On the Realization of Line Arrangements as Multiplier Ideals

by

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To my wife, Mina

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CHAPTER I

Introduction

1.1 Introduction to the Main Theorem

The main purpose of this thesis is to deal with the question of which ideals can or cannot be realized as multiplier ideals. We introduce now the main context of this problem.

Let X be a smooth complex algebraic variety, and let $\mathfrak{b} \subset \mathcal{O}_X$ be an ideal sheaf. Given a rational number or real number $c > 0$, one can construct the multiplier ideal

$$\mathcal{J}(\mathfrak{b}^c) = \mathcal{J}(X, \mathfrak{b}^c) \subset \mathcal{O}_X$$

of $\mathfrak{b}$ with weighting coefficient c. This new ideal measures the singularities of functions $f \in \mathfrak{b}$ in a subtle manner. In recent years, multiplier ideals have found many applications in local and global geometry due to the fact that they satisfy vanishing theorems.

Because of their importance, there has been considerable interest in understanding which ideals can occur as multiplier ideals. It has long been known that multiplier ideals are integrally closed. It is then natural to ask whether conversely every integrally closed ideal is actually a multiplier ideal.

In dimension 2, Favre-Jonsson [6] and Lipman-Watanabe [24] showed that every integrally closed ideal can locally be realized as a multiplier ideal. However the corresponding statement in dimension greater than 2 was open for a number of years. I studied the question of whether the ideal $I_r \subset \mathcal{O}_{\mathbb{C}^3}$ of r very general lines passing through the origin can be realized as a multiplier ideal $\mathcal{J}(\mathfrak{b}^c)$.

The main theorem of this thesis is given below. The precise definition of a rationally defined cone-like divisor will be given in Chapter 4. Vaguely speaking, it is a certain type of function which behaves in some sense naturally along r very general lines, and which varies rationally as the lines are deformed.

Theorem 1 (Main Theorem)**.** *When $r \leq 10$, I_r is a multiplier ideal (of a rationally defined cone-like divisor). When $r \gg 0$, I_r cannot be realized as a multiplier ideal of a rationally defined cone-like divisor.*

It is possible that the condition on the divisor is unnecessary, but the computations are very involved and it is not yet certain that they handle the case of arbitrary divisor.

This work for lines suggested that there must be an obstruction to realizing integrally closed ideals as multiplier ideals in dimension ≥ 3. Subsequently Lazarsfeld and I proved that the minimal syzygies of any multiplier ideal satisfy rather strong conditions of an algebraic nature. It is amusing that the ideal of the union of lines, which is unlikely to be a multiplier ideal, satisfies the necessary conditions.

This thesis is organized as follows. In Chapter 2, we define multiplier ideals and give a quick snapshot of how we approach to the proof of the main theorem. Chapter 3 is devoted to develop the main tools used in the proof of the main theorem. Specifically we obtain some bounds on colengths of multiplier ideals on surfaces. Chapter 3 also deals with t-multiplicity. Here we have strong bounds on colengths of multiplier ideals and intersection multiplicities of two curves. Finally, in Chapter 4 we carry out the proof of the main theorem by using induction on the order of blow-ups. We start by assuming that I_r can be realized as a multiplier ideal of a rationally defined cone-like divisor, and show that a desired log resolution would not stop in finitely many times, which is a contradiction.

Besides the material in this thesis, I have completed some other works, some of which are related and some not. In particular, I have completed several projects touching on syzygies of multiplier ideals, cores of ideals, polynomial interpolation, and Hilbert schemes. Before turning to the main content of this thesis, I will say a few words about these other works.

1.2 Other Work

Here I briefly explain other work that I have completed besides this thesis.

1.2.1 Local syzygies of multiplier ideals

In a joint work with Lazarsfeld, we showed that syzygies of any multiplier ideals satisfy rather strong conditions of an algebraic nature. We work in the local ring $(\mathcal{O}, \mathfrak{m})$ of a smooth complex variety X at a point $x \in X$, and as above $d = \dim X$.

Theorem 1.2.1. *Let $\mathcal{J} = \mathcal{J}(\mathfrak{b}^c)_x \subseteq \mathcal{O}$ be (the germ at x of) any multiplier ideal. If $p \geq 1$, then no minimal p^{th} syzygy of $\mathcal{J}$ vanishes modulo $\mathfrak{m}^{d+1-p}$.*

Let us explain the statement more precisely. For the case $p = 1$, fix minimal generators $f_1, \ldots, f_b \in \mathcal{J}$, and let $g_1, \ldots, g_b \in \mathfrak{m}$ be functions giving a minimal syzygy

$$\sum g_i f_i \;=\; 0$$

among the f_i. Then the claim is that

$$\mathrm{ord}_x(g_i) \leq d - 1$$

for at least one index i. In general, consider a minimal free resolution

$$\ldots \xrightarrow{u_3} \mathcal{O}^{b_2} \xrightarrow{u_2} \mathcal{O}^{b_1} \xrightarrow{u_1} \mathcal{O}^{b_0} \longrightarrow \mathcal{J} \longrightarrow 0$$

of $\mathcal{J}$, where each u_p is a matrix of elements in $\mathfrak{m}$ whose columns minimally generate the module of p^{th} syzygies of $\mathcal{J}$. The assertion of the theorem is that no column of u_p (or any $\mathbf{C}$-linear combination thereof) can consist entirely of functions vanishing to order $\geq d+1-p$ at x. Equivalently, no minimal generator of the p^{th} syzygy module

$$\mathrm{Syz}_p(\mathcal{J}) \ =_{\text{def}}\ \mathrm{Im}(u_p) \ \subseteq\ \mathcal{O}^{b_{p-1}}$$

of $\mathcal{J}$ lies in $\mathfrak{m}^{d+1-p} \cdot \mathcal{O}^{b_{p-1}}$.

The theorem implies that if $d \geq 3$, then many integrally closed ideals cannot arise as multiplier ideals. For example consider $2 \leq m \leq d-1$ functions

$$f_1, \ldots, f_m \ \in\ \mathcal{O}$$

vanishing to order $\geq d$ at x. If the f_i are chosen generally, then the complete intersection ideal $\mathcal{I} = (f_1, \ldots, f_m)$ that they generate will be radical, hence integrally closed. On the other hand, the Koszul syzygies among the f_i violate the condition in Theorem 1.2.1, and hence $\mathcal{I}$ is not a multiplier ideal.

Theorem 1.2.1 follows from a more technical statement involving the vanishing of a map on Koszul cohomology groups. Specifically, let $h_1, \ldots, h_r \in \mathfrak{m}$ be any collection of non-zero elements generating an ideal $\mathfrak{a} \subseteq \mathcal{O}$, and let $K_\bullet(h_1, \ldots, h_r)$ be the Koszul complex on the h_i. We prove:

Theorem 1.2.2. *For every $0 \leq p \leq r$, the natural map*

$$H_p\big(K_\bullet(h_1, \ldots, h_r) \otimes \mathfrak{a}^{r-p}\mathcal{J}(\mathfrak{b}^c)\big) \longrightarrow H_p\big(K_\bullet(h_1, \ldots, h_r) \otimes \mathcal{J}(\mathfrak{b}^c)\big)$$

vanishes.

Now fix generators $z_1, \ldots, z_d \in \mathfrak{m}$, and write $\mathbf{C} = \mathcal{O}/\mathfrak{m}$ for the residue field at x, viewed as an $\mathcal{O}$-module. Taking $r = d$ and $h_i = z_i$, the theorem implies

Corollary 1.2.3. *The natural maps*

$$\operatorname{Tor}_p\big(\mathfrak{m}^{d-p}\mathcal{J}, \mathbf{C}\big) \longrightarrow \operatorname{Tor}_p\big(\mathcal{J}, \mathbf{C}\big)$$

vanish for all $0 \leq p \leq d$.

Theorem 1.2.1 is deduced from this statement. As for Theorem 1.2.2, the proof is simply to note that an exact "Skoda complex" [19, Section 9.6.C] sits inbetween the two Koszul complexes in question.

Variant 1.2.4. In the situation of Theorem 1.2.1, suppose that $\mathcal{J}$ has a minimal p^{th} syzygy vanishing to order a_p at x, and for $1 \leq i \leq p-1$ denote by

$$\varepsilon_i \ = \ \varepsilon_i(\mathcal{J})$$

the least order of vanishing at x of all non-zero entries in the matrix u_i appearing in the minimal resolution of $\mathcal{J}$. Then

$$(*) \qquad a_p + \varepsilon_{p-1} + \ldots + \varepsilon_1 \ \leq \ d-1. \quad \square$$

For example, consider when $d = 4$ the complete intersection ideal

$$\mathcal{I} \ = \ (f_1, f_2, f_3) \ \subseteq \ \mathcal{O}$$

generated by three functions vanishing to order 2 at the origin. Then $a_2 = \varepsilon_1 = 2$, so $\mathcal{I}$ cannot be a multiplier ideal, but this does not follow from the statement of Theorem 1.2.1 alone. We do not know how close (*) comes to being optimal. However we construct an example lying on the boundary of Corollary 1.2.3.

Example 1.2.5. Let $\mathcal{J} = \mathcal{J}((z_1, z_2, ..., z_{d-1}, z_d^d)^d) = (z_1, z_2, ..., z_{d-1}, z_d^d)$. Then the natural map

$$\mathrm{Tor}_p\big(\mathfrak{m}^{d-1-p}\mathcal{J}, \mathbf{C}\big) \longrightarrow \mathrm{Tor}_p\big(\mathcal{J}, \mathbf{C}\big)$$

does not vanish for any $0 \le p \le d$.

Remark 1.2.6. In [20], we say that the ideal of a suitable number of general lines through the origin in $\mathbf{C}^3$ couldn't arise as a multiplier ideal. However the correct statement is the one which appears in Theorem 1.

1.2.2 Containment of cores of ideals

In [21], I study cores of ideals in a regular local ring R. Given an ideal $I \subset R$, the core of I is defined to be the intersection of all the reductions of I. Huneke and Swanson [16] raised the question of whether, given integrally closed ideals $I \subset I'$ in a ring R, it is necessarily true that $\mathrm{core}(I) \subset \mathrm{core}(I')$. I show that cores of ideals do not preserve the inclusion.

Theorem 1.2.7. *Let $R = k[x, y, z, w]_{(x,y,z,w)}$ with k a field of characteristic zero and let $\mathfrak{m}$ denote the maximal ideal of R. Let $I = I_2 + \mathfrak{m}^3$, where*

$$I_2 = (x^2 + yw, y^2 + zw, z^2 + xw).$$

Then $I \subset \mathfrak{m}^2$ but $\mathrm{core}(I) \not\subset \mathrm{core}(\mathfrak{m}^2)$.

This theorem motivates the following conjecture.

Conjecture 1.2.8. *Let $R = k[x_1, ..., x_n]_{(x_1,...,x_n)}$ with k a field of characteristic zero and let $\mathfrak{m}$ denote the maximal ideal of R. Let $I = I_d + \mathfrak{m}^{d+1}$, where I_d is a radical complete intersection ideal of s homogeneous polynomials of degree d $(1 \le s < n)$. Let $b = \lfloor \frac{dn-s+1}{d+1} \rfloor$ and $a = dn - s + 1 - (d+1)b$. Then*

$$\mathrm{core}(I) = \mathfrak{m}^a I^b. \tag{1.2.1}$$

Theorem 1.2.9. *The conjecture holds true for $d = 1$.*

1.2.3 Interpolation problem in characteristic 2

It is natural and interesting to ask whether, given (d, n, m), there is a plane curve of degree d passing through n general points with multiplicity at least m. If there is no such curve, then the vector space V generated by all the monomials of degree $\leq d$ can be (non-canonically) decomposed into n subspaces

$$V = \oplus_{\alpha=1}^{n} V_\alpha,$$

where each of V_α is generated by monomials and does not contain any curve passing through a general point with multiplicity at least m. There has been some interest in understanding which subspaces satisfy such conditions.

In [22], I work on a bivariate polynomial interpolation problem in characteristic 2. Given a nonnegative integer t, I describe all the sub-linear systems generated by monomials, in which there is no curve passing through a general point with multiplicity at least 2^t. This interpolation problem is of inductive nature on t.

Given a fixed set S of distinct lattice points $(i, j), i, j \geq 0$, the sub-linear system $\mathcal{P}(S)$ with respect to S consists of all the polynomials of the form

$$P(x, y) = \sum_{(i,j)\in S} a_{i,j} x^i y^j \in \mathbb{K}[x, y].$$

Let T_m be the triangle of all (i, j) with $i+j \leq m-1$. T_m contains $|T_m| = \frac{1}{2}m(m+1)$ lattice points.

For a set of n distinct interpolation knots $Z = \{z_q := (x_q, y_q)\}_{q=1}^{n}$ in $\mathbb{K}^2$, one can consider (sub-)linear systems of plane curves passing through Z with multiplicity $\geq m_q$ at each point z_q. To put it in another way, we are interested in solving the

Hermite interpolation problem

$$\frac{1}{\alpha!\beta!}\cdot\frac{\partial^{\alpha+\beta}P}{\partial x^\alpha \partial y^\beta}\Big|_{z_q}=0, \qquad (\alpha,\beta)\in T_{m_q}, \qquad q=1,\cdots,n. \qquad (*)$$

Note that we do not necessarily require $|S|=\sum_{q=1}^{n}|T_{m_q}|$. We say that an interpolation scheme is *almost surely solvable* or *almost regular* if $(*)$ is solvable for almost all $Z\in(\mathbb{K}^2)^n$. Since the right hand sides in $(*)$ are 0, our interpolation problem is almost regular if and only if $(*)$ has only the trivial solution for almost all Z.

Since it is natural to ask which (sub-)linear systems are almost regular, there has been some interest in trying to understand it. But up to now, even in the case $n=|Z|=1$ there have been no explicit criterions in positive characteristic, and no criterions in characteristic 0 other than Bezout-Dumnicki Lemma and Petrakiev's Lemma, which give sufficient but not necessary conditions for a (sub-)linear system to be almost regular.

We completely solve the interpolation problem in characteristic 2 in the case when $n=|Z|=1$ and $m=m_1=2^t$ $(t\in\mathbb{N})$. In other words, given $t\in\mathbb{N}$, we describe all the sub-linear systems generated by monomials, in which there is no curve passing through a general point with multiplicity $\geq 2^t$. This case is already interesting in its own right, and is indispensable for dealing with the cases of $n\geq 2$ knots. Because an n-generic-points interpolation problem can be reduced to n one-generic-point interpolation problems on linear systems defined by adequate sets of monomials. We remark that the asymptotic behaviors of certain interpolation problems, such as the nef cone of the blown-up space of $\mathbf{P}^2$ at a given number $(n>9)$ of very general points, can be determined by infinitely many interpolation problems. Hence for some purposes it would be enough to look at interpolation problems with $m=m_1=\cdots=m_n=2^t$ for $t\in\mathbb{N}$.

When $n = 1$, our interpolation scheme $\langle S, T_m \rangle$ becomes

$$\frac{1}{\alpha!\beta!} \cdot \frac{\partial^{\alpha+\beta} P}{\partial x^\alpha \partial y^\beta}\bigg|_{z_1} = 0, \qquad (\alpha, \beta) \in T_m \qquad\qquad (**).$$

The main result in [22] is the following.

Theorem 1.2.10. *The determination of the solvability of a bivariate Hermite interpolation problem with a point of multiplicity 2^{t+1} can be reduced to solving three interpolation problems, each with a point of multiplicity 2^t.*

1.2.4 Hilbert schemes of points

We work over $\mathbb{C}$. The maximal ideals in a polynomial ring are very basic objects, and their deformations are easy to understand. However very little is known about the family of the ideals that can be deformed to the square of a maximal ideal. Its existence and connectedness [13] are well known. In [23], I study its dimension.

We consider the Hilbert scheme $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ of $(d+1)$ points in affine d-space $\mathbb{C}^d$, $d \geq 3$ (for general introduction to the Hilbert schemes of points, see [25, §18.4]). It parametrizes the ideals I of colength $(d+1)$ in $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, ..., x_d]$. As with any moduli problem, it is natural to ask whether $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is irreducible. It is already interesting because $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is irreducible for $d \leq 3$ but reducible for $d \geq 12$.[1]

Theorem 1.2.11. *The dimension of* $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ *is greater than* $k\binom{d+1-k}{2}$ *for any* $k = 0, 1, ..., d-1$.

Theorem 1.2.12. $\mathrm{Hilb}^n(\mathbb{C}^d)$ *is reducible for* $n > d \geq 12$.

Our purpose is to describe equations for the most symmetric affine open subscheme of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$. We let $U \subset \mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ denote the affine open subscheme consisting of all ideals $I \in \mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ such that $\{1, x_1, ..., x_d\}$ is a $\mathbb{C}$-basis of $\mathbb{C}[\mathbf{x}]/I$. We will

[1] Iarrobino [17] showed that $\mathrm{Hilb}^n(\mathbb{C}^d)$ is reducible for $d > 5$ and $n > (1+d)(1+d/4)$.

call U *the symmetric affine subscheme*. We note that the square of any maximal ideal in $\mathbb{C}[\mathbf{x}]$ belongs to the symmetric affine subscheme.

We give an elementary description of the coordinate ring of the symmetric affine subscheme U. For a $\mathbb{C}$-vector space V and a partition λ, the module $\mathbb{S}_\lambda V$ is defined by the Schur-Weyl construction. By abuse of notation, the quotient ring given by the ideal generated by $\mathbb{S}_\lambda V$ in the ring $\mathrm{Sym}^\bullet(\mathbb{S}_\mu V)$ for some partitions λ and μ will be denoted by $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_\mu V)}{<\mathbb{S}_\lambda V>}$.

Theorem 1.2.13. *Let $d \geq 3$. Let U be the symmetric affine open subscheme of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$. Then U is isomorphic to*

$$\mathbb{C}^d \times \mathrm{Spec}\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{< \mathbb{S}_{(4,3,2,\cdots,2,1)}V >},$$

where V is a d-dimensional $\mathbb{C}$-vector space, $(3,1,1,\cdots,1,0)$ is a partition of $(d+1)$ and $(4,3,2,\cdots,2,1)$ is of $(2d+2)$.

Let us explain the notation more precisely. There is an injective homomorphism

$$j : \mathbb{S}_{(4,3,2,\cdots,2,1)}V \hookrightarrow \mathrm{Sym}^2(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)$$

of Schur modules. Then j induces natural maps

$$\begin{aligned} &\mathbb{S}_{(4,3,2,\cdots,2,1)}V \otimes \mathrm{Sym}^{r-2}(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) \\ &\hookrightarrow \mathrm{Sym}^2(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) \otimes \mathrm{Sym}^{r-2}(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) \\ &\to \mathrm{Sym}^r(\mathbb{S}_{(3,1,1,\cdots,1,0)}V), \qquad\qquad r \geq 2, \end{aligned}$$

which define the quotient ring $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{<\mathbb{S}_{(4,3,2,\cdots,2,1)}V>}$.

Corollary 1.2.14. *Let $H(r)$ be the Hilbert function of $\frac{\mathrm{Sym}^\bullet(\mathbb{S}_{(3,1,1,\cdots,1,0)}V)}{<\mathbb{S}_{(4,3,2,\cdots,2,1)}V>}$. Let*

$$\mathrm{Sym}^r(\mathbb{S}_{(3,1,1,\cdots,1,0)}V) = \bigoplus_{|\lambda|=r(d+1)} \mathbb{S}_\lambda^{\oplus m_\lambda},$$

where $m_\lambda \in \mathbb{Z}_{\geq 0}$ *and* $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ *is a partition of* $r(d+1)$*, i.e.,* $\sum_{i=1}^{d} \lambda_i = r(d+1)$ *and* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$*. Then*

$$H(r) \geq \sum_{\substack{|\lambda|=r(d+1) \\ \lambda_{d-k}+\cdots+\lambda_d \leq rk}} m_\lambda (\dim_{\mathbb{C}} \mathbb{S}_\lambda), \tag{1.2.2}$$

for any $r \geq 2$ *and any* $k = 0, ..., d-1$*.*

Corollary 1.2.14 is an elementary consequence of the combinatorial Littlewood-Richardson rule (for example, see [8, Appendix]). In fact any $\mathbb{S}_\lambda$ appearing in the decomposition of $\mathbb{S}_{(4,3,2,\cdots,2,1)}V \otimes (\mathbb{S}_{(3,1,1,\cdots,1,0)}V)^{\otimes(r-2)}$ satisfies $\lambda_{d-k} + \cdots + \lambda_d \geq rk+1$, for any $r \geq 2$ and any $k = 0, ..., d-1$.

