have difficulties applying definitions of shapes and shape properties. For example, the relationship between a square and a rectangle was problematic with most teachers seeing them as distinct shapes. In fact, a square is a special case of a rectangle where all sides are of equal length. A square is a special case of a parallelogram, a quadrilateral, a trapezoid, and a kite (Weisstein, 2008). Classifying shapes in multiple ways makes demands on teachers' knowledge, in particular their knowledge of definitions and properties of shapes. For simplicity, many textbooks introduce shapes discretely, often with inadequate or no definitions. A related issue is that textbooks may introduce stereotypical examples of shapes, such as using illustrations of a regular hexagon and not qualifying it with reference to its regular quality. Such simplification may initially help students learn shape properties but it quickly becomes inadequate when students begin to investigate relationships among shapes, or test their understanding of shapes with non-examples. Teachers' mathematical knowledge is a necessary resource to prevent students acquiring misconceptions about shapes and to support students who become confused about whether a shape belongs or does not belong in a specific category. It is an area of MKT that many Irish teachers need to acquire.

Knowledge of geometrical properties can be helpful when using materials in mathematics class. The Irish curriculum suggests using geoboards to teach topics such as two-dimensional shapes, symmetry, and square and rectangular numbers (Government of Ireland, 1999a). Geoboards can be used to teach perimeter and an item on this topic was difficult for Irish teachers. The context was a classroom where students had been asked to make shapes with perimeters of 12 cm on geoboards with pegs spaced one centimeter apart. The teacher was checking the work and one student had made a right angled triangle with sides of 3 and 4 centimeters. Although the length of the third side could not be figured out empirically, the Pythagorean Theorem[88] could be applied to determine that the side length was 5 centimeters and therefore, the total perimeter was 12 centimeters. Most Irish teachers, however, responded either that the perimeter does not equal 12 cm or there was not enough information to determine the perimeter. Most teachers encounter the Pythagorean Theorem in secondary school so why did they not apply it when responding to the item? It may be because they had forgotten it or it may be because they did not recognize the situation as one where the

[88] In any right angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides. Common examples are triangles of side lengths 3, 4 and 5 or 5, 12 and 13.

theorem may be applied. Interviews with teachers about their responses would be needed to determine the actual reason. Whatever the reason, it is an example of knowledge that is not part of the primary school curriculum but which is useful knowledge for a teacher when setting tasks for students relating to perimeter.

Identifying and applying properties of numbers and operations.

Irish teachers had difficulty identifying and applying properties of operations and properties of number. Many teachers appeared to lack the knowledge needed to evaluate rules of thumb frequently given to students, such as not taking a larger number from a smaller number. This type of task can be illustrated with an example. A teacher may be asked to consider the rule of thumb that "the sum of two numbers always results in a bigger number." If this rule of thumb is applied to counting numbers (i.e. 1, 2, 3, 4, 5…) it is clearly true. The smallest counting number is 1 and if one adds the two smallest counting numbers possible, 1 + 1, the sum is 2, a bigger number. But if the rule of thumb is applied to whole numbers (0, 1, 2, 3, 4, 5…), it is no longer true. Adding 0 + 0 equals 0 and this is not a bigger number. The sum of 5 + 0 is 5 and this number is not bigger than 5. If the numbers are extended to the integers the rule is untrue because adding -3 and -4 is -7 and – 7 is smaller than both – 3 and – 4. Therefore, despite the intuitive logic that adding produces a bigger number, as a rule of thumb it is not always mathematically true. If students internalize such a rule it may cause problems when they work with negative numbers in fifth and sixth class because they may think that say, -7 is greater than -3.

One reason why Irish teachers may have had problems evaluating properties of numbers and operations is that the teachers may have restricted the numbers they considered to counting numbers, which is the first set of numbers introduced in primary school. This is likely because a third of teachers agreed that it is always true that a larger number cannot be taken from a smaller number. Most if not all of these teachers know about integers from their study of mathematics in secondary school and possibly even from teaching the topic in fifth or sixth class. In addition, a couple of teachers annotated their answers with comments such as "For whole numbers?" (JPM21D) or "Are we talking about whole numbers or fractions?" (GM2F). Knowing the subset of the number system being drawn is part of the subject matter knowledge of teaching (Leinhardt & Smith, 1985). Another reason why these items were difficult for Irish teachers may be that they are not familiar with choosing key numbers on which to test such rules. For example, choosing numbers such as 0, 1, fractions or negative numbers can be useful for evaluating whether rules apply to numbers generally. Knowing properties and rules in relation to different sets of numbers and being able to choose useful examples for testing properties is important for primary teachers because by the end of primary school students have encountered whole numbers, integers, rational numbers and probably at least one irrational number (π). If students find rules they were taught in younger classes no longer make sense as they move through the school, they may perceive mathematics as a subject with arbitrary and incomplete rules. Such a perception is unlikely to contribute to students' understanding or to provide a strong foundation for future learning. A teacher who knows number and operation properties and who is clear about the number sets to which particular rules apply is well placed to prevent students acquiring such misplaced ideas about mathematics. Such a teacher can be comfortable discussing with students when and why mathematical rules and properties apply, making the students more mathematically discriminating, opening up for them a vista of the mathematical horizon (Ball, 1993).

Attending to explanations and evaluating understanding.

The next area Irish teachers' found difficult was in attending to student explanations and evaluating their understanding. I begin by creating a context for this finding. Interestingly, the *Primary School Curriculum: Mathematics* (Government of Ireland, 1999a) refers to explaining only a handful of times in the document and only once in the curriculum objectives. Despite this, I observed several teachers requesting and following explanations from students in the video study. The curriculum does, however, include communicating and expressing mathematical ideas as a practice or skill to be developed in mathematics. It is likely that the curriculum assumes student explanations to be included as part of communicating and expressing "mathematical ideas, processes and results in oral and written form" (p. 12). In addition, when students explain mathematical ideas or procedures, teachers need to be able to evaluate student understanding because teachers are expected to identify "incomplete understanding of mathematical terminology or processes" (p. 116) when students discuss their work. Many tasks of teaching, including attending to explanations are implicit in communication. Lampert (2001) identifies over twenty "teaching and studying" events that occur in her class in one ten-minute discussion. Each event makes demands on the teacher's MKT. The teacher formulates and asks a question, a student makes an assertion, the teacher represents what the student says, the student interprets the representation, the teacher highlights patterns, the teacher asks for an explanation and links the explanation to the representation and so on (pp. 143-177). These descriptions of the tasks, taken out of context, cannot do justice to Lampert's rich description of the complex work of teaching or its mathematical demands but they give a sense of some of what a teacher does to promote the practice of communicating in mathematics class.

Attending to explanations and evaluating understanding may be difficult because many teachers have learned mathematics procedurally in school. Further, given the complexity of the tasks of communicating in mathematics class it should come as no surprise that attending to explanations and evaluating understanding is difficult for teachers generally. Irish teachers are no exception. When teachers were presented with student explanations and asked to evaluate a student's explanation for evidence of understanding they found it difficult. Figure 6.7 contains one problem that was difficult for Irish teachers. The item centers on a pattern on the 100 square which has the quality that anywhere a plus sign, three squares wide and three squares tall, is shaded, the sum of numbers on the row equals the sum of numbers on the column. Students are asked to explain why this is true for all similar signs. The task for teachers is to identify which explanation shows sufficient understanding of why the pattern is true for all similar plus signs.

13. Ms. Walker's class was working on finding patterns on the 100's chart. A student, LaShantee, noticed an interesting pattern. She said that if you draw a plus sign like the one shown below, the sum of the numbers in the vertical line of the plus sign equals the sum of the numbers in the horizontal line of the plus sign (i.e., 22 + 32 + 42 = 31 + 32 + 33). Which of the following student explanations shows sufficient understanding of <u>why this is true</u> for all similar plus signs? (Mark YES, NO or I'M NOT SURE for each one.)

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

		Yes	No	I'm not sure
a) The average of the three vertical numbers equals the average of the three horizontal numbers.		1	2	3
b) Both pieces of the plus sign add up to 96.		1	2	3
c) No matter where the plus sign is, both pieces of the plus sign add up to three times the middle number.		1	2	3
d) The vertical numbers are 10 less and 10 more than the middle number.		1	2	3

Figure 6.7

Problem 13 from Form B_01.[89]

[89] Downloaded from http://sitemaker.umich.edu/lmt/files/LMT_sample_items.pdf

Four student explanations are presented. The first one states that in any plus sign shape on the 100-square the average of the three vertical numbers is the same as the average of the three horizontal numbers. If the averages of two equal size sets of numbers are equal then it follows that the sums of both sets of numbers are equal. This response shows evidence of understanding why the pattern is true. The second response simply makes a statement about the specific plus sign shaded on the 100 square. It gives the specific details that the vertical and the horizontal lines are equal by adding them. Nothing said explains why this might be true in other parts of the 100 square and the statement does not move much beyond the original student's claim. The third student explanation uses another relationship between the row and the column to explain why the pattern is true. The student notes that the three numbers on both row and column add up to three times the number in the middle. This observation, which is generalized to "no matter where the plus sign is", shows understanding of why the pattern holds in every case: if the three numbers add up to three times the middle number and the middle number is the same for the row and the column, the sums of the row and the column will be equal. The fourth response shows insufficient understanding of why the pattern applies. The statement is true but it refers only to the numbers in the vertical column, not to the numbers in the horizontal row. In order to show understanding, an explanation must show a relationship that exists between the vertical and horizontal rows.

Irish teachers found the 100-square item difficult, especially parts (b) and (d) where they frequently accepted statements as showing understanding which did not meet the standards of understanding required. Other items requiring evaluation of student explanations were also difficult. Items included explanations of the decomposition algorithm for subtraction and why reducing fractions produces an equivalent fraction. The difficulties Irish teachers had with these items demonstrate that attending to a student explanation (orally or in writing) is difficult. The teacher needs to know what constitutes an adequate explanation of the particular mathematical idea; the teacher needs to be able to interpret what the student produces and compare the two before evaluating the student's understanding. This task calls on knowledge the teacher needs to respond to the student or to ask for further elaboration. The teacher does not have time to check facts in a book and respond later. Even if a book is consulted, mathematical judgment will always be required because the form and content of student explanations are frequently unorthodox and rarely predictable. Being able to follow and evaluate a mathematical explanation made by a student draws on a teachers' KCS.

Linking number and word problems.

Many studies of pre-service teachers have expressed concern about the depth of their understanding of arithmetic operations (e.g. Chapman, 2007), and this understanding can be particularly shallow when operations with fractions are involved (e.g. Borko et al., 1992). This knowledge becomes important in the work of teaching when teachers need to determine what meaning of an operation is implied in a particular word problem or when a teacher needs to write a word problem for a test. Irish teachers found it particularly difficult to match a word problem to the fraction problem $\frac{1}{2} - \frac{1}{3}$.[90] The advice of a teacher from the video study about multiplying and dividing with fractions comes to mind. He advised students not to

> be confused by the fact that it's quarters or fractions that are being mentioned. If they were whole numbers, what would you do? And I bet you'll find an answer fairly quickly. (SDVS9, C, 10)

Although this advice is helpful, what makes writing and interpreting word problems based on fractions difficult is the notion of what constitutes a whole. For example, a word problem such as "Mary had a $\frac{1}{2}$ box of sweets and she gave $\frac{1}{3}$ of the sweets to her brother. What fraction of her sweets were left?" may at first glance appear to match the problem. It mentions both numbers and the word "gave" implies subtraction. But a more detailed look at the question reveals that for the half, the whole is the box of sweets; and for the third the whole is the half box of sweets. Therefore, that word problem is not a good match for the number problem $\frac{1}{2} - \frac{1}{3}$. The word problem as written would be solved using the numbers $\frac{1}{2} - \frac{1}{6}$ where both fractions refer to the whole box of sweets. This item was a testlet where three items shared a common stem. Another item in the testlet in which the word problem referred, incorrectly, to $\frac{1}{3}$ of one half was easier for

[90] In Chapter 4 I noted that this was one task of teaching identified in the U.S. MKT literature but not observed in ten Irish lessons observed by me. It is possible that Irish teachers do not relate fractions to word problems but it seems as if doing so would enhance the achievement of a curriculum objective such as "add and subtract simple fractions and simple mixed numbers" (Government of Ireland, 1999a, p. 89).

teachers. Because the operation was changed,[91] but the whole stayed the same, teachers identified this wording as problematic. It seems, therefore, that the key difficulty for teachers with this task was the subtle change in the whole unit. Matching word problems and fraction calculations, and drawing attention to the relevant whole unit, is important for Irish teachers because the curriculum wants children to see mathematics as "*practical* and *relevant*" (Government of Ireland, 1999a, p. 15, italics in original) but popular Irish textbooks present no worked examples of fraction computations in practical contexts (Delaney et al., 2007). This task drew on teachers' CCK and it is knowledge that many Irish teachers do not currently hold.

Summary of Irish Teachers' Mathematical Knowledge for Teaching

This study has shown that many Irish teachers scored highly on the survey items and on the whole Irish teachers found more items easy than difficult. Among teachers generally, however, MKT varies widely, a variation which is reduced slightly when number items alone are considered. Teachers exhibited strong MKT across all algebra items, including generalizing mathematical processes and finding functional relationships. Performance on geometry (shape and space) ranged more widely than algebra but overall, teachers performed less well on this strand. Teachers had difficulties in applying properties and definitions of two-dimensional shapes. Taken as a whole, performance on the number strand was stronger than geometry but not as strong as algebra. This overall finding, however, conceals marked differences in domains of knowledge. Teachers were strongest on SCK items and the most difficult items of all were those tapping into CCK. KCS was in between. In SCK teachers performed well on knowledge of different graphical representations of fractions. In CCK, teachers had few problems identifying a fractional part of an unequal area shape but applying number properties and operations to test rules and matching a fraction calculation to a word problem (especially when the whole unit changes) was more difficult. In their KCS teachers had few problems identifying and classifying student mistakes, but attending to explanations and evaluating student understanding was problematic.

Early in this chapter I listed ways in which MKT can enhance and constrain mathematical instruction. Several Irish teachers performed well on the measures of MKT in this study but many students are taught by teachers who responded incorrectly to many items. Shortcomings in teachers' mathematical knowledge did not appear

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91 $\frac{1}{3}$ of one half implies $\frac{1}{3} \times \frac{1}{2}$

overnight and raising the mean and reducing the variation of knowledge held will require determined effort. The variation and difficulties in teachers' mathematics today are understandable because little was known about the resource of MKT generally or specifically about Irish teachers' MKT. That is no longer true.

Chapter 7

A Discussion of Adapting U.S. Measures for Use in Ireland

The final chapter takes the form of a hypothetical discussion among people interested in my findings. I am joined by an educational policymaker, a primary school teacher, a mathematics teacher educator, a comparative psychologist and an educational researcher. The policymaker, the teacher and the teacher educator work in Ireland; the educational researcher and the comparative psychologist are based in the United States. Each has read the first six chapters of the dissertation, and drawing from their individual experiences and perspectives they wish to raise questions about the methods used and the findings. I open the discussion with an overview of what I have learned from using the construct of MKT to study Irish teachers' mathematical knowledge. I begin with an overview of the construct followed by a summary of the process used for applying it in Ireland. The findings about Irish teachers' MKT are presented next, followed by a response to the research question.

