A NEW STANDARD IN PROCESS CAPABILITY ANALYSIS

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Abstract

In this paper, a new process capability index C_{pp} based on the process yield is presented. This new index and the k index are recommended here as the standard measures for process performance in place of the traditional C_{pk} and C_{pm} indices. This standard is consistent, easily interpreted, and can be directly extended to nonstandard tolerancing situations and multidimensional distributions. Examples on using C_{pp} and k to analyze process capability requirements for assemblies and hole location problems are discussed. Statistical testing procedures for both C_{pp} and k are derived. Tables on minimum sample size to achieve appropriate probability levels of Type I and Type II errors are also presented.

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Introduction

Integrated manufacturing systems require that accurate information be exchanged and analyzed freely between many different departments. It is this need for improved communication that motivated the development of process capability indices. According to Kane (1986), process capability indices are intended to provide a common, easily understood language for quantifying the performance of a process. Intuitively, the performance of a process should reflect how well the process produces parts which conform to some predetermined engineering design specification. In the standard one-dimensional case, bilateral design specifications denote the minimum and maximum acceptable values for the characteristic of interest and the nominal target value is usually centered between the specifications. This characteristic is generally assumed to be normally distributed and thus the process performance is a function of the process mean and standard deviation.

Process capability indices have become widely used as an attempt to convey the process performance in a dimensionless, easily interpreted manner. The C_p index reported in Kane (1986) has become a fixture in industry for measuring the potential capability of a process with perfect centering. Kane states that the benchmark of 1.0 was chosen to relate C_p to the standard six sigma spread used on control charts. Therefore, a capable process with $C_p = 1.0$ with an underlying stable normal distribution will result in 0.27 percent of parts beyond the specification limits assuming the process mean is centered at the target value. Other authors have recommended that an ongoing production process have a minimum value of $C_p = 1.33$ (e.g., Juran, Gryna, and Bingham [1979 p. 9-22]). Furthermore, it is not uncommon for industries to have corporate goals of $C_p = 1.67$ or higher for every process in their manufacturing system. This standardized interpretation has spread and many engineers and managers responsible for quality gauge their processes in comparison to this 1.33 benchmark.

One major use of process capability indices is enable decision makers with little statistical training to make informed statistical decisions regarding the purchase or removal of long term capital equipment. An informed decision regarding the capability of a process must then consider both the potential and actual performance of the process. While process potential is well summarized by C_p, there exists some debate on how to effectively measure the actual process performance. The two most common measures that have been introduced are C_{pk} and C_{pm}. These indices simultaneously account for both the mean and standard deviation of the process in an attempt to reflect the actual process performance. Unfortunately, neither of these two indices have direct physical meaning and this makes interpretation difficult.

The actual performance of a process is frequently judged on two separate criteria. One criterion is the ability of the process to minimize the difference between the process mean and the design target. This is known as process centering. The second criterion is the ability of the process to produce parts which conform to the design specifications. This is known as the process yield. It is well understood that C_{pk} is an approximate measure of the yield, but is a poor measure of process centering (Boyles [1991]).

C_{pm} was proposed independently by both Chan, Cheng, and Spiring (1988), and Boyles (1991). Boyles (1991) proposed use of C_{pm} because it is a better measure of process centering and it is also consistent with the Taguchi approach to quality improvement in which reduction of variation from the target value is the guiding principle. Chan, Cheng, and Spiring (1988) favored C_{pm} over C_{pk} because C_{pm} has a more tractable sampling distribution than does the C_{pk} index. From a statistical perspective, C_{pm} seems desirable because of the simplicity in its sampling distribution and its ability to summarize information on process centering and process yield into one index. However, engineers and managers who must interpret and make process control and improvement decisions based on C_{pm} are unlikely to easily understand what a given value of C_{pm} represents. In particular, examples will be given that illustrate a larger value of C_{pm} does not necessarily imply that the process mean is closer to target or the process yield is higher. This is not surprising since C_{pm} tries to summarize two distinct process performance criteria into one measure.

We approach the problem from an engineering perspective and believe that the most important measure of a process capability index is its ability to convey process information accurately and consistently. An easily interpreted reliable index will allow engineers to make decisions which promote continuous improvement. It is therefore unwise in our opinion to combine process centering and process yield information into one index such as C_{pk} or C_{pm}. For some processes, conformance to specification limits or process yield may be the most important factor in order to ensure product functionality. For other processes, small deviation from the nominal target may be essential. By combining these two distinct criteria together into one index, engineers are unable to make rational comparisons or determine appropriate tradeoffs. Therefore, our recommendation is that process centering and process yield be separately reported as unitless process capability indices.

In this paper, we define a new process capability index C_{pp}. This new index is uniquely defined based on the actual process yield. The definition of C_{pp} is inspired by noting that C_p is a measure of the potential process yield if the process is centered at the target value. Since C_p and its interpretation as potential yield are widely accepted and understood in industry, it is natural to define the process performance capability, C_{pp}, as a measure of the actual process yield. It should be noted that the calculation of C_{pp} requires a computer and unlike C_{pm}, the sampling distribution of C_{pp} does not have a simple functional form. However, the sampling distribution of C_{pp} can be computed rapidly and accurately using numerical methods and existing computer technology. So while use of C_{pp} in place of C_{pk} or C_{pm} will require increased computational efforts, the result will be a more consistent and easily interpreted index with direct physical meaning.

In order to measure the process centering criterion, we propose that the k index defined in Kane (1986) be used. The k index is a unitless linear measure describing the amount the process mean is off-center or off-target and is therefore an appropriate measure of process centering. Under the assumption that the underlying process can be modeled by a normal distribution, we will show later that once C_p and C_{pp} are given, the k index is also uniquely defined. We propose that in practice three indices should be reported in order to consistently, effectively, and completely communicate the ability of a process to meet design specifications. These three indices are C_p (a measure of the potential process yield), C_{pp} (a measure of the actual process yield), and k (a measure of the process centering).

In the following sections, we define the C_{pp} index and compare this new index with C_{pk} and C_{pm}. Examples on using these new indices to analyze process capability requirements for assemblies and hole location problems are discussed. Statistical testing procedures of the new C_{pp} index and the k index are presented. Finally, we summarize the results in the paper and discuss future extensions.

Process Performance based on Proportion of Yield

An analysis of the potential capability of a process must answer how well a process might conform to its specifications under ideal centering and control conditions. C_p fulfills this need with a consistent and easily understood unitless index. However, an analysis of actual performance of a process must answer both how well that process is conforming to its specifications, and how well the process is centered with respect to the target value. In this section, we motivate the development of a new index by demonstrating the deficiencies inherent in C_{pk} and C_{pm} as measures of process centering and process yield. A new process capability index C_{pp} is then derived as a measure of the actual process yield. The k index defined in Kane (1986) is suggested here as a measure of process centering.

Let L and U be the lower and upper specification limits on the characteristic X of each item. Assume that X is normally distributed with mean μ and standard deviation σ. μ and σ are assumed to be unknown. Let p be the proportion of yield and $T = (U - L)/2$. We define C_p, CPU, CPL and C_{pk} indices as given in Kane (1986).

$$C_p = \frac{U - L}{6\sigma}, \tag{1}$$

$$CPU = \frac{U - \mu}{3\sigma},$$

$$CPL = \frac{\mu - L}{3\sigma},$$

and

$$C_{pk} = \min(CPL, CPU) = C_p(1 - k)$$

where

$$k = \frac{|T - \mu|}{(U - L)/2}. \tag{2}$$

Note that k is an index describing the amount the process mean is off-center standardized by the width of the design specifications. When X is normally distributed, the proportion of yield can be written as

$$p = \Phi(3CPU) - \Phi(-3CPL) = \Phi[3C_p(1 - k)] - \Phi[-3C_p(1 + k)] \tag{3}$$

where $\Phi(x)$ is the integral of a standardized normal density from minus infinity to any real number x.

Boyles (1991) notes that C_{pk} is an unsuitable measure of process centering since a large value of C_{pk} does not in itself say anything about the distance between μ and T. He shows that given C_{pk}, we can compute an upper and a lower bound for the actual process yield. The upper and lower bounds are very tight for most practical engineering purposes especially when C_{pk} is greater than 1. However, use of C_{pk} to compare the process performance between two or more processes can lead to counterintuitive results. Consider the decision rule stating that when C_p or C_{pk} falls below some critical value c_0 the process should be stopped and corrective actions should be taken. This decision rule is quite natural and is indeed used in industry to control the output quality from processes. It would seem that use of this rule would guarantee some minimum level of process performance. However, as illustrated by the examples given in Table 1, because of the fact that C_{pk} is not a consistent measure of the actual process yield, this rule can lead to bad decisions.

INSERT TABLE 1 HERE

Assume that a critical value of $c_0 = 0.60$ has been specified. Admittedly, this is much lower than most situations would dictate, but it serves as an appropriate example here. A control decision on processes A, B, and C would find that all three have C_p values greater than or equal to the critical value of 0.60. However, process A has a C_{pk} value which is well below the critical value and thus process A is stopped and investigated. The irony here is that process A has the highest actual process yield and yet it is the only one of the three which was stopped for correction. Obviously, C_{pk} cannot be used as an independent measure of process performance for these types of decisions. Process D is included to demonstrate that C_{pk} is also not a measure of process centering since process B and D have identical C_{pk} values but substantially different departure from target.

