PROPORTIONAL MEAN RESIDUAL LIFE

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SUMMARY

Consider two survivor functions $\overline{F}$ and $\overline{G}$ with proportional mean residual life, we identify classes of life distributions and conditions such that properties of F are inherited by G and vice versa. The statistical tests for certain life distribution properties of a system with finite identical components connected in series are now reduced to the corresponding statistical tests for the same properties of life distribution of the individual components.

Some key words: Harmonic new better than used in expectation; Increasing failure rate; Increasing failure rate average; Mean residual life; New better than used; New better than used in expectation; New better than used of specified age; Superposition of independent renewal processes.

1. INTRODUCTION

Let C_0 be a set of life distributions such that $F \in C_0$ if and only if it is a probability distribution function of a continuous nonnegative random variable and $F(0) = 0$. Let $\overline{F} = 1 - F$. For all $x \geq 0$, the mean residual life of a continuous nonnegative random variable with survivor function $\overline{F}$ and finite mean μ_F is given by

$$e_F(x) = \begin{cases} \{\overline{F}(x)\}^{-1} \int_x^\infty \overline{F}(u)\, du & \overline{F}(x) > 0 \\ 0 & \overline{F}(x) = 0 \end{cases} \tag{1}$$

From Cox (1962), p. 128, we can recover $\overline{F}$ from e_F by the inversion formula

$$\overline{F}(x) = \{e_F(0)/e_F(x)\} \exp\left[-\int_0^x \{e_F(u)\}^{-1}\, du\right]. \tag{2}$$

Given $F, G \in C_0$, the two survivor functions $\overline{F}$ and $\overline{G}$ are said to have proportional mean residual life if and only if for all $x \geq 0$, $e_F(x) = p\, e_G(x)$ where $p > 0$. From Oakes and Dasu (1990) Equation (6), F can now be expressed in term of G and vice versa. More specifically,

$$\overline{F}(x) = \overline{G}(x)\left[\int_x^\infty \{\overline{G}(u)/\mu_G\}\, du\right]^{1/p-1} \tag{3}$$

and

$$\overline{G}(x) = \overline{F}(x)\left[\int_x^\infty \{\overline{F}(u)/\mu_F\}\, du\right]^{p-1} \tag{4}$$

where $\mu_F = e_F(0)$ and $\mu_G = e_G(0)$. Note that when $p \geq 1$, given any survivor function $\overline{F}$, $\overline{G}$ defined in (4) is always a survivor function. From Oakes and Dasu (1990), a necessary and sufficient condition for $\overline{G}$ to be a survivor function for all $p > 0$ is that e_F is nondecreasing. Similar results hold for Equation (3).

Conversely, suppose that Equation (4) holds for two survivor functions $\overline{F}$ and $\overline{G}$, using the equality

$$\int_x^\infty \overline{G}(y)\,dy = (\mu_F/p)\left[\int_x^\infty \left\{\overline{F}(y)/\mu_F\right\} dy\right]^p, \tag{5}$$

it is easily verified that for all $x \geq 0$, $e_G(x) = e_F(x)/p$ and hence Equation (3) also holds for the two survivor functions $\overline{F}$ and $\overline{G}$. We now have the following characterization.

THEOREM 1. Given $p > 0$ and $F, G \in C_0$ $e_G(x) = e_F(x)/p$ for all $x \geq 0$ if and only if (4) holds. Furthermore, we can invert $\overline{F}$ from $\overline{G}$ using Equation (3).

Theorems 2 and 3 below are immediate from Theorem 1.

THEOREM 2. Given $p > 0$, $F, G \in C_0$ and they are related by (3) or (4), F has nonincreasing (nondecreasing) mean residual life if and only if G has nonincreasing (nondecreasing) mean residual life.

A life distribution F is new better (worse) than used in expectation if and only if $e_F(x) \leq (\geq) e_F(0)$.

THEOREM 3. Given $p > 0$, $F, G \in C_0$ and they are related by (3) or (4), F is new better (worse) than used in expectation if and only if G is new better (worse) than used in expectation.

Given $p > 0$, $F, G \in C_0$ and they are related by (3) or (4) ($\overline{F}$ and $\overline{G}$ have proportional mean residual life), it is natural to ask if G inherits other properties of F and vice versa. In §2, we identify some of these properties. In §3, we discuss our results.