My goal in this research was to study Irish teachers' MKT. The measures developed at the University of Michigan appealed to me because they were grounded in the work of teaching and I wanted to investigate knowledge needed by Irish teachers to do the work of teaching. The problem was that the measures were grounded in tasks of teaching identified in U.S. mathematics education literature and in a limited sample of U.S. teaching; I did not know if the U.S. teaching that informed the construct was similar to or different from the work of teaching identified in a sample of Irish lessons. Stigler and Hiebert's (1999) work suggested that teaching may be a cultural activity and consequently, the work of teaching may differ in Ireland and the United States. Therefore, before using the measures to study Irish teachers' knowledge, I needed to establish the extent of construct equivalence between the U.S. construct of MKT and the construct in Ireland. Furthermore, I needed to validate the use of the measures for studying Irish teachers' MKT. In the dissertation I described the process used to establish the relevant aspects of construct equivalence and the validity of the use of the measures, before presenting my findings about Irish teachers' MKT.

In the first chapter I made the case for MKT being a resource that could contribute to increasing mathematical skills of students in Ireland and elsewhere. Chapter 2 provides some background to the development of the theory and construct of MKT and summarizes other studies of teachers' mathematical knowledge. It describes attempts to measure teachers' knowledge on a large scale and provides an overview of what is known about primary teachers' knowledge in Ireland. In Chapter 3 the data used in the dissertation and the modes of analysis are described. The main data are responses from a national sample of 501 teachers to a survey and 40 videotaped lessons taught by ten teachers. Analyses include open coding of videotapes, video coding using an instrument developed by the Learning Mathematics for Teaching Project, and IRT and factor analyses of the survey responses. Chapter 4 outlines Singh's (1995) stages of construct equivalence and describes how I established conceptual equivalence, factorial similarity and factorial equivalence for the construct of MKT in Ireland and the United States. In Chapter 5, I established that the measures were valid for use with large groups of teachers and that a challenge of establishing validity was the relatively narrow spread of teachers who participated in the video study. Chapter 6 describes the variation in levels of MKT among Irish teachers and refers to aspects of MKT that were more and less difficult for teachers.

Overview of Mathematical Knowledge for Teaching

MKT conceptualizes a particular type of mathematical knowledge needed by teachers in order to do the work of teaching. The construct of MKT was developed at the University of Michigan where teaching was studied from a disciplinary mathematics perspective. Tasks of teaching were identified and these tasks formed the basis of the construct. The development of the construct was further informed by literature on teachers' knowledge of mathematics, and the design and administration among U.S. teachers of measures of MKT. MKT is thought to consist of at least four domains: CCK, SCK, KCS and KCT (Ball et al., in press). Researchers in the United States have found teachers' MKT to be a good predictor of both the mathematical quality of instruction observed in mathematics classes and growth in student achievement (Hill et al., 2007). Teachers with high MKT scores tend to display better mathematical quality of instruction than their low-scoring peers and their students make stronger gains in performance on standardized mathematics exams than students taught by teachers with low MKT scores. Because the construct of MKT was developed by researchers in the United States, it was possible that the construct was specific to that country. Nevertheless, both the construct and the associated measures appeared to me to have potential for use in Ireland to increase understanding of the mathematical work of teaching there and to identify areas of knowledge where Irish teachers were more and less proficient. This study was designed to test the premise that the measures could be used in this way.

A Process to Apply Mathematical Knowledge for Teaching in a setting outside the United States

In order to use the U.S. measures of MKT in Ireland, I proposed and followed a three-stage process. The stages were to: (a) establish the relevant steps of construct equivalence described by Singh (1995); (b) adapt the U.S. measures for Ireland; and (c) validate, at the ecological level, the use of the measures for Ireland. The first issue was to consider construct equivalence. Three aspects of construct equivalence were studied in this research, namely conceptual equivalence, factorial similarity and factorial equivalence. I claimed by logical argument that functional equivalence applied in both countries. Evidence for instrument equivalence was taken from findings in the pilot study (Delaney et al., in press) and measurement equivalence was not necessary because a cross national study of teacher knowledge was not involved. The most complex step was establishing conceptual equivalence.

MKT is a practice-based construct which means that it is grounded in the practice of teaching. The basis of the U.S. construct of MKT is the knowledge demanded by the work of teaching. I argued that if tasks of teaching observed in Irish lessons were similar to tasks that informed the construct in the United States, I could establish conceptual equivalence. In other words, based on the theory that the work of teaching in any setting determines the knowledge needed, if the tasks of teaching were similar the construct would likely be equivalent in each country. This is because the construct began with close scrutiny of the work of teaching, but analysis of the work extended beyond what the teacher actually did and knowledge the teacher used, to identify knowledge the teacher *needed* or *might have used* to do particular work (Ball & Bass, 2003b). Therefore, the construct of MKT is specific to the United States to the extent that the work of U.S. teaching is its starting point. It is plausible, however, that tasks of mathematics teaching in the United States may differ from tasks of mathematics teaching in other countries because several scholars consider teaching to be a cultural activity (e.g. Anderson-Levitt, 2002).

Tasks of teaching were identified in ten Irish lessons and I looked for similarities and differences between these tasks and tasks that contributed to the development of

MKT. There was substantial overlap between tasks of teaching identified in Ireland and tasks that informed the construct of MKT, suggesting that the construct is conceptually equivalent in both countries. But the lack of a comprehensive description of tasks of teaching made this technique difficult and led to a probable underestimation of the actual extent of conceptual equivalence.

Multiple-group confirmatory factor analysis, a popular method for comparing measures in different groups (Stein, Lee, & Jones, 2006), was used to establish factorial similarity and factorial equivalence of the construct in both countries. That evidence further supported the notion that the construct of MKT was substantially similar in the United States and Ireland.

Once I established that the construct of MKT was similar in Ireland and the United States, attention turned to the MKT measures which are based on the U.S. construct of MKT. The measures were adapted for use in Ireland. Adaptation was relatively minor because English is the predominant language of schooling in each setting. Nevertheless guidelines developed in the pilot study were applied. They included adjustments such as (a) making names culturally familiar, (b) changing U.S.–English spellings to British–English spellings, (c) adapting non-mathematical language and culturally specific activities to local circumstances, (d) changing language of schooling and education to local terms, (e) changing units of measurement (imperial – metric, money) where necessary, (f) changing school mathematical language, and (g) changing culturally specific representations. These guidelines have been documented in more detail elsewhere (Delaney et al., in press).

The adapted measures were validated for use in Ireland. An instrument to study the mathematical quality of instruction developed by the Learning Mathematics for Teaching Project at the University of Michigan was used to code 40 lessons taught by the ten Irish teachers. The codes included a global lesson score to reflect each teacher's overall knowledge of mathematics as exhibited in the lesson, and the following metacodes: (a) making mathematical connections, (b) responding to students appropriately, (c) appropriate use of language, (d) total errors, (e) using rich mathematics, and (f) equity. Each teacher in the video study had previously completed the multiple-choice survey and an IRT score was calculated for each teacher. I studied correlations between the IRT score, the global score and the metacode scores above. I focused on the extent to which MKT scores predicted the mathematical quality of instruction and found a reasonable relationship between them. The correlations were lower than expected but this is likely because six of the ten video study teachers were in the top quartile of Irish teachers in terms of the MKT scores and at least one of the lower scoring teachers was an outlier in that the quality of mathematical instruction she demonstrated was higher than her MKT score predicted. Nevertheless, teachers with high MKT generally exhibited instruction with fewer errors, more precise use of language and more mathematical explanations.

This three stage process can be developed further if it is applied in other countries. For example, compiling a comprehensive description of mathematical tasks of teaching in the United States (or elsewhere) would be useful for other researchers who wish to establish conceptual equivalence by comparing tasks. A list of tasks may not be the best way to conceive of the tasks but a more explicit account of how the work of teaching informed the construct of MKT would be helpful. I used MKT literature to identify tasks even though they were not intended to be used in that way. In the literature tasks had been identified to illustrate various aspects of the construct, rather than to be compared to tasks of teaching in Ireland or elsewhere. Consequently, they vary in detail, in grain size and in scope. I responded to this by providing extensive detail of the Irish tasks when I considered similarities and differences with tasks that had informed the construct of MKT. Additional guidelines would be needed if item translation was required for a country where a language of instruction other than English is used. Validity may be addressed at the elemental and structural levels to establish that teachers use MKT when they respond to items and that their responses draw on knowledge used in teaching (Schilling & Hill, 2007). This would be particularly necessary in a country where respondents were unfamiliar with multiple-choice measures. In the current study, I established that the construct of MKT in the United States and Ireland is similar and I validated the measures for use with Irish teachers. This means that the measures could be used to describe Irish teachers' MKT.

Summary of Mathematical Knowledge for Teaching Findings for Ireland

Overall, Irish teachers' MKT varied substantially with the highest performing teachers responding correctly to around 60% more items than the lowest performing teachers. On specific topics performance in algebra was strong, and although overall geometry was difficult for teachers difficulty levels varied more than in algebra. Among number items SCK was stronger than CCK, thanks perhaps to knowledge acquired in methods courses or through teaching experience. Teachers knew how to identify and classify student mistakes; and knowledge of graphical representations of fractions was good. Areas of difficulty included knowledge of applying definitions and properties of shapes, number and operations: teachers tended to overgeneralize properties of counting numbers to all subsets of the number system. Attending to student explanations and evaluating student understanding was difficult, as was linking fraction number and word problems. Having applied the various techniques and studied Irish teachers' knowledge I now return to the research question that guided this study.

To what extent and how can measures of MKT developed in the United States be used to study MKT held by Irish primary teachers? I established that the construct of MKT is similar in both countries. A set of U.S. items from one form was selected for this study, adapted for use in Ireland, and validated for their connection to the mathematical quality of teachers' instruction. A process to establish construct equivalence, adapt the measures, and validate interpretation of the measures was followed and documented. Based on the similarity of the construct and the validity of the measures for describing Irish teachers' knowledge, I concluded that the measures of MKT developed in the United States can be used to study MKT held by Irish teachers generally. Furthermore, the knowledge captured by the measures is linked to the work of teaching in Ireland. Therefore, the findings about teachers' knowledge in this study can be used in future research in Ireland. The findings can be used to inform mathematics teacher education for prospective and practicing teachers; it can inform policy with regard to mathematics requirements for pre-service teacher education; it can be used to study how teachers acquire MKT because some teachers have acquired it either through other mathematics study, reflection on practice or in some other way. The measures themselves can be used to evaluate growth in MKT through either professional development or through pre-service education. The measures can be used to study teacher knowledge in Ireland to understand its relationship to student achievement.

I would like now to open the conversation to my guests. I see that the primary school teacher has a hand raised and will ask the first question.

Primary school teacher: Seán, I have read through your dissertation and I appreciate your recognition of the mathematical demands of the work of primary teaching. Too often people who are not teachers think that anyone can do the job, but you have shown that it does involve some expert knowledge. At the same time, I wonder if you may have taken it too far. Does it really impact negatively on students' learning if a teacher says you cannot take 6 from 2 or if a teacher draws shapes free-hand on the board?

Seán: Well, such lack of care does affect the mathematical quality of instruction and consequently, could affect students' learning by restricting opportunities students have to learn mathematics. Students have limited time in elementary school in which to acquire mathematical concepts and practices. A teacher is responsible for introducing mathematics to students and inferior and inappropriate representations or imprecise language can hamper students' opportunities to learn. Students may become frustrated or disinterested over the course of a year if they find ideas make little or no sense. Lack of care can also adversely affect what students take from instruction. If a teacher is unclear about what the equal sign means in early years of primary school, for example, misunderstandings can develop which cause problems later when the student starts learning algebra (Knuth, Stephens, McNeil, & Alibali, 2006). Another way to think about the question is to ask if you would be concerned if a teacher mispronounced words in English or misspelt words written on the board. Like in mathematics, this would be considered unacceptable because it reduces students' opportunities to learn and may negatively impact on subsequent learning.

Primary school teacher: Of course I agree that teachers should not misspell or mispronounce words but it is interesting that you mentioned language arts. When I taught younger students I often introduced the letter 'c' to the students telling them that "c has a /k/ sound" (as in cat) but I did not tell them initially that in other contexts and letter combinations it also has a /tʃ/ sound (as in change) or an /s/ sound (as in nice). Surely telling students that they can't take 6 from 2 is similar: not the full picture but a step on the way to more developed understanding?

Seán: When I talk about teachers' mathematical knowledge I am talking about what the teacher knows or needs to know. In Chapter 4 I listed tasks of teaching, things the teacher does, but these were used merely as a way to identify demands on teachers' knowledge. When a teacher tells students that "c has a /k/ sound," most people would be surprised if the teacher did not know about other sounds the letter c makes. In this study, however, a sizeable group of teachers responded to the effect that it was "always true" that you cannot subtract 6 from 2, indicating that they did not know that 6 can be taken from 2. What the teacher does with the knowledge is the next step. Surely it cannot hurt to say to students "I cannot take 6 from 2 using whole numbers" which is mathematically correct, just as a teacher might say, "the 'c' sound we are learning about today is the /k/ sound." When concepts and ideas in mathematics (and presumably in reading) are presented to students explicitly, students are better enabled to see how ideas grow and connect. Consequently, they should be less likely to see ideas in the discipline as fragmented and random. But if a teacher is to present material explicitly, the teacher must first know the material.

Educational policymaker: I'd like to raise a related question. I agree that teachers need to know these things but surely many of the tasks of teaching in Chapter 4 and in Appendix 4.4 draw on trivial mathematical knowledge, knowledge held by lots of people not just by teachers. I mean, everybody knows the properties of two-dimensional shapes for example.

Seán: I agree that some MKT (primarily CCK) is likely to be held by many people, not just teachers, but remember that in 2007 10% of Irish students failed mathematics in the Leaving Certificate examinations and the minimum requirement to enter a teacher preparation course is only one notch above a fail. We cannot presume that teachers will automatically possess the knowledge you describe as trivial. You mention two-dimensional shapes and I recall observing in one classroom where the teacher explained that a two-dimensional shape is called two-dimensional because it has two dimensions, the front and the back. That might seem like a trivial mistake to make, yet where is a teacher supposed to learn that the dimensions refer to length and width? In fact, when considered outside the context of three-dimensional shapes, the teacher's interpretation was a good guess. But if the teacher cannot specify the three dimensions in three-dimensional shapes, students will likely leave that classroom with misconceptions about the dimensions of shapes. In contrast, another teacher took an apparently trivial idea, the commutative property of addition and used it to enhance her teaching of the number seven to senior infant students. I would, therefore, advise being careful neither to underestimate the power for teaching of apparently trivial mathematical knowledge, nor to overestimate the amount of mathematical knowledge held by the population generally.

Educational policymaker: If teaching is as difficult as you claim or if it requires specialized knowledge, do you think there is a role for specialist mathematics teachers in primary schools as currently exists in secondary schools?

Seán: This is an interesting question and one that has practical dimensions as well as theoretical ones. I have shown that teachers have widely varying levels of MKT, and MKT levels are related to instruction. It seems probable that the mathematical quality of instruction is associated with higher student achievement. Therefore, if Teacher A in a school has substantially higher MKT than Teacher B, more students would benefit from higher mathematical quality of instruction if Teacher A taught Teacher B's class for mathematics. In many schools, however, such an arrangement may be difficult to organize. It might work well in a large school, for example, if one teacher with high MKT taught mathematics to fifth and sixth classes, and another teacher taught, say, English to both class levels. In smaller schools such an arrangement may not be practical.