It is tedious but entirely possible to compute the right hand side of (1.2.2) for small r. These computations suggest that the Hilbert function $H(r)$ grows faster than $\mathcal{O}\left(r^{k\binom{d-k}{2}}\right)$ for any $k = 0, ..., d-1$. So Corollary 1.2.14 suggests that, for sufficiently large d, the symmetric open subscheme U of $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ has dimension greater than $d(d+1)$, which implies that $\mathrm{Hilb}^{d+1}(\mathbb{C}^d)$ is reducible. To prove Theorem 1.2.11, we actually find large dimensional families of ideals in a very explicit way.

CHAPTER II

Realization of I_r as a multiplier ideal for small r

2.1 Overview of multiplier ideals

In this section we give brief definitions and properties of multiplier ideals. For a general introduction, we refer to [19]. Let X be a smooth complex algebraic variety of dimension d. To define the multiplier ideals, we need log resolutions.

Definition 2.1.1 (Simple normal crossing support). Let Y be a smooth complex variety with dimension $\dim Y = d$ and $E \subset Y$ be a hypersurface (effective Weil divisor). Write $E = a_1E_1 + \cdots + a_rE_r$, where the E_i are distinct irreducible components of E and each $a_i > 0$. Then E has *simple normal crossing support* (abbreviated SNC support) if the E_i are all smooth, and if $E_1 + \cdots + E_r$ is defined in a neighborhood of any point by an equation in local analytic coordinates of the type

$$z_1 \cdot \ldots \cdot z_k = 0$$

for some $k \leq d$.

Definition 2.1.2 (Log resolution). Let $\mathfrak{b} \subset \mathcal{O}_X$ be an ideal sheaf. A *log resolution* of $\mathfrak{b}$ is a projective birational map

$$\mu : X' \to X,$$

with X' non-singular, such that

$$\mathfrak{b} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F),$$

where F is an effective divisor on X', and $F + \mathrm{except}(\mu)$ has simple normal crossing support.

Example 2.1.3. Let

$$\mathfrak{b} = (x^3, y^2) \subset \mathbb{C}[x, y].$$

Then a log resolution $\mu : X' \to X$ of $\mathfrak{b}$ is obtained by the sequence of three blowings up at singular points. It gives an embedded resolution of the cuspidal cubic $\{y^2 = x^3\}$. In this case we have

$$\mathfrak{b} \cdot \mathcal{O}_{X'} \;=\; \mathcal{O}_{X'}(-2E_1 - 3E_2 - 6E_3).$$

Definition 2.1.4 (**Relative canonical divisor**)**.** Given any log resolution $\mu : X' \to X$, we define the relative canonical divisor of X' over X by

$$K_{X'/X} \;=\; K_{X'} - \mu^* K_X.$$

This is naturally defined as an effective divisor supported on the exceptional locus of μ. In fact $\det(d\mu)$ gives a local equation for the effective divisor $K_{X'/X}$.

Example 2.1.5. Let $\mu : X' \to X$ be the log resolution of $\mathfrak{b} = (x^3, y^2)$ as in Example 2.1.3. Then

$$K_{X'/X} = E_1 + 2E_2 + 4E_3. \qquad \square$$

Now we define the multiplier ideals associated to an ideal sheaf.

Definition 2.1.6 (**Multiplier ideal**)**.** Let $\mathfrak{b} \subset \mathcal{O}_X$ be an ideal sheaf. Choose local generators $f_1, \cdots, f_r \in \mathfrak{b}$. Given $c > 0$, we can define the *multiplier ideal* $\mathcal{J}(\mathfrak{b}^c)$ by

$$\mathcal{J}(\mathfrak{b}^c) :=_{\text{locally}} \left\langle h \;\middle|\; \frac{|h|^2}{(|f_1|^2 + \cdots + |f_r|^2)^c} \text{ is locally integrable} \right\rangle.$$

Equivalently, let $\mu : X' \to X$ be a log resolution, and let $\mathfrak{b} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$. Then

$$\mathcal{J}(\mathfrak{b}^c) := \mu_* \mathcal{O}_{X'}(K_{X'/X} - \lfloor cF \rfloor).$$

The equivalence is established in [19, 9.3.D].

Remark 2.1.7. $\mathcal{J}(\mathfrak{b}^c)$ is independent of the log resolution μ. This is proven in [19, 9.2.A].

Example 2.1.8. Let $\mathfrak{b} = (x_1^{e_1}, \ldots, x_d^{e_d}) \subset \mathbb{C}[x_1, \ldots, x_d]$. Then

$$\mathcal{J}(\mathfrak{b}^c) = \left(x_1^{m_1}, \ldots, x_d^{m_d} \Big| \sum_i \frac{m_i + 1}{e_i} > c \right). \qquad \square$$

Multiplier ideals satisfy vanishing theorems. Here we introduce Nadel's vanishing theorem which we will use in Section 3.4.

Theorem 2.1.9 (Nadel vanishing theorem). *Let X be a smooth complex projective variety, let D be any $\mathbf{Q}$-divisor on X, and let L be any integral divisor such that $L - D$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0 \quad \text{for} \quad i > 0. \qquad \square$$

We present a well-known property of multiplier ideals.

Definition 2.1.10 (Integrally closed ideal). An ideal $\mathfrak{a} \subset \mathcal{O}_X$ is *integrally closed* if it satisfies either of the following:

(1) If $f_1, \ldots, f_r \in \mathfrak{a}$ are local generators, and $g \in \mathcal{O}_X$ satisfies

$$|g(z)| \leq_{\text{loc}} C \cdot \sum |f_i(z)|,$$

for some $C > 0$, then $g \in \mathfrak{a}$.

(2) There is a birational proper map $\nu : X^+ \to X$, with X^+ normal, and an effective Cartier divisor F such that $\mathfrak{a} = \nu_* \mathcal{O}_{X'}(-F)$.

Corollary 2.1.11. *Multiplier ideals are integrally closed.*

Conversely one can ask the following question.

Question 2.1.12. Which integrally closed ideal is a multiplier ideal?

Example 2.1.13. Any radical ideal defining the union of disjoint smooth subvarieties on X can be realized as a multiplier ideal.

2.2 How to realize I_r as a multiplier ideal for small r

Let $I_r \subset \mathcal{O}_{\mathbb{C}^3}$ be the ideal of r very general lines passing through the origin. Here we illustrate how I_r can be realized as a multiplier ideal for small r, and give a brief intuition on why it cannot be generalized to $r \gg 0$.

It is convenient to describe r very general lines passing through O as follows. Fix r very general points $q_1, \cdots, q_r$ on $\mathbf{P}^2$ and think of I_r as the ideal of the affine cone over the r points. Let $C_d \subset \mathbf{P}^2$ be a smooth curve of degree d passing through $q_1, \cdots, q_r$ on $\mathbf{P}^2$ and let $f_d \in \mathbb{C}[x, y, z]$ denote the corresponding homogeneous polynomial.

The following type of singularities will be necessary in the proof of the first statement of the main theorem.

Definition 2.2.1. Two smooth curves meeting a point q are said to have the same n-th order tangent direction at q if exactly n successive blow-ups are needed to resolve the singularity at q in a minimal way. More concretely, two curves have the same n-th order tangent direction at q if their union near q is analytically isomorphic to $\{y(y - x^n) = 0\} \subset \mathbb{A}^2$.

We deal with the first statement of the main theorem.

Theorem 2.2.2. *When $r \leq 10$, I_r is a multiplier ideal (of a rationally defined cone-like divisor).*

Below we give an idea of how to prove this theorem. The precise proof will be given in Theorem 2.2.4.

Sketch of Proof. It is easy to realize I_r as a multiplier ideal of an ideal with weighting coefficient 2, when $r \leq 5$, in other words, there is a curve of degree 1 or 2 passing through $q_1, \cdots, q_r$. We have

$$I_1 = \mathcal{J}(X, 2 \cdot (f_1, f_1')$$

$$I_2 = \mathcal{J}(X, 2 \cdot (f_1, f_2'))$$

$$I_3 = \mathcal{J}(X, 2 \cdot (f_2, f_2', f_2''))$$

$$I_4 = \mathcal{J}(X, 2 \cdot (f_2, f_2'))$$

$$I_5 = \mathcal{J}(X, 2 \cdot (f_2, f_3', f_3'')),$$

where f' and f'' correspond to general curves meeting C transversally at each of $q_1, \cdots, q_r$, and the subscripts indicate the degrees.

However as soon as $r = 6$ it is impossible to realize I_r as a multiplier ideal of an ideal $\mathfrak{b}$ with weighting coefficient 2. Equivalently I_6 is not an adjoint ideal. Because if it were an adjoint ideal, $\mathfrak{b}$ would vanish along 6 general lines so $\mathrm{ord}_o\mathfrak{b} \geq 3$ hence

$$\mathcal{J}(X, 2 \cdot \mathfrak{b}) \subset \mathbf{m}^4$$

but $I_6 \not\subset \mathbf{m}^4$.

Therefore it is natural to use a smaller weighting coefficient in order to realize I_6 as a multiplier ideal. But to have a smaller coefficient, worse singularities are required. We have

$$I_6 = \mathcal{J}(X, \frac{3}{2} \cdot (f_3, f_5')),$$

where f' corresponds to a general curve having the same tangent direction as C_3 at each of $q_1, \cdots, q_6$.

As r increases up to 10, we keep making coefficients smaller and singularities worse. We realize I_{10} as a multiplier ideal as follows.

We have

$$I_{10} = \mathcal{J}(X, \frac{5}{4} \cdot (f_4, f'_{11})),$$

where f' corresponds to a general curve having the same 4-th order tangent direction as C_4 at each of $q_1, \cdots, q_{10}$. $\square$

The main observation is that the ideals which are used in the above examples have somewhat similar log resolutions. They consist of blowing-up at the origin, followed by blowing-up along a curve in the first exceptional divisor, followed by blowing-up along a curve in the second exceptional divisor, and so on. Therefore we may guess that if I_r were realized as a multiplier ideal of an ideal $\mathfrak{b}$ then a log resolution of $(X, \mathfrak{b})$ would be obtained by the same process. But this will not work for sufficiently large r. To give the reader a quick intuition, we present the following theorem, which essentially says that the above examples cannot be generalized to large r.

Theorem 2.2.3. *For sufficiently large r, the ideal I_r cannot be realized as a multiplier ideal of an ideal $\mathfrak{b}$ satisfying the following properties $(*)$:*

- *$\mathfrak{b}$ is a homogeneous ideal,*
- *$\mathfrak{b}$ is generated by two elements f_{d_1} and f'_{d_2} having degree d_1 and d_2 respectively,*
- *and two curves C_{d_1} and C_{d_2} defined by f_{d_1} and f'_{d_2} respectively are smooth and have the same n-th order tangent direction at each of $q_1, \cdots, q_r$ for some n.*

Proof. Let $d_1 \leq d_2$. Suppose to the contrary that

$$I_r = \mathcal{J}(X, c \cdot \mathfrak{b}),$$

for some $c > 0$ and for some ideal $\mathfrak{b}$ satisfying the stated properties.

Let S be a finite set of several general complex numbers. We consider the product of several general elements in $\mathfrak{b}$:

$$g = \prod_{\lambda \in S}(f_{d_1}(x,y,z) + \lambda f'_{d_2}(x,y,z)).$$

Then after blow up at the origin, on a suitable affine chart, the proper transform of g becomes

$$\prod_{\lambda \in S}(f_{d_1}(x,y,1) + \lambda z^{d_2-d_1} f'_{d_2}(x,y,1)).$$

There are two distinguished singularities along curves $\{f_{d_1} = z^{d_2-d_1} = 0\}$ and $\{f_{d_1} = f'_{d_2} = 0\}$. We first resolve the singularity along $\{f_{d_1} = z^{d_2-d_1} = 0\}$, and then resolve the singularity along the proper transform of $\{f_{d_1} = f'_{d_2} = 0\}$.

Let ψ be the minimal resolution of $\mathfrak{b}$, i.e.,

$$\psi = \sigma_n \circ \cdots \circ \sigma_1 \circ \pi_{d_2-d_1} \circ \cdots \circ \pi_1 \circ \pi_0,$$

where $\pi_0 : X_0 \to X$ is the blow-up of X at the origin O, $\pi_i : X_i \to X_{i-1}$ $(1 \le i \le d_2 - d_1)$ is the blow-up of X_i along the intersection of the exceptional divisor and the proper transform of the affine cone over C_{d_1}, and $\sigma_i : X_{i+d_2-d_1} \to X_{i+d_2-d_1-1}$ $(1 \le i \le n)$ is the blow-up along the intersection of the proper transforms of the affine cones over C_{d_1} and C_{d_2}.

Let $X' = X_{n+d_2-d_1+1}$. Then we have

$$\begin{aligned} K_{X'/X} &= 2E_0 + 3E_1 + \cdots + (d_2 - d_1 + 2)E_{d_2-d_1} \\ &\quad + E_{d_2-d_1+1} + \cdots + nE_{d_2-d_1+n}, \\ \mathfrak{b} \cdot \mathcal{O}_{X'} &= \mathcal{O}_{X'}\Big(- d_1E_0 - (d_1+1)E_1 - \cdots - d_2E_{d_2-d_1} \\ &\quad - E_{d_2-d_1+1} - \cdots - nE_{d_2-d_1+n}\Big). \end{aligned}$$

Then since $\mathcal{J}(X, c\cdot\mathfrak{b})$ vanishes along the given r lines, we have $c \geq \frac{n+1}{n}$. Then

$$\begin{aligned}\mathcal{J}(X, c\cdot\mathfrak{b}) &\subset \psi_*\mathcal{O}_{X'}\Big((d_2-d_1+2)E_{d_2-d_1} - \Big\lfloor \frac{n+1}{n}d_2\Big\rfloor E_{d_2-d_1}\Big)\\ &= \psi_*\mathcal{O}_{X'}\Big(\big(-d_1+2-\Big\lfloor\frac{d_2}{n}\Big\rfloor\big)E_{d_2-d_1}\Big).\end{aligned}$$

On the other hand, since C_{d_1} and C_{d_2} have the same n-th order tangent direction at each of r points, the intersection number of C_{d_1} and C_{d_2} on $\mathbf{P}^2$ becomes

$$C_{d_1}\cdot C_{d_2} = d_1d_2 \geq nr.$$

Thus $\frac{d_2}{n} \geq \frac{r}{d_1}$, which implies

$$-d_1+2-\Big\lfloor\frac{d_2}{n}\Big\rfloor \leq -d_1+2-\Big\lfloor\frac{r}{d_1}\Big\rfloor \leq -\lfloor 2\sqrt{r}\rfloor + 3.$$

Let $d = \mathrm{ord}_O I_r$. Then $\binom{d+1}{2} \leq r < \binom{d+2}{2}$. Let D be a general element in $I_r \cap \mathbf{m}^{d+1}$. Then $\mathrm{ord}_{E_{d_2-d_1}}\psi^*D = d+1 < \lfloor 2\sqrt{r}\rfloor - 3$ for sufficiently large r. So

$$D \notin \psi_*\mathcal{O}_{X'}\Big(\big(-\lfloor 2\sqrt{r}\rfloor+3\big)E_{d_2-d_1}\Big),$$

which is a contradiction. $\square$

We now formally prove the realization of I_r for $r \leq 10$. We prove in fact that I_r can be realized by using the sort of resolution we have just described.

Theorem 2.2.4. *For $r \leq 10$, the ideal I_r can be realized as a multiplier ideal of an ideal $\mathfrak{b}$ satisfying the three properties $(*)$ in Theorem 2.2.3. In particular, I_r is a multiplier ideal of a rationally defined cone-like divisor.*

Proof. We keep using notations and log resolutions in the proof of Theorem 2.2.3. We show that

$$I_r = \mathcal{J}(X, c\cdot\mathfrak{b}) = \mathcal{J}(X, \frac{n+1}{n}\cdot\mathfrak{b}),$$

for some ideal $\mathfrak{b} = (f_{d_1},\ f'_{d_2})$ satisfying $(*)$.

r	d_1	d_2	n	c
1	1	1	1	2
2	1	2	1	2
3	2	4	2	$\frac{3}{2}$
4	2	2	1	2
5	2	6	2	$\frac{3}{2}$
6	3	5	2	$\frac{3}{2}$
7	3	5	2	$\frac{3}{2}$
8	3	11	4	$\frac{5}{4}$
9	3	7	2	$\frac{3}{2}$
10	4	11	4	$\frac{5}{4}$

For each of the 10 cases, the existence of an ideal $\mathfrak{b}$ satisfying the properties $(*)$ can be checked by using Macaulay 2. Then we have

$$
\begin{aligned}
&\mathcal{J}(X, c\cdot\mathfrak{b})\\
&= \mathcal{J}\Big(X, \frac{n+1}{n}\cdot\mathfrak{b}\Big)\\
&= \mathcal{J}\Big(X, \frac{n+1}{n}\cdot(f_{d_1},\ f'_{d_2})\Big)\\
&= \psi_*\mathcal{O}_{X'}\Big(K_{X'/X} - \big\lfloor cd_1E_0 + c(d_1+1)E_1 + \cdots + cd_2E_{d_2-d_1}\\
&\qquad + cE_{d_2-d_1+1} + \cdots + cnE_{d_2-d_1+n}\big\rfloor\Big)\\
&= \psi_*\mathcal{O}_{X'}\Big(2E_0 + 3E_1 + \cdots + (d_2-d_1+2)E_{d_2-d_1}\\
&\qquad + E_{d_2-d_1+1} + \cdots + nE_{d_2-d_1+n}\\
&\qquad - \big\lfloor cd_1E_0 + c(d_1+1)E_1 + \cdots + cd_2E_{d_2-d_1}\\
&\qquad + cE_{d_2-d_1+1} + \cdots + cnE_{d_2-d_1+n}\big\rfloor\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \psi_* \mathcal{O}_{X'}\Big((2-\lfloor cd_1\rfloor)E_0 + (3-\lfloor c(d_1+1)\rfloor)E_1 + \cdots + (d_2-d_1+2-\lfloor cd_2\rfloor)E_{d_2-d_1} \\
&\qquad - E_{d_2-d_1+n}\Big) \\
&= \psi_* \mathcal{O}_{X'}\Big(\big(-d_1+2-\lfloor \frac{d_2}{n}\rfloor\big)E_{d_2-d_1} - E_{d_2-d_1+n}\Big) \\
&= I_r.
\end{aligned}
$$

$\square$

CHAPTER III

Curves on a smooth surface

In this chapter we develop the main tools which will be used in the proof of the main theorem. In particular we obtain some bounds on colengths of multiplier ideals on surfaces. If we specify t-multiplicity, we have strong bounds on colengths of multiplier ideals and intersection multiplicities of two curves.

3.1 Preliminaries

We study curves on a smooth surface, their multiplier ideals and log canonical thresholds. The theorems presented in this section will be proved in the following sections. Throughout this thesis, we use the same notations for multiplier ideals and log canonical thresholds as in [19].

We now introduce some definitions for a curve on a smooth surface. Let S be a complex smooth surface, and $C \subset S$ an effective (possibly non-reduced or reducible) divisor. Fix $q \in S$.

First,

Definition 3.1.1 (Component multiplicity). The *component multiplicity* of C at q is the maximum coefficient of any component in C that passes through q, i.e., if

$C = \sum a_i C_i$ with C_i prime divisors, then

$$\text{comp-mult}_q(C) = \max\{a_i | C_i \ni q\}.$$

Our next invariant measures the maximal order of contact of C with any of its tangent lines.

Definition 3.1.2 (t-multiplicity)**.** Let

$$\eta : \mathrm{Bl}_q(S) = \tilde{S} \longrightarrow S$$

be the blow up of S at q, with exceptional divisor E, and denote by $\tilde{C}$ the proper transform of C. The *t-multiplicity* of C at q is

$$\text{t-mult}_q(C) = \max_{p\in E}(\tilde{C}\cdot E)_p,$$

where $(\tilde{C}\cdot E)_p$ denotes the intersection multiplicity of $\tilde{C}$ and E at a point $p \in E$.

