ERRATA

Page	Line	Should Read
(iii)	9	$u = (u^1, \ldots, u^m)$
1	10	
8	1. f. b.	existence
10	6	$\tilde{f}(t, x, u)$
20	8	$\ldots a + (\bar{t}, \bar{x}), \ldots$
20	9	$\delta > 0$
23	13 f. b.	(ix)
	8 f. b.	(x)
	2 f. b.	$f_o + f_o$
24	7	(xi)
	7 f. b.	Statement (x) applies
25	11	Fillipov
35	2	$\lvert y(t) - y(t') \rvert \leq \epsilon/2, \ldots$
39	3 f. b.	$(t_1, x \lvert t_1), t_2, x(t_2))$
42	2 f. b.	. . . when A is not compact. . .
43	8 f. b.	i is finite
	2 f. b.	$t_{1k} \leq t \leq t_{2k}$
45	8 f. b.	$x_k{}^o(t) = \int_{t_{1k}}^{t_{2k}} f_o(T, X_k(T), U_k(T) \lvert dT, t_{1k} \leq t \leq t_{2k}$
46	7	$(y_k^+ - y_k^-) + y_k^- = x_k^o + y_k^-$
55	13	. . . and (ix) of No. 5.

Page	Line	Should Read
55	1 f. b.	$\ldots 0 \leq \zeta < +\infty$
57	6	By lemma (x_i) of . . .
61	4	. . . condition (α) implies
	5	. . . show that (α) is . . .
64	10 f. b.	immediate
74	4 f. b.	. . . , $C_0 \geq 0$, $D_0 \geq 0$, with $\phi(\xi)$ non-
79	3	$\gamma_i \geq \phi_i(\beta_i)$.
83	5 f. b.	$\int_{t_{1k}}^{t_{2k}} [f_0(t, x_k(t), u_k(t)) + \bar{\mu}]dt$
92	10 f. b.	. . . , and infinitely . . .
118	9	$x^2u^2 + u$
	10	for $u \not\leq 0$, . . .
121	2	. . . Linear in u
125	4 f. b.	$\ldots (1 - \alpha_k)u_k''] + \ldots$
132	5 f. b.	conditions (b) and (d) hold.
133	7 f. b.	from Chapter II, as is the case for A closed and contained in a slus as above.
	6 f. b.	. . . A closed, but not contained . . .
134	11 f. b.	. . . A is closed, but not contained . . .
	8, 9, 10 f. b.	Delete these lines.
135	9	conditions (b) and (d) hold.
138	4 f. b.	. . . Liapunov . . .

THE UNIVERSITY OF MICHIGAN

SYSTEMS ENGINEERING LABORATORY

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College of Engineering

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EXISTENCE ANALYSIS FOR OPTIMAL CONTROL PROBLEMS WITH EXCEPTIONAL SETS

by

James R. La Palm

November 1967

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ABSTRACT

The purpose of this thesis is to prove by direct methods existence theorems for optimal solutions in problems of optimal control and the calculus of variations. Indeed in each existence theorem we prove the existence of at least one element, in any given nonempty complete class Ω of admissible elements, which minimizes a given cost functional

$$I[x,u] = \int_{t_1}^{t_2} f_o(t, x(t), u(t))dt.$$

An admissible element is here a pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, of vector valued functions $x(t) = (x^1, \ldots, x^n)$, $u(t) = (u^1, \ldots, u^n)$, $x(t)$ a trajectory and $u(t)$ a control function, or strategy, for which the following requirements are made: (a) $x(t)$ is absolutely continuous in $[t_1, t_2]$; (b) $u(t)$ is measurable in $[t_1, t_2]$; (c) the pair x, u satisfies a given system of ordinary differential equations

$$dx^i/dt = f_i(t, x(t), u(t)), \qquad i=1, \ldots, n, \qquad \text{or}$$

$$dx/dt = f(t, x(t), u(t)), \qquad t_1 \leq t \leq t_2,$$

in the sense of Carathéodory, $f(t, x, u) = (f_1, \ldots, f_n)$ being a given vector function; (d) $x(t)$ satisfies a constraint on the time and space variables t and x of the form $(t, x(t)) \in A$ for all $t \in [t_1, t_2]$, where A is a given fixed subset of the tx-space $E_1 \times E_n$; (e) $u(t)$ satisfies a

As Cesari proved his Theorem I in 1966 by finally extending to Lagrange problems a Tonelli-Nagumo Theorem (1915-29) for free problems and n=1 together with Filippov's statement (1959) for Pontryagin problems, so our Theorems II, III and IV extend to Lagrange problems analogous theorems of Tonelli for free problems and n=1.

In Chapter III we deduce from Theorems II, III and IV as corollaries analogous existence theorems for problems with f linear in u.

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INTRODUCTION

The purpose of this thesis is to prove by direct methods existence theorems for optimal solutions in problems of optimal control and the calculus of variations. Indeed in each existence theorem we prove the existence of at least one element, in any given nonempty complete class Ω of admissible elements, which minimizes a given cost functional

$$I[x, u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t))dt.$$

An admissible element is here a pair $x(t), u(t)$, $t_1 \leq t \leq t_2$, of vector valued functions $x(t) = (x^1, \ldots, x^n)$, $u(t) = (u^1, \ldots, u^n)$, $x(t)$ a trajectory and $u(t)$ a control function, or strategy, for which the following requirements are made: (a) $x(t)$ is absolutely continuous in $[t_1, t_2]$; (b) $u(t)$ is measurable in $[t_1, t_2]$; (c) the pair x, u satisfies a given system of ordinary differential equations

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in the sense of Carathéodory, $f(t, x, u) = (f_1, \ldots, f_n)$ being a given vector function; (d) $x(t)$ satisfies a constraint on the time and space variables t and x of the form $(t, x(t)) \in A$ for all $t \in [t_1, t_2]$, where A is a given fixed subset of the tx-space $E_1 \times E_n$; (e) $u(t)$ satisfies a constraint of the form $u(t) \in U(t, x(t))$ for almost all $t \in [t_1, t_2]$, where $U(t,x)$ is a given subset of the u-space E_m, and where $U(t,x)$ may depend on both t and x; (f) the pair x, u is such that $f_o(t, x(t), u(t))$ is L-integrable in $[t_1, t_2]$. Usually, complete classes Ω of admissible pairs are obtained by considering all admissible pairs x(t), u(t), $t_1 \leq t \leq t_2$, whose trajectories x(t) satisfy given boundary conditions of a rather general type. The boundary conditions can be written in the McShane's form $(t_1, x(t_1), t_2, x(t_2)) \in B$, where B is a given closed subset of the $t_1x_1t_2x_2$-space E_{2n+2}.

In Chapter I, for the convenience of the reader and for the necessary references, we repeat with variants an Existence Theorem I due to Cesari (1966), where a growth condition with respect to u on the scalar function f_o is assumed.

Our results are contained in Chapters II and III. In Chapter II we first prove an Existence Theorem II which is equivalent to Theorem I and in which the growth condition is expressed in terms of a uniform limit $f_o/|u| \to +\infty$ as $|u| \to +\infty$. By means of counter-examples we show the need of the uniformity in our statement II. In the same Chapter II we prove then Existence Theorems III and IV, where we show that the growth condition can be relaxed in a closed exceptional subset E of A provided either an additional condition is satisfied at every point of E (Theorem III), or no additional condition is satisfied but E is slender according to a suitable definition

(Theorem IV).

As Cesari [4a] proved his Theorem I in 1966 by finally extending to Lagrange problems a Tonelli-Nagumo Theorem (1915-29) [17, 23a] for free problems and $n=1$ together with Filippov's statement (1959) for Pontryagin problems [8], so our Theorems II, III and IV extend to Lagrange problems analogous theorems of Tonelli for free problems and $n=1$. Theorems II and IV which had been proved by Tonelli for free problems, $n=1$, and f_0 of class C^1, had been extended by L. Turner [24] to free problems, any $n \geq 1$, and f_0 of class C^0. Our extensions to Lagrange problems are also of class C^0. We owe to L. Turner the concept of slenderness we use in Theorem IV. In Chapter III we deduce from Theorems II, III and IV as corollaries analogous existence theorems for problems with f linear in u.

As in Cesari's work we assume A to be closed, and $U(t,x)$ also closed for every $(t,x) \in A$ (though not necessarily compact) and satisfying Kuratowski's condition of upper semicontinuity (property (U)). Analogously, for the sets $Q(t,x)$ and $\tilde{Q}(t,x)$, which are the images of $U(t,x)$ under f and $\tilde{f} = (f_0, f)$, and for the derived sets $\tilde{\tilde{Q}}(t,x)$, we assume with Cesari that the modified Kuratowski upper semicontinuity condition for convex sets is satisfied (property (Q)). As in Tonelli's and Cesari's works we first prove the existence theorems for A compact, and then we extend them to the case of A closed under usual additional assumptions.

Our proof of Theorem II in Chapter II is given by means of five lemmas. By these lemmas we prove first the equivalence of the conditions in Theorems I and II, and then the statement that the uniformity requested in our Theorem II need not be explicitly verified for free problems, since for these problems the uniformity is a consequence of the remaining hypotheses. This last statement is relevant since in the corresponding Tonelli's Theorem II such a uniformity was not requested. In such a way we can conclude that our Theorem II contains as a particular case the corresponding Tonelli statement for free problems and $n=1$, as well as Turner's statement for free problems and any $n \geq 1$.

In order to describe our proofs of Theorems III and IV we must first summarize Cesari's scheme for the proof of Theorem I. This proof of Cesari is a modification of the usual direct method in the calculus of variations for free problems, in the sense that an application of Helly's Theorem and suitable modifications of a closure theorem due to A. F. Filippov replace Tonelli's lower semi-continuity argument. We describe Cesari's proof only for the case A compact.

The first step is to show that the infinium i of $I[x, u]$ is finite, where the infimum is taken over all pairs x, u of the given complete class Ω of admissible pairs. Thus, there exists a minimizing sequence $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, of pairs in Ω, that is, a sequence such that $I[x_k, u_k] \to i$ as $k \to +\infty$. The second step consists in showing that the trajectories $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\dots$, of such a minimizing sequence are equiabsolutely continuous and also equibounded. Then, Ascoli's Theorem guarantees the existence of some subsequence of integers k and of some continuous vector function $x(t)$, $t_1 \leq t \leq t_2$, such that $t_{1k} \to t_1$, $t_{2k} \to t_2$ and $x_k(t) \to x(t)$ in an obvious modification of the uniform topology as $k \to +\infty$ along the extracted subsequence. The equiabsolute continuity of the trajectories $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\dots$, and the uniform convergence $x_k \to x$ guarantee then that $x(t)$ is also absolutely continuous in $[t_1, t_2]$.

The third step concerns the sequence $x_k^o(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\dots$, with

$$x_k^o(t) = \int_{t_{1k}}^{t} v_k(t)dt = \int_{t_{1k}}^{t} f_o(\tau, x_k(\tau), u_k(\tau)d\tau$$

for which no equicontinuity can be proved at this point, neither under the conditions of Theorem I, nor under the conditions of Theorems II, III and IV. Instead of the functions $x_k^o(t)$, the two functions

$$Y_k^-(t) = -\int_{t_{1k}}^{t} v_k^-(t)dt, \quad Y_k^+(t) = \int_{t_{1k}}^{t} v_k^+(t)dt, \quad t_{1k} \leq t \leq t_{2k},$$

are taken into consideration, where as usual

$$v_k^+ = (|v_k| + v_k)/2, \quad v_k^- = (|v_k| - v_k)/2,$$

and hence

$$v_k^+, v_k^- \geq 0, \quad v_k = v_k^+ - v_k^-, \quad |v_k| = v_k^+ + v_k^-$$

and

$$x_k^o(t) = Y_k^-(t) + Y_k^+(t), \quad t_{1k} \leq t \leq t_{2k}, \quad k=1,2,\ldots$$

Now the functions $Y_k^-(t)$ are monotone nonincreasing and uniformly Lipschitzian, and the functions $Y_k^+(t)$ are nonnegative, monotone nondecreasing and uniformly bounded. By applying Helly's Theorem to the sequence $Y_k^+(t)$ and then Ascoli's Theorem to $Y_k^-(t)$, further successive extractions are obtained such that $x_k^o(t) = Y_k^-(t) + Y_k^+(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\ldots$, converges for each t, $t_1 < t < t_2$, toward $x^o(t) = Y(t) + Z(t)$, where $Y(t)$ is a (scalar) absolutely continuous function, and $Z(t)$ is a nonnegative monotone nondecreasing function with $Z'(t) = 0$ almost everywhere in $[t_1, t_2]$, and $Y(t_1) = Z(t_1) = 0$. Since

$$Y_k^+(t_{2k}) + Y_k^-(t_{2k}) = x_k^o(t_{2k}) = I[x_k, u_k] \to i \quad \text{as } k \to +\infty,$$

then by taking the limit along the last extracted subsequence one obtains $Y(t_2) + Z(t_2) = i$ where $Z(t_2) \geq 0$, and hence $Y(t_2) \leq i$.

The fourth step consists in applying a closure theorem which is a generalization by Cesari of one due to A. F. Filippov. This guarantees the existence of a measurable control $u(t)$, $t_1 \leq t \leq t_2$, such that $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, is an admissible pair for the problem, and $I[x,u] = Y(t_2) \leq i$.

The fifth and final step consists in a simple application of the completeness property of Ω in order to conclude that the pair x, u belong to the class Ω and hence $i \leq I[x, u]$. Therefore, $I[x, u] = i$, and the proof is complete, in the case A is compact.

The proof of Existence Theorem III in Chapter II repeats essentially the same steps, but the proof that the trajectories $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\ldots$, are equiabsolutely continuous is much more complicated based as it is on the growth property at the points not on the exceptional set E and on the assumed additional property at the points of E. This part of the proof is inspired by the corresponding part of the analogous theorem of Tonelli for free problems.

The proof of Existence Theorem IV in Chapter II also differs from Cesari's proof in step two. First the trajectories $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\ldots$, are proved to be equicontinuous and of uniform bounded variation. Thus the limit vector function given by Ascoli's Theorem is continuous and of bounded variation. Finally a proof by contradiction shows that x(t) is also an absolutely continuous vector function. The other steps in the proof are essentially the same as in Cesari's proof. The long and difficult argument replacing step two again is modeled on the corresponding part of the analogous argument of L. Turner for free problems.

Theorems I to IV extend the analogous theorems of Tonelli for free problems with the exception of Theorem IV where an additional hypothesis is being made.

Although the present work concerns only usual solutions, we can say that existence theorems for generalized solutions can easily be derived from Theorems I to IV. Indeed, as pointed out by R. V. Gamkrelidze [9], the generalized solutions of a given problem can be thought of as usual solutions of an analogous problem obtained by a suitable relaxation of the given problem and with an enlarged set of control variables. Thus, as shown by Cesari [4b] for Theorem I, also Theorems II, III and IV yield analogous existence theorems for weak solutions. Generalized solutions, which had been introduced for free problems by L. C. Young [27], have been studied extensively for free and for Lagrange problems by E. J. McShane [16], J. Warga [25], A. Plis [19], R. V. Gamkrelidze [9], and Cesari [4b].

There are various other approaches for obtaining existence theorems for problems of optimal control and the calculus of variations. A few of these are briefly described below. Nonlinear existence theorems have been proven by E. O. Roxin [22] who employed the concept of the attainable set and also by E. Lee and L. Markus [14]. L. Neustadt [18] has proven an existence theorem for the Pontryagin problem, where the functions f and f_0 are assumed to be linear in the state x and U is a fixed compact subset of the u-space E_m. In an existance theorem due to L. W. Neustadt no convexity condition is required. For the proof he employs a theorem by A. Lyapunov on the convexity of the range of a vector measure, which was previously used by H. Halkin [10] to derive necessary conditions for optimal control problems, and the concept of the attainable set, which was previously used by E. O. Roxin [22]. D. Blackwell [2], H. Chernoff [5], H. Hermes [12] and P. R. Halmos [11] have given alternate proofs or extensions of the theorem due to A. Lyapunov on the convexity of the range of a vector measure. A. V. Balakrishnan [1] has treated optimal control problems in abstract function spaces. E. H. Rothe [21] has proven an existence theorem for multidimensional free problems of the calculus of variation by utilizing Sobolev's imbedding theorems.

For problems of optimal control with U(t) depending on t only, L. S. Pontryagin [20] gave his now famous principle of maximum as a wide ranging necessary condition. This principle has been extended in many ways by R. V. Gamkrelidze, H. Halkin, L. Neustadt, and others.

Chapter I

Statement of the Optimal Control Problem, Closure Theorems, and Existence Theorem I

1. Usual solutions.

Let A be a closed subset of the tx-space $E_1 \times E_n$, $t \in E_1$, $x = (x^1, \dots, x^n) \in E_n$, and for each $(t, x) \in A$, let $U(t, x)$ be a closed subset of the u-space E_m, $u=(u^1, \dots, u^m)$. We do not exclude that A coincides with the whole tx-space and that U coincides with the whole u-space. Let M denote the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$. Let $f(t, x, u) = (f_o, f) = (f_o, f_1, \dots, f_n)$ be a continuous vector function from M into E_{n+1}. Let B be a closed subset of points (t_1, x_1, t_2, x_2) of E_{2n+2}, $x_1 = (x_1^1, \dots, x_1^n)$, $x_2 = (x_2^1, \dots, x_2^n)$. We shall consider the class of all pairs $x(t), u(t)$, $t_1 \leq t \leq t_2$, of vector functions $x(t), u(t)$ satisfying the following conditions:

(a) $x(t)$ is absolutely continuous (AC) in $[t_1, t_2]$;

(b) $u(t)$ is measurable in $[t_1, t_2]$;

(c) $(t, x(t)) \in A$ for every $t \in [t_1, t_2]$;

(d) $u(t) \in U(t, x(t))$ almost everywhere (a. e.) in $[t_1, t_2]$;

(e) $f_o(t, x(t), u(t))$ is L-integrable in $[t_1, t_2]$;

(f) $dx/dt = f(t, x(t), u(t))$ a. e. in $[t_1, t_2]$;

(g) $(t_1, x(t_1), t_2, x(t_2)) \in B$.

By (f) we mean that the n ordinary differential equations

$$dx^i/dt = f_i(t, x(t), u(t)), \qquad i=1,2,\ldots,n, \tag{1}$$

are satisfied a. e. in $[t_1, t_2]$. Since x(t) is AC, that is, each component $x^i(t)$ of x(t) is AC, we conclude that all $f_i(t, x(t), u(t))$, $i=1,2,\ldots,n$, are L-integrable in $[t_1, t_2]$ as f_o.

A pair x(t), u(t) satisfying (abcdefg) is said to be admissible and for such a pair x(t) is called a trajectory and u(t), a strategy, control, or steering function. As usual, U(t, x) is said to be the control space at the time t and space point x. The functional

$$I[x, u] = \int_{t_1}^{t_2} f_o(t, x(t), u(t))dt \tag{2}$$

is called the cost functional, and we seek the minimum of I[x, u] in the total class Ω of admissible pairs x(t), u(t), or in some well defined subclass of Ω.

In the particular case where U(t, x) is a compact subset of E_m for every $(t, x) \in A$, the problem of the minimum of I[x, u] is called a <u>Pontryagin problem</u> of optimal control theory. The general case above, where U(t, x) is a closed but not necessarily compact subset of E_m for every $(t, x) \in A$ will be denoted as a <u>Lagrange problem with unilateral constraints</u> or as the optimal control problem. The classical Lagrange problem corresponds essentially to the case where

$U = E_m$ is the whole u-space, with the side conditions being here differential equations in normal form.

In the particular case in which $f_0=1$, then $I[x,u] = t_2 - t_1$, and the problem of minimization under consideration is then called a problem of minimum transfer time (from the state $x(t_1)$ to the state $x(t_2)$).

There is another particular case of the Lagrange problem which shall be taken into consideration, namely $m=n$, $U=E_m$, and the vector function $f(t,x,u)$ given by $f(t,x,u) = u$, or $f_i(t,x,u) = u^i$, $i=1,\dots,m=n$, and hence $\tilde{f}(t,x,u) = (f_0, u)$. Then the differential system (1) reduces to $dx^i/dt = u^i$, $i=1,\dots,n$, and the cost functional becomes

$$I[x] = \int_{t_1}^{t_2} f_0(t, x(t), x'(t))dt. \qquad (3)$$

This problem is called a free problem.

2. Generalized solutions.

Often a given problem has no optimal solution, but the mathematical problem and the corresponding concept of solution can be modified in such a way that an optimal solution exists and yet neither the system of trajectories, nor the corresponding values of the cost functional are essentially modified. The modified (or generalized) problem and its solutions are of interest in themselves, and have relevant physical interpretations. Essentially, we consider a finite system of distinct strategies which are thought of as being used at the same time according to some probability distribution.

Instead of considering the usual cost functional, differential equations and constraints

$$I[x, u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t))dt,$$

$$dx/dt = f(t, x(t), u(t)), \qquad f = (f_1, \ldots, f_n), \tag{4}$$

$$(t, x(t)) \in A, \qquad u(t) \in U(t, x(t)),$$

we consider a new cost functional, differential equations and constraints

$$J(x, p, v) = \int_{t_1}^{t_2} g_0(t, x(t), p(t), v(t))dt,$$

$$dx/dt = g(t, x(t), p(t), v(t)), \qquad g = (g_1, \ldots, g_n), \tag{5}$$

$$(t, x(t)) \in A, \quad v(t) \in V(t, x(t)), \quad p(t) \in \Gamma.$$

Precisely, $v(t) = (u^{(1)}, \ldots, u^{(\gamma)})$ represents a finite system of $\gamma \geq n+1$ ordinary strategies $u^{(1)}, \ldots, u^{(\gamma)}$, each $u^{(j)}$ having its values in $U(t, x(t)) \subset E_m$. Thus, we think of $v = (u^{(1)}, \ldots, u^{(\gamma)})$ as a vector variable whose γ components $u^{(1)}, \ldots, u^{(\gamma)}$, are themselves vectors u with values in $U(t, x)$. In other words

$$v = (u^{(1)}, \ldots, u^{(\gamma)}), \quad u^{(j)} \in U(t, x), \quad j=1, \ldots, \gamma,$$

or

$$v \in V(t, x) = [U(t, x)]^{\gamma} = U \times \ldots \times U \subset Em\gamma, \qquad (6)$$

where the last term is the product space of U by itself taken γ times, and thus V is a subset of the Euclidean space $E_{m\gamma}$. In (5) $p = (p_1, \ldots, p_\gamma)$ represents a probability distribution. Hence, p is an element of the simplex Γ of the Euclidean space E_γ defined by $p_j \geq 0$, $p_1 + \ldots + p_\gamma = 1$. Finally, in (5) the new control variable is (p, v), with values $(p, v) \in \Gamma \times V(t, x) \subset E_{\gamma+m\gamma}$. In (5) $g = (g_1, \ldots, g_n)$, and all $g_0, g_1, \ldots, g_n$ are defined by

$$g_i(t, x, p, v) = \sum_{j=1}^{\gamma} p_j f_i(t, x, u^{(j)}), \qquad i=0, 1, \ldots, n. \qquad (7)$$

As usual we shall require that the functions p(t), v(t), $t_1 \leq t \leq t_2$, are measurable and that x(t), $t_1 \leq t \leq t_2$, is absolutely continuous. As in No. 1, we shall require as usual that x(t) satisfies boundary conditions of the type $(t_1, x(t_1), t_2, x(t_2)) \in B \subset E_{2n+2}$ where B is a given closed subset of E_{2n+2}. As in No. 1, we require $g_0(t, x(t), p(t), v(t))$ to be L-integrable in $[t_1, t_2]$.

We shall say that [p(t), v(t)] is a generalized strategy, that $p(t) = (p_1, \ldots, p_\gamma)$ is a probability distribution, and that

$v(t) = (u^{(1)}, \ldots, u^{(\gamma)})$ is a finite system of (ordinary) strategies. We shall say that x(t) is a generalized trajectory.

It is important to note that any (ordinary) strategy u(t) and corresponding (ordinary) trajectory x(t) (thus, satisfying (1)) can be interpreted as a generalized strategy and generalized trajectory, by taking $v(t) = (u^{(j)}(t),\ j=1, \ldots, \gamma)$ and $p(t) = (p_j(t),\ j=1, \ldots, \gamma)$ defined by $u^{(j)}(t) = u(t)$, $p_j(t) = 1/\gamma$, $j=1, \ldots, \gamma$. Then relations (5) reduce to relations (1).

Instead of the usual set M we shall now consider the set $N \subset E_{1+n+\gamma+m\gamma}$ of all (t, x, p, v) with $(t, x) \in A$, $p \in \Gamma$, $v \in V(t, x)$. As usual, we shall assume that A is a closed subset of $E_1 \times E_n$, and that $\tilde{f} = (f_0, f_1, \ldots, f_n)$ is a continuous vector function from M into E_{n+1}.

Under hypotheses which are often satisfied, any generalized trajectory can be approached as closely as we wish by means of usual solutions, and correspondingly the value of the cost J[x, p, v] can be approached as closely as we wish by the value of the usual cost I[x, u]. In this sense we shall understand that the usual solutions and the corresponding values of the cost functional are not essentially modified by the introduction of generalized solutions. The existence theorems of the present thesis apply to generalized as well as to usual solutions, provided g_0 and g are replaced for f_0 and f. More details concerning the application of Cesari's Existence

Theorem I to generalized solutions are given in [4b].

3. The distance function ρ.

If we denote by X the space of all continuous vector functions $x(t) = (x^1, \dots, x^n)$, $a \leqq t \leqq b$, from arbitrary finite intervals $[a, b]$ to E_n, it is convenient to define a distance function $\rho(x, y)$ for elements $x(t)$, $a \leqq t \leqq b$, and $y(t)$, $c \leqq t \leqq d$, of X, so as to make X a metric space. For this purpose we extend x(t) in all $(-\infty, +\infty)$ by defining it equal to x(a) for $t \leqq a$ and equal to x(b) for $t \leqq b$, and, analogously, for y(t). We then define

$$\rho(x, y) = |a-c| + |b-d| + \max |x(t) - y(t)|,$$

where the maximum is taken for all t, $-\infty < t < +\infty$. Then ρ is a distance function and X is a metric space.

Given functions $x_k(t)$, $a_k \leq t \leq b_k$, $k=1, 2, \dots$, and $x(t)$, $a \leq t \leq b$, we shall say therefore that $x_k \to x$ as $k \to \infty$ in the ρ-metric if $\rho(x_k, x) \to 0$ as $k \to \infty$. If the interval $[a, b]$ is fixed, then this reduces to the usual uniform convergence.

For any admissible pair $[x(t), u(t)]$ the trajectory x(t) is an element of X, but of course an element of X may not be the trajectory of an admissible pair.

A class Ω of admissible pairs is said to be complete provided it satisfies the following property: If $x_k(t), u_k(t)$, $t_{1k} \leqq t \leqq t_{2k}$, $k=1, 2, \dots$, and $x(t), u(t)$, $t_1 \leqq t \leqq t_2$, are all admissible pairs, if

$x_k(t) \to x(t)$ as $k \to \infty$ in the ρ-metric, and if all pairs $x_k(t), u_k(t)$, $k=1,2,\dots$, belong to Ω, then $x(t), u(t)$ also belongs to Ω. The classes usually taken into consideration in applications are complete. The class of all admissible pairs (satisfying (abcdefg)) is certainly complete.

4. Upper semicontinuity of variable sets.

Using the same notations of No. 1 we shall denote by $Q(t,x)$ the set

$$\begin{aligned} Q(t,x) &= f(t,x,U(t,x)) \\ &= [z \in E_n \,|\, z = f(t,x,u),\ u \in U(t,x)] \subset E_n. \end{aligned}$$

Here $Q(t,x)$ is the image in E_n of the set $U(t,x)$ in the mapping $U(t,x) \to E_n$ defined by $z = f(t,x,u)$, $u \in U(t,x)$. If f is continuous, as assumed in No. 1 and if $U(t,x)$ is compact for every $(t,x) \in A$, then also $Q(t,x)$ is compact.

Given any point $(t_o, x_o) \in A$ and $\delta > 0$, we denote by $N_\delta(t_o, x_o)$ the set of all $(t,x) \in A$ at a distance $\leq \delta$ from (t_o, x_o) and denote by $N^o_\delta(t_o, x_o)$ those points of A at a distance $< \delta$ from (t_o, x_o). The set $U(t,x)$ is said to be an upper semicontinuous function of (t,x) in A provided for every $(t_o, x_o) \in A$ there is a $\delta > 0$ such that

$$U(t,x) \subset [U(t_o, x_o)]_\epsilon$$

for every $(t,x) \in N_\delta(t_o, x_o)$, where U_ϵ denotes the closed ϵ-neighborhood of U in E_m.

If U(t, x) is compact for every $(t,x) \in A$ and an upper semicontinuous function of (t, x) in the closed set A, then Cesari [4a] has proven that Q(t, x) is also compact for every $(t,x) \in A$ and an upper semicontinuous function of (t, x) in A.

5. Properties (U) and (Q) of variable sets.

If E denotes any subset of E_n we shall denote by cl E the closure of E and by co E the convex hull of E. Thus cl co E denotes the closure of the convex hull of E, or briefly, the closed convex hull of E.

Let U(t, x), $(t,x) \in A$, be a variable subset of E_m. For every $\delta > 0$, let $U(t,x,\delta) = \cup U(t',x')$, where the union is taken for all $(t',x') \in N_\delta(t,x)$. We shall say that U(t, x) has property (U) at $(\bar{t},\bar{x}) \in A$ if

$$U(\bar{t},\bar{x}) = \bigcap_{\delta>0} \text{cl}\, U(\bar{t},\bar{x},\delta) = \bigcap_{\delta>0} \text{c}\ell \bigcup_{(t,x)\in N_\delta(\bar{t},\bar{x})} U(t,x).$$

We shall say that U(t, x), $(t,x) \in A$, has property (U) in A if U(t, x) has property (U) at every point $(t,x) \in A$. If a set U(t, x) has property (U), say at $(\bar{t},\bar{x})$, then obviously $U(\bar{t},\bar{x})$ is closed for it is the intersection of closed sets.

Let Q(t, x), $(t,x) \in A$, be a variable subset of E_n. For every

$(t,x) \in A$ and $\delta > 0$, let $Q(t,x,\delta) = \cup Q(t',x')$ where the union is taken for all $(t',x') \in N_\delta(t,x)$. We shall say that $Q(t,x)$ has property (Q) at $(\bar{t},\bar{x}) \in A$ if

$$Q(\bar{t},\bar{x}) = \bigcap_{\delta>0} \text{cl co } Q(\bar{t},\bar{x},\delta) = \bigcap_{\delta>0} \text{cl co} \bigcup_{(t,x)\in N_\delta(\bar{t},\bar{x})} Q(t,x).$$

We shall say that $Q(t,x)$ has property (Q) in A if $Q(t,x)$ has property (Q) at every point $(t,x) \in A$. If a set $Q(t,x)$ has property (Q), say at $(\bar{t},\bar{x})$, then obviously $Q(\bar{t},\bar{x})$ is closed and convex, as the intersection of closed and convex sets.

Property (U) is Kuratowski's concept of upper semicontinuity [13] used also by Choquet [6] and Michael [15].

The following statements (i) - (viii) and their proofs are given in [4a].