2. OTHER CHARACTERIZATIONS

A life distribution F is said to have increasing (decreasing) failure rate if and only if

$$\overline{F}(x \mid t) = \overline{F}(x+t)/\overline{F}(t) \text{ is decreasing (increasing) in } t \geq 0 \text{ for each } x \geq 0. \tag{6}$$

When the density of F exists, (6) is equivalent to the failure rate $r(t) = F'(t)/\overline{F}(t)$ is increasing (decreasing) in $t \geq 0$. Using (1), inversion formula (2) and (4), it is easily verified that for all $p > 0$,

$$\overline{G}(x \mid t) = \overline{F}(x \mid t) \exp\left[-(p-1)\int_t^{x+t} \{e_F(u)\}^{-1}\, du\right]. \tag{7}$$

Note that F is an increasing (decreasing) failure rate distribution implies that e_F is a decreasing (increasing) function. We now have the following result.

THEOREM 4. If $p \geq 1$, $F \in C_0$ and it has increasing (decreasing) failure rate, then G defined in (4) also has increasing (decreasing) failure rate.

If the density of F exists, then G defined in (4) is differentiable and for all $x \geq 0$,

$$r_G(x) = r_F(x) + (p-1)/e_F(x).$$

Obviously, for $p \geq 1$, if r_F is increasing (decreasing) in $x \geq 0$, then e_F is a decreasing (increasing) function and r_G is therefore increasing (decreasing).

A life distribution F is said to be increasing (decreasing) failure rate average if and only if $-(1/x)\log \overline{F}(x)$ is increasing (decreasing) in $x \geq 0$. Put $t = 0$, take natural log on both sides of (7) and then multiply the equation by $-(1/x)$, we have for all $p > 0$,

$$-(1/x)\log \overline{G}(x) = -(1/x)\log \overline{F}(x) + \{(p-1)/x\} \int_0^x \{e_F(u)\}^{-1}\,du.$$

Let

$$\theta(t,x) = \int_t^{x+t} \{e_F(u)\}^{-1}\,du$$

and $\theta(x) = \theta(0,x)$. The following two theorems are now immediate.

THEOREM 5. If $p \geq 1$, $F \in C_0$, F has increasing (decreasing) failure rate average and $(1/x)\theta(x)$ is increasing (decreasing) in $x \geq 0$, then G defined in (4) also has increasing (decreasing) failure rate average.

THEOREM 6. Given $1 > p > 0$, $F, G \in C_0$ and they are related by (3) or (4), if F has increasing (decreasing) failure rate average and $(1/x)\theta(x)$ is decreasing (increasing) in $x \geq 0$, then G also has increasing (decreasing) failure rate average.

A life distribution F is new better (worse) than used if and only if $\overline{F}(x \mid t) \leq (\geq) \overline{F}(x)$ for all $x, t \geq 0$. From (7), we have the following characterizations.

THEOREM 7. If $p \geq 1$, $F \in C_0$, F is new better (worse) than used and $\theta(t,x) \geq (\leq) \theta(x)$ for all $x, t \geq 0$, then G defined in (4) is also new better (worse) than used.

THEOREM 8. Given $1 > p > 0$, $F, G \in C_0$ and they are related by (3) or (4), if F is new better (worse) than used and $\theta(t,x) \leq (\geq) \theta(x)$ for all $x, t \geq 0$, then G is also new better (worse) than used.

Note that $e_F(x)$ is decreasing (increasing) in $x \geq 0$ is a sufficient condition to ensure that $(1/x)\theta(x)$ is increasing (decreasing) in $x \geq 0$ and $\theta(t,x) \geq (\leq)\theta(x)$ for all $x, t \geq 0$ in Theorems 5 to 8.

From Ebrahimi and Habibullah (1990), a life distribution F is new better (worse) than used of age $t_0 \geq 0$ if and only if for all $x \geq 0$,

$$\overline{F}(x \mid t_0) \leq (\geq)\overline{F}(x). \tag{8}$$

From (7), it is obvious that if $p \geq 1$, $F \in C_0$, F is new better (worse) than used of age $t_0 \geq 0$ and $\theta(t_0, x) \geq (\leq)\theta(x)$ for all $x \geq 0$, then G defined in (4) is also new better (worse) than used of age t_0. Similarly, given $1 > p > 0$, $F, G \in C_0$ and they are related by (3) or (4), if F is new better (worse) than used of specified age $t_0 \geq 0$ and $\theta(t_0, x) \leq (\geq)\theta(x)$ for all $x \geq 0$, then G is also new better (worse) than used of specified age t_0.