Educational policymaker: I'd like to ask a different question. I notice that in Chapter 5, the validity chapter, you report on teachers' scores on the measures and their performance in the classroom. In some cases there is a mismatch between the knowledge teachers appear to hold and what they do in the classroom. This works both ways. Some teachers coordinated good instruction with apparently less knowledge and some teachers coordinated poor instruction with higher knowledge as measured by the MKT items. Do you believe this is evidence of a knowledge-practice gap, a breach between knowing what to do and being able to do it in practice?

Seán: I can see why this question is of interest and although my research looked at knowledge and not at practice, I can offer some general thoughts. No direct line exists between teacher knowledge and what happens in practice. No one can claim that a teacher who answers discrete questions on a multiple-choice survey instrument possesses knowledge in a way that is usable in practice or even that the teacher will use the knowledge in practice. The instruments that have been developed to study MKT – the survey measures and the video coding instrument – make it possible to study the relationship between knowledge and practice. Using these instruments and teacher interviews, Hill and her colleagues (in press) identify some mediating factors that may affect how a teacher's mathematical knowledge is deployed in instruction: teacher beliefs about teaching mathematics and how to make it fun, beliefs about how to use curriculum materials and the availability of such materials.

However, teaching is a multifaceted endeavor where problems are rarely easy to understand and solve. Lampert (2001) expresses this well in an elaborated model of teaching at the end of her book which shows how complex the interactions are between teacher, students and content in teaching. I think the best way for me to capture where knowledge fits in her view of teaching is to read a brief passage from her book:

> For the teacher, working in relation to multiple, complicated, and changing students and multiple, complicated, and changing contents may be compared to navigating an unwieldy ship on a large and tumultuous body of water. There are shifting winds and current to take account of, there are obstructions that are not obvious, and sometimes it is foggy. With the appropriate tools and knowledge, you can usually determine where you are, where you need to go, and where everyone else is in relation to where they need to go, but not always. (p 446)

This metaphor seems like a compelling portrayal of the practice of teaching and where knowledge fits in teaching. Subject matter knowledge is part of the knowledge that usually helps practice, but not always. Figuring out why knowledge does not always help practice is part of the challenge. For those of us who study subject matter knowledge, it is good to remember that teaching is a complex endeavor in which elements can be, and possibly have to be, dismantled and studied separately. Ultimately, however, they need to be reassembled in real classroom instruction.

Comparative psychologist: Lampert's metaphor sounds like an interesting one for teaching. But I'd like to move the conversation in a different direction now and address more directly the cross-national aspect of your work. You chose to study MKT between two relatively prosperous countries, in which English is the predominant language of instruction. Moreover, ideas can travel quickly and easily between the United States and Ireland. You might be challenged that the techniques used are limited and would not work with more diverse countries. How would you respond to such a suggestion?

Seán: What you say about the countries is true although I disagree that the techniques used are limited in their applicability. The countries were chosen because MKT originated in the United States, and I am from Ireland. Rather than being a limitation, however, I believe that the choice of countries provided a relatively controlled environment in which to first test how measures based on MKT might be adapted for use in another country. Previous researchers have often assumed that if a construct applies in one setting, it applies elsewhere. I made no such assumption and deliberately and rigorously set out to test if the construct of MKT as conceptualized in the United States captured knowledge required to teach in Ireland. I would not describe the attempt as limited based on the countries chosen, because similar techniques could be used to assess construct equivalence and validity of the measures, no matter what two countries were involved. Indeed my detailed video analysis of Irish tasks of teaching demonstrates that despite the possible similarities based on each country's prosperity and language spoken, I made no prior assumptions about how MKT would be similar or different between the countries. If other countries with different languages were involved, additional translation issues would arise, but working with languages that are generally similar made it easier in this initial study to evaluate the success of the techniques.

Comparative psychologist: Apart from the language issues can you envisage other issues that might arise and steps you might take if two more diverse countries were participating in the study?

Seán: Several potential issues may arise. The content and the format of the items would have to be evaluated for their suitability in the new countries. I should point out to my guests that in a pilot study prior to this larger study (reported in Delaney et al., in press), I established that the content of the measures was familiar to Irish teachers. This was done by interviewing a focus group of teachers and eliciting their comments about the suitability of item adaptations for use in Ireland. In addition, I interviewed five teachers who had completed the questionnaire and asked them if the items seemed authentic in light of their work as teachers. The teachers considered the items to be authentic and their mathematical content to reflect the kind of mathematics encountered by Irish teachers in their work. Evaluating the authenticity of the item scenarios would be particularly important if more diverse countries were involved.

Another issue to consider relates to the multiple-choice format of the questions. Multiple-choice formats are not familiar in all settings and establishing what Schilling and Hill (2007) call elemental validity would be important in settings where they are not widely used, to establish that teachers' responses are consistent with their reasoning about individual items (van de Vijver & Hambleton, 1996).

The process to be followed when adapting the measures for use in two different countries would depend on the purpose of the adaptation. If I wanted to learn about the MKT held by teachers in a given country outside the United States, the steps outlined above would likely be sufficient. If, however, my goal were to compare teachers' knowledge across two or more countries, additional steps would be required and the guidelines issued by the International Testing Commission would be helpful in this regard (e.g. Hambleton & de Jong, 2003).

Educational policymaker: If I can interrupt here. What you have said about comparing teachers' knowledge makes me want to ask, how does knowledge held by Irish teachers compare to knowledge held by U.S. teachers?

Seán: I had expected that question. Unfortunately it is not one I can answer right now for several reasons. First, the focus of my study was on studying Irish teachers' knowledge and although I compared the construct as elaborated in the United States with MKT required in Irish lessons, I did not compare or aspire to compare knowledge held by Irish and U.S. teachers. Second, the sample of Irish teachers in this study is a national sample chosen from a random representative sample of Irish schools. None of the groups of elementary school teachers studied to date in the United States was randomly selected or representative of all U.S. elementary teachers.[92] In many cases teachers were chosen because they were attending mathematics professional development institutes (Hill & Ball, 2004). A third reason why comparison is not possible is that in many cases, questionnaires were administered differently in both countries. Some forms were distributed by mail to teachers and the forms were mailed back whereas others were administered in the more formal "test-like" conditions that applied in Ireland. The main issue, however, with comparison across countries relates to equivalence.

In this study I established conceptual equivalence, which has also been labeled as both construct equivalence and structural equivalence (van de Vijver & Leung, 1997). However, scalar equivalence would be needed if scores are to be compared across countries. This occurs "when the measurement instrument is on the same ratio scale in each cultural group" (p. 8). Measures of height in inches or centimeters, or weight are examples of measures that have scalar equivalence across countries. This is a very difficult type of equivalence to establish and a pre-requisite is that the scales first be very precise in each setting (pp. 144-145).

Comparative psychologist: Yes, many steps need to be followed to be able to legitimately compare results across countries.

Educational policymaker: Sorry to interrupt again but I was looking online at documents relating to the *Teacher Education and Development Study in Mathematics (TEDS-M)* (Tatto et al., 2008).[93] This research project is studying teacher education cross-nationally and the researchers use multiple-choice and open ended items to measure mathematics content knowledge and mathematics pedagogical content knowledge. If I'm not mistaken, items from the MKT bank of items are included in their measures. Why can they use such items to compare student teacher knowledge across countries and you claim it's so difficult?

Seán: TEDS-M is a study for which data are currently being collected in 18 countries around the world. The TEDS-M researchers have taken several steps to make cross-national comparisons possible. National probability samples of future teachers are drawn transparently and consistently across countries. Detailed manuals were prepared on all aspects of the study, including its administration so that it is done consistently from

[92] A national study of U.S. teachers' MKT is currently underway. The process of survey administration, however, is different to that used in Ireland.
[93] See https://teds.educ.msu.edu/20080318_TEDS-M_CF.pdf. Accessed on April 11, 2008.

country to country. Items were selected from several research projects (including MKT items) and from several countries. Extensive piloting and trialing of the items and the measures were completed. Furthermore, expert panels examined the items for clarity, correctness, cultural relevance, relevance to teacher preparation and curricular level, and close collaboration on the development of materials was maintained with teams in each country.

Some features of TEDS-M make it different to the study described in this dissertation. The framework of mathematics content knowledge and the mathematics pedagogical content knowledge being tested are based not on the work of teaching but on the content teachers teach. Specifically, the mathematics knowledge to be tested is the mathematics content up to two grades beyond the grade level the future teachers are required to teach. The grade level content is based not on the curriculum in any given country but on the content used in the Trends in International Mathematics and Science Study (TIMSS). The mathematics pedagogical content framework includes tasks of teaching that require mathematical knowledge. An assumption underlying the knowledge framework used in TEDS-M is that the work of teaching and consequently its knowledge demands are similar across countries, which may or may not be the case.

The other difference in TEDS-M is that the Conceptual Framework (Tatto et al., 2008) outlines no plan for validating the use of the measures for making claims about the mathematical quality of instruction. The researchers may not be able to indicate how high-achieving student teachers coordinate instruction compared to their low-achieving counterparts. In short, many procedures have been put in place to enable TEDS-M to make comparisons across countries, but it will be more difficult to establish a connection between future teachers' knowledge scores on the measures and the instruction they are required to implement. Moreover, demonstrating that the framework is equally relevant to the work of teaching in every country in the study will be difficult, given that teaching may be a cultural activity.

Comparative psychologist: Yes, in all such studies certain compromises need to be made, but if every country were similar we would have little to learn from cross-national studies. One thing I wondered briefly about was that you took a lot of care to investigate conceptual equivalence for the survey measures of MKT Seán, but you seemed to accept the video-coding instrument, developed in the United States, as you found it. Can you explain why?

Seán: Like the multiple-choice MKT measures, the video-coding instrument is based on the construct of MKT. The same tasks of teaching are assumed and when conceptual equivalence of the construct was established for the survey measures it applied equally to the video-codes. The tasks of teaching are possibly more transparent in the video-codes than in the MKT literature or items because many codes refer explicitly to tasks of teaching such as selecting "correct manipulatives, and other visual and concrete models to represent mathematical ideas," eliciting "student explanation," and talking explicitly "about the meaning and use of mathematical language." Most of the tasks included in the video-coding instrument were observed in the Irish teaching studied.

Comparative Psychologist: Before we leave the topic of equivalence, I have one more question. Stigler and Hiebert, and other researchers have claimed that teaching is a cultural activity and it differs across countries. They made this claim by studying videotapes of mathematics teaching in several countries. In contrast, you are saying that the tasks of mathematics teaching are similar in the United States and Ireland. I wonder could you say how you interpret your findings in light of previous research such as that reported by Stigler and Hiebert (1999) and the 1999 TIMSS Video Study (Hiebert, Gallimore et al., 2003).

Seán: The TIMSS video studies and the current study differ in several respects with regard to focus, unit of analysis, sampling and grade level. One difference between my work and that of the 1999 TIMSS video study is that although our foci overlapped, they were different. TIMSS researchers looked at teaching methods across countries and I looked at tasks of teaching. Whereas they were interested in what happened generally in the classroom during mathematics lessons, my interest was on what the teacher was doing to coordinate mathematics instruction. A second difference is that TIMSS researchers used a lesson as their unit of analysis whereas I studied individual tasks of teaching within lessons. A third difference is that their data constituted a national probability sample in each country whereas the teaching that informed the construct of MKT in the United States was not a random sample of U.S. teaching and similarly the Irish video study teachers were not randomly selected in Ireland. Finally, in the TIMSS video study most teachers were certified to teach eighth grade mathematics and in the Irish study most teachers were certified to teach up to sixth grade. Each of these differences helps explain variations in our findings.

Caution was required when importing MKT measures to Ireland from the United States based on the TIMSS claim that teaching is a cultural activity. Insufficient detail, however, was supplied by the TIMSS study to conclude definitively that the work of teaching differed across countries. The 1999 TIMSS study identified similarities and differences in teaching practices across countries. Although this included mathematical tasks of teaching such as using representations in problem solving and having students examine alternative solution methods, it included other factors not directly related to the tasks of teaching such as lesson length. Furthermore some tasks of teaching, such as the use of mathematical terms, received no specific mention. I needed to specifically compare tasks of teaching identified in the sample of ten Irish lessons with tasks that underlay the U.S. construct of MKT. In this study I found the two sets of tasks of teaching to be largely similar but this finding may be consistent with the TIMSS findings of teaching as cultural for the following reasons.

First, TIMSS researchers studied and found variations in the extent to which a given country differed significantly from all the other countries. Japan differed most from other countries on particular features of teaching (e.g. percentage of problems containing drawings/diagrams) whereas Australia and Switzerland differed least from the other countries. It is possible that Ireland and the United States are two countries where few differences exist in tasks of teaching. This would be consistent with two countries where a common language is spoken and ideas on many topics are frequently exchanged.

A second reason why my findings are consistent with TIMSS is that Hiebert and his colleagues looked on lesson interactions as ingredients which can be combined differently to produce different types of lessons. My focus of analysis was on individual tasks of teaching whereas TIMSS looked at lessons, in which interactions are integrated, combined and nested in different ways. Therefore, although mathematical tasks of teaching may be common to two countries, the tasks may be combined in ways that produce lessons that differ. Stigler and Hiebert's (1999) claim of teaching as a cultural activity is based on different emphases or arrangements of teaching ingredients in different countries. In other words, countries use the same ingredients but recipes vary. My dissertation focused on the ingredients whereas TIMSS focused on the recipe.

My focus on individual tasks is justifiable because if a mathematical task is done several times a day or only once a week, it still draws on teachers' knowledge. This approach is consistent with the approach adopted by the Learning Mathematics for Teaching Project when developing the construct. It can be argued that what teachers need to know should be prioritized according to the frequency with which teachers draw on particular knowledge. To prioritize knowledge in this way would require us to know more about the work of teaching than is currently known. Such prioritization would require careful consideration about its implications for the construct of MKT. Assuming that tasks of teaching could be prioritized on this basis, a probability sample of lessons would be required in every country to estimate frequency of tasks. Such an investigation may show differences in how frequently tasks of teaching occur in the United States and in Ireland.

Educational researcher: Have you any thoughts about how you might develop the work done in this study?

Seán: Certainly! A broad range of techniques were applied in this study relating to my interests in moving a construct across countries, and in probing the knowledge demands of Irish teaching. Several areas of possible further study come to mind such as more work on validity, investigation of lower secondary school teachers' mathematical knowledge in Ireland and studying teachers' MKT in other European countries. But I would like to outline in more detail four specific areas of study that lead from the work reported here.

One interesting area of research would be to examine the relationship between Irish teachers' performance on MKT measures and their students' mathematics learning. This study has shown an association between teachers' MKT and the mathematical quality of instruction but if MKT were demonstrated to be related to student achievement, it would provide further confirmation that developing the construct of MKT is a productive means of improving mathematics education in at least one country outside the United States. A study could be designed where both students' growth on standardized test scores and their growth in conceptual understanding are investigated in association with teachers' MKT.