(i) If A is closed and $U(t,x)$ is any variable set which is a function of (t,x) in A and has property (U) in A, then the set M of all $(t,x,u) \in A \times E_m$ with $u \in U(t,x)$, $(t,x) \in A$ is closed.

(ii) If the set $U(t,x)$ is closed for each $(t,x) \in A$ and is an upper semicontinuous function of (t,x) in A, then $U(t,x)$ has property (U) in A.

Thus, for closed sets the upper semicontinuity property implies property (U) but the converse is not true, that is, the upper semicontinuity property for closed sets in more restrictive than property (U). This is shown by an example in [4a].

(iii) If A is compact, if U(t, x) is compact for every $(t,x) \in A$ and is an upper semicontinuous function of (t, x) in A, then M is compact.

(iv) Property (Q) at some $(\bar{t}, \bar{x})$ implies property (U) at the same $(\bar{t}, \bar{x})$, and

$$U(\bar{t},\bar{x}) = \bigcap_{\delta > 0} \text{cl co } U(\bar{t},\bar{x},\delta) = \bigcap_{\delta > 0} \text{cl } U(\bar{t},\bar{x},\delta)$$

$$= \bigcap_{\delta > 0} U(\bar{t},\bar{x},\delta).$$

Analogously, if U(t, x) has property (U) at (t, x), then

$$U(\bar{t},\bar{x}) = \bigcap_{\delta > 0} \text{cl } U(\bar{t},\bar{x},\delta) = \bigcap_{\delta < 0} U(\bar{t},\bar{x},\delta).$$

(v) If for each $(t,x) \in A$ the set U(t, x) is closed and convex, and U(t, x) is an upper semicontinuous function of (t, x) in A, then U(t, x) has property (Q) in A.

Let us now consider the sets $Q(t,x) = f(t,x,U(t,x))$, $(t,x) \in A$, $Q(t,x) \subset E_n$, which are the images of sets $U(t,x) \subset E_m$, for each $(t,x) \in A$.

The hypothesis that A is compact, that f is continuous on M, that U(t, x) has property (Q) [or (U)] in A, and that Q(t, x) is convex for each $(t,x) \in A$ does not imply that Q(t, x) has property (Q) [or (U)] in A. Even the stronger hypothesis that A is compact, that f is continuous on M, that U(t, x) has property (Q) in A, and that Q(t, x) is compact and convex for each $(t,x) \in A$, does not imply that Q(t, x) has property (Q) in A. These statements are shown by two examples in [4a]. However, the following statement is valid.

(vi) If A is closed and f is continuous on M, if U(t, x) is compact for each $(t, x) \in A$ and U(t, x) is an upper semicontinuous function of (t, x) in A, then Q(t, x) possesses the same property, and also has property (U) in A. If we know that Q(t, x) is convex, then Q(t, x) also has property (U) in A. If we know that Q(t, x) is convex, then Q(t, x) also has property (Q) in A.

REMARK: The statements above show that properties (U) and (Q) are generalizations of the concept of upper semicontinuity for closed, or closed and convex sets, respectively.

(vii) If A is a closed subset of the tx-space $E_1 \times E_n$, if $U(t, x), (t, x) \in A$, $U(t, x) \subset E_m$ is a variable subset of E_m satisfying property (U) in A, if M denotes the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$, if $f_o(t, x, u)$ is a continuous scalar function from M into the reals, if $\tilde{U}(t, x)$ denotes the variable subset of E_{m+1} defined by $\tilde{U} = [\tilde{u} = (u^o, u) \in E_{m+1} \mid u^o \geq f_o(t,x,u),\ u \in U(t,x)]$, then $\tilde{U}(t, x)$ satisfies property (U) in A.

An example is given in [4a] which shows that $\tilde{U}(t,x)$ of statement (vii) does not necessarily have property (Q) in A even if we assume that U(t, x) has property (Q) in A and $f_o(t, x, u)$ is convex in u for each $(t, x) \in A$.

However a slightly stronger statement does hold. It is necessary to introduce a new property for this statement.

A scalar function $f_o(t, x, u)$, $(t, x, u) \in M$ is said to be quasi-normally convex in u at $(t_o, x_o, u_o) \in M$ provided, given $\epsilon > 0$, there are a number $\delta = \delta(t_o, x_o, u_o, \epsilon) > 0$ and a linear scalar function $z(u) = r + b \cdot u$, $b = (b_1, \ldots, b_m)$, r, $b_1, \ldots, b_m$, real such that

(a) $f_o(t, x, u) \geq z(u)$ for all $(t, x) \in N_\delta(t_o, x_o)$, $u \in U(t, x)$

(b) $f_o(t, x, u) \leq z(u) + \epsilon$ for all $(t, x) \in N_\delta(t_o, x_o)$, $u \in U(t, x)$, $|u - u_o| \leq \delta$.

The scalar function $f_o(t, x, u)$ is said to be quasi-normally convex in u, if it has this property at each $(t_o, x_o, u_o) \in M$.

(viii) If A is a closed subset of the tx-space $E_1 \times E_n$, if $U(t, x)$, $(t, x) \in A$, $U(t, x) \subset E_m$, is a variable subset of E_m satisfying property (Q) in A, if M denotes the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$, if f_o is a continuous scalar function on M, which is convex in u for each $(t, x) \in A$, if either (α) the sets $U(t, x)$ are all contained in a fixed solid sphere S of E_m, or (β) the function $f_o(t, x, u)$ is quasi-normally convex in u at every $(t_o, x_o, u_o) \in M$, then the set $\tilde{U}(t, x)$ of statement (vii) has property (Q) in A.

A function $f_o(t, x, u)$ is said to be normally convex in u at $(t_o, x_o, u_o) \in M$, if for each $(t_o, x_o, u_o) \in M$ and for every $\epsilon > 0$ there are constants $\delta = \delta(t_o, x_o, u_o, \epsilon) > 0$, $\nu = \nu(t_o, x_o, u_o, \epsilon) > 0$ and a function $z(u) = c + d \cdot u$ such that

(a) $f_o(t,x,u) \geq z(u) + \nu |u - u_o|$, for each $(t,x) \in N_\delta(t_o, x_o)$, $u \in U(t,x)$

and (b) $f_o(t,x,u) \leq z(u) + \epsilon$ for each $(t,x) \in N_\delta(t_o,x_o)$, $u \in U(t,x)$, $|u-u_o| \leq \delta$.

The scalar function $f_o(t,x,u)$ is said to be normally convex in u, if it has this property for each $(t_o, x_o, u_o) \in M$.

We shall need the following statement due to Tonelli [23a] and L. Turner [24].

(viii) $f_o(t,x,u)$ is normally convex in $A \times E_m$ if and only if $f_o(t,x,u)$ is a convex function of u for each $(t,x) \in A$, and for no points $(t_o,x_o) \in A$, $u_o, u_1 \in E_m$ with $|u_1| \neq 0$ it is true that for all real λ

$$\frac{1}{2}\{f_o(t_o,x_o,u_o + \lambda u_1) + f_o(t_o,x_o,u_o - \lambda u_1)\} = f_o(t_o,x_o,u_o). \quad (8)$$

(ix) If $f_o(t,x,u)$ is a convex function of u for each $(t,x) \in A$ and $f_o(t,x,u)|u|^{-1} \to +\infty$ as $|u| \to +\infty$ for each $(t,x) \in A$, then $f_o(t,x,u)$ is normally convex in $A \times E_m$.

PROOF: By statement (ix), it suffices to prove that there exists no points $(t_o,x_o) \in A$, $u_o, u_1 \in E_m$ with $|u_1| \neq 0$, for which relation (8) holds. Suppose such points exist. Then,

$$\frac{1}{2}\{f_o(t_o,x_o,u_o + \lambda u_1) - f_o(t_o,x_o,u_o - \lambda u_1)\} |u_o + \lambda u_1|^{-1}$$
$$= f_o(t_o,x_o,u_o)|u_o + \lambda u_1|^{-1}$$

when $u_o + \lambda u_1 \neq 0$. Now $f_o(t_o, x_o, u_o) \, |u_o + \lambda u_1|^{-1} \to 0$ as $\lambda \to +\infty$. As $f(t,x,u) \, |u|^{-1} \to +\infty$ as $|u| \to +\infty$ for each $(t,x) \in A$, this statement implies that $f_o(t_o, x_o, u_o + \lambda u_1) \, |u_o + \lambda u_1|^{-1} \to +\infty$ as $\lambda \to +\infty$. Thus, $f_o(t_o, x_o, u_o - \lambda u_1) \, |u_o + \lambda u_1|^{-1} = f_o(t_o, x_o, u_o - \lambda u_1) \cdot$ $\cdot \, |u_o - \lambda u_1|^{-1}(|u_o - \lambda u_1| \, |u_o + \lambda u_1|^{-1}) \to +\infty$ as $\lambda \to +\infty$. This is a contradiction and the statement is proven.

(x) If $f_o(t,x,u)$ is a convex function of u for each $(t,x) \in A$, and there exists a function $\Phi(z)$, $0 \leq z < +\infty$, such that

$$f_o(t,x,u) \geq \Phi(|u|) \text{ for each } (t,x,u) \in A \times E_m$$

and $\Phi(z)z^{-1} \to +\infty$ as $z \to +\infty$, then $f_o(t,x,u)$ is normally convex in $A \times E_m$.

PROOF: One has

$$f_o(t,x,u) \, |u|^{-1} \geq \Phi(|u|) \, |u|^{-1} \text{ for } |u| \neq 0$$

and $(t,x,u) \in A \times E_m$. As $\Phi(z) \, z^{-1} \to +\infty$ and $z \to +\infty$, therefore $f_o(t,x,u) \, |u|^{-1} \to +\infty$ as $|u| \to +\infty$. Statement (ix) applies and this statement is proven.

6. Closure Theorem I

(Cesari [4a]). Let A be a closed subset of $E_1 \times E_n$, let $U(t,x)$ be a closed subset of E_m for every $(t,x) \in A$, let $f(t,x,u) = (f_1, \ldots, f_n)$ be a continuous vector function on the set $M = \{(t,x,u) \mid (t,x) \in A, \ u \in U(t,x)\}$ into E_n, and let

$Q(t, x) = f(t, x, U(t, x))$ be a convex subset of E_n for every $(t, x) \epsilon A$. Assume that $U(t, x)$ has property (U) in A, and that $Q(t, x)$ has property (Q) in A. Let $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1, 2, \ldots$, be a sequence of trajectories, which is convergent in the metric ρ toward an absolutely continuous function $x(t)$, $t_1 \leq t \leq t_2$. Then $x(t)$ is a trajectory.

REMARK: If we assume that $U(t, x)$ is compact for every $(t, x) \epsilon A$, and that $U(t, x)$ is an upper semicontinuous function of (t, x) in A, then the set $Q(t, x)$ has the same property, $U(t, x)$ has property (U), $Q(t, x)$ has property (Q), and Closure Theorem I reduces to one of A. F. Flipppov [8] (not explicitly stated in [8] but contained in the proof of his existence theorem for the Pontryagin problem with $U(t, x)$ always compact).

PROOF: The vector functions

$$\phi(t) = x'(t), \qquad t_1 \leq t \leq t_2,$$

$$\phi_k(t) = x'_k(t) = f(t, x_k(t), u_k(t)), \qquad t_{1k} \leq t \leq t_{2k},$$

$$k=1, 2, \ldots, \tag{9}$$

are defined almost everywhere and are L-integrable. We have to prove that $(t, x(t)) \epsilon A$ for every $t_1 \leq t \leq t_2$, and that there is a measurable control function $u(t)$, $t_1 \leq t \leq t_2$, such that

$$\phi(t) = x'(t) = f(t, x(t)\ u(t)), \qquad u(t) \ \epsilon\ U(t, x(t)) \tag{10}$$

for almost all $t \epsilon [t_1, t_2]$.

First, $\rho(x_k, x) \to 0$ as $k \to \infty$, hence $t_{1k} \to t_1, t_{2k} \to t_2$. If $t \in (t_1, t_2)$ or $t_1 < t < t_2$, then $t_{1k} < t < t_{2k}$ for all k sufficiently large and $(t, x_k(t)) \in A$. Since $x_k(t) \to x(t)$ as $k \to \infty$ and A is closed, we conclude that $(t, x(t)) \in A$ for every $t_1 < t < t_2$. Since x(t) is continuous, and hence continuous at t_1 and t_2, we conclude that $(t, x(t)) \in A$ for every $t_1 \leq t \leq t_2$.

For almost all $t \in [t_1, t_2]$ the derivative x'(t) exists and is finite. Let t_o be such a point with $t_1 < t_o < t_2$. Then there is a $\sigma > 0$ with $t_1 < t_o - \sigma < t_o + \sigma < t_2$, and, for some k_o and all $k \geq k_o$, also $t_{1k} < t_o - \sigma < t_o + \sigma < t_{2k}$. Let $x_o = x(t_o)$.

We have $x_k(t) \to x(t)$ uniformly in $[t_o - \sigma, t_o + \sigma]$ and all functions $x(t), x_k(t)$ are continuous in the same interval. Thus, they are equicontinuous in $[t_o - \sigma, t_o + \sigma]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that $t, t' \in [t_o - \sigma, t_o + \sigma]$, $|t-t'| \leq \delta$, $k \geq k_o$, imply

$$|x(t) - x(t')| \leq \epsilon/2, \quad |x_k(t) - x_k(t')| \leq \epsilon/2.$$

We can assume $0 < \delta < \sigma$, $\delta \leq \epsilon$. For any h, $0 < h \leq \delta$, let us consider the averages

$$\begin{aligned} m_h &= h^{-1} \int_o^h \phi(t_o + s)ds = h^{-1}[x(t_o+h) - x(t_o)], \\ m_{hk} &= h^{-1} \int_o^h \phi_k(t_o+s)ds = h^{-1}[x_k(t_o+h) - x_k(t_o)]. \end{aligned} \tag{11}$$

Given $\eta > 0$ arbitrary, we can fix h, $0 < h \leq \delta < \sigma$, so small that

$$|m_h - \phi(t_o)| \leq \eta. \qquad (12)$$

Having so fixed h, let us take $k_1 \geq k_o$ so large that

$$|m_{hk} - m_h| \leq \eta, \quad |x_k(t_o) - x(t_o)| \leq \epsilon/2 \qquad (13)$$

for all $k \geq k_1$. This is possible since $x_k(t) \to x(t)$ as $k \to \infty$ both at $t = t_o$ and $t = t_o + h$. Finally, for $0 \leq s \leq h$, $k \geq k_1$,

$$|x_k(t_o+s) - x(t_o)| \leq |x_k(t_o+s) - x_k(t_o)| + |x_k(t_o) - x(t_o)|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon,$$

$$|(t_o + s) - t_o| \leq h \leq \delta \leq \epsilon,$$

and

$$f(t_o + s, x_k(t_o + s), u_k(t_o + s)) \in Q(t_o + s, x_k(t_o + s)).$$

Hence, by the definition of $Q(t_o, x_o, 2\epsilon)$, also

$$\phi_k(t_o + s) = f(t_o + s, x_k(t_o + s), u_k(t_o + s)) \in Q(t_o, x_o, 2\epsilon).$$

The second integral relation (11) shows that we have also

$$m_{hk} \in \text{cl co}\, Q(t_o, x_o, 2\epsilon),$$

since the latter is a closed convex set. Finally by relations (12) and (13), we deduce

$$|\phi(t_o) - m_{hk}| \leq |\phi(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta,$$

and hence

$$\phi(t_o) \in [\text{cl co } Q(t_o, x_o, 2\epsilon)]_{2\eta}.$$

Here $\eta > 0$ is an arbitrary number, and the set in brackets is closed. Hence,

$$\phi(t_o) \in \text{cl co } Q(t_o, x_o, 2\epsilon),$$

and this relation holds for every $\epsilon > 0$. By property (Q) we have

$$\phi(t_o) \in \bigcap_{\epsilon > 0} \text{cl co } Q(t_o, x_o, 2\epsilon) = Q(t_o, x_o),$$

where $x_o = x(t_o)$, and $Q(t_o, x_o) = f(t_o, x_o, U(t_o, x_o))$. This relation implies that there are points $\bar{u} = \bar{u}(t_o) \in U(t_o, x_o)$ such that

$$\phi(t_o) = f(t_o, x(t_o), \bar{u}(t_o)). \tag{14}$$

This holds for almost all $t_o \in [t_1, t_2]$, that is, for all t of a measurable set $I \subset [t_1, t_2]$ with meas $I = t_2 - t_1$. If we take $I_o = [t_1, t_2] - I$, then meas $I_o = 0$. Hence, there is at least one function $\bar{u}(t)$, defined almost everywhere in $[t_1, t_2]$, for which relation (14) holds a. e. in $[t_1, t_2]$. We have to prove that there is at least one such function which is measurable. For every $t \in I$, let $P(t)$ denote the set

$$P(t) = [u \mid u \in U(t, x(t)),\ \phi(t) = f(t, x(t), u)] \subset U(t, x(t)) \subset E_m.$$

We have proved that $P(t)$ is not empty.

For every integer $\lambda = 1, 2, \dots,$ there is a closed subset C_λ of I, $C_\lambda \subset I \subset [t_1, t_2]$, with meas $C_\lambda > \max[0, t_2 - t_1 - 1/\lambda]$, such that $\phi(t)$ is continuous on C_λ. Let W_λ be the set

$$W_\lambda = [(t,u) \mid t \in C_\lambda,\ u \in P(t)] \subset E_1 \times E_m.$$

Let us prove that the set W_λ is closed. Indeed, if $(\bar{t}, \bar{u})$ is a point of accumulation of W_λ, then there is a sequence (t_s, u_s), $s=1,2,\ldots$, with $(t_s, u_s) \in W_\lambda$, $t_s \to \bar{t}$, $u_s \to \bar{u}$. Then, $t_s \in C_\lambda$ and $\bar{t} \in C_\lambda$ since C_λ is closed. Also $x(t_s) \to x(\bar{t})$, $\phi(t_s) \to \phi(\bar{t})$, and since $(t_s, x(t_s)) \in A$, $\phi(t_s) = f(t_s, x(t_s), u(t_s))$, $(t_s, x(t_s), u(t_s)) \in M$, we have also $(\bar{t}, x(\bar{t})) \in A$, $(\bar{t}, x(\bar{t}), \bar{u}) \in M$, because A and M are closed, and $\phi(\bar{t}) = f(\bar{t}, x(\bar{t}), \bar{u})$ because f is continuous. Thus, $\bar{u} \in P(\bar{t})$, and $(\bar{t}, \bar{x}) \in W_\lambda$. This proves that W_λ is a closed set.

For every integer ℓ let $W_{\lambda\ell}$, $P_\ell(t)$ be the sets

$$W_{\lambda\ell} = [(t,u) \mid (t,u) \in W_\lambda,\ |u| \leq \ell] \subset W_\lambda \subset E_1 \times E_m,$$

$$P_\ell(t) = [u \mid u \in P(t),\ |u| \leq \ell] \subset P(t) \subset U(t, x(t)) \subset E_m,$$

$$C_{\lambda\ell} = [t \mid (t,u) \in W_{\lambda\ell} \text{ for some } u] \subset C_\lambda \subset I \subset [t_1, t_2].$$

Obviously, $W_{\lambda\ell}$ is compact, and so is $C_{\lambda\ell}$ as its projection on the t-axis. Also $\bigcup_\ell C_{\lambda\ell} = C_\lambda$, and $W_{\lambda\ell}$ is the set of all (t,u) with $t \in C_{\lambda\ell}$, $u \in P_\ell(t)$. Thus, for $t \in C_{\lambda\ell}$, $P_\ell(t)$ is a compact subset of $U(t, x(t))$.

For $t \in C_{\lambda\ell}$ and ℓ large enough, the set $P_\ell(t)$ is a nonempty compact subset of all $u = (u^1, \ldots, u^m) \in U(t, x(t))$ with $f(t, x(t), u) = \phi(t)$ and $|u| \leq \ell$. Let P_1 be the subset of P_ℓ with u^1 minimum, let P_2 be the subset of P_1 with u^2 minimum, ..., let P_m be the subset of P_{m-1} with u^m minimum. Then P_m is a single point $u = u(t) \in U(t, x(t))$ with $u(t) = (u^1, \dots, u^m)$, $t \in C_{\lambda\ell}$, $|u(t)| \leq \ell$, and $f(t, x(t), u(t)) = \phi(t)$. Let us prove that $u(t)$, $t \in C_{\lambda\ell}$, is measurable. We shall prove this by induction on the coordinates. Let us assume that $u^1(t), \dots, u^{s-1}(t)$ have been proved to be measurable on $C_{\lambda\ell}$ and let us prove that $u^s(t)$ is measurable. For $s=1$ nothing is assumed, and the argument below proves that $u^1(t)$ is measurable. For every integer j there are closed subsets $C_{\lambda\ell j}$ of $C_{\lambda\ell}$ with $C_{\lambda\ell j} \subset C_{\lambda\ell, j+2}$, meas $C_{\lambda\ell j} > \max[0, \text{meas } C_{\lambda\ell} - 1/j]$, such that $u^1(t), \dots, u^{s-1}(t)$ are continuous on $C_{\lambda\ell j}$. The function $\phi(t)$ is already continuous on C_λ and hence $\phi(t)$ is continuous on every set $C_{\lambda\ell}$ and $C_{\lambda\ell j}$. Let us prove that $u^s(t)$ is measurable on $C_{\lambda\ell j}$. We have only to prove that, for every real a, the set of all $t \in C_{\lambda\ell j}$ with $u^s(t) \leq a$ is closed. Suppose that this is not the case. Then there is a sequence of points $t_k \in C_{\lambda\ell j}$ with $u^s(t_k) \leq a$, $t_k \to \bar{t} \in C_{\lambda\ell j}$, $u^s(\bar{t}) > a$. Then $\phi(t_k) \to \phi(\bar{t})$, $u^\alpha(t_k) \to u^\alpha(\bar{t})$ as $k \to \infty$, $\alpha = 1, \dots, s-1$. Since $|u^\beta(t_k)| \leq \ell$ for all k and $\beta = s, s+1, \dots, m$, we can select a subsequence, say still $[t_k]$, such that $u^\beta(t_k) \to \tilde{u}^\beta$ as $k \to \infty$, $\beta = s, s+1, \dots, m$, for some real numbers $\tilde{u}^\beta$. Then $t_k \to \bar{t}$, $x(t_k) \to x(\bar{t})$, $u(t_k) \to \tilde{u}$, where

$$\tilde{u} = (u^1(\bar{t}), \dots, u^{s-1}(\bar{t}), \tilde{u}^s, \dots, \tilde{u}^m).$$

Then, given any number $\eta > 0$, we have

$$u(t_k) \in U(t_k, x(t_k)) \subset \text{cl } U(\bar{t}, x(\bar{t}), \eta)$$

for all k sufficiently large, and, as $k \to \infty$, also

$$\tilde{u} \in \operatorname{cl} U(\bar{t}, x(\bar{t}), \eta).$$

By property (U) we have

$$\tilde{u} \in \bigcap_{\eta > 0} \operatorname{cl} U(\bar{t}, x(\bar{t}), \eta) = U(\bar{t}, x(\bar{t})).$$

On the other hand $\phi(t_k) = f(t_k, x(t_k), u(t_k))$, $u^s(t_k) \leq a$, yield as $k \to \infty$,

$$\phi(\bar{t}) = f(\bar{t}, x(\bar{t}), \tilde{u}), \; \tilde{u}^s \leq a, \tag{15}$$

while $\bar{t} \in C_{\lambda\ell}$ implies

$$\phi(\bar{t}) = f(\bar{t}, x(\bar{t}), u(\bar{t})), \; u^s(\bar{t}) > a. \tag{16}$$

Relations (15) and (16) are contradictory, because of the minimum property with which $u^s(\bar{t})$ has been chosen. Thus, $u^s(t)$ is measurable on $C_{\lambda\ell j}$ for every j, and then $u^s(\bar{t})$ is also measurable on $C_{\lambda\ell}$. By induction, all components $u^1(t), \dots, u^m(t)$ of u(t) are measurable on $C_{\lambda\ell}$, hence, u(t) is measurable on $C_{\lambda\ell}$. Since $\bigcup_\ell C_{\lambda\ell} = C_\lambda$, meas $C_\lambda >$ meas $I - 1/\lambda$, we conclude that u(t)) is measurable on every set C_λ and hence on I, with meas $I = t_2 - t_1$. Thus u(t) is defined a. e. on $[t_1, t_2]$, $u(t) \in U(t, x(t)$ and $f(t, x(t), u(t)) = \phi(t)$ a. e. on $[t_1, t_2]$. Closure Theorem I is thereby proved.

REMARK: The last part of the proof of Closure Theorem I concerning the existence of at least one measurable function u(t) is a modification, for U(t, x) closed and satisfying property (U), of the analogous argument of A. F. Filippov [8] for the case where U(t, x) is an upper semicontinuous compact subset of the Euclidean space E_m. A different argument - again concerning only the last part of the proof - has been devised by C. Castaing [3]. His result concerns a multi-valued map Γ: $t \to \Gamma(t)$, with $\Gamma(t)$ depending on t only. If $\Gamma(t) = U(t, x(t))$ and $U(t) = U(t, x(t))$ is an upper semicontinuous function of the time t, then Castaing's result provides a different argument for the second part of the closure theorem.

7. Another closure theorem

Let us denote by $y = (x^1, \dots, x^s)$ the s-vector made up of certain components, say $x^1, \dots, s^s$, $0 \leq s \leq n$, of $x = (x^1, \dots, x^n)$, and by z the complementary (n-s)-vector $z = (x^{s+1}, \dots, x^n)$ of x, so that $x = (y, z)$. Let us assume that f(t, y, u) depends only on the coordinates $x^1, \dots, x^s$ of x. If x(t), $t_1 \leq t \leq t_2$, is any vector function, we shall denote by $x(t) = [y(t), z(t)]$ the corresponding decomposition of x(t) in its coordinates $y(t) = (x^1, \dots, x^s)$ and $z(t) = (x^{s+1}, \dots, x^n)$.

We shall denote by A_o a closed subset of points $(t, x^1, \dots, x^s)$, that is, a closed subset of the ty-space $E_1 \times E_s$, and let $A = A_o \times E_{n-s}$. Thus, A is a closed subset of the tx-space $E_1 \times E_n$.

Closure Theorem II. (Cesari [4a].) Let A_o be a closed subset of the ty-space $E_1 \times E_s$, and then $A = A_o \times E_{n-s}$ is a closed subset of the tx-space $E_1 \times E_n$. Let U(t, y) denote a closed subset of E_m for every $(t, y) \in A_o$, let M_o be the set of all $(t, y, u) \in E_{1+s+m}$ with $(t, y) \in A_o$, $u \in U(t, y)$, and let $f(t, y, u) = (f_1, \ldots, f_n)$ be a continuous vector function from M into E_n. Let Q(t, y) = f(t, y, U(t, y)) be a closed convex subset of E_n for every $(t, y) \in A_o$. Assume that U(t, y) has property (U) in A_o and that Q(t, y) has property (Q) in A_o. Let $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1, 2, \ldots$, be a sequence of trajectories, $x_k(t) = (y_k(t), z_k(t))$, for which we assume that the s-vector $y_k(t)$ converges in the ρ-metric toward an AC vector function y(t), $t_1 \leq t \leq t_2$, and that the (n-k)-vector $z_k(t)$ converges (pointwise) for almost all $t_1 < t < t_2$, toward a vector z(t) which admits of a decomposition z(t) = Z(t) + S(t) where Z(t) is an AC vector function in $[t_1, t_2]$, and $S'(t) = 0$ a. e. in $[t_1, t_2]$ (that is, S(t) is a singular function). Then, the AC vector X(t) = [y(t), Z(t)], $t_1 \leq t \leq t_2$, is a trajectory.

REMARK: For s=n, this theorem reduces to Closure Theorem I.

PROOF: The vector functions

$$\phi(t) = X'(t) = (y'(t), Z'(t)), \quad t_1 \leq t \leq t_2,$$

$$\phi_k(t) = x'_k(t) = (y'_k(t), z'_k(t)) = f(t, y_k(t), u_k(t)), \quad t_{1k} \leq t \leq t_{2k}$$

$$k=1, 2, \ldots, \tag{17}$$

are defined almost everywhere and are L-integrable. We have to prove that $[t, y(t), Z(t)] \epsilon A$ for every $t_1 \leq t \leq t_2$, and that there is a measurable control function $u(t)$, $t_1 \leq t \leq t_2$, such that

$$\phi(t) = X'(t) = (y'(t), Z'(t)) = f(t, y(t), u(t)), \qquad (18)$$
$$u(t) \epsilon U(t, y(t)),$$

for almost all $t \epsilon [t_1, t_2]$.

First, $\rho(y_k, y) \to 0$ as $k \to 0$; hence $t_{1k} \to t_1, t_{2k} \to t_2$. If $t \epsilon (t_1, t_2)$, or $t_1 < t < t_2$, then $t_{1k} < t < t_{2k}$ for all k sufficiently large, and $(t, y_k(t) \epsilon A_o$. Since $y_k(t) \to y(t)$ as $k \to \infty$ and A_o is closed, we conclude that $(t, y(t)) \epsilon A_o$ for every $t_1 < t < t_2$, and finally $(t, y(t), Z(t)) \epsilon A_o \times E_{n-s}$, or $(t, X(t)) \epsilon A$, $t_1 \leq t \leq t_2$.

For almost all $t \epsilon [t_1, t_2]$ the derivative $X'(t) = [y'(t), Z'(t)]$ exists and is finite, $S'(t)$ exists and $S'(t) = 0$, and $z_k(t) \to z(t)$. Let t_o be such a point with $t_1 < t_o < t_2$. Then there is a $\sigma > 0$ with $t_1 < t_o - \sigma < t_o + \sigma < t_2$, and, for some k_o and all $k \geq k_o$, also $t_{1k} < t_o - \sigma < t_o + \sigma < t_{2k}$. Let $x_o = X(t_o) = (y_o, Z_o)$, or $y_o = y(t_o)$, $Z_o = Z(t_o)$. Let $z_o = z(t_o)$, $S_o = S(t_o)$. We have $S'(t_o) = 0$, hence $z'(t_o)$ exists and $z'(t_o) = Z'(t_o)$. Also, we have $z_k(t_o) \to z(t_o)$.

We have $y_k(t) \to y(t)$ uniformly in $[t_o - \sigma, t_o + \sigma]$, and all functions $y(t)$, $y_k(t)$ are continuous in the same interval. Thus, they are equicontinuous in $[t_o - \sigma, t_o + \sigma]$. Given $\epsilon > 0$, there is a $\delta > 0$ such that

$t, t' \in [t_o - \sigma, t_o + \sigma]$, $|t-t'| \leq \delta$, $k \geq k_o$, implies

$|y(t) - y(t')| \; \epsilon/2$, $|y_k(t) - y_k(t')| \leq \epsilon/2$.

We can assume $0 < \delta < \sigma$, $\delta \leq \epsilon$. For any h, $0 < h \leq \delta$, let us consider the averages

$$m_h = h^{-1} \int_0^h \phi(t_o + s)ds = h^{-1}[X(t_o + h) - X(t_o))],$$

$$m_{hk} = h^{-1} \int_0^h \phi_k(t_o + s)ds = h^{-1}[x_k(t_o + h) - x_k(t_o))], \tag{19}$$

where $X = (y, Z)$, $x_k = (y_k, z_k)$.