C_{pm} was proposed by Chan, Cheng, and Spiring (1988) as an alternative measure of process performance largely because it has a more tractable sampling distribution than C_{pk}. The definition for C_{pm} is given by

$$C_{pm} = \frac{U - L}{6\sqrt{\sigma^2 + (\mu - T)^2}}.$$

Boyles promotes the use of C_{pm} because it is a more accurate measure of process centering than C_{pk}. He also points out that the definition of the C_{pm} index fits nicely with the loss function approach championed by Hsiang and Taguchi (1985), and Taguchi (1986) for quality improvement in which reduction of variation from the target value is the guiding principle. However as shown below, C_{pm} is an extremely inconsistent measure of process yield and therefore it is a misleading indicator of actual process performance. Boyles presents a graph displaying the upper and lower bounds on process yield as a function of C_{pk}. For the sake of comparison and to display the difficulty in interpreting individual values of C_{pm} as measures of process performance, we derive here the upper and lower bounds on process yield for C_{pm}. First note that the C_{pm} index can be written in terms of C_p and k as in Equation (4). Using this equation, we can express the k index as a function of C_p and C_{pm} as in Equation (5) below.

$$C_{pm} = \left(\frac{1}{C_p^2} + 9k^2\right)^{-1/2} \tag{4}$$

and hence

$$k = \frac{1}{3C_{pm}}\left[1 - \left(\frac{C_{pm}}{C_p}\right)^2\right]^{1/2}. \tag{5}$$

From Equations (4) and (5), the following results are obvious.

$$\begin{aligned} 0 < \; & C_{pm} \leq C_p, \\ 0 \leq \; & k \leq \frac{1}{3C_{pm}}, \end{aligned} \tag{6}$$

and

$$\lim_{C_p \to \infty} k = \frac{1}{3C_{pm}}.$$

Equation (6) is the same as Equation (18) in Boyles (1991). It means that given C_{pm}, the process mean is guaranteed to lie within the middle $3C_{pm}$th of the specification range. For example if C_{pm} is equal to 1, then the process mean lies within the middle third of the specification range. Boyles (1991) suggests that such statements provide a concrete interpretation of C_{pm} as a measure of process centering. However, the k index is superior here since it is an exact measure of the deviation of the process mean from nominal. From Equation (4), for fixed C_{pm} and k such that $0 < C_{pm} < 1/3$ and $0 \leq k \leq 1$,

$$C_p = \left(\frac{1}{C_{pm}^2} - 9k^2\right)^{-1/2} \leq \left(\frac{1}{C_{pm}^2} - 9\right)^{-1/2}.$$

From Equation (3),

$$\Phi[3C_p(1-k)] + \Phi(3C_p) - 1 \;\leq\; p \;\leq\; \min\{\Phi[3C_p(1-k)], 2\Phi(3C_p) - 1\}.$$

Hence the bounds on the process yield as a function of C_{pm} are given by

$$\max[\Phi(3C_{pm} - 1) + \Phi(3C_{pm}) - 1, 0] \;\leq\; p \;\leq\; \begin{cases} 2\Phi\left[3\left(\frac{1}{C_{pm}^2} - 9\right)^{-1/2}\right] - 1 & \text{if } C_{pm} < 1/3 \\ 1 & \text{if } C_{pm} \geq 1/3 \end{cases}.$$

These bounds are plotted in Figure 1 as a function of C_{pm}. Some examples from Figure 1 are presented in Table 2 to demonstrate that C_{pm} is an inconsistent measure of process yield.

INSERT FIGURE 1 HERE

INSERT TABLE 2 HERE

From Table 2, it is clear that larger values of C_{pm} do not imply that the process yield is higher. In fact, for certain values of C_{pm} (for example, $C_{pm} = 0.3$), the process yield can be drastically different. While it is apparent that C_{pm} is a poor measure of process yield, it is also true that C_{pm} is an inconsistent measure of process centering as well. The examples presented in Table 3 highlight the fact that increasing values of C_{pm} do not necessarily indicate improved process centering.

INSERT TABLE 3 HERE

C_{pm} and C_{pk} fail as measures of process performance for two important reasons. First of all, they have no direct physical meaning which makes interpretation a very subjective process. Secondly, their inconsistency results from the fact that they are both trying to capture two distinct parameters of process performance (centering and yield) into one measure. Both of these measures therefore fail to accomplish Kane's goal of providing a common, easily understood language for quantifying the performance of a process.

Cpp and *k* Indices

The solution to this problem is to simply separate the description of process centering from the description of process yield. An easily interpreted index of process centering is k as introduced by Kane (1986) which is also given in Equation (2) above. This measure is simply a ratio which can be interpreted as the proportion of the tolerance zone consumed by the deviation of the process mean from target T. Thus a value of $k = 0$ indicates perfect centering while $k = 0.50$ indicates that the mean off-target and is equidistant from the target and the specification limit. This value is instantly recognizable and provides a nice graphical picture in the mind as to exactly where the process is actually centered.

To develop an easily interpreted index of actual process yield which is consistent with the widely used and understood C_p index, we propose the following. First, we redefine the C_p index as follows. If p' is the potential yield of the process,

$$C_p = \frac{1}{3}\Phi^{-1}\left(\frac{p'+1}{2}\right). \tag{7}$$

Note that when the underlying process distribution is univariate normal, this definition is consistent with the definition given in Equation (1) above. Furthermore, this new definition of C_p is extremely general since it is independent of the underlying distribution of the process. The distribution can be multidimensional and even non-normal. This definition of the C_p index also has natural extensions to unilateral, circular, or other higher dimensional tolerances. Therefore, this new definition allows the expression of process capability in a consistent easily understood manner for a much wider variety of process situations than was previously available.

Based on this generalization of the C_p index, it is natural to define a new capability index C_{pp} based on the actual yield in order to measure the process performance capability. If p is defined as the actual proportion of yield for the process, then we define C_{pp} as

$$C_{pp} = \frac{1}{3}\Phi^{-1}\left(\frac{p+1}{2}\right). \tag{8}$$

Note that $C_{pp} = C_p$ only when the process is centered at the target value T. By definition, $C_{pp} \leq C_p$ and if $C_{pp} < C_p$ then the process is known to be off-center. In the case when the underlying distribution for the process is normal, C_{pp} can be written as

$$C_{pp} = \frac{1}{3}\Phi^{-1}\left\{\frac{\Phi[3C_p(1-k)] + \Phi[3C_p(1+k)]}{2}\right\}. \tag{9}$$

Given C_p and C_{pp}, the k index is therefore uniquely defined by Equation (9). Given C_{pp} the proportion of actual yield is always given by

$$p = 2\Phi(3C_{pp}) - 1.$$

This function is plotted in Figure 2. This one to one mapping between the process yield and C_{pp} indicates that for any value of $C_{pp} = c$ the process is producing $100 \times \{2[1 - \Phi(3c)]\}\%$ of nonconforming parts independent of the underlying distribution of the process. In particular, a C_{pp} of 1.0 means that the process is producing 0.27% of parts outside of the specification limits.

Since the C_p index is the most widely used and understood measure of yield in industry today, this mapping will be easily interpreted by engineers. It is this property that makes the C_{pp} index more consistent, interpretable, and extendible than other indices for measuring process performance capability.

INSERT FIGURE 2 HERE

Just like the C_{pk} index, the C_{pp} index defined here depends on both μ and σ. The contour plot given in Figure 3 illustrates the nature of this dependency. Figure 3 can be compared to the contour plot of C_{pk} and C_{pm} given by Boyles (1991). The five contours representing C_{pp} values of 1/3, 2/3, 1, 4/3 and 5/3 are chosen here for direct comparisons with the results given in Boyle (1991). Similar to the contour plot of C_{pk}, for any fixed σ, C_{pp} attains its maximum value when $\mu = T$, and decreases as μ approaches either U or L.

INSERT FIGURE 3 HERE

It is also obvious from Figure 3 that C_{pp} is definitely not a measure of process centering. Similar examples as given in Figure 2 of Boyles (1991) can be constructed to show that while C_{pp} remains constant, the process mean can be shifted arbitrarily close to the upper or lower specification limit by varying the standard deviation appropriately. Unlike C_{pm}, a given value of C_{pp} does not ensure that the process mean will be within a certain middle range of the specification limit. This is why we encourage the use of k as the measure of process centering since k precisely pinpoints the location of the mean within the specification limit.

Examples

Component Capability Requirements for Assemblies

Boyles (1991) demonstrates the importance of process centering with an airfoil assembly example. In this subsection, we consider the same example and show how we can use the C_p, C_{pp} and k indices to ensure the final assembly meets certain requirements.

Let X denote an airfoil dimensional characteristic, n of which are used in the manufacture of an assembly. Recall from Boyles (1991) that $Y = f(X_1, \ldots, X_n)$ where $X_1, \ldots, X_n$ represent the X characteristics of n airfoils produced from a common process and f is a symmetric function of its argument because of the symmetric airfoil configuration. Let U_x and L_x be the upper and lower specification limits for the individual airfoils. Similarly, let U_y and L_y be the upper and lower specification limits of the final assembly. Using a first order Taylor approximation applied to $f(X_1, \ldots, X_n)$, Boyles (1991) shows that

$$\mu_y \approx T_y + n\beta(\mu_x - T_x) \tag{10}$$

and

$$\sigma_y \approx \sqrt{n}\beta\sigma_x \tag{11}$$

where μ_x and μ_y are the process mean and σ_x and σ_y are the process standard deviation of X and Y respectively. $T_x = (U_x + L_x)/2$, $T_y = (U_y + L_y)/2$ and $\beta = \partial f / \partial x_i(T_x, \ldots, T_x) > 0$. For the remainder of this subsection, we assume that Equations (10) and (11) are exact.

Let C_p^x, C_{pp}^x and k_x represent capability indices for the airfoil manufacturing process and C_p^y, C_{pp}^y and k_y represent capability indices for the final assembly. From Equation (11), since the process standard deviation of the final assembly process is a constant multiple of the standard deviation of the airfoil process it is reasonable for us to set

$$U_y - L_y = \sqrt{n}\beta\,(U_x - L_x). \tag{12}$$

In this case $C_p^x = C_p^y$. From Equations (10), (11) and (12), it is clear that

$$k_y = \sqrt{n}k_x. \tag{13}$$

Suppose $k_{\max}$ is the maximum allowable deviation from the target for the final assembly, i.e., $k_y \leq k_{\max}$. Also, let c_p and c_{pp} be the minimum acceptable values for the potential and actual capability. From (13), we can conclude that provided $k_x \leq k_{\max}/\sqrt{n}$ for the airfoil component process, the final assembly process will meet the required centering criterion. Furthermore under the assumption of normality,

$$\begin{aligned} C_{pp}^y &= \frac{1}{3}\Phi^{-1}\left\{\frac{\Phi[3C_p^y(1-k_y)] + \Phi[3C_p^y(1+k_y)]}{2}\right\} \\ &\geq \frac{1}{3}\Phi^{-1}\left\{\frac{\Phi[3c_p(1-k_{\max})] + \Phi[3c_p(1+k_{\max})]}{2}\right\}. \end{aligned}$$

To ensure that $C_{pp}^{y} \geq c_{pp}$, we solve for $k_{\max}$ in

$$\Phi^{-1}\left\{\frac{\Phi[3c_p(1-k_{\max})]+\Phi[3c_p(1+k_{\max})]}{2}\right\} = 3c_{pp}$$

with an iterative root finding method. Hence, provided that $C_p^x \geq c_p$ and $k_x \leq k_{\max}/\sqrt{n}$, we know that $C_p^y \geq c_p$, $C_{pp}^y \geq c_{pp}$ and $k_y \leq k_{\max}$ where c_p, c_{pp} and $k_{\max}$ are prespecified criteria for a capable final assembly process.

