REPORT NO. M720-1 R 41
UNCLASSIFIED
COPY NO. ____________

ONE PARAMETER SOLUTION OF A GAME OF PURSUIT

BY JESUS GIL DE LAMADRID

DIRECTOR OF PROJECT

A. H. COPELAND, SR.
PROFESSOR OF MATHEMATICS

CONTRACT N6 ONR 232-1

PROJECT M720-1 AUGUST, 1953

DISTRIBUTION LIST

Office of Naval Research
Navy Department
Air Branch, Code 461
T/3 Building, Room 2609
Washington 25, D. C. ....Copies 1 - 10

Office of Naval Research
New York Branch Office
346 Broadway
New York 10, New York....................................Copy 11

Director, Office of Naval Research
Chicago Branch Office
The John Crerar Library Building
Tenth Floor, 86 E. Randolph Street
Chicago 1, Illinois....................................Copy 12

Director, Office of Naval Research
San Francisco Branch Office
1000 Geary Street
San Francisco 9, California....................................Copy 13

Operations Evaluation Group (Op 374)
Office of the Chief of Naval Operations
Navy Department
Washington 25, D. C. ....Copies 14 - 15

Director, Naval Research Laboratory
Washington 25, D. C....................................Copy 16

Commander, Operational Development Force
USS Adirondack, U. S. Naval Base
Norfolk, Virginia....................................Copy 17

Chief of Staff, U. S. Army
Attn: DC/S P & R
3-D-640 Pentagon
Washington 25, D. C. ....Copy 18

Naval Research Laboratory
Washington 20, D. C. ....Copy 19

Mr. R. A. Bailey
Military Operations Research Division
Lockheed Aircraft Corporation
Burbank, California....................................Copy 20

Chief, Bureau of Aeronautics
Navy Department
Washington 25, D. C.
Attn: Aer - TD - 414 ....Copies 21- 22

Chief, Bureau of Aeronautics
Navy Department
Washington 25, D. C.
Attn: Mr. B. A. Wiener....................................Copy 23

Chief, Bureau of Ordnance
Navy Department
Washington, D. C.
Attn: Re 9....................................Copy 24

Chief, Bureau of Ordnance
Navy Department
Washington, D. C.
Attn: Re 2....................................Copy 25

Mr. James V. Burke
HQ WADC
Wright-Patterson Air Force Base, Ohio
Attn: WCEGO....................................Copy 26

Research and Development Board
Library Branch
Room 3 - E - 1065
The Pentagon
Washington 25, D. C. ....Copies 27 - 28

Chief of Staff, U. S. Air Force
Washington, D. C. ....Copy 29

Chief of Staff, U. S. Army
Office of Chief of Ordnance
Washington, D. C. ....Copy 30

Professor J. LaSalle
Department of Mathematics
University of Notre Dame
South Bend, Indiana....................................Copy 31

Professor T. H. Hildebrandt, Chairman
Department of Mathematics
University of Michigan
Ann Arbor, Michigan ....Copy 32

Professor Richard Courant
New York University
New York 3, New York....................................Copy 33

Professor S. Lefschetz
Princeton University
Princeton, New Jersey................................Copy 34

Professor William Prager
Brown University
Providence 12, Rhode Island............................Copy 35

Dr. Richard Bellman
The Rand Corporation
1700 Main Street
Santa Monica, California..............................Copy 36

Professor Edwin W. Titt
Indiana University
Bloomington, Indiana..................................Copy 37

Professor G. E. Uhlenbeck
Department of Physics
University of Michigan
Ann Arbor, Michigan...................................Copy 38

Dr. Andrew Sobczyk
3007 Villa Street
Los Alamos, New Mexico................................Copy 39

Library
U. S. Naval Ordnance Plant
Indianapolis 18, Indiana..............................Copy 40

Cornell Aeronautical Laboratory
4455 Genessee Street
Buffalo 5, New York
Attn: Mr. R. Shatz....................................Copy 41

Professor A. W. Tucker
Fine Hall, Box 708
Princeton University
Princeton, New Jersey.................................Copy 42

Mr. Harry Goode
Willow Run Research Center
Willow Run, Michigan..................................Copy 43

Mr. Leonard Gillman
Mathematics Department
Purdue University
Lafayette, Indiana....................................Copy 44