At a practical level, research-based initiatives could be designed to develop in-service and pre-service teachers' MKT in Ireland. Teachers' subject matter knowledge in mathematics has been neglected in pre-service and in-service courses in the past. Furthermore, teachers have been critical of mathematics professional development for practicing teachers (Shiel & Kelly, 2001) and many beginning teachers consider themselves poorly prepared to teach mathematics (Department of Education and Science, 2005a). The planning, content and delivery of initiatives to develop MKT would be based on existing international research, adapted for the needs of Irish teacher education. Rigorous monitoring of the effectiveness of such measures would be built into the programs.

Another area of research would be to study factors associated with teachers' knowledge, such as teaching experience, mathematics courses taken and classes taught. Such data were collected with the survey measures in the current study. Hypotheses might be developed from such a study as to how teachers who have high levels of MKT developed it. Further studies could be designed to investigate these hypotheses.

A fourth area of research would be to develop further the glossary of mathematical tasks of teaching in Ireland. Further examples of practice could be studied to identify tasks of teaching. Tasks observed in Ireland and tasks identified in U.S. literature could be used to further refine the descriptions of tasks of teaching. Sample tasks and mathematical knowledge demanded by the tasks could be included to strengthen the list. Such a list of well-articulated tasks would be useful to international researchers because it would enable them to label tasks of teaching in their countries and they would then be better able to compare MKT in the United States with tasks of MKT in their own country.

Educational researcher: But wait a minute here Seán, I don't get this. Why would researchers, say in France, use a list of tasks developed in Ireland to compare tasks in their own country with tasks of teaching in the United States?

Seán: Sorry. I need to fill in some blanks here. Something that would help researchers to study MKT across countries is a relatively comprehensive list of the mathematical tasks of teaching. I found no such list in the United States or elsewhere. A list of tasks of teaching observed in Ireland would be useful for a researcher, say in France, who was studying tasks of teaching mathematics in France because it would provide a clear description of tasks observed in one country, Ireland. Many of the Irish tasks, though granted not all, appeared in the U.S. literature. The French researcher could use the glossary compiled in Ireland to identify tasks common to both Ireland and France and supplement the list of common tasks with tasks observed only in France. The French researcher would then have a list of tasks that could be compared to tasks identified in literature in the United States. Of course, it would be even better if a researcher in the United States had already compared the work of U.S. teaching to the Irish list and identified tasks common to the United States and Ireland and named new ones specific to the United States. My point here is that such a list, developed in any country, would be helpful. It does not have to be developed in Ireland, but I am suggesting Ireland because Appendix 4.2 contains the seeds of such a list for Ireland.

Educational researcher: I see. So the advantage of such a list is that it would make it easier to recognize and identify mathematical tasks of teaching in any country and when tasks are identified it would be easier to compare them across countries. Does the kind of research you describe have any relevance for other subject areas?

Seán: An approach similar to that used by Hill, Schilling and Ball (2004) has been used in the United States to develop measures of teachers' content knowledge for the teaching of reading (Phelps & Schilling, 2004). It would be interesting to see if the reading measures could be adapted for use in Ireland.

Apart from adapting measures the approach of studying practice to determine teacher knowledge might be applied to knowledge required for teaching other subjects in Ireland. For example, the Minister for Education and Science has expressed an interest in raising teachers' proficiency in speaking the Irish language.[94] It is likely, however, that the proficiency teachers need in speaking Irish differs from the language proficiency needed by, say a prospective writer, broadcaster, translator, or historian working through the medium of the Irish language. Teachers need to be able to select vocabulary that provides learners with high leverage in speaking the language early on; they need to be able to sequence vocabulary and grammatical structures to be taught; they need to anticipate common errors students make; they need to know how to express common classroom phrases accurately in Irish; they need to be able to present rules in understandable ways; they need to be able to select contexts in which the language can be practiced and so on. Just like MKT, this seems to me to require a special type of knowledge of the Irish language. This could be studied by carrying out a form of task analysis of the work of teaching the Irish language, similar to the analysis done for mathematics by Ball and Bass and it could yield fruitful results for understanding the Irish language knowledge that would be of most help to teachers.

Educational Researcher: In addition to possible future avenues of research, I am sure you have thought about some limitations of the current study.

Seán: Yes, as with any study there were some limitations. The first was the relatively narrow range of MKT scores held by the teachers in the video study. No teacher in the video study was placed below the 36th percentile of teachers nationally, which meant that the 33% of Irish teachers with the lowest MKT scores were not represented in the

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[94] See, for example, this press release from April 2006 which lists three initiatives aimed at developing teachers' language fluency: http://www.education.ie/home/home.jsp?maincat=&pcategory=10861&ecategory=40280&sectionpage=12251&language=EN&link=link001&page=20&doc=30795 Accessed on April 8, 2008.

validation part of the study. I would like to recruit additional teachers for the video part of the study, especially teachers with lower levels of MKT. By expanding the range of teachers I would get a better idea of how well the MKT measures correlated with the mathematical quality of instruction along the full range of the MKT scale. It is possible, for example, that the measures are better discriminating among teachers with lower MKT than among teachers with high MKT.

A second limitation is that some mathematics topics relevant to the Irish curriculum (measures and data) were not included in the questionnaire because items on these topics were not developed when the survey was conducted. But this did allow a greater representation of items from the number, algebra and geometry strands than would otherwise have been possible.

A third limitation is that neither the Irish tasks of teaching nor the list of tasks that undergirded the U.S. measures is complete and mathematical tasks of teaching not yet identified may exist in both lists. This affects how effectively conceptual equivalence can be assessed qualitatively. A more systematic study of the tasks of teaching is needed.

Finally, any attempt to improve the mathematical quality of instruction must acknowledge that important as mathematical knowledge is, it is only one factor that affects the mathematical quality of instruction. In this study it seems possible that teacher preparation and precision with language generally contributed to higher mathematical quality of instruction in one classroom whereas a distracting topic and lack of care about language lowered the mathematical quality observed in another. In a study by Hill et al (in press) the effect of teachers' knowledge on instruction was found to be mediated by factors such as teachers' beliefs and the use and availability of materials.

Primary school teacher: I spoke to one teacher who completed your questionnaire. She teaches an infants' class and she said that the measures in the study do not reflect the mathematical knowledge she uses when teaching. Is that not another limitation of the study?

Seán: This is an interesting issue and one that was raised by many teachers of junior classes, especially in schools containing only junior classes, when I was administering the survey. The teachers claimed the knowledge being tested applied to teachers of senior classes in the school rather than to teachers of junior classes. I agree that specific mathematical demands exist at the junior end of primary school and that these have not been comprehensively documented. The tasks of teaching at this level require further study and analysis. Nevertheless, when Hill and her colleagues (2005) studied gains made by students in their scores on standardized mathematics tests, they found that teacher knowledge, as measured by items similar to those used in this study, made a difference with first grade students, the youngest age group studied. This suggests that teachers' levels of MKT make a difference when teaching young students even if the topics and tasks of teaching appear to relate to more senior class levels.

An assumption in your question is that teachers gain MKT as a result of teaching experience in senior primary school classes. I am not sure if that assumption is an accurate one. It certainly merits further investigation.

Your point about the mathematical knowledge needed by teachers of junior classes fits into a broader discussion of whether teachers should be certified to teach only junior classes. All certified teachers in the study were certified to teach primary class levels (up to age 12) even if they have never done so, or have not done so for many years. If teachers are certified to teach up to sixth grade, is it not reasonable that they be required to have the knowledge needed to do so for all subjects?

Educational researcher: Rather than answer that question, I'd like to ask another. Seeing that we have discussed the limitations of the study, what contribution do you think your work makes to the field?

Seán: I believe the most significant contribution made by the dissertation is to show that the construct of MKT applies in at least one setting outside the United States. It was possible that because the construct is deliberately grounded in U.S. practice it would not travel well beyond the United States. I have shown that MKT applies beyond the setting for which it was initially intended. This is good news for mathematics educators in many countries who are interested in studying teachers' mathematical knowledge because it provides evidence for a construct and associated measures that may be used to better understand such knowledge in other countries. Second, I have used and documented a rigorous process using several research techniques that can be used to establish conceptual equivalence of a construct across countries and to validate measures for the purpose of describing mathematical knowledge held by large groups of teachers. This is a process that can be repeated and developed by others who wish to study teachers' mathematical knowledge in settings outside the United States. Third, I have identified strengths and shortcomings in Irish teachers' MKT. This is the first step to developing the mathematical knowledge held by Irish teachers.

Educational researcher: Did any aspect of your research point to areas where the U.S. research program may need to rethink aspects of MKT?

Seán: The single most important area where I think the U.S. research program can learn from my findings relates to conceptions of the work of teaching. Central to the construct of MKT is the idea that the knowledge needed for teaching is the knowledge required to do the work of teaching. The research team has directly and indirectly studied the work of teaching in the United States but conceptions of the work of teaching remain under-specified. As can be seen in Appendix 4.3, tasks of teaching vary in terms of grain-size and specificity. More than once in the dissertation, including in this discussion, I have called for a list of tasks of teaching that would help another researcher like me to compare tasks of teaching. But a list may not be the best format. I think, for example, of Lampert's (2001) difficulties in conceiving of mathematical topics to be learned by students in terms of lists and she adopts instead Vergnaud's idea of conceptual fields. Developing the equivalent of conceptual fields for the work of teaching may help the research program to be more explicit and precise in terms of both what the mathematical work of teaching is, and consequently what its knowledge demands are.

A related area to be addressed is that of relating the knowledge needed to the work of teaching. I am thinking in particular here of the work of Hiebert and his colleagues who found that few practices in mathematics teaching in the countries they studied were exclusive to a particular country. Differences found among countries related to how frequently teachers engaged in particular tasks and how tasks were combined in lessons. Hiebert and his colleagues wrote about the seven countries they studied that "there are many similarities across countries, especially in the basic ingredients used to construct eighth grade mathematics lessons….However, mathematics teachers in the different countries used these ingredients with different emphases and arranged them in different ways" (2003, pp. 150-151). At present the existence of a task of teaching in the United States is sufficient for its knowledge demands to be part of the construct but attention may need to be paid to deciding whether the knowledge demands of tasks that are more widespread in the setting should be prioritized in the construct.

Two other factors worth considering relate to the knowledge demands of teaching younger students and the future development of MKT measures. First, I believe that the mathematical demands of teaching younger students (i.e. 4 – 7-year-olds) needs to be studied because mathematical tasks of teaching may be present in these class levels that have not yet been identified. Second, as more countries become interested in using the measures it is likely that researchers in other countries may want to develop additional measures of MKT. The process of developing measures, however, is a demanding, rigorous one that draws on substantial resources of expertise and time. The research team may need to be more explicit about describing the process by which measures are developed, refined and approved. It would also be good to develop criteria by which new measures could be evaluated. Care needs to be taken that the quality of measures is maintained and improved, especially if comparing knowledge across countries is to be considered in the future.

Primary school teacher: A few moments ago you identified some benefits of the study. None of the benefits mentioned seemed to apply to classroom teachers. Do you see your research as being relevant to classroom teachers?

Seán: Yes. I hope classroom teachers will see several benefits of this research. First, it should lead to an improvement in professional development because a new means of evaluating professional development in mathematics for Irish teachers – multiple-choice measures – is described in the study. Second, by detailing samples of mathematical tasks of teaching that have been identified in two settings, the study provides ways for teachers to think and learn about the work involved in teaching mathematics in other countries, which could contribute to more professional mobility among teachers. Finally, I hope that it will contribute to raising teachers' mathematical knowledge and that teachers will find the teaching of mathematics more stimulating and professionally fulfilling and that their students will benefit from higher quality mathematics instruction leading to higher achievement in mathematics.

There we must leave the conversation. I am sure it is one that will continue formally and informally over the coming weeks, months and years. I look forward to that.

Appendices

Appendix 2.1

Sample items from Begle's (1972) test of teacher knowledge.

1. Using the least number of properties, which of the following must be used in showing that *a* (*b* + *c*) and (*c* + *b*) *a* are numerals for the same number?

   I. Commutative property of addition

   II. Commutative property of multiplication

   III. Distributive property of multiplication over addition

2. If x < 0, $\sqrt{x^2} = (?)$

   (A) $-x^2$

   (B) $-x$

   (C) $-|x|$

   (D) *x*

   (E) None of these

3. Which of the following statements is (are) true?

   I. No irrational number has a rational square root

   II. No rational number has an irrational square root

   III. The square of every rational number is rational

   IV. The square of every irrational number is irrational

   (A) I only

   (B) IV only

   (C) II and III only

   (D) I and III only

   (E) II and IV only

Appendix 3.1
Consent letter signed by teachers participating in the video study.

Coláiste Mhuire Marino,
Griffith Avenue,
Dublin 9
December 2007

Dear Teacher,

I am writing to ask for your help with a mathematics study that investigates the mathematical knowledge that matters for primary school teaching and how teachers develop this sort of mathematical knowledge. The research project involves learning more about the mathematical knowledge that Irish teachers use when teaching mathematics. I hope that the findings of the study will inform teacher educators and policy makers about how teachers can be prepared to do the work of teaching mathematics.

There are three parts to the study: (i) a questionnaire (ii) an interview based on the questionnaire and (iii) videotaping of mathematics lessons. The questionnaire has two parts. In the first, I ask you to respond to questions about common mathematics problems in primary school classrooms – for instance examining unusual solution methods, evaluating students' mathematical statements and determining how to best represent material or generate examples. The second part asks some general questions about your background and teaching. This data will NOT be used to evaluate your own knowledge of mathematics. Instead, I will analyze responses from all teachers participating in this project to identify the best questions for use in future studies of teacher learning and to inform future pre-service and in-service mathematics preparation of teachers. In total it takes between 60 and 90 minutes to complete the questionnaire. You are under no obligation to complete the questionnaire, or to answer all questions presented in it. If you come to a question you do not wish to answer, simply skip it.

When you have completed the questionnaire, I would like to interview you about some of the answers that you gave to the questions. In particular, I would like to ask you why you chose particular answers and why you did not pick alternative answers. This will help me to determine if the questions are written as clearly as possible and if the answers are reasonable and precise. The interview would last for 60 minutes or less and it would be recorded on audiotape.

The third phase of the study involves videotaping you teaching four mathematics lessons to your class. I would like to use the videotapes of the lessons to study, in a different way, the mathematics that a sample of Irish teachers uses in their teaching. I want to see if there is a relationship between the mathematics teachers use when teaching and the answers that they give to the multiple-choice questions. The videotapes would be very helpful for studying mathematical knowledge for teaching in several ways. In particular I would like to be able to use the videotapes in the following ways:

(a)To investigate the relationship between the mathematics teachers use when teaching and the answers the teachers give to the questionnaire.
(b)To study other aspects of mathematical knowledge for teaching that arise when viewing the videotapes.
(c) To show members of a research team of which I am a member at the University of Michigan. This research group studies the mathematical knowledge that teachers in the United States use in

teaching and they would be interested in viewing Irish teachers' mathematics lessons to advance their thinking about the knowledge used by teachers when teaching mathematics. Neither your name nor your geographic location will be revealed to the research group.
(d)To investigate teaching issues in pre-service and in-service courses for teachers. In my work as a teacher educator I teach mathematics and mathematics methods courses. I would like permission to be able to use the tapes as teaching tools in these courses. Neither your name nor your geographic location will be revealed in the classes.
(e)To present at conferences to prompt discussion about teaching and learning. It would be helpful to me to provide examples of teaching to audiences at conferences such as those organized by the American Educational Research Association or the Educational Studies Association of Ireland. Neither your name nor your geographic location will be revealed at these conferences.