Hole Location Problems

The improvements in consistency and interpretation are strong reasons to adopt C_{pp} and k as the standard measure of process performance. Adding strength to this argument is the direct extendibility of C_{pp} to measure many unique tolerancing situations which have yet to be well defined by process capability indices. One of the more common of these situations is the case of a bivariate location characteristic in a plane. While this issue has received relatively little attention in the literature, it has been a persistent problem in industry since bivariate characteristics are common and management requires process performance to be measured in terms of process capability indices. Typically, process capability indices for bivariate characteristics have been calculated in practice by compressing the two-dimensional data into one-dimension and applying the traditional formulas to this transformed data. This transformation is an inherent loss of information and frequently results in misleading representations of the true bivariate capability. In this subsection we discuss some examples highlighting the need for a consistent two-dimensional process capability measure and show how C_p, C_{pp}, and k are extended to achieve this.

Consider a manufacturing process which is responsible for drilling a hole in a single plane. A nominal location (x_0, y_0) for the center of the hole will generally be specified where x_0 and y_0 are measured from some fixed reference point on the plane. The specification region for these characteristics is frequently in the form of a circular or elliptical region.

There are two common methods in use to compress the two-dimensional data down to one-dimension so that the traditional capability formulas can be applied. *True position deviation* measures the absolute deviation from the nominal location. Thus, the true position deviation for a part is defined as

$$d = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

where (x, y) is the hole location relative to same fixed reference point on the plane. In practice, the mean and standard deviation of d are then used to compute C_p and C_{pk}. One immediate drawback with this method is that d is strictly nonnegative and hence is not likely to be modeled well by a normal distribution. Hence, many of the well known statistical results to estimate C_p based on independent and identically distributed normal samples no longer apply.

A second popular method, *part center radial deviation*, addresses these issues by measuring the radial deviation from a reference point on the plane outside of the specification zone. Generally, this reference point is the part center or a locating pin position on the part. If this reference point is defined to be (x_0', y_0') then the part center radial deviation is given by

$$d' = \sqrt{(x - x_0')^2 + (y - y_0')^2}.$$

This data compression method has an advantage over the true position deviation method since d' is usually relatively much larger than 0 and most of the time, d' can be approximated fairly well by a univariate normal distribution with some large positive mean. However as illustrated in the examples below, both of these methods can produce extremely inconsistent and inappropriate results. In the examples, we assume that the hole location process (X, Y) relative to the fixed reference point on the plane follows the bivariate normal distribution with process mean $\underset{\sim}{\mu} = (\mu_x, \mu_y)^t$ and variance covariance matrix Σ. Furthermore, the specification is circular with radius r.

Consider two location hole processes (X_A, Y_A) and (X_B, Y_B). Assume that (X_A, Y_A) is bivariate normal with

$$\underset{\sim}{\mu}_A = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad \text{and} \qquad \Sigma_A = \begin{bmatrix} \frac{r^2c^2}{9} & 0 \\ 0 & \frac{r^2c^2}{9} \end{bmatrix}$$

where c is a positive constant. Also, assume that (X_B, Y_B) is bivariate normal with

$$\underset{\sim}{\mu}_B = \begin{bmatrix} x_0 - \frac{r}{2} \\ y_0 \end{bmatrix} \qquad \text{and} \qquad \Sigma_B = \begin{bmatrix} \frac{r^2}{36} & 0 \\ 0 & \delta^2 \end{bmatrix}$$

where δ is a positive constant. The contour plots for the density function of process A and B are given in Figure 4. Let

$$D_A = \sqrt{(X_A - x_0)^2 + (Y_A - y_0)^2}$$

and

$$D_B = \sqrt{(X_B - x_0)^2 + (Y_B - y_0)^2}.$$

It can be verified easily that

$$\text{VAR}(D_A) = \frac{r^2c^2}{9}\left(2 - \frac{\pi}{2}\right).$$

For process B, it is clear that $\text{VAR}(D_B)$ is approximately equal to $\text{VAR}(X_B) = (r/6)^2$ when δ is small. If $c = 0.75$ and $r = 3$, then $\text{VAR}(D_B)$ is approximately equal to 0.25 and $\text{VAR}(D_A)$ is equal to $0.24143 < 0.25$. This means that provided that δ is small enough, the C_p of process A is strictly greater than the C_p of process B. Thus the true position method produces results which claim that the potential capability of process A is greater than the potential capability of process B. However, examination of Figure 4 indicates that process B has a far better potential for producing conforming parts.

INSERT FIGURE 4 HERE

Similarly, we can construct examples to show that comparing processes by using a C_p computed based on the part center radial deviation can be misleading. Assume $x_0' = 0$ in the following example and let R be the distance between the reference point $(0, y_0')$ and the center of the circular tolerance. Let us compare processes (X_C, Y_C) and (X_D, Y_D). Process (X_C, Y_C) is assumed to follow a bivariate normal with

$$\underset{\sim}{\mu}_C = \begin{bmatrix} 0 \\ y_0' + R \end{bmatrix} \quad \text{and} \quad \Sigma_C = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix},$$

and process (X_D, Y_D) is bivariate normal with

$$\underset{\sim}{\mu}_D = \begin{bmatrix} 0 \\ y_0' + R \end{bmatrix} \quad \text{and} \quad \Sigma_D = \begin{bmatrix} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{bmatrix}.$$

where σ_1 is much larger than σ_2. An example of this situation is illustrated in Figure 5. The part center radial deviation method defines

$$D_C = \sqrt{X_C^2 + (Y_C - y_0')^2}$$

and

$$D_D = \sqrt{X_D^2 + (Y_D - y_0')^2}.$$

Since the potential capability for processes C and D is directly proportional to the inverse of the standard deviation of D_C and D_D, it is obvious from Figure 5 that the C_p for process C is much larger than the C_p for process D. However, the variance covariance matrices of the two process are identical except for a 90 degree rotation of the principal axis. From a manufacturing standpoint process C and process D are equivalent, yet the part center radial deviation method reports that process C has a much greater potential capability than process D. From these examples, it should be clear that it is inappropriate to calculate process capability for two dimensional characteristics using either of these methods.

INSERT FIGURE 5 HERE

The solution to this problem is a direct extension from the definition of C_p and C_{pp} in Equations (7) and (8) respectively. Note that p' and p now represent the potential and actual yield of a bivariate normal density over a circular specification zone. Furthermore, the multidimensional indices have exactly the same meaning and interpretation as the one-dimensional C_p and C_{pp}. However, it should be noted that it is likely to be more difficult to achieve the standard goals of 1.0 or 1.33 in the multidimensional case. The equivalent of the process centering index k in two-dimensions can be easily obtained by recognizing that Equation (2) is simply a ratio of the distance the process mean is off-center with respect to the length of the tolerance zone. For the case of a bivariate characteristic and a circular specification zone of radius r, we can define

$$k = \frac{\sqrt{(\mu_x - x_0)^2 + (\mu_y - y_0)^2}}{r}.$$

We present a capability analysis of the four processes here to demonstrate the effectiveness of the proposed standard in conveying relevant process information. The C_p, C_{pp}, and k indices have been computed and the results appear in Table 4. Note that the index values nicely confirm the results of a visual inspection of Figures 4 and 5. In a comparison of processes A and B for example, one notes that process B is potentially superior to process A since C_p for process B is greater than C_p for process A. However, process A is actually producing a greater proportion of parts which conform to the tolerance (C_{pp} for process A is greater than C_{pp} for process B) due to the fact that the mean for process B is off-center by 50% of the total tolerance limit (k index for process B is 0.50). Similarly, a comparison of the capability values for processes C and D correctly indicates that the processes are equivalent.

INSERT TABLE 4 HERE

The computation of p' and p requires integration of the bivariate elliptical normal density function over an offset circle. Groenewoud, Hoaglin, and Vitalis (1967) published tables of the bivariate normal probabilities over offset circles using a recursion formula which resulted in an extremely rapid iterative technique. They document the necessary formulas for encoding this computational algorithm. This fast and accurate computational technique is recommended for determining the potential and actual yield of the process over the circular specification zone.

Statistical Testing Procedures for C_{pp} index

Consider a random sample of n observations $X_1, \cdots, X_n$ from a normal population, let $\bar{X}$ and S be the sample mean and sample standard deviation respectively. Let $\hat{C}_{pp}$ be an estimator for C_{pp} using $\bar{X}$ and S to estimate the population mean μ and population variance σ respectively. Given any $\alpha \in [0,1]$, C_p and C_{pp}, define $c_\alpha(C_p, C_{pp})$ to be the $100(1-\alpha)$th percentile of the distribution of $\hat{C}_{pp}$. That is,

$$\mathcal{P}[\hat{C}_{pp} > c_\alpha(C_p, C_{pp}) \mid C_p, C_{pp}] = \alpha. \tag{14}$$

Suppose we want to test for process capability $C_{pp} > c_0 > 0$,

$$H_0 : C_{pp} \le c_0 \text{ (process is not capable)}$$
$$H_1 : C_{pp} > c_0 \text{ (process is capable).}$$

The critical region of this test is of the form $\hat{C}_{pp} > c_\alpha(c_0)$ where $c_\alpha(c_0)$ is the critical value of the test. α is the probability of Type I error. The critical value is a function of c_0 and α. By choosing

$$c_\alpha(c_0) = \max_{C_p \in [c_0, \infty)} c_\alpha(C_p, c_0), \tag{15}$$

the probability of Type I error when $C_{pp} = c_0$ is no greater than α. Given any c_0, α and sample size n, $c_\alpha(c_0)$ can be computed numerically. The detailed procedures are given in the appendix. The critical values for different sample size n and c_0 are tabulated in Tables 5 and 6 below for $\alpha = 0.01$ and 0.05 respectively. For example, when $c_0 = 1.67$, $n = 100$ and $\alpha = 0.05$, $\hat{C}_{pp}$ computed from a sample must be at least 1.886 in order to conclude that the process is capable with probability of Type I error at most 0.05. Type I error means that we conclude that process is capable when in fact $C_{pp} \leq c_0$ (not capable). c_0 here is also referred as the rejectable quality level or RQL (see Kane [1986]), i.e., it is a sufficiently low process capability such that we would like to reject processes with capabilities below the RQL. Let c_1 represent an acceptable quality level (AQL), i.e., it is a sufficiently high process capability such that we would like to accept processes with capabilities above the AQL.