Given $\eta > 0$ arbitrary, we can fix h, $0 < h \leq \delta < \sigma$, so small that

$$|m_h - \phi(t_o)| \leq \eta,$$

$$|S(t_o + h) - S(t_o)| < \eta h/4 .$$

This is possible since $h^{-1} \int_0^h \phi(t_o + s)ds \to \phi(t_o)$ and $[S(t_o + h) - S(t_o)] h^{-1} \to 0$ as $h \to 0+$. Also, we can choose h, in such a way that $z_k(t_o + h) \to z(t_o + h)$ as $k \to +\infty$. This is possible since $z_k(t) \to z(t)$ for almost all $t_1 < t < t_2$.

Having so fixed h, let us take $k_1 \geq k_o$ so large that

$$|y_k(t_o) - y(t_o)|, \; |y_k(t_o + h) - y(t_o + h)| \leq \min[\eta h/4, \epsilon/2],$$

$$|z_k(t_o) - z(t_o)|, \; |z_k(t_o + h) - z(t_o + h)| \leq \eta h/8.$$

This is possible since $y_k(t) \to y(t)$, $z_k(t) \to z(t)$ both at $t = t_o$ and $t = t_o + h$. Then we have

$$|h^{-1}[y_k(t_o + h) - y_k(t_o)] - h^{-1}[y(t_o + h) - y(t_o)]|$$

$$\leq |h^{-1}[y_k(t_o + h) - y(t_o + h)]| + |h^{-1}[y_k(t_o) - y(t_o)]|$$

$$\leq h^{-1}(\eta\, h/4) + h^{-1}(\eta\, h/4) = \eta/2.$$

Analogously, since $z = Z + S$, we have

$$|h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[Z(t_o + h) - Z(t_o)]|$$

$$= |h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[z(t_o + h) - z(t_o)]$$

$$+ h^{-1}[S(t_o + h) - S(t_o)]|$$

$$\leq |h^{-1}[z_k(t_o + h) - z(t_o + h)]| + |h^{-1}[z_k(t_o) - z(t_o)]|$$

$$+ |h^{-1}[S(t_o + h) - S(t_o)]|$$

$$\leq h^{-1}(\eta\, h/8) + h^{-1}(\eta\, h/8) + h^{-1}(\eta\, h/4) = \eta/2.$$

Finally, we have

$$|m_{hk} - m_h| = |h^{-1}[x_k(t_o + h) - x_k(t_o)]$$

$$- h^{-1}[X(t_o + h) - X(t_o)]|$$

$$< |h^{-1}[y_k(t_o + h) - y_k(t_o)] - h^{-1}[y(t_o + h) - y(t_o)]| +$$

$$+ |h^{-1}[z_k(t_o + h) - z_k(t_o)] - h^{-1}[Z(t_o + h) - Z(t_o)]|$$

$$\leq \eta/2 + \eta/2 = \eta.$$

We conclude that for the chosen value of h, $0 < h \leq \delta < \sigma$, and every $k \geq k_1$ we have

$$|m_h - \phi(t_o)| \leq \eta, \; |m_{hk} - m_h| \leq \eta,$$

$$|y_k(t_o) - y(t_o)| \leq \epsilon/2. \qquad (20)$$

For $0 \leq s \leq h$ we have now

$$|y_k(t_o + s) - y(t_o)| \leq |y_k(t_o + s) - y_k(t_o)| + |y_k(t_o) - y(t_o)|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon,$$

$$|(t_o + s) - t_o| \leq h \leq \delta \leq \epsilon$$

$$f(t_o + s, \; y_k(t_o + s), \; u_k(t_o + s)) \in Q(t_o + s, \; y_k(t_o + s)).$$

Hence, by definition of $Q(t_o, y_o, 2\epsilon)$, also

$$\phi_k(t_o + s) = f(t_o + s, \; y_k(t_o + s), \; u_k(t_o + s)) \in Q(t_o, y_o, 2\epsilon).$$

The second integral relation (19) shows that we have also

$$m_{hk} \in \text{cl co } Q(t_o, y_o, 2\epsilon),$$

since the latter is a closed convex set. Finally, by relation (20), we deduce

$$|\phi(t_o) - m_{hk}| \leq |\phi(t_o) - m_h| + |m_h - m_{hk}| \leq 2\eta,$$

and hence

$$\phi(t_o) \in [\text{cl co } Q(t_o, y_o, 2\epsilon)]_{2\eta}.$$

Here $\eta > 0$ is an arbitrary number, and the set in brackets is closed. Hence

$$\phi(t_o) \in \text{cl co } Q(t_o, y_o, 2\epsilon),$$

and this relation holds for every $\epsilon > 0$. By property (Q) we have

$$\phi(t_o) \in \bigcap_{\epsilon > 0} \text{cl co } Q(t_o, y_o, 2\epsilon) = Q(t_o, y_o),$$

where $y_o = y(t_o)$, and $Q(t_o, y_o) = f(t_o, y_o, U(t_o, y_o))$. This relation implies that there are points $\bar{u} = \bar{u}(t_o) \in U(t_o, y_o)$ such that

$$\phi(t_o) = f(t_o, y(t_o), \bar{u}(t_o)).$$

This holds for almost all $t_o \in [t_1, t_2]$. Hence, there is at least one function $\bar{u}(t)$, defined a. e. in $[t_1, t_2]$ for which relation (18) holds a. e. in $[t_1, t_2]$. We have to prove that there is at least one such function which is measurable. The proof is exactly as the one for Closure Theorem I, where we write y, y_k instead of x, x_k, and will not be repeated here. Closure Theorem II is thereby proved.

REMARK: The proof of Closure Theorem I is a modification due to Cesari [4a] of the proof of an analogous statement by A. F. Filippov [8] for compact instead of closed sets. Both the statement and proof of Closure Theorem II bearing on singular functions are due to Cesari [4a].

8. Notations for the optimal control problem.

We shall again use the notations of No. 1. It will be convenient to write the problem in a slightly different form. First we introduce the auxiliary variable x^o satisfying the differential equation and initital value

$dx^o/dt = f_o(t, x(t), u(t)),\ x^o(t_1) = 0,\ x^o(t)$ AC in $[t_1, t_2]$.

Then

$$x^o(t_2) = \int_{t_1}^{t_2} f_o(t, x(t), u(t))dt = I[x, u]. \qquad (21)$$

If we now denote by $\tilde{x}$ the (n+1)-vector $\tilde{x} = (x^o, x^1, \dots, x^n)$, and by $\tilde{f}(t, x, u)$ the (n+1)-vector function $\tilde{f}(t, x, u) = (f_o, f_1, \dots, f_n)$, then the problem of minimum discussed in No. 1 reduces to the determination of a pair $[\tilde{x}(t), u(t)]$, $t_1 \leq t \leq t_2$, satisfying the differential system

$$d\tilde{x}/dt = \tilde{f}(t, x(t), u(t)) \text{ a. e. in } [t_1, t_2], \qquad (22)$$

the boundary conditions

$$(t_1, x(t_1),\ t_2,\ (t_2)) \in B,\ x^o(t_1) = 0, \qquad (23)$$

and the constraints

$$(t, x(t)) \in A,\ u(t) \in U(t, x(t)),\quad t \in [t_1, t_2]$$

for which $x^o(t_2)$ has its minimum value. Here $x(t) = (x^1, \ldots, x^n)$, $\tilde{x}(t) = (x^o, x^1, \ldots, x^n)$, and the present formulation corresponds to a transformation of the Lagrange type problem of No. 1 into a problem of the Mayer type.

We shall now consider for each $(t, x) \in A$, the sets $Q(t, x)$, $\tilde{Q}(t,x)$, $\tilde{\tilde{Q}}(t, x)$ defined as follows:

$$Q(t, x) = f(t, x, U(t, x)) = [z \mid z = f(t, x, u),\ u \in U(t, x)] \subset E_n,$$

$$\begin{aligned}\tilde{Q}(t, x) &= \tilde{f}(t, x, U(t, x)) = [\tilde{z} = (z^o, z) \mid \tilde{z} = \tilde{f}(t, x, u),\ u \in U(t, x)] \\ &= [\tilde{z} = (z^o, z) \mid z^o = f_o(t, x, u),\ z = f(t, x, u),\ u \in U(t, x)] \\ &\qquad \subset E_{n+1}\end{aligned}$$

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o, z) \mid z^o \geq f_o(t, x, u),\ z = f(t, x, u),\ u \in U(t, x)] \subset E_{n+1}.$$

The main hypothesis of the existence theorems which we shall state and prove below is that the set $\tilde{\tilde{Q}}(t, x)$ is convex for each $(t, x) \in A$.

For free problems the sets Q, $\tilde{Q}$, $\tilde{\tilde{Q}}$ — thought of as subsets of the $z^o u$-space are

$$Q(t, x) = E_n$$

$$\tilde{Q}(t, x) = [z = (z^o, u) \mid z^o = f_o(t, x, u),\ u \in E_n] \subset E_{n+1},$$

$$\tilde{\tilde{Q}}(t, x) = [z = (z^o, u) \mid z^o \geq f_o(t, x, u),\ u \in E_n] \subset E_{n+1}.$$

Thus, the convexity of $\tilde{\tilde{Q}}$ reduces to the usual convexity condition of $f_0(t, x, u)$ as a function of u in E_n — a condition which is familiar in the calculus of variation for free problems. The proof of this equivalence is to be found in [7].

We mention here that a function $\phi(u)$, $u \in E_n$, is said to be convex in u, provided $u, v \in E_n$, $0 \leq \alpha \leq 1$, implies $\phi(\alpha u + (1 - \alpha)v) \leq \alpha\, \phi(u) + (1 - \alpha)\, \phi(v)$.

9. Statement of existence theorem I

Existence Theorem I (Cesari [4a])

Let A be any compact subset of the tx-space $E_1 \times E_n$, and for every $(t, x) \in A$ let $U(t, x)$ be a closed subset of the u-space E_m. Let M be the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$, and let $\tilde{f}(t, x, u) = (f_0, f) = (f_0, f_1, \ldots, f_n)$ be a continuous vector function on M. Assume that for every $(t, x) \in A$ the set

$$\tilde{\tilde{Q}}(t, x) = \{\tilde{z} = (z^0, z) \mid z^0 \geq f_0(t, x, u),$$

$$z = f(t, x, u), \quad u \in U(t, x)\} \subset E_{n+1}$$

is convex. Assume that $U(t, x)$ satisfies property (U) in A, and that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) in A. Assume that there is a continuous scalar function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, with $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, such that $f_0(t, x, u) \geq \Phi(|u|)$ for all $(t, x, u) \in M$, and that there are constants C, $D \geq 0$ such that

$|f(t, x, u)| \leq C + D|u|$ for all $(t, x, u) \in M$. Then the cost functional $I[x, u] = \int_{t_1}^{t_2} f_0(t, x, u)dt$ has an absolute minimum in any nonempty complete class Ω of admissible pairs $x(t), u(t)$.

If A is not compact, but closed and contained in a slab $[t_0 \leq t \leq T,\ x \in E_n]$, t_0, T finite, then Existence Theorem I still holds under the additional hypotheses that

(a) $x \circ f = x^1 f_1 + \dots + x^n f_n \leq N[(x^1)^2 + \dots + (x^n)^2 + 1]$ for all $(t, x, u) \in M$ and some constant $N \geq 0$, and

(b) each trajectory $x(t)$ of the class Ω contains at least one point $(t^*, x(t^*))$ on a given compact subset P of A (for instance, the initial point $(t^*, x(t^*))$ is fixed, or the end point is fixed).

If A is not compact, nor contained in a slab as above, but A is closed, then Theorem I still holds if the hypotheses (a), (b) and (c) are satisfied.

(c) $f_0(t, x, u) \geq \mu > 0$ for all $(t, x, u) \in M$ with $|t| \geq R$ and some constants $\mu > 0$, $R \geq 0$.

Finally, condition (a) can be replaced in any case by condition (d).

(d) $f_0(t, x, u) \geq E\ |f(t, x, u)|$ for all $(t, x, u) \in M$ with $|x| \geq F$ and for some constants $E > 0$ and $F \geq 0$.

Furthermore, when A is compact but closed, the conditions $f_0 \geq \Phi(|u|)$, $|f|, \leq C + D|u|$ above can be replaced by the following condition (g):

(g) for every compact subset A_0 of A there are functions Φ_0 as above and constants $C_0 \geq 0$, $D_0 \geq 0$ (all may depend on A_0) such that $f_0 \geq \Phi_0(|u|)$, $|f| \leq C_0 + D_0 |u|$ for all $(t, x, u) \in M$ with $(t, x) \in A_0$.

PROOF: We have $\Phi(\zeta) \geq -M_0$ for some number $M_0 \geq 0$, hence $\Phi(\zeta) + M_0 \geq 0$ for all $\zeta \geq 0$, and $f_0(t, x, u) + M_0 \geq 0$ for all $(t, x, u) \in M$. Let D be the diameter of A. Then for every pair $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, of Ω we have

$$I[x, u] = \int_{t_1}^{t_2} f_0 \, dt \geq \int_{t_1}^{t_2} \Phi(|u|) dt \geq -DM_0 > -\infty \tag{25}$$

Let $i = \inf I[x, u]$, where inf is taken over all pairs $(x, u) \in \Omega$. Then i if finite.

Let $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, be a sequence of admissible pairs, all in Ω, such that $I[x_k, u_k] \to i$ as $k \to +\infty$. We may assume

$$i \leq I[x_k, u_k] = \int_{t_{1k}}^{t_{2k}} f_0(t, x_k(t), u_k(t)) dt \leq i + k^{-1}$$

$$\leq i + 1, \quad k = 1, 2, \ldots$$

Let us prove that the AC vector functions $x_k(t)$, $t_{1k} \leq t_{2k}$, $k = 1, 2, \ldots$ are equiabsolutely continuous. Let $\epsilon > 0$ be any given number, and let $\sigma = 2^{-1} \epsilon \, (DM_0 + |i| + 1)^{-1}$.

Let $N > 0$ be a number such that $\Phi(\zeta)/\zeta > \sigma^{-1}$ for $\zeta \geq N$. Let E be any measurable subset of $[t_{1k}, t_{2k}]$ with meas $E < \eta = \epsilon\, 2^{-1}N^{-1}$. Let E_1 be the subset of all $t \in E$ where $u_k(t)$ is finite and $|u_k(t)| \leq N$, and let $E_2 = E - E_1$. Then $|u_k(t)| \leq N$ in E_1 and $\Phi(|u_k(t)|)\,|u_k(t)|^{-1} \geq \sigma^{-1}$, or $|u_k| \leq \sigma\Phi(|u_k|)$, a.e. in E_2. Hence

$$
\begin{aligned}
(E)\int |u_k(t)|\,dt &= (E_1 + E_2)\int |u_k(t)|\,dt \\
&\leq N \text{ meas } E_1 + \sigma(E_2)\int \Phi(|u_k(t)|)dt \\
&\leq N \text{ meas } E + \sigma(E_2)\int [\Phi(|u_k(t)|) + M_o]dt \qquad (26)\\
&\leq N\eta + \sigma\int_{t_{1k}}^{t_{2k}} [\Phi(|u_k(t)|) + M_o]dt
\end{aligned}
$$

as $\Phi(\zeta) + M_o \geq 0$ for $0 \leq \zeta < +\infty$. As $f_o \geq \Phi(|u|)$ for each $(t, x, u) \in M$, one has that

$$
\begin{aligned}
(E)\int |u_k(t)|\,dt &\leq N\eta + \sigma\int_{t_{1k}}^{t_{2k}} [f_o(t, x_k(t), u_k(t)) + M_o]dt \\
&\leq N\eta + \sigma(DM_o + |i| + 1) \\
&\leq \epsilon/2 + \epsilon/2 = \epsilon .
\end{aligned}
$$

This proves that the vector functions $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\ldots$ are equiabsolutely integrable. From here we deduce

$$(E) \int |x'_k(t)|\, dt = (E) \int |f(t, x_k(t), u_k(t))|\, dt$$

$$\leq (E) \int [C + D|u_k(t)|]\, dt$$

$$\leq C \text{ meas } E + D \ (E) \int |u_k(t)|\, dt$$

as $|f(t, x, u)| \leq C + D|u|$ for each $(t, x, u) \in M$ and this inequality and the equiabsolute integrability of the vector functions $u_k(t)$, $t_1 \leq t \leq t_2$, prove the equiabsolute continuity of the vector functions $x_k(t)$, $t_1 \leq t \leq t_2$.

Now let us consider the sequence of AC scalar functions $x_k^o(t)$ defined by

$$x_k^o(t) = \int_{t_{1k}}^{t} f_o(\tau, x_k(\tau), u_k(\tau))d\tau, \quad t_1 \leq t \leq t_{2k}. \tag{27}$$

Then

$$x_k^o(t_{1k}) = 0, \quad x_k^o(t_{2k}) = I[x_k, u_k] \to i \text{ as } k \to +\infty$$

and $i \leq x_k^o(t_{2k}) \leq 1 + k^{-1} \leq i + 1$, $k=1, 2, \ldots$. If $v_k^o(t) = f_o(t, x_k(t), u_k(t))$, $t_{1k} \leq t \leq t_{2k}$, then we define the functions $v_k^-(t)$, $v_k^+(t)$ as follows:

$$v_k^-(t) = - M_o, \ v_k^+(t) = v_k^o + M_o = f_o(t, x_k(t), u_k(t)) + M_o \geq 0.$$

Then $v_k^-(t) \leq 0$, $v_k^+(t) \geq 0$ a.e. in $[t_{1k}, t_{2k}]$, and we define

$$y_k^-(t) = \int_{t_{1k}}^{t} v_k^-(t)dt, \quad y_k^+(t) = \int_{t_{1k}}^{t} v_k^+(t)dt$$

$$t_{1k} \le t \le t_{2k}, \qquad k=1,2,\ldots .$$

Since $v_k^-(t) = -M_o$, we have $y_k(t) = -M_o(t-t_{1k}) \le 0$, and the functions $y_k^-(t)$ are monotone nonincreasing and uniformly Lipschitzian with constant M_o. On the other hand, the functions $y_k^+(t)$ are nonnegative, monotone nondecreasing, and uniformly bounded since

$$0 \le y_k^+(t_{2k}) = (y_k^+(t_{2k}) - y_k^-(t_{2k})) - y_k^-(t_{2k}) = x_k^o(t_{2k}) - y_k^-(t_{2k})$$
$$\le i + 1 + M_o(t_{2k} - t_{1k}) \le DM_o + |i| + 1.$$

By Ascoli's Theorem we first extract a sequence for which $(x_k(t),\ y_k^-(t))$, $t_{1k} \le t \le t_{2k}$, converges in the ρ-metric toward a continuous vector function $(x(t),\ Y^-(t))$, $t_1 \le t \le t_2$. Here $x(t)$ is AC because of the equiabsolute continuity of the vector functions $x_k(t)$, $t_{1k} \le t \le t_{2k}$ and $Y^-(t) = -M_o(t-t_1)$, $Y^-(t_1) = 0$. Then we apply Helly's Theorem to the sequence $y_k^+(t)$, $t_{1k} \le t \le t_{2k}$, $k=1,2,\ldots,$ and we perform a successive extraction so that the corresponding sequence of $y_k^+(t)$ converges for every $t_1 < t < t_2$ toward a function $Y_o^+(t)$, $t_1 < t < t_2$, which is nonnegative, monotone nondecreasing, but necessarily continuous. We define $Y_o^+(t)$ at t_1 by taking $Y_o^+(t_1) = 0$, and at t_2 by continuity at t_2, because of its monotoneity. Thus

$$0 \leq Y_o^+(t) \leq DM_o + |i| + 1, \qquad t_1 \leq t \leq t_2.$$

Finally, $Y_o^+(t)$ admits of a unique decomposition $Y_o^+(t) =$ $= Y^+(t) + Z(t)$, $t_1 \leq t \leq t_2$ with $Y^+(t_1) = 0$, where both $Y^+(t)$, $Z(t)$ are nonnegative monotone nondecreasing, where $Y^+(t)$ is AC and $Z'(t) = 0$ a. e. in $[t_1, t_2]$. If $Y(t) = Y^-(t) + Y^+(t)$, we see that $y_k^o(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1, 2, \ldots,$ converges for all $t_1 < t < t_2$, toward $x^o(t) = Y(t) + Z(t)$, where $Y(t)$ is a (scalar) AC function, $-DM_o \leq Y(t) \leq DM_o + |i| + 1$, $Y(t_1) = 0$. Let us prove that $Y(t_2) \leq i$. For the subsequence $[k]$ we have extracted last, we have

$$t_{2k} \to t_2, \ x_k^o(t_{2k}) \to i, \ x_k^o(t_{2k}) = y_k^-(t_{2k}) + y_k^+(t_{2k}).$$

If $\bar{t}_2$ is any point $t_1 < \bar{t}_2 < t_2$, $\bar{t}_2$ as close as we wish to t_2, then $\bar{t}_2 < t_{2k}$ for all k sufficiently large (of the extracted sequence), since $t_{2k} \to t_2$. We can assume k so large that $\bar{t}_2 < t_{2k}$,

$$|\bar{t}_2 - t_{2k}| < 2|\bar{t}_2 - t_2|.$$

Then

$$|y_k^-(\bar{t}_2) - y_k^-(t_{2k})| = M_o|\bar{t}_2 - t_{2k}| \leq 2M_o|\bar{t}_2 - t_2|.$$

Since $y_k^+(t)$ is nondecreasing, we have $y_k^+(\bar{t}_2) \leq y_k^+(t_{2k})$, and finally

$$y_k^-(\bar{t}_2) + y_k^+(\bar{t}_2) \leq y_k^-(\bar{t}_2) + y_k^+(t_{2k})$$
$$\leq y_k^-(t_{2k}) + y_k^+(t_{2k}) + |y_k^-(\bar{t}_2) - y_k^-(t_{2k})|$$
$$\leq y_k^o(t_{2k}) + 2M_o|\bar{t}_2 - t_2|,$$

where $y_k^o(t_{2k}) \to i$ as $k \to +\infty$ and $y_k^o(t_{2k}) < i + k^{-1}$. Hence

$$y_k^-(\bar{t}_2) + y_k^+(\bar{t}_2) < i + 2M_o|\bar{t}_2 - t_2| + k^{-1}.$$

As $k \to +\infty$ (along the extracted sequence), we have

$$Y^-(\bar{t}_2) + Y_o^+(\bar{t}_2) \leq i + 2M_o|\bar{t}_2 - t_2|, \quad \text{or}$$

$$Y^-(\bar{t}_2) + Y^+(\bar{t}_2) + Z(\bar{t}_2) \leq i + 2M_o|\bar{t}_2 - t_2|,$$

where the third term in the first member is ≥ 0. Thus

$$Y(\bar{t}_2) = Y^-(\bar{t}_2) + Y^+(\bar{t}_2) \leq i + 2M_o|\bar{t}_2 - t_2|.$$

As $\bar{t}_2 \to t_2 - 0$, we obtain $Y(t_2) \leq i$, since Y is continuous at t_2.

We shall apply below Closure Theorem II to an auxiliary problem, which we shall now define. Let $\tilde{u} = (u^o, u) = (u^o, u^1, \ldots, u^m)$, let $\tilde{U}(t,x)$ be the set of all $\tilde{u} \in E_{m+1}$ with $u = (u^1, \ldots, u^m) \in U(t,x)$, $u^o \geq f_o(t,x,u)$, let $\tilde{x} = (x^o, x) = (x^o, x^1, \ldots, x^n)$, let $\tilde{f} = \tilde{f}(t,x,u)$ $= (\tilde{f}_o, f) = (\tilde{f}_o, f_1, \ldots, f_n)$ with $\tilde{f}_o = u^o$. Thus, $\tilde{f}$ depends only on $(t, x, \tilde{u}$ (instead of $(t, \tilde{x}, \tilde{u})$, and $\tilde{U}$ depends only on t, x (instead of $(t, \tilde{x})$. Finally, we consider the differential system

$$d\tilde{x}/dt = \tilde{f}(t,x,u)$$

or

$$dx^o/dt = u^o(t), \quad dx^i/dt = f_i(t,x,u),$$
$$i = 1, 2, \ldots, n,$$

with the constraints

$$\tilde{u}(t) \in \tilde{U}(t, x(t)),$$

or

$$u^0(t) \geq f_0(t, x(t), u(t)), \; u(t) \in U(t, x(t)),$$

a.e. in $[t_1, t_2]$, with moreover $x^0(t_1) = 0$, $(t, x(t)) \in A$, and $(x, u) \in \Omega$. We have here the situation discussed in Closure Theorem II, where $\tilde{x}$ replaces x, x replaces Y, x^0 replaces Z, n+1 replaces n, n replaces s, hence (n+1) - n = 1 replaces n-s. For the new auxiliary problem the cost functional is

$$J[x, u] = \int_{t_1}^{t_2} \tilde{f}_0 \, dt = \int_{t_1}^{t_2} u^0(t) dt = x^0(t_2).$$

Note that the set $\tilde{\tilde{Q}}(t, x) = \tilde{\tilde{f}}(t, x, \tilde{U}(t, x))$ of the new problem is the set of all $\tilde{z} = (z^0, z) \in E_{n+1}$ such that $z^0 = u^0$, since $\tilde{f}_0 = u^0$, $z = f(t, x, u)$, $u^0 \geq f_0(t, x, u)$, $u \in U(t, x)$. Thus, the sets $\tilde{U}, \tilde{\tilde{Q}}$ for this auxiliary problem are the sets $\tilde{U}, \tilde{\tilde{Q}}$ considered before.

We consider now the sequence of trajectories $\tilde{x}_k(t) = [x_k^0(t), x_k(t)]$, $t_{1k} \leq t \leq t_{2k}$, for the problem $J[\tilde{x}, \tilde{u}]$ corresponding to the control function $u_k(t) = [u_k^0(t), u_k(t)]$, with $u_k^0(t) = f_0(t, x_k(t), u_k(t))$, $u_k(t) \in U(t, x_k(t))$, and hence $\tilde{u}_k(t) \in \tilde{U}(t, x_k(t))$, $t_{1k} \leq t \leq t_{2k}$, k=1, 2, ... The sequence $[x_k(t)]$ converges in the metric ρ toward the AC vector function x(t), while $x_k^0(t) \to x^0(t)$ as $k \to +\infty$ for all $t \in (t_1, t_2)$, and

$x^o(t) = Y(t) + Z(t)$, where $Y(t)$ is AC in $[t_1, t_2]$ and $Z'(t) = 0$ a. e. in $[t_1, t_2]$.

By Closure Theorem II we conclude that $X(t) = [Y(t), x(t)]$ is a trajectory for the problem. In other words, there is a control function $\tilde{u}(t)$, $t_1 \leq t \leq t_2$, $\tilde{u}(t) = (u^o(t), u(t))$, with

$$
\begin{aligned}
dY/dt = u^o(t) &\geq f_o(t, x(t), u(t)), \qquad u(t) \in U(t, x(t)), \\
dx/dt &= f(t, x(t), u(t)),
\end{aligned}
\tag{28}
$$

a. e. in $[t_1, t_2]$, and

$$i \geq Y(t_2) = J[\tilde{x}, \tilde{u}] = \int_{t_1}^{t_2} u^o(t)dt. \tag{29}$$

First of all $[x(t), u(t)]$ is admissible for the original problem and hence belongs to Ω, since by hypothesis Ω is complete. From this remark, and relations (28) and (29) we deduce

$$i \leq I[x, u] = \int_{t_1}^{t_2} f_o(t, x(t), u(t))dt \leq \int_{t_1}^{t_2} u^o(t)dt \leq i.$$

Hence all $\leq$ signs can be replaced by $=$ signs, $u^o(t) = f_o(t, x(t), u(t))$ a. e. in $[t_1, t_2]$, and $I[x, u] = i$. This proves that i is attained in Ω. Thus, Existence Theorem I is proved in the case that A is compact.

Let us assume now that A is not compact but closed, that A is contained in a slab $[t_o \leq t \leq T,\ x \in E_n]$, t_o, T, finite, and that conditions (a) and (b) hold. If Z(t) denotes the scalar function $Z(t) = |x(t)|^2 + 1$, then the condition $x \cdot f \leq N(|x|^2 + 1)$ implies $Z' \leq 2NZ$ and hence, by integration from t^* to t, also

$$1 \leq Z(t) \leq Z(t^*) \exp \{2N|t^* - t|\}.$$

Since $[t^*, x(t^*)] \in P$ where P is a compact subset of A, there is a constant N_o such that $|x| \leq N_o$ for every $x \in P$, hence $1 \leq Z(t^*) \leq N_o^2 + 1$, and $1 \leq Z(t) \leq (N_o^2 + 1) \exp \{2NT - t_o)\}$. Thus, for $t_o \leq t \leq T$, Z(t) remains bounded, and hence $|x(t)| \leq D$ for some constant D. We can now restrict ourselves to the consideration of the compact part A_o of all points $(t, x) \in A$ with $t_o \leq t \leq T$, $|x| \leq D$. Thus, Theorem I is proved for A closed and contained in a slab as above, and under the hypothesis (a) and (b).

Let us assume that A is not compact, nor contained in a slab as above, but closed, and that hypothesis (a), (b) and (c) hold. First, let us take an arbitrary element $(\bar{x}(t), \bar{u}(t)) \in \Omega$ and let $j = I[\bar{x}, \bar{u}]$. Then we consider a bounded interval [a, b] of the t-axis containing the entire projection P_o of P on the t-axis, as well as the interval [-R, +R]. In the slab $[a \leq t \leq b,\ x \in E_n]$ conditions (a) and (b) hold and hence by the previous argument there is a D (constant) such that $|x(t)| \leq D$ and we can confine our attention from

$A \cap [a \leq t \leq b,\ x \in E_n]$ to some compact part of this set, say A_o. For A_o there is a function $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, and constants C_o, $D_o \geq 0$ such that $f_o(t, x, u) \geq \Phi(|u|)$ and $|f| \leq C_o + D_o|u|$ for each $(t, x, u) \in M$ with $(t, x) \in A_o$. Therefore by previous reasoning, there is an M_o (this argument still holds even if Φ_o, C_o, D_o and M_o depend on A_o) such that $f_o(t, x, u) \geq \Phi(|u|) \geq - M_o$ and $|f| \leq C_o + D_o|u|$ for each $(t, x, u) \in M$ with $(t, x) \in A_o$. Condition (c) guarantees that for each $(t, x, u) \in M$ with $(t, x) \notin A \cap [a \leq t \leq b,\ x \in E_n]$ and hence for each $(t, x, u) \in M$ with $(t, x) \notin A_o$, there is a $\mu > 0$ such that $f_o(t, x, u) \geq \mu > 0$. Now let $\ell = \mu^{-1}[|j| + 1 + (b-a)M_o]$, let $[a', b']$ denote the interval $[a-\ell, b+\ell]$. Then for any admissible pair (if any) $(x(t), u(t))$, $t_1 \leq t \leq t_2$, of the class Ω, whose interval $[t_1, t_2]$ is not contained in $[a', b']$, there is at least one point $t^* \in [t_1, t_2]$ with $(t^*, x(t^*)) \in P$, $a < t^* < b$, and a point $\bar{t} \in [t_1, t_2]$ outside $[a', b']$. Hence $[t_1, t_2]$ contains at least one subinterval, say E, outside $[a, b]$, of measure $\geq \ell$. Then $I[x, u] \geq \ell\mu - (b-a)M_o = |j| + 1 \geq i + 1$. Obviously, we may disregard any pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, whose interval $[t_1, t_2]$ is not contained in $[a', b']$. In other words we can limit ouselves to the closed part A' of all $(t, x) \in A$ with $a' \leq t \leq b'$. We are in the previous situation, and Theorem I is proved for any closed set A under the hypothesis (a), (b) and (c). Finally, we have to show that condition (a) can be replaced by condition (d) in any case. It is enough to prove Theorem I under the hypothesis that A is closed and contained in a slab $t_o \leq t \leq T$, t_o, T finite as above, and hypothesis (b) and (d).