Dr. Seymour Sherman
Navy Research Group Dept. 72-25
Lockheed Aircraft Corporation
Burbank, California...................................Copy 45

The Rand Corporation
1700 Main Street
Santa Monica, California
Attn: Mr. Frank R. Collbohm ....Copy 46

Engineering Research Institute
University of Michigan
Ann Arbor, Michigan...................................Copy 47

Dr. Max A. Woodbury
Department of Statistics
Wharton School
University of Pannsylvania
Philadelphia 4, Pennsylvania..........................Copy 48

Bell Aircraft Corporation
P. O. Box 1
Buffalo 5, New York
Attn: Dorothy Zeman...................................Copies 49 - 50

Mathematical Reviews
80 Waterman Street
Providence 12, Rhode Island...........................Copy 51

Armed Services Technical Information Agency
Document Service Center
Knott Building
Dayton 2, Ohio
Attn: DSC-SA..Copies 52 - 56

Professor Oskar Morgenstern
Princeton University
Princeton, New Jersey.................................Copy 57

Dean Walter Bartkey
Air Weapons Research Center
University of Chicago
Chicago, Illinois.....................................Copy 58

Guided Missiles Library, 22-001
Massachusetts Institute of Technology
Cambridge 39, Massachusetts...........................Copies 59 - 60

Dr. E. D. McAlister
Eastman Kodak Company
Navy Ordnance Division
121 Lincoln Avenue
Rochester 11, New York................................Copy 61

Dr. W. L. Barrow, Chief Engineer
Sperry Gyroscope Company
Great Neck, New York.................................Copy 62

Professor G. P. Wadsworth, Room 2-285
Massachusetts Institute of Technology
Cambridge, Massachusetts..............................Copy 63

Commanding Officer, 3151st Electronics Station
Watson Laboratories, AMC
Red Bank, New Jersey
Attn: WLEPL - 1......................................Copy 64

S. B. Littauer, Project Director
NONR 266(04)
Department of Industrial Engineering
Engineering Building
Columbia University
New York 27, New York................................Copy 65

Professor Abraham Charnes
Carnegie Institute of Technology
Pittsburgh, Pennsylvania..............................Copy 66

Dr. Howard Raiffa
Department of Mathematical Statistics
Columbia University
New York 27, New York................................Copy 67

Dr. B. O. Koopman
Columbia University
New York 27, New York................................Copy 68

U. S. Department of Commerce
National Bureau of Standards
Washington 25, D. C.
Attn: 11.3/A-5RA.....................................Copy 69

Introduction

R. P. Isaacs formulated and solved* the game of pursuit outlined below. In it the probabilities pertaining to the chance outcomes are assigned definite values (a probability of 1/3 to each outcome). The purpose of the present work is to assign to these probabilities values depending on one parameter a, to study the effect of this on the game, and to obtain a solution (as far as is possible) in terms of a. The problem is not completely solved. Most of the features of the solution which are direct extensions of the case $a = 1/3$ (the game of RM-791), are obtained together with others. There are still others whose treatment requires a deeper analysis. These were foregone for lack of time. I have tried to indicate where the gaps lie, unless they seem quite evident. Following is an outline of the original game. For further details, see RM-791.

It is a two-person zero-sum game. One player is called the pursuer, P; the other, the evader, E. Both move in a discrete lineal set. This set may be denoted by the sequence of integers ..., -2, -1, 0, 1, 2, ... At each move, if E is at point i, he can make any of three moves, to $i + 1$, $i - 1$, or remain stationary. If P is at point i, he can make any of four moves, to $i + 1$, to $i - 1$, to $i + 2$, or to $i - 2$. He may not remain stationary. E is completely informed of P's position, and P's state of information. On the other hand, P's information about E's position is incomplete. After each one of E's moves, there is a signal indicating E's position to within three points (each with probability 1/3). P's information

---

* R. P. Isaacs, A Pursuit Game with Incomplete Information, Project RAND Research Memorandum RM-791. Constant reference will be made to this paper which is my sole direct source, and throughout this report it will be referred to as RM-791.

is based entirely on all signals given. The cycle of moves is as follows:

1. E moves.

2. Signal

3. P moves.

A description of a signal is as follows: If E is at point n one of the following signals may occur:

$$\sigma_1 = (n - 2, n - 1, n)$$

$$\sigma_2 = (n - 1, n, n + 1)$$

$$\sigma_3 = (n, n + 1, n + 2)$$

each with probability 1/3. Each signal contains the information that E is at one of the points indicated in it.