For each $t_0 \geq 0$, let $C_1(t_0) = \{F : \overline{F}(x + t_0) = \overline{F}(x)\overline{F}(t_0) \text{ for all } x \geq 0\}$. $C_1(t_0)$ is therefore the boundary class of members of the new better than used of age t_0 obtained by insisting on equality in (8) above. Hollander, Park and Proschan (1986) showed that the following distributions are the only distributions in $C_1(t_0)$.

$$\overline{F}(x) = \sum_{j=0}^{\infty} \{\overline{H}_F(t_0)\}^j \,\overline{H}_F(x - jt_0) I_{[jt_0,(j+1)t_0)}(x)$$

where H_F is a distribution function defined for $x \geq 0$ and $I_{[jt_0,(j+1)t_0)}(x) = 1$ if $x \in [jt_0, (j+1)t_0)$ and 0 otherwise. Note that (i) if $\overline{H}_F(x) = \exp(-\lambda x)$, $x \geq 0$ and $\lambda > 0$, then

$\overline{F}(x) = \exp(-\lambda x)$, (ii) if $\overline{H}_F(t_0) = 0$, then $\overline{F}$ is a distribution function with $\overline{F}(t_0) = 0$. The following theorem can be verified easily.

THEOREM 9. Given $p > 0$, $F, G \in C_0$ and they are related by (3) or (4), $F \in C_1(t_0)$ if and only if $G \in C_1(t_0)$ with

$$\overline{H}_G(x) = \overline{H}_F(x)\left[\int_x^{t_0} \{\overline{H}_F(u)/\mu_F\}\,du + \overline{H}_F(t_0)\right]^{p-1}$$

for all $0 \leq x \leq t_0$ and

$$\mu_F = \int_0^{t_0} \overline{H}_F(u)\,du/\{1 - \overline{H}_F(t_0)\}.$$

A life distribution F is harmonic new better (worse) than used in expectation if and only if

$$\int_x^{\infty} \overline{F}(u)\,du \leq \mu_F \exp(-x/\mu_F)$$

for every $x \geq 0$. Using equality (5) and note that $\mu_G = \mu_F/p$, the following theorem is immediate.

THEOREM 10. Given $p > 0$, $F, G \in C_0$ and they are related by (3) or (4), F is harmonic new better (worse) than used in expectation if and only if G is harmonic new better (worse) than used in expectation.

Consider the family of distributions with survivor functions

$$\overline{F}(x) = \begin{cases} [B_F/(A_F x + B_F)]_+^{1/A_F+1} & \text{if } A_F \neq 0, A_F > -1 \\ \exp(-x/B_F) & \text{if } A_F = 0 \end{cases}$$

for all $x \geq 0$. From Hall and Wellner (1981) or Oakes and Dasu (1990), F belongs to the family (9) if and only if $e_F(x) = A_F x + B_F$ for all $x \geq 0$. Using Theorem 1, we now have the following characterization.

THEOREM 11. Given any $p > 1$, $F \in C_0$ and G as defined in (4), F is a member of the family (9) if and only if G belongs to the same family of distributions with $A_G = A_F/p$ and $B_G = B_F/p$.

Note that when $1 > p > 0$, if $F \in C_0$ and it is a member of the family (9) with $A_F/p \leq -1$, then $\overline{G}$ calculated from Equation (4) is no longer a distribution function. A necessary and sufficient conditions for Theorem 5 to hold for all $p > 0$ is that e_F is nondecreasing.

3. DISCUSSION

Given any positive integer valued p, Equation (4) can arise in the following situation. Suppose that there are p independent ordinary renewal processes in operation simultaneously, all with the same probability distribution $F(x)$ of failure time. From Cox and Smith (1954), it is well known that when the superposed process is in equilibrium, the survivor function of the interval between successive events in the superposed process is given by Equation (4) above. The superposed process described here can be used to model the following system. The system consists of p component positions in series, each containing a component. Each failed component is immediately replaced. All life lengths are mutually independent and identically distributed. Theorem 1 tells us that the survivor function of waiting times between failures of this series system and the survival function of failure times of components have proportional mean residual life. This means that we can estimate the mean residual life of a system with identical components connected in series by estimating the mean residual life of the individual component and vice versa. Estimation of e_F on the basis of samples from F were studied by Yang (1978).