First let us take F so large that the projection P* of P on the x-space is completely in the interior of the solid sphere $|x| \leq F$, and also so large that $F \geq T - t_o$. Let $\bar{x}(t)$, $\bar{u}(t)$, $\bar{t}_1 \leq t \leq \bar{t}_2$ be any arbitrary pair contained in Ω, and let j denote the corresponding value of the cost functional. Let $L = E^{-1}[FM_o + |j| + 1]$, and let us take $F_o = F + L$. If any admissible pair x(t), u(t), $t_1 \leq t \leq t_2$, of Ω possesses a point $(t_o, x(t_o))$ with $|x(t_o)| \geq F_o$, then x(t) possesses also a point $(t^*, x(t^*)) \in P$ with $|x(t_*)| \leq F$. Thus, there is at least a subarc Γ: $x = x(t)$, $t' \leq t \leq t''$, of x(t) along which $|x(t)| \geq F$ and x(t) passes from the value F to the value $F_o = F + L$. Such an arc Γ has a length $\geq L$. If $G = [t_1, t_2] - [t', t'']$, then for $t \in G$, $|x| \geq F$, $(t, x) \in A$, $f_o(t, x, u) \geq 0$ and letting A_o be the compact part of A with $t \in G$, $|x| \leq F$ we have, as before, an $M_o \geq 0$ such that $f_o + M_o \geq 0$ for each $(t, x, u) \in M$, $(t, x) \in A_o$. Thus, one has $f_o \geq - M_o$ for each $(t, x, u) \in M$ with $t \in G$. Then

$$I[x,u] = \int_{t_1}^{t_2} f_o dt = (G)\int f_o dt + \int_{t'}^{t''} f_o dt$$

$$\geq -(T - t_o)M_o + \int_{t'}^{t''} E|f|\,dt$$

$$\geq -FM_o + E\int_{t'}^{t''} |f|\,dt \geq -FM_o + EL$$

$$\geq |j| + 1 \geq i + 1.$$

As before we can restrict ourselves to the compact part A'_o of all points $(t, x) \in A$ with $t_o \leq t \leq T$, $|x| \leq F$. The case where A is closed, A is not contained in any slab as above, but conditions (b), (c) and (d) hold can be treated as before. The case where A is not compact and condition (g) holds, also can be treated as before. Theorem I is thereby completely proved.

10. A few corollaries.

Corollary 1 (A. F. Filippov's existence theorem for Pontryagin's problems). As in Theorem I, if $A = E_1 \times E_n$, $\tilde{f}(t, x, u) = (f_o, f) = (f_o, f_1, \dots, f_n)$ is continuous on M, $U(t, x)$ is compact for every (t, x) in A, $U(t, x)$ is an upper semicontinuous function of (t, x) in A, $\tilde{Q}(t, x) = \tilde{f}(t, x, U(t, x))$ is a convex subset of E_{n+1} for every (t, x) in A, conditions (a) and (c) are satisfied, and the class Ω of all admissible pairs for which $x(t_1) = x_1$, $x(t_2) = x_2$,

t_1, x_1, x_2 fixed, t_2 undetermined, is not empty, then $I[x, u]$ has an absolute minimum in Ω.

PROOF: This statement is a corollary of Theorem I. Indeed, under hypothesis (c) we can restrict A to the closed part A_0 of all $(t, x) \in A$ with $a' \leq t \leq b'$, and $|x| \leq N$ for some large N. If M_0 is the part of all (t, x, u) of M with $(t, x) \in A_0$, then the hypothesis that U(t, x) is compact and an upper semicontinuous function of (t, x) in A_0 certainly implies that U(t, x) satisfies property (U) in A_0 and that M_0 is compact (No. 5, (ii) and (iii)). Also, since $\tilde{Q}(t, x)$ is convex for every (t, x) by hypothesis, we deduce that Q(t, x) is an upper semicontinuous function of (t, x) and satisfies property (Q) (No. 5, (v) and (vi)). Also, $\tilde{\tilde{Q}}(t, x)$ is closed, convex, and satisfies property (Q) by force of Lemmas (viii) and (x) of No. 4.

Finally, since M_0 is compact, the growth condition $f_0 \geq \Phi$ and the remaining condition $|f| \leq C + D|u|$ are trivially satisfied. Thus, all conditions of Theorem I are satisfied, and Filippov's theorem is proved to be a particular case of Theorem I.

Corollary 2 (the Nagumo-Tonelli existence theorem for free problems). If A is a compact subset of the tx-space $E_1 \times E_n$, $f_0(t, x, u)$ is a continuous scalar function on the set $M = A \times E_n$, for every $(t, x) \in A$, $f_0(t, x, u)$ is convex as a function of u in A and there is a continuous scalar function $\Phi(\zeta)$, $0 \leq \zeta + \infty$, with $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$ such that $f_0(t, x, u) \geq \Phi(|u|)$ for all

$(t, x, u) \in M$, then the cost functional

$$I[x] = \int_{t_1}^{t_2} f_o(t, x(t), x'(t))dt$$

has an absolute minimum in any nonempty complete class Ω of absolutely continuous vector functions $x(t)$, $t_1 \leq t \leq t_2$, for which $f_o(t, x(t), x'(t))$ is L-integrable in $[t_1, t_2]$.

If A is not compact, but closed and contined in a slab $[t_o \leq t \leq T,\ x \in E_n]$, t_o, T finite, then the statement still holds under the additional hypotheses $(\tau_1) f_o \geq C|u|$ for all $(t, x, u) \in M$ with $|x| \geq D$ and convenient constants $C > 0$, $D \geq 0$; (τ_2) every trajectory $x(t)$ of Ω possesses at least one point $(t^*, x(t^*))$ on a given compact subset P of A. If A is not compact, nor contained in a slab as above, but A is closed, then the statement still holds under the additional hypotheses (τ_1), (τ_2), and (τ_3) $f_o(t, x, u) \geq \mu > 0$ for all $(t, x, u) \in M$ with $|t| \geq R$, and convenient constants $\mu > 0$ and $R \geq 0$.

PROOF: The free problem under consideration can be written as an optimal control problem with $m = n$, $f_i = u_i$, $i=1, \ldots, n$, $U(t, x) = E_m = E_n$, so that the differential system reduces to $dx/dt = u$, and the cost functional is

$$I[x, u] = \int_{t_1}^{t_2} f_o(t, x(t), u(t))dt = \int_{t_1}^{t_2} f_o(t, x(t), x'(t))dt.$$

First assume A to be compact. Then the set $\tilde{\tilde{Q}}(t, x)$ reduces here to the set of all $\tilde{z} = (z^o, z) \in E_{n+1}$ with $z^o \geq f_o(t, x, z)$, $z \in E_n$, where f_o is convex in z, and satisfies the growth condition

$$f_o \geq \Phi(|u|) \text{ with } \Phi(\zeta)/\zeta \to +\infty \text{ as } \zeta \to +\infty .$$

By lemma (x) of No. 5, f_o is normally convex in u, hence quasi-normally convex, and by lemma (viii), part (β) of No. 5, $\tilde{\tilde{Q}}$ satisfies property (Q) in A. Thus, all hypotheses of Theorem I of No. 9 are satisfied. If A is closed but contained in a slab as above then condition (a) of Theorem I reduces to $u \cdot x \leq C(|x|^2 + 1)$ which cannot be satisfied since we have no bound on u. On the other hand, the condition (d) $f_o \geq E|f|$ for some $E > 0$ reduces here to requirement (τ_1) and condition (b) to requirement (τ_2). Finally, if A is not compact, nor contained in a slab as above, but A is closed, then requirement (c) of Theorem I reduces to requirement (τ_3). All conditions of Theorem I are satisfied, and the cost functional $I[x, u] = I[x, x']$ has an absolute minimum in Ω.

Chapter II

Further Existence Theorems

11. An existence theorem with uniform growth of f_o.

Let us assume for the moment that A is compact. The condition of Theorem I

(α) $f_o(t,x,u) \geq \Phi(|u|)$ for each $(t,x,u) \in M$ where $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, is a continuous scalar function satisfying $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$; $|f(t,x,u)| \leq C + D|u|$ for all $(t,x,u) \in M$ and some constants $C, D \geq 0$;

is usually called a "growth condition." This condition (α) obviously implies

(β) $f_o(t,x,u)\,|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t,x)$ for each $(t,x) \in A$; $|f(t,x,u)| \leq C + D|u|$ for all $(t,x,u) \in M$ and some constants $C, D \geq 0$.

This condition (β) is also a "growth condition", and examples show that—in general—(β) is a weaker condition than (α). An example of this is given below. Nevertheless, there are situations where (α) and (β) (for A compact) are equivalent. One of these situations concerns free problems, that is, m=n, f=u, $U=E_n$. For these problems, with f_o convex in u for every (t, x) in A, conditions (α) and (β) are equivalent (Tonelli [23a] for f_o of class C^1; L.Turner [24] for f_o only continuous as assumed here). This case of equivalence, together with another relevant case of equivalence will be obtained below as a consequence of a number of lemmas.

Example: Let $m = n = 1$, $A = [(t,x) | 0 \leq t \leq 1,\ 0 \leq x \leq 1]$,

$$
\begin{aligned}
U(t,x) &= E_1 \text{ for each } (t,x) \in A,\ f(t,x,u) = u \text{ and} \\
f_o(t,x,u) &= tu^2 \text{ for } 0 \leq u \leq 2^{-1}t(1-t)^{-1},\ 0 \leq t \leq 1 \\
&= t[u-t(1-t)^{-1}]^2 \text{ for } u \geq 2^{-1}\, t(1-t)^{-1},\ 0 \leq t < 1 \\
&= u^2 \text{ for } 0 \leq u < +\infty,\ t=1 \\
f_o(t,x,u) &= f_o(t,x,-u) \text{ for } -\infty < u < +\infty,\ 0 \leq t \leq 1.
\end{aligned}
$$

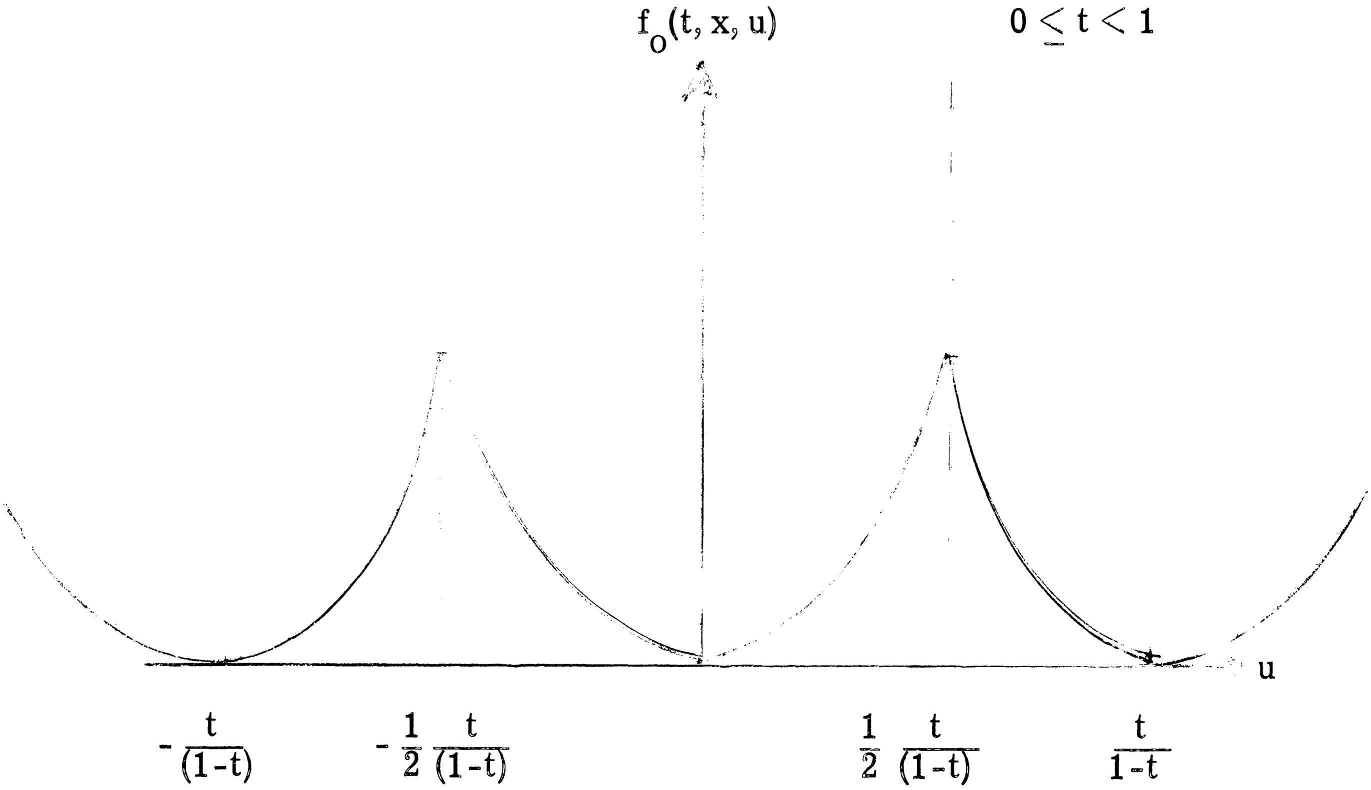

Figure 1 An example where $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in E_1$ pointwise in A, but not uniformly.

Hence

$$f_o(t, x, o) = f_o(t, x, t(1-t)^{-1}) = 0$$

$$f_o(t, x, 2^{-1}t(1-t)^{-1}) = 2^{-2}t^3(1-t)^{-2} \text{ for } 0 \leq t < 1, \ 0 \leq x \leq 1,$$

and f_o is a continuous function of (t, x, u) in $A \times E_1$. The second part of both conditions (α) and (β) is satisfied by $C = 0$, $D = 1$ as $|f| = |u|$ in $A \times E_1$. For $t = 1$, we have $f_o = u^2$ and for $0 \leq t < 1$, we have $f_o = (u - t(1-t)^{-1})^2$ for $|u|$ sufficiently large. Hence $f_o/|u| \to +\infty$ as $|u| \to +\infty$, for every $(t, x) \in A$, and condition (β) is satisfied. On the other hand, for every $|u|$, there are $(t, x) \in A$ with $f_o = 0$, namely all (t, x) with $|u| = t(1-t)^{-1}$, or $t = |u|(|u| + 1)^{-1}$. Hence, a relation $f_o(t, x, u) \geq \Phi(|u|)$ can be satisfied for all $(t, x, u) \in A \times E_1$, only with $\Phi \leq 0$, and condition (α) is not satisfied. In this example f_o is not convex in u, and $U(t, x) = E_1$.

An analogous example with f_o convex in u can be obtained by taking A, f as above, by taking $U(t, x) = [u \mid t(1-t)^{-1} \leq u < +\infty]$ for $0 \leq t < 1$, $U(1, x) = [u \mid 0 \leq u < +\infty]$ and by defining f_o as above. Then, for every $(t, x) \in A$, f_o is convex in u for $u \in U(t, x)$, condition (β) is satisfied and condition (α) is not.

A condition slightly stronger than (β) has been taken into consideration, say (γ), and we state it here again for A compact.

(γ) $f_o(t, x, u)\, |u|^{-1} \to +\infty$ uniformly for $(t, x) \in A$ as $|u| \to +\infty$, $u \in U(t, x)$; $|f(t, x, u)| \leq C + D|u|$ for each $(t, x, u) \in M$ and some constants C, $D \geq 0$.

Obviously, condition (a) implies condition (γ). The following three lemmas show that (a) is equivalent to (γ) when A is compact.

Lemma 1: Let A be a fixed compact subset of the tx-space $E_1 \times E_n$, and for every $(t, x) \in A$ let $U(t, x)$ be any subset of the u-space E_m. Let $f_o(t, x, u)$ be a continuous scalar function on the set M of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$, and let Z be the subset of all nonnegative z with $z = |u|$ for some $(t, x, u) \in M$. Then $|u|^{-1} f_o(t, x, u) \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$, uniformly on A, if and only if there is a scalar function $\Phi(z)$, $z \in Z$, bounded below, with $\Phi(z)|z \to +\infty$ as $z \to +\infty$, $z \in Z$, and $f_o(t, x, u) \geq \Phi(|u|)$ for each $(t, x, u) \in M$.

PROOF: Suppose such a $\Phi(z)$ exists with the above properties. Then for $u \neq 0$, $f_o|u|^{-1} \geq \Phi(|u|)|u|^{-1}$ and hence $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$, uniformly on A. Suppose $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$ uniformly on A. For each fixed $z \in Z$, let $\Phi(z) = \inf f_o(t, x, u)$ for all $(t, x) \in A$, $|u| = z$, $u \in U(t, x)$. If $\Phi(z)/z$ does not approach $+\infty$ as $z \to +\infty$, $z \in Z$, then there exists an $N > 0$ and a sequence $(t_\nu, x_\nu, u_\nu) \in M$ such that

$$f_o(t_\nu, x_\nu, u_\nu)|u_\nu|^{-1} < N, \quad |u_\nu| \to +\infty \text{ as } \nu \to +\infty.$$

As $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$, uniformly on A, then for $N^* = N + 1$ there exists a constant $\Lambda > 0$ such that $f_o |u|^{-1} \geq N^* = N + 1$ for all $u \in U(t, x)$, $|u| \geq \Lambda$ and this last inequality holds for any $(t, x) \in A$. Therefore, $N + 1 = N^* \leq f_o(t_\nu, x_\nu, u_\nu) |u_\nu|^{-1} \leq N$ for ν large enough so that $|u_\nu| \geq \Lambda$. This is a contradiction and thus, $\Phi(z)/z \to +\infty$ as $z \to +\infty$, $z \in Z$. Clearly $f_o(t, x, u) \geq \Phi(|u|)$ for each $(t, x, u) \in M$ by the definition of $\Phi(z)$, $z \in Z$.

Since $\Phi(z)/z \to +\infty$ as $z \to +\infty$, $z \in Z$, we also conclude that $\Phi(z) \to +\infty$ as $z \to +\infty$, $z \in Z$ and hence there is some z_o such that $\Phi(z) \geq 0$ for all $z \geq z_o$, $z \in Z$. Now the set S' of all points $(t, x, u) \in M$ where u belongs to a sphere $S = [u | \, |u| \leq z_o]$ is a compact set, as the intersection of the closed set M with the compact set $A \times S$. Thus, f_o is continuous on S', hence bounded there, say $|f_o| \leq M_o$, and thus $\Phi(z) \geq -M_o$ for each $z \in Z \cap [o, z_o]$. This proves that $\Phi(z)$ is bounded below in Z. Lemma 1 is proven.

<u>Lemma 2:</u> $\Phi_1(z)$ is a scalar function of z on a subset Z of $[o, +\infty)$, if $\Phi_1(z)$ is bounded below, and if $\Phi_1(z)/z \to +\infty$ as $z \to +\infty$, $z \in Z$, there is also a scalar function $\Phi(z)$, continuous on $[o, +\infty)$ such that $\Phi_1(z) \geq \Phi(z)$ for each $z \in Z$ and $\Phi(z)/z \to +\infty$ as $z \to +\infty$.

PROOF: Let Λ_o be a bound below for $\Phi_1(z)$, $z \in \mathbf{Z}$. Now for each n=1, 2, ... there exists a $\Lambda(n) > 0$ such that $\Phi_1(z)|z \geq n$ when $z \geq \Lambda(n)$, $z \in Z$ and we can assume that $\Lambda(n) < \Lambda(n+1)$, n=1, 2, ... Let $\Phi_2(z)$, $-1 \leq z < +\infty$, be the function defined by

$$\Phi_2(z) = \Lambda_o \text{ when } z \in [-1, \Lambda(1))$$

$$\Phi_2(z) = nz \text{ when } z \in [\Lambda(n), \Lambda(n+1))$$

$$\text{for } n=1, 2, \ldots$$

Clearly $\Phi_1(z) \geq \Phi_2(z)$ for $z \in Z$ and $\Phi_2(z)/z \to +\infty$ as $z \to +\infty$, and $\Phi_2(z)$ is monotone nondecreasing in $[-1, +\infty)$. For any compact subset K of $[-1, +\infty)$, $\Phi_2(z)$ is Lebesque integrable.

Let $\Phi(z)$, $0 \leq z < +\infty$, be the function defined by

$$\Phi(z) = \int_{z-1}^{z} \Phi_2(z')\,dz', \qquad 0 \leq z < +\infty$$

As $[z-1, z]$ is a compact subset of $[-1, +\infty)$ for each $z \in [0, +\infty)$, $\Phi(z)$ exists for each $z \in [0, +\infty)$. On the other hand,

$$\Phi(z)/z = z^{-1} \int_{z-1}^{z} \Phi_2(z')dz' \geq (z-1)z^{-1}(z-1)^{-1}\Phi_2(z-1)$$

$$\geq \frac{1}{2} n \text{ for all } z > 2$$

with $z \in [\Lambda(n) + 1, \Lambda(n+1) + 1)$ Thus $\Phi(z)/z \to +\infty$ as $z \to +\infty$. Finally, by the monotoneity of Φ_2 we have

$$\Phi(z) = \int_{z-1}^{z} \Phi_2(z')dz' \leq \Phi_2(z) \leq \Phi_1(z)$$

for all $z \in Z$.

Since $\Phi_2(z)$ is L-integrable in any finite interval of $[-1, +\infty)$, we conclude that $\Phi(z)$ is a continuous function in $[0, +\infty)$.

Lemma 3: Let A be a fixed compact subset of the tx-space $E_1 \times E_n$. If $f_o(t, x, u)$ is a continuous scalar function on M, the set of all (t, x, u) with $(t, x) \in A$, $u \in U(t, x)$, then $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$, uniformly on A, if and only if there exists a continuous scalar function $\Phi(z)$ on $[0, +\infty)$ such that $\Phi(z)/z \to +\infty$ as $z \to +\infty$ and $f_o(t, x, u) \geq \Phi(|u|)$ for each $(t, x, u) \in M$.

PROOF: The sufficiency is obvious. The necessity is an immediage consequence of lemmas 1 and 2.

Having proved that properties (α) and (γ) are equivalent for the general optimal control problem formulated as a Lagrange problem, we proceed now to the consideration of two special cases, where it is possible to replace property (α) by the weaker property (β) in the statement of Existence Theorem II. These two cases will be the object of lemmas 4 and 5 below.

In both lemmas 4 and 5 we shall assume that the control space is a fixed closed subset of the u-space E_m, and therefore that M has the form $M = A \times U$ where A is a given compact subset of the tx-space $E_1 \times E_n$. Let $f_o(t, x, u)$, $(t, x, u) \in A \times U$ be a given scalar function. We shall say that $f_o(t, x, u)$, $(t, x, u) \in A \times U$ is a uniformly continuous function of (t, x) in A with respect to u, if for each $\epsilon > 0$ and $(t_o, x_o) \in A$ there is a $\delta = \delta(t_o, x_o, \epsilon) > 0$ such that

$$|f_o(t, x, u) - f_o(t_o, x_o, u)| < \epsilon$$

for all $\quad (t, x) \in A, \; u \in U, \; (t, x) \in N_\delta(t_o, x_o)$. $\quad$ (30)

Lemma 4: Let A be a compact subset of the tx-space $E_1 \times E_n$, let U be a fixed closed subset of the u-space E_m, and let $f_o(t, x, u)$, $(t, x, u) \in A \times U$, be a continuous scalar function, continuous on M and a uniformly continuous function of (t, x) in A with respect to u. Then $f_o(t, x, u)/|u| \to +\infty$ as $|u| \to +\infty$, $u \in U$, for every $(t, x) \in A$, if and only if $f_o(t, x, u)/|u| \to +\infty$ as $|u| \to +\infty$, $u \in U$, uniformly on A.

PROOF: The sufficiency is obvious.

Suppose $f_o/|u|$ does not tend to $+\infty$ as $|u| \to +\infty$, $u \in U$, uniformly on A. Then there is a sequence $(t_\nu, x_\nu, u_\nu) \in M = A \times U$ with $|u_\nu| \to +\infty$ as $\nu \to +\infty$ and $f_o(t_\nu, x_\nu, u_\nu)|u_\nu|^{-1} \leq N$ for some constant N. The sequence (t_ν, x_ν), $\nu = 1, 2, \ldots$ has a convergent subsequence, as A is compact. Let $(t_o, x_o) \in A$ be the limit of this convergent subsequence.

As $f_o(t, x, u)$ is a continuous function on M and a uniformly continuous function of (t, x) in A with respect to u, for $\epsilon = 1$, there is a $\delta = \delta(t_o, x_o, 1) > 0$ such that

$$|f_o(t,x,u) - f_o(t_o,x_o,u)| \leq 1$$

when $(t,x,u) \in M$ with $(t,x) \in N_\delta(t_o,x_o)$. Given any constant $L \geq 1$, then $|u| \geq L$ implies that $|u| \geq L \geq 1$ and

$$|f_o(t,x,u)|u|^{-1} - f_o(t_o,x_o,u)|u|^{-1}| \leq |u|^{-1} \leq L^{-1} \leq 1 \qquad (31)$$

when $(t,x,u) \in M$ with $(t,x) \in N_\delta(t_o,x_o)$.

Since $(t_\nu, x_\nu) \to (t_o, x_o)$ as $\nu \to +\infty$ then $(t_\nu, x_\nu) \in N_\delta(t_o,x_o)$ for all ν sufficiently large, and $f_o(t_\nu, x_\nu, u_\nu)/|u_\nu| \leq N$ together with (31) implies that $f_o(t_o, x_o, u_\nu)|u_\nu|^{-1} \leq N+1$ for all ν sufficiently large, and as $|u_\nu| \to +\infty$ as $\nu \to +\infty$, $u_\nu \in U$, this contradicts the hypothesis $f_o(t_o,x_o,u)|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$. Lemma 4 is proven.

We have already shown by one of the examples in No. 11 that there are functions $f_o(t,x,u)$, $(t,x,u) \in A \times U$, for which $f_o(t,x,u)|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$, for every $(t,x) \in A$ (compact), but for which the same limit does not occur uniformly in A. Obviously, in the example given in No. 11 f_o is continuous on M, but is not a uniformly continuous function of (t,x) in A with respect to u. On the other hand, lemma 4 concerns a class of functions which is not empty. Below we give a class of functions f_o satisfying all conditions of lemma 4.

Example: Let A be a fixed, arbitrary, compact subset of the tx-space $E_1 \times E_n$ and let $U(t, x) = U$ be some fixed, closed (but not necessarily bounded) subset of E_m. Define $f_o(t, x, u) = g(t, x)h(u) + k(u)$ where $g(t, x)$ is a continuous function on A, $h(u)$ is a bounded, continuous function on U, and $k(u)$ is a continuous function on U such that $k(u)|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$. Thus, $f_o(t, x, u)$ is a uniformly continuous function of (t, x) in A with respect to u and $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$ pointwise, and therefore by force of lemma 4, uniformly in A.

Lemma 5 will allow us to replace property (α) in Existence Theorem I by the weaker property (β) instead of property (γ) in the special case of the free problems of the calculus of variations. Actually, lemma 5 is slightly more general than needed for free problems. Its proof is due to L. Turner [24].

Lemma 5: Let A be a fixed compact subset of the tx-space $E_1 \times E_n$, and let $U(t, x) = E_m$ for each $(t, x) \in A$. Let $f_o(t, x, u)$ be a continuous scalar function on $A \times E_m$ and a convex function of u for each $(t, x) \in A$. Then, $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in E_m$ pointwise in A if and only if the same limit holds uniformly in A.

PROOF: The sufficiency of the condition is trivial.

Let us prove its necessity. Suppose $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$ pointwise in A, but not uniformly. Then, there exists some $N > 0$ and a sequence $(t_\nu, x_\nu, u_\nu) \in A \times E_m$, $\nu = 1, 2, \ldots$ such that $f_o(t_\nu, x_\nu, u_\nu)|u_\nu|^{-1} \leq N$, $\nu = 1, 2, \ldots$ and $|u_\nu| \to +\infty$ as $\nu \to +\infty$. Without loss of generality, suppose $(t_\nu, x_\nu, u_\nu|u_\nu|^{-1})$, $\nu = 1, 2, \ldots,$ is convergent to $(t_o, x_o, u_o) \in A \times E_m$ with $|u_o| = 1$. Now $f_o(t_o, x_o, u)|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$. Since A is compact, $f_o(t, x, o)$ is bounded in A, say $|f_o(t, x, o)| \leq C$, there is a constant λ_o such that

$$\lambda^{-1} f_o(t_o, x_o, \lambda u_o) > N + C \quad \text{when } \lambda \geq \lambda_o$$

(one can clearly assume that $\lambda_o \geq 1$). By continuity, there is a $\alpha > 0$ such that $\lambda_o^{-1} f_o(t, x, \lambda_o u) > N + C$ if

$$(t, x) \in N_\delta(t_o, x_o), \quad u, u_o \in E_m, \quad |u - u_o| < \delta .$$

Let ν_o be large enough so that $(t_\nu, x_\nu) \in N_\delta(t_o, x_o)$, $|u_o - u_\nu|u_\nu|^{-1}| < \delta$ and $|u_\nu| > \lambda_o$ for each $\nu > \nu_o$. Then

$$\lambda_o u_\nu |u_\nu|^{-1} = (1 - \lambda_o |u_\nu|^{-1})o + \lambda_o |u_o|^{-1} u_\nu$$

is a convex combination for $\nu > \nu_o$ and

$$\begin{aligned} f_o(t_\nu, x_\nu, \lambda_o |u_\nu|^{-1} u_\nu) &\leq (1 - \lambda_o |u_\nu|) f_o(t_\nu, x_\nu, o) + \lambda_o |u_\nu|^{-1} f_o(t_\nu, x_\nu, u_\nu) \\ &\leq C + \lambda_o |u_\nu|^{-1} f_o(t_\nu, x_\nu, u_\nu) \text{ for } \nu > \nu_o. \end{aligned}$$

Therefore

$$f_o(t_\nu, x_\nu, u_\nu)|u_\nu|^{-1} \geq \lambda_o^{-1} f_o(t_\nu, x_\nu, \lambda_o |u_\nu|^{-1} u_\nu) - C\lambda_o^{-1}$$
$$\geq \lambda_o^{-1} f_o(t_\nu, x_\nu, \lambda_o |u_\nu|^{-1} u_\nu) - C$$

as λ_o can be assumed to be ≥ 1. Thus, $f_o(t_\nu, x_\nu, u_\nu)|u_\nu|^{-1} > N$ which is a contradiction and lemma 5 is proven.