Finally recall that the Arnold multiplicity of C at q is

$$\text{Arnold-mult}(C; q) = \frac{1}{\mathrm{lct}(C; q)}.$$

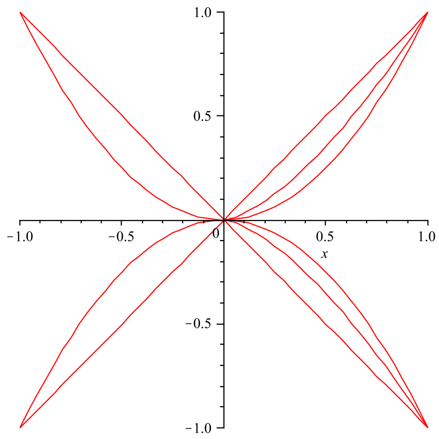
Example 3.1.3. Let $C = \{(y-x)(y+x)(y+x^2)(y-x^2)(y^2-x^3) = 0\} \subset \mathbb{C}^2$. Then $\text{t-mult}_O(C) = 4$. (See Figure 3.1.)

We develop a new reduction method from the general case to the toric case. Before stating the theorems, we need some definitions.

Definition 3.1.4. A permutation $[s_1, s_2, \cdots, s_n]$ of a sequence $[1, 2, \cdots, n]$ of length n is called *upper unimodal* or simply *unimodal* if there exists w, $1 \leq w \leq n$ such that

$$1 = s_1 < s_2 < \cdots < s_w > s_{w+1} > s_{w+2} > \cdots > s_n.$$

We always assume that $s_1 = 1$.



$$C = \{(y-x)(y+x)(y+x^2)(y-x^2)(y^2-x^3) = 0\} \subset \mathbb{C}^2$$

$$\text{t-mult}_O(C) = 4$$

Figure 3.1: An example of t-multiplicity

Definition 3.1.5. Let S be a smooth complex surface. Let $\psi : \tilde{S} \to S$ be a proper birational morphism which is a sequence of n smooth blow-ups. By abuse of notations, let F_i be (the proper transform of) the exceptional curve of the i-th blow-up. Suppose that the exceptional locus of ψ is a linear chain of n smooth rational curves. The chain (illustrated diagrammatically)

$$F_1 = F_{s_1} \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\, F_{s_2} -\!\!\!-\!\!\!- \cdots -\!\!\!-\!\!\!-\!\!\bullet\, F_{s_n}$$

will be said to be *unimodal* if $s_1 = 1$ and the sequence $s_1, s_2, \cdots, s_n$ is unimodal.

The following three theorems will be very useful in the proof of Theorem 1.

Theorem 3.1.6. *Let S, $\tilde{S}$ be smooth complex surfaces. Let $\psi : \tilde{S} \to S$ be a proper birational morphism whose exceptional locus forms a unimodal linear chain. Let F be the exceptional locus of ψ, and let $p = \psi(F)$. Then there are analytic coordinates (x, y) at p, depending on ψ, such that we can associate to any effective divisor $C \subset S$ a unique integrally closed monomial ideal J_C of the type $\prod_j \overline{(x^{a_j}, y^{b_j})}$ $(a_j \geq b_j \geq 1$ for* *any j) satisfying the following properties:*

(*i*) $C \mapsto J_C$ *is additive, i.e.* $J_{C_1+C_2} = J_{C_1} \cdot J_{C_2}$ *for any effective divisors* C_1 *and* C_2*,*

(*ii*) $$\mathrm{mult}_p C = \mathrm{mult}_p C', \text{ where } C' \text{ is a general element in } J_C,$$

(*iii*) $$\mathcal{J}(S, c \cdot C) \subset \mathcal{J}(S, c \cdot C') \text{ for any } 0 \leq c < 1,$$

(*iv*) *and if* F *contains a place of log canonical singularities of* C *then*

$$\mathrm{lct}(C; p) = \mathrm{lct}(C'; p).$$

In the 2-dimensional case, the monomial ideal J_C in the theorem may give more precise information about C than a term ideal [19, 9.3.C] or an initial monomial ideal [4, Chapter 15], [3, proof of Theorem 1.1], in the sense that J_C satisfies (*iv*) while the others do not in general. As an application, we obtain a bound of the colength of the multiplier ideal of an effective $\mathbb{Q}$-divisor on a surface.

Theorem 3.1.7. *Let* S *be a smooth surface,* C *an effective divisor on* S*, and* p *a point on* C*. Let* $m = \mathrm{mult}_p C$ *and* $l = \mathrm{lct}(C; p)$*. Then for any* $0 \leq c < 1$*, the inequality*

$$\text{colength } \mathcal{J}(S, c \cdot C)_p \geq \left\lfloor \frac{cm-1}{lm-1} \right\rfloor$$

holds.

Thanks to the Nadel vanishing theorem [19, Theorem 9.4.8], Theorem 3.1.7 can provide better bounds on cohomologies, especially dimensions of global sections, of sheaves on smooth projective surfaces. This is related to Nagata's theorem [26], which stimulated research on the existence of curves of a given degree with prescribed singularities on the projective plane (for example, see [1],[9],[10],[11],[27] and references therein). Along these lines we consider a variant of Nagata's theorem where the log canonical threshold is prescribed at given points. Before stating Theorem 3.1.9, We define a notion of *equisingularity* at the points.

Definition 3.1.8 (Almost-equisingularity)**.** Let S be a smooth surface, C an effective divisor on S, and p a point on C. If C has the same multiplicity, the same log canonical threshold and the same 1-dimensional (component) multiplicity at each of several points $p_1, \cdots, p_r$, then we say that C is *almost-equisingular* at $p_1, \cdots, p_r$.

Theorem 3.1.9. *Fix $r > 9$ very general points $p_1, \cdots, p_r$ on $\mathbf{P}^2$. Suppose that a curve C of degree d is almost-equisingular at p_i for $1 \leq i \leq r$, and that the LC-loci of C at p_i are 0-dimensional. Let $l \in \mathbb{Q}$ denote the log canonical threshold of C at p_i. Then the inequality*

$$d \geq \frac{3}{2l} \lfloor \sqrt{r} \rfloor$$

holds.

3.2 Proof of Theorem 3.1.6 : Plane curves and monomialization

In this section we prove Theorem 3.1.6.

Theorem 3.2.1. *Same assumptions as in Theorem 3.1.6. Then there are analytic coordinates (x, y) at p, depending on ψ, such that we can associate to any effective divisor $C \subset S$ a unique integrally closed monomial ideal J_C of the type $\prod_j \overline{(x^{a_j}, y^{b_j})}$ ($a_j \geq b_j \geq 1$ for any j) satisfying the following properties:*

(i) $J_{C_1+C_2} = J_{C_1} \cdot J_{C_2}$ *for any effective divisors* C_1 *and* C_2*,*

(v) ψ *is a log resolution of* J_C*,*

(vi) and

$$\mathrm{ord}_{F_i}\psi^*C = \mathrm{ord}_{F_i}\psi^*C' \text{ for any irreducible exceptional curve } F_i \subset \tilde{S},$$

where C' *is a general element in* J_C*.*

Lemma 3.2.2. *Theorem 3.2.1 implies Theorem 3.1.6.*

Proof. The property *(ii)* follows from *(vi)*:

$$\mathrm{mult}_p C = \mathrm{ord}_{F_1}\psi^*C = \mathrm{ord}_{F_1}\psi^*C' = \mathrm{mult}_p C'.$$

Let $0 \le c < 1$. Since ψ is a log resolution of J_C hence of C', the exceptional divisors of ψ are the only ones contributing to $\mathcal{J}(S, c \cdot C')$. Applying birational transformation rule [19, Proposition 9.2.33] of multiplier ideals to ψ, we get

$$\begin{aligned}\mathcal{J}(S, c \cdot C) &= \psi_*\left(\mathcal{J}(\tilde{S}, \psi^*cC) \otimes \mathcal{O}_{\tilde{S}}(K_{\tilde{S}/S})\right)\\ &\subset \psi_*\left(\mathcal{J}(\tilde{S}, \psi^*cC') \otimes \mathcal{O}_{\tilde{S}}(K_{\tilde{S}/S})\right) = \mathcal{J}(S, c \cdot C'),\end{aligned}$$

because

$$\begin{aligned}\mathcal{J}(\tilde{S}, \psi^*cC) &= \mathcal{J}\left(\tilde{S}, c\sum_i (\mathrm{ord}_{F_i}\psi^*C)F_i + c(\text{proper transform of } C)\right)\\ &\subset \mathcal{J}\left(\tilde{S}, c\sum_i (\mathrm{ord}_{F_i}\psi^*C)F_i\right)\\ &= \mathcal{J}\left(\tilde{S}, c\sum_i (\mathrm{ord}_{F_i}\psi^*C')F_i\right) \qquad \text{(by } (vi))\\ &= \mathcal{O}_{\tilde{S}}\left(-\lfloor c\sum_i (\mathrm{ord}_{F_i}\psi^*C')F_i\rfloor\right) \qquad \text{(by } (v))\\ &= \mathcal{O}_{\tilde{S}}\left(-\lfloor c\sum_i (\mathrm{ord}_{F_i}\psi^*C')F_i\rfloor - \lfloor c(\text{proper transform of } C')\rfloor\right)\\ &\qquad \text{(by } (v) \text{ and } 0 \le c < 1)\\ &= \mathcal{J}(\tilde{S}, \psi^*cC'),\end{aligned}$$

where we have used the fact that a general element in a monomial ideal of the type $\prod_j \overline{(x^{a_j}, y^{b_j})}$ ($a_j \geq b_j \geq 1$ for any j) is reduced.

It remains to show (iv), but it follows from

$$\begin{aligned} \mathrm{lct}(S, C; p) &= \min_i \frac{\mathrm{ord}_{F_i} K_{\tilde{S}/S} + 1}{\mathrm{ord}_{F_i} \psi^* C} \\ &= \min_i \frac{\mathrm{ord}_{F_i} K_{\tilde{S}/S} + 1}{\mathrm{ord}_{F_i} \psi^* C'} = \mathrm{lct}(C'; p). \end{aligned}$$

□

We actually prove the following

Theorem 3.2.3. *Same assumptions as in Theorem 3.1.6. Then there are analytic coordinates* (x, y) *at* p *such that each analytically irreducible curve* C *gives rise to* J_C *satisfying* (v) *and* (vi).

Lemma 3.2.4. *Theorem 3.2.3 implies Theorem 3.2.1.*

Proof. Any effective divisor C in a suitable analytic neighborhood of p can be expressed as

$$C = \sum_{i=1}^{n} m_i C_i,$$

where C_i are analytically irreducible components of effective divisors and m_i are positive integers. Then we define J_C by

$$J_C := \prod_{i=1}^{n} \overline{(J_{C_i})}^{m_i}.$$

We recall a fact that a general element in J_C is locally the product of each general element in J_{C_i} (cf. [5, p.197], [28, p.332], [30, p.386, Thm 3]). Then it is straightforward to check that (i), (v) and (vi) are satisfied. □

The following lemma, which deals with the singular case, plays a key role.

Lemma 3.2.5. *Same assumptions as in Theorem 3.1.6. Then there are analytic coordinates* (x, y) *at* p *such that each analytically irreducible singular curve* C *at* p *gives rise to* J_C *satisfying* (v) *and* (vi).

The rest of this section is devoted to the proof of Lemma 3.2.5. We recall the Puiseux parametrization of an analytically irreducible curve.

Proposition 3.2.6 ([2], p386)**.** *Suppose that* $V \subset U \subset \mathbb{C}^2$ *is an irreducible complex analytic subset of dimension 1 where* U *is a domain. Suppose that* $0 \in V$. *Then there exists an analytic (holomorphic) map* $f : \mathbb{D} \to V$, *where* $\mathbb{D}$ *is the unit disc, such that* $f(0) = 0$ *and* $f(\mathbb{D}) = N$ *where* $N \subset V$ *is a neighbourhood of* 0 *in* V, f *is one to one, and further* $f|_{\mathbb{D}\setminus\{0\}}$ *is a biholomorphism onto* $N \setminus \{0\}$. *In fact there exist suitable local coordinates* (x, y) *in* $\mathbb{C}^2$ *such that* f *is then given by* $\xi \to (x, y)$ *where* $x = \xi^l$, $y = \sum_{n=m}^{\infty} c_n \xi^n$ *where* $m > l$. *We call this a Puiseux parametrization of* V.

Lemma 3.2.7. *Same hypotheses and notations as in Lemma 3.2.5. Let* C *be a locally analytically irreducible singular curve* C *at* p, *and consider a Puiseux parametrization of* $C : x' = \xi^l$, $y' = \sum_{n=m}^{\infty} c_n \xi^n$ *where* $m > l$. *Then*

$$\mathrm{ord}_{F_i}\psi^* C = \mathrm{ord}_{F_i}\psi^* \tilde{C} \text{ for any irreducible exceptional curve } F_i \text{ of } \psi,$$

where $\tilde{C} := \{(x')^m - \omega(y')^l = 0,\ \omega$ *is a general element in* $\mathbb{C}\}$.

Proof. We recall that if C is singular, then its first Puiseux pair (m, l) is uniquely determined [2, p.406]. Let $\tilde{\psi}$ be a minimal log resolution of $\tilde{C}$. Then the exceptional locus of $\tilde{\psi}$ forms another unimodal linear chain $\tilde{F}$. Kuwata [18, pp.710-711, 715-716] showed that, loosely speaking, C and $\tilde{C}$ behave the same along $\tilde{F}$ hence any part of $\tilde{F}$. In particular they behave the same along $F \cap \tilde{F}$ where we abuse notations. Precisely speaking, let

$$\psi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_t, \quad \tilde{\psi} = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_u,$$

where φ_i and ϕ_i are single blow-ups. Let v, $1 \leq v \leq t$ be the largest number such that

$$\varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_v = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_v =: \mu.$$

Then

$$\mathrm{ord}_{F_i} \mu^* C = \mathrm{ord}_{F_i} \mu^* \tilde{C} \text{ for any irreducible exceptional curve } F_i \text{ of } \mu,$$

which implies

$$\mathrm{ord}_{F_i} \psi^* C = \mathrm{ord}_{F_i} \psi^* \tilde{C} \text{ for any irreducible exceptional curve } F_i \text{ of } \psi.$$

□

The following is a toric geometric fact which is known by experts. Since the author cannot find a reference, he includes a proof. He would like to thank Mattias Jonsson and Howard Thompson for valuable discussions.

Lemma 3.2.8. *Same hypotheses and notations as in Lemma 3.2.5. There are analytic coordinates (x, y) at p and a pair $(a, b), a \geq b$ of positive integers such that ψ is a minimal log resolution of (x^a, y^b).*

Proof. The pair $(a, b), a \geq b$ of two relatively prime integers is obtained by Lemma 3.5.3. Suppose that the exceptional locus F of ψ is F_{s_1}-F_{s_2}-$\cdots$-F_{s_t}. Let q_0 be the length of the first consecutively increasing part of $[s_1, ..., s_t]$ so that $s_1 = 1$, $s_2 = 2$, $\cdots$, $s_{q_0} = q_0$, and $s_t = q_0 + 1$. Take general coordinates $(\tilde{x}, \tilde{y})$ centered at p. Then there is a sequence $[\varepsilon_1, \cdots, \varepsilon_{q_0}]$ of q_0 complex numbers such that after change of coordinates

$$x = \tilde{x}, \quad y = \tilde{y} + \varepsilon_1 \tilde{x} + \cdots + \varepsilon_{q_0} \tilde{x}^{q_0},$$

ψ is a log resolution of (x^{q_0+1}, y). In these coordinates (x, y), ψ is a minimal log resolution of (x^a, y^b). In fact, after $(q_0 + 1)$-th blow-up, each blow-up takes place only at an intersection of two irreducible exceptional curves, which can be considered as an origin of an affine chart at each step of the well-known resolution process of singularities of a monomial ideal (x^a, y^b). $\square$

Definition 3.2.9. Let $\psi : \tilde{S} \to S$ be a proper birational morphism whose exceptional locus forms a unimodal linear chain of t smooth rational curves. For any curve D, there is a uniquely determined sequence $(m_i(D))_{i=1}^t$ of multiplicities (of the proper transform of D) at the center of the i-th blow-up. We call it *the multiplicity sequence* of D with respect to ψ. (cf. [2, p.503])

Example 3.2.10. Let $\psi : \tilde{S} \to \mathbb{C}^2$ be the minimal log resolution of $\{x^3 - y^2 = 0\}$. Let $D : x = \xi^{10}, y = \xi^{40} + \xi^{45} + \xi^{47}$. Then the multiplicity sequence of D is $(10, 10, 0)$ with respect to ψ. Note that $\psi^* D = 10F_1 + 20F_2 + 30F_3 +$ (proper transform of D), and that ψ is not a resolution of D.

Lemma 3.2.11. *Let $\psi : \tilde{S} \to S$ be a proper birational morphism whose exceptional locus forms a unimodal linear chain. Let (x', y') be any analytic coordinates on S. Let D be a general member in a monomial ideal of the type $({x'}^{\ell}, {y'}^{\wp}) \subset \mathbb{C}[x', y']$. Then the multiplicity sequence of D with respect to ψ is a partial Euclidean sequence with respect to $(\ell, \wp)$ (see Definition 3.5.2).*

Proof. This follows from the well-known resolution process of singularities of a monomial ideal $({x'}^{\ell}, {y'}^{\wp})$. Note that ψ is a sequence of smooth blow-ups. As long as the the centers of the blow-ups lie in the proper transforms of D, the multiplicity sequence of D with respect to ψ is determined by the Euclidean algorithm. But as soon as the center of a blow-up is away from the proper transform of D, the Euclidean algorithm stops contributing to the multiplicity sequence. In fact since ψ is a linear chain, if a blow-up takes place away from the proper transform of D, then all the following blow-ups never hit the proper transform of D. □

Proof of Lemma 3.2.5. By Lemma 3.2.7, we may assume that C is defined by $\{(x')^\ell - \omega(y')^\wp = 0,\ \omega$ is a general element in $\mathbb{C}\}$. Thanks to Lemma 3.2.11, the multiplicity sequence of C with respect to ψ is a partial Euclidean sequence with respect to $(\ell, \wp)$. It is a sum of finitely many, say n, full Euclidean sequences with respect to $(\ell_i, \wp_i)_{i=1}^n$ by Lemma 3.5.4. Lemma 3.2.8 gave us coordinates (x, y) so that the multiplicity sequence of a general member in $(x^{\ell_i}, y^{\wp_i})$ agrees with the full Euclidean sequences with respect to $(\ell_i, \wp_i)$, hence that ψ is a log resolution of $\overline{(x^{\ell_i}, y^{\wp_i})}$. So we can take $J_C = \prod_{i=1}^n \overline{(x^{\ell_i}, y^{\wp_i})}$. The uniqueness follows from the construction.

□

Example 3.2.12. Let $\psi : \tilde{S} \to \mathbb{C}^2$ be the minimal embedded resolution of $\{x^4 - y^3 = 0\}$. Let $(3, 1, 1, 0)$ be a multiplicity sequence of some curve with respect to ψ. Then $(3, 1, 1, 0) = (2, 1, 1, 0) + (1, 0, 0, 0)$ is the multiplicity sequence of a general member in $\overline{(x^3, y^2)(x, y)}$.

To complete the proof of Theorem 3.2.3, it remains to treat the smooth case.

Lemma 3.2.13. *Same assumptions as in Theorem 3.1.6. Then there are analytic coordinates (x, y) at p such that each analytically irreducible smooth curve C at p gives rise to J_C satisfying (v) and (vi). In particular $J_C = \overline{(x^j, y)}$ for some j.*

Proof. Suppose that the exceptional locus F of ψ is F_{s_1}-F_{s_2}-$\cdots$-F_{s_t}. Let q_0 be the length of the fist consecutively increasing part of $[s_1, ..., s_t]$ so that $s_1 = 1$, $s_2 = 2$, $\cdots$, $s_{q_0} = q_0$, and $s_t = q_0 + 1$. Since C is smooth at p, after the i-th blow-up the proper transform of C meets F_i transversally or does not meet F_i at all. So the proper transform of C under ψ meets one, say F_j, of $F_1, F_2, ..., F_{q_0}, F_{q_0+1}$ transversally. Then a monomial ideal $\overline{(x^j, y)}$ is the desired one. □

3.3 Proof of Theorem 3.1.7 : Morphism of Log Canonical Singularities

In this section we prove Theorem 3.1.7.