INSERT TABLE 5 HERE

INSERT TABLE 6 HERE

Operating Characteristic Curve and Sample Size Consideration

Let $\beta(C_{pp})$ represent the operating characteristic curve (OCC) defined as the probability that the null hypothesis (the process is not capable) is accepted given $C_p = C_{pp}$ values. That is,

$$\beta(C_{pp}) = \mathcal{P}\left[\hat{C}_{pp} \leq c_\alpha(c_0) \mid C_p = C_{pp}\right].$$

Note that the probability of Type II error should be a function of both C_p and C_{pp}. In practice, by choosing $C_p = C_{pp}$, we have found that this gives an upper bound to the probability of Type II error. The function $\beta(C_{pp})$ is plotted in Figures 6 and 7 for $c_0 = 1$, $\alpha = 0.01, 0.05$, and $n = 10, 25, 75, 150$.

INSERT FIGURE 6 HERE

INSERT FIGURE 7 HERE

Given fixed $\alpha = \beta(c_1) = \beta$, the required sample size to ensure that $\alpha = \beta = 0.01$ and 0.05 are given in Tables 7 and 8 respectively for different combinations of c_0 and c_1. For example, if $c_0 = 1.33$, $c_1 = 1.67$ and $\alpha = \beta = 0.05$, then from Table 8, the required sample size is $n = 113$. From Table 6, the critical value $c_\alpha(c_0)$ of the test lies between 1.491 and 1.508 corresponding to sample size $n = 100$ and $n = 120$ respectively. The conservative choice is to pick $c_\alpha(c_0) = 1.508$.

INSERT TABLE 7 HERE

INSERT TABLE 8 HERE

Statistical Testing Procedures for k Index

Suppose we want to test for process centering $k < k_{\max}$,

$$H_0 : k \geq k_{\max} \text{ (process is off-centered)}$$
$$H_1 : k < k_{\max} \text{ (process is centered)}$$

where $k_{\max}$ is the maximum allowable derivation from the target. Consider a critical region of the form $3\hat{C}_p\hat{k}\sqrt{n} < k_0$ where k_0 is the critical value of the test, and $\hat{k}$ is the estimator for k using $\bar{X}$ to estimate the process mean μ. The probability of making Type I error when $k = k_{\max}$ and Type II error when $k = 0$ are given by

$$\alpha = \mathcal{P}\left(3\hat{C}_p\hat{k}\sqrt{n} < k_0 \mid k = k_{\max}\right), \tag{16}$$

and

$$\beta = \mathcal{P}\left(3\hat{C}_p\hat{k}\sqrt{n} \geq k_0 \mid k = 0\right). \tag{17}$$

Note that when $k = 0$, $3\hat{C}_p\hat{k}\sqrt{n} \geq k_0$ if and only if $\mid \bar{X} - \mu \mid /(S/\sqrt{n}) \geq k_0$. Hence, Equation (17) holds whenever $k_0 = t_{\beta/2,n-1}$ where $t_{\beta/2,n-1}$ represents the $(1-\beta/2)$th quantile of the t distribution with $n-1$ degrees of freedom. Also, note that Equation (16) can be rewritten as

$$\alpha = \mathcal{P}\left(\frac{\mid \bar{X} - T \mid}{S/\sqrt{n}} < t_{\beta/2,n-1} \middle| 3C_p k = 3C_p k_{\max}\right).$$

This is equal to the probability of making Type II error of the well studied two-sided t test of a normal population mean based on one sample at a significance level β. This is because the usual two-sided t test is defined as testing $H_0 : \mu = T$ against the alternative $H_1 : \mu = T + (U - L)k_{\max}/2$ at a significance level β. The required sample size to achieve various levels of probability of making Type I and Type II errors are well known (for example, see Milton and Arnold (1990)) and is tabulated in Table 9 here for easy reference.

In the test above, Type I error means that we conclude the process mean is acceptably close to the target when in fact $k \geq k_{\max}$. Again, we can refer to $k_{\max}$ as the RQL, the point where the process is unacceptably off-center and we would like to conclude that the process is not capable. When $k = 0$, the process is perfectly centered, in this case our test above ensures the probability of concluding the process is off-center is equal to β. For example, if $\alpha = \beta = 0.05$, $k_{\max} = 1/3$ and $C_p = 1$, from Table 9, the required sample size is 16 and the critical value of the test is given by $k_0 = t_{0.025,15} = 2.131$. Hence, provided that $3\hat{C}_p\hat{k} < 0.53275$, we conclude that the process is centered and the significance level of the test is $\alpha = 0.05$. Note that in using Table 9, we assume that C_p is known. In practice when C_p is unknown, a conservative (small) guess for C_p will yield a conservative (large) estimate of the sample size necessary for prescribed α and β.

INSERT TABLE 9 HERE

Note that in this paper, we have only presented statistical testing procedures for C_{pp} and k indices. Similar statistical analysis for C_p can be found in Kane (1986).

Summary and Discussion

Process capability indices are widely used in industry in an attempt to describe relevant process information for decision makers in a unitless easily interpreted index. We have demonstrated the potential inconsistencies in using C_{pk} and C_{pm} as measures of process performance. A new capability index C_{pp} based on process yield has been defined as a solution to this problem. We have shown that C_p, C_{pp} and k are reliable and easily understood indices for summarizing process information. Use of these indices to quantify the relative process capability of two important industrial problems has been presented.

It is important to note that C_{pp} is a function of the actual yield which is defined here based on zero-one yield function meaning that all characteristics that lie within the specification range are considered to be of equal quality. Several authors have pointed out that focusing only on conformance to specifications may actually discourage efforts aimed at continuous process improvement. Recognizing that a zero-one yield function may not be appropriate for all situations, an index based on more general yield functions that is consistent with the interpretation of C_{pp} is currently under development.

Further extensions of this research will expand the scope of the tolerancing situations which can be described by this proposed standard. We intended to generalize the C_{pp} and k indices reported here to cases in which the target T is not located in the middle of specifications, and to characteristics which are toleranced with a one-sided specification limit. In this paper we have presented statistical testing procedures for C_{pp} based on a univariate normal characteristic. Similar statistical procedures are being investigated at the moment for bivariate normal characteristics.

Appendix

Given any positive constant c, define

$$p = \Phi(3c) - \Phi(-3c) = 2\Phi(3c) - 1,$$

and

$$\hat{p} = \Phi(3\hat{C}_{pp}) - \Phi(-3\hat{C}_{pp}) = 2\Phi(3\hat{C}_{pp}) - 1.$$

Let $K : (0,\infty) \times [0,1] \to (-\infty,\infty)$ be a function of two variables which satisfies

$$\Phi[K(a,b)] - \Phi[K(a,b) - 6a] = b. \tag{18}$$

Since the normal distribution is symmetric, for each fixed pair of (a,b), Equation (18) can have two solutions $K_1(a,b)$ and $K_2(a,b)$. If $K_1(a,b) \leq K_2(a,b)$, then

$$K_2(a,b) = -K_1(a,b) + 6a.$$

Note that $\Phi^{-1}(b) \leq K_1(a,b) \leq 3a$. When a is large and b is close to 1, it is more accurate numerically to solve for $K_1(a,b)$ than to solve for $K_2(a,b)$ since the normal tails decay exponentially and $\Phi(x)$ is very flat when x is large. Hence, we pick $K(a,b) = K_1(a,b)$, the smaller of the two solutions.

Define $Y = (n-1)S^2/\sigma^2$, and $Z = \sqrt{n}(\bar{X} - \mu)/\sigma$. It can be verified easily that

$$\hat{C}_p = \frac{C_p}{\sqrt{Y/(n-1)}},$$

and

$$3\hat{CPU} = \frac{3CPU - Z/\sqrt{n}}{\sqrt{Y/(n-1)}}.$$

Using the notations defined above, given any positive constant c, we have

$$\begin{aligned}
& \mathcal{P}(\hat{C}_{pp} > c \mid C_p, C_{pp}) \\
= \; & \mathcal{P}(\hat{p} > p \mid C_p, C_{pp}) \\
= \; & \mathcal{P}[\Phi(3\hat{CPU}) - \Phi(-3\hat{CPL}) > p \mid C_p, C_{pp}] \\
= \; & \mathcal{P}\left[\hat{C}_p > c \text{ and } K\left(\hat{C}_p, p\right) < 3\hat{CPU} < -K\left(\hat{C}_p, p\right) + 6\hat{C}_p \middle| C_p, C_{pp}\right] \\
= \; & \mathcal{P}\left[Y < \frac{(n-1)C_p^2}{c^2} \text{ and } A_1 < Z < A_2 \middle| C_p, C_{pp}\right]
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \sqrt{n}\left[-3CPL + K\left(\frac{C_p}{\sqrt{Y/(n-1)}}, p\right)\sqrt{\frac{Y}{n-1}}\right] \\
A_2 &= \sqrt{n}\left[3CPU - K\left(\frac{C_p}{\sqrt{Y/(n-1)}}, p\right)\sqrt{\frac{Y}{n-1}}\right].
\end{aligned}$$

We know that Y and Z are statistically independent. Y follows the chi-squared distribution with $n-1$ degrees of freedom and Z is a standard normal. Given any $\alpha \in [0,1]$ and $c_0 > 0$, we can find numerically $c_\alpha(c_0)$ such that Equations (14) and (15) hold.