Lemmas 3, 4 and 5 have immediate consequences when combined with Existence Theorem I. We give below Existence Theorem II, which is an equivalent form of Existence Theorem I, together with a few corollaries to Existence Theorem II.

When A is not compact but closed, condition (γ) needs to be slightly altered to the new condition $(\gamma)_m$ defined by

$(\gamma)_m$ $|u|^{-1} f_o(t, x, u) \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$,

uniformly for (t, x) in any compact part A_o of A. For every compact part A_o of A there are constants $C_o, D_o \geq 0$ such that $|f(t, x, u)| \leq C_o + D_o|u|$ for each $(t, x, u) \in M$ with $(t, x) \in A_o$.

<u>Existence Theorem II</u> (Equivalent form of Existence Theorem I). Theorem II is the same as Existence Theorem I where the growth condition (α) is replaced by (γ) if A is compact.

If A is not compact but closed and contained in a slab $[t_o \leq t \leq T, \ x \in E_n]$, t_o, T finite, then Existence Theorem II still holds if (a), (b) hold, and (α) is replaced by $(\gamma)_m$. If A is not compact nor contained in a slab as above, but A is closed, then

Theorem II still holds if (a), (b), (c) hold and (α) is replaced by $(\gamma)_m$. Finally, condition (a) can be replaced in any case by condition (d).

This theorem is a consequence of Existence Theorem I and lemma 3.

Corollary 1: This is the same as Existence Theorem II where for each $(t, x) \in A$ $U(t, x) = U$ is a fixed, closed subset of the u-space E_m, condition (γ) and $(\gamma)_m$ are replaced by the assumptions that (i) $f_o(t, x, u)$ is a uniformly continuous function of (t, x) in A with respect to u, and (ii) $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$ pointwise in A.

It is clear that if U is a fixed, closed subset of E_m, then U satisfies property (U) in A.

PROOF: This corollary follows from Existence Theorem II and lemma 4.

Corollary 2: This is the same as Theorem II where for each $(t,x) \in A$ $U(t, x) = E_m$ is the whole u-space E_m, condition (γ) and $(\gamma)_m$ are replaced by the assumptions (i) $f_o(t, x, u)$ is a convex function of u for each $(t, x) \in A$, and (ii) $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in E_m$ pointwise in A. (It is clear that $U = E_m$ satisfies property (U) in A.)

In addition, if m = n and $f_i(t, x, u) = u^i$, $i=1, 2, \ldots, n$, then the condition that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) in A can be relaxed in the above case.

PROOF: The first statement follows from Theorem II and lemma 5. The last statement follows from lemma 5 of Chapter II and lemma 7 of Chapter III.

Theorem II extends to Lagrange problems and problems of optimal control the analogous Existence Theorem II for free problemas proved by Tonelli in [23a] for n=1 and f_0 of class C^1, and proved by L. Turner in [24] for any $n \geq 1$ and f_0 of class C^0. Although condition (α) appears stronger than that given by Tonelli or L. Turner, lemma 5 shows that it is not stronger in the case of free problems.

Let us give an example of an optimal control problem to which Existence Theorems I and II apply.

Example 1: Let $m = n = 1$, $A = [0,1]^2$, for each $(t,x) \in A$ let $U(t,x) = E_1$, let $f(t,x,u) = u$, $f_0(t,x,u) = u^2$ and boundary conditions $x(0) = x(1) = 0$. Then $U(t,x)$ satisfies property (U) in A and $\tilde{\tilde{Q}}(t,x) = \tilde{\tilde{Q}} = \{(z_0, z) | \ z_0 \geq z^2, \ z \in E_1\}$ is a fixed closed, convex subset of $E_1 \times E_1$ and obviously satisfies property (Q) in A. Now $f_0/|u| = u^2/|u| = |u| \to +\infty$ as $|u| \to +\infty$, $u \in E_1$ and the limit is uniform in A. It is clear that Existence Theorem II applies. If one chooses, $\Phi(z) = z^2$ for each $z \in [0, +\infty)$, then $f_0(t,x,u) = u^2 \geq |u|^2 = \Phi(|u|)$ for each $(t,x,u) \in [0,1]^2 \times E_1$ and $\Phi(z)/z \to +\infty$ as $z \to +\infty$, $z \in E_1$, $z > 0$. Thus, Existence Theorem I also applies to this example as the equivalence of Existence Theorems I and II imply.

However, Existence Theorems I and II do not apply to the following example.

Example 2: Let $m = n = 1$, $A = [0, 1]^2$, for each $(t, x) \in A$ let $U(t, x) = E_1$, let $f(t, x, u) = u$, $f_o(t, x, u) = tu^2$ and let the boundary conditions be $x(0) = x(1) = 0$. Then $U(t, x)$ satisfies property (U) for each $(t, x) \in A$ and $\tilde{\tilde{Q}}(t, x) = \{(z_o, z) \mid z_o \geq tz^2, \ z \in E_1\}$. Now,

$$\tilde{\tilde{Q}}(o, x, \delta) = \{(z_o, z) \mid z_o \geq 0, \ z \in E_1\} = \tilde{\tilde{Q}}(o, x).$$

Hence, $\tilde{\tilde{Q}}(o, x) = \bigcap_{\delta} \text{cl co}\, \tilde{\tilde{Q}}(o, x, \ )$ for each $x \in [0, 1]$. Therefore $\tilde{\tilde{Q}}(o, x)$ satisfies property (Q) for each $(t, x) \in \{0\} \times [0, 1]$. Since $\tilde{\tilde{Q}}(t, x)$ obviously satisfies property (Q) for each $(t, x) \in (0, 1] \times [0, 1]$, we conclude that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) in A. Although $f_o/|u| \to +\infty$ as $|u| \to +\infty$, $u \in E_1$ uniformly on each compact subset of $(0, 1] \times [0, 1]$, $f_o(t, x, u)|u|^{-1}$ does not approach $+\infty$ as $|u| \to +\infty$ for any $(t, x) \in A$ with $t = 0$. Hence $f_o/|u|$ does not tend to $+\infty$ as $|u| \to +\infty$, $u \in E_1$ uniformly on A and neither Existence Theorem I nor II applies to this example.

Nevertheless this example possesses an obvious optimal solution given by $x(t) = 0$, $u(t) = 0$, $0 \leq t \leq 1$. Let us consider the same example with slightly modified boundary condition $x(0) = 1$, $x(1) = 0$. Theorems I and II still do not apply and the new problem is known to have no optimal solution. Indeed let $i = \inf I[x, u]$, where the infimum is taken over the class Ω of all admissible pairs, that is the class of all pairs $x(t)$, $u(t)$ such that $x(t)$, $0 \leq t \leq 1$ is an AC scalar function with $x(0) = 1$, $x(1) = 0$, with tx'^2 L-integrable in $[0, 1]$ and $u(t) = x'(t)$ almost everywhere. Then $I[x, u] \geq 0$ in Ω, and therefore $i \geq 0$. If we consider the sequence $x_k(t)$, $u_k(t)$, $0 \leq t \leq 1$, defined by $x_k(t) = 1$, $u_k(t) = 0$ for $0 \leq t \leq k^{-1}$, $x_k(t) = -(\log t)(\log k)^{-1}$, $u_k(t) = -t^{-1}(\log k)^{-1}$ for $k^{-1} < t \leq 1$, and $k = 2, 3, \ldots$, we have $I[x_k, u_k] = (\log k)^{-1}$ and therefore $I[x_k, u_k] \to 0$ as $k \to +\infty$. Thus, $i = 0$. But $I[x, u] = 0$ implies that $t u^2 = 0$ almost everywhere, and hence $u(t) = 0$ almost everywhere and $x(t) =$ constant on $[0, 1]$. This is impossible as $x(0) = 1$, $x(1) = 0$. Thus, the problem above with boundary condition $x(0) = 1$, $x(1) = 0$ has no optimal solution and Theorems I and II do not apply to either of these examples.

12. An existence theorem with exceptional points.

For free problems it was shown by Tonelli [23a] that the growth condition (α) can be dispensed with at the points (t, x) of an exceptional subset E of A provided some additional mild hypothesis is satisfied at the points of E, or E is a suitable "slender" set. This situation recurs for Lagrange problems as we shall state in Theorems III and IV below.

Let us consider again general Lagrange problems (optimal control problems) as stated in No. 1, and let E be a given subset of A. We shall need a new condition, say (γ^*), another modification of condition (γ) of no. 11.

(γ^*) $\quad |u|^{-1} f_o(t,x,u) \to +\infty$ as $|u| \to +\infty$, $u \in U(t,x)$

uniformly for (t,x) in any compact part A_o of A-E; for every compact part A_o of A-E there are constants $C_o, D_o \geq 0$ such that $|f(t,x,u)| \leq C_o + D_o|u|$ for all $(t,x) \in A_o$, $u \in E(t,x)$.

We shall need also a local property concerning the relative behavior of f_o and f in the neighborhood of given points $(t_o, x_o) \in A$. We shall denote it property (T), since it is modeled on an analogous condition used by Tonelli [23a] for free problems, n=1 and f_o of class C^1.

We shall denote by $N_\delta^o(t,x)$ the open neighborhood of radius δ of (t,x) in A, that is, the set of all points $(t',x') \in A$ at a distance $< \delta$ from (t,x). A point $(t_o, x_o) \in A$ is said to possess property (T) provided there is a neighborhood $N_\delta^o(t_o, x_o)$ in A, two functions

$$\phi(\xi), \quad 0 < \xi \leq 1, \quad \Psi(\zeta), \quad 0 < \zeta < +\infty$$

and five constants $\ell > 0$, $\alpha > 0$, μ real, $C_o \geq 0$, with $\phi(\xi)$ nonnegative, $\phi(0+) = +\infty$, ϕ integrable in $(0, \ell)$, $\Psi(\zeta)$ nonnegative, nondecreasing such that $\xi\, \phi(\xi)\Psi(\phi(\xi)) \to +\infty$ as $\xi \to 0+$ and such that $(t,x) \in N_\delta^o(t_o, x_o)$, $u \in U(t,x)$ implies that

$$f_0(t,x,u) \geq |t-t_0|^{\alpha} |u|^{1+\alpha} \Psi^{\alpha}(|u|) + \mu,$$
$$|f(t,x,u)| \leq C_0 + D_0|u| \quad \text{for each } (t,x) \in N^0_{\delta}(t_0,x_0), \; u \in U(t,x) \tag{32}$$

For instance condition (T) is certainly satisfied at the point $(t_0, x_0) \in A$ if there is a neighborhood $N^0_\delta(t_0, x_0)$ in A, and constants $k > 0$, $\alpha > 0$, $\sigma > 0$, μ real, $C_0 \geq 0$, $D_0 \geq 0$, such that

$$f_0(t,x,u) \geq k|t-t_0|^{\alpha} |u|^{1+\alpha+\sigma} + \mu,$$
$$|f(t,x,u)| \leq C_0 + D_0|u| \quad \text{for each } (t,x) \in N^0_{\delta}(t_0,x_0), \; u \in U(t,x). \tag{33}$$

Indeed we have only to take $\ell = 1$, choose a number β such that $\alpha(\alpha+\sigma)^{-1} < \beta < 1$, and select

$$\phi(\xi) = \xi^{-\beta}, \quad 0 < \xi \leq 1,$$

$$\Psi(\zeta) = k^{1/\alpha} \zeta^{\sigma/\alpha}, \quad 0 < \zeta < +\infty$$

Here ϕ is nonnegative, $\phi(0+) = +\infty$ since β is greater than zero, ϕ is integrable in $(0,1)$ since β is between zero and one, Ψ is nonnegative, nondecreasing since $\alpha > 0$, and

$$\begin{aligned} \xi\, \phi(\xi)\, \Psi(\phi(\xi)) &= \xi\, \xi^{-\beta}\, k^{1/\alpha} (\xi^{-\beta})^{\sigma/\alpha} \\ &= k^{1/\alpha} \xi^{1-\beta-\beta\sigma/\alpha} \\ &= k^{1/\alpha} \xi^{1-\beta(\alpha+\sigma)/\alpha} \end{aligned}$$

and $\xi\,\phi(\xi)\Psi(\phi(\xi)) \to +\infty$ as $\xi \to 0+$ since $1 - \beta(\alpha+\sigma)/\alpha < 0$

On the other hand

$$f_0(t,x,u) \geq k\,|t-t_0|^{\alpha}\,|u|^{1+\alpha+\sigma} + \mu =$$
$$= |t-t_0|^{\alpha}\,|u|^{1+\alpha}(k^{1/\alpha}|u|^{\sigma/\alpha})^{\alpha} + \mu =$$
$$= |t-t_0|^{\alpha}\,|u|^{1+\alpha}\,\Psi^{\alpha}(|u|) + \mu$$

In the Existence Theorems III and IV below concerning Lagrange problems, viewed as optimal control problems, E is a closed subset of $E_1 \times E_n$. Although this condition was not explicitly stated in Tonelli's papers [23] concerning free problems, we shall show that E can always be assumed to be a closed set for free problems in each of the cases corresponding to Existence Theorems III and IV, and, therefore, Tonelli's Theorems satisfy this hypothesis in Theorems III and IV below. We shall prove this property of the set E after each of the theorems.

Existence Theorem III.

Theorem III is the same as Existence Theorem I, where a closed exceptional subset E of A is given, condition (T) holds at every point of E and condition (α) is replaced by (γ*) if A is compact.

If A is not compact, but closed and contained in a slab $[t_0 \leq t \leq T, x \in E_n]$, t_0, T finite, then Theorem III still holds if (a) and (b) hold (and (α) is replaced by (γ*) and condition (T) holds on the exceptional set E).

If A is not compact, nor contained in a slab as above, but A is closed, then Theorem III still holds if (a), (b) and (c) hold (and (α) is replaced by (γ*) and condition (T) holds on the exceptional set E). Finally, condition (a) can be replaced in any case by condition (d).

PROOF: Let us prove the result for the case when A is compact. If A is compact, then $E \subset A$ is compact. Because of property (T), to each point $(t, x) \epsilon E$ there is an open neighborhood $N_\delta^o(t, x)$ with the properties described in property (T). Therefore, by the compactness of E, a finite number of the neighborhoods $N_\delta^o(t, x)$, say $A_i = N_{\delta_i}^o(t_i, x_i)$, $i=1, 2, \ldots, \ell$, and $(t_i, x_i) \epsilon E$ for each i, cover E. Consider the set $A_o = A - \bigcup_{i=1}^{\ell} A_i$ which is clearly a compact subset of A - E. By property (T) in each $N_{\delta_i}^o(t_i, x_i)$, and by the uniform growth condition in the compact set A_o, there exists real numbers μ_i, $i=0, 1, \ldots, \ell$, such that

$$f_o(t, x, u) \geq \mu_i \text{ for each } (t, x, u) \epsilon M \text{ with } (t, x) \epsilon A_i, \quad i=0, 1, 2, \ldots, \ell.$$

We shall denote by "sub i" any of the elements defined by property (T) relative to (t_i, x_i), $i=1, 2, \ldots, \ell$. Letting $\bar{\mu} = \max(|\mu_o|, \ldots, |u_\ell|)$, one has $f_o(t, x, u) \geq -\bar{\mu}$ for each $(t, x, u) \epsilon M$. Therefore,

$$I[x, u] = \int_{t_1}^{t_2} f_o(t, x, u)dt \geq -D\bar{\mu},$$

where D is the diameter of A and hence the infinum i of $I[x, u]$ in Ω is finite.

Let $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$ be a sequence of admissible pairs such that $I[x_k, u_k] \to i$ as $k \to +\infty$. We may well assume that

$i \leq I[x_k, u_k] \leq i + k^{-1} \leq i+1$, $k=1,2,\ldots$. Let us prove that $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\ldots$, are equiabsolutely continuous vector functions. For each k, denote by T_{ik} the set $\{t \mid (t, x_k(t)) \in \in A_i, \ t_{1k} \leq t \leq t_{2k}\}$, $i=0,\ldots,\ell$; $k=1,2,\ldots$. On A_o the usual growth condition (α) holds, and therefore, by the same argument used in Theorems I and II, the vector functions $x'_k(t)$ are equiabsolutely integrable on T_{ok}; in other words, given $\epsilon > 0$ there is a $\delta_o = \delta(\epsilon, A_o) > 0$ such that for any subset H_o of T_{ok} of measure $< \delta_o$ we have

$$(H_o) \int |x'_k(t)| \, dt < \epsilon \ 4^{-1} (\ell+1)^{-1}, \quad k=1,2,\ldots .$$

Now, if H is an given subset of $[t_{1k}, t_{2k}]$ with meas $H < \delta_o$, then the subset $H_{ok} = H \cap T_{ok}$ also has measure $< \delta_o$ and we have

$$(H \cap T_{ok}) \int | x'_k(t) | \, dt < \epsilon \ 4^{-1} (\ell+1)^{-1}. \tag{34}$$

Let $\bar{C} = \max[1, C_1, \ldots, C_\ell]$, $\bar{D} = \max[1, D_1, \ldots, D_\ell]$, $\mu_* = \min[1, \mu_1, \ldots, \mu_\ell]$. For each $i=1,2,\ldots,\ell$, let us determine a number β_i, $0 < \beta_i < \delta_i$, small enough so that

$$\int_0^{\beta_i} \phi_i(z) \, dz < \epsilon \ 8^{-1} \bar{D}^{-1} (\ell+1)^{-1} \tag{35}$$

and so that

$$z^{\alpha_i}\phi_i^{\alpha_i}(z)\Psi_i^{\alpha_i}(\phi_i(z)) > 8\,\mu_*\bar{D}\,\epsilon^{-1} \tag{36}$$

for all z with $0 < z \leq \beta_i$. Also, again for each $i=1, 2, \ldots, \ell$, let us choose a number $\gamma_i \quad \phi_i(\beta_i)$. Then

$$\beta_i^{\alpha_i}\,\gamma_i^{\alpha_i}\,\Psi_i^{\alpha_i}(\gamma_i) > 8\,\mu_*\bar{D}\,\epsilon^{-1},\ i=1, 2, \ldots, \ell. \tag{37}$$

Let $\bar{\gamma} = \max[1, \gamma_1, \ldots, \gamma_\ell]$. For each $i=1, 2, \ldots, \ell$, and $k=1, 2, \ldots,$ let us divide the set T_{ik} into four disjoint subsets E_{ikj}, $j=1, 2, 3, 4$, as follows. Let $t_i(\beta_i) = [t_i-\beta_i, t_i+\beta_i]$, and let

$$E_{ik1} = \{t \mid t \in T_{ik},\ t \in t_i(\beta_i),\ |u_k(t)| \leq \phi_i(|t-t_i|)\},$$

$$E_{ik2} = \{t \mid t \in T_{ik},\ t \in t_i(\beta_i),\ t \notin E_{ik1}\}.$$

$$E_{ik3} = \{t \mid t \in T_{ik},\ t \notin t_i(\beta_i),\ |u_k(t)| \leq \gamma_i\},$$

$$E_{ik4} = \{t \mid t \in T_{ik},\ t \notin t_i(\beta_i),\ t \notin E_{ik3}\}.$$

In E_{ik1} we have $|u_k(t)| \leq \phi_i(|t-t_i|)$ and hence, by force of (35),

$$(H \cap \bigcup_{i=1}^{\ell} E_{ik1}) \int |u_k(t)|\,dt \leq \sum_{i=1}^{\ell} (E_{ik1}) \int |u_k(t)|\,dt$$

$$\leq \sum_{i=1}^{\ell} (e_{ik1}) \int \phi_i(|t-t_i|)\,dt \leq$$

$$\leq \sum_{i=1}^{\ell} 2 \int_0^{\beta_i} \phi_i(z)dz \leq 2\ell \cdot \epsilon\, 8^{-1} \bar{D}^{-1}(\ell + 1)^{-1}.$$

In E_{ik2} we have $|u_k(t)| \geq \phi_i(|t-t_i|)$, hence

$$|u_k(t)|\phi_i^{-1}(|t-t_i|) \geq 1.$$

Now, $\Psi(|u_k(t)|) \geq \Psi_i(\phi_i(|t-t_i|)$, and then

$$|u_k(t)| \leq |u_k(t)| \cdot |u_k(t)|^{\alpha_i} \phi_i^{-\alpha_i}(|t-t_i|)\Psi_i^{\alpha_i}(|u_k(t)|)\Psi_i^{-\alpha_i}(\phi_i(|t-t_i|))$$

$$= |t-t_i|^{\alpha_i}|u_k(t)|^{1+\alpha_i}\Psi_i^{\alpha_i}(|u_k(t)|)\cdot$$

$$\cdot [|t-t_i|^{\alpha_i}\phi_i^{\alpha_i}(|t-t_i|)\, \Psi_i^{\alpha_i}(\phi_i(|t-t_i|))]^{-1} \tag{39}$$

where $0 < |t-t_i| \leq \beta_i$ since $t \in E_{ik2}$ and $t=t_i$ is obviously in E_{ik1}. By (36) we have

$$|t-t_i|^{\alpha_i} \phi_i^{\alpha_i}(|t-t_i|)\Psi_i^{\alpha_i}(\phi_i(|t-t_i|)) \geq 8\, \mu_* \bar{D}\, \epsilon^{-1},$$

and (39) yields

$$|u_k(t)| \leq (8^{-1}\mu_*^{-1}\bar{D}^{-1} \epsilon)|t-t_i|^{\alpha_i}|u_k(t)|^{1+\alpha_i}\Psi_i^{\alpha_i}(|u_k(t)|).$$

By (32) we have then

$$|u_k(t)| \leq (8^{-1} \mu_*^{-1}\bar{D}^{-1}\epsilon)[f_o(t, x_k(t), u_k(t)) - \mu_i]$$

for every $t \in E_{ik2}$, and hence also

$$|u_k(t)| \leq (8^{-1}\mu_*^{-1}\bar{D}^{-1}\epsilon)\,[f_o(t, x_k(t), u_k(t)) + \bar{\mu}]. \tag{40}$$

In E_{ik3} we have $|u_k(t)| \leq \gamma_i$ and hence since $\bar{\gamma} = \max[1, \gamma_1, \ldots, \gamma_\ell]$ we have also

$$|u_k(t)| \leq \bar{\gamma} \quad \text{for } t \in \bigcup_{i=1}^{\ell} E_{ik3} \tag{41}$$

In E_{ik4} we have

$$|u_k(t)| \geq \gamma_i, \quad |t-t_i| \geq \beta_i, \quad \Psi_i(|u_k(t)|) \geq \Psi_i(\gamma_i),$$

$$f_o(t, x_k(t), u_k(t)) - \mu_i \geq |t-t_i|^{\alpha_i}|u_k(t)|^{1+\alpha_i}\Psi_i^{\alpha_i}(|u_k(t)|),$$

and hence

$$|u_k(t)| \leq [f_o(t, x_k(t), u_k(t)) - \mu_i]\,|t-t_i|^{-\alpha_i}|u_k(t)|^{-\alpha_i}\Psi_i^{-\alpha_i}(|u_k(t)|)$$

$$\leq [f_o(t, x_k(t), u_k(t)) + \bar{\mu}]\beta_i^{-\alpha_i}\gamma_i^{-\alpha_i}\Psi_i^{-\alpha_i}(\gamma_i)$$

and by force of (37) also for $t \in E_{ik4}$,

$$|u_k(t)| \leq (8^{-1}\mu_*^{-1}\bar{D}^{-1}\epsilon)[f_o(t, x_k(t), u_k(t)) + \bar{\mu}]. \tag{42}$$

For every measurable set $H \subset [t_{1k}, t_{2k}]$ with meas $H < \delta_o$ we have now

$$(H)\int |x'_k(t)|\,dt = (H\cap\bigcup_{i=0}^{\ell} T_{ik})\int |x'_k(t)|\,dt$$

$$\leq (H\cap T_{ok})\int |x'_k(t)|\,dt + (H\cap\bigcup_{i=1}^{\ell} T_{ik})\int |x'_k(t)|\,dt$$

and by force of (34), also

$$(H)\int |x_k'(t)|\,dt \leq \epsilon\ 4^{-1}(\ell+1)^{-1} + (H\cap\bigcup_{i=1}^{\ell} T_{ik})\int |x'_k(t)|\,dt.$$

We have now, almost everywhere in T_{ik},

$$|x'_k(t)| = |f(t,x_k(t),u_k(t))| \leq C_i + D_i\,|u_k(t)| \leq \bar{C} + \bar{D}\,|u_k(t)|,$$

and hence

$$(H)\int |x'_k(t)|\,dt \leq \epsilon\ 4^{-1}(\ell+1)^{-1} + (H\cap\bigcup_{i=1}^{\ell} T_{ik})\int (\bar{C} + \bar{D}\,|u_k(t)|)\,dt$$

$$\leq \epsilon\ 4^{-1}(\ell+1)^{-1} + \bar{C}\text{ meas } H + \bar{D}\,(H\cap\bigcup_{i=1}^{\ell} T_{ik})\int |u_k(t)\,dt$$

$$\leq \epsilon\ 4^{-1}(\ell+1)^{-1} + \bar{C}\text{ meas } H +$$

$$+\,\bar{D}\left(\sum_{j=1}^{4}\ (H\cap\bigcup_{i=1}^{\ell} E_{ikj})\int |u_k(t)|\,dt\right).$$

By force of (38), (40), (41) and (42) we have now

$$(H) \left| \int x'_k(t) \right| dt \leq \epsilon\, 4^{-1} (\ell+1)^{-1} + \bar{C} \text{ meas } H + \bar{D}\ell \cdot \epsilon\, 4^{-1} \bar{D}^{-1} (\ell+1)^{-1}$$

$$+ \bar{D} (8^{-1} \mu_*^{-1} \bar{D}^{-1} \epsilon) \left(H \cap \bigcup_{i=1}^{\ell} E_{ik2}\right) \int [f_o(t, x_k(t), u_k(t)) + \bar{\mu}] dt$$

$$+ \bar{D} \left(H \cap \bigcup_{i=1}^{\ell} E_{ik_3}\right) \int \bar{\gamma}\, dt$$

$$+ \bar{D} (8^{-1} \mu_*^{-1} \bar{D}^{-1} \epsilon) \left(H \cap \bigcup_{i=1}^{\ell} E_{ik4}\right) \int [f_o(t, x_k(t), u_k(t)) + \bar{\mu}] dt.$$

Since $f_o + \bar{\mu} \geq 0$ for all $t \in [t_{1k}, t_{2k}]$, we have

$$(H) \int, x'_k(t) | dt \leq \epsilon\, 4^{-1} (\ell+1)^{-1} + (\bar{C} + \bar{D}\bar{\gamma}) \text{ meas } H + \epsilon\, 4^{-1} \ell(\ell+1)^{-1}$$

$$+ 2\bar{D} (8^{-1} \mu_*^{-1} \bar{D}^{-1} \epsilon) \int_{t_{1k}}^{t_{2k}} [f_o(t, x_k(t)) + \bar{\mu}] dt$$

$$\leq 4^{-1} \epsilon + (\bar{C} + \bar{D}\bar{\gamma}) \text{ meas } H + 4^{-1} \mu_*^{-1} \epsilon (|i| + 1 + \bar{\mu})$$

$$2^{-1} \epsilon + (\bar{C} + \bar{D}\bar{\gamma}) \text{ meas } H.$$

If we take $\delta = \min [\delta_o, (\bar{C} + \bar{D}\bar{\gamma})^{-1} 2^{-1} \epsilon]$, then for every measurable set $H \subset [t_{ik}, t_{2k}]$ with meas $H < \delta$ we have

(H) $\int |x'_k(t)| dt \leq 2^{-1}\epsilon + 2^{-1}\epsilon = \epsilon.$

The vector functions $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\ldots$ are equiabsolutely continuous.

If one recalls that for this case there is a constant $\bar{\mu} > 0$ such that $f_o(t,x,u) \geq -\bar{\mu}$ for each $(t,x,u) \in M$ and the fact that $x_k(t)$, $t_{1k} \leq t \leq t_{2k}$, $k=1,2,\ldots$ are equiabsolutely continuous, then one may consider the function $x^o_k(t)$ as in Existence Theorem I and the proof proceeds just as before. The proof is complete for the case that A is compact.

If A is not compact, let us consider an arbitrary compact subset $A_o \subset A$, and its respective parts $A_o \cap E$ and $A - A_o \cap E$. The set $A_o \cap E$ is a compact part of A_o on which condition (T) holds. Therefore, by the same reasoning as in the beginning of the proof of this theorem there is a $\bar{\mu}_o \geq 0$ such that $f_o(t,x,u) \geq -\bar{\mu}_o$ for all $(t,x) \in A_o$, $u \in U(t,x)$ (where $\bar{\mu}_o$ may depend on A_o). This fact, the assumptions given for the compact case of A, and the reasoning given for the case of A not compact in Theorem I reduces these cases to the compact case. The theorem is completely proven.

REMARKS: 1. There exists a function $f_o(t,x,u)$ satisfying all conditions of Tonelli 's Theorem III, but, for which the sets $\tilde{\tilde{Q}}$ do not satisfy everywhere property (Q).

Take $m = n = 1$, $A = [0 \leq t \leq 1,\ -1 \leq x \leq 1]$, $f = u$, $U = [-\infty < u < +\infty]$ and $f_0(t, x, u) = |tu^3| + \max[0,\ 1 - xu]$. Then, $f_0(t, x, u) \geq |tu^3| \geq 0$ for all $(t, x, u) \in A \times U$. For $t \neq 0$, f_0 satisfies the growth condition. The set $E = \{(t, x) | t = 0,\ -1 \leq x \leq +1\}$ is the exceptional set, and condition (T) is satisfied at every point of E since $f_0 \geq |t|\ |u|^3$, and condition (24) (Tonelli, Opere Scelte, p. 216) is satisfied with $t_0 = 0$, $k = 1$, $\alpha = 1$, $\sigma = 1$, $\mu = 0$. The function $f_{01} = |tu^3| = |t|\,|u|^3$ is obviously convex in u for each t, and the function $f_{02} = \max[0, 1 - xu]$ is also convex in u for each x. Hence $f_0(t, x, u) = f_{01} + f_{02}$ is certainly convex in u for each $(t, x) \in A$. Now we have

$$\tilde{\tilde{Q}}(t, x) = \{(z^0, u) | z^0 \geq f_0(t, x, u),\ -\infty < u < +\infty\},$$

and hence

$$\tilde{\tilde{Q}}(0, 0) = \{(z^0, u) | z^0 \geq 1,\ -\infty < u < +\infty\}.$$

On the other hand, for $0 < \delta \leq 1$,

$$\tilde{\tilde{Q}}(0, \delta) = \{(z^0, u) | z^0 \geq \max[0, 1 - \delta u],\ -\infty < u < +\infty\}$$

$$\tilde{\tilde{Q}}(0, -\delta) = \{(z^0, u) | z^0 \geq \max[0, 1 + \delta u],\ -\infty < u < +\infty\}.$$

Hence $(0, \delta^{-1}) \in \tilde{\tilde{Q}}(0, \delta)$, $(0, -\delta^{-1}) \in \tilde{\tilde{Q}}(0, -\delta)$, and finally $(0, \delta^{-1})$, $(0, -\delta^{-1}) \in \tilde{\tilde{Q}}(0, 0; \delta)$. As a consequence

$$(0, 0) \in \text{co}\,\tilde{\tilde{Q}}(0, 0; \delta),\ (0, 0) \in \bigcap_{\delta} \text{cl co}\,\tilde{\tilde{Q}}(0, 0; \delta),$$

while $(0, 0) \notin \tilde{\tilde{Q}}(0, 0)$. We have here

$\tilde{\tilde{Q}}(0, 0) \neq \bigcap_{\delta} \text{cl co } Q(0, 0; \delta)$.