Let X be a smooth complex surface, p a point on X, and $C \subset X$ a (possibly reducible and non-reduced) curve passing through p. Let S be a local analytical neighborhood of X at p. Suppose that $\mathcal{J}(S, c'C) := \mathcal{J}(X, c'C)|_S$ is co-supported at p for some $c' \in \mathbb{Q}$. Let r be the maximum coefficient of any component of C passing through p. Consider the minimal log resolution $\mu : S' \to S$ of (S, C), which consists of smooth blow-ups. Such a resolution exists and is uniquely obtained when we blow up as few times as possible ([2, p.498]). Let

$$K_{S'/S} = \sum_i a_i F_i \qquad \text{and} \qquad \mu^* C = \sum_i b_i F_i + (\text{proper transform of } C),$$

where $i's$ denote the order of smooth blow-ups and F_i are the corresponding irreducible exceptional divisors. Then we have

$$\mathrm{lct}(S, C) = \min\Big\{\min_i \frac{a_i + 1}{b_i}, \frac{1}{r}\Big\}.$$

If $\mathrm{lct}(S, C) = \frac{1}{r}$ then the LC-locus is 1-dimensional, while if $\mathrm{lct}(S, C) = \min_i \frac{a_i+1}{b_i} < \frac{1}{r}$ then the LC-locus is 0-dimensional.

Definition 3.3.1. We assume that the LC-locus is 0-dimensional. If $\frac{a_j+1}{b_j} = \mathrm{lct}(S, C)$ then F_j is called a *place of log canonical singularities.* It may be possible that there are many places of log canonical singularities.

Definition 3.3.2. For any irreducible exceptional curve F_t, there is a unique proper birational morphism $\psi_t : S_t \to S$ such that μ factors through ψ_t and that the push-forward of F_t is the only one irreducible exceptional (-1)-curve on S_t. In fact, ψ is obtained by keeping contracting (-1)-curves until there is no other (-1)-curve than (the push-forward of) F_t. By abuse of notation, we denote the push-forward of F_t by F_t.

If F_t is a place of log canonical singularities, then we call ψ_t *the morphism of log canonical singularities* with respect to F_t. By reordering blow-ups, we may assume that the exceptional locus of ψ_t is $\{F_1, F_2, \cdots, F_t\}$.

Proposition 3.3.3 ([18])**.** *The exceptional locus of the morphism ψ_t of log canonical singularities is a unimodal linear chain.*

Proof. Since μ is a minimal log resolution, we can apply Kuwata's result [18, pp 715–716]. □

Remark 3.3.4. Mattias Jonsson pointed out that Proposition 3.3.3 follows from [6, Lemma 2.11] as well.

Proof of Theorem 3.1.7. Let ψ be a morphism of log canonical singularities. Then by Proposition 3.3.3, the exceptional locus of ψ forms a unimodal linear chain. So we can apply Theorem 3.1.6 and get

$$\mathcal{J}(S, c \cdot C) \subset \mathcal{J}(S, c \cdot C'),$$

where C' is a general element in J_C. Hence

$$\text{colength } \mathcal{J}(S, c \cdot C) \geq \text{colength } \mathcal{J}(S, c \cdot C').$$

Since $\text{mult}_p C = \text{mult}_p C'$ and $\text{lct}(C; p) = \text{lct}(C'; p)$, it is enough to prove for C' hence for monomial ideals. But this is an easy application of Howald's theorem [15]. In fact $\mathcal{J}(\mathbb{C}^2, (x^m, y^{(lm-1)m})^{\frac{c}{lm-1}})$ has the smallest colength in the collection of multiplier ideals

$$\{\mathcal{J}(cw \cdot I) \mid w \in \mathbb{R}_{>0}, \quad I : \text{monomial ideal},$$
$$m = w \cdot \text{ord}_p I, \quad l = \inf_{r \in \mathbb{R}} \langle \mathcal{J}(rw \cdot I) \subseteq \mathfrak{m}_p \rangle\}.$$

We note that the edge of the Newton polytope corresponding to $(x^m,\ y^{(lm-1)m})^{\frac{1}{lm-1}}$ passes through $(0,m)$ and $(\frac{1}{l},\frac{1}{l})$.

Let $N(c,m,l)$ denote the number of non-negative lattice points in the quadrilateral region with vertices $(-1,-1),(-1,\frac{c}{l}-1),(\frac{c}{l}-1,\frac{c}{l}-1)$ and $(\frac{cm}{lm-1}-1,-1)\in\mathbb{R}^2$ (including boundaries). Then for any $0\le c<1$, the inequality

$$\text{colength}\ \mathcal{J}(S,c\cdot C)_p \ge N(c,m,l)$$

holds. It is elementary to check that

$$N(c,m,l)\ge\left\lfloor\frac{cm-1}{lm-1}\right\rfloor.$$

□

3.4 Proof of Theorem 3.1.9: Vanishing theorem

In this section we prove Theorem 3.1.9.

As a matter of notation, let $(C_1\cdot C_2)_p$ denote the intersection multiplicity of C_1 and C_2 at p.

Proposition 3.4.1. *Let S be a smooth surface and p a point on S. Let $m_1\in\mathbb{N}$. Let C_1 be a smooth effective divisor and C_2 be another effective divisor whose support does not contain C_1. Assume $p\in C_1\cap C_2$. Then the following are equivalent.*
(i) The multiplier ideal $\mathcal{J}(S,c(m_1C_1+C_2))_p$ is co-supported at $\{p\}$ for some $c\in\mathbb{Q}_{>0}$.
(ii) $(C_1\cdot C_2)_p>m_1$.

Proof. (i) $\Rightarrow$ (ii).
Since $\mathcal{J}(S,c(m_1C_1+C_2))_p$ is co-supported at $\{p\}$ but not along C_1, we have $cm_1<1$. Since $\mathcal{J}(S,cC_2+cm_1C_1)$ is non-trivial and C_1 is smooth, Inversion of adjuction [19, Corollary 9.5.11] implies that $\mathcal{J}(C_1,c(C_2)_{C_1})$ is non-trivial, which means

$$c(C_1\cdot C_2)_p\ge 1.$$

The desired inequality follows from $cm_1 < 1$.

(ii) $\Rightarrow$ (i).

Since the intersection multiplicity is an integer, we have $(C_1 \cdot C_2)_p \geq m_1 + 1$. Take a minimal resolution $\mu : \tilde{S} \to S$ of singularities of $(S, m_1C_1 + C_2)$ at p. Let F_j be the exceptional divisor of μ, with which the proper transform of C_1 meets. As in Definition 3.3.2, there is a unique proper birational morphism $\psi_j : S_j \to S$ such that μ factors through ψ_j and that the push-forward of F_j is the only one irreducible exceptional (-1)-curve on S_j. Since C_1 is smooth, the exceptional locus of ψ_j forms a linear chain. Suppose that the chain consists of j irreducible exceptional divisors. Then $(C_1 \cdot C_2)_p = \mathrm{ord}_{F_j}\psi_j^*C_2$. Since $\mathrm{ord}_{F_j}\psi_j^*C_1 = j$ and $(C_1 \cdot C_2)_p \geq m_1 + 1$, we have

$$\mathrm{lct}(S, m_1C_1 + C_2) \leq \frac{1 + \mathrm{ord}_{F_j}K_{S_j/S}}{\mathrm{ord}_{F_j}\psi_j^*(m_1C_1 + C_2)} \leq \frac{j+1}{jm_1 + m_1 + 1} < \frac{1}{m_1}.$$

So if $c = \mathrm{lct}(S, m_1C_1 + C_2)$ then $\mathcal{J}(S, c(m_1C_1 + C_2))_p$ is co-supported at $\{p\}$. $\square$

Proposition 3.4.2. *Fix r distinct points $p_1, \cdots, p_r$ on a smooth projective surface S. Assume that an effective divisor C is almost-equisingular at p_i for $1 \leq i \leq r$, and that the LC-loci of (S, C) at p_i are 0-dimensional. Let $m = mult_{p_i}C$. Fix a real number m' with $m/2 < m' < m$. Suppose that*

$$(C - uC') \cdot C' < (u+1)r$$

for any u, $m' \leq u \leq m-1$ and for any curve $C' \leq C$ passing through $p_1, \cdots, p_r$.

Then there is no non-reduced curve $D \leq C$, which has multiplicity $\geq m'$ and passes through $p_1, \cdots, p_r$. In particular, the local multiplier ideal $\mathcal{J}(S, (1/m')C)_{p_i}$ is trivial or co-supported at p_i.

Proof. Suppose that there is a non-reduced curve $D \leq C$, which has multiplicity $u \geq m'$ and passes through $p_1, \cdots, p_r$. If D_{red} is singular at p_i, i.e. $\mathrm{mult}_{p_i}D_{\mathrm{red}} \geq 2$, then

$$m = \mathrm{mult}_{p_i} C \geq \mathrm{mult}_{p_i} D \geq \mathrm{mult}_{p_i} u D_{\mathrm{red}} \geq 2u \geq 2m' > m,$$

which is absurd. So D_{red} is smooth at p_i. By Proposition 3.4.1 the intersection multiplicity of D_{red} and $C - uD_{\mathrm{red}}$ at each p_i is at least $(u+1)$.

Hence the global intersection number

$$(C - uD_{\mathrm{red}}) \cdot D_{\mathrm{red}} \geq (u+1)r,$$

but this contradicts our assumption. $\square$

Corollary 3.4.3. *Same assumptions as in Proposition 3.4.2. Write $C = C_{<m'} + C_{\geq m'}$, where $C_{<m'}$ (resp. $C_{\geq m'}$) is the union of irreducible components of C with multiplicity $< m'$ (resp. $\geq m'$). Let $L \subset S$ a divisor such that $L - (1/m')C_{<m'}$ is nef and big. Then*

$$\chi(K_S + L) \geq \sum_{i=1}^{r} \dim_{\mathbb{C}} \Big(\mathcal{O}_{S,p_i} / \mathcal{J}\big(S, \frac{1}{m'} C\big)_{p_i} \Big).$$

Proof. Due to Proposition 3.4.2, any component of $C_{\geq m'}$ does not pass through any p_i. So

$$\mathcal{J}\Big(S, \frac{1}{m'} C\Big)_{p_i} = \mathcal{J}\Big(S, \frac{1}{m'} C_{<m'}\Big)_{p_i}.$$

Since the global multiplier ideal $\mathcal{J}(\frac{1}{m'} C_{<m'})$ defines at most zero-dimensional scheme, we get

$$\begin{aligned} \sum_{i=1}^{r} \dim_{\mathbb{C}} \Big(\mathcal{O}_{S,p_i} / \mathcal{J}\big(\frac{1}{m'} C_{<m'}\big)_{p_i} \Big) &\leq \dim_{\mathbb{C}} \Big(\mathcal{O}_S / \mathcal{J}\big(\frac{1}{m'} C_{<m'}\big) \Big) \\ &= \chi \Big(\mathcal{O}_S / \mathcal{J}\big(\frac{1}{m'} C_{<m'}\big) \Big). \end{aligned}$$

Consider the exact sequence

$$0 \longrightarrow \mathcal{J}\Big(\frac{1}{m'} C_{<m'}\Big) \longrightarrow \mathcal{O}_S \longrightarrow \frac{\mathcal{O}_S}{\mathcal{J}(\frac{1}{m'} C_{<m'})} \longrightarrow 0.$$

Twisting by $\mathcal{O}_S(K_S + L)$ and taking the Euler characteristic, we have

$$\chi(K_S + L) - \chi\Big(\frac{\mathcal{O}_S}{\mathcal{J}(\frac{1}{m'} C_{<m'})} \Big) = \chi\Big(\mathcal{O}_S(K_S + L) \otimes \mathcal{J}((1/m') C_{<m'}) \Big).$$

Applying Nadel vanishing theorem to

$$\mathcal{O}_S(K_S+L)\otimes\mathcal{J}\big((1/m')C_{<m'}\big),$$

we get

$$\begin{aligned}&\chi(K_S+L)-\sum_{i=1}^{r}\dim_{\mathbb{C}}\big(\mathcal{O}_{S,p_i}/\mathcal{J}((1/m')C_{<m'})_{p_i}\big)\\&\geq\chi(K_S+L)-\dim_{\mathbb{C}}\big(\mathcal{O}_S/\mathcal{J}((1/m')C_{<m'})\big)\\&=\chi\Big(\mathcal{O}_S(K_S+L)\otimes\mathcal{J}((1/m')C_{<m'})\Big)\\&=h^0\Big(\mathcal{O}_S(K_S+L)\otimes\mathcal{J}((1/m')C_{<m'})\Big)\geq 0.\end{aligned}$$

□

Proof of Theorem 3.1.9. Let m be the multiplicity of C at p_i. Suppose $l\geq\frac{3}{2m}$. Then we get

$$d\geq m\lfloor\sqrt{r}\rfloor\geq\frac{3}{2l}\lfloor\sqrt{r}\rfloor,$$

where the first inequality follows from [26, p.767] or [10, pp.692-694].

Suppose $\frac{3}{2m}>l$. Thanks to Theorem 3.1.7, we have

$$\text{colength }\mathcal{J}(\mathbb{P}^2,\frac{4l}{3}\cdot C)_{p_i}\geq N(\frac{4l}{3},m,l),$$

where $N(\frac{4l}{3},m,l)$ denotes the number of non-negative lattice points in the quadrilateral region with vertices $(-1,-1),(-1,\frac{4l/3}{l}-1),(\frac{4l/3}{l}-1,\frac{4l/3}{l}-1)$ and $(\frac{4lm/3}{lm-1}-1,-1)\in\mathbb{R}^2$ (including boundaries). It is easy to check that $N(\frac{4l}{3},m,l)\geq 2$, which yields that

$$\text{colength }\mathcal{J}(\mathbb{P}^2,\frac{4l}{3}\cdot C)_{p_i}\geq 2. \qquad (*)$$

Suppose to the contrary that $d<\frac{3}{2l}\lfloor\sqrt{r}\rfloor$. Let $C_{<\frac{3}{4l}}$ be the union of components of C with multiplicity $<\frac{3}{4l}$. Set $L=2\lfloor\sqrt{r}\rfloor H$, with H being the hyperplane divisor on $\mathbb{P}^2$, so that $L-\frac{4l}{3}C$ is big and nef hence so is $L-\frac{4l}{3}C_{<\frac{3}{4l}}$.

We want to apply Proposition 3.4.2 and Corollary 3.4.3. Let $m' = \frac{3}{4l}$ so $m/2 < m' < m$. Then for any u, $m' \le u \le m-1$ and for any curve C' of degree d' passing through $p_1, \cdots, p_r$, we get

$$\begin{aligned}(C - uC') \cdot C' &= (d - ud')d' \\ &< \left(\frac{3}{2l}\lfloor\sqrt{r}\rfloor - \frac{3}{4l}d'\right)d' \\ &= \frac{3}{4l}\Big(2\lfloor\sqrt{r}\rfloor - d'\Big)d' \\ &< (u+1)(2\lfloor\sqrt{r}\rfloor - d')d' \\ &\le (u+1)r,\end{aligned}$$

where we have assumed $d - ud' > 0$ (otherwise $(d-ud')d' \le 0 < (u+1)r$ is obvious), and the last inequality follows from $(d' - \sqrt{r})^2 \ge 0$. So the assumptions of Proposition 3.4.2 and Corollary 3.4.3 are satisfied.

Due to $(*)$, we have

$$\begin{aligned}\chi(K_{\mathbb{P}^2} + L) &= \binom{2\lfloor\sqrt{r}\rfloor - 1}{2} \\ &< 2r \\ &\le \sum_{i=1}^{r} \dim_{\mathbb{C}}(\mathcal{O}_{p_i}/\mathcal{J}\Big(\frac{4l}{3}\cdot C)_{p_i}\Big),\end{aligned}$$

which contradicts Corollary 3.4.3. □

3.5 Some Combinatorial Lemmas

This section includes, for the convenience of the reader, two combinatorial results which are elementary in nature but necessary for the completeness of the proofs.

Definition 3.5.1. A permutation $[s_1, s_2, \cdots, s_t]$ of a sequence $[1, 2, \cdots, t]$ of length t is called *upper unimodal* or simply *unimodal* if there exists w, $1 \le w \le t$ such that

$$1 = s_1 < s_2 < \cdots < s_w > s_{w+1} > s_{w+2} > \cdots > s_t.$$

We always assume that $s_1 = 1$.

Definition 3.5.2. Let $(\ell, \wp)$ be any pair of positive integers with $\ell \geq \wp$. Then the Euclidean algorithm allows us to define a sequence $r_0, q_0, \cdots, r_k, q_k$ in the following way:

$$\begin{aligned}
&r_0 = \wp \\
&\ell = q_0 r_0 + r_1 \quad (0 < r_1 < r_0) \\
&r_0 = q_1 r_1 + r_2 \quad (0 < r_2 < r_1) \\
&\quad \vdots \\
&r_{k-2} = q_{k-1} r_{k-1} + r_k \quad (0 < r_k < r_{k-1}) \\
&r_{k-1} = q_k r_k.
\end{aligned} \tag{$\ddagger$}$$

Then we define a unique sequence of integers by

$$(\underbrace{r_0, ..., r_0}_{q_0}, \underbrace{r_1, ..., r_1}_{q_1}, ..., \underbrace{r_k, ..., r_k}_{q_k}), \tag{$\diamondsuit$}$$

and we call it *the full Euclidean sequence* with respect to $(\ell, \wp)$.

For any n, $1 \leq n \leq \sum_{i=0}^{k} q_i$, a sequence consisting of the first n terms in $(\diamondsuit)$ is called *a partial Euclidean sequence.*

Lemma 3.5.3. *There is a canonical bijection between the set of unimodal permutation sequences and the set of full Euclidean sequences with respect to pairs of relatively prime integers.*

Proof. Any unimodal permutation sequence $[s_1, s_2, \cdots, s_t]$ uniquely defines a sequence $\{q_0, q_1, \cdots, q_k\}$ of positive integers such that

$$s_1 = 1,\ s_2 = 2, \cdots,\ s_{q_0} = q_0,$$

$$s_{q_0+1} = q_0 + q_1 + 1,\ s_{q_0+2} = q_0 + q_1 + 2,\ \cdots,\ s_{q_0+q_2} = q_0 + q_1 + q_2,$$

$$s_{q_0+q_2+1} = 1 + \sum_{i=0}^{3} q_i,\ s_{q_0+q_2+2} = 2 + \sum_{i=0}^{3} q_i,\ \cdots,\ s_{q_0+q_2+q_4} = q_4 + \sum_{i=0}^{3} q_i,$$

and so on.

Note that q_0 is the length of the first consecutively increasing part of $[s_1, s_2, \cdots, s_t]$, that q_1 is the length of the last consecutively decreasing part, that q_2 is the length of the second consecutively increasing part, and so forth.

If we let $r_k = 1$, then $r_{k-1}, \cdots, r_1, r_0$ are (reverse-)inductively defined as in (‡). Hence a full Euclidean sequence can be determined. The inverse is straightforward.

$\square$

Lemma 3.5.4. *Any partial Euclidean sequence can be expressed as a sum of finitely many full Euclidean sequences.*

Proof. Any partial Euclidean sequence with respect to $(\ell, \wp)$ is of the form

$$(\underbrace{r_0, ..., r_0}_{q_0}, \underbrace{r_1, ..., r_1}_{q_1}, ..., \underbrace{r_k, ..., r_k}_{q_k}, 0, ..., 0), \cdots\cdots (*)$$

where

$$\begin{aligned}
&r_0 = \wp \\
&\ell = q_0 r_0 + r_1 \quad (0 < r_1 < r_0) \\
&r_0 = q_1 r_1 + r_2 \quad (0 < r_2 < r_1) \\
&\quad \vdots \\
&r_{k-2} = q_{k-1} r_{k-1} + r_k \quad (0 < r_k < r_{k-1}) \\
&r_{k-1} \geq q_k r_k.
\end{aligned} \qquad (\dagger)$$

Note that the last relation in (†) is an inequality.