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Key Words: C_p *index*, C_{pk} *index*, C_{pm} *index*, C_{pp} *index*, k *index*, *Process Capability*, *Process Potential*, *Process Performance*, *Statistical Testing Procedure*, *Hole Location Capability*, *Component Capability Requirements for Assemblies.*

Table 1. Examples to Show that C_{pk} is an Inconsistent Measure of Process Performance

Table 2. Examples to Show that C_{pm} is not a Measure of Process Yield

Table 3. Examples to Show that C_{pm} is not a Measure of Process centering

Table 4. C_p, C_{pp} and k Indices for Hole Location Processes

Table 5. Minimum Value for $\hat{C}_{pp}$ such that the Probability of Type I Error, α, is no Greater than 0.01

Table 6. Minimum Value for $\hat{C}_{pp}$ such that the Probability of Type I Error, α, is no Greater than 0.05

Table 7. Sample Size Required to Achieve $\alpha = \beta = 0.01$ in Testing the C_{pp} Index

Table 8. Sample Size Required to Achieve $\alpha = \beta = 0.05$ in Testing the C_{pp} Index

Table 9. Sample Size Required to Achieve Different Combination of α and β in Testing the k Index

Figure 1. Upper and Lower Bounds on Process Yield as a Function of C_{pm}

Figure 2. Process Yield as a Function of C_{pp}

Figure 3. Five Contours of C_p and C_{pp} in the (μ, σ) Plane

Figure 4. Counterexample to the True Position Deviation Method for Bivariate Characteristics

Figure 5. Counterexample to the Part Center Radial Deviation Method for Bivariate Characteristics

Figure 6. Operating Characteristic Curves for Testing $H_0 : C_{pp} \leq 1$ vs. $H_1 : C_{pp} > 1$ at a Significance Level $\alpha = 0.01$

Figure 7. Operating Characteristic Curves for Testing $H_0 : C_{pp} \leq 1$ vs. $H_1 : C_{pp} > 1$ at a Significance Level $\alpha = 0.05$

Process	C_{pk}	C_p	p	k
A	0.55	1.0	0.951	0.45
B	0.60	0.60	0.928	0
C	0.65	0.65	0.949	0
D	0.60	2.4	0.964	0.75

Table 1: Examples to Show that C_{pk} is an Inconsistent Measure of Process Performance

Process	C_{pm}	C_p	p
A	0.3	3	0.1711
B	0.3	0.3	0.6318
C	0.5	3	0.9990
D	0.75	1.667	0.9987
E	1	1	0.9973

Table 2: Examples to Show that C_{pm} is not a Measure of Process Yield

Process	C_{pm}	C_p	k
A	0.8	0.8	0
B	1	1.25	0.2
C	1.2	$12/\sqrt{19}$=2.753	0.25

Table 3: Examples to Show that C_{pm} is not a Measure of Process centering

Process	C_p	C_{pp}	k
A	1.2	1.2	0
B	2.0	1.07	0.50
C	1.05	1.05	0
D	1.05	1.05	0

Table 4: C_p, C_{pp} and k Indices for Hole Location Processes

	n														
c_0	10	20	30	40	50	60	70	80	90	100	120	140	160	180	200
0.70	1.401	1.082	0.984	0.934	0.902	0.881	0.864	0.852	0.842	0.833	0.820	0.810	0.802	0.795	0.790
0.80	1.611	1.240	1.126	1.068	1.032	1.007	0.988	0.974	0.962	0.952	0.937	0.926	0.916	0.909	0.903
0.90	1.822	1.399	1.269	1.203	1.162	1.134	1.112	1.096	1.083	1.072	1.054	1.041	1.031	1.023	1.016
1.00	2.032	1.557	1.412	1.338	1.292	1.260	1.237	1.218	1.203	1.191	1.172	1.157	1.146	1.137	1.129
1.10	2.242	1.716	1.556	1.474	1.423	1.387	1.361	1.341	1.324	1.311	1.289	1.273	1.261	1.250	1.242
1.20	2.453	1.875	1.699	1.609	1.553	1.514	1.486	1.463	1.445	1.430	1.407	1.389	1.375	1.364	1.355
1.30	2.662	2.034	1.842	1.744	1.683	1.641	1.610	1.586	1.566	1.550	1.525	1.505	1.490	1.478	1.468
1.33	2.732	2.086	1.890	1.789	1.727	1.684	1.652	1.627	1.606	1.590	1.564	1.544	1.529	1.516	1.506
1.40	2.872	2.192	1.985	1.880	1.814	1.768	1.735	1.708	1.687	1.670	1.642	1.622	1.605	1.592	1.581
1.50	3.081	2.351	2.128	2.015	1.944	1.895	1.859	1.831	1.808	1.789	1.760	1.738	1.720	1.706	1.694
1.60	3.291	2.509	2.271	2.150	2.075	2.022	1.984	1.953	1.929	1.909	1.877	1.854	1.835	1.820	1.807
1.67	3.430	2.615	2.367	2.240	2.162	2.107	2.067	2.035	2.010	1.989	1.956	1.931	1.912	1.896	1.883
1.70	3.499	2.668	2.414	2.285	2.205	2.149	2.108	2.076	2.050	2.029	1.995	1.970	1.950	1.934	1.920
1.80	3.709	2.826	2.558	2.421	2.336	2.277	2.233	2.199	2.171	2.148	2.113	2.086	2.065	2.048	2.033
1.90	3.918	2.985	2.701	2.556	2.466	2.404	2.357	2.321	2.292	2.268	2.231	2.202	2.180	2.162	2.147
2.00	4.127	3.143	2.844	2.691	2.596	2.531	2.482	2.444	2.413	2.388	2.348	2.318	2.295	2.276	2.260

Table 5: Minimum Value for $\hat{C}_{pp}$ such that the Probability of Type I Error, α, is no Greater than 0.01

	n														
c_0	10	20	30	40	50	60	70	80	90	100	120	140	160	180	200
0.70	1.118	0.994	0.886	0.855	0.835	0.821	0.811	0.802	0.796	0.790	0.781	0.775	0.770	0.765	0.762
0.80	1.284	1.081	1.013	0.978	0.955	0.939	0.927	0.917	0.910	0.903	0.893	0.886	0.880	0.875	0.871
0.90	1.450	1.219	1.142	1.101	1.075	1.057	1.043	1.032	1.024	1.017	1.005	0.997	0.990	0.984	0.979
1.00	1.616	1.356	1.270	1.224	1.195	1.175	1.160	1.148	1.138	1.130	1.117	1.108	1.100	1.094	1.088
1.10	1.782	1.494	1.398	1.348	1.316	1.293	1.276	1.263	1.252	1.243	1.229	1.219	1.210	1.203	1.197
1.20	1.948	1.632	1.527	1.471	1.436	1.411	1.393	1.378	1.366	1.357	1.341	1.330	1.320	1.313	1.306
1.30	2.114	1.769	1.655	1.595	1.556	1.530	1.509	1.494	1.481	1.470	1.453	1.441	1.431	1.422	1.416
1.33	2.169	1.815	1.698	1.636	1.597	1.569	1.548	1.532	1.519	1.508	1.491	1.478	1.467	1.459	1.452
1.40	2.280	1.907	1.783	1.718	1.677	1.648	1.626	1.609	1.595	1.584	1.565	1.552	1.541	1.532	1.525
1.50	2.446	2.045	1.912	1.842	1.797	1.766	1.743	1.724	1.709	1.697	1.678	1.663	1.651	1.642	1.634
1.60	2.611	2.182	2.040	1.965	1.918	1.884	1.859	1.840	1.824	1.811	1.790	1.774	1.761	1.751	1.743
1.67	2.722	2.274	2.126	2.047	1.998	1.963	1.937	1.917	1.900	1.886	1.864	1.848	1.835	1.824	1.815
1.70	2.777	2.320	2.168	2.089	2.038	2.003	1.976	1.955	1.938	1.924	1.902	1.885	1.872	1.861	1.852
1.80	2.942	2.457	2.297	2.212	2.158	2.121	2.093	2.070	2.052	2.037	2.014	1.996	1.982	1.971	1.961
1.90	3.108	2.595	2.425	2.336	2.279	2.239	2.209	2.186	2.167	2.151	2.126	2.107	2.092	2.080	2.070
2.00	3.273	2.732	2.553	2.458	2.399	2.357	2.326	2.301	2.281	2.264	2.238	2.218	2.203	2.190	2.179

Table 6: Minimum Value for $\hat{C}_{pp}$ such that the Probability of Type I Error, α, is no Greater than 0.05

	c_0								
c_1	1.00	1.10	1.20	1.30	1.33	1.40	1.50	1.60	1.67
1.20	325								
1.30	159	387							
1.33	133	293	967						
1.40	98	188	453						
1.50	68	115	219	527	774				
1.60	52	80	133	253	328	604			
1.67	44	65	103	178	220	358	967		
1.70	41	60	92	153	186	289	688		
1.80	34	47	69	105	123	174	328	777	
1.90	29	39	54	78	89	119	197	369	631
2.00	25	33	44	61	69	88	134	221	328
2.25	19	24	30	39	43	51	69	97	124
2.50	15	19	23	28	30	35	45	58	69
2.75	13	16	18	22	23	27	32	40	46
3.00	11	13	16	18	19	21	25	30	34

Table 7: Sample Size Required to Achieve $\alpha = \beta = 0.01$ in Testing the C_{pp} Index

c_1	c_0 1.00	1.10	1.20	1.30	1.33	1.40	1.50	1.60	1.67
1.10	593								
1.20	165	709							
1.30	82	196	841						
1.33	68	149	488						
1.40	51	96	231	982					
1.50	36	59	112	268	393				
1.60	27	42	69	129	166	307			
1.67	23	34	53	91	113	182	490		
1.70	22	31	48	79	95	148	249		
1.80	18	25	36	54	64	90	167	394	924
1.90	16	21	29	41	46	62	101	187	320
2.00	14	18	22	32	36	46	69	113	167
2.25	11	13	16	21	23	27	36	50	64
2.50	9	10	13	15	16	19	24	30	36
2.75	8	9	10	12	13	14	17	20	25
3.00	7	8	9	10	11	12	14	16	18