I hope you will be willing to participate because your responses are important and a valued part of the study. Your participation will remain strictly confidential. Your name will not be attached to any of the data you provide. You are welcome to discontinue participation in the study at any time, should you wish to do so. The risks of participation in the study are very low and of a social or reputational nature. There is a chance, for example, that someone who views the video may recognise you. However, the video will be kept in a secure location without your name on it and the intended audiences for viewing the tapes are those who are learning about or interested in the teaching of mathematics. The videotapes will benefit future learners and teachers of mathematics by providing information about the mathematical knowledge that teachers use when teaching and on how that knowledge can be measured. There are no risks or direct benefits in completing the questionnaire or participating in the cognitive interview. You will be asked to sign forms (below) indicating agreement to participate in the different parts of the study.

If you agree to participate please contact me in one of the following ways: by phone 01 – 805 7722 (office), 01 5212242 (home), 086 8962665 (mobile for calls or text messages); e-mail: sean.delaney@mie.ie; or by post c/o Marino Institute of Education, Griffith Avenue, Dublin 9. If you are willing to participate, it would help me greatly to know this as soon as possible so that your participation can begin as soon as possible.

Your participation in this project is sincerely appreciated. I understand that your time is valuable and as a token of appreciation all teachers who participate in all parts of the study will receive a gift token for €200. (€30 of this is for completing the questionnaire, €30 for completing the interview and €35 for each of the four videotape recordings). The voucher may be for Eason's shops, for mathematics teaching materials or for a local restaurant, depending on your preference.

Thank you for volunteering to participate in this research. Should you have questions regarding your participation, please contact Seán Delaney (sean.delaney@mie.ie or at 086 8962665). You may also contact my advisor for the project, Professor Deborah Ball of the University of Michigan (deborahball@umich.edu). Should you have questions regarding your rights as a research participant, please contact the Behavioral Sciences Institutional Review Board, 540 East Liberty, Suite 202, Ann Arbor, MI 48104-2210, 734-936-0933, email: irbhsbs@umich.edu.

Yours faithfully,

____

Seán Delaney

You will be given a copy of this information to keep for your records.

Statement of Consent:
Please read the questions below and indicate whether or not you would be willing to participate in the study as described.

Do you consent to participate in the study by completing the questionnaire described above?	Yes	No
Do you consent to be interviewed based on your questionnaire answers and to have the interview audiotaped?	Yes	No
Do you consent to have four mathematics lessons videotaped in your classroom for the purposes of studying the relationship between the mathematics used in teaching and the mathematics used in answering the questions on the questionnaire?	Yes	No
May I use the videotapes to study other aspects of mathematics teaching that arise when viewing them?	Yes	No
May I use the videotapes to show them to other members of the research team at the University of Michigan?	Yes	No
May I use the videotapes in pre-service and in-service teacher education courses to investigate teaching issues?	Yes	No
May I use the tapes to present at conferences to prompt discussion?	Yes	No

Signature:_____________________________________ Date: __________________

Signature of Investigator:_________________________ Date: __________________

Appendix 3.2
Consent form signed by school principal giving consent for research to be conducted in the school.

December 2007

Re: Video Study of Irish Teachers' Mathematical Knowledge for Teaching (Application: HUM00011619)

<u>TO WHOM IT CONCERNS</u>

____
(Insert name and address of school)

is a primary school in Ireland. This is to confirm that Seán Delaney, as Principal Investigator on the above-named project, has permission to conduct part of his study on this site. If you require any further information please do not hesitate to contact me.

Yours faithfully,

____
Signature

Name in block letters: __________________________

Position: _____________________________________

Date: ____

Appendix 3.3
Consent letter completed by parents to allow their son or daughter to be filmed or not to be filmed.

May 2007

Dear Parent/Guardian,

Your child's teacher has agreed to participate in a research study about mathematical knowledge for teaching. The study looks at how primary teachers improve their understanding of the mathematics they use in teaching. As part of his/her participation in the study, your child's teacher will be videotaped teaching mathematics to your child's class. I am requesting your consent to allow your child to be videotaped as part of this project.

If you decide not to allow your child to be videotaped, he or she will still participate in the classroom lesson, but will simply be asked to sit outside the range of the video camera.

If you agree to allow your child to be videotaped, your child's identity will remain completely confidential. His or her name will not be attached to any information I collect nor will these videotapes be used by anyone other than qualified researchers working on this study.

For more information about the study please contact Seán Delaney by e-mail at sean.delaney@mie.ie or by phone at 01 805 7722. Should you have questions regarding your child's participation in the research you may also contact my advisor for the project, Professor Deborah Ball of the University of Michigan (deborahball@umich.edu). Should you have questions regarding your rights as a research participant, please contact the Behavioral Sciences Institutional Review Board, 540 East Liberty, Suite 202, Ann Arbor, MI 48104-2210, 734-936-0933, email: irbhsbs@umich.edu.

Yours sincerely

Seán Delaney

Please complete **one** of the two options below:

1. ___ I **do** consent to allow my child ______________________________ to be videotaped.
(Print child's name)

2. ___ I do **not** wish my child __________________________________ to be videotaped.
(Print child's name)

Parent/Guardian Signature: ____

Appendix 3.4
Oral script for contacting principal teachers or other contact within school to inform them about the study and to notify them about sending information about the study to teachers in the school.

Hello,

My name is Seán Delaney, from Coláiste Mhuire Marino. I am phoning you about some research I am doing about teachers' mathematical knowledge. Many teachers complain that the in-service maths days and the maths courses in the colleges of education don't prepare them very well for teaching maths in school. I am trying to learn from teachers about the mathematical work that is involved in teaching so that pre-service and in-service education can better meet the needs of teachers. Towards this end, I would like to ask you to complete a questionnaire on the topic.

I am using questions that were developed in the U.S. and I am trying to see how well they relate to the work that Irish teachers do. I am interested in how the items work in Ireland rather than in the answers given by any individual teacher. The questions relate to the maths that teachers use in their work and I think you would find them interesting.

A number of schools have been selected at random from all the schools in Ireland and your school was one of those selected. Is this a good time to tell you some more about the study?

If no: I have some information about the study that I would like to send you for your consideration. I would like to send this information to the teachers in the school. Can you give me their names? Write separately to each teacher and ask them if they will complete the questionnaire. Ask them to make contact with me directly.
If yes: Continue as below.

I have a questionnaire that I would like to ask the teachers in your school to complete. It consists of (a) questions about common mathematics situations that occur in primary school classrooms and (b) questions about teachers' backgrounds. It takes between 60 and 90 minutes to complete the survey. The survey would take place at a time that is convenient to the teachers (e.g. before school, after school, evening, etc.). As a token of appreciation every teacher who participates will be given a gift token of €30. It could be for a bookshop, a theatre, a department store or a local restaurant for the end of year staff night out.

I would like to send a letter to every teacher in the school with more details and to invite them to take part in the study. How many teachers do you have in the school?

Would it be better to send the letter by e-mail or by regular post?

If it were possible for all teachers in the school to do the questionnaire at the same time, that would be really convenient.

Would it be all right to contact you again in two days (5 days if postal mail is suggested), to see if the teachers are interested in participating? And if so, to schedule a time that would be suitable?

Follow-up Oral Script

This is Seán Delaney from Marino. I spoke to you (2, 5…) days ago about a mathematical study that I am conducting. Did you receive the letter with further information that I sent you?

Are the teachers interested in participating? Is there a particular time that is convenient for the teachers? Set up a date and time.

Times suggested by schools when the survey might be administered:

- Before school
- After school
- During a staff meeting
- Any other time that suited all members of the school (e.g. if all teachers in the school are attending the same summer course).

Appendix 3.5
Letter sent to principal and teachers asking for their participation in the study.

June 8, 2006

Dear Principal,

Following our phone call this morning, I am writing to ask for your help with a mathematics survey that investigates the mathematical knowledge that matters for primary school teaching and how teachers develop this sort of mathematical knowledge. I would be grateful if you could bring this letter to the attention of your staff. The research project, funded in part by the Department of Education and Science, is developing a questionnaire that focuses on mathematical problems that arise in the course of teaching children. The questionnaire will eventually be used to help evaluate and enhance professional development programmes that are meant to improve teachers' ability to solve such problems.

There are two parts to the questionnaire. In the first, I ask you to respond to questions about common mathematics problems in primary school classrooms – for instance examining unusual solutions methods, evaluating students' mathematical statements and determining how to best represent material or generate examples. The second part asks some general questions about your background and teaching. This data will NOT be used to evaluate your own knowledge of mathematics. Instead, I will analyze responses from all teachers participating in this project to identify the best questions for use in future studies of teacher learning and to inform future pre-service and in-service mathematics preparation of teachers. I hope you will be willing to participate because your responses are important and a valued part of the study.

Your response to the 60 – 90 minute questionnaire will remain strictly confidential. Your name will not be attached to the information you provide. You are under no obligation to complete the questionnaire, or to answer all questions presented in it. If you come to a question you do not wish to answer, simply skip it. There are no risks or direct benefits to taking part in this study. You will be asked to sign a form (overleaf) indicating agreement to participate in the study.

If you agree to participate please contact me in one of the following ways: by phone 01 805 7722 (office), 01 8572086 (home), 086 8962665 (mobile for calls or text messages); e-mail: sean.delaney@mie.ie; or by post c/o Coláiste Mhuire, Marino Institute of Education, Griffith Avenue, Dublin 9. If you are willing to participate, it would help me greatly to know this as soon as possible so that you can complete the questionnaire during the month of June. I can then arrange for you to answer the questionnaire along with some of your colleagues or at a time that is convenient for you.

Your participation in this project is sincerely appreciated, especially at this busy time of the year. I understand that your time is valuable and as a token of appreciation all teachers who participate in the study will receive a gift token for €30. The voucher may be for Eason's shops or for a local restaurant or for mathematics teaching resources, depending on your preference.

Thank you for volunteering to participate in this research. Should you have questions regarding your participation, please contact Seán Delaney (sean.delaney@mie.ie or at 086 8962665). You may also contact my advisor for the project, Professor Deborah Ball of the University of Michigan (deborahball@umich.edu). Should you have questions regarding your rights as a research participant, please contact the Behavioral Sciences Institutional Review Board, Kate Keever, 540 East Liberty, Suite 202, Ann Arbor, MI 48104-2210, 734-936-0933, email:

irbhsbs@umich.edu.

Yours faithfully,

____

Seán Delaney

You will be given a copy of this information to keep for your records.

Statement of Consent:

I have read the information overleaf. I have asked questions and have received answers. I consent to participate in this study by completing the survey described overleaf.

Signature:____ Date: ____________________

Signature of Investigator:__________________________ Date: ____________________

Appendix 3.6
Number of teachers in each stratum chosen for the sample.

Stratum	Dublin	Leinster (ex. Dublin)	Munster	Connacht/ Ulster	Total
Breaking the Cycle (Urban)	1	0	0	1	2
Breaking the Cycle (Rural)	0	0	0	1	1
Disadvantaged	3	2	2	1	8
Gaeltacht School	0	1	1	3	5
Gaelscoil	1	1	2	1	5
None of the above categories	7	19	22	18	66
Total	12	23	27	25	87

Appendix 3.7
Guidelines issued to those who helped administer the survey to teachers.

Mathematical Knowledge for Teaching:
Notes for Administering the Survey

General

- Many teachers feel that their college of education course does not prepare them well to teach mathematics. This study of teachers' mathematical knowledge for teaching is designed to learn from classroom practitioners about the mathematical demands of teaching. This will help inform how teachers can be best prepared to teach mathematics.
- The survey items are confidential. Please do not copy them or allow them to be copied because if they are circulated among teachers this could affect the validity of the items in the future. The questionnaires should be completed in your presence.
- If you need to contact me at any stage during the administration of the study my numbers are 086 8962 665 (mobile), 01 805 7722 (office), 01 8572086 (home) or by e-mail: sean.delaney@mie.ie

Setting up the appointment

- Once a school has been selected, the design of the survey requires every teacher in the school to be given the opportunity to participate. That includes the principal, those with no formal teaching qualification etc.
- If you know someone in the school, try and make contact with them directly and outline the study to them and ask them to mention it or outline it to their colleagues and encourage them to participate in it. The phone script (attached) will help with this. If you know no-one in the school, I will try to make contact with the principal and ask them if they would be interested in participating. Even if the principal declines to participate individual teachers may be happy to participate.
- Ask for the names of the teachers in the school. A letter will then be sent to each teacher and they will be asked to contact me to arrange a suitable time. A follow-up phone-call can be made to the contact-teacher a few days after sending out the letter.
- I would like teachers to be given their gift token on completion of the survey. If possible, try to find out which token they choose in advance of administering the survey.
- We will also try and establish with the principal or contact-teacher where the best place is to complete the questionnaire. It may be in the staff room, in a classroom, in the local education centre, etc.

Prior to the administration of the survey

- You will be given copies of the survey form and copies of the consent letter. Each teacher should already have received one copy of the consent letter.
- Ask each teacher to sign the consent letter. Collect the signed letter and give a copy of the letter to any teacher who has not already received one.
- Inform the teachers that the selected answers are indicated by circling the relevant number or letter. Pen or pencil may be used.
- Rough work can be done on the page

During the administration of the survey

- The survey needs to be administered under similar conditions everywhere it is administered. That means that it needs to be done in your presence. However, you are not in the position of a "test supervisor." Teachers may talk, (but preferably not discuss the items!), eat, leave the room etc. It might also be good for you to bring along a book, newspaper, crossword puzzle or sudoku etc. to do while the teachers are completing the survey.
- There is no time limit to completing the questionnaire but a general guideline is that it takes between 60 and 90 minutes.
- If teachers ask questions about what particular terms mean, please keep a record of the questions, especially the terms that they ask about and the number of people who ask about the terms. At

your discretion you may decide to answer or not answer. If you are unsure of a definition etc. say that (i.e. there is no need to look up terms in advance).

- If a teacher begins the survey and decides not to continue it that is the teacher's right. They still should be given a token of appreciation for participating.

Following the administration of the survey

- In analyzing the data I want to be able to know which teachers were in the same school. However, I do not need to be able to identify the specific school. One way to do this is to put the administrator's initials on the cover of the questionnaire and a randomly chosen number for the school. It would also be helpful if you could note the classes that are in the school e.g. J.I. to 6th, J.I to 2nd or 3rd to 6th etc.
- Please complete the Survey Administrator's summary document about the number of teachers who took the questionnaire.
- To claim payment complete a form similar to the one that is completed for teaching practice expenses and return that to me. The rates that apply are similar to teaching practice: €60 per hour. Travel expenses are reimbursed at the rate of €1.03 per mile. Receipts may be submitted for lunch (up to €12 per day), for dinner (up to €25 per day) and for overnight accommodation (up to €85 per night). A payment form is attached. The Expenditure codes are: 5010 for professional fees, 5004 for travel and subsistence. The Department code is 202. The Description is "Mathematical Knowledge for Teaching Study" and details refer to the item of expenditure (e.g. administering survey, travel, accommodation etc.). The form should be signed and returned with the questionnaires and the signed consent letters to Seán Delaney, Coláiste Mhuire, Marino Institute of Education, Griffith Avenue, Dublin 9.