We define a sequence

$$(\underbrace{s_0, ..., s_0}_{q_0}, \underbrace{s_1, ..., s_1}_{q_1}, ..., \underbrace{s_k, ..., s_k}_{q_k}, 0, ..., 0). \cdots\cdots (**)$$

by

$$s_k = r_k$$

$$s_{k-1} = q_k r_k$$

$$s_{k-2} = q_{k-1} s_{k-1} + s_k$$

$$\vdots$$

$$s_0 = q_1 s_1 + s_2,$$

so that this sequence is a full Euclidean sequence with respect to $(q_0 s_0 + s_1,\ s_0)$.

Subtracting $(**)$ from $(*)$, we get

$$(\underbrace{r_0 - s_0, ..., r_0 - s_0}_{q_0}, ..., \underbrace{r_{k-1} - s_{k-1}, ...r_{k-1} - s_{k-1}}_{q_{k-1}}, 0, ..., 0)$$

This sequence satisfies (†) with k being replaced by a smaller number, because

$$r_i - s_i = (q_{i+1} r_{i+1} + r_{i+2}) - (q_{i+1} s_{i+1} + s_{i+2})$$

$$= q_{i+1}(r_{i+1} - s_{i+1}) + (r_{i+2} - s_{i+2}),$$

for $0 \leq i \leq k-2$. Repeating this, we get the desired set of full Euclidean sequences.

□

3.6 Ruled surfaces

In this section we give a variant of Corollary 3.4.3 on ruled surfaces.

Lemma 3.6.1. *Let T be a smooth curve and let $E = \mathbb{P}(\mathcal{O}_T \oplus L)$, where L is a line bundle on T with deg $L = -w$. Let $\varpi : E \to T$ be the natural surjective morphism.*

Fix r distinct points $q_1, \cdots, q_r$ on E with $\varpi(q_i) \neq \varpi(q_j)$ for any $i \neq j$. Let C be an effective divisor in the numerical equivalent class of $a\mathfrak{s} + b\mathfrak{f}$, where $\mathfrak{s}$ denotes the negative section and $\mathfrak{f}$ denotes the fiber class. Assume that any component of C does not contain $\mathfrak{s}$.

Let u be a positive integer. Suppose that $u > a/2$, $aw > b - ur$, and

$$C = u \cdot C' + C'',$$

where C' passes through $q_1, \cdots, q_r$. Then

$$C' \cong_{\text{num}} \mathfrak{s} + x\mathfrak{f}$$

for some $x \geq w$.

Proof. Suppose that C' is the union of some fibers on the ruled surface E, i.e., $C' \cong_{\text{num}} x \cdot \mathfrak{f}$ for some x. Since C' passes through r points all of which lie in different fibers, $x \geq r$. By assumption, we have

$$C'' = C - uC' \in |a\mathfrak{s} + (b - ux)\mathfrak{f}|.$$

Since any effective divisor on a ruled surface is nef if and only if it does not contain a negative section, any component of C is nef. Then C'', and hence $a\mathfrak{s} + (b - ux)\mathfrak{f}$ should be nef, which implies $aw \leq b - ux$. But this contradicts $aw > b - ur \geq b - ux$.

It follows that

$$C' \cong_{\text{num}} y\mathfrak{s} + x \cdot \mathfrak{f}$$

for some $y \in \mathbb{N}_{>0}$. This yields

$$C'' = C - uC' \in |(a - uy)\mathfrak{s} + (b - ux)\mathfrak{f}|.$$

If $y \geq 2$ then $a - uy \leq a - 2u < 0$ by assumption. Then C'' cannot be effective. $\square$

Lemma 3.6.2. *Same assumptions as above. Fix a positive rational number $\gamma > 0$. Suppose that $\mathcal{J}(E, \gamma C)_{q_\alpha}$ is trivial or co-supported at q_α for each point q_α. Then*

$$\chi\Big(K_E + \lceil \gamma a \rceil \mathfrak{s} + (w + \lceil \gamma b \rceil)\mathfrak{f}\Big) > \sum_{\alpha=1}^{r} \dim_{\mathbb{C}}\Big(\mathcal{O}_{E,q_\alpha}/\mathcal{J}(E, \gamma C)_{q_\alpha}\Big).$$

Proof. The proof is essentially the same as that of Lemma 3.4.3. □

3.7 t-multiplicity

In this section we show that Theorem 3.1.6 can be generalized to obtain some inequalities on intersection multiplicities and t-multiplicities.

We recall definition of t-multiplicity of a curve at a point on a smooth surface.

Definition 3.7.1 (t-multiplicity)**.** Let

$$\eta : \mathrm{Bl}_q(S) = \tilde{S} \longrightarrow S$$

be the blow up of S at q, with exceptional divisor E, and denote by $\tilde{(C)}$ the proper transform of C. The *t-multiplicity* of C at q is

$$\text{t-mult}_q(C) = \max_{p \in E}(\tilde{C} \cdot E)_p,$$

where $(\tilde{C} \cdot E)_p$ denotes the intersection multiplicity of $\tilde{C}$ and E at a point $p \in E$.

We recall Theorem 3.1.6.

Theorem 3.7.2 (3.1.6)**.** *Let S, $\tilde{S}$ be smooth complex surfaces. Let $\psi : \tilde{S} \to S$ be a proper birational morphism whose exceptional locus forms a unimodal linear chain. Let F be the exceptional locus of ψ, and let $p = \psi(F)$. Then there are analytic coordinates (x, y) at p, depending on ψ, such that we can associate to any effective divisor $C \subset S$ a unique integrally closed monomial ideal J_C of the type $\prod_j \overline{(x^{a_j}, y^{b_j})}$ ($a_j \geq b_j \geq 1$ for any j) satisfying the following properties:*

(i) $C \mapsto J_C$ *is additive, i.e.* $J_{C_1+C_2} = J_{C_1} \cdot J_{C_2}$ *for any effective divisors* C_1 *and* C_2*,*

(ii) $$\mathrm{mult}_p C = \mathrm{mult}_p C',$$

(iii) $\mathcal{J}(S, c \cdot C) \subset \mathcal{J}(S, c \cdot C')$ *for any* $0 \le c < 1$*,*

(iv) and if F *contains a place of log canonical singularities of* C *then*

$$\mathrm{lct}(C;p) = \mathrm{lct}(C';p),$$

where C' *is a general element in* J_C*.*

Theorem 3.7.2 followed from Theorem 3.2.1. We recall Theorem 3.2.1.

Theorem 3.7.3 (3.2.1)**.** *Same assumptions as in Theorem 3.7.2. Then there are analytic coordinates* (x, y) *at p, depending on* ψ*, such that we can associate to any effective divisor* $C \subset S$ *a unique integrally closed monomial ideal* J_C *of the type* $\prod_j \overline{(x^{a_j}, y^{b_j})}$ *(*$a_j \ge b_j \ge 1$ *for any* j*) satisfying the following properties:*

(i) $J_{C_1+C_2} = J_{C_1} \cdot J_{C_2}$ *for any effective divisors* C_1 *and* C_2*,*

(v) ψ *is a log resolution of* J_C*,*

(vi) and

$$\mathrm{ord}_{F_i}\psi^* C = \mathrm{ord}_{F_i}\psi^* C' \text{ for any irreducible exceptional curve } F_i \subset \tilde{S},$$

where C' *is a general element in* J_C*.*

Lemma 3.7.4. *The map* $C \mapsto J_C$ *in the Theorem 3.7.2 satisfies more properties :*

(vii) if C_1 *is smooth then* $(C_1 \cdot C_2)_p \ge (C_1' \cdot C_2')_p$*,*

(viii) $\text{t-mult}_p C \ge \text{t-mult}_p C'$.

Proof. We stick to the notations in Theorem 3.2.1. If C_1 is smooth then $\mathrm{mult}_p C_1 = 1$, hence the property (ii) implies that $J_{C_1} = (x^j, y)$ for some j. Then we have

$$(C_1 \cdot C_2)_p \geq \mathrm{ord}_{F_j} \psi^* C_2,$$

and

$$(C_1' \cdot C_2')_p = \mathrm{ord}_{F_j} \psi^* C_2'.$$

(cf. [9, Proof of Lemma 2.16]) Due to Theorem 3.2.1, we get

$$\mathrm{ord}_{F_j} \psi^* C_2 = \mathrm{ord}_{F_j} \psi^* C_2',$$

which implies (vii).

It remains to prove $(viii)$. Let $\tilde{C}$ (resp. $\tilde{C'}$) be the proper transform of C (resp. C') under the single blow-up at p. Suppose that the exceptional locus of ψ is a linear chain $F_{s_1} - F_{s_2} - \cdots$ $(s_1 = 1)$. Let $\tilde{p}$ denote the intersection point of F_1 and F_{s_2}. Then we have

$$\begin{aligned} \text{t-mult}_p C &\geq (F_1 \cdot \tilde{C})_{\tilde{p}} \\ &= \mathrm{ord}_{F_{s_2}} \psi^* C - (s_2 - 1)\mathrm{ord}_{F_1} \psi^* C \\ &= \mathrm{ord}_{F_{s_2}} \psi^* C' - (s_2 - 1)\mathrm{ord}_{F_1} \psi^* C' \\ &= (F_1 \cdot \tilde{C'})_{\tilde{p}} = \text{t-mult}_p C', \end{aligned}$$

where we have used Theorem 3.2.1. □

3.8 t-multiplicity and colength

In this section we give a stronger bound on the colengths of multiplier ideals on surfaces.

We recall Theorem 3.1.7.

Theorem 3.8.1 (3.1.7)**.** *Let S be a smooth surface, C an effective divisor on S, and p a point on C. Let $m = \mathrm{mult}_p C$ and $l = \mathrm{lct}(C;p)$. Then for any $0 \le c < 1$, the inequality*

$$\text{colength } \mathcal{J}(S, c\cdot C)_p \ge \left\lfloor \frac{cm-1}{lm-1} \right\rfloor$$

holds.

If t-multiplicity is specified, then we have a stronger inequality.

Proposition 3.8.2. *Let S be a smooth surface, C a curve on S and p a point on C. Let $m = \mathrm{mult}_p C$, $t = \text{t-mult}_p C$, and $l = \mathrm{lct}(C;p)$. Suppose that $\mathcal{J}(S, c\cdot C)_p$ is nontrivial for some $0 < c < 1$. Then the inequality*

$$\text{colength } \mathcal{J}(S, c\cdot C) \ge \left\lfloor \frac{(2t-m)(c-l)}{lt-1} \right\rfloor + 1$$

holds.

Proof. Let ψ be a morphism of log canonical singularities of (S, C) at p. Then the exceptional locus of ψ forms a unimodal linear chain. So we can apply Theorem 3.7.2 and Lemma 3.7.4. If the exceptional locus of ψ has only one irreducible exceptional divisor, then $l = \frac{2}{m}$. In this case it is straightforward to check the inequality.

Suppose that the exceptional locus of ψ consists of more than one irreducible exceptional divisors. First we prove the inequality in question for a general element C' in J_C. Let $t' = \text{t-mult}_p C'$.

By Theorem 3.7.2, $\mathrm{mult}_p C = \mathrm{mult}_p C'$ and J_C is of the form $\prod_j \overline{(x^{a_j}, y^{b_j})}$ ($a_j \ge b_j \ge 1$ for any j). Since $m = \mathrm{mult}_p C'$ and $t' = \text{t-mult}_p C'$, we get

$$J_C = (x,y)^{(m-t')} \prod_j \overline{(x^{a_j}, y^{b_j})} \qquad (\sum_j b_j = t', \quad a_j > b_j \ge 1). \tag{$*$}$$

Since $\mathrm{lct}(C;p) = \mathrm{lct}(C';p)$, one of the edges of the Newton polytope corresponding to J_C must pass through $(\frac{1}{l}, \frac{1}{l})$. $(**)$

Then the convex Newton polytope corresponding to any integrally closed ideal satisfying $(*)$ and $(**)$ is contained in the unbounded region given by the x-axis, the y-axis, the line segment joining $(0, \frac{c}{l})$ and $(\frac{c}{l}, \frac{c}{l})$, and the line segment joining $(\frac{c}{l}, \frac{c}{l})$ and $(\frac{c(2t'-m)}{lt'-1}, 0)$. So, due to Howald's theorem [15],

$$\text{colength } \mathcal{J}(S, c \cdot C') \geq N(c, m, t', l),$$

where $N(c, m, t, l)$ denotes the number of non-negative lattice points in the quadrilateral region with vertices $(-1, -1), (-1, \frac{c}{l} - 1), (\frac{c}{l} - 1, \frac{c}{l} - 1)$ and $(\frac{c(2t-m)}{lt-1} - 1, -1) \in \mathbb{R}^2$ (including boundaries).

Now we are ready to prove the general case. Recall (iii) in Theorem 3.7.2: $\mathcal{J}(S, c \cdot C) \subset \mathcal{J}(S, c \cdot C')$ for any $0 \leq c < 1$. So we have

$$\text{colength } \mathcal{J}(S, c \cdot C) \geq \text{colength } \mathcal{J}(S, c \cdot C').$$

On the other hand, $\text{t-mult}_p C \geq \text{t-mult}_p C'$ by Lemma 3.7.4. Since $t \geq t'$ and $l \leq \frac{2}{m}$, we get $\frac{c(2t-m)}{lt-1} - 1 \leq \frac{c(2t'-m)}{lt'-1} - 1$, which implies $N(c, m, t', l) \geq N(c, m, t, l)$. It is elementary to check

$$N(c, m, t, l) \geq \left\lfloor \frac{(2t-m)(c-l)}{lt-1} \right\rfloor + 1.$$

Therefore the desired inequality follows. $\square$

3.9 t-multiplicity and intersection multiplicity

In this section we give another bound on intersection multiplicities between two curves on surfaces.

We recall Proposition 3.4.1.

Proposition 3.9.1 (3.4.1). *Let S be a smooth surface. Let $m_1 \in \mathbb{N}$. Let C_1 be a smooth effective divisor passing through $p \in S$ and C_2 be another effective divisor.*

Assume that the multiplier ideal $\mathcal{J}(S, c(m_1C_1 + C_2))_p$ is co-supported at $\{p\}$ for some $c \in \mathbb{Q}_{>0}$. Then

$$(C_1 \cdot C_2)_p > m_1.$$

If t-multiplicity is specified, then we have another inequality.

Proposition 3.9.2. *Let S be a smooth surface. Let C_1 be a smooth effective divisor passing through $p \in S$, C_2 another arbitrary effective divisor, and $C = m_1C_1 + C_2$. Let $m = \mathrm{mult}_p C$ and $t = \text{t-mult}_p C$. Suppose that $m_1 \geq \frac{m}{2}$. Assume that the multiplier ideal $\mathcal{J}(S, c \cdot C)_p$ is co-supported at $\{p\}$ for some $c \in \mathbb{Q}$. Then*

$$(C_1 \cdot C_2)_p \geq \frac{(t - m_1)(1 - c(m - t))}{ct - 1} + m - t.$$

Proof. We take the minimal resolution $\mu : \tilde{S} \to S$ of (S, C) near p. Let $F_t \subset \tilde{S}$ be a place of log canonical singularities, and ψ the morphism of log canonical singularities (see Definition 3.3.2). Then, thanks to Proposition 3.3.3, the exceptional locus of ψ forms a unimodal linear chain. Applying Theorem 3.7.2 and Lemma 3.7.4, it is enough to prove for C', i.e. for a general element in the monomial ideal J_C.

By Theorem 3.7.2, we have local coordinates (x, y). Since C_1 is smooth, we have $\mathrm{mult}_p C_1 = 1$, hence the property (ii) in Theorem 3.7.2 implies that

$$J_{C_1} = (x^u, y)$$

for some u. Since J_{C_2} is of the type $\prod_j \overline{(x^{a_j}, y^{b_j})}$ ($a_j \geq b_j \geq 1$ for any j), let

$$J_{C_2} = (x, y)^{m_3} \prod_{i=1}^{n} \overline{(x^{a_i}, y^{b_i})},$$

where $a_i > b_i$ for any $1 \leq i \leq n$. The property (i) in Theorem 3.7.2 gives

$$J_C = (J_{C_1})^{m_1} J_{C_2}.$$

Let $m_2 = \sum_{i=1}^{n} b_i$. Then a general element C_2' in J_{C_2} has multiplicity $(m_2 + m_3)$ at p, so $m = m_1 + m_2 + m_3$. If ψ were a single blow-up, i.e., the first exceptional divisor F_1 were a place of log canonical singularities, then

$$\text{(3.9.1)} \qquad \operatorname{lct}(S, C'; p) = \frac{2}{m}.$$

But since $\mathcal{J}(S, c(m_1C_1 + C_2))_p$ vanishes not along C_1 but at $\{p\}$, we get

$$\text{(3.9.2)} \qquad \frac{1}{m_1} > c \geq \operatorname{lct}(S, C; p) = \operatorname{lct}(S, C'; p).$$

Then (3.9.1) and (3.9.2) would contradict $m_1 \geq \frac{m}{2}$. So ψ consists of more than one blow-ups.

Lemmas 3.9.3 and 3.9.4 will complete the proof. □

For the proof of Lemmas 3.9.3 and 3.9.4, we will use only Howald's theorem [15] and Inversion of adjuction [19, Corollary 9.5.11]. The rest will be merely delicate computations.

Lemma 3.9.3. $\operatorname{lct}(S, C'; p)(m_1 + m_2) - 1 \geq 0$.

Proof. Let $c' = \operatorname{lct}(S, C'; p)$. If $m_3 = 0$ then, by [19, Proposition 9.5.13], $c' \geq \frac{1}{m} = \frac{1}{m_1+m_2}$. Suppose $m_3 > 0$. Let H be a general element in (x, y), D in $\prod_{i=1}^{n} \overline{(x^{a_i}, y^{b_i})}$, and C_1' in J_{C_1}.

Since $\mathcal{J}(S, c' \cdot J_C)$ is non-trivial, neither is $\mathcal{J}(S, c'(m_1C_1' + D) + c'm_3H)$. Since $m = m_1 + m_2 + m_3$ and $m_1 \geq \frac{m}{2}$, we have $m_1 \geq m_2 + m_3$. Putting $m_1 \geq m_2 + m_3 \geq m_3$ and $\frac{1}{m_1} > c \geq c'$ together, we get $c'm_3 < 1$. Inversion of adjuction [19, Corollary 9.5.11] implies that $\mathcal{J}(H, c'(m_1C_1' + D)_H)$ is non-trivial, which means

$$c'(m_1 + m_2) \geq 1.$$

□

Lemma 3.9.4. $(C_1 \cdot C_2)_p \geq \frac{m_2(1-cm_3)}{c(m_1+m_2)-1} + m_3$.

Proof. Since $\mathcal{J}(S, c(m_1C_1 + C_2))_p$ is co-supported at $\{p\}$ and

$$\frac{m_2(1-cm_3)}{c(m_1+m_2)-1} + m_3 \leq \frac{m_2(1-c'm_3)}{c'(m_1+m_2)-1} + m_3 \qquad \text{for any } c' \leq c,$$

we may assume that $c = \mathrm{lct}(C; p)$.