Table 8: Sample Size Required to Achieve $\alpha = \beta = 0.05$ in Testing the C_{pp} Index

	β = 0.01			β = 0.02			β = 0.05			β = 0.1			
$\alpha =$	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1	
0.25												139	0.25
0.30									119		122	97	0.30
0.35			125			109		109	88		90	72	0.35
0.40		115	97		101	85	117	84	68	101	70	55	0.40
0.45		92	77	110	81	68	93	67	54	80	55	44	0.45
0.50	100	75	63	90	66	55	76	54	44	65	45	36	0.50
0.55	83	63	53	75	55	46	63	45	37	54	38	30	0.55
0.60	71	53	45	63	47	39	53	38	32	46	32	26	0.60
0.65	61	46	39	55	41	34	46	33	27	39	28	22	0.65
0.70	53	40	34	47	35	30	40	29	24	34	24	19	0.70
0.75	47	36	30	42	31	27	35	26	21	30	21	17	0.75
0.80	41	32	27	37	28	24	31	22	19	27	19	15	0.80
0.85	37	29	24	33	25	21	28	21	17	24	17	14	0.85
0.90	34	26	22	29	23	19	25	19	16	21	15	13	0.90
0.95	31	24	20	27	21	18	23	17	14	19	14	11	0.95
1.00	28	22	19	25	19	16	21	16	13	18	13	11	1.00
$3C_pk =$													
1.1	24	19	16	21	16	14	18	13	11	15	11	9	1.1
1.2	21	16	14	18	14	12	15	12	10	13	10	8	1.2
1.3	18	15	13	16	13	11	14	10	9	11	8	7	1.3
1.4	16	13	12	14	11	10	12	9	8	10	8	7	1.4
1.5	15	12	11	13	10	9	11	8	7	9	7	6	1.5
1.6	13	11	10	12	10	9	10	8	7	8	6	6	1.6
1.7	12	10	9	11	9	8	9	7	6	8	6	5	1.7
1.8	12	10	9	10	8	7	8	7	6	7	6		1.8
1.9	11	9	8	10	8	7	8	6	6	7	5		1.9
2.0	10	8	8	9	7	7	7	6	5	6			2.0
2.1	10	8	7	8	7	6	7	6		6			2.1
2.2	9	8	7	8	7	6	7	6		6			2.2
2.3	9	7	7	8	6	6	6	5		5			2.3
2.4	8	7	7	7	6	6	6						2.4
2.5	8	7	6	7	6	6	6						2.5
3.0	7	6	6	6	5	5	5						3.0
3.5	6	5	5	5									3.5
4.0	6												4.0

Table 9: Sample Size Required to Achieve Different Combination of α and β in Testing the k Index