2. In the proof of Theorems I - IV we may disregard property (Q) at the points (t, x) of a subset of A of the form $\{(t, x) | (t, x) \in A,\ t \in \{t\}\}$, where $\{t\}$ is a given set of linear measure zero.

The proof is the same. A set of points t of linear measure zero is always disregarded in the proof.

Corollary 1: This is the same as Existence Theorem III where for each $(t, x) \in A$ $U(t, x) = U$ is a fixed, closed subset of the u-space E_m, and the part of condition (γ*) where $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$ uniformly on a compact subset A_o of A - E is replaced by the conditions (i) $f_o(t, x, u)$ is a uniformly continuous function of (t, x) in A_o and (ii) $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$ pointwise in A - E.

PROOF: This statement is a consequence of Theorem III and lemma 4.

Corollary 2: This is the same as Existence Theorem III where for each $(t, x) \in A$ $U(t, x) = E_m$ is the whole u-space E_m, the function $f_o(t, x, u)$ is convex in u for each $(t, x) \in A - E$, $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in E_m$ pointwise in A - E, and the uniform growth condition of $f_o(t, x, u)$ on each compact part A_o of A - E, given in (γ*) is omitted.

In addition, if $m = n$, $f(t, x, u) = u$, $f_o(t, x, u)$ is a convex function of u for each $(t, x) \in E$, and hence for each $(t, x) \in A$, then the conditions that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) in A and E is a closed set can be omitted if the exceptional set E is chosen to be the subset of A at which condition (T) holds and condition (α) does not hold.

PROOF: The control set $U(t, x) \equiv E_m$ for each $(t, x) \in A$ obviously satisfies property (U) in A. On each compact subset A_o of $A - E$, the convexity of f_o in u for each $(t, x) \in A_o$, the pointwise growth of $f_o |u|^{-1}$ to $+\infty$ as $|u| \to +\infty$, and lemma 5, guarantee the uniform growth of $f_o |u|^{-1}$ to $+\infty$ as $|u| \to +\infty$ and hence that the condition (γ^*) is satisfied. Theorem III applies and the first part of the statement is proven.

We shall first show that under the assumptions of the second part of the statement E is a closed subset of the tx-space $E_1 \times E_n$. If $(t, x) \in E_{n+1} - A$, then (t, x) has a finite distance to the closed set A. Thus, there is an open neighborhood of (t, x) which lies entirely in the set $E_{n+1} - A$ and hence in $E_{n+1} - E$. If $(t, x) \in A-E$, then f_o is a convex function of u and $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$ at (t, x). Lemma 5 implies, under the assumption of the second part of the statement, that in some neighborhood of $(t, x) \in A - E$, $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$ uniformly. Thus for each point in this neighborhood of uniform growth one has $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$ pointwise and therefore this neighborhood of uniform growth lies entirely in E_{n+1} - E as at points of E condition (α) is not satisfied. One has that E_{n+1} - E is an open set and that E is closed in E_{n+1}.

We shall now show that under the assumptions in the second statement the hypothesis that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) in A can be omitted from the first part of the statement. We note that the proof of Theorem III for A not compact is reduced by additional assumptions to the proof for the compact A case applied to some fixed compact subset A_0 of A. In order to omit the condition that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) in A under the assumptions of the second part of the statement, this latter fact and remark 2 to Theorem III imply that it suffices to prove that, under the conditions of the second part of this statement, $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) on any fixed compact subset A_0 of A except for a set A'_0 of the form $\{(t, x) | (t, x) \in A_0, \ t \in \{t\}\}$ where $\{t\}$ is a given set of linear measure zero. We shall show that under the assumptions of the second part of this statement that the set $\{t\}$ can be taken to be finite.

Consider a fixed compact subset A_0 of A. Then $f_0 |u|^{-1} \to +\infty$ as $|u| \to +\infty$ at each point of the set $B = A_0 \cap (A-E)$ and f_0 is a convex function of u for each $(t, x) \in B$. As each point $(t, x) \in B$ has a finite distance from the set $A_0 \cap E$, lemmas 5 and 7 imply that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) at each point of the set B. Condition (T) holds at each point $(t, x) \in A_0 \cap E$.

The neighborhoods given by condition (T) cover the compact set $A_o \cap E$. Consider a finite subcover of $A_o \cap E$. Then there are a finite number of t_i, $i=1, 2, \ldots, \ell$ such that

$$f_o(t, x, u) \geq |t-t_i|^{\alpha_i} |u|^{1+\alpha_i} \Psi_i^{\alpha_i}(|u|) + \mu_i$$

for each $(t, x, u) \in N_{\delta_i}(t_i, x_i) \times E_m$ where α_i, μ_i and Ψ_i are given in condition (T). As $\Psi_i(z)$ is a nondecreasing function on $0 \leq z \leq +\infty$ and $\alpha_i > 0$, $i=1, 2, \ldots, \ell$, the function $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$ at each point $(t, x) \in A_o \cap E$ except when $t = t_i$ for some i, $i=1, 2, \ldots, \ell$. Each point $(t, x) \in A_o \cap E$ with $t \neq t_i$ for any $i=1, 2, \ldots, \ell$, has a finite distance from the closed set $\{(t, x) | (t, x) \in A_o \cap E, \; t = t_i \text{ for some } i=1, 2, \ldots, \ell\}$. Therefore lemmas 5 and 7 guarantee that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) at each point $(t, x) \in A_o \cap E$ with $t \neq t_i$ for any $i=1, 2, \ldots, \ell$. Thus, under the assumptions of the second part of the statement, $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) on A_o except for the set $\{(t, x) | (t, x) \in A_o, \; t=t_i \text{ for some } i, \; i=1, 2, \ldots, \ell\}$ and the set $\{t | t=t_i, \; i=1, 2, \ldots, \ell\}$ is a finite set and clearly has linear measure zero. The second part of the statement is entirely proven.

Example 3: Theorem III applies to the following example to which neither Existence Theorem I nor II applies.

Let $m = n = 1$, $A = [0, 1]^2$, for each $(t, x) \in A$, let $U(t, x) = E_1$, let $f(t, x, u) = u$, $f_o(t, x, u) = t^2 u^4$ and boundary conditions $x(0) = a$,

$x(1) = b$ where a, b are fixed real numbers. Then $U(t, x)$ satisfies property (U) for each $(t, x) \in [0, 1]^2$,

$$\tilde{\tilde{Q}}(t, x) = \{(z_0, z) | z_0 \geq t^2 u^4, \ z = u, \ u \in E_1\}$$
$$= \{z_0, z) | z_0 \geq t^2 z, \ z \in E_1\} \text{ and}$$

$\tilde{\tilde{Q}}(0, x, \delta) = \{(z_0, z) | \ z_0 \geq 0, \ z \in E_1\} = \tilde{\tilde{Q}}(0, x)$ which is a closed, convex set. Hence, $\tilde{\tilde{Q}}(0, x) = \bigcap_{\delta} \text{cl co } \tilde{\tilde{Q}}(0, x, \delta)$ for each $x \in [0, 1]$. Therefore, $\tilde{\tilde{Q}}(0, x)$ satisfies property (Q) for each $(t, x) \in \{0\} \times [0, 1]$. The set $\tilde{\tilde{Q}}(t, x, \delta) = \{(z_0, z) | z_0 \geq (t-\delta)^2 z^4, z \in E_1\}$ if $t \neq 0$ and $0 < \delta \leq |t|$. The latter set is a closed, convex set in E_2. Hence, $\tilde{\tilde{Q}}(t, x, \delta) = \text{cl co } \tilde{\tilde{Q}}(t, x, \delta)$ if $t \neq 0$ and $0 \leq \delta \leq |t|$. Therefore, $\tilde{\tilde{Q}}(t, x) = \bigcap_{\delta} \tilde{\tilde{Q}}(t, x, \delta) = \bigcap_{\delta} \text{cl co } \tilde{\tilde{Q}}(t, x, \delta)$ and $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) for each $(t,x) \in (0, 1] \times [0, 1]$ and also for each $(t, x) \in A$.

Although $f_0/|u| \to +\infty$ as $|u| \to +\infty$, $u \in E_1$ uniformly on each compact subset $(0, 1] \times [0, 1]$, $f_0(t, x, u)/|u|$ does not tend to $+\infty$ as $|u| \to +\infty$, $u \in E_1$ for $(t, x) \in \{0\} \times [0, 1]$. As a result $f_0/|u|$ does not tend to $+\infty$ as $|u| \to +\infty$, $u \in E_1$ uniformly in A and both Existence Theorems I and II fail to apply. Yet Existence Theorem III applies. Let $E = \{0\} \times [0, 1]$ which is a closed subset of E_2. Then $f_0/|u| \to +\infty$ as $|u| \to +\infty$, $u \in E_1$ uniformly on any compact subset of $A - E$. Also, condition (T) is satisfied at each point of E. Let $(0, x_0)$ be an arbitrary point in E. Thus,

$$f_o(t, x, u) = t^2u^4 = t^2u^{1+2+1} \geq |t|^{\alpha} |u|^{1+\alpha} |u|$$

where $\alpha = 2$. Consider an arbitrary neighborhood of $(0, x_o)$, say $N_\delta^o(0, x_o)$. Let

$\ell = 1$, $\alpha = 2$, $\phi(z) = z^{-3/4}$ for $0 < z \leq 1$, $\Psi(z) = z^{1/2}$ for $0 < z < +\infty$ in this neighborhood. Then $\phi(z)$ is a nonnegative integrable function on $(0, 1]$ with $\phi(0+) = +\infty$ and $\Psi(z)$ is a nonnegative, nondecreasing function on $(0, +\infty)$ such that $z\,\phi(z)\,\Psi(\phi(z)) = z \cdot z^{-3/4} \cdot z^{-3/8} = z^{-1/8} \to +\infty$ as $z \to 0+$. It is also clear that

$$f_o(t, x, u) = t^2u^4 \geq |t|^{\alpha} |u|^{1+\alpha} \Psi^{\alpha}) |u|)$$

for each $(t, x) \in N_\delta^o(0, x_o)$, $u \in E_1$ where $\ell, \alpha, \mu, \phi(z)$ and $\Psi(z)$ have been chosen as indicated. Indeed, $|f| = |u| \leq C_o + D_o|u|$ for each $(t, x) \in N_\delta^o(0, x_o)$, $u \in E_1$ where $C_o = 0$, $D_o = 1$. This analysis shows that condition (T) is satisfied at $(0, x_o) \in E$. As $(0, x_o)$ is an arbitrary point of E, condition (T) is satisfied at each point of E. Finally, Existence Theorem III applies to this example.

Note that the present example verifies (33) at the points $(0, x)$ with $k = 1$, $\alpha = 2$, $\sigma = 1$, $\mu = 0$, $C_o = 0$, $D_o = 1$.

Note that Theorems I, II and III do not apply to the examples 1 and 2.

13. An existence theorem with a "slender" exceptional set.

If the exceptional set E is suitably "slender", then property (T) at the points $(t_0, x_0) \in E$ is not needed. Let E be any subset of A. Then for any subset H of the t-axis we shall denote by $E^i(H)$ the set of points ξ of the x^i-axis, $i=1, 2, \dots, n$, defined as follows

$$E^i(H) = \{\xi \mid \text{there exists a point } (t, x^1, \dots, x^n) \in E, \text{ with } t \in H, \ x^i = \xi\}.$$

We shall denote by $\mu^* [E^i(H)]$ the one dimensional outer measure of $E^i(H)$, and we shall require below that $\mu^*[E^i(H)] = 0$, $i=1, 2, \dots, n$, for every subset H of the t-axis of measure zero.

For instance, any set E contained in countably many straight lines parallel to the t-axis, and to finitely many (nonparametric) curves $x^i = \phi_i(t)$, $i=1, \dots, n$, $t' \leq t \leq t''$, $\phi_i(t)$ AC in $[t', t'']$, certainly possesses the property above. The property above was proposed and used by L. Turner to extend to free problems in E_n results of Tonelli for free problems in E_1.

Existence Theorem IV:

If A is compact, Theorem IV is the same as Existence Theorem I where a closed exceptional set E is given, condition (α) is replaced by (γ^*), and (L_1): $\mu^*[E^i(H)] = 0$, $i=1, 2, \dots, n$, for every subset of the t-axis of measure zero;

(L_2): for every $(t_o, x_o) \in A$ there are numbers

$$\delta = \delta(t_o, x_o) > 0, \quad \gamma = \gamma(t_o, x_o) > 0, \quad r = r(t_o, x_o) \text{ real},$$

$$b_i = b_i(t_o, x_o) \text{ real}, \quad i=1, 2, \ldots, n,$$

such that

$$f_o(t, x, u) \geq r + \sum_{i=1}^{n} b_i f_i(t, x, u) + \gamma | f(t, x, u) |$$

for each $(t, x) \in N_\delta^o(t_o, x_o)$, $u \in U(t, x)$;

(L_3): for each compact subset A_o of A there is a constant $M_o \geq 0$ such that

$$f_o(t, x, u) \geq -M_o \quad \text{for each } (t, x) \in A_o, \quad u \in U(t, x).$$

If A is not compact, but closed and contained in a slab $[t_o \leq t \leq T, \ x \in E_n]$, to, T, finite, then Theorem IV still holds if (a), (b) hold [and (α) is replaced by (γ^*), provided (L_1), (L_2) and (L_3) hold].

If A is not compact, nor contained in a slab as above, but A is closed, then Theorem IV still holds if (a), (b) and (c) hold [and (α) is replaced by (γ^*) provided (L_1), (L_2), and (L_3) hold]. Finally, condition (a) can be replaced in any case by condition (d).

PROOF: Let us consider an arbitrary admissible pair x(t), u(t), $t_1 \leq t \leq t_2$ from the complete, nonempty admissible class Ω and its restriction to a subinterval of $[t_1, t_2]$, i.e. x(t), u(t),

$t'_1 \leq t \leq t'_2$ where $t_1 \leq t'_1 \leq t'_2 \leq t_2$. Denote by Ω' the set of all such possible restrictions of admissible pairs from Ω. Then Ω is contained in Ω'.

Let us assume that A is compact. We shall first prove that $\inf_{\Omega'} I[x, u]$ is finite. Thus, $\inf_{\Omega} I[x, u]$ is finite as the inequality $\inf_{\Omega'} I[x, u] \leq \inf_{\Omega} I[x, u]$ follows from $\Omega \subset \Omega'$. As the set A is compact by (L_3) there is a constant $M_0 \geq 0$ such that $f_0(t, x, u) \geq -M_0$ for each $(t, x) \in A$, $u \in U(t, x)$. Therefore,

$I[x, u] = \int_{t_1}^{t_2} f_0(t, x(t), u(t))dt \geq -M_0\bar{D}$ for each $(x, u) \in \Omega'$ where $\bar{D}$ = diameter of the set A. Thus $\inf_{\Omega'} I[x, u]$ is finite and so is $\inf_{\Omega} I[x, u]$.

Fix a real number $M \geq 0$, consider the set $\{(x, u) | (x, u) \in \Omega', I[x, u] \leq M\}$ and denote it by $\{(x, u)\}_M$. Then, we shall prove that the vector functions x(t) from $\{(x, u)\}_M$ have uniformly bounded variation when A is compact. This last statement will be shown by finding two constants $\bar{a}$ real, $\bar{b}$ real and $\bar{b} \geq 0$ such that

$M \geq I[x, u] \geq \bar{a} + \bar{b}BV|x(t)|$ for each $(x, u) \in \Omega'$ with $I[x, u] \leq M$. The last inequality is proven after a special partition of the tx-space is constructed and the trajectory $x(t)$, $t'_1 \leq t \leq t'_2$ is divided in a special way into subtrajectories $x_{pk}(t)$, $t'_{1pk} \leq t \leq t'_{2pk}$. We shall now obtain this special partition.

Condition (L_2) guarantees that to each point $(t, x) \in A$ there is a linear scalar function $z(v) = r + b \cdot v$ where $v = (v_1, \ldots, v_n)$,

$b = (b_1, \ldots, b_n)$ or $z(v) = r + \sum_{i=1}^{n} b_i v_i$ ($r, b_1, \ldots, b_n$ all real), a number $\nu > 0$, and a constant $\bar{\delta} > 0$ such that

$$f_o(t, x, u) \geq z(f(t, x, u)) + \nu |f(t, x, u)|,$$

$$\text{or } f_o \geq z(f) + \nu(f) \tag{43}$$

for each $(t, x) \in N^o_{3\sqrt{n+1}\,\bar{\delta}}(t_o, x_o)$, $u \in U(t, x)$, and hence for each $(t, x) \in \bar{N}^o_{3\bar{\delta}}(t_o, x_o)$, $u \in U(t, x)$ where

$\bar{N}^o_{3\bar{\delta}}(t_o, x_o) = \{t, x|, |t-t_o| < 3\bar{\delta}, |x^i - x^i_o| < 3\bar{\delta}$ for $i=1, 2, \ldots, n\}$.

A finite number of the above open cubes, $\bar{N}^o_{3\bar{\delta}_i}(t_i, x_i)$, $i=1, 2, \ldots, i_o$, cover A, because A is compact. Divide E_{n+1} into cubes E^{pq} whose sides have length $\delta > 0$ by taking

$$E^{pq} = \{(t, x) | (p-1)\delta \leq t \leq p\delta, \ (q_i-1)\delta \leq x^i \leq q_i\delta, \ i=1, 2, \ldots, n\}$$

where p, $q_1, \ldots, q_n$ are integers, $q=(q_1, \ldots, q_n)$, $\delta \leq \min(\bar{\delta}_1, \ldots, \bar{\delta}_{i_o})$ and where δ is chosen so small that each cube E^{pq} which has a nonempty intersection with the set A, and its $3^{n+1} - 1$ adjacent cubes, are completely contained within one $\bar{N}^o_{3\bar{\delta}_i}(t_i, x_i)$, $i=1, \ldots, i_o$. (This can be done as A is a compact set and is covered by the $\bar{N}^o_{3\bar{\delta}}(t_i, x_i)$, $i=1, \ldots, i_o$.)

Hence, one can associate to each cube E^{pq}, which has a nonempty intersection with A, a linear scalar function

$$z(v) = r + \sum_{i=1}^{n} b_i v_i, \quad v = (v_1, \ldots, v_n)$$ and number $\nu > 0$, such that

$$f_o(t, x, u) \geq z(f(t, x, u)) + \nu |f(t, x, u)|$$

for each $(t, x, u) \in M$ with $(t, x) \in E^{pq}$ or any of the $3^{n+1} - 1$ cubes which are adjacent to E^{pq}. The t-coordinates of the vertices of these cubes define a partition of the t-axis.

Thus, there are two integers p_o, q_o such that each vertex of the previously mentioned cube has t-coordinates of the form $p\delta$ with $p_o \leq p \leq q_o$. For each $t_o \in [p_o\delta, q_o\delta]$ the set $E(\{t_o\})$, $i=1, \ldots, n$, has measure zero. Given η such that $0 < \eta < \delta/2$ for each $i=1, 2, \ldots, n$ the set $E^i(\{t_o\})$ can be covered by an open set F^i of linear measure less than η. The cartesian product $F_o = \{t_o\} \times F_1 \times F_2 \times \ldots \times F_n$ of the set $\{t_o\}$ and these open sets are open in the hyperplane $H(t_o) = \{(t, x) | (t, x) \in E_{n+1}, t = t_o\}$. Now, $(H(t_o) - F_o) \cap A$ is a compact subset of A since it is the intersection of the closed set $H(t_o) - F_o$ and the compact set A.

As E is a closed subset of the compact set A there is a $\rho > 0$ such that the set $N_{\rho\sqrt{n+1}}(H(t_o) - F_o) = \{(t, x) | \text{dist}((t, x), H(t_o) - F_o) \leq \rho\sqrt{n+1}\}$ where dist (b, B) is the distance between a point $b \in E_{n+1}$ and a set B in E_{n+1}, has an empty intersection with the closed set E. This last statement follows by the argument below. Indeed, F_o is an open covering of the intersection $E(t_0) = E \cap H(t_o)$.

Thus, $H(t_o) - F_o$ is a closed set, and must have a positive distance from the compact set E, since in the opposite case, the set $H(t_o)-F_o$ and hence $(H(t_o) - F_o) \cap A$ would contain points of accumulation of E, and thus would contain points of E since E is closed, but this is impossible because F_o is an open cover of all points $E(t_o)$, that is of all points of E which are on the hyperplane $H(t_o)$.

Thus, for each point $(t_o, x_o) \in (H(t_o) - F_o) \cap A$ there is a cube $\bar{N}_\rho(t_o, x_o) = \{(t, x) | \ |t-t_o| \leq \rho, \ |x^i - x^i_o| \leq \rho$ for $i=1, \ldots, n\}$ and the open cubes $\bar{N}^o_\rho(t_o, x_o)$ cover the compact set $(H(t_o) - F_o) \cap A$. There exists a finite subcover $\bar{N}^o_\rho(t_o, x_i)$, $i=1, 2, \ldots, \tau$, for $(H(t_o) - F_o) \cap A$. We note that the compact set $\bigcup_{i=1}^{\tau} \bar{N}_\rho(t_o, x_i)$ covers $(H(t_o) - F_o) \cap A$ and has an empty intersection with E as it is contained in the closed set $N_{\rho\sqrt{n+1}}(H(t_o) - F_o)$ which has an empty intersection with the set E.

As $B_o = \bigcup_{i=1}^{\tau} \bar{N}_\rho(t_o, x_i)$ is a compact subset of $E_{n+1} - E$ and hence $B^A_o = B_o \cap A$ is a compact subset of $A - E$ as both B_o and A are compact sets of E_{n+1}, now $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$ uniformly on B^A_o and there exists constants $C(t_o) \geq 0$, $D(t_o) \geq 0$ such that $|f(t, x, u)| \leq C(t_o) + D(t_o)|u|$ for each $(t, x, u) \in M$ with $(t, x) \in B^A_o$ by condition (γ^*). If S is an open n-dimensional cube such that for each $(t, x) \in A$, $x \in S$, then the complement I_o of the compact set B_o in $[(t_o-\rho, t_o+\rho) xS]$ is not only a bounded open set, but it is the union of a finite number of disjoint open intervals.

Also, the sets I_o and $I_o^A = I_o \cap A$ have the property that each of their projections on the x^i-axis, $i=1,\ldots,n$ has linear measure less than η, as the following argument shows. Indeed, if $(t,x) \in I_o$ then $|t-t_o| < \rho$ and $(t_o, x) \notin H(t_o) - F_o$ since (t,x) would then belong to B_o and not to I_o as above. Thus, for any $(t,x) \in I$, we have $(t_o, x) \in F_o$, and the projections on each of the x^i-axis have linear measure less than η.

In this manner we have associated an open interval $(t_o-\rho, t_o+\rho)$ with each $t_o \in [p_o\delta, q_o\delta]$. By taking a finite subcovering, and then a suitable contraction of the corresponding intervals we define a partition

$$P: \; p_o\delta = t_o < t_1 < \ldots < t_R = q_o\delta$$

of $[p_o\delta, q_o\delta]$ and it may be assumed without loss of generality that the points $p\delta$ for $p_o \leq p \leq q_o$ are all used in this partition. Refine the previous partition of E_{n+1} into intervals $Q^{\alpha q}$ by means of the hyperplanes $t = t_j$, $j=1,\ldots,R$. Thus, the new intervals are of the form

$$Q^{\alpha q} = \{(t,x) \mid t_{\alpha-1} \leq t \leq t_\alpha, \; (q_i - 1)\delta \leq x^i \leq q_i\delta, \; i=1,\ldots,n\},$$

$\alpha = 1,\ldots,R$, $q = (q_1,\ldots,q_n)$ and $t_\alpha - t_{\alpha-1} \leq \delta$. Let $z_{\alpha q}(v)$ be the linear scalar function and $\nu_{\alpha q} > 0$, the real number associated with the cube of the former partition which contains $Q^{\alpha q}$.

Summarizing we have the two following partitions:

1. A subdivision of the slab $[p_o\delta \leq t \leq q_o\delta,\ x \in E_n]$ into hypercubes E^{pq}, $p = p_o, p_o + 1, \dots, q_o - 1$, $q = (q_1, \dots, q_n)$, q_i integers, of side length δ, with corresponding linear functions $z_{pq}(v)$ and real numbers $\nu_{pq} > 0$.

2. Given $\eta > 0$, there is a linear subdivision of the same slab into intervals $Q^{\alpha q} = [t_{\alpha-1} \leq t \leq t_\alpha,\ (q_i-1)\delta \leq x^i \leq q_i\delta]$, $\alpha=1, 2, \dots, R$, whose edges along the x^i-axis, $i=1, 2, \dots, n$, still have length δ, independent of η, such that

$$f_o(t, x, u) \geq z_{\alpha q}(f(t, x, u)) + \nu_{\alpha q}|f(t, x, u)|$$

for each $(t, x, u) \in M$ with $(t, x) \in Q^{\alpha q} \cap A$ and for each α, $\alpha=1, 2, \dots, R$. In addition each slab $[t_{\alpha-1} \leq t \leq t_\alpha,\ x \in E_n] \cap S$ is divided into two disjoint sets B_α and I_α, each made up of finitely many intervals whose edge along the t-axis of total length $< \eta$. There exist constants $C_\alpha \geq 0$, $D_\alpha \geq 0$ such that $|f(t, x, u)| \leq C_\alpha + D_\alpha|u|$ for each $(t, x, u) \in M$ with $(t, x) \in B_\alpha$, and $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$ uniformly on $B_\alpha \cap A$. Thus, the set

$$A_\alpha = \{(t, x) | (t, x) \in A,\ t_{\alpha-1} \leq t \leq t_\alpha\} = B^A_\alpha \cup I^A_\alpha$$

where

$$B^A_\alpha = B_\alpha \cap A,\ I^A_\alpha = I_\alpha \cap A,\ B^A_\alpha \cap I^A_\alpha = B_\alpha \cap I_\alpha \cap A = \phi \cap A = \phi$$

and I^A_α has projections on each of the x^i-axis of linear measure less than η. Also the exceptional set E is contained in $\bigcup_{\alpha=1}^{R} I_{\alpha}^{A}$. Moreover, the constants $\nu_{\alpha q}$, $b_{\alpha q}$, C_{α}, and D_{α} are independent of η. If $C_0 = \max(C_1, \ldots, C_R)$ and $D_0 = \max(D_1, \ldots, D_R)$, then $|f(t, x, u)| \leq C_0 + D_0 |u|$ for each $(t, x, u) \in M$ with $(t, x) \in \bigcup_{\alpha=1}^{R} B_{\alpha}^{A}$. We may clearly assume that $D_0 > 0$ without loss of generality.

Let $r = \max |r_{\alpha q}|$, $b = \max |b_{\alpha q}|$, $\nu = \min \nu_{\alpha q}$

where the maximum and minimum are taken over all (α, q) for which $Q^{\alpha q}$ has a nonempty intersection with A. Also, take a real number $N > 2b(1 + 4\sqrt{n+1})D_0$. Now, suppose $x(t)$, $t'_1 \leq t \leq t'_2$ is the AC trajectory from some admissible pair $(x, u) \in \Omega'$. Let $x_{\alpha} = x(t)$, $t_{\alpha-1} \leq t \leq t_{\alpha}$, be the part of $x(t)$, $t'_1 \leq t \leq t'_2$ (if any) defined on $[t_{\alpha-1}, t_{\alpha}]$. Divide x_{α} into more subtrajectories $x_{\alpha 1}, \ldots, x_{\alpha T_{\alpha}}$ as follows:

The first end point of $x_{\alpha 1}$ is $x(t_{\alpha-1})$ (or $x(t'_1)$ if $t_{\alpha-1} < t'_1 < t_{\alpha}$); the second end point is either the x-component of the first point where $(t, x_{\alpha}(t))$ leaves one of the $3^{n+1} - 1$ intervals in the section $A_{\alpha} = \{ (t, x) | (t, x) \in A,\ t_{\alpha-1} \leq t \leq t_{\alpha} \}$ adjacent to any one of the at most 2^{n+1} intervals containing $(t_{\alpha-1}, x(t_{\alpha-1}))$, or $x(t_{\alpha})$ if $(t, x_{\alpha}(t))$ does not leave these $3^{n+1} - 1$ intervals (or $x(t'_2)$ if $t_{\alpha-1} < t'_2 < t_{\alpha}$). The first point of $x_{\alpha 2}$ is the end point of $x_{\alpha 1}$ and the end point of $x_{\alpha 2}$ is either the x-component of first point of $(t, x_{\alpha}))$ which leaves the $3^{n+1} - 1$ intervals adjacent to the at most

2^{n+1} intervals containing the end point of $(t, x_{\alpha 1}(t))$ or $x(t_\alpha)$ if $(t, x_\alpha(t))$ does not leave these $3^{n+1} - 1$ intervals (or $x(t'_2)$ if $t_{\alpha-1} < t'_2 < t_\alpha$). Continuing in this manner, $x_\alpha = x(t)$, $t_{\alpha-1} \leq t \leq t_\alpha$, is broken up into subtrajectories $x_{\alpha k}$, $k=1, \ldots, T_\alpha$. This process must terminate after a finite number of steps since each subtrajectory $x_{\alpha k}$ except the last has length $> \delta$. Thus, the domain of $x_{\alpha k}$ is an interval, say $\Delta_{\alpha k}$.

Let $\Lambda_{\alpha k}$ be the set of all t in $\Delta_{\alpha k}$ where $(t, x(t)) \in I_\alpha$ and let $\Lambda'_{\alpha k}$ be the complement of $\Lambda_{\alpha k}$ in $\Delta_{\alpha k}$. Thus, $\Delta_{\alpha k} = \Lambda_{\alpha k} \cup \Lambda'_{\alpha k}$. Consider any $(x, u) \in \{(x, u)\}_M$. Then

$$I[x, u] = \int_{t'_1}^{t'_2} f_o(t, x(t), u(t)dt \geq$$

$$\geq \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \{(\Lambda_{\alpha k}) \int [r_{\alpha k} + \sum_{i=1}^{n} b^i_{\alpha k} f_i(t, x(t), u(t))$$

$$+ \nu_{\alpha k} | f(t, x(t), u(t)) |]dt + (\Lambda'_{\alpha k}) \int f_o(t, x(t), u(t))dt\} \geq \tag{44}$$

$$\geq -r\bar{D} + \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \{-b | (\Lambda_{\alpha k}) \int f(t, x(t), u(t))dt |$$

$$+ \nu(\Lambda_{\alpha k}) \int | f(t, x(t), u(t) | dt + (\Lambda'_{\alpha k}) \int f_o(t, x(t), u(t))dt$$

where $\bar{D}$ denotes the diameter of A.