Let

$$a = \sum_{i,\ a_i \leq ub_i} a_i, \qquad b = \sum_{i,\ a_i \leq ub_i} b_i,$$
$$a' = \sum_{i,\ a_i > ub_i} a_i, \qquad b' = \sum_{i,\ a_i > ub_i} b_i.$$

Since

$$J_C = \overline{(x^u, y)^{m_1}(x,y)^{m_3}} \prod_{i=1}^{n} \overline{(x^{a_i}, y^{b_i})}$$
$$\supset \overline{(x^u, y)^{m_1}(x,y)^{m_3}(x^a, y^b)(x^{a'}, y^{b'})} =: J'_C,$$

we have

$$\mathrm{lct}\ J_C \geq \mathrm{lct}\ J'_C.$$

So we may assume that $c = \mathrm{lct}\ J'_C$. Since

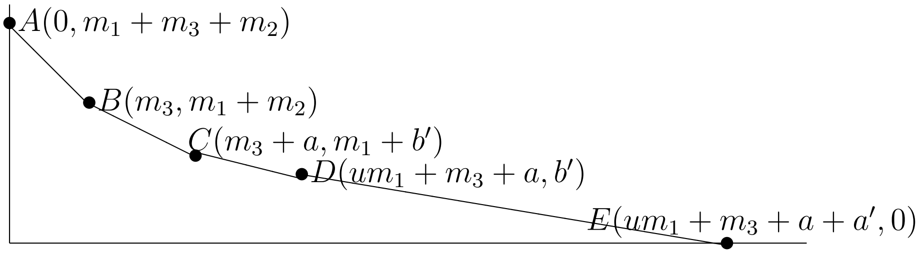
$$\mathrm{mult}_p \overline{(x^a, y^b)(x^{a'}, y^{b'})} = b + b' = \sum_{i=1}^{n} b_i = m_2,$$

we may replace J_C by $J_{C'}$.

We note that

$$(C'_1 \cdot C'_2)_p \geq m_3 + \sum_{i=1}^{n} \min(a_i, u \cdot b_i)$$
$$= m_3 + a + ub'.$$

In order to use Howald's theorem [15], we illustrate the Newton polytope corresponding to J'_C.



The line $y = x$ does not meet the line segment AB because $m_3 < m_1+m_2$. Neither does DE because $m_1 > m_2 = b+b'$ implies $um_1+m_3+a > b'$. Hence it meets BC or CD, in other words, lct J'_C is determined by BC or CD. It is tedious to check that

$$m_3 + a + ub' \geq \frac{m_2(1 - cm_3)}{c(m_1 + m_2) - 1} + m_3, \qquad (*)$$

which implies the desired inequality. We present the computations for $(*)$ in the next subsection. $\square$

3.9.1 Computations : Completion of the proof of Lemma 3.9.4

In order to complete Lemma 3.9.4, we need to show that

$$m_3 + a + ub' \geq \frac{m_2(1 - cm_3)}{c(m_1 + m_2) - 1} + m_3.$$

We separate into two cases : $m_3 + a \geq m_1 + b'$ or $m_3 + a < m_1 + b'$.

Lemma 3.9.5. *If* $m_3 + a \geq m_1 + b'$ *then*

$$m_3 + a + ub' \geq \frac{m_2(1 - cm_3)}{c(m_1 + m_2) - 1} + m_3.$$

Proof. We have

$$c = \text{lct } J'_C = \frac{m_2 + a - b'}{a(m_1 + m_2) + m_3(m_2 - b')}, \tag{3.9.3}$$

equivalently

$$a(c(m_1 + m_2) - 1) = m_2 - b' - cm_2m_3 + cb'm_3,$$

so

$$a = \frac{(m_2 - b')(1 - cm_3)}{c(m_1 + m_2) - 1}.$$

Then it is enough to show that

$$u \geq \frac{1 - cm_3}{c(m_1 + m_2) - 1},$$

equivalently

$$c \geq \frac{u + 1}{u(m_1 + m_2) + m_3}.$$

From (3.9.3), we need to show

$$\frac{m_2 + a - b'}{a(m_1 + m_2) + m_3(m_2 - b')} \geq \frac{u + 1}{u(m_1 + m_2) + m_3}. \tag{3.9.4}$$

Recall $b + b' = m_2$. So (3.9.4) becomes

$$\frac{a + b}{a(m_1 + m_2) + bm_3} \geq \frac{u + 1}{u(m_1 + m_2) + m_3}. \tag{3.9.5}$$

Recall that

$$a = \sum_{i,\ a_i \leq ub_i} a_i, \qquad b = \sum_{i,\ a_i \leq ub_i} b_i.$$

So $a \leq ub$. Since $m_1 + m_2 \geq m_3$, we get (3.9.5). $\square$

Lemma 3.9.6. *If $m_3 + a < m_1 + b'$ then*

$$m_3 + a + ub' \geq \frac{m_2(1 - cm_3)}{c(m_1 + m_2) - 1} + m_3.$$

Proof. We have

$$c = \operatorname{lct}\ J'_C = \frac{u + 1}{m_3 + a + u(m_1 + b')}. \tag{3.9.6}$$

We want to show that

$$\frac{u + 1}{m_3 + a + u(m_1 + b')} \geq \frac{m_2 + (a + ub')}{(m_1 + m_2)(a + ub') + m_2 m_3},$$

equivalently

$$(3.9.7) \qquad (m_3 + a + ub' - m_1)um_2 \geq (m_3 + a + ub' - m_1)(a + ub').$$

Since $1 > cm_1$, we have

$$\frac{1}{m_1} > c = \frac{u+1}{m_3 + a + u(m_1 + b')},$$

which gives

$$m_3 + a + ub' - m_1 > 0.$$

On the other hand, since $a \leq ub$ and $b + b' = m_2$, we get $um_2 \geq a + ub'$. So (3.9.7) follows.

Therefore

$$c \geq \frac{m_2 + (a + ub')}{(m_1 + m_2)(a + ub') + m_2 m_3},$$

equivalently

$$a + ub' \geq \frac{m_2(1 - cm_3)}{c(m_1 + m_2) - 1}.$$

$\square$

CHAPTER IV

The Main Theorem

In this chapter, we prove Theorem 1. We start by giving definitions of a cone-like and rationally defined divisor, and carry out the proof of the main theorem by using induction on the order of blow-ups.

4.1 Rationally defined cone-like divisor

Here we define a rationally defined cone-like divisor on $\mathbb{A}^3$. First a cone-like divisor is defined.

Definition 4.1.1 (Cone-like divisor)**.** Let $X = \mathbb{A}^3$. Let

$$\pi_0 : X_0 \longrightarrow X$$

be the blowing-up along O, with the exceptional divisor E_0.

An effective divisor D on X will be said to be a *cone-like divisor* if it satisfies the following property :

In a neighborhood of any point in E_0, each analytic branch of the proper transform of D, if not empty, is smooth and meets E_0 transversally.

Example 4.1.2. Any affine cone over unions of smooth curves on $\mathbb{P}^2$ is cone-like.

Next, we define a notion of being rationally defined. The intuition is that a rationally defined divisor varies rationally as points are deformed and switched.

Definition 4.1.3 (Rationally defined divisor)**.** Fix a set Z of r distinct points on $\mathbb{P}^2$. We say that an effective divisor $D \subset X(= \mathbb{A}^3)$ is *rationally defined* with Z if there are a smooth variety U, a point $u \in U$, an effective divisor $\mathcal{D} \subset X \times U$, and a variety $\mathcal{Z} \subset \mathcal{E}_0(:=$ the exceptional divisor of the blow-up π_0 of $X \times U$ along $O \times U = \mathbb{P}^2 \times U)$ satisfying the following properties :

(1) $\mathcal{Z}$ is irreducible and smooth,

(2) $\mathcal{Z}$ is contained in the proper transform of $\mathcal{D}$ under π_0,

(3) Z is the fiber of $\mathcal{Z} \to U$ over u,

(4) D is the fiber of $\mathcal{D} \to U$ over u, and

(5) $\mathcal{Z}$ is flat and finite of degree r over U.

Example 4.1.4. Let U be (any smooth open set of) the Hilbert scheme $\mathrm{Hilb}^r(\mathbb{P}^2)$ of r points on $\mathbb{P}^2$, $\mathcal{Z} \subset U \times \mathbb{P}^2$ the universal family, and let $\mathcal{D} \subset X \times U$ be any effective divisor whose proper transform contains $\mathcal{Z}$. Then a general fiber of $\mathcal{D} \to U$ is rationally defined.

As we will see, any rationally defined divisor over r very general points has similar behaviors near each of the r points.

Here is the statement of the Main Theorem.

Theorem 4.1.5 (Main Theorem)**.** *Let $X = \mathbb{A}^3$. Let $r \gg 0$ be a sufficiently large integer. Let $I_r \subset \mathcal{O}_X$ be the defining ideal of the affine cone over r very general points on $\mathbb{P}^2$.*

Then I_r cannot be realized as a multiplier ideal of a rationally defined cone-like divisor.

4.2 Set up

Let $I_r \subset \mathcal{O}_X$ be the defining ideal of the affine cone over r very general points. We will assume that

$$I_r = \mathcal{J}(X, c \cdot D), \tag{4.2.1}$$

for some integral rationally-defined cone-like divisor $D \subset X$ and some $c \in \mathbb{Q}$. The plan is to start by blowing up at O, and restrict to the exceptional divisor.

Set-up 4.2.1. *Let*

$$\pi_0 : X_0 \longrightarrow X$$

be the blowing-up at the origin O, with the exceptional divisor $E_0 \cong \mathbb{P}^2 \subset X_0$. Let

$$D_0 = \text{ proper transform of } D, \tag{4.2.2}$$

$$C_0 = D_0 | E_0. \tag{4.2.3}$$

Note that D_0 contains the proper transforms $(l_\alpha)_0$ of the r lines l_α. Write

$$q_\alpha = (l_\alpha)_0 \cap E_0, \tag{4.2.4}$$

equivalently, $q_1, \cdots, q_r \in E_0 \cong \mathbb{P}^2$ are points in $\mathbb{P}^2$ determined in a natural way by lines $l_\alpha \subset \mathbb{C}^3$. (See Figure 4.1.)

One of the advantages of dealing with general lines and a rationally defined divisor is that we can assume that D_0 looks similar at all of the r points $q_\alpha \in E_0$.

Definition 4.2.2. Let S be a smooth surface, C an effective divisor on S, and $q_1, ..., q_r$ points on C. If C has the same multiplicity, the same t-multiplicity, the same log canonical threshold and the same 1-dimensional (component) multiplicity at each of several points $q_1, \cdots, q_r$, then we say that C is *almost-equisingular* at $q_1, \cdots, q_r$.

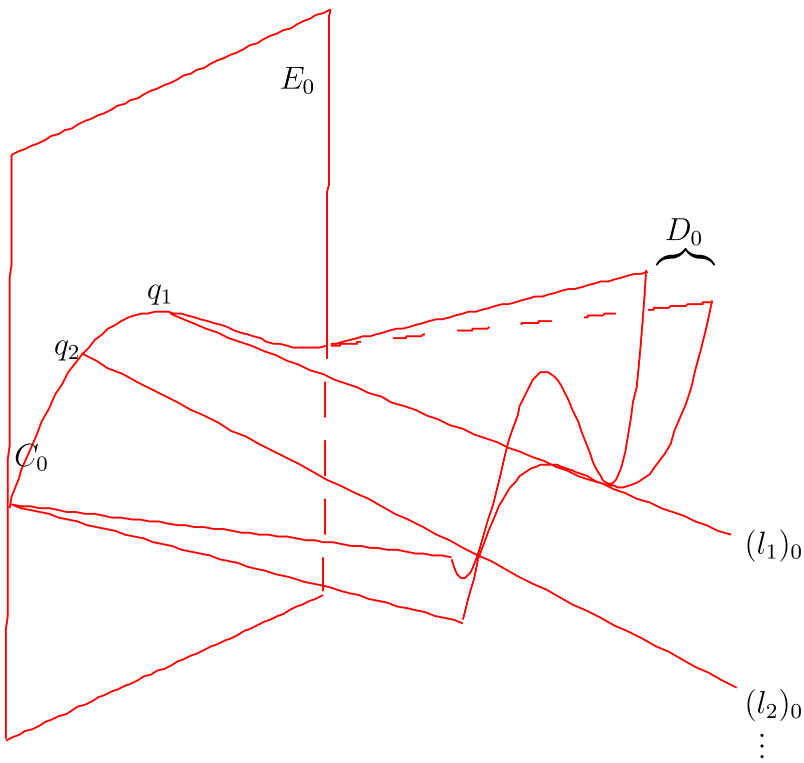


Figure 4.1: The blown-up space of X at the origin O

Lemma 4.2.3. *If D is rationally defined over r very general points, $D_0|E_0$ is almost-equisingular at $q_1, \cdots, q_r$.*

Proof. The uniform invariant property that we will describe now is motivated by [14],[7].

By Definition 4.1.3, there are a smooth variety U, a point $u \in U$, an effective divisor $\mathcal{D} \subset X \times U$, and a variety $\mathcal{Z} \subset \mathcal{E}_0$(:= the exceptional divisor of the blow-up π_0 of $X \times U$ along $O \times U = \mathbb{P}^2 \times U$) satisfying the following properties :

(1) $\mathcal{Z}$ is irreducible and smooth,

(2) $\mathcal{Z}$ is contained in the proper transform of $\mathcal{D}$ under π_0,

(3) Z is the fiber of $\mathcal{Z} \to U$ over u,

(4) D is the fiber of $\mathcal{D} \to U$ over u, and

(5) $\mathcal{Z}$ is flat and finite of degree r over U.

The key point is that $\mathcal{Z}$ is irreducible.

We just blew up $X \times U$ along $O \times U$. Let $\mathcal{D}_0$ be the proper transform of $\mathcal{D}$, and $\mathcal{L}_0$ the proper transform of the universal family of affine cones over r points. $\mathcal{E}_0$ and $\mathcal{L}_0$ meet transversally, and their intersection $\mathcal{Z}$ is irreducible.

Let $\mathcal{C}_0 := \mathcal{D}_0|\mathcal{E}_0$ and denote by $\mathcal{C}_{0u}$ the fiber of the morphism $\mathcal{C}_0 \to U$ at $u \in U$. Let invariant$(\mathcal{C}_{0u}; (q_\alpha)_u)$ be one of multiplicity, t-multiplicity, Arnold multiplicity, or component multiplicity of $\mathcal{C}_{0u}$ at $(q_\alpha)_u$. Choose the largest possible rational number p such that

$$\text{invariant}(\mathcal{C}_{0u}; (q_\alpha)_u) \geq p$$

for any $u \in U$ and any $(q_\alpha)_u$. By semi-continuity, the set

$$\{(q_\alpha)_u \in \mathcal{Z} \mid \text{invariant}(\mathcal{C}_{0u}; (q_\alpha)_u) > p\}$$

is of codimension ≥ 1 in $\mathcal{Z}$, hence its dimension is less than $\dim U (= \dim \mathcal{Z})$. Therefore there is $u \in U$ such that invariant$(\mathcal{C}_{0u}; (q_\alpha)_u)$ is the same $(= p)$ for every $\alpha = 1, ..., r$. $\square$

4.3 The base step of the Induction

In this section we prove the base step of the induction (see Section 4.4.1). We return to Set-up 4.2.1. Recall that we are assuming

$$I_r = \mathcal{J}(X, c \cdot D).$$

Let

$$d = \text{ord}_O I_r, \tag{4.3.1}$$

equivalently, d is the least degree of curves in $\mathbb{P}^2$ passing through r points

$$q_1, \cdots, q_r \in \mathbb{P}^2$$

determined by lines. As in Set-up 4.2.1, consider the blowing-up $\pi_0 : X_0 \longrightarrow X$ at the origin O, with the exceptional divisor E_0.

Lemma 4.3.1. *We have*

$$\frac{d+1+\mathrm{ord}_{E_0} K_{X_0/X}}{\mathrm{ord}_{E_0}\pi_0{}^*D} = \frac{d+3}{\mathrm{ord}_{E_0}\pi_0{}^*D} > c. \tag{4.3.2}$$

Proof. Let $\mathbf{m}$ be the maximal ideal corresponding to O. Then

$$\mathcal{J}(X, c\cdot D) \subset \pi_{0*}\mathcal{O}_{X_0}((\mathrm{ord}_{E_0} K_{X_0/X} - \lfloor c\cdot \mathrm{ord}_{E_0}\pi_0{}^*D\rfloor)E_0),$$

equivalently

$$\mathcal{J}(X, c\cdot D) \subset \mathbf{m}^{\lfloor c\cdot \mathrm{ord}_{E_0}\pi_0{}^*D\rfloor - \mathrm{ord}_{E_0} K_{X_0/X}}.$$

Since d is the order of $\mathcal{J}(X, c\cdot D)$ at the origin,

$$\mathcal{J}(X, c\cdot D) \subset \mathbf{m}^d$$

but

$$\mathcal{J}(X, c\cdot D) \not\subset \mathbf{m}^{d+1}.$$

Thus we have

$$\lfloor c\cdot \mathrm{ord}_{E_0}\pi_0{}^*D\rfloor - \mathrm{ord}_{E_0} K_{X_0/X} \le d.$$

□

Lemma 4.3.2. *Recall the situation of Set-up 4.2.1. We have*

$$\frac{d+3}{\mathrm{ord}_{E_0}\pi_0{}^*D} > c \ge \mathrm{lct}(E_0, C_0; q_\alpha).$$

Proof. Since $I_r = \mathcal{J}(X, c\cdot D)$, the multiplier ideal $\mathcal{J}(X, c\cdot D)$ vanishes along the r lines. Under π there is an isomorphism $D - O \cong D_0 - \pi_0^{-1}(O)$. So the multiplier ideal $\mathcal{J}(X_0, c \cdot D_0)$ vanishes on the proper transform $(l_\alpha)_0$ of r lines, in particular, $\mathcal{J}(X_0, c \cdot D_0)$ vanishes at q. Then, by the restriction theorem, we have

$$\mathcal{J}(E_0, c \cdot D_0|E_0) \subseteq \mathcal{J}(X_0, c \cdot D_0)|_{E_0}.$$

Hence $\mathcal{J}(E_0, c \cdot C_0)$ vanishes at q. This implies

$$\frac{d+3}{\mathrm{ord}_{E_0}{\pi_0}^* D} >_{\text{Lemma 4.3.1}} c \geq \mathrm{lct}(E_0, C_0; q_\alpha).$$

□

Lemma 4.3.3. *Let q_α ($\alpha = 1, 2, \cdots, r$) be the intersection points of E_0 and the proper transform of r lines. For every $\alpha = 1, 2, \cdots, r$, the LC-locus of (E_0, C_0) at q_α is 1-dimensional.*

Proof. Let e denote the degree of C_0 on $E_0 \cong \mathbb{P}^2$. We observe that $e = \mathrm{ord}_{E_0}{\pi_0}^* D$. The Proposition 4.3.2 shows

$$d + 3 > e \cdot \mathrm{lct}(E_0, C_0; q_\alpha). \tag{4.3.3}$$

Since d is the least degree of curve in $\mathbb{P}^2$ passing through r general points, we have

$$\binom{d+1}{2} \leq r, \tag{4.3.4}$$

so

$$d + 3 < \frac{3}{2}\lfloor\sqrt{r}\rfloor \qquad \text{for } r > 1600.$$

In particular we have

$$\frac{3}{2}\lfloor\sqrt{r}\rfloor > e \cdot \mathrm{lct}(E_0, C_0; q_\alpha).$$

But we saw in Theorem 3.1.9 that if C_0 were a curve of degree e passing through r very general points q_α, with 0-dimensional LC-locus, then we would have $\frac{3}{2}\lfloor\sqrt{r}\rfloor \leq e\cdot\mathrm{lct}(E_0, C_0; q_\alpha)$. Therefore the LC-locus of (E_0, C_0) at q_α should be 1-dimensional. □

Remark 4.3.4. Assuming 0-dimensional LC-locus as in Theorem 3.1.9, one can easily obtain $e \cdot \mathrm{lct} \geq (d+1)$. In fact if $e \cdot \mathrm{lct} < (d+1)$ then Nadel vanishing theorem would imply $H^i(\mathbb{P}^2, \mathcal{O}(K_{\mathbb{P}^2} + (d+1)H) \otimes \mathcal{J}(l \cdot C_0)) = 0$, $i > 0$, which contradicts Riemann-Roch. But this weaker bound cannot prove Lemma 4.3.3.