Accompanying documents

- Letters of consent for teachers to sign (and copies for teachers who have not received one)
- Oral script for contacting teachers known to you
- Survey administrator's summary document
- Expenses sheet

For the administration of the survey you will need

- Copies of the questionnaire
- Some spare pencils or biros
- Gift tokens

Appendix 4.1 Lesson Table for Studying the Mathematical Work of Teaching in Ireland

Teacher ID:SDVS9	**Lesson ID**:C	**Strand**: **<u>N</u>** A SS M D **Strand Unit**: Fractions	**Topic**: Dividing a whole number by a unit fraction.
Clip No.	**Task of Teaching**	**Dialogue from Transcript**	**Comment from Researcher**
1	Connecting what will be done in a lesson to what was done in a previous lesson.	T: the last day we were working, we were working on fractions; we were working on multiplication of fractions … today what I want to do is something a little bit different, it's related but it's just a little step on again. You've done a little bit of work, I spoke to you about it earlier, and you did a little bit of it, when you were in fifth class, some of it will probably come back to you. But let's have a little look and see.	Involves knowing what students did in a previous class and also how a mathematical topic progresses.
1	Writing number and operation sign on the board	T: Before I start that I just want to ask you a question	Teacher writes 78 ÷ on the board
1	Chooses an example to fit the purpose of the lesson	T: better get a number I can divide first	Changes the 78 ÷ to 72 ÷9
1	Asking a question to elicit the meaning of an operation	T: What would I be asking you to do? Don't give me an answer; I'm not interested in the answer, ok? What would I be asking you to do? If I was asking you to do that sum? In English.	Use of phrases "I'm not interested in the answer" and "In English" suggests that the teacher is interested in the meaning rather than a mathematical phrase. Note that in Ireland teachers refer to any calculation generically as a "sum" whether it is addition, multiplication, subtraction or as here division. However, "Find the sum of x and y" means to add x and y.
1	Asking a student to expand on an incomplete response to a question (evaluating a student's response)	S: Divide T: Divide, so you do know something about it. What else springs to mind? Ultan? You were going to say divide, does anything else come to mind?	Here the teacher seems to be looking for a response that gets closer to explaining division. It becomes clear later in the clip that the teacher wants the

			student to refer to something along the lines of "dividing seventy-two into bundles of nine"
1	Asking a student to clarify an unexpected response to a question (evaluating a student's response)	S: Multiplication. T: Multiplication comes to your mind, why does multiplication come to your mind?	
1	Teacher follows student explanation. Teacher helps the student construct the explanation Teacher writes a number sentence on the board Teacher corrects a misunderstanding of a word	S: You have to see how many times you multiply nine and it still fits into 72. T: Ok yeah. Because multiplication and division S: Are the same T: They're the same Jack? S: Almost. Well the basics are. T: Ok, what do you mean by that? S: Because really all you're doing is turning the sum around and swapping …ok so you could have eight times nine equal 72, but in that case you just swap the sum around and 72 divided by nine equals eight. T: Could you add anything else? If you kept going in that plan, going off the track here a little bit but? Yes? S: There's a word to describe it equivalent, because like… T: Mmm, would it be equivalent? S: No, not really T: I know what you're thinking, and I can understand where you're coming from, I don't think equivalent is the right word	It would have been helpful here for the teacher to have explained what inverse operations are. A student explains the inverse relationship of multiplication and division. The teacher knows that equivalent is not the correct word but cannot suggest another. Teacher writes up 8 x 9 = 72 When the teacher refers to "those two things" he is referring to the two number sentences 72 ÷ 9 = 8 and 8 x 9 = 72 which are written on the board. Teacher then writes 9 x 8 = 72 and 72 ÷ 8 = 9.

		though, because when we talk about equivalence, we're actually talking S: It's fractions T: Well it mightn't necessarily just be fractions, but we're talking about things that are equal, aren't we? You couldn't really say that those two things are equal, they are related certainly. They have something in common. It's related as well, isn't it? And what about? They're four tables aren't they? Four for the price of one.	

Appendix 4.2 Glossary to Explain Tasks of Mathematics Teaching Identified in Ten Irish Lessons

Task of Teaching	Description of Task (as it could happen but may have happened differently in example)	Sample MKT Demands
Connect a mathematics problem to a skill for living	Teacher relates a mathematics problem to an activity related to life outside school (e.g. managing a budget).	• Know how mathematics can be applied in society for a citizen's benefit
Apply mathematics in the students' environment (in school and out of school)	Teacher uses mathematical examples from the students' environment. For example, the teacher points out that an item in the students' environment is an example of a particular shape, or property of a shape or that it is a quantity of a particular size (length, capacity, area etc.) Alternatively, the teacher must decide if what a student calls an example of a shape (or property of a shape) is a correct example. Teacher asks questions which students can answer using information given in class but where the context in which the information must be applied is different.	• Know names and properties of shapes in the curriculum • Determine if a particular shape fits a category (e.g. if one student suggests a door as a rectangle and another suggests it as a rectangular prism) • Know benchmarks for common measures • Recognize contexts where students can apply mathematics taught in class.
Tell students what they will be working on in a lesson	Teacher outlines what students will be working on in the lesson. It could also occur if there is a transition in a lesson or if the problem type changes.	• Know how to present a topic in a way that will be comprehended by students and that will stimulate their interest
Tell students why they are doing an activity	Teacher states specifically what students will learn or practice by doing a particular task or game in class.	• Know the mathematical purpose behind tasks sourced in textbooks, teacher courses, from other teachers, from the internet.
Tell students what they have been doing in a lesson or activity	Teacher tells students what they have learned without stating the key points again.	• Know what the mathematical focus of the lesson is
Identify salient information in a lesson or topic	Teacher points out to students the most important aspects of a topic (a shape, a definition, an algorithm etc.) to which they should attend	• Know the key points in a given topic • Know which aspects of a topic will help future learning
Decide not to pursue a topic in a lesson	Teacher decides not to pursue a topic that is introduced by a student.	• Know which aspects of a topic will be productive in terms of mathematics learning and which will not
Choose numerical or geometric examples for the lesson	Teacher chooses examples relevant to what is being taught and that work. Examples need to be appropriate for the age and stage of the children.	• How to calculate the answers to the examples • Know the range of numeric or geometric examples that

	For example, to teach division with no remainder the numbers 78÷9 would not be good, or to teach subtraction without regrouping 72 – 24 would not work. Similarly an equilateral triangle would not be good if you wanted to study right-angled triangles, or a circle if you wanted to study polygons.	are available for selection
Connect current topic to material students will work on in the future	A teacher explicitly relates content being taught to something students will learn in the future, most probably not in the current class level. Casual references to something that will be done or finished tomorrow are not included.	• How “rules can change”: e.g. in second grade saying that one cannot take 6 from 2 and expecting students to do that in sixth grade. • Continuum of a topic
Connect current lesson topic with material learned in a previous class level	A teacher explicitly relates a topic to something studied in a previous class level.	• Knowledge of the mathematics curriculum outside the current class taught • Links between and among different mathematics topics (e.g. fractions and decimals) • Continuum of one topic (e.g. multiplication of fractions and division of fractions)
Connect current lesson topic with work done in a previous lesson in current class level	A teacher explicitly relates a topic to a previous lesson done any time before in the current class level.	• How a topic is sequenced • How a topic links to other topics • How a previous topic can help or hinder understanding of a new topic (e.g. decimals and money)
Ask a mathematical question on a topic not taught in the lesson (but which at least some students are expected to know)	The teacher asks the students a mathematical question on a topic that does not feature in the current lesson. Teacher asks students to compute using an operation not in the lesson.	• Know what prior mathematical knowledge students can be expected to have • Know how students’ prior mathematical knowledge can be incorporated into a new topic
Recap on mathematics practiced so far in lesson	The teacher summarizes what mathematics has been done so far in the lesson.	• Know what the key points of the lesson are
Ask questions to revise material in lesson	Teacher asks questions to revise material covered in the lesson. The answers require repetition of mathematics information already presented in class. [Excludes questions asked to elicit names or properties of shapes]	• Know what the key points of the lesson are
Respond to a	Teacher responds to a mathematical question that	• Know what the student is trying to understand

mathematical question from a student	students ask.	• Frame an answer in a way that is comprehensible to the student • Know what resources can be accessed to assist in responding to questions when the teacher does not know the answer
Help or prompt a student who is stuck or incorrect (E.g. giving a clue or a suggestion)	Teacher responds to a student who has an incorrect answer or who is not making progress on work by giving some form of support. It might be in the form of a question or a clue as to the answer or to change the context of the problem. This refers to a short, focused one or two sentence intervention.	• Identify what caused the student's error or what is preventing the child from continuing to work • Know what question or clue could be most productive in advancing the student's mathematics learning
Explain mathematical ideas	Teacher explains a mathematical idea to students using words, pictures, examples or other materials. This task is distinct from a teacher eliciting an explanation from or co-constructing an explanation with students.	• Understand the idea • Know the key parts of the idea (including required background knowledge) and sequence them appropriately • Know how to communicate the idea to elementary students
Help a student describe a mathematical procedure	Teacher helps the student to construct the description by questioning the student or by supplementing the description with necessary information.	• Understand what the student is describing • Have the knowledge necessary to supplement any relevant information omitted by the student
Anticipate ideas that may be confused by students	Teacher anticipates what may cause difficulties for students when teaching a topic. This may be done by pointing out common errors, highlighting important differences or by giving students time to understand one idea before introducing the second.	• Know common student errors (e.g. that if the ones digit in the minuend is less than the ones digit in the subtrahend, students are likely to take the minuend from the subtrahend)
Elicit the meaning of an operation	The teacher asks questions so that students will state at least one meaning of a number operation.	• Know the different meanings of number operations (e.g. regrouping and equal addition for subtraction; partitive and measurement for division)
Teach students how to write numerals or other mathematical symbols	Teacher gives students specific guidance on how to write numerals for small and large numbers or other mathematical notation. Examples might include teaching young students a single numeral or differentiating algebraic *x* and multiplication sign for older students.	• Know how numerals/symbols are written • Knowing difficulties students are likely to have in writing numerals or symbols

Ask students how numerals or mathematical symbols should be written.	Teacher does not ask students to write but asks how certain numbers would be written. For example large numbers (say ten thousand and fifty or half a litre as a decimal)	• Know what numerals students find difficult to write. • Know conventional forms of saying numbers (e.g. “Three point twenty,” or “three and twenty hundredths”)
Write numerals and operation signs on the board	Teacher writes numerals and other mathematical notation signs on the board.	• Avoid errors that are commonly made by teachers (e.g. 3 + 4 = 7 + 5 = 12)
Record work done in lesson on board or poster	Teacher records work done in class publicly for students to see.	• Know which work from the class is the most important to place on the record to reinforce student learning or to use in a future lesson
Use correct and appropriate mathematical terms	Teacher uses mathematical terms to describe various mathematical concepts. The terms are used precisely and terms that have non-mathematical meanings (e.g. face, odd) are differentiated from the mathematical meaning.	• Know terms that are used when teaching the primary school curriculum • Know what words are acceptable as synonyms and which are not. E.g. one teacher used “plan” and “pattern” as synonyms for the net of a shape. • Know which words are mathematical and which are made up e.g. is “unparallel” a mathematical term?
Elicit a mathematical term (including name of shape or number)	Use a stimulus so that students will use appropriate mathematical terms. These will generally be terms that the teacher believes some or most students already know.	• Know terms that students are likely to mix up (e.g. multiple and factor; faces and sides) • Know what prompts will help (e.g. if a student responds “rectangle” when the required term is “cuboid”)
Define and/or explain mathematical terms	The teacher uses words, pictures and examples to define or explain a mathematical term. A teacher may also give a larger context such as the origin of the word or the plural.	• Know definitions that are mathematically accurate and understandable by students in the class level. • Have alternative ways to explain words that may be difficult for students to learn (e.g. state what the dimensions are in a 2-D shape)
Elicit the meaning of a mathematical term	Teacher uses a stimulus to prompt a student to explain the meaning of a word. One student may give the meaning or several students may contribute.	• Know what the term means so that the student’s response can be evaluated for its accuracy and completeness
Describe or identify properties of shapes	The teacher describes the properties of a shape for the class or identifies an instance of a property of a shape. Some properties might require the teacher to give justification (e.g. a shape that is a polygon because it is closed and has straight sides).	• Know the names and properties of shapes on the primary school curriculum • Know definitions of shape properties in order to resolve disputes about properties such as the number of sides on

		a circle, the number of edges on a cylinder and whether or not a cone has a vertex.
Elicit properties of shapes	Teacher uses various stimuli (e.g. chart, game, open-ended or closed questions) to elicit properties of shapes. The teacher may require a student to give a specific property of one shape or several properties of that shape.	• Know common errors made by students (e.g. finding 24 corners on a cube)
Compare or differentiate between/among shapes or categories of shapes	Teacher chooses to discuss shapes in relation to one another or to discuss 2-D shapes alongside 3-D shapes. Sometimes students can appreciate particular properties of shapes when they are compared or contrasted with other shapes.	• Know various ways in which shapes can be compared to and contrasted with one another • Know interesting patterns in properties of shapes (e.g. Euler's polyhedral formula)
Collect data from students	Teacher decides to collect data from students (e.g. letters in their name, favorite color) in order to represent it when teaching students about data collection.	• Know about different stages of data collection: posing a question, collecting and recording the data, organizing the data and representing the data.
Compare or differentiate between/among different ways of representing data	Teacher discusses with students different ways of presenting data: e.g. bar charts, multiple bar charts, pie-charts, trend graph.	• Know different means of representing data and the benefits and limitations of each
Illustrate a property of an operation	Teacher shows students instances of a property such as the commutative property of addition. The teacher may or may not use the term "commutative."	• Know properties of operations and the number sets to which they apply • Know how to present the properties in ways that are comprehensible to primary school students
Illustrate a property of a number	Teacher shows students properties of numbers (e.g. odd, even, prime, square). Students may also be given the opportunity to test other numbers for the same property.	• Know properties of numbers that are relevant to primary school students • Know how to test for properties of numbers (e.g. prime number tests).
Use representations to explain operations, or other mathematical ideas	Teacher uses a representation to help students understand an operation (e.g. multiplication, division by fractions). The representation may be in the textbook, drawn by the teacher or by a student. A math sentence may be linked to the representation.	• Understand representations that are commonly used in schools to explain operations. • Match a math sentence to the representation. • Know ways to represent equivalence of fractions, decimals and percentages
Make a mathematically accurate representation	Teacher produces a representation on the board that is accurate and that achieves its purpose of	• Know how to use the available resources to produce a useful representation

	promoting understanding. This requires using appropriate resources.	
Teach students to make accurate representations	Teacher shows students, through instruction and/or modeling how to make accurate representations either on the board or in students' notebooks.	• Know what resources are available to students and what difficulties they have in making representations (e.g. in terms of scale or orientation)
Choose an appropriate representation for a situation	Different situations require different representations. These may vary in shape (e.g. circles or rectangles), in orientation (portrait or landscape) depending on the operation being represented, the purpose of the representation (e.g. comparison) or on the available space.	• Know what advantages different formats of representations offer and determine which one would be best for illustrating a concept.
Follow student explanation	Teacher listens to a student explaining a mathematical idea. The teacher may highlight aspects of the explanation or respond in other ways such as completing missing details.	• Know what a mathematical explanation is in general • Know what would be a good explanation in this case
Follow student description	Teacher listens to a student describe a feature of a shape or a procedure used or to be used. The description may be supported by reference to a picture or representation.	• Know the terms or other supports that can help a student to give a clear description which other students can follow
Respond to a mathematical comment, statement or conjecture from a student	Teacher responds to a mathematical utterance from a student that is related to the lesson in question or may not be. The student (and possibly others) will understand the point better after the response.	• Know what mathematical point lies behind the utterance • Relate the point to the students' mathematical knowledge
Ask other students to comment on a response or a statement made by one student	Teacher asks other students to respond to one student's comment or answer to a question.	• Know if the initial student's comment or response is accurate or inaccurate
Ask a student to justify an answer or statement	Teacher responds to a student's answer to a question or problem by asking the student to justify the answer. Questions used may be: How do you know? Why? Why not? Are you sure? What do you think?	• Know what would serve as a mathematical justification of an answer
Ask student to expand on a response	Teacher asks a student to give a more detailed response. Typical questions might be "can you say some more about that?" or "what else springs to mind?"	• Know that a student's response is incomplete and what the response needs to be complete