As $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t,x)$ uniformly on each B^A_α, $\alpha=1,\ldots,R$, $f_o|u|^{-1} \to +\infty$, as $|u| \to +\infty$, $u \in U(t,x)$ uniformly on $\bigcup_{\alpha=1}^{R} B^A_\alpha$. We also note that there exists constants $C_o \geq 0$, $D_o \geq 0$ such that $|f(t,x,u)| \leq C_o + D_o|u|$ for each $(t,x,u) \in M$, $(t,x) \in \bigcup_{\alpha=1}^{R} B^A_\alpha$. Then, for the $N > 0$ given previously there exists a $Y > 0$ such that $f_o|u|^{-1} \geq N$ for $|u| \geq Y$, $u \in U(t,x)$ and each $(t,x) \in \bigcup_{\alpha=1}^{R} B^A_\alpha$. Let $\bar{\bar{D}} = \{\sup[N|u| - f_o(t,x,u)] \mid (t,x,u) \in M,\ |u| \leq Y\}$. Since $N|u| - f_o \leq \bar{\bar{D}}$ for $t \in \Lambda'_{\alpha k}$, $u \in U(t,x)$, $|u|$ Y, we conclude that $f_o + \bar{\bar{D}}$ $N|u|$ for all $t \in \Lambda'_{\alpha k}$, $u \in U(t,x)$. We have

$$I[x,u] \lesssim -(r + |\bar{\bar{D}}|)\bar{D} +$$

$$+ \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \{-b|(\Lambda_{\alpha k}) \int |(t,x(t),u(t))|\,dt + \nu(\Lambda_{\alpha k}) \int |f(t,x(t),u(t))|\,dt$$

$$+ (\Lambda'_{\alpha k}) \int [f_o(t,x(t),u(t)) + \bar{\bar{D}}]\,dt\} \geq -(r + |\bar{\bar{D}}|)\bar{D} +$$

$$+ \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \{-b(\Lambda_{\alpha k}) \int |f(t,x(t),u(t))|\,dt + \tag{45}$$

$$+ \nu(\Lambda_{\alpha k}) \int |f(t,x(t),u(t))|\,dt + N(\Lambda'_{\alpha k}) \int |u(t)|\,dt\}$$

Since $|f| \leq C_o + D_o|u|$ for all $t \in \Lambda'_{\alpha k}$ with $D_o > 0$, we have

$$|u| \geq (|f| - C_o)D_o^{-1} \quad \text{for all } (t,x,u) \in M \text{ with } t \in \Lambda'_{\alpha k}. \tag{46}$$

The relations (45) and (46) yield

$$I[x,u] \geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1} \bar{D} +$$

$$+ \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \{- b(\Lambda_{\alpha k}) \int |f(t, x(t), u(t))| dt + \nu(\Lambda_{\alpha k}) \int |f(t, x(t), u(t)| dt +$$

$$+ N D_o^{-1} (\Lambda'_{\alpha k}) \int |f(t, x(t), u(t))| dt\}. \tag{47}$$

Now $|(\Lambda_{\alpha k} \cup \Lambda'_{\alpha k}) \int f(t, x(t), u(t)) dt| =$

$$= |(\Delta_{\alpha k}) \int f(t, x(t), u(t)) dt| = |(\Delta_{\alpha k}) \int x'(t) dt| \leq 2\, \delta \sqrt{n+1}$$

and therefore

$$|(\Lambda_{\alpha k}) \int f(t, x(t), u(t)) dt| - |(\Lambda'_{\alpha k}) \int f(t, x(t), u(t)) dt| \leq$$

$$|(\Lambda_{\alpha k} \cup \Lambda'_{\alpha k}) \int f(t, x(t), u(t)) dt| \leq 2\, \delta \sqrt{n+1}\,.$$

We have

$$|(\Lambda_{\alpha k}) \int f(t, x(t), u(t)) dt| \leq 2\delta \sqrt{n+1} + |(\Lambda'_{\alpha k}) \int f(t, x(t), u(t)) dt|$$

$$\leq 2\delta \sqrt{n+1} + (\Lambda'_{\alpha k}) \int |f(t, x(t), u(t))| dt \tag{48}$$

for each $\alpha = 1, 2, \ldots, R$; $k = 1, 2, \ldots, T_\alpha$.

Relations (47) and (48) yield

$$I[x, u] \geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} +$$

$$+ \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \{-b(2\delta\sqrt{n+1} + (\Lambda'_{\alpha k}) \int |f(t, x(t), u(t))| dt) +$$

$$+ \nu(\Lambda_{\alpha k}) \int |f(t, x(t), u(t))| dt + N D_o^{-1} (\Lambda'_{\alpha k})$$

$$\int |f(t, x(t), u(t))| dt\}. \qquad (49)$$

If we let $\lambda_{\alpha k} = (\Lambda_{\alpha k}) \int |f(t, x(t), u(t))| dt,$

$$\lambda'_{\alpha k} = (\Lambda'_{\alpha k}) \int |f(t, x(t), u(t))| dt,$$

$$\lambda = \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \lambda_{\alpha k} \text{ and } \lambda' = \sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha} \lambda'_{\alpha k},$$

we have

$$I[x, u] \geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} + \nu\lambda + N D_o^{-1} \lambda' +$$

$$- b(\sum_{\alpha=1}^{R} \sum_{k=1}^{T_\alpha - 1} (2\delta\sqrt{n+1} + \lambda'_{\alpha k}) - b(2\delta\sqrt{n+1} + \lambda'_{\alpha T_\alpha}). \qquad (50)$$

But

$$\lambda'_{\alpha} = (\Lambda'_{\alpha k}) \int |f(t, x(t), u(t))| dt \geq (\Lambda'_{\alpha k}) \int |f_i(t, x(t), u(t))| dt \geq$$

$$\geq |(\Lambda'_{\alpha k}) \int f_i(t, x(t), u(t)) dt| \geq$$

$$\geq \delta - \eta > \delta - \delta/2 = \delta/2$$

for each $\alpha = 1, 2, \ldots, R$; $k=1, \ldots, T_{\alpha}-1$ and for $i=1, 2, \ldots, n$. Also, $N > 2b(1 + 4\sqrt{n+1})D_o$. Therefore

$$I[x, u] \geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} + \nu\lambda + N D_o^{-1}\lambda'$$

$$- b\Big(\sum_{\alpha=1}^{R} \sum_{k=1}^{T_{\alpha}-1} (4\lambda'_{\alpha k}\sqrt{n+1} + \lambda'_{\alpha k}\Big)$$

$$- b\Big[\sum_{\alpha=1}^{R} (2\delta\sqrt{n+1} + \lambda'_{\alpha T_{\alpha}})\Big] \geq$$

$$\geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} - 2\delta bR\sqrt{n+1} + \nu\lambda + N D_o^{-1}\lambda'$$

$$- b(1 + 4\sqrt{n+1} \sum_{\alpha=1}^{R} \sum_{k=1}^{T_{\alpha}} \lambda'_{\alpha k} \geq$$

$$\geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} - 2\delta bR\sqrt{n+1} + \nu\lambda +$$

$$+ [N D_o^{-1} - b(1 + 4\sqrt{n+1})]\lambda'$$

$$\geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} - 2\delta bR\sqrt{n+1}$$

$$+ \nu\lambda + b(1 + 4\sqrt{n+1})\lambda'. \qquad (51)$$

If we let $\bar{\nu} = \min(\nu, b(1 + 4\sqrt{n+1}))$, we have from (51)

$$I[x, u] \geq -(r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} - 2\delta bR\sqrt{n+1} + \bar{\nu}(BV|x(t)|).$$

Hence,

$$BV|x(t)| \leq [I[x, u] + (r + |\bar{\bar{D}}| + C_o D_o^{-1})\bar{D} + 2\delta bR\sqrt{n+1}]\,\bar{\nu}^{-1}.$$

But $(x, u) \in \{(x, u)\}_M$ and so $I[x, u] \leq M$. Therefore the vector functions $x(t)$ from $\{(x, u)\}_M$ have uniform bounded variation. The same holds when Ω replaces Ω' as $\Omega \subset \Omega'$.

Let us prove that the vector functions $x(t)$, $t'_1 \leq t \leq t'_2$, of the family $\{x(t), u(t)\}_M$ are equicontinuous. Suppose they are not equicontinuous. Then, there is an $\epsilon > 0$ such that for each nonnegative integer j there is a trajectory $x = x_j(t)$, $t_{1j} \leq t \leq t_{2j}$, and two points t'_{1j}, t'_{2j} such that $t_{1j} \leq t'_{1j} \leq t'_{2j} \leq t_{2j}$, x_j is a trajectory from $\{x, u\}_M$, $|x_j(t'_{j1}) - x_j(t'_{j2})| > \epsilon$, and $0 < t'_{j2} - t'_{j1} < j^{-1}$; letting u_j denote the control corresponding to x_j, $I[x_j, u_j] \leq M$. Without loss of generality we can assume that $t'_{j1} \to t_o$, $t'_{j2} \to t_o$, $x_j(t'_{j1}) \to x_1$, $x_j(t'_{j2}) \to x_2$ as $j \to +\infty$, and then necessarily $|x_2 - x_1| \geq \epsilon$. As the trajectories $x_j(t)$, $t_{1j} \leq t \leq t_{2j}$, of the family $\{x(t), u(t)\}_M$ have uniformly bounded variation by what was proven before, let us denote one such bound by U_M.

The sets $E^i(\{t_o\})$ have measure zero. Hence, as before, they can be covered by open sets of measure $< \eta$ for arbitrary $\eta > 0$. If η is not less than $\epsilon/2$, choose η so that it is less than $\epsilon/2$. Let $F_o = \{t_o\}x, F^o_1 x, \ldots, xF^o_n$; then F_o is an open set in the hyperplane $H(t_o)$.

By the same analysis we used in forming the partition of E_{n+1} into intervals $Q^{\alpha q}$, there is a real number $\rho > 0$, two sets B_o and I_o such that the set I_o is the complement of B_o in $(t_o - \rho, t_o + \rho) \times S$ where S is an n-dimensional open sphere such that for each $(t, x) \in A$, $x \in S$, $(t_o-\rho, t_o+\rho) \times S = B_o \cup I_o \cap S$, the set I_o is a finite union of disjoint open intervals, the sets I_o and $I^A_o = I_o \cap A$ have the property that each of its projections on the x^i-axis, $i=1, \ldots, n$, has linear measure less than η, there are constants $C(t_o) \geq 0$, $D(t_o) \geq 0$ such that $|f(t, x, u)| \leq C(t_o) + D(t_o)|u|$ for each $(t, x, u) \in M$ with $(t, x) \in B^A_o = B_o \cap A$, $f_o|u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$ uniformly on B^A_o and $E(t_o) = E \cap \{t_o\} \times E_n$ is contained in I^A_o.

Let $N > D(t_o)\, 4\epsilon^{-1}[M + 1 + |2Z - r - bU_M|]$ where r, δ, b are the constants defined above and $Z = \inf_{\Omega'} I[x, u]$. Then, there is a $Y > 0$ such that $f_o|u|^{-1} \geq N$ when $(t, x, u) \in M$, $(t, x) \in B^A_o$, $|u| \geq Y$. Divide the set $[t'_{j1}, t'_{j2}]$ into three subsets Λ_{j1}, Λ_{j2} and Λ_{j3} in the following manner. Let

$\Lambda_{j1} = \{t | t \in [t'_{j1}, t'_{j2}], (t, x_j(t)) \in B^A_o, \ u_j(t)$ exists and $|u_j(t)| \geq Y\}$,

$\Lambda_{j2} = \{t | t \in [t'_{j1}, t'_{j2}], (t, x_j(t)) \in B^A_o, \ t \notin \Lambda_{j1}\}$

and $\Lambda_{j3} = \{t | t \in [t'_{j1}, t'_{j2}], t \notin \Lambda_{j1} \cup \Lambda_{j2}\}$. Then the set $[t'_{j1}, t'_{j2}] = \Lambda_{j1} \cup \Lambda_{j2} \cup \Lambda_{j3}$.

Clearly there exists a real number $j_o > 0$ such that for $j \geq j_o$ each $t \in [t'_{j1}, t'_{j2}]$ satisfies the inequality $|t - t_o| < \rho$. We now have, for $j \geq j_o$

$$I[x_j, u_j] = \int_{t_{j1}}^{t'_{j1}} f_o(t, x_j(t), u_j(t))dt + \int_{t'_{j1}}^{t'_{j2}} f_o(t, x_j(t), u_j(t))dt$$

$$+ \int_{t'_{j2}}^{t_{j2}} f_o(t, x_j(t), u_j(t))dt$$

$$\geq 2Z + \int_{t'_{j1}}^{t'_{j2}} f_o(t, x_j(t), u_j(t))dt$$

$$\geq 2Z + (\Lambda_{j1}) \int f_o(t, x_j(t), u_j(t)dt + (\bigcup_{k=2}^{3} \Lambda_{jk}) \int f_o(t, x_j(t), u_j(t))dt$$

$$\geq 2Z + N(\Lambda_{j1}) \int |u_j(t)| dt + (\bigcup_{k=2}^{3} \Lambda_{jk}) \int (-r - b | f(t, x_j(t), u_j(t)|)dt$$

by the growth condition of f_o and the inequality $f_o \geq -r - b|f|$ for each $(t, x, u) \in M$ as proven above. As $\bigcup_{k=2}^{3} \Lambda_{jk}$ is contained in

$\bigcup_{k=1}^{3} \Lambda_{jk} = [t'_{j1}, t'_{j2}]$, we obtain for $j \geq j_o$.

$$I[x_j, u_j] \geq 2Z - r(t'_{j2} - t'_{j1}) - bU_M + N(\Lambda_{j1}) \int |u_j(t)|\,dt$$

$$\geq 2Z - rj^{-1} - bU_M + N(\Lambda_{j1}) \int |u_j(t)|\,dt$$

$$\geq 2Z - r - bU_M + N(\Lambda_{j1}) \int |u_j(t)|\,dt.$$

Now $(\Lambda_{j2})\int |u_j(t)|\,dt \leq Y(t'_{j2} - t'_{j1}) \leq Yj^{-1}$. Also, letting $C_o = C(t_o)$, $D_o = D(t_o)$ and assuming that $D_o > 0$ (there is no loss in generality), we have $|u_j(t)| \geq |f(t, x_j(t), u_j(t))| D_o^{-1} - C_o D_o^{-1}$ for each $(t, x) \in B_o^A$, $u \in U(t, x)$. In addition

$$|(\bigcup_{k=2}^{2} \Lambda_{jk}) \int |f(t, x_j(t), u_j(t))|\,dt \geq (\bigcup_{k=1}^{2} \Lambda_{jk}) \int |f_i(t, x_j(t), u_j(t))|\,dt$$

$$\geq |(\bigcup_{k=1}^{2} \Lambda_{jk}) \int f_i(t, x_j(t), u_j(t))dt|$$

$$= |(\bigcup_{k=1}^{3} \Lambda_{jk}) \int f_i(t, x_j(t), u_j(t))dt - (\Lambda_{j3}) \int f_i(t, x_j(t), u_j(t))dt|$$

$$\geq |(\bigcup_{k=1}^{3} \Lambda_{jk}) \int f_i(t, x_j(t), u_j(t))dt| - |(\Lambda_{j3}) \int f_i(t, x_j(t), u_j(t))dt|$$

$$\geq |x_j(t'_{j2}) - x_j(t'_{j1})| - \eta \geq \epsilon - \eta > \epsilon - \epsilon/2 = \epsilon/2$$

as $\eta < \epsilon/2$, for each $i=1, 2, \ldots, n$. Thus,

$$(\Lambda_{j1}) \int |u_j(t)|\,dt = (\bigcup_{k=1}^{2} \Lambda_{jk}) \int |u_j(t)|\,dt - (\Lambda_{j2}) \int |u_j(t)|\,dt$$

$$\geq D_o^{-1}(\bigcup_{k=1}^{2} \Lambda_{jk}) \int |f(t, x_j(t), u_j(t))|\,dt - (C_o D_o^{-1} + Y)j^{-1}$$

$$\geq D_o^{-1} \epsilon/2 - D_o^{-1} \epsilon/4 = D_o^{-1} \cdot \epsilon/4$$

by the above inequalities and for each $j \geq j_m = \max(j_o, D_o 4\epsilon^{-1} \cdot \cdot (C_o D_o^{-1} + Y))$.

But this last inequality implies that

$$I[x_j, u_j] \geq 2Z - r - bU_M + N(\Lambda_{j1}) \int |u_j(t)|\,dt$$

$$\geq 2Z - r - bU_M + N D_o^{-1} \epsilon/4 \text{ for } j \geq j_m.$$

Hence,

$$I[x_j, u_j] \geq 2Z - r - bU_M + (M+1) + |2Z - r - bU_M|$$

$$\geq M + 1 > M \quad \text{for } j \geq j_m \quad \text{as}$$

$$N > D_o\, 4\epsilon^{-1}[M + 1 + |2Z - r - bU_M|].$$

This is a contradiction. Therefore, the vector functions x(t), $t'_1 \leq t \leq t'_2$ of the family $\{(x, u)\}_M$ are equicontinuous when A is compact. Hence, these same vector functions are equicontinuous when A is compact if Ω replaces Ω'.

Now, let $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, be a sequence of admissible pairs, each from Ω, such that $I[x_k, u_k] \to i$ as $k \to +\infty$. We may assume that

$$i \leq I[x_k, u_k] = \int_{t_{1k}}^{t_{2k}} f_o(t, x_k(t), u_k(t))dt \leq i + 1/k \quad k=1, 2, \ldots$$

Therefore, the previous result implies that $\{x_k(t)\}$ are equicontinuous as $I[x_k, u_k] \leq i + 1/k \leq i + 1$ for $k=1, 2, \ldots$. As $x_k(t)$, $u_k(t)$, $t_{1k} \leq t \leq t_{2k}$, is an admissible pair from Ω and A is a compact subset of E_{n+1}. There exists a constant Λ such that $|x_k(t)| \leq \Lambda$ for $t \in [t_{1k}, t_{2k}]$, and $k=1, 2, \ldots$. Thus, there is a continuous function $x_o(t)$, $t_1 \leq t \leq t_2$ such that $\rho(x_k, x_o) \to 0$ as $k \to +\infty$. $x_o(t)$, $t_1 \leq t \leq t_2$ is not only a continuous vector function but is also a BV vector function.

For each $\epsilon > 0$ there exists an ℓ and a partition $t_1 = t'_1 \leq t'_2 \leq \ldots \leq t'_\ell = t_2$ such that either

$$\sum_{i=1}^{\ell-1} |x_o(t'_{i+1}) - x_o(t'_i)| \geq BV|x_o(t)| - \epsilon$$

or

$$\sum_{i=1}^{\ell} |x_o(t'_{i+1}) - x_o(t'_i)| \geq \epsilon^{-1}$$

accordingly as BV $|x_o(t)|$ is finite or not. If we choose k_o so large that $k \geq k_o$ implies that $\rho(x_k, x_o) \leq \epsilon$, we have that either

$$BV|x_o(t)| - \epsilon \leq \sum_{i=1}^{\ell-1} |\bar{x}_k(t'_{i+1}) - \bar{x}_k(t'_i)| + 2(\ell-1)\epsilon$$

$$\leq BV|x_k(t)| + 2(\ell-1)\epsilon \quad \text{for } k \geq k_o$$

or

$$\epsilon^{-1} \leq \sum_{i=1}^{\ell} |\bar{x}_k(t'_{i+1}) - \bar{x}_k(t'_i)| + 2(\ell-1)\epsilon$$

$$\leq BV\, |x_k(t)| + 2(\ell-1)\epsilon \quad \text{for } k \geq k_o$$

where $\bar{x}_k(t)$ is $x_k(t)$, $t_1 \leq t \leq t_2$, extended to $(-\infty, +\infty)$ by constancy. Since BV $|x_k(t)| \leq U_{i+1} < +\infty$ the second alternative is contradictory for ϵ sufficiently small. As a result $BV|x_o(t)| < +\infty$ and $x_o(t)$ is a measure induced on $[t_1, t_2]$.

We shall now show that $x_o(t)$, $t_1 \leq t \leq t_2$ is not only a continuous BV vector function but that it is also an AC vector function. Suppose it is not an AC vector function, then, there exists a Borel measurable set $B \subset [t_1, t_2]$ which has zero Lebesque measure, but positive x_o measure, i. e. $x_o(B) > 0$. Now, every set $E^i(B)$, $i=1, 2, \ldots, n$, has zero Lebesque measure. Thus if $E_o = \{(t, x) | t \in B,\ x = x_o(t),\ t_1 \leq t \leq t_2\}$, the set $E_o \cap [B \times E^1(B) \times E^2(B) \times \ldots \times E^n(B)]$ has projections of zero measure on each x^i-axis, $i=1, 2, \ldots, n$. Then, there is a closed set $B' \subset B$ such that $x_0(B') > 3L/4$ where $L = x_0(B)$ and $(t, x_0(t)) \notin [B \times E^1(B) \times \ldots \times E^n(B)]$ for each $t \in B'$.

Now $P' = \{(t,x) | t \in B', x = x_0(t)\}$ is a compact subset of A-E, and hence the set $P' \cap E$ is empty. Thus, there are two dusjoint open sets O_1 and O_2 such that $P' \subset O_1$ and $E \subset O_2$. Consequently $A \cap (E_{n+1} - O_2)$ is a compact subset of A-E and contains P'. Indeed, there is a $\rho_0 > 0$ such that the set $N_{\rho_0}(P') = \{(t,x) | \text{dist}((t,x), P') \leq \rho_0\}$ is contained in $A \cap (E_{n+1} - O_2)$. For $N > 4D_0[|i| + 2 + r\bar{D} + bU_{i+1}]L^{-1}$ where $i = \inf_\Omega I[x, u]$, there is a $Y > 0$ such that $f_0(t, x, u) \geq N|u|$ for each $(t,x) \in A \cap (E_{n+1} - O_2)$, $u \in U(t,x)$ and hence for each $(t,x) \in N_{\rho_0}(P')$, $u \in U(t,x)$. Let $C_0, D_0 \geq 0$ denote the constants (and D_0 can clearly be assumed to be greater than 0) such that $|f| \leq C_0 + D_0|u|$ for each $(t, x, u) \in M$ with $(t,x) \in A \cap (E_{n+1} - O_2)$ and hence for each $(t,x) \in N_{\rho_0}(P')$, $u \in U(t,x)$. We may assume without loss of generality that $Y \geq C_0$. Since B' is compact, there are infinitely many open disjoint intervals (α_j, β_j), $j=1, \ldots, r$, such that B' is contained in $\bigcup_{j=1}^{r} (\alpha_j, \beta_j)$, $x_0(\bigcup_{j=1}^{r} (\alpha_j, \beta_j)) < x_0(B') + L\, 8^{-1} Y^{-1} D_0^{-1}$ and $(t, x_0(t))$ maps $\bigcup_{j=1}^{r} (\alpha_j, \beta_j)$ into $N_{\rho_0}(P')$. We can clearly assume without loss of generality that

$$\sum_{j=1}^{r} (\beta_j - \alpha_j) < \min (L8^{-1}Y^{-1}D_o^{-1}, \rho_o/2)$$

The following argument shows this.

Now B' is a compact set with $\mu^*(B') = 0$. For each $\epsilon > 0$ there exists a bounded open set O'_ϵ which contains B' and such that $\mu^*(O'_\epsilon) < \epsilon$. Choose $\epsilon_o < \max(L8^{-1}Y^{-1}D_o^{-1}, \rho_o^{-1}2)$. As O'_{ϵ_o} is a bounded open set it consists of a finite or denumerable number of open subintervals, say O_j, j=1, 2, ... These subintervals O_j cover the compact set B' and hence a finite number of them also cover B'. Denote this finite subcover by O_j, $j=1, 2, \ldots, j_o$. Form the open intervals $(\alpha_1, \beta_1) \cap O_1, \ldots, (\alpha_r, \beta_r) \cap O_1, \ldots, (\alpha_1, \beta_1) \cap O_{j_o}, \ldots, (\alpha_r, \beta_r) \cap O_{j_o}$. These intervals have the required properties if the original intervals $(\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)$ do not.

Then, we choose k_o so large that

$$\sum_{j=1}^{r} ((\alpha_j, \beta_j) \cap [t_{1k}, t_{2k}]) \int |f(t, x_k(t), u_k(t))| \, dt > L/2$$

and $(t, x_k(t)) \in N_{\rho_o}(P')$ for $t \in [\bigcup_{j=1}^{r} (\alpha_j, \beta_j)] \cap [t_{1k}, t_{2k}]$ and all $k \geq k_o$. Thus

$$\sum_{j=1}^{r} ((\alpha_j, \beta_j) \cap [t_{1k}, t_{2k}]) \int |u_k(t)|\, dt \geq$$

$$[(\sum_{j=1}^{r} ((\alpha_j, \beta_j) \cap [t_{1k}, t_{2k}]) \int |f(t, x_k(t), u_k(t))|\, dt)] D_o^{-1}$$

$$- C_o (\sum_{j=1}^{r} (\beta_j - \alpha_j)) D_o^{-1} \qquad \geq$$

$$L\, 2^{-1} D_o^{-1} - C_o\, L\, 8^{-1} Y^{-1} D_o^{-1} \geq L\, 2^{-1} D_o^{-1} - L\, 8^{-1} D_o^{-1} =$$

$$= 3\, L\, 8^{-1} D_o^{-1}.$$

Let $B_k = \{ t : t \in \bigcup_{j=1}^{r} [\alpha_j, \beta_j],\ |u_k(t)| \leq Y\}$ and

$B'_k = \bigcup_{j=1}^{r} (\alpha_j, \beta_j) - B_k$. Then

$$(B_k) \int |u_k(t)|\, dt \leq YL/8YD_o = L/8D_o.$$

Hence

$$(B'_k) \int |u_k(t)|\, dt \geq 3L/8D_o - L/8D_o = L/4D_o > 0.$$

Therefore

$$I[x_k, u_k] \geq -r\bar{D} - bU_{i+1} + (B'_k) \int f_o(t, x_k(t), u_k(t))dt \geq$$

$$\geq -r\bar{D} - bU_{i+1} + N(B'_k) \int |u_k(t)| dt$$

$$\geq -r\bar{D} - bU_{i+1} - [4D_o(|i| + 2 + r\bar{D} + bU_{i+1})/L]L/4D_o$$

$$\geq |i| + 2 > |i| + 1 \geq i + 1$$

and hence $I[x_k, u_k] > i + 1$ which is a contradiction. Therefore $x_o(t)$, $t_1 \leq t \leq t_2$, is an AC vector function.

If we utilize the fact that (i) $x_k(t) \rightarrow x_o(t)$ which is an AC vector function in the ρ-metric and (ii) property (L_3) holds, (L_3) $f_o(t, x, u) \geq - M_o$ for each $(t, x, u) \in M$ for some $M_o \geq 0$ when A is compact, and if we apply the reasoning given in Existence Theorem I after the equi-AC of the sequence $\{x_k(t)\}$ is proven, then we obtain the result that there is a measurable control function such that $x_o(t)$, $u_o(t)$, $t_1 \leq t \leq t_2$ is an admissible pair from Ω and $I[x_o, u_o] \leq i$. As $i \leq I[x_o, u_o]$, one has that the absolute minimum if $I[x, u]$ exists and is taken on in Ω.

Thus, the theorem is proven in the case that A is compact.

When A is not compact, the reasoning given in Existence Theorem I can be applied with the various conditions stated in this theorem and utilizing property (L_3) to reduce these cases to the case where A is compact. The theorem is completely proven.

Corollary 1: This is the same as Existence Theorem IV where for each $(t, x) \in A$ $U(t, x) = U$ is a fixed, closed subset of the u-space E_m, and the part of condition (γ^*) where $f_0 |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U(t, x)$ uniformly on any compact subset A_0 of A-E is replaced by the conditions (i) and (ii) of Corollary 1 to Theorem III.

PROOF: This statement is a consequence of Theorem IV and lemma 4.

Corollary 2: This is the same as Theorem IV where for each $(t, x) \in A$ $U(t, x) = E_m$ is the whole u-space E_m, the function $f_0(t, x, u)$ is convex in u for each $(t, x) \in A-E$, $f_0 |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in E_m$ pointwise in A-E, and the uniform growth condition of $f_0(t, x, u)$ on each compact part A_0 of A-E, given in (γ^*) is omitted.

In addition for free problems, that is, when $m = n$, $f_i(t, x, u) = u^i$, $i=1, 2, \ldots, n$, $f_0(t, x, u)$ is a convex function of u for each $(t, x) \in E$, then the condition that $\tilde{\tilde{Q}}(t, x)$ satisfy property (Q) in A and that E is a closed set can be omitted if E is assumed to be the minimal exceptional set ($f_0 |u|^{-1}$ does not approach $+\infty$ as $|u| \to +\infty$ at points of E) and f_0 is a normally convex function of u in A.

PROOF: The first part of the statement follows by the same argument of the first part of the statement of corollary 2 to

Theorem III.

The statement that E is a closed set in the second part of the statement above follows by the reasoning used to prove the same property in corollary 2 to Theorem III. $\tilde{\tilde{Q}}(t,x)$ satisfies property (Q) in A as f_o is normally convex in u (and hence quasi-normally convex), f = u, m = n and statement (viii) in Chapter I applies.

Example 4: Let m = n = 1, A = $[0,1]^2$, for each $(t,x) \in A$ let $U(t,x) = E_1$, let $f(t,x,u) = u$,

$$f_o(t,x,u) = x^2u^2 = u \qquad \text{for } u \geq 0,\ u \in E_1$$

$$= x^2u^2 \qquad \text{for } u \geq 0,\ u \in E_1$$

and boundary conditions x(0) = x(1) = 0. Then, U(t,x) satisfies property (U) for each (,x) $\in$ A and $\tilde{\tilde{Q}}(t,x) = \{(z_o, z) \mid z_o \geq x^2z^2 + z \text{ for } z \geq 0,\ z_o \geq x^2z^2 \text{ for } z \leq 0\}$. Now, $\tilde{\tilde{Q}}(t,o,\delta) = \{(z_o, z) \mid z_o \geq z \text{ for } z \geq 0, \text{ and } z_o \geq 0 \text{ for } z \geq 0\} = \tilde{\tilde{Q}}(t,o)$. Hence $\tilde{\tilde{Q}}(t,o) = \bigcap_\delta \operatorname{cl} \operatorname{co} \tilde{\tilde{Q}}(t,o,\delta)$ for each $t \in [0,1]$. Therefore $\tilde{\tilde{Q}}(t,o)$ satisfies property (Q) for each $(t,x) \in [0,1] \times \{0\}$. As $\tilde{\tilde{Q}}(t,x)$ also satisfies property (Q) for each $(t,x) \in [0,1] \times (0,1]$, we conclude that $\tilde{\tilde{Q}}(t,x)$ satisfies property (Q) in A. Although $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in E_1$ uniformly on each compact subset of $[0,1] \times (0,1]$, $f_o(t,x,u)|u|^{-1}$ does not approach $+\infty$ as $|u| \to +\infty$ for any $(t,x) \in A$ with $x = 0$. Hence, $f_o\ |u|^{-1}$ does not tend to $+\infty$ as $|u| \to +\infty$, $u \in E_1$ uniformly on A and both Existence Theorem I and II do not apply to this example. Indeed, even Existence Theorem III does not apply to this problem. Clearly, the closed exceptional set E for Theorem III must contain the set $[0, 1] \times \{0\}$ where the growth condition fails, but condition (T) is obviously not satisfied on this set. Therefore, Theorem III does not apply.