In Theorems 2 to 11, we studied conditions such that different properties of F are preserved under superposition. Once we test that F has the required properties as stated in the theorem, we can conclude that the distribution function G defined in (4) also has the same properties. Statistical tests on monotone increasing failure rate, increasing failure rate average, new better than used, new better than used in expectation and decreasing mean residual life properties of life distributions have been investigated in the literature. Tests for monotone increasing failure rate were studied by Barlow and Proschan (1969), Bickel and Doksum (1969), Bickel (1969), Ahmad (1975), and Gerlach (1987). Deshpande (1983), Tiwari, Jammalamadaka and Zalkikar (1989), and Gerlach (1989) investigated tests for increasing failure rate average distributions. Hollander and Proschan (1972), and Chen, Hollander and Langberg (1983) proposed tests for new better than used for life distributions. New better than used in expectation and decreasing mean residual life tests were presented in Koul (1978) and Hollander and Proschan (1975) respectively. Statistical tests for survival distribution is new better than used of specified age were studied in Hollander, Park and

Proschan (1986), and Ebrahimi and Habibullah (1990). These test procedures can now be applied to either a random sample of F or G. The results in this paper are especially useful since the collection of a random sample from G may be difficult for the following two reasons. (i) The interevent times of the superposed process is not independent and (ii) in testing a relatively new system of components in series, each with large mean life, the duration of the test may not be great enough to allow the assumption that the system age is relatively large to the system mean life.

REFERENCES

AHMAD, J. A. (1975). A nonparametric test for monotonicity of a failure rate function. *Commun. Statist.* **4**, 967-74.

BARLOW, R. E. and PROSCHAN, F. (1969). A note on tests for monotone failure rate based on incomplete data. *Ann. Math. Statist.* **40**, 595-600.

BICKEL, P. J. and DOKSUM, K. A. (1969). Tests for monotone failure rate based on normalized spacings. *Ann. Math. Statist.* **40**, 1216-35.

BICKEL, P. J. (1969). Tests for monotone failure rate II. *Ann. Math. Statist.* 40, 1250-60.

CHEN, Y., HOLLANDER, M. and LANGBERG, N. A. (1983). Testing whether new is better than used with randomly censored data. *Ann. Statist.* **11**, 267-74.

COX, D. R. (1962). *Renewal Theory.* London: Chapman and Hall.

COX, D. R. and SMITH, W. L. (1954). On the superposition of renewal processes. *Biometrika* **41**, 91-9.

DESHPANDE, J. V. (1983). A class of tests for exponentiality against increasing failure rate average alternative. *Biometrika* **70**, 514-8.

EBRAHIMI, N. and HABIBULLAH, M. (1990). Testing whether the survival distribution is new better than used of specified age. *Biometrika* **77**, 212-5.

GERLACH, B. (1987). Testing exponentiality against increasing failure rate with randomly censored data. *Statistics* **18**, 275-86.

GERLACH, B. (1989). Tests for increasing failure rate average with randomly right censored data. *Statistics* **20**, 287-95.

HALL, W. J. and WELLNER, J. A. (1981). Mean residual life. In *Proceedings of the International Symposium on Statistics and Related Topics,* Ed. M. Csórgő, D. A. Dawson, J. N. K. Rao and A. K. Md. E. Saleh, pp169-84. Amsterdam: North Holland.

HOLLANDER, M,, PARK, D. H. and PROSCHAN, F. (1986). A class of life distributions for aging. *J. Am. Statist. Assoc.* **81**, 91-5.

HOLLANDER, M. and PROSCHAN, F. (1972). Testing whether new is better than used. *Ann. Math. Statist.* **43**, 1136-46.

HOLLANDER, M. and PROSCHAN, F. (1975). Tests for mean residual life. *Biometrika* **62**, 585-93.

KOUL, H. I. (1978). Testing for new is better than used in expectation. *Commun. Statist.* A **7**, 685-701.

OAKES, D. and DASU, T. (1990). A note on residual life. *Biometrika* **77**, 409-10.

TIWARI, R. C., JAMMALAMADAKA, S. R. and ZALKIKAR, J. N. (1989). Testing an increasing failure rate average distributions with censored data. *Statistics* **20**, 279-86.

YANG, G. L. (1978). Estimation of a biometric function. *Ann. Statist.* **6**, 112-6.