We have established that C_0 has 1-dimensional LC-locus in a neighborhood of each q_α, and by almost-equisingularity in Lemma 4.2.3, it has the same component multiplicity at each q_α. So we can write

$$C_0 = mC_0' + C_0'', \tag{4.3.5}$$

where C_0' is a reduced curve passing through each q_α, and

$$\mathcal{J}(E_0, \mathrm{lct}(C_0) \cdot C_0) = \mathcal{O}_{E_0}(-C_0') \tag{4.3.6}$$

in a neighborhood of each q_α. In particular,

$$\mathrm{lct}(C_0; q_\alpha) = \frac{1}{m}. \tag{4.3.7}$$

Lemma 4.3.5. *C_0' is smooth at every q_α.*

Proof. Recall the fact that an one dimensional LC-locus of an effective divisor on a surface is locally either a node or smooth. If C_0' has a node at some q_α, then $C_0 = mC_0'$ in a neighborhood of q_α. Let $H \subset X$ be a general effective divisor whose proper transform under π_0 passes through q_α. If we replace D by $D + \epsilon H$ $(0 < \epsilon \ll 1)$, then we still have $I_r = \mathcal{J}(X, c \cdot D)$ but the LC-locus of $(E_0, D_0|_{E_0})$ at q_α is not 1-dimensional. □

Next,

Lemma 4.3.6. *C_0' must be irreducible.*

Proof. If C_0' were not irreducible, then by monodromy argument as in Lemma 4.2.3, it would have to have at least one irreducible component passing through each q_α so deg $C_0' \geq r$, which gives

$$e = \mathrm{ord}_{E_0}{\pi_0}^* D = \deg C_0 \geq mr.$$

But this contradicts Lemma 4.3.2. $\square$

Lemma 4.3.7. *Let δ be the degree of C_0'. Then δ is equal to d, $(d+1)$ or $(d+2)$.*

Proof. By (4.3.3), we get $\delta < d+3$. On the other hand, since C_0' passes r general points, we have $\delta \geq d$. $\square$

We observe that c is quite tightly bounded.

Lemma 4.3.8. *We have*

$$\frac{d+3}{e} > c \geq \mathrm{lct}(E_0, C_0; q_\alpha) = \frac{1}{m} \geq \frac{d}{e}. \tag{4.3.8}$$

Proof. We recall that m is the multiplicity of C_0 along C_0'. The first and second inequalities follow from Proposition 4.3.1 and Proposition 4.3.2. The equality is due to Lemma 4.3.3. The last inequality follows from

$$e = \deg C_0 \geq m \cdot \deg C_0' \geq m\delta \geq md.$$

$\square$

Recall that we are assuming that, in a neighborhood of any point of C_0', each analytic component of D_0 is smooth and meets E_0 transversally.

Definition 4.3.9 (Desingularization away from q_α)**.** We obtained an irreducible curve C_0' which is smooth at q_α. But C_0' may have singularities away from q_α. So we take a desingularization $\pi_0^\varsigma : X_0^\varsigma \to X_0$ in such a way that π_0^ς is a proper birational morphism and is isomorphic over some open set containing q_α, that the indeterminacy locus of ${\pi_0^\varsigma|_{E_0}}^{-1}$ is 0-dimensional, and that $(C_0^\varsigma)'$ is smooth, where we denote the proper transform of D_0 by D_0^ς, the proper transform of E_0 by E_0^ς, and $D_0^\varsigma|E_0^\varsigma$ by C_0^ς and we write $C_0^\varsigma = m(C_0')^\varsigma + (C_0'')^\varsigma$.

4.4 Induction : Blow-ups along Curves

We recall the notations we are using. Let $X = \mathbb{A}^3 = \operatorname{Spec} \mathbb{C}[x, y, z]$. Let r be a sufficiently large integer and consider r very general lines passing through $O := (0, 0, 0) \in X$. We denote by I_r the radical ideal corresponding to the union of r lines. Let

$$d = \operatorname{ord}_O I_r.$$

Throughout the induction process below, we assume that

$$I_r = \mathcal{J}(X, c \cdot D)$$

for some $c \in \mathbb{Q}_{>0}$ and some effective rationally defined cone-like divisor D on X.

Notation 4.4.1. For any birational morphism $Z' \to Z$ and any divisor $W \subset Z$, we denote by W^p the proper transform of W on Z'.

4.4.1 Induction Hypotheses and Notations

We present the induction hypotheses and the notations we will use. The hypotheses are labeled [**I**], [**II**], [**III**] and [**IV**].

[**I**] Let

$$\pi_i : X_i \to X_{i-1}^\varsigma$$

be the smooth blow-up along $(C_{i-1}')^\varsigma$ ($X_{-1}^\varsigma := X$, $(C_{-1}')^\varsigma :=$ the origin O). Denote $\pi_0 \circ \pi_0^\varsigma \circ \pi_1 \circ \pi_1^\varsigma \circ \cdots \circ \pi_i$ by σ_i. Let D_i be the proper transform of D under σ_i, and

E_i the exceptional divisor with respect to π_i. Let

(4.4.1) $$C_i = D_i|E_i.$$

Remark 4.4.2. We notice that $E_0 \cong \mathbb{P}^2$ but that E_i is a ruled surface over $(C^{\varsigma}_{i-1})'$ for every $i \geq 1$.

[**II**] Each of the proper transforms of the given r lines meets E_i. By abuse of notation, let q_α $(1 \leq \alpha \leq r)$ be the intersection points of E_i and the proper transforms of r lines. We assume that there is an effective irreducible reduced divisor C_i' on E_i such that

- $C_i = m_{i+1}C_i' + C_i''$ for some $m_{i+1} \in \mathbb{Z}_{>0}$,
- $\mathcal{J}(E_i, \mathrm{lct}(C_i) \cdot C_i) = \mathcal{O}_{E_i}(-C_i')$ in a neighborhood of each q_α,
- C_i' is smooth at q_α, $1 \leq \alpha \leq r$.
- C_i' belongs to the numerically equivalent class

$$\Big| -(\delta + i - 1 - \varepsilon_i)E^p_{i-1} - (\delta + i - \varepsilon_i)E_i\Big|_{E_i} \quad (E^p_{-1} := \emptyset)$$

for some $\varepsilon_i \geq 0$, where

$$\delta = \deg C_0'$$

on $E_0 \cong \mathbf{P}^2$.

Let

$$\pi_i^{\varsigma} : X_i^{\varsigma} \to X_i$$

be a desingularization of C_i' in the sense of Definition 4.3.9. Let D_i^{ς} be the proper transform of D_i under π_i^{ς}, and E_i^{ς} be the proper transform of E_i under π_i^{ς}. Let

$$C_i^{\varsigma} = D_i^{\varsigma}|E_i^{\varsigma}.$$

The proper transform of C_i' is smooth and will be denoted by $(C_i')^\varsigma$.

[**III**] Let $e = \deg C_0$ on $E_0 \cong \mathbf{P}^2$. We have

$$\frac{d+4+i}{e+s_i} > c \geq \frac{1}{m_{i+1}} \geq \cdots \geq \frac{1}{m_1} \geq \frac{d}{e}, \tag{4.4.2}$$

where $s_0 := 0, s_i := \sum_{j=1}^{i} m_j$.

[**IV**] We have

$$\mathrm{ord}_{E_i} K_{X_i/X} = i+2, \tag{4.4.3}$$

and

$$\mathrm{ord}_{E_i} (\sigma_i)^* D = e + s_i. \tag{4.4.4}$$

On the other hand, $(C_i')^\varsigma \cap E_k^p = \emptyset$ for all $k < i$.

Remark 4.4.3. It follows that

$$\mathrm{ord}_{E_i^\varsigma} K_{X_i^\varsigma/X} = i+2, \tag{4.4.5}$$

and

$$\mathrm{ord}_{E_i^\varsigma} (\sigma_i \circ \pi_i^\varsigma)^* D = e + s_i. \tag{4.4.6}$$

Since we have a long list of induction hypotheses, it might be helpful to illustrate them in Figure 4.2.

4.5 The Proof of the Induction : Non-termination of resolution

In Section 4.3, we blew up X at the origin O and defined [**I**] for $i = 0$ and proved [**II**], [**III**] and [**IV**] for $i = 0$.

$$
\begin{tikzcd}
\vdots & \vdots \arrow[dr, "\pi_{i+1}"] & & \\
 & & X_i^{\varsigma} \arrow[dl, "\pi_i^{\varsigma}"'] & \\
E_i,\ E^p_{i-1}, \ldots,\ E^p_0,\ D_i \arrow[r, hook] & X_i \arrow[dr, "\pi_i"] & & C_i := D_i|E_i \\
 & & X_{i-1}^{\varsigma} \arrow[dl, "\pi_{i-1}^{\varsigma}"'] & C_i = m_{i+1}C_i' + C_i'' \\
\vdots & \vdots & & \\
E_1,\qquad E^p_0,\ D_1 \arrow[r, hook] & X_1 \arrow[dr, "\pi_1"] & & \\
 & & X_0^{\varsigma} \arrow[dl, "\pi_0^{\varsigma}"'] & \\
E_0,\ D_0 \arrow[r, hook] & X_0 \arrow[d, "\pi_0"'] & & \\
D \arrow[r, hook] & X & &
\end{tikzcd}
$$

Figure 4.2: The induction process

Using induction on i, we will prove that [I],[II],[III] and [IV] are valid for every $i \geq 0$. Concretely speaking, the Subsections 4.5.1, 4.5.2, 4.5.3 and 4.5.4 will show that

$$\begin{gathered} \text{[I],[II],[III],[IV] for } i = j \\ \Downarrow \\ \text{[I],[II],[III],[IV] for } i = j+1. \end{gathered}$$

Our induction proceeds as follows:

[[I],[II],[III],[IV] for $i = 0$] $\Rightarrow$ Lemma 4.5.3 $\Rightarrow$ [[I],[II],[III],[IV] for $i = 1$] $\Rightarrow \cdots \Rightarrow$ [[I],[II],[III],[IV] for $i = j$] $\Rightarrow$ Lemma 4.5.3 $\Rightarrow$ [[I],[II],[III],[IV] for $i = j+1$] $\Rightarrow \cdots$.
Therefore we have the following statement.

Proposition 4.5.1. *If $I_r \subset \mathcal{O}_X$ were realized as a multiplier ideal of a rationally defined cone-like divisor D, then a sequence of blow-ups given by the induction would be infinite.*

The idea is that we try to resolve the singularities of D_0 at a generic point of C_0' in X_0. We state the theorem of Beppo Levi in the following form:

Theorem 4.5.2 ([29], Theorem 6)**.** *A sequence of surfaces obtained from D_0 by blow-ups along curves having multiplicity > 1 is necessarily finite.*

We prove the Main Theorem as follows.

Proof of Main Theorem. Suppose that $I_r \subset \mathcal{O}_X$ could be realized as a multiplier ideal of a rationally defined cone-like divisor D. Then our inductive steps would produce an infinite sequence of blow-ups along curves having multiplicity > 1, by Lemma 4.5.3. But this contradicts Theorem 4.5.2. This means that the singularities of D_0 at a generic point of C_0' cannot be resolved. This is an ultimate contradiction. Therefore I_r cannot be realized as a multiplier ideal of a rationally defined cone-like divisor. $\square$

4.5.1 The reason that we blow up along $(C_j')^\varsigma$ and define [I] for $i = j+1$

The following lemma explains why we are forced to blow up X_j^ς along $(C_j')^\varsigma$.

Lemma 4.5.3. *$D_j^\varsigma \cup E_j^\varsigma$ does not have a SNC singularity at a generic point of $(C_j')^\varsigma$. In particular, D_j^ς has multiplicity > 1 along $(C_j')^\varsigma$.*

Proof. Suppose that $D_j^\varsigma \cup E_j^\varsigma$ has a SNC singularity at a generic point of $(C_j')^\varsigma$. Then since

$$C_j = m_{j+1} C_j' + C_j'',$$

we have

$$D_j^\varsigma = m_{j+1} D_j^{\varsigma\prime} + D_j^{\varsigma\prime\prime},$$

where $D_j^{\varsigma\prime}$ is smooth along $(C_j')^\varsigma$. Then it follows from $c \geq \frac{1}{m_{j+1}}$ (the induction hypothesis [III]) that

$$\begin{aligned}\mathcal{J}(X, c\cdot D) &\subset \mathcal{J}\Big(X, \frac{1}{m_{j+1}}\cdot D\Big)\\ &\subset (\sigma_j)_* O_{X_j}\Big(\Big\lfloor -\frac{1}{m_{j+1}} D_j^\varsigma \Big\rfloor\Big) = (\sigma_j)_* O_{X_j}(-D_j^{\varsigma\prime}).\end{aligned}$$

But the last ideal defines a 2-dimensional subscheme in X. Therefore $\mathcal{J}(X, c\cdot D)$ cannot be I_r which defines 1-dimensional lines. This establishes the first statement.

The second statement follows from the assumption that D is cone-like. $\square$

4.5.2 The Proof of [IV] for $i = j+1$

As soon as we blow up along $(C_j')^\varsigma$ as in [I], we immediately have

$$\mathrm{ord}_{E_{j+1}} K_{X_{j+1}/X} = (j+1) + 2.$$

For $\mathrm{ord}_{E_{j+1}} (\sigma_{j+1})^* D$, it is enough to observe that E_j^ς intersects transversally with any analytic branch of D_j^ς, which follows from D being cone-like.

4.5.3 The Proof of the first inequality in [III] for $i = j+1$

Lemma 4.5.4. *Let q_α be the intersection points of E_{j+1} and the proper transform of r lines. Then we have*

$$\frac{d+(j+1)+4}{e+s_{j+1}} > \mathrm{lct}(E_{j+1}, C_{j+1}; q_\alpha).$$

Proof. Let $\mathbf{m}$ be the maximal ideal corresponding to O. Then

$$\mathcal{J}(X, c\cdot D) \subset {\sigma_{j+1}}_* \mathcal{O}_{X_{j+1}}\Big((\mathrm{ord}_{E_{j+1}} K_{X_{j+1}/X} - \lfloor c \cdot \mathrm{ord}_{E_{j+1}} {\sigma_{j+1}}^* D \rfloor) E_{j+1}\Big),$$

which, by [IV], yields

$$I_r = \mathcal{J}(X, c\cdot D) \subset {\sigma_{j+1}}_* \mathcal{O}_{X_{j+1}}\Big(((j+1)+2 - \lfloor c(e+s_{j+1}) \rfloor) E_{j+1}\Big).$$

For a general element G in I_r, we have

$$\mathrm{ord}_{E_{j+1}} {\sigma_{j+1}}^* G \leq d+1,$$

because d is the vanishing order of I_r at the origin, and on E_1 the proper transform of G does not contain $(C_1')^\varsigma$. Thus we have

$$(j+1)+2-\lfloor c(e+s_{j+1})\rfloor \geq -d-1.$$

Now it remains to show

$$c \geq \mathrm{lct}(E_{j+1}, C_{j+1}; q_\alpha),$$

but its proof is essentially the same as that of Lemma 4.3.2. □

The following bound will be useful later.

Lemma 4.5.5. *We have*

$$\frac{1}{m_{j+1}} \cdot \frac{d+(j+1)+4}{d+(j+1)} > \mathrm{lct}(E_{j+1}, C_{j+1}; q_\alpha).$$

Proof. It follows from $e \geq m_1 d$ and $m_1 \geq m_2 \geq \cdots \geq m_{j+1}$ (induction hypothesis [III]) that

$$\frac{1}{m_{j+1}} \cdot \frac{d+(j+1)+4}{d+(j+1)} \geq \frac{d+(j+1)+4}{e+\sum_{k=1}^{j+1} m_k} = \frac{d+(j+1)+4}{e+s_{j+1}} > \mathrm{lct}.$$

$\square$

4.5.4 The Proof of [II] and [III] for $i = j+1$

Here we prove [II] and [III] for $i = j+1$. The following lemma is the key one. Its entire proof is rather long, so we present it in Section 4.6.

Lemma 4.5.6. *The LC-loci of (E_{j+1}, C_{j+1}) at q_α are 1-dimensional. Hence the LC-loci of $(E^\varsigma_{j+1}, C^\varsigma_{j+1}; q_\alpha)$ are 1-dimensional.*

Proof. See Section 4.6. $\square$

We will frequently use intersection theory on E_{j+1}. The following Lemmas 4.5.7, 4.5.8 and 4.5.10 are all straightforward.

Lemma 4.5.7. *We have the following three intersection numbers on E_1:*

$(E^p_0)^2|_{E_1} = -\delta(\delta+1+\varepsilon_1)$,

$E^p_0 \cdot E_1|_{E_1} = \delta(\delta+\varepsilon_1)$,

and $(E_1)^2|_{E_1} = -\delta(\delta-1+\varepsilon_1)$, *for some* $\varepsilon_1 \geq 0$.

Proof. First we treat with the case that C'_0 is nonsingular. Then we get $(E^p_0)^3 = (-1-\delta)^2$ and $(E^p_0)^2(E_1) = -\delta^2 - \delta$ by using

$$\mathcal{N}_{E^p_0/X_1} = \pi_0^* \mathcal{N}_{E_0/X_0}(-E^p_0 \cdot E_1).$$

It follows from

$$0 \to \mathcal{N}_{C'_0/E_0} \to \mathcal{N}_{C'_0/X_0} \to \mathcal{N}_{E_0/X_0}|_{C'_0} \to 0$$

that $(E_1)^3 = -\delta^2 + \delta$. To get $(E_0^p)(E_1)^2 = \delta^2$, we use

$$1 = (E_0)^3 = (E_0^p + E_1)^3 = (E_0^p)^3 + 3(E_0^p)^2(E_1) + 3(E_0^p)(E_1)^2 + (E_1)^3.$$

If C_0' is singular, then we take a desingularization $\pi_0^\varsigma : X_0^\varsigma \to X_0$ as in Notation 4.3.9. Then $(E_0^p)^2(E_1)$ is less than $-\delta^2 - \delta$. Other intersection numbers can be similarly computed. □

Lemma 4.5.8. *We have the following three intersection numbers on E_{j+1}:*

$(E_j^p)^2|_{E_{j+1}} = -\delta \cdot (\delta + j + 1 + \varepsilon_{j+1})$,

$E_j^p \cdot E_{j+1}|_{E_{j+1}} = \delta \cdot (\delta + j + \varepsilon_{j+1})$,

and $(E_{j+1})^2|_{E_{j+1}} = -\delta \cdot (\delta + j - 1 + \varepsilon_{j+1})$, *for some* $\varepsilon_{j+1} \geq 0$.

In particular,

$$(-xE_j^p - yE_{j+1}|_{E_{j+1}})^2 = (y - x)\delta(2y - (y - x)(\delta + j + 1 + \varepsilon_{j+1})).$$

Proof. Once we have

$$C_i^\varsigma \equiv -(\delta + i - 1 + \varepsilon_i)E_{i-1}^p - (\delta + i + \varepsilon_i)E_i|_{E_i}$$

and use induction on i, the computation is straightforward as above. In particular, $(E_j^p)^2|_{E_{j+1}}$ can be obtained from

$$\mathcal{N}_{E_j^p/X_{j+1}} = \pi_{j+1}^* \mathcal{N}_{E_j^\varsigma/X_j}(-E_j^p \cdot E_{j+1}).$$

Again the extra term ε_{j+1} results from a desingularization $\pi_j^\varsigma : X_j^\varsigma \to X_j$. □

Remark 4.5.9. Those who are familiar with notations $\mathfrak{s}$ and $\mathfrak{f}$ in [12, V.2.] can easily check that

$$\mathfrak{s} = E_{i-1}^p|_{E_i}$$

and

$$\mathfrak{f} = -(1/\delta)E^p_{i-1} - (1/\delta)E_i|_{E_i}.$$

In particular, $\mathfrak{s}$ denotes the section whose self-intersection number is negative, and $\mathfrak{f}$ stands for the fiber class.

Lemma 4.5.10. *On E_{j+1}, the divisor class $-xE^p_j - yE_{j+1}|_{E_{j+1}}$ is ample if and only if*

$$1 < \frac{y}{x} < \frac{\delta + j + 1 + \varepsilon_{j+1}}{\delta + j + \varepsilon_{j+1}}.$$

Moreover, the integral nef cone is generated by

$$-(\delta + j + \varepsilon_{j+1})E^p_j - (\delta + j + 1 + \varepsilon_{j+1})E_{j+1}|_{E_{j+1}}$$

and

$$-(1/\delta)E^p_j - (1/\delta)E_{j+1}|_{E_{j+1}}.$$

Proof. In the situation of Remark 4.5.9, apply [12, V. Proposition 2.20]. □

Now we are ready to complete the proof of [II] and [III] for $i = j + 1$.