Ask student to clarify a response	Teacher asks a student to be clearer in the response. This may happen if a student offers a response that is difficult to follow or contradictory. A student may also be asked to specify a unit of measurement.	• Recognize when an answer is unclear and know what will make it clear
Direct students to a mathematical definition	The teacher refers a student to a mathematical definition in response to a question from a student or to encourage independent work in mathematics.	• Know a source of definitions that are accurate and comprehensible to the students.
Present a mathematics task or game to students	Teacher presents a task to students making it clear to them what they are required to do and how to do it. This includes setting the conditions for the task and setting up the necessary materials. This may also include sequencing the presentation of the task so that students can complete one step before progressing to the next step. This task also covers the choice a teacher makes about how students will work on an activity. Teacher decides if students will work alone, in pairs, as a class-group when completing a task. Although other considerations may come into play in this task (e.g. layout of the room, attentiveness of the children etc.), part of the decision is mathematical.	• Anticipate the quantities of materials required so that all students can participate as required (e.g. whether or not the large cube “thousands” block is needed if using base ten materials) • Know the conditions that need to be specified to maximize mathematics learning • Adjust the conditions (e.g. number and type of shapes in a feely bag) to maximize the cognitive demand of the task for students. • Recognize the mathematical skills that can be developed in different group formations (e.g. explaining a mathematical idea may be more likely if students work in groups than if they work alone). • Judge if the demands of the task are such that students can complete it alone or if some collaboration is needed • Know the mathematical content of the game (e.g. properties of shapes for a shape “feely bag” activity)
Draw students’ attention to a pattern that leads to a procedure	Teacher gives students various examples to complete (e.g. multiplying numbers by 10) and after students have completed several of them the teacher asks students if they have noticed a pattern.	• Know how to pick numbers that make the pattern obvious • Know what procedures can be taught in this way
Enable students to check if a procedure works (a) in a specific case (b) in general	Teacher asks students to test a procedure to determine the cases in which it works and does not work.	• Know what procedures are useful for primary students to learn • Know when procedures apply and when they do not • Know what cases might be particularly helpful for checking to test a procedure
Give students a formal	Teacher gives students an algorithm that they can	• Know commonly-used algorithms for different operations

algorithm to help them with calculations and explain how it works	apply to compute operations efficiently. The teacher may be asked by students why the algorithm works.	(with whole numbers, integers, fractions and decimals) • Understand how and why the algorithms work
Demonstrate how to apply an informal algorithm or procedure to compute an answer	The teacher specifically demonstrates how to do a problem on the board or in a student's notebook. The teacher may ask questions of the student(s) while demonstrating the procedure	• Know what informal algorithms can be useful for particular numbers. • Know when students are ready for informal algorithms.
Observe and/or help publicly (e.g. on the board) a student apply an algorithm or procedure	Teacher requests a student to do an algorithm/procedure so that all students can see it. The teacher may observe, comment to highlight features of the procedure or help the student to complete it.	• Know features of algorithms that cause difficulties for students • Know what language to use to help students apply and follow the algorithm/procedure • Know what mathematical benefits can be expected to accrue to students from the activity
Give students a means to check answers	Teacher gives students either a criterion against which to judge their answers to questions or the teacher enables students to check their answers in a practical way (e.g. using a right-angle tester).	• Know range of numbers in which answers to a set of problems will fall (e.g. when dividing a whole number by a unit fraction). • Know ways of checking answers for different primary school mathematics problems (e.g. estimation, inverse operations)
Elicit or present strategies that can be used for problem solving generally	Teacher explicitly shares generic problem solving strategies with students that can be used for solving mathematics problems.	• Know what problem-solving strategies are helpful at primary school level • Know difficulties primary school students have in applying problem-solving strategies
Elicit or present methods (including alternative methods) for solving specific problems	The teacher presents specific problems to the class and discusses how they will be solved. The students may be asked to do the problems immediately after the discussion or at a subsequent time for independent work (as in a multi-grade class).	• Know what strategies are likely to be useful for specific topics (so that some omitted by students can be included). • Know how students respond to the topic so that the challenge in the problem is not diminished
Help students convert measurement quantities	Teacher assists students with problems that require changing from milliliters to liters; centimeters to millimeters to meters; grams to kilograms and so on.	• Know what prior knowledge students need to be able to convert measures • Know how to sequence instruction so that easier problems and examples precede more difficult ones
Ask students to estimate or predict what an answer	The teacher asks students to predict or estimate an answer before working it out in any formal way.	• Know different strategies for estimation that are used at primary school level.

will be	They may be also asked if their estimate is likely to be higher or lower than the actual answer.	• Know benchmarks for common measures
Ask students to solve a problem or to calculate mentally	The teacher asks students to do a problem or to calculate an answer mentally in class.	• Know how to calculate mentally • Know strategies that can be used to calculate mentally
Select suitable exercises for students to attempt	Teacher selects exercises related to what is being taught. Exercises may be selected from the class textbook, an alternative textbook or from another source.	• Know which exercises students can attempt with ease and which are likely to be more challenging • Know which exercises are likely to result in optimal student learning
Assign homework	Teacher assigns exercises for students to complete outside of school.	• Judge which exercises will reinforce what was learned in class and will be challenging enough but not too challenging for the students.
Modify exercises in a textbook	Teacher may supplement or omit part of the exercises in the students textbook.	• Know when a change is desirable and why and how to achieve maximum benefit for the students' learning with the change
Devise supplementary exercise for students	Teacher prepares a worksheet for students to work on in the lesson	• Know how to prepare the activity so that all students will learn some mathematics and achieve success
Provide work for students who finish early	Teacher assigns work to students who successfully finish class work while other students are still completing the class work.	• Know what would be a suitable extension of the main topic in the lesson
Indicate to a student that an answer is correct	Teacher evaluates an answer to a question or to a problem and tells the student that the answer is correct. This may be accompanied by a compliment or an instruction to keep going.	• Know or work out the answer to the question or problem
Indicate to a student that an answer is incorrect	Teacher evaluates an answer to a question or to a problem and tells the student that the answer is incorrect and does not follow-up the answer in any mathematical way.	• Know or work out the answer to the question or problem
Tell or show students the answer to a question	Teacher tells students the answer to a question or shows it to them in the form of a picture, diagram or object.	• Know how to get the answer • Know the importance of the answer in relation to the solution of the particular problem (Lampert, 1990)
Share one student's (or one group's) work with the rest of the class	Teacher directs all students' attention to work done by one student. This may be because the student has used a novel approach or because the student has done particularly good work. Alternatively it may	• Recognize what constitutes mathematical work that could benefit other students' learning through sharing.

	be because the student has made a common error which other students should avoid making.	
Check if a student understands	Teacher checks if an individual student understands a term or concept or a procedure that arises in class by giving a task or asking a question (other than “Do you understand?”). The term or concept or procedure may be one that arose informally in the lesson.	• Know what constitutes understanding of the term or concept • Know what would be a suitable task to assess understanding
Demonstrate how to investigate properties of a shape	Teacher shows students ways to investigate the properties of shapes. These may include using equipment (e.g. right-angle testers, or a ruler) or using a system (e.g. to count the number of edges on a rectangular prism).	• Know ways of investigating properties of shapes that are appropriate for use with primary school students.
Use class materials or activities to model a mathematical concept	The teacher uses activities or materials to explicitly model a mathematical concept.	• Know which materials and activities are best for teaching specific concepts • Know how to link the concept and the materials
State the purpose and use of mathematics education equipment	Teacher knows a wide range of materials that are available for teaching mathematics. These include materials that are available in the school. They also include materials that are not in the school because a student may see reference to them in a textbook and ask what they are	• Know a wide range of materials that are available to support teaching mathematics in primary school • Know the strengths and limitations of different materials for use in teaching various topics
Identify appropriate equipment for doing a mathematical task	Teacher knows materials that students use when doing primary school mathematics. This includes items such as compasses and protractors.	• Know what materials can be used for particular mathematics tasks in primary school.
Introduce materials or visual aids to the students	The teacher gives students an overview of the mathematics materials, pointing out the key features. There may also be time for students to freely explore the materials.	• Know the key features of the materials about which students need to know
Ask students to use materials in a specific way for a specific purpose	The teacher gives an instruction to students to perform a specific activity using the materials, directed towards learning an aspect of a mathematical topic.	• Know how materials can be used to help teach mathematics concepts
Explain inadequacies in materials or drawings being used	The teacher may not always have the ideal equipment for the task in hand and, therefore, may need to explain to students why and the way in	• Assess the shortcomings of the available materials • Judge whether the inadequate materials are preferable to using no materials

	which the materials are inadequate.	
Draw shapes on the board or on a poster	Teacher uses available resources (e.g. rulers and markers or interactive whiteboard) to produce clear shapes for class discussion.	• Know how to use the available resources to produce the shapes • Avoid choosing stereotypical shapes, e.g. an equilateral triangle, unless it is specifically required
Use materials or a picture to confirm, question or understand a student response	Teacher uses materials or a picture with the students to either confirm the student's answer, to question it or to understand why or how the student came up with the answer.	• Know how to connect a written or oral answer with concrete materials

Tasks I Have Done as a Teacher That Would Not Be Seen in Videotapes of Lessons

Sample Dimensions of Work of Teaching	Description of Task	Sample MKT Demands
Teach number facts to students	Children need to remember basic calculations in all four operations (within twenty for addition and subtraction and within one hundred for multiplication and division)	• Know number and operation properties (e.g. commutative property; additive identity property) that make learning the number facts easier.
Correct (mark) students' homework or class work in mathematics	The teacher collects students' mathematics copies and marks them after school or at home. Students are given feedback on their work and the teacher can evaluate how well students have grasped a mathematical idea.	• Know the answers to the problems assigned • Work out students' strategies or errors when students are not present • Identify common patterns of errors • Know what feedback will be helpful to students
Contribute to writing a school plan in mathematics	Schools are required to have a written plan stating how each subject is taught throughout the school and although they are prepared at school level, the aspiration is that all teachers contribute to the plans.	• Know how each strand (number, algebra, shape and space, measures and data) of the mathematics curriculum develop throughout students' years of primary school
Contribute to staff discussions about mathematics	Discussions about mathematics can be informal or formal and the topics can be wide-ranging from figuring out a solution to recommending materials for teaching a particular topic to a problem to the language that is used when teaching subtraction.	• Know how to solve primary school mathematics problems • Know about materials that are suitable for teaching various topics • Know language to talk about teaching mathematics
Recommend a textbook to be adopted by the school	Select a suitable mathematics textbook for use in the school. Typically about three options are available at a given time and once chosen textbooks can be used in a school for several years.	• Know how to source and use frameworks for evaluating textbooks (e.g. for sequence and presentation of topics, cognitive demands of tasks and so on).
Write long-term and short-term plans for teaching mathematics	Plan the material that will be taught during the specified time period. This will include topics to be taught, exercises to be used and plans for assessment	• Source and choose problems that will help students learn the planned content
Keep a record of mathematics taught to students	Record the material that has been taught in a specified period of time.	• Know how to document mathematical learning in a way that is useful to colleagues
Purchase and make manipulative materials and	Choose materials that will help to teach particular topics.	• Know what features of materials are mathematically sound and will help students acquire the desired

visual aids		concepts
Attend professional development sessions on the teaching of mathematics	Required and optional workshops are available to teachers.	• Know the language of teaching mathematics to follow the content of the workshop and to contribute and to ask questions
Design tests to assess students' progress	Many teachers give mathematics tests regularly throughout the school year and especially at periods such as the end of term. Many of these are designed by the teacher.	• Know how to write items that will test what students have learned and through which all students can achieve some success
Administer a standardized mathematics test to students each year	Many schools administer standardized mathematics tests to their students once a year. This is used to monitor learning and to identify students who may need additional support in learning mathematics.	• Know how to interpret the result of the test for each student in light of their work in mathematics during the year
Document a child's progress in mathematics in a school report or discuss progress at an individual parent-teacher meeting	School reports vary but most reports require at least a box to be ticked summarizing a student's progress in mathematics. A meeting with a student's parent(s) requires more detail about mathematics learning.	• Know how to summarize concisely what a student has learned in mathematics during the year. • Know strategies to recommend for the student to make further progress in mathematics
Answer parents' questions about mathematics teaching	Many teachers meet parents at the start of the school year to outline their expectations and plans for teaching all subjects (including mathematics) during the year. This is an opportunity for parents to ask questions about approaches that will be used.	• Explain teaching approaches and the rationale for them in language that parents understand.

Appendix 4.3 Conceptions of the Work of Teaching that Informed the U.S. Construct of MKT.