However, Theorem IV applies. Let $E = [0, 1] \times \{0\}$. Then $\mu^*[E(H)] = 0$ for every subset of the t-axis of measure zero (in fact for any subset $H \subset [0, 1]$) and (L_1) holds. If $r=0$, $b_1 = \frac{1}{2}$, $\gamma = \frac{1}{2} > 0$, then

$$f_0(t, x, u) = x^2u^2 + u \geq 0 + 2^{-1} \cdot u + 2^{-1} \cdot |u|$$
$$\geq u \text{ for } u \geq 0, \ (t, x) \in A$$
$$= x^2u^2 \geq 0 + 2^{-1} \cdot u + 2^{-1}|u| = 0$$
$$\text{for } u \leq 0, \ (t, x) \in A$$

and (L_2) holds. Also, $f_0(t, x, u) \geq 0$ and (L_3) holds. Now the growth condition on $f_0(t, x, u)$ of property (γ^*) was shown to be valid on any compact part A_0 of $A-E$. Letting $C=0$, $D=1$, $|f(t, x, u)| = |u| \leq 0 + 1 \cdot |u|$ for each $(t, x, u) \in A \times E_1$ and (γ^*) is valid. Therefore Theorem IV applies, but Theorems I, II and III do not.

REMARK: Theorem IV is similar to a theorem of Tonelli [23a] for free problems, $n=1$, and f_0 continuously differentiable. Tonelli's statement has been extended to free problems with $n \geq 1$

and f_o not necessarily differentiable by L. Turner [24]. The present theorem differs from that given by Tonelli and L. Turner in two ways. Condition (L_3) was not required, but normal convexity of $f_o(t, x, u)$ was required in place of condition (L_2); hence one condition was added and one condition greatly weakened from the case of the free problems. Examples 1 and 2 are due to Tonelli.

Chapter III

Optimal Control Problems where f is Linear

14. A few lemmas.

We shall now consider the case where all functions $f_i(t, x, u)$, $i=1, 2, \ldots, n$, are linear in u, and the control space is a fixed closed convex subset of E_m for each (t, x) of A. Precisely, we shall consider the optimal control problem

$$I[x, u] = \int_{t_1}^{t_2} f_o(t, x, u)dt = \text{minimum}, \tag{52}$$

$$dx^i/dt = \sum_{j=1}^{m} g_{ij}(t, x)u^j + g_i(t, x), \quad i=1, 2, \ldots, n, \tag{53}$$

where $x = (x^1, \ldots, x^n) \in E_n$, and $f_o(t, x, u)$ is a convex function of u, $u \in U$, for each fixed $(t, x) \in A$. If H(t, x) denotes the $n \times m$ matrix $(g_{ij}(t, x))$, and h(t, x) the n-vector $(g_i(t, x))$, then the differential system (53) becomes

$$dx/dt = H(t, x)u + h(t, x).$$

We shall, henceforth, assume that $g_{ij}(t, x)$, $g_i(t, x)$, $i=1, 2, \ldots, n$; $j=1, 2, \ldots, m$ are continuous bounded functions on A.

The sets Q(t, x), $\tilde{\tilde{Q}}(t, x)$ relative to the above problem are

$$Q(t,x) = [z \mid z = H(t,x)u + h(t,x),\ u \in U] \subset E_n \qquad (54)$$

$$\tilde{\tilde{Q}}(t,x) = [\tilde{z} = (z^o, z) \mid z^o \geq f_o(t,x,u),$$

$$z = H(t,x)u + h(t,x),\ u \in U] \subset E_{n+1}$$

We shall need a few lemmas concerning the sets $Q(t,x)$ and $\tilde{\tilde{Q}}(t,x)$. The proofs of the next two lemmas are due to Cesari [4a].

Lemma 6: Let the set A be a fixed closed subset of the tx-space $E_1 \times E_n$, $U(t,x) = U$ be a fixed, closed, convex subset of E_m for each $(t,x) \in A$, $f_o(t,x,u)$, $(t,x,u) \in A \times U$, be a continuous scalar function on $M = A \times U$ which is also a convex function of u for each $(t,x) \in A$, and the differential system be given by (53). Then, both sets $Q(t,x)$ and $\tilde{\tilde{Q}}(t,x)$, which are defined by (54), are convex for each $(t,x) \in A$.

PROOF: The set $Q(t,x)$ is obviously convex for each $(t,x) \in A$. Let us now give the proof for the set $\tilde{\tilde{Q}}(t,x)$. Let $\tilde{p} = (p^o, p)$, $\tilde{q} = (q^o, q)$ be any two points of $\tilde{\tilde{Q}}(t,x)$, let $0 \leq \alpha \leq 1$, and $\tilde{z} = (z^o, z) = \alpha\tilde{p} + (1-\alpha)\tilde{q}$. Then for some vectors $u, v \in U$ we have

$$p^o \geq f_o(t,x,u), \qquad p = Hu + h$$
$$q^o \geq f_o(t,x,v), \qquad q = Hv + h,$$

$$\tilde{z} = \alpha\tilde{p} + (1-\alpha)\tilde{q}, \qquad z^o = \alpha p^o + (1-\alpha)q^o, \qquad z = \alpha p + (1-\alpha)q.$$

Now the vector $w = \alpha u + (1-\alpha)v \in U$ as U is a convex subset of E_m.

We have

$$z = \alpha p + (1-\alpha)q = \alpha(Hu + h) + (1-\alpha)(Hv + h) =$$

$$= H(\alpha u + (1-\alpha)v) + h = Hw + h,$$

$$z^o = \alpha p^o + (1-\alpha)q^o \geq \alpha\, f_o(t,x,u) + (1-\alpha)f_o(t,x,v) \geq$$

$$\geq f_o(t,x,\ \alpha u + (1-\alpha)v) = f_o(t,x,w).$$

Thus, $\tilde{z} = (z^o, z) \in \tilde{\tilde{Q}}(t,x)$ and $\tilde{\tilde{Q}}(t,x)$ is a convex set for each (t,x) $\in A$.

<u>Lemma 7:</u> Let the set A be a fixed closed subset of the tx-space $E_1 \times E_n$, $U(t,x) = U$ be a fixed, closed, convex subset of E_m for each $(t,x) \in A$, $f_o(t,x,u)$, $(t,x,u) \in A \times U$, be a continuous scalar function in $M = A \times U$ which is also a convex function of u for each $(t,x) \in A$, and for each compact part A_o of A let $\Phi_o(z)$ be a continuous scalar function in the set $Z = [z \mid z = |u|$ for some $u \in U]$ such that $f_o(t,x,u) \geq \Phi_o(|u|)$ for each $(t,x,u) \in A_o \times U$ and $\Phi_o(z) \to +\infty$ as $z \to +\infty$, and let the differential system be given by (53). Then, the set $\tilde{\tilde{Q}}(t,x)$, which is defined by (54), satisfies property (Q) in A.

PROOF: We have to prove that $\tilde{\tilde{Q}}(\bar{t},\bar{x}) = \bigcap_\delta \operatorname{cl}\operatorname{co} \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta)$. It is enough to prove that $\bigcap_\delta \operatorname{cl}\operatorname{co} \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta) \subset \tilde{\tilde{Q}}(\bar{t},\bar{x})$ as the opposite inclusion is trivial. Let us assume that a given point $\tilde{z} = (\bar{z}^o, \bar{z}) \in \bigcap_\delta \operatorname{cl}\operatorname{co} \tilde{\tilde{Q}}(\bar{t},\bar{x},\delta)$ and let us prove that $\tilde{z} = (\bar{z}^o, \bar{z}) \in \tilde{\tilde{Q}}(\bar{t},\bar{x})$. For every $\delta > 0$ we have

$\tilde{z} = (\bar{z}^o, \bar{z}) \in \text{cl co}\, \tilde{\tilde{Q}}(\bar{t}, \bar{x}, \delta)$, and thus, for every $\delta > 0$, there are points $\tilde{z} = (z^o, z) \in \text{co}\, \tilde{\tilde{Q}}(\bar{t}, \bar{x}, \delta)$ at as small a distance as we want from $\tilde{z} = (\bar{z}^o, \bar{z})$. Thus, there is a sequence of points $\tilde{z}_k = (z^o{}_k, z_k) \in \text{co}\, \tilde{\tilde{Q}}(\bar{t}, \bar{x}, \delta_k)$ and a sequence of numbers $\delta_k > 0$ such that $\delta_k \to 0$, $\tilde{z}_k \to \tilde{z}$ as $k \to +\infty$. In other words, for every integer k, there are some pairs (t'_k, x'_k), (t''_k, x''_k), corresponding points $\tilde{z}'_k = (z^{o'}{}_k, z'_k) \in \tilde{\tilde{Q}}(t'_k, x'_k)$, $\tilde{z}''_k = (z^{o''}{}_k, z''_k) \in \tilde{\tilde{Q}}(t''_k, x''_k)$, points u'_k, $u''_k \in U$, and numbers α_k, $0 \leq \alpha_k \leq 1$, such that

$$\tilde{z}_k = \alpha_k \tilde{z}'_k + (1-\alpha_k)\tilde{z}''_k,$$

$$z^o_k = \alpha_k z^{o'}_k + (1-\alpha_k) z^{o''}_k, \qquad z_k = \alpha_k z'_k + (1-\alpha_k) z''_k,$$

$$z^{o'}_k \geq f_o(t'_k, x'_k, u'_k), \qquad z'_k = H(t'_k, x'_k)u'_k + h(t'_k, x'_k)$$

$$z^{o''}_k \geq f_o(t''_k, x''_k, u''_k), \qquad z''_k = H(t''_k, x''_k)u''_k + h(t''_k, x''_k) \quad (55)$$

and such that

$$t'_k \to \bar{t},\ x'_k \to \bar{x},\ t''_k \to \bar{t},\ x''_k \to \bar{x},\ \tilde{z}_k \to \tilde{z},\ z^o_k \to \bar{z}^o,\ z_k \to \bar{z}$$

as $k \to +\infty$. Because of these limit properties, there is some closed ball $B \subset E_{n+1}$ such that $(t'_k, x'_k) \in B$, $(t''_k, x''_k) \in B$ and $(t, x) \in B$. Let $A_o = B \cap A$ and consider Φ_o on this set.

The second relation (55) shows that of the two numbers $z^{o'}_k$, $z^{o''}_k$ one must be $\leq z^o_k$. It is not restrictive to assume that $z^{o'}_k \leq z^o_k$ for all k. Then the fourth relation (55) together with the lower bound for f_o yields

$$z_k^o \geq z_k^{o'} \geq f_o(t'_k, x'_k, u'_k) \geq \Phi_o(|u'_k|),$$

where $z_k^o \to \bar{z}^o$, and hence $[z_k^o]$ is a bounded sequence. Thus, $\Phi_o(|u'_k|) \leq z^o{}_k$ and the boundedness of the sequence $[z^o{}_k]$ together with this previous inequality and the limit property and continuity of $\Phi_o(z)$ implies that $[\Phi_o(|u'_k|)]$ is also a bounded sequence. Finally, $[u'_k]$ is also a bounded sequence because of this last result and the growth property of Φ_o. We can select a subsequence, say still $[u'_k]$, which is convergent, say $u'_k \to \bar{u}' \in U$ as $k \to +\infty$. The sequence $[\alpha_k]$ is also bounded; hence we can select a further subsequence, which also we shall call $[\alpha_k]$, for which $\alpha_k \to \bar{\alpha}$ as $k \to +\infty$ with $0 \leq \bar{\alpha} \leq 1$. Let $u_k \in U$ be the point $u_k = \alpha_k u'_k + (1-\alpha_k)u''_k$ (this can clearly be done as U is a convex set in E_m). Then,

$$\begin{aligned}
z_k &= \alpha_k z'_k + (1-\alpha_k)z''_k = \\
&= \alpha_k[H(t'_k, x'_k)u'_k + h(t'_k, x'_k)] + (1-\alpha_k)[H(t''_k, x''_k)u''_k \\
&\qquad\qquad + h(t''_k, x''_k)] \\
&= H(t''_k, x''_k)\,[\alpha_k u'_k + (1-\alpha_k)u''_k) + h(t''_k, x''_k) + \\
&\quad + \alpha_k\{[H(t'_k, x'_k) - H(t''_k, x''_k)]u'_k + \\
&\quad + [h(t'_k, x'_k) - h(t''_k, x''_k)]\} \\
&= H(t''_k, x''_k)u_k + h(t''_k, x''_k) + \Delta_k, \qquad (56)
\end{aligned}$$

$$z_k^o = \alpha_k z_k^{o'} + (1-\alpha_k) z_k^{o''} \geq$$

$$\geq \alpha_k f_o(t'_k, x'_k, u'_k) + (1-\alpha_k) f_o(t''_k, x''_k, u''_k)$$

$$\geq \alpha_k f_o(t''_k, x''_k, u'_k) + (1-\alpha_k) f_o(t''_k, x''_k, u''_k) +$$

$$+ \alpha_k [f_o(t'_k, x'_k, u'_k) - f_o(t''_k, x''_k, u'_k)]$$

$$\geq f_o(t''_k, x''_k, \alpha_k u'_k + (1-\alpha_k) u''_k) +$$

$$+ \alpha_k [f_o(t'_k, x'_k, u'_k) - f_o(t''_k, x''_k, u'_k)]$$

$$\geq f_o(t''_k, x''_k, u_k) + \Delta_k^o$$

where Δ_k and Δ_k^o have the property $\Delta_k \to 0$, $\Delta_k^o \to 0$ and $h(t''_k, x''_k) \to h(\bar{t}, \bar{x})$ as $k \to +\infty$ since $t'_k \to \bar{t}$, $x'_k \to \bar{x}$, $t'_k \to \bar{t}$, $x''_k \to \bar{x}$ and $u'_k \to \bar{u}'$ as $k \to +\infty$. Again we can conclude that $[\Phi(|u_k|)]$ is a bounded sequence, and so is $[u_k]$. Hence, we can further select a convergent subsequence, say still $[u_k]$, with $u_k \to \bar{u} \in U$. Relations (56) now imply as $k \to +\infty$

$$\bar{z} = H(\bar{t}, \bar{x})\bar{u} + h(\bar{t}, \bar{x})$$

$$z^o \geq f_o(\bar{t}, \bar{x}, \bar{u}).$$

Thus, $\tilde{z} = (\bar{z}^o, \bar{z}) \in \tilde{\tilde{Q}}(\bar{t}, \bar{x})$, and lemma 7 is proven.

The following example shows that $\tilde{\tilde{Q}}(t, x)$ does not necessarily satisfy property (Q) for each possible optimal control problem. Lemma 7 does not apply to this case as $f_o(t, x, u)$ fails to satisfy the "uniform growth" property implied by the function $\Phi(z)$ such that

$\Phi(z) \to +\infty$ as $z \to +\infty$, $z \geq 0$ and $f_0(t, x, u) \geq \Phi(|u|)$ for each $(t, x, u) \in A \times U$. This example was given by Cesari [4a].

Example 5: Let $m = n = 1$, $U = E_1$, $A = [0, 1]^2$, $f_0(t, x, u) = t^3u^2$ and $f(t, x, u) = tu$. Then,

$$\tilde{\tilde{Q}}(t, x) = \tilde{\tilde{Q}}(t) = [\tilde{z} = (z^0, z) \mid z^0 \geq t^3u^2,\ z = tu,\ u \in E_1].$$

Thus, $\tilde{\tilde{Q}}(0, x) = \tilde{\tilde{Q}}(0) = [(z^0, z) \mid z^0 \geq 0,\ z = 0]$ and for $t \neq 0$, $\tilde{\tilde{Q}}(t) = [(z^0, z) \mid z^0 \geq tz^2,\ z \in E_1]$ and cl co $\tilde{\tilde{Q}}(0, \delta)$ is the entire half plane $[(z^0, z) \mid z^0 \geq 0,\ z \in E_1]$. Therefore $\tilde{\tilde{Q}}(t, x) = \tilde{\tilde{Q}}(t)$ does not satisfy property (Q) at $t = 0$.

However, there are optimal control problems for which the "uniform growth condition on $f_0(t, x, u)$ can be relaxed and the set $\tilde{\tilde{Q}}(t, x)$ still satisfies property (Q). The following lemma states such a case. But before we proceed we shall define a new property, called (SB), for the control set U.

(SB): A fixed control set U in E_m is said to satisfy property (SB) if for each i, $i=1, 2, \ldots, m$, there is a real number a_i such that either $u_i \leq a_i$ for each $u \in U$ or $u_i \geq a_i$ for each $u \in U$.

Examplesof such control sets are the sets

$U = [(u_1, u_2) \mid u_1 \geq 0,\ u_2 \geq 1]$ where $a_1 = 0$, $a_2 = 1$ and

$U = [(u_1, u_2) \mid u_1 \geq 1,\ u_2 \leq -1]$ where $a_1 = 1$, $a_2 = -1$.

Lemma 8: Let the set A be a fixed closed subset of the tx-space $E_1 \times E_n$, $U(t,x) = U$ be a fixed, closed, convex subject of the u-space E_m for each $(t,x) \in A$ which also satisfies property (SB), let $f_o(t,x,u)$ be a continuous scalar function on $M = A \times U$ which is also a convex function of u for each $(t,x) \in A$, $m = n$ and $f(t,x,u) = u$. Then, the set $\tilde{\tilde{Q}}(t,x)$, which is defined by (54), satisfies property (Q) in A.

PROOF: Proceed as in lemma 7 until relations (55) are reached. The new relations (55) are as follows:

$$\tilde{z}_k = \alpha_k \tilde{z}'_k + (1-\alpha_k)\tilde{z}''_k,$$

$$z_k^o = \alpha_k z_k^{o'} + (1-\alpha_k) z_k^{o''}, \qquad z_k = \alpha_k z'_k + (1-\alpha_k) z''_k,$$

$$z_k^{o'} \geq f_o(t'_k, x'_k, u'_k), \qquad z'_k = u'_k$$

$$z_k^{o''} \geq f_o(t''_k, x''_k, u''_k), \qquad z''_k = u''_k \tag{57}$$

and such that $t'_k \to \bar{t}$, $x'_k \to \bar{x}$, $t''_k \to \bar{t}$, $x''_k \to \bar{x}$, $\tilde{z}_k \to \tilde{\bar{z}}$, $z_k^o \to \bar{z}^o$, $z_k \to \bar{z}$ as $k \to +\infty$. As U is a convex set, $z_k = \alpha_k u'_k + (1-\alpha_k)u''_k \in U$. Also $\bar{z} \in U$ as $z_k \to \bar{z}$ as $k \to +\infty$ and U is a closed set. If we write the third relation of (57) in component form with the help of relations five and seven of (57), we have

$$z_k^{(i)} = \alpha_k u'^{(i)}_k + (1-\alpha_k) u''^{(i)}_k \quad \text{for } i = 1, 2, \ldots, m.$$

But $z_k^{(i)} \to \bar{z}^{(i)}$. Now either there are only a finite number of $\alpha_k = 0$ or 1 or there are not. Suppose there are only a finite number of $\alpha_k = 0$ or 1. Then $u'^{(i)}_k$ and $u''^{(i)}_k$ are both bounded or both unbounded as $0 \leq \alpha_k \leq 1$ for each k, $\alpha_k \geq 0$, $1-\alpha_k \geq 0$ for each k and their convex sum $z_k^{(i)} \to \bar{z}^{(i)}$ as $k \to +\infty$ where $\bar{z}^{(i)}$ is some finite number. In fact, if $u'^{(i)}_k$ and $u''^{(i)}_k$ are both unbounded, they are unbounded with opposite signs. As u'_k, $u''_k \in U$, and U satisfies property (SB), this cannot happen. Therefore, both $u'^{(i)}_k$ and $u''^{(i)}_k$, $i=1, 2, \ldots, m$, are both bounded and so are u'_k and u''_k bounded m-vectors. Suppose there are an infinite number of k-values for which $\alpha_k = 0$ or $\alpha_k = 1$. Consider $\alpha_k = 0$ an infinite number of times. Then, $z_k = u''_k$ and $z_k \to \bar{z}$ implies that $u''_k \to \bar{z} \in U$ for these values of k. Re-label this new subsequence with the same index k and we obtain new relations (57) with $u''_k \to \bar{z} \in U$. For $\alpha_k = 1$ an infinite number of times, the same logic yields new relations (57) for which $u'_k \to \bar{z} \in U$. Now,

$$\begin{aligned} z_k &= \alpha_k u'_k + (1-\alpha_k)u''_k \\ &= u_k \quad \text{where } u_k = \alpha_k u'_k + (1-\alpha_k)u''_k, \end{aligned} \tag{58}$$

$$z_k^o = \alpha_k z_k^{o'} + (1-\alpha_k) z_k^{o''} \geq$$

$$\geq \alpha_k f_o(t'_k, x'_k, u'_k) + (1-\alpha_k) f_o(t''_k, x''_k, u''_k)$$

$$\geq \alpha_k f_o(t''_k, x''_k, u'_k) + (1-\alpha_k) f_o(t''_k, x''_k, u''_k)$$

$$+ \alpha_k [f_o(t'_k, x'_k, u'_k) - f_o(t''_k, x''_k, u''_k)]$$

$$\geq f_o(t''_k, x''_k, u_k) + \Delta_k^o$$

where $\Delta_k^o = \alpha_k [f_o(t'_k, x'_k, u'_k) - f_o(t''_k, x''_k, u'_k)]$. In the first and third cases (58) is valid and u'_k approaches a finite limit, say $\bar{u}'$, as $k \to +\infty$. Thus as $t'_k \to \bar{t}$, $t''_k \to \bar{t}$, $x'_k \to \bar{x}$, $x''_k \to \bar{x}$ and α_k is bounded between 0 and 1, $\Delta_k^o \to 0$ as $k \to +\infty$. Hence, the relations (58) now imply as $k \to +\infty$.

$$\bar{z} = \bar{u}$$

$$\bar{z}^o \geq f_o(\bar{t}, \bar{x}, \bar{u}).$$

Thus, $\tilde{z} = (z^o, z) \in \tilde{\tilde{Q}}(\bar{t}, \bar{x})$, and lemma (8) is proven for the first and third cases.

Consider the second case where (57) is valid and u''_k and u_k approach a finite limit in U as $k \to +\infty$. Again,

$$z_k = \alpha_k u'_k + (1-\alpha_k)u''_k$$

$$= u_k,$$

$$z^o_k = \alpha_k z^{o'}_k + (1-\alpha_k)z^{o''}_k \geq$$

$$\geq \alpha_k f_o(t'_k, x'_k, u'_k) + (1-\alpha_k)f_o(t'_k, x'_k, u''_k)$$

$$+ (1-\alpha_k)[f_o(t''_k, x''_k, u''_k) - f_o(t'_k, x'_k, u''_k)]$$

$$z^o_k \geq f_o(t'_k, x'_k, u_k) + \bar{\Delta}^o_k$$

where $\bar{\Delta}^o_k = (1-\alpha_k)[f_o(t''_k, x''_k, u''_k) - f_o(t'_k, x'_k, u''_k)]$. Thus, in the second case, with $t'_k \to \bar{t}$, $x'_k \to \bar{x}$, $t''_k \to \bar{t}$, $x''_k \to \bar{x}$ as $k \to +\infty$ and $0 \leq 1 - \alpha_k \leq 1$ for each k, $\bar{\Delta}^o_k \to 0$ as $k \to +\infty$. Hence the last relations imply that

$$\bar{z} = \bar{u}$$

$$\bar{z}^o \geq f_o(\bar{t}, \bar{x}, \bar{u}).$$

Therefore $\tilde{z} = (z^o, z) \in \tilde{\bar{Q}}(\bar{t}, \bar{x})$ as before and lemma 8 is completely proven.

Example 5, which precedes this lemma, shows that property (SB) cannot be relaxed in lemma 8.

15. Existence theorems where f is linear in u.

It is now possible to state several existence theorems for optimal control problems in which the differential equation is linear in the control variable.

Existence Theorem V: Let us consider the optimal control problem described in relations (52) and (53). Let the set A be a fixed compact subset of the tx-space $E_1 \times E_n$, let $U(t, x) = U$ be a fixed, closed, convex subset of the u-space E_m for each $(t, x) \in A$, let $f_o(t, x, u)$, $(t, x, u) \in A \times U$, be a continuous scalar function on $M = A \times U$, which is also a convex function of u for each $(t, x) \in A$. Let $\Phi(\zeta)$, $0 \leq \zeta < +\infty$, be a given continuous function of ζ such that $\Phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$ and $f_o(t, x, u) \geq \Phi(|u|)$ for each $(t, x, u) \in A \times U$. Let Ω be the class of all pairs $x(t)$, $u(t)$, $t_1 \leq t \leq t_2$, $x(t)$ absolutely continuous, $u(t)$ measurable satisfying (53) a.e.. Then, the optimal control problem has an optimal solution.

If A is not compact, but closed and contained in a slab $[t_o \leq t \leq T, x \in E_n]$, t, T finite, then the result still holds if condition (b) holds.

If A is closed, but not contained in a slab as above, then Theorem I still holds if conditions (b) and (c') hold:

(c') $f_o(t, x, u) \geq \mu > 0$ for each $(t, x, u) \in A \times U$ where μ is a positive constant.

Finally, for A not compact, the condition $f_0 \geq \Phi(|u|)$ can be replaced by conditions (d) and (e):

(e) for every compact subset A_0 of A there is a function Φ_0 as above such that $f_0 \geq \Phi_0(|u|)$ for each $(t, x, u) \in A_0 \times U$.

PROOF: By lemmas 6 and 7 of this chapter, the set $\tilde{\tilde{Q}}(t, x)$ is convex for each $(t, x) \in A$ and satisfies property (Q) in A, where A is a fixed closed subset of $E_1 \times E_n$. The set U is a fixed, closed, convex subset of E_m for each $(t, x) \in A$ and obviously satisfies property (U) in A for each case.

Now g_{ij} and g_i are bounded, continuous functions of (t, x) on A and hence there is a $C_0 \geq 0$ such that $|g_{ij}| \leq C_0$, $|g_i| \leq C_0$ for each $(t, x) \in A$ and

$$|f| \leq |Hu + h| \leq |H|\ |u| + |h| \leq n^2C_0|u| + nC_0$$

for each $(t, x, u) \in A \times U$ for each case (A compact or A not compact). The case for A compact is now just a special case of Theorem I from Chapter II.

The case for A not compact, but contained in a slab is also proven if condition (d) is satisfied, i. e., if $f_0(t, x, u) \geq E|f(t, x, u)|$ for all $(t, x, u) \in A \times U$ with $|x| \geq F$ and for some constants $E > 0$, $F > 0$. Let us prove that condition (d) is satisfied. Indeed, $\Phi(\zeta)/\zeta \geq 1$ for all $|\zeta| \geq D$ for some constant $D \geq 0$. Then, for $|u| \geq D$ we have $|u| \leq \Phi(|u|)$ and hence $|u| \leq D + \Phi(|u|)$ for each

$u \in U$. Now for each $(t, x, u) \in A \times U$ we have

$$|f| = |Hu + h| \leq |H|\ |u| + |h| \leq |H|(D + \Phi(|u|)) + |h|$$

$$\leq |H|\ \Phi(|u|) + (D|H| + |h|)$$

$$\leq [|H| + (D|H| + |h|)\mu^{-1}]\Phi(|u|) \text{ as } \Phi(|u|) \geq \mu > 0,$$

$$\leq [|H| + (D|H| + |h|)\mu^{-1}]f_0 .$$

Thus $f_0 \geq [|H| + (D|H| + |h|\mu^{-1}]\ |f| \geq E|f|$ for some constant $E > 0$ and for each $(t, x, u) \in A \times U$ as $|H|$ and $|h|$ are bounded on A since g_{ij}, g_i are continuous bounded functions on A. Thus, the case where A is not compact, but contained in a slab is proven.

Suppose A is closed, not compact, nor contained in a slab as before. Then, the conditions of Theorem I in Chapter II are verified. This case is also proven.

The last statement follows as Theorem I from Chapter II only required this to hold, and neither lemma 6 nor lemma 7 require more than condition (e). The theorem is proven.

Existence Theorem VI': Let us consider the optimal control problem described in relations (52) and (53). Let the set A be a fixed compact subset of the tx-space $E_1 \times E_n$, let $U(t, x) = U$ be a fixed, closed, convex subset of the u-space E_m for each $(t, x) \in A$, let

$f_o(t, x, u)$, $(t, x, u) \in A \times U$, be a continuous scalar function on $M = A \times U$ which is a convex function of u for each $(t, x) \in A$ and is such that $|u|^{-1} f_o(t, x, u) \to +\infty$ as $|u| \to +\infty$, $u \in U$ uniformly in A. Let Ω be the class of all pairs x(t), u(t), $t_1 \leq t \leq t_2$, x(t) absolutely continuous, u(t) measurable satisfying (53) a. e.. Then the optimal control problem has an optimal solution.

If A is not compact, but closed and contained in a slab $[t_o \leq t \leq T,\ x \in E_n]$, t_o, T, finite, then the result still holds if condition (b) holds.

If A is closed, but not compact, nor contained in a slab as above, then Theorem VI still holds if (b) and (c') hold.

Finally, for A not compact, the condition $f_o |u|^{-1} \to +\infty$ as $|u| \to +\infty$, $u \in U$ uniformly on A can be replaced by conditions $(\gamma)_m$ and (d).

PROOF: The proof follows as a result of the reasoning given in Theorem I of Chapter III and lemmas 3 and 7 in Chapters II and III, respectively.

Existence Theorem VII: This theorem is the same as theorem V, where a closed exceptional subset E of A is given, condition (T) holds at each point of E, and condition (α) is replaced by (γ^*) if A is compact (the inequality on f in condition (γ^*) is obviously satisfied in this case).

If A is not compact, but closed and contained in a slab $[t_o \leq t \leq T, x \in E_n]$, t_o, T, finite, then theorem VII still holds if conditions (b) and (d) hold (and (α) is replaced by (γ*) and condition (T) holds on the exceptional set E).

If A is not compact, nor contained in a slab as above, but A is closed, then theorem VII still holds if conditions (b), (c') and (d) hold.

PROOF: This theorem follows from Theorem III by the reasoning contained in the proofs of Theorem V and corollary 2 of Theorem III.

Existence Theorem VIII: This theorem is the same as Theorem V, where a closed exceptional set E is given, condition (α) is replaced by (γ*), conditions (L_1), (L_2), and (L_3) hold, and E is a set on which condition (α) is not satisfied (the inequality on f in condition (γ*) is obviously satisfied in this case). We shall also assume that $\tilde{\tilde{Q}}(t, x)$ satisfies property (Q) on the set E.

If A is not compact, but closed and contained in a slab $[t_o \leq t \leq T, x \in E_n]$, t_o, T, finite, then Theorem VIII still holds if conditions (b) and (d) hold (and (α) is replaced by (γ*), provided (L_1), (L_2) and (L_3) hold).

If A is not compact, nor contained in a slab as above, but A is closed, then theorem VIII still holds if condition (b), (c') and (d) hold (and (α) is replaced by (γ^*), provided (L_1), (L_2), and (L_3) hold).

In addition, if m=n, f(t, x, u) = u and the set U satisfies property (SB) then the condition that $\tilde{\tilde{Q}}(t,x)$ satisfies property (Q) on E can be omitted in each of the above cases.

PROOF: The first statements of this theorem follow from Theorem IV and the reasonings contained in Theorem V and corollary 2 to Theorem IV. An application of lemma 8 proves the last statement.

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