Lemma 4.5.11. *Let ${C_{j+1}}'$ be the union of the LC-loci of (E_{j+1}, C_{j+1}) at q_α's. Then the curve ${C_{j+1}}'$ is irreducible and belongs to*

$$\Big| - (\delta + j + \varepsilon_{j+1})E^p_j - (\delta + (j + 1) + \varepsilon_{j+1})E_{j+1}\Big|_{E_{j+1}}$$

for some $\varepsilon_{j+1} \in \frac{1}{\delta}\mathbb{Z}_{\geq 0}$.

Proof. By the same argument in the proof of Lemma 4.3.5, we may assume that an 1-dimensional LC-locus is smooth in a neighborhood of α.

We can compute the numerically equivalent class of C_{j+1} on the ruled surface E_{j+1}. Indeed, by Lemma 4.6.4, we have

$$C_{j+1} \cong_{num} m_{j+1}\mathfrak{s} + (\delta(e + s_{j+1}) - \epsilon)\mathfrak{f}$$

for some $\epsilon \geq 0$. Then it is elementary to check that the induction hypothesis [III] satisfies the assumptions in Lemma 3.6.1. If ${C_{j+1}}' \cong_{\mathrm{num}} a\mathfrak{s} + b\mathfrak{f}$ for some $a \geq 2$ and b, then this would contradict Lemma 3.6.1. Therefore $a \leq 1$.

If ${C_{j+1}}'$ were reducible, then for some α, the LC-locus of $(E_{j+1}, C_{j+1}; q_\alpha)$ would be a fiber $\mathfrak{f}$ on the ruled surface E_{j+1}. Then by the monodromy argument as in Lemma 4.2.3, for all α, the LC-loci of $(E_{j+1}, C_{j+1}; q_\alpha)$ would be fibers. But again this would contradict Lemma 3.6.1. $\square$

Therefore all the statements for $i = j+1$ in [II] follow from Lemmas 4.5.6 and 4.5.11. They in turn give $c \geq \mathrm{lct}(E_{j+1}, C_{j+1}; q_\alpha) = \frac{1}{m_{j+2}}$. It remains to show $m_{j+2} \leq m_{j+1}$, but this is clear from construction (or see Lemma 4.6.1).

4.6 The Proof of Lemma 4.5.6

In this section we prove Lemma 4.5.6.

4.6.1 Bound of Multiplicity

In this subsection, we will compute upper bounds on the multiplicity and t-multiplicity of C_{j+1} at q_α.

Lemma 4.6.1. *The t-multiplicity (see Definition 3.1.2) of C_{j+1} at q_α is at most m_{j+1}.*

This is a local statement. Before we give a proof, we describe D_j with analytic local coordinates near $q_\alpha \in X_j$.

Notation 4.6.2. We define analytic coordinates (u, v, z) at $q_\alpha \in X_j$ as follows.

The set $\{z = 0\}$ locally defines E_j, and $\{u = 0\}$ corresponds to any fixed smooth surface which meets E_j transversally along the smooth curve $C_j^{\mathfrak{c}}$. We call another coordinate v.

To simplify notations, we let $m = m_{j+1}$. Since the multiplicity of $C_j^\varsigma \subset E_j$ is m, we may locally define D_j by an equation of the form $P(u,v)u^m + \sum_{s\geq 1} z^s Q_s(u,v)$ near q_α, where $P(u,v), Q_s(u,v) \in k\{\{u,v\}\}$.

Lemma 4.6.3. $\mathrm{mult}_0 P(0,v) \leq m$.

Proof. Note that $C_j = D_j|E_j$ is locally defined by $\{P(u,v)u^m = 0\}|_{\{z=0\}}$ and the LC-locus of $(E_j, C_j; q_\alpha)$ is C_j^ς, that is to say locally, the LC-locus of $\{P(u,v)u^m = 0\}$ on $\{z = 0\}$ is $\{u = z = 0\}$. Then Proposition 3.4.1 implies

$$\mathrm{mult}_0 P(0,v) = (C_j^\varsigma \cdot (C_j - mC_j^\varsigma))_{q_\alpha} \leq m.$$

□

Proof of Lemma 4.6.1. D being cone-like implies that every analytic branch of D_j and E_j meet transversally along $\{u = z = 0\}$. Then, for each $1 \leq s \leq m$, $z^s Q_s(u,v)$ contains a factor $z^s u^x$ where $x \geq m - s$. Because if $x < m - s$ then D_j and $E_j = \{z = 0\}$ would share some tangent vectors. So $Q_s(u,v) = u^{m-s} Q'_s(u,v)$ for $1 \leq s \leq m$, which means that in local coordinates, D_j is actually defined by

$$P(u,v)u^m + \sum_{1\leq s\leq m} z^s u^{m-s} Q'_s(u,v) + \sum_{s>m} z^s Q_s(u,v).$$

Since we carried out blowing up along $C_j^\varsigma = \{u = z = 0\}$ by

$$z \mapsto z_1, u \mapsto z_1 u_1 \text{ and } v \mapsto v_1,$$

the pullback of D_j locally becomes

$$P(z_1u_1, v_1)(z_1u_1)^m + \sum_{s=1}^{m} (z_1)^s (z_1u_1)^{m-s} Q'_s(z_1u_1, v_1) + (z_1)^{m+1} R$$

$$= (z_1)^m \Big[P(z_1u_1, v_1)(u_1)^m + \sum_{s=1}^{m} (u_1)^{m-s} Q'_s(z_1u_1, v_1) + (z_1)^1 R \Big],$$

where E_{j+1} locally corresponds to $\{z_1 = 0\}$. Hence $C_{j+1}(= D_{j+1}|E_{j+1})$ is locally defined by

$$(\ddagger) \qquad P(0,v_1)(u_1)^m + \sum_{1 \le s \le m} (u_1)^{m-s} Q'_s(0,v_1).$$

The Lemma 4.6.3 implies that $\mathrm{mult}_0 P(0,v_1) \le m$.

Consider the blow up $\widetilde{E_{j+1}} \to E_{j+1}$ of E_{j+1} at $q_\alpha = \{u_1 = v_1 = 0\}$ and let F be the exceptional curve. Then the intersection multiplicity at any point $x \in F$ between F and the proper transform C^p_{j+1} of C_{j+1} is less than or equal to

$$\max\{\mathrm{mult}_0 P(0,v_1), \mathrm{mult}_0 (u_1)^{m_{j+1}}\} = m = m_{j+1}.$$

Note that the result does not depend on the choice of coordinates. □

Before we give a bound on $\mathrm{mult}_{q_\alpha} C_{j+1}$, it will be more convenient to describe the numerically equivalent class of C_{j+1} in terms of $\mathfrak{s}$ and $\mathfrak{f}$.

Lemma 4.6.4. *Let E' be the union of proper transforms of exceptional loci $Exc(\pi_i^{\varsigma})$ $(i = 0, ..., j)$, counting multiplicities appearing in $\sigma^*_{j+1}D$. Then*

$$C_{j+1} \cong_{num} \Big| -(e+s_j)E^p_j - (e+s_{j+1})E_{j+1} - E' \Big|_{E_{j+1}}.$$

Equivalently,

$$C_{j+1} \cong_{num} m_{j+1}\mathfrak{s} + (\delta(e+s_{j+1}) - \epsilon)\mathfrak{f}$$

for some $\epsilon \ge 0$.

Proof. Since the induction hypothesis [IV] implies

$$\sigma^*_{j+1}D = D_{j+1} + \sum_{i=0}^{j-1}(e+s_i)E^p_i + (e+s_j)E^p_j + (e+s_{j+1})E_{j+1} + E',$$

we have

$$\begin{aligned} C_{j+1} = D_{j+1}|E_{j+1} &\in \Big|\sigma^*_{j+1}D - \sum_{i=0}^{j-1}(e+s_i)E^p_i - (e+s_j)E^p_j - (e+s_{j+1})E_{j+1} - E'\Big|_{E_{j+1}} \\ &= \Big| -(e+s_j)E^p_j - (e+s_{j+1})E_{j+1} - E'\Big|_{E_{j+1}}, \end{aligned}$$

where the last equality follows from the induction hypothesis [IV].

Then Lemma 4.5.9 implies

$$\begin{aligned} C_{j+1} &\in \left|(e+s_{j+1}-e-s_j)\mathfrak{s}+\delta(e+s_{j+1})\mathfrak{f}-E'_{E_{j+1}}\right| \\ &= \left|m_{j+1}\mathfrak{s}+\delta(e+s_{j+1})\mathfrak{f}-E'\cap E_{j+1}\right|. \end{aligned}$$

Since $E'\cap E_{j+1}$ is the union of some fibers, we get the last statement. □

We will need later the following result.

Lemma 4.6.5. *We have*

$$\mathrm{mult}_{q_\alpha} C_{j+1} \le m_{j+1} + \frac{4\delta}{r} m_{j+1}.$$

Proof. Let $m = m_{j+1}$. By Lemma 4.6.4 and the induction hypothesis [II], we have

$$C_j = -(e+s_{j-1})E_{j-1} - (e+s_j)E_j|_{E_j} - \epsilon\mathfrak{f},$$

for some $\epsilon \ge 0$, and

$$C_j{}' = -(\delta+j-1+\varepsilon_j)E^p_{j-1} - (\delta+j+\varepsilon_j)E_j|_{E_j}.$$

Then

$$
\begin{aligned}
&(C_j - m{C_j}') \cdot {C_j}' \\
&= \; mC_j \cdot {C_j}' - m({C_j}')^2 \\
&\leq \; (-(e+s_{j-1})E^p_{j-1} - (e+s_j)E_j) \cdot (-(\delta+j-1+\varepsilon_j)E^p_{j-1} - (\delta+j+\varepsilon_j)E_j)|_{E_j} \\
&\quad - m(-(\delta+j-1+\varepsilon_j)E^p_{j-1} - (\delta+j+\varepsilon_j)E_j)^2|_{E_j} \\
&= \; -(e+s_{j-1})(\delta+j-1+\varepsilon_j)\delta(\delta+j+\varepsilon_j) \\
&\quad + (e+s_{j-1})(\delta+j+\varepsilon_j)\delta(\delta+j-1+\varepsilon_j) \\
&\quad + (e+s_j)(\delta+j-1+\varepsilon_j)\delta(\delta+j-1+\varepsilon_j) \\
&\quad - (e+s_j)(\delta+j+\varepsilon_j)\delta(\delta+j-2+\varepsilon_j) \\
&\quad - m\delta(2(\delta+j+\varepsilon_j) - (\delta+j+\varepsilon_j)) \qquad \text{by Lemma 4.5.8} \\
&= \; (e+s_j)(\delta) - m(\delta)(\delta+j+\varepsilon_j) \\
&\leq \; m(\delta)(\delta+j+4) - m(\delta)(\delta+j+\varepsilon_j) \qquad \text{by induction hypothesis [III]} \\
&\leq \; 4\delta m.
\end{aligned}
$$

Since we may assume singularities at q_α are almost-equisingular, the intersection multiplicity at each q_α of $(C_j - m{C_j}')$ and ${C_j}'$ is no greater than $\frac{4\delta m}{r}$. Using notations above, we have

$$\mathrm{mult}_0 P(0, v_1) = \mathrm{mult}_0 P(0, v) = (C'_j \cdot (C_j - m{C_j}'))_{q_\alpha} \leq \frac{4\delta}{r} m_{j+1}.$$

So

$$\mathrm{mult}_{q_\alpha} C_{j+1} \leq \mathrm{mult}_0 P(0, v_1)(u_1)^{m_{j+1}} \leq m_{j+1} + \frac{4\delta}{r} m_{j+1}.$$

□

4.6.2 Completion of the Proof of Lemma 4.5.6

We complete the Proof of Lemma 4.5.6.

Lemma 4.6.6. *Suppose that the LC-loci of* (E_{j+1}, C_{j+1}) *at* q_α *are 0-dimensional. Suppose that*

$$C_{j+1} = u \cdot C'_{j+1} + C''_{j+1},$$

where C'_{j+1} *is a reduced effective divisor passing through* $q_1, \cdots, q_r$*. Then*

$$u < 0.7 m_{j+1}.$$

Proof. Suppose to the contrary that $u \geq 0.7 m_{j+1}$.

It is elementary to check that the induction hypothesis [III] satisfies the assumptions in Lemma 3.6.1. Then, by Lemma 3.6.1, we have

$$C'_{j+1} \cong_{\text{num}} \mathfrak{s} + x\mathfrak{f}$$

for some $x \geq \delta(\delta + j + 1)$.

We will compute the intersection number of C'_{j+1} and C''_{j+1} This is a merely elementary computation.

$$\begin{aligned}
& C'_{j+1} \cdot C''_{j+1} \\
&= C'_{j+1} \cdot (C_{j+1} - u \cdot C'_{j+1}) \\
&= (\mathfrak{s} + x\mathfrak{f}) \cdot ((m_{j+1} - u)\mathfrak{s} + (\delta(e + s_{j+1}) - \epsilon - ux)\mathfrak{f}) \\
&= -(m_{j+1} - u)\delta \cdot (\delta + j + 1) + (m_{j+1} - 2u)x + \delta(e + s_{j+1}) - \epsilon \\
&\leq -(m_{j+1} - u)\delta(\delta + j + 1) + (m_{j+1} - 2u)x + \delta(e + s_{j+1}) \\
&\leq -(m_{j+1} - u)\delta(\delta + j + 1) + (m_{j+1} - 2u)\delta(\delta + j + 1) + \delta(e + s_{j+1}) \\
&= -u\delta \cdot (\delta + j + 1) + \delta(e + s_{j+1}) \\
&= \delta\Big[(e + s_{j+1}) - u(\delta + j + 1)\Big]
\end{aligned}$$

On the other hand, Lemma 4.6.4 implies

$$u \leq m_{j+1}.$$

Multiplying by $(\frac{r}{4\delta} - \frac{d+j+5}{d+j+1} - 1)(d+j+1)$ gives

$$u\left(\frac{r}{4\delta} - \frac{d+j+5}{d+j+1} - 1\right)(d+j+1) \;\leq\; \left(\frac{r}{4\delta} - \frac{d+j+5}{d+j+1} - 1\right) m_{j+1}(d+j+1).$$

Then

$$\begin{aligned}
& u\Big[\frac{r}{4\delta}(d+j+1)\left(1 - \frac{d+j+5}{d+j+1}\frac{4\delta}{r}\right) - (\delta+j+1)\Big] \\
\leq\;& u\Big[\frac{r}{4\delta}(d+j+1)\left(1 - \frac{d+j+5}{d+j+1}\frac{4\delta}{r}\right) - (d+j+1)\Big] \\
\leq\;& \frac{r}{4\delta}(d+j+1)\left(1 - \frac{d+j+5}{d+j+1}\frac{4\delta}{r}\right) m_{j+1} + 4m_{j+1} - m_{j+1}(d+j+4) - m_{j+1},
\end{aligned}$$

equivalently

$$m_{j+1}(d+j+4)+m_{j+1}-u(\delta+j+1) \leq \frac{r}{\delta}\Big[\frac{m_{j+1}-u}{4}(d+j+1)\left(1 - \frac{d+j+5}{d+j+1}\frac{4\delta}{r}\right)+\frac{4\delta}{r}m_{j+1}\Big]$$

or

$$\delta\Big[m_{j+1}(d+j+4)+m_{j+1}-u(\delta+j+1)\Big] \leq r\Big[\frac{m_{j+1}-u}{4}(d+j+1)\left(1 - \frac{d+j+5}{d+j+1}\frac{4\delta}{r}\right)+\frac{4\delta}{r}m_{j+1}\Big].$$

Hence

$$\begin{aligned}
& C'_{j+1}\cdot C''_{j+1} \\
\leq\;& \delta\Big[(e+s_{j+1}) - u(\delta+j+1)\Big] \\
=\;& \delta\Big[(e+s_j) + m_{j+1} - u(\delta+j+1)\Big] \\
<\;& \delta\Big[m_{j+1}(d+j+4) + m_{j+1} - u(\delta+j+1)\Big] \qquad \text{by induction hypothesis [III]} \\
\leq\;& r\Big[\frac{m_{j+1}-u}{4}(d+j+1)\left(1 - \frac{d+j+5}{d+j+1}\frac{4\delta}{r}\right) + \frac{4\delta}{r}m_{j+1}\Big] \\
=\;& r\Big[\frac{(m_{j+1}-u)\left(1 - \frac{1}{m_{j+1}}\frac{d+j+5}{d+j+1}\frac{4\delta}{r}m_{j+1}\right)}{\frac{1}{m_{j+1}}\frac{d+j+5}{d+j+1}m_{j+1} - 1} + \frac{4\delta}{r}m_{j+1}\Big] \\
<\;& r\Big[\frac{(m_{j+1}-u)(1 - l\frac{4\delta}{r}m_{j+1})}{lm_{j+1}-1} + \frac{4\delta}{r}m_{j+1}\Big] \qquad \text{by Lemma 4.5.5 } (l := \mathrm{lct}(E_{j+1}, C_{j+1}; q_\alpha)) \\
\leq\;& r\Big[\frac{(t-u)(1-l(m-t))}{lt-1} + m - t\Big] \qquad (m := \mathrm{mult}_{q_\alpha} C_{j+1},\; t := \text{t-mult}_{q_\alpha} C_{j+1}),
\end{aligned}$$

but this contradicts Proposition 3.9.2. $\square$

Finally we complete the proof of Lemma 4.5.6.

Proof of Lemma 4.5.6. Suppose to the contrary that the LC-loci of (E_{j+1}, C_{j+1}) at q_α are 0-dimensional. Then Lemma 4.6.6 implies that $\mathcal{J}(E_{j+1}, \frac{1}{0.7m_{j+1}} C_{j+1})_{q_\alpha}$ is trivial or co-supported at q_α for each point q_α. Applying Lemma 3.6.2, we get

$$\chi\left(K_{E_{j+1}} + \lceil \frac{1}{0.7m_{j+1}} m_{j+1} \rceil \mathfrak{s} + \Big(\delta(\delta + j + 1 + \varepsilon_{j+1}) + \lceil \frac{1}{0.7m_{j+1}} (\delta(e + s_{j+1} - \epsilon) \rceil\Big) \mathfrak{f}\right)$$

$$> \sum_{\alpha=1}^{r} \dim_{\mathbb{C}} \Big(\mathcal{O}_{E_{j+1}, q_\alpha} / \mathcal{J}(E_{j+1}, \frac{1}{0.7m_{j+1}} C_{j+1})_{q_\alpha}\Big),$$

but this contradicts Proposition 3.8.2.

More precisely, since Lemma 4.5.8 provides all the intersection numbers on E_{j+1}, the left hand side of the above inequality can be explicitly computed by Riemann-Roch formula, and is approximately equal to

$$\left(K_{E_{j+1}} + \lceil \frac{1}{0.7m_{j+1}} m_{j+1} \rceil \mathfrak{s} + \Big(\delta(\delta + j + 1 + \varepsilon_{j+1}) + \lceil \frac{1}{0.7m_{j+1}} (\delta(e + s_{j+1} - \epsilon) \rceil\Big) \mathfrak{f}\right)$$

$$\cdot \left(\lceil \frac{1}{0.7m_{j+1}} m_{j+1} \rceil \mathfrak{s} + \Big(\delta(\delta + j + 1 + \varepsilon_{j+1}) + \lceil \frac{1}{0.7m_{j+1}} (\delta(e + s_{j+1} - \epsilon) \rceil\Big) \mathfrak{f}\right),$$

which is less than

$$3\delta(\delta + j + 1).$$

On the other hand, Proposition 3.8.2 implies that the right hand side is greater than

$$r \frac{\big(2m_{j+1} - (m_{j+1} + \frac{4\delta}{r} m_{j+1})\big)\big(\frac{1}{0.7m_{j+1}} - l\big)}{l m_{j+1} - 1} \approx \frac{0.3}{4} r(d + j + 1).$$

Since $r \approx \mathcal{O}(d^2) \gg d \approx \delta$, it is easy to see

$$3\delta(\delta + j + 1) < \frac{0.3}{4} r(d + j + 1)$$

for sufficiently large r, which is a contradiction. $\square$

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