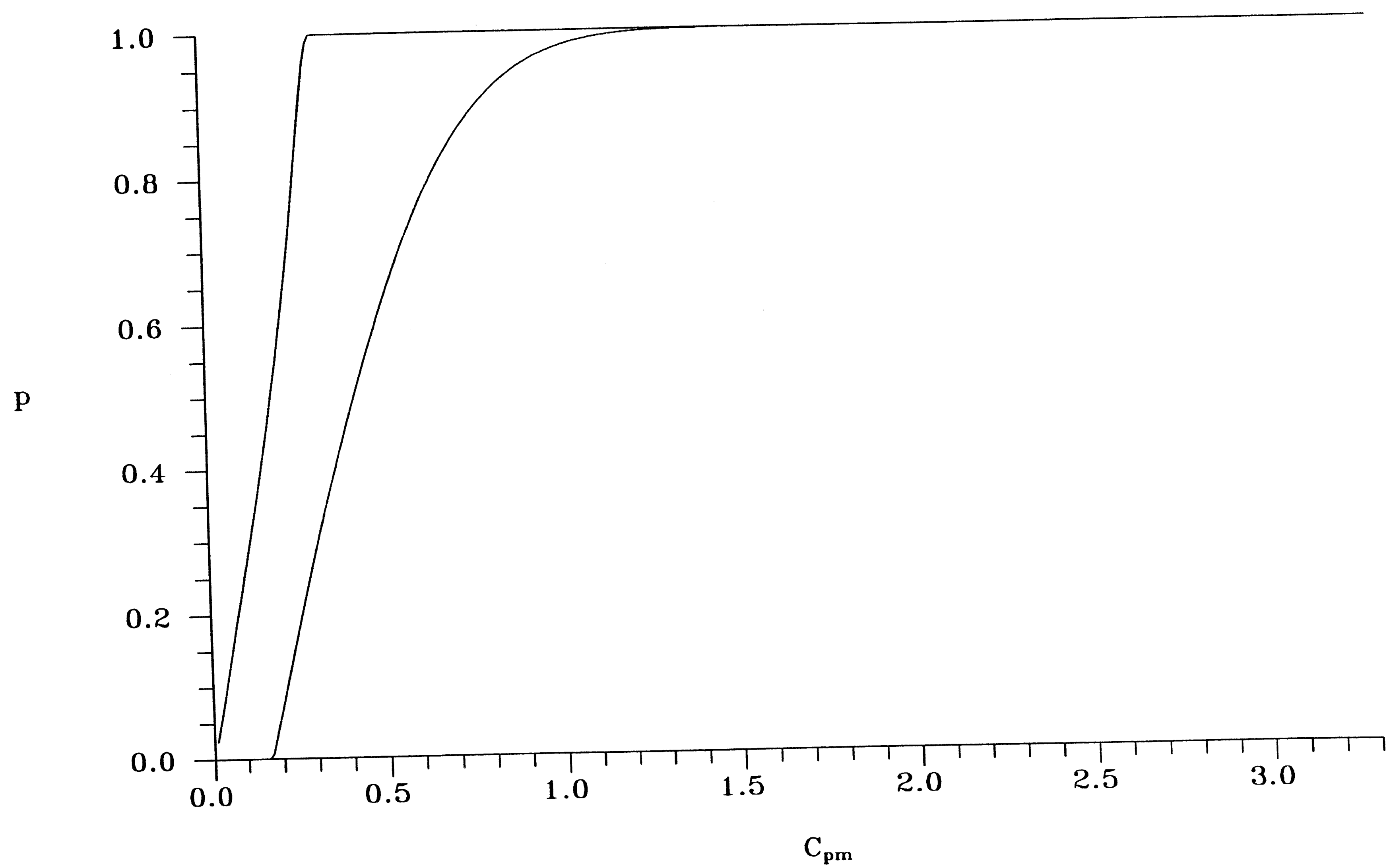

Figure 1 : Upper and Lower Bounds on Process Yield as a Function of C_{pm}

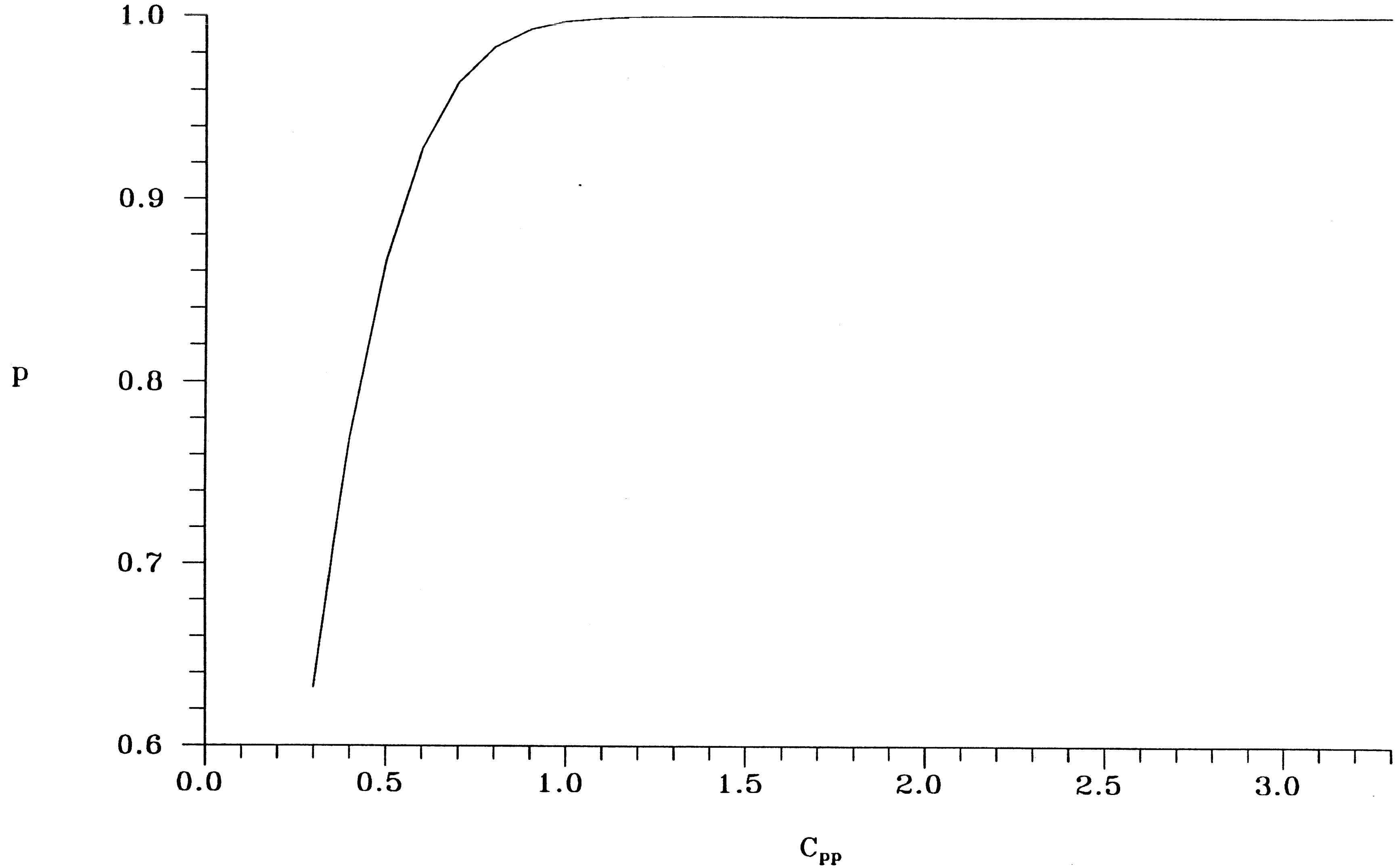

Figure 2: Process Yield as a Function of C_{pp}

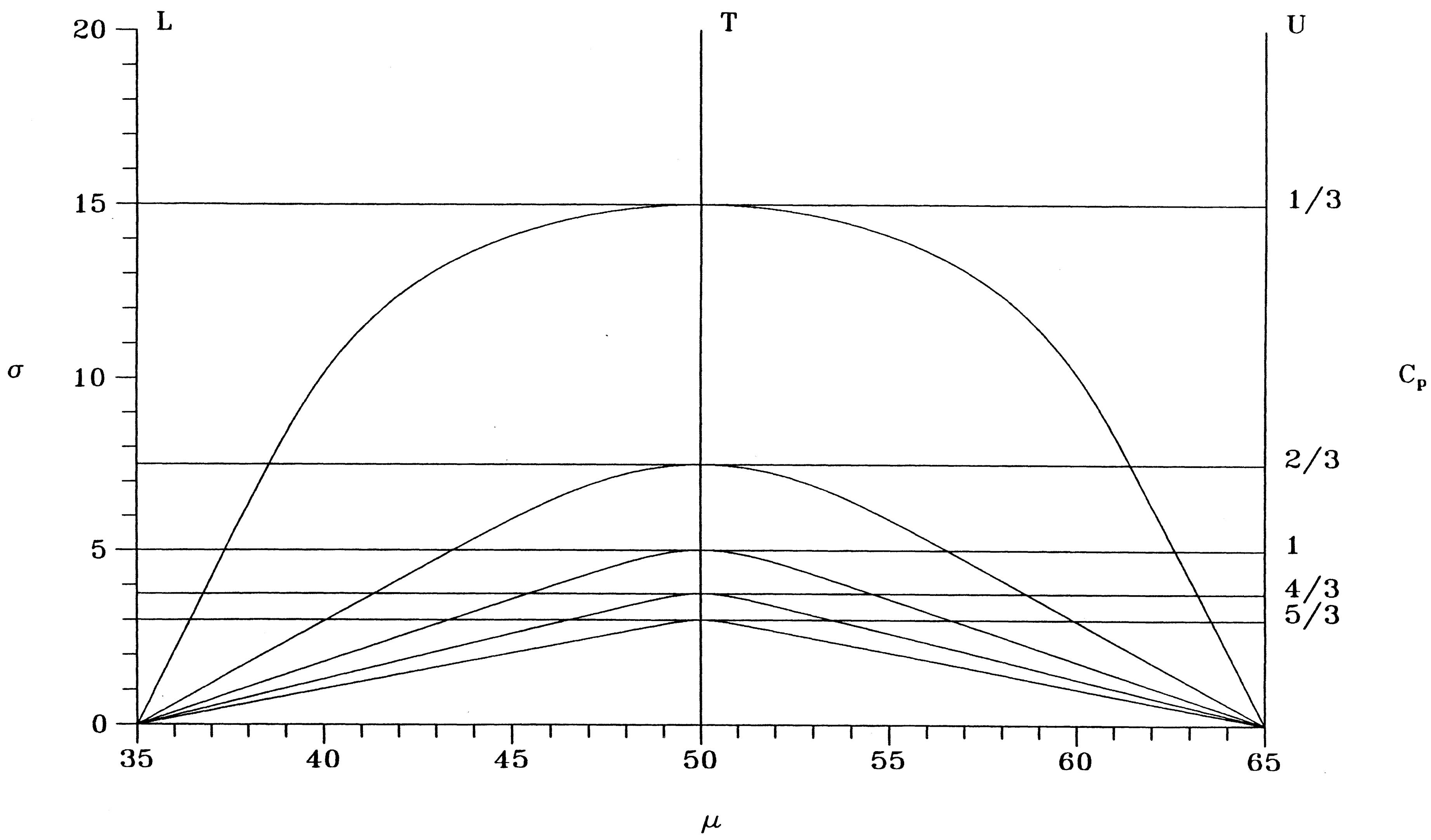

Figure 3: Five Contours of C_p and C_{pp} in the (μ,σ) Plane

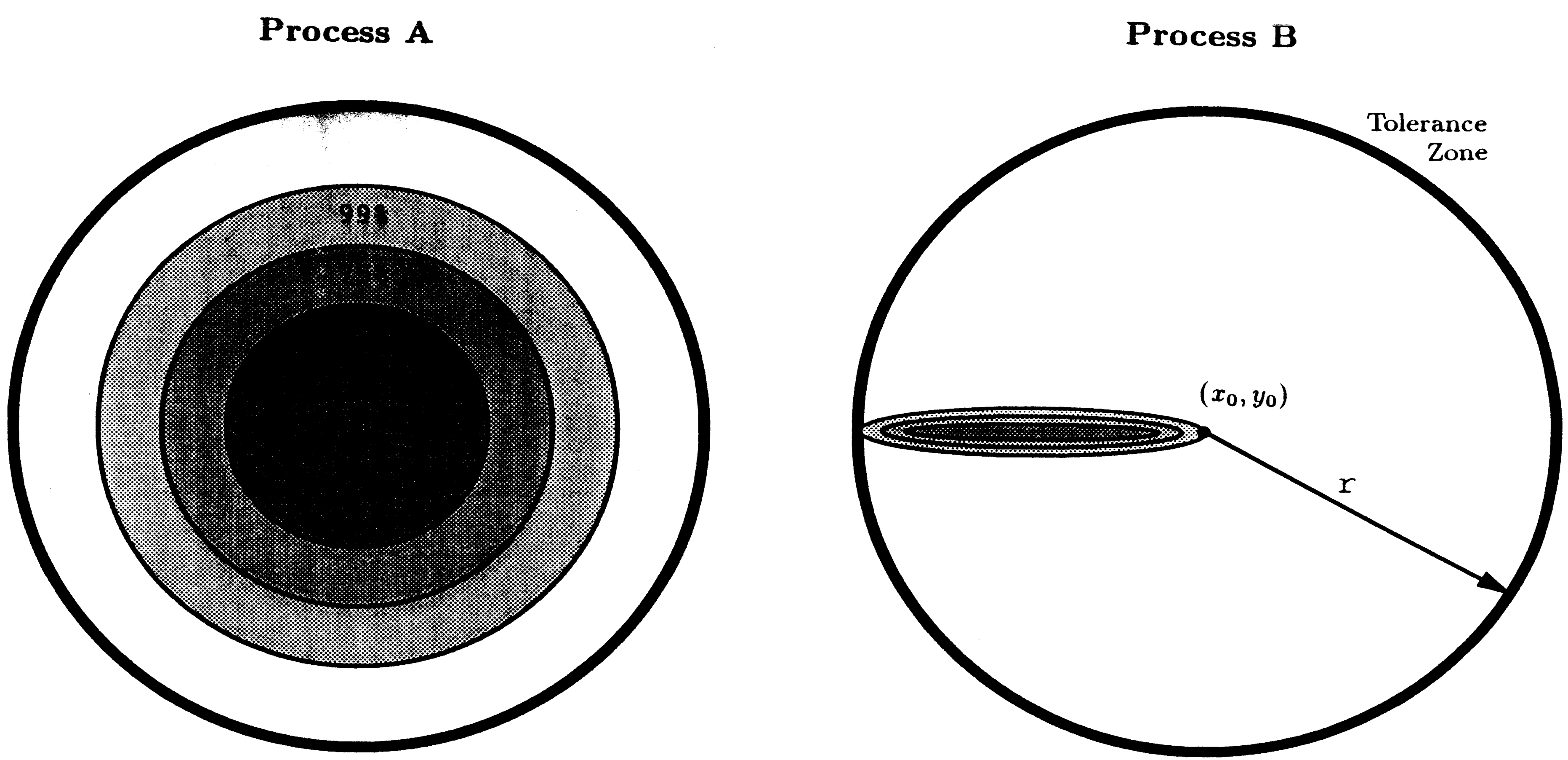

Figure 4. Counterexample to the True Position Deviation Method for Bivariate Characteristics