Source	**Mathematical work of teaching**	**Type**
(Ball & Bass, 2003b)	• Provide a correct representation for division of fractions	p. 4
(Ball & Bass, 2003b)	• Explain why an algorithm works (Ms. Daniels in Borko et al., 1993)	p. 4
(Ball & Bass, 2003b)	• Create a word problem for a fraction division problem (Ms. Daniels in Borko et al., 1993)	p. 4
(Ball & Bass, 2003b)	• Explain how to find equivalent fractions	p. 5
(Ball & Bass, 2003b)	• Answer student questions about primes or factors	p. 5
(Ball & Bass, 2003b)	• Represent place value	p.5
(Ball & Bass, 2003b)	• Represent and make mathematical ideas available to students (Represent ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process")	p. 6; 11.
(Ball & Bass, 2003b)	• Attend to, interpret and handle students' oral and written productions	p. 6
(Ball & Bass, 2003b)	• Give and evaluate mathematical explanations and justifications ("Design mathematically accurate explanations that are comprehensible and useful for students")	p. 6; 11
(Ball & Bass, 2003b)	• "Establish and manage the discourse and collectivity of the class for mathematics learning"	p. 6
(Ball & Bass, 2003b)	• Understanding and responding to a student's unfamiliar solution "What, if any, is the method and will it work for all cases?"	p. 6; p. 6-7
(Ball & Bass, 2003b)	• Keep the classroom orderly	p. 6
(Ball & Bass, 2003b)	• Keep track of students' progress	p. 6
(Ball & Bass, 2003b)	• Communicate with parents	p. 6
(Ball & Bass, 2003b)	• Build relationships with students	p. 6
(Ball & Bass, 2003b)	• Select and modify instructional tasks ("Make judgments about the mathematical quality of instructional materials and modify as necessary")	p. 6; 11
(Ball & Bass, 2003b)	• Make up quizzes	p. 6
(Ball & Bass, 2003b)	• Manage discussions	p. 6
(Ball & Bass, 2003b)	• Interpret and use curriculum materials	p. 6
(Ball & Bass, 2003b)	• Pose questions ("Pose good mathematical questions and problems that are productive for students' learning")	p. 6
(Ball & Bass, 2003b)	• Evaluate student answers	p. 6
(Ball & Bass, 2003b)	• Decide what to take up and what to leave	p. 6
(Ball & Bass, 2003b)	• Inspect alternative methods, examine their mathematical structure and principles and judges whether or	p. 7

	not they can be generalized.	
(Ball & Bass, 2003b)	• Select “definitions that are mathematically appropriate and also usable by students at a particular level” (“mathematically appropriate and comprehensible)	p. 7-8; 11
(Ball & Bass, 2003b)	• “Choose a task to assess student understanding” (“Assess students’ mathematics learning and take next steps”) Monitor whether or not students are learning.	p. 9; 11
(Ball & Bass, 2003b)	• “Interpret and evaluate students’ non-standard mathematical ideas”	p.. 9 – 10
(Ball & Bass, 2003b)	• “Make and evaluate explanations”	p. 10
(Ball & Bass, 2003b)	• Interpret and make pedagogical judgments about students’ questions, solutions, problems, and insights (both predictable and unusual)	p. 11
(Ball & Bass, 2003b)	• “Be able to respond productively to students’ mathematical questions and curiosities”	p. 11
(Ball & Bass, 2003b)	• Help students connect ideas they are learning (e.g. geometry to arithmetic – example of multiplication given)	p. 11
(Ball & Bass, 2003b)	• “Make connections across mathematical domains” (“Help students build links and coherence in their knowledge” and “seeing themes”)	p. 12
(Ball & Bass, 2003b)	• Anticipate how mathematical ideas change and grow	p. 12
(Ball & Bass, 2003b)	• Attend to mathematical practices as a component of mathematical knowledge	p. 12
(Ball & Bass, 2003b)	• Make sense of methods and solutions different form one’s own	p. 13
(Ball & Bass, 2003b)	• Size up other methods, determine their adequacy and compare them	p. 13
(Ball, 1999)	• Figure out what students know	p. 15
(Ball, 1999)	• Choose and manage representations of mathematical ideas	p. 15
(Ball, 1999)	• Appraise, select and modify textbooks	p. 15
(Ball, 1999)	• Decide among alternative courses of action	p. 15
(Ball, 1999)	• Respond to student questions	p. 26
(Ball, 1999)	• Plan lessons	p. 28
(Ball, 1999)	• Listen to children	p. 28
(Ball, 1999)	• Ask questions	p. 28
(Ball, 1999)	• Pose examples or counterexamples	p. 28
(Ball, 1999)	• Help develop, validate, and justify mathematical definitions, claims, and methods	p. 28
(Ball, 1999)	• Compare different solution strategies and solutions	p. 28
(Ball, 1999)	• Respond to an unexpected student assertion or idea	p. 34

(Ball, 1999)	• Interpret and decide what to do with unexpected student comments	p. 36
(Ball, 2000)	• Solve a mathematics problem	p. 242
(Ball, 2000)	• Figure out how to organize solutions	p. 242
(Ball, 2000)	• Select tasks for students	p. 242
(Ball, 2000)	• Know what a problem is asking	p. 242
(Ball, 2000)	• Identifying the mathematical potential of a task	p. 242
(Ball, 2000)	• Anticipate difficulties students might have with a problem	p. 242
(Ball, 2000)	• Rescale a problem for younger or older learners; to make it easier or more challenging	p. 242, p. 244.
(Ball, 2000)	• Modify a problem	p. 243
(Ball, 2000)	• Collect students' work and peruse it	p. 243
(Ball, 2000)	• Hold class discussion	p. 243
(Ball, 2000)	• Formulate probes	p. 243
(Ball, 2000)	• Push students	p. 243
(Ball, 2000)	• Offer hints	p. 243
(Ball, 2000)	• Provide explanations	p. 243
(Ball, 2000)	• Remobilize students who get stuck	p. 243
(Ball, 2000)	• Hear students flexibly	p. 243
(Ball, 2000)	• Select good tasks	p. 243
(Ball, 2000)	• Help all their students learn	p. 243
(Ball, 2000)	• Represent ideas in multiple ways	p. 243
(Ball, 2000)	• Connect content to contexts effectively	p. 243
(Ball, 2000)	• Think about things in ways other than their own	p. 243
(Ball, 2000)	• Size up textbooks and adapt them effectively	p. 243
(Ball, 2000)	• Probe the task	p. 244
(Ball, 2000)	• Consider how a task might be done by particular students	p. 244
(Ball, 2000)	• Figure out what students know	p. 244, 246
(Ball, 2000)	• Choose and managing representations of ideas	p. 244
(Ball, 2000)	• Appraise, select, and modify textbooks	p. 244,

		246
(Ball, 2000)	• Decide among alternative courses of action	p. 244
(Ball, 2000)	• See and hear from someone else's perspective	p. 245
(Ball, 2000)	• Make sense of a student's apparent error	p. 245
(Ball, 2000)	• Appreciate a student's unconventionally expressed insight	p. 245
(Ball, 2000)	• Produce a comprehensible explanation	p. 245
(Ball, 2000)	• Pose questions	p. 246
(Ball, 2000)	• Explain the curriculum to parents	p. 246
(Ball, 2002a)	• Ask students to explain their reasoning	p. 2
(Ball, 2002a)	• Ask students to respond to ideas of others	p. 2
(Ball, 2002a)	• Select, adapt, use representations for mathematical ideas	p. 4
(Ball, 2002a)	• Size up the validity of a child's non-standard procedure: understand and appraise it and make a decision about its significance and generality.	p. 4
(Ball, 2002a)	• Judge if an idea is mathematically significant and worth taking up	p. 4
(Ball, Hill et al., 2005)	• Plan lessons	p. 17
(Ball, Hill et al., 2005)	• Evaluate students' work	p. 17
(Ball, Hill et al., 2005)	• Write and grade assessments	p. 17
(Ball, Hill et al., 2005)	• Explain class work to parents	p. 17
(Ball, Hill et al., 2005)	• Make and manage homework	p. 17
(Ball, Hill et al., 2005)	• Attend to concerns for equity	p. 17
(Ball, Hill et al., 2005)	• Deal with the building principal who has strong views about the mathematics curriculum	p. 17
(Ball, Hill et al., 2005)	• Do problems	p. 17
(Ball, Hill et al., 2005)	• Explain	p. 17
(Ball, Hill et al., 2005)	• Listen	p. 17
(Ball, Hill et al., 2005)	• Examine students' work	p. 17
(Ball, Hill et al., 2005)	• Choose useful models and examples	p. 17
(Ball, Hill et al., 2005)	• See and size up a typical wrong answer	p. 17
(Ball, Hill et al., 2005)	• Analyze the source of errors in students' work	p. 17
(Ball, Hill et al., 2005)	• Grade students' homework at home	p. 20

(Ball, Hill et al., 2005)	• Use representations	p. 20
(Ball, Hill et al., 2005)	• Explain the basis for an algorithm in words that children can understand and show why it works	p. 20
(Ball, Hill et al., 2005)	• Carry out and understand multi-step problems	p. 20
(Ball, Hill et al., 2005)	• Make judgments about how to define terms and whether to permit informal language or introduce and use technical vocabulary grammar and syntax	p. 20
(Ball, 2002b)	• Size up the mathematical validity of alternative approaches	p. 12
(Ball, 2002b)	• Make sound decisions about what to take up and what to deal with individually	p. 12
(Ball, 2002b)	• Appraise and use curriculum materials: size up their mathematical quality, perceive and compensate for distortions, transform weak presentations, learn from unfamiliar but promising representations or approaches	p. 13
(Ball & Bass, 2002)	• Find out what students know how to do	p. 83
(Ball & Bass, 2002)	• Teach students ways of working that the teacher will use during that school year	p. 83

Appendix 4.4
Additional common tasks observed in Ireland and documented in the United States

Task of Teaching	Observed Example from Irish Lesson	Warrant from U.S. Literature or Items
Record work done in lesson on the board or on a poster	“A polygon is a flat shape with straight sides” (SDVS4, C, 1)	Teacher makes “mathematical knowledge and language public” (Ball & Bass, 2003a, p. 41).
Making students’ work public	The student did something interesting the teacher wanted other students to hear about (SDVS7, C, 6; SDVS5, C, 3) Teacher highlights a desired way to do a task (SDVS7, C, 10).	Teacher makes students’ ideas “accessible for others’ consideration” (Ball & Bass, 2003a, p. 42).
Ask other students to comment on a response or a statement made by one student	Teacher says “Have a close look. Hands up. What do you think” of Simon’s answer? (SDVS7, C, 1	Teacher asks students to respond to other’s ideas (Ball, 2002a) (Form B_01, Item 12)
Make a mathematically accurate representation	“I’m still not getting these even. There’s a bigger piece at the end;” or “There’s a shaky pizza line. They’re all [supposed to be] equal now, ok?” (SDVS9, C)	“Representing and making mathematical ideas available to students” (Ball & Bass, 2003b, p. 6)
Represent ideas in multiple ways.	Teacher taught students to divide and used a word problem, lollipop sticks, pictures and numerals to model the problems (SDVS5, C)	Teacher represents ideas in multiple ways (Ball, 2000) Item 17 on Form B_01
Teach students to make accurate representations	Teacher 9 suggested partitioning a circle by starting with a centre point and later pointed his arms in the air to show a student how to divide a circle into thirds	Form B_01, Item 1
Evaluate student representations	e.g. SDVS9, C, 5	Form B_01, Item 1
Explain the basis for an algorithm and how it works	Teacher had opportunity to do it as part of the following description: “Will I tell you what I learned in school about this? Not a lot. Turn it upside down and multiply. When we learned how to	Teacher “explains the basis for an algorithm in words that children can understand and showing why it works.” Ball, Hill and Bass (2005)

	divide fractions that's what we were told to do. Turn the number you're dividing by, upside down, and multiply. (SDVS9, C, 9)	
Elicit and evaluate student explanations	"What do you mean by that?" and "Could you add anything else" – SDVS9, C, 1 "Ah yes, she's right, because look this is a face" pointing to the curved face connecting the parallel circular faces (SDVS4, C, 6)	Form: B_01 Item 27 (a), (b), (c)
Select suitable exercises for students to attempt	Frequent examples of choosing exercises for class work and occasional references to homework.	Teachers are responsible for "making and managing homework" (Ball, Hill et al., 2005, p. 17)
Appraise and select and modify content of textbooks.	Decisions about textbooks are made at school level. In one case a teacher asked students to draw a picture to illustrate each problem (SDVS9, C) and in another the teacher asked a student to omit parts of an exercise (SDVS1, C, 6)	Teachers "size up the mathematical quality of alternative materials, perceive and compensate for distortions, transform weak presentations,[and] learn from unfamiliar but promising representations or approaches" (Ball, 2002b)
Introduce materials to students	One teacher introduced students to the names of base ten materials: units, and tens or longs. The teacher checked that students had the board turned correctly with "the units on your right" (SDVS7, C, 2).	Several examples possible.
Present a task to students and/or explain assignments to students	I observed teachers communicating to students what is needed to do a task, how or where to get it, what the task is and how to work on it. In some cases the teacher responded to questions from students and sometimes the teacher gave a clue or a suggestion for how to do the work	Teachers need to source and choose problems for students (Form B_01, item 16) or design a task (Ball & Bass, 2000a, p. 88) and they need to identify the mathematical potential of a task (Ball, 2000, p. 242)

	Examples observed of teachers telling students what they will work on in a lesson; telling students what they have been doing in a lesson or activity; and telling students why they are doing an activity	
Build on students' prior knowledge	Teachers link to work done previously in the current class level or in a previous class level.	Figure out what students know (Ball, 1999)
Evaluate student understanding.	Teachers asked questions to revise material taught in the lesson so far.	Form B_01, Item 26
Evaluate and respond to students' work	Teachers asked students to justify an answer, asked the student to expand on or to clarify a response or directed the student to a definition of a relevant term.	(Ball & Bass, 2003b)
Write and grade assessments	The curriculum refers to the useful planning information that can be obtained from the regular use of teacher-designed tasks and tests (Government of Ireland, 1999a)	"Assess students' mathematics learning and take [the] next steps" (Ball & Bass, 2003b, p. 11) Also (Ball, Hill et al., 2005)
Explaining the mathematics curriculum to parents	Formal and informal meetings with individual and groups of parents occur in schools.	(Ball, Hill et al., 2005)
Attend to mathematical practices	The Irish curriculum identifies mathematical skills (practices) students are expected to develop: applying and problem solving, communicating and expressing, integrating and connecting, reasoning, implementing, understanding and recalling (Government of Ireland, 1999a).	Attend to "mathematical practices as a component of mathematical knowledge." Students are engaged in "representing ideas, developing and using definitions, interpreting and introducing notation, figuring out whether a solution is valid and noticing patterns" (Ball & Bass, 2003b, p. 12)
Decide among alternative courses of action	Irish teachers do this when they select exercises for students, choose representations, record work on the board and define mathematical terms.	Teachers "decide among alternative courses of action" (e.g. Ball, 1999)
Teaching students how to	This is a pre-requisite for	Teaching students ways of

be learners of mathematics.	any teaching but it is difficult to know how regularly it is as deliberate as some U.S. teachers make it (e.g. Lampert, 2001).	working that will be used by the teacher (Ball & Bass, 2002)
Discussing mathematical issues with colleagues	Formal discussions about mathematical topics at staff meetings. Informal discussions with colleagues about how to solve specific mathematics problems.	A teacher may explain to a colleague how to solve a word problem and collaborate with a colleague to solve a problem (Form B_01, items 3, 5). A teacher may also need to is to deal with a principal with strong views about the teaching of mathematics (2005, p. 17)
Teach students to write numerals and other mathematical symbols	E.g. SDVS8, B, 1. Students practice writing the number seven	No specific mention found. May be considered trivial.
Engage in activities to support student problem solving		Know what a problem is asking (Ball, 2000) Solve and understand multi-step problems (Ball, Hill et al., 2005)
Checking student work	Examples of this task occurred frequently in the Irish lessons. In addition, the task of “correcting copies” in mathematics and in other subjects has long been considered one of the tasks of teaching in Ireland.	Ball (2000) mentions tasks involved in perusing students work: “the teacher may grade it, determine where her students are, or decide to go further” (p. 243). Another factor that makes this work more difficult is that sometimes it is done at home (Ball, Hill et al., 2005) or without the student present.
Establishing and managing the discourse and collectivity of the class	Many aspects of this were observed in the Irish lessons	“Establishing and managing the discourse and collectivity of the class for mathematics learning” (Ball & Bass, 2003b, p. 6). Many tasks of teaching stem from this such as managing discussions, posing good mathematical questions (Ball & Bass, 2003b) and formulating probes (Ball, 2000).
Help students connect ideas in mathematics	The curriculum aspires to have students use “their knowledge of one area of	One example is to use an area model (geometry) to explain multiplication

	mathematics to explore another" (Government of Ireland, 1999a, p. 15)	(number) which will later support learning about algebra (2003b).

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