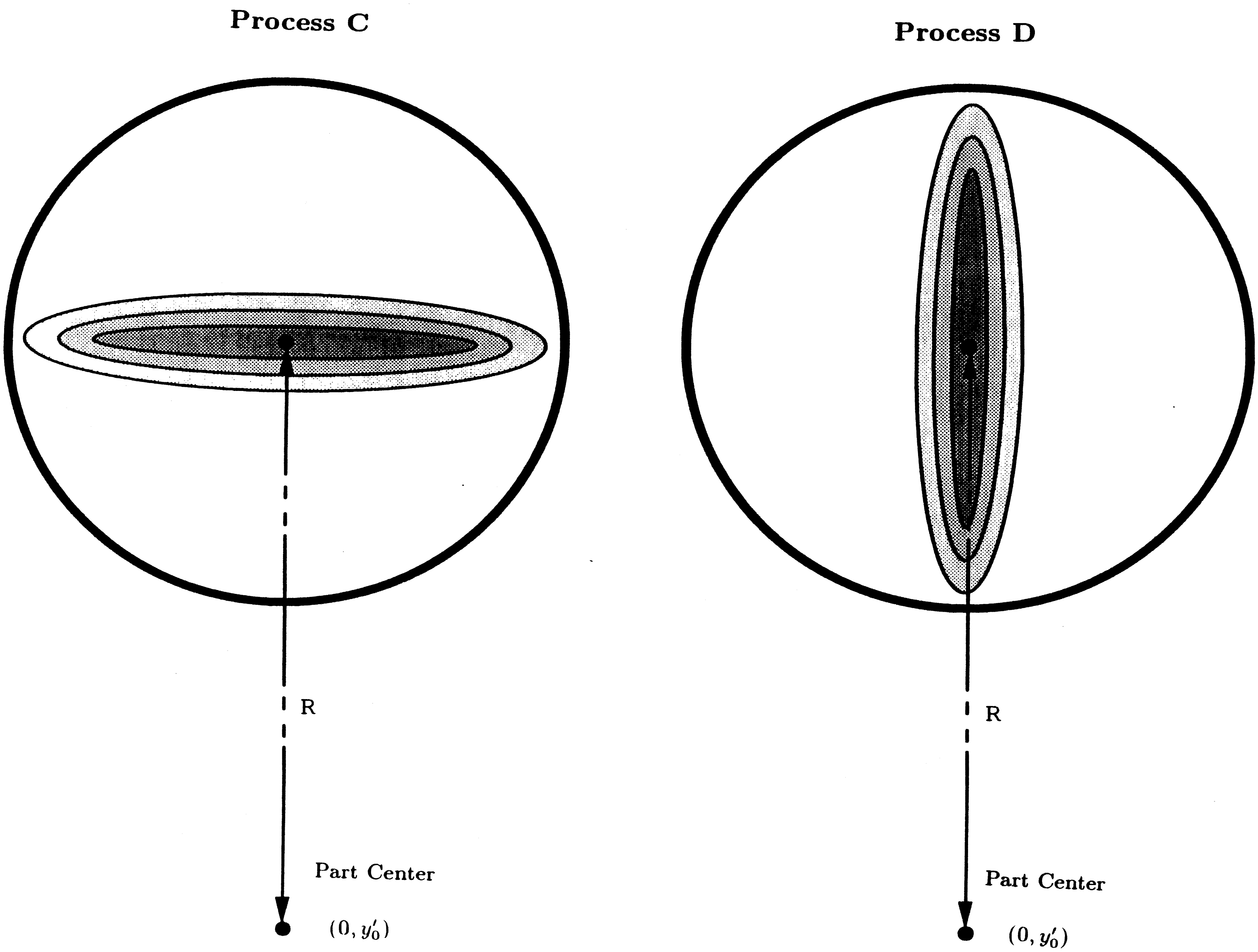

Figure 5. Counterexample to the Part Center Radial Deviation Method for Bivariate Characteristics

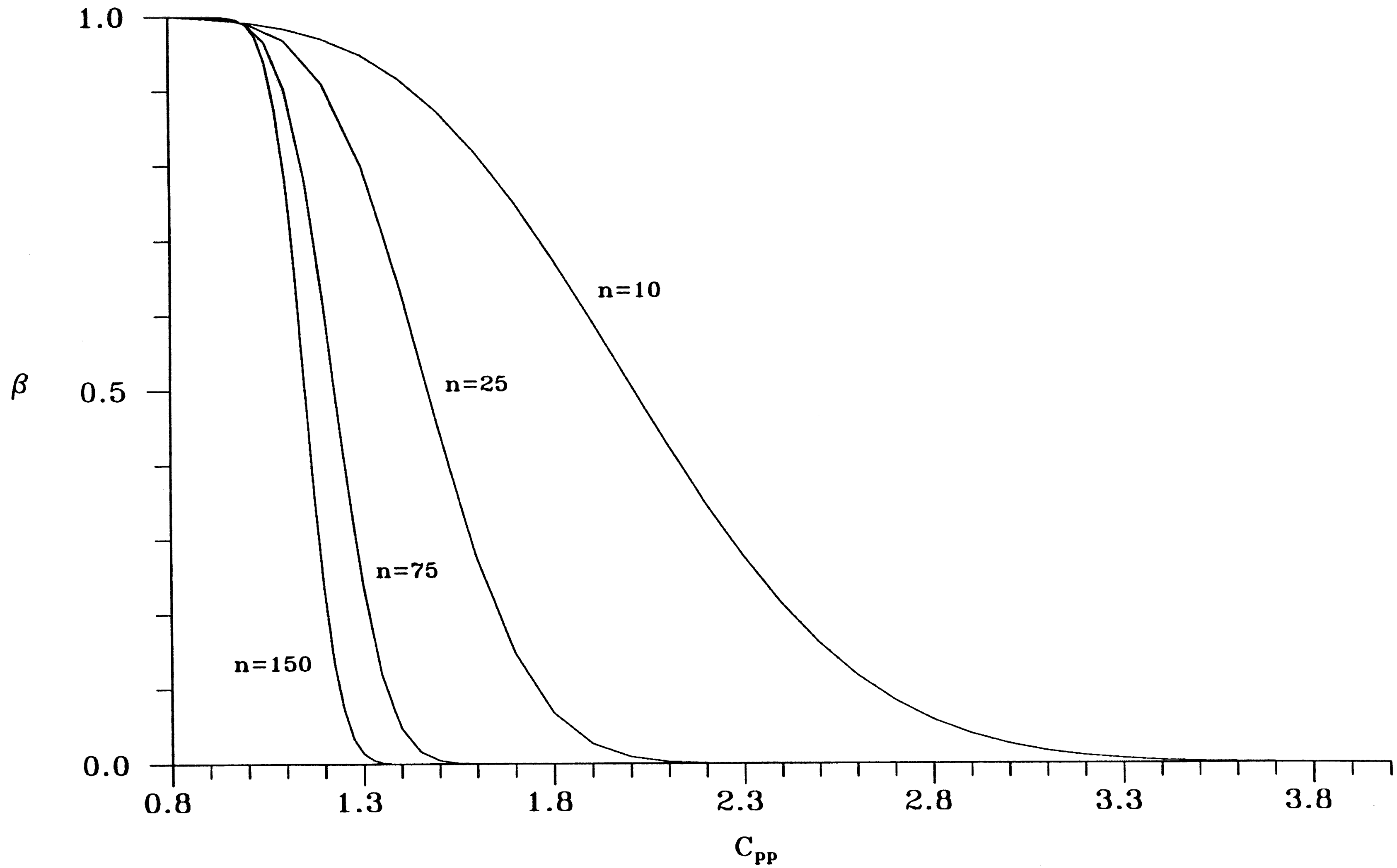

Figure 6: Operating Characteristic Curves for Testing $H_0:C_{pp}\leq 1$ vs. $H_1:C_{pp}>1$ at a Sigificance Level $\alpha=0.01$

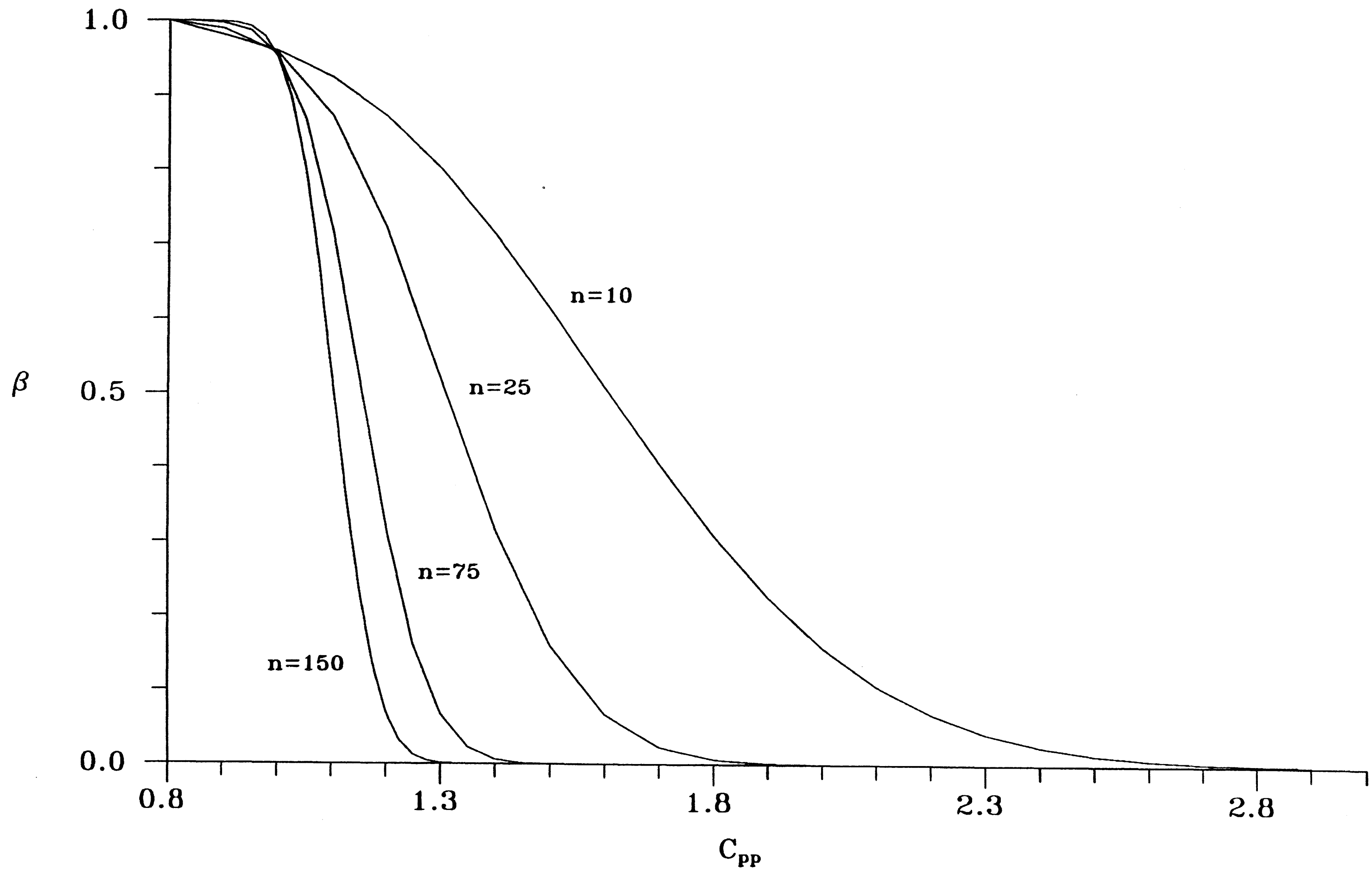

Figure 7: Operating Characteristic Curves for Testing $H_0: C_{pp} \leq 1$ vs. $H_1: C_{pp} > 1$ at a Sigificance Level $\alpha = 0.05$