Capture occurs when P and E are at the same point. The payoff to E is the number of cycles before capture.

After several moves P can come and maintain himself within at most two moves of E. When this occurs two configurations can describe the relative positions of E and P. These two configurations (Configuration I and Configuration II) are shown respectively in Figures 1 and 2.

Each represents the given configuration and P's knowledge of it. The point marked with a square indicates the position of P. E may be at either of the 2 points marked with probabilities S and 1 - S in Figure 1, T and 1 - T in Figure 2. For example in Configuration I (Figure 1), S is the probability that E is at 3; and 1 - S the probability that E is at 2. Now if E moves, a signal is given. The figures also give the set of all possible signals for each configuration, denoted by the σ's.

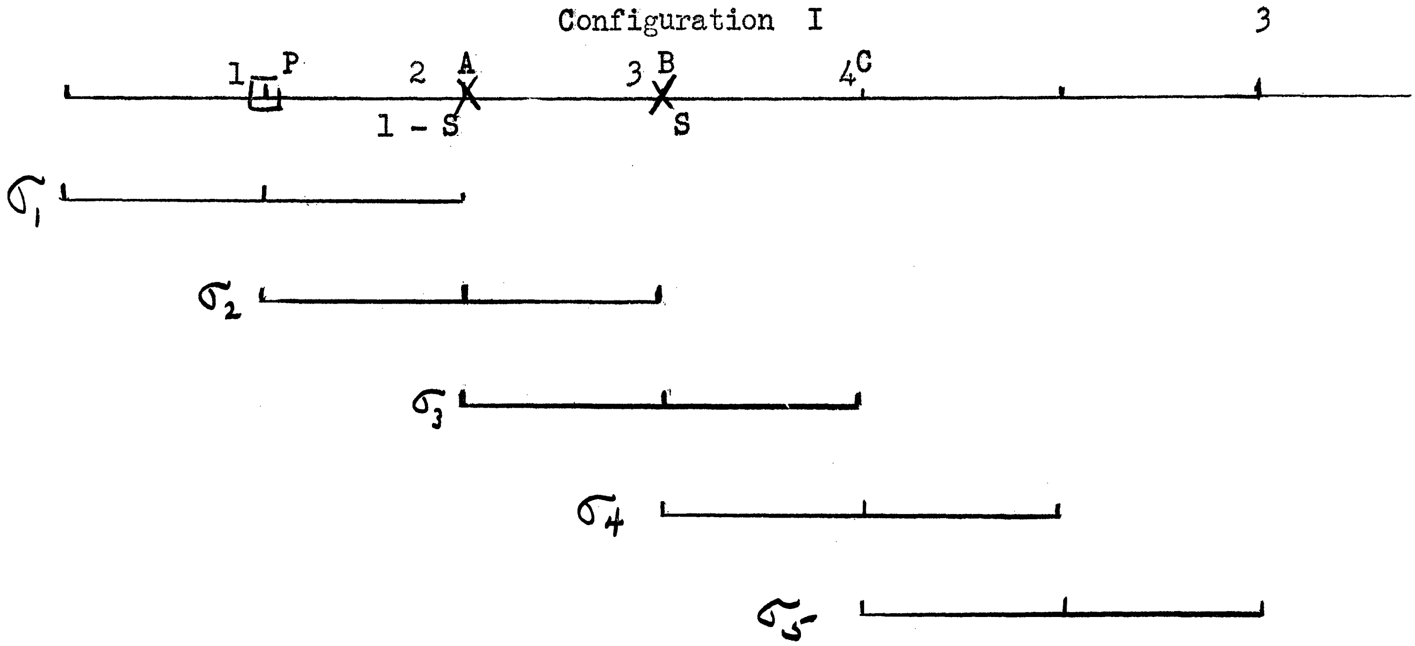


FIGURE 1

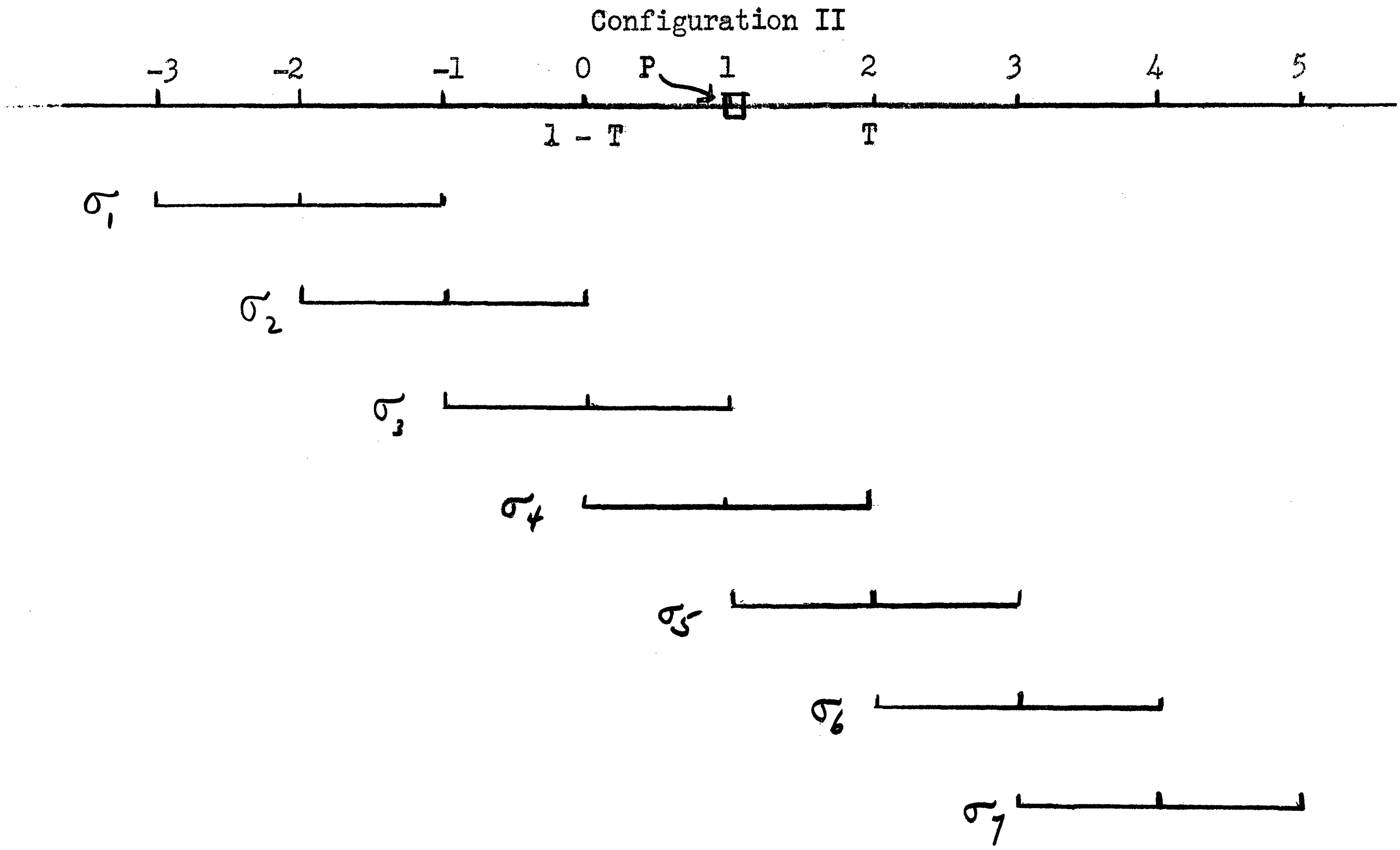


FIGURE 2

These two configurations have the property that starting from either, P can move in such a way as to establish one of them, or capture, regardless of what E does. That is, for the rest of the game one of these two configurations will prevail at any time if P so chooses, regardless of E's strategies.

Suppose the game has been played up to the time when one of the configurations has been established, and that E is to move next. Consider the following new game. It consists of 3 moves as follows: (1) E moves, (2) A chance move consisting of a signal, (3) P moves. The payoff is the expected number of cycles before capture counting the cycle of the new game, assuming P and E behave optimally for the rest of the game. So for each configuration and each value of S and T correspondingly, we have one of these games. Solving the larger game is equivalent to solving these games for each value of S and T. Isaac's paper presents such a solution of the game. The value for configuration I with probability S is denoted by $\mathcal{E}(S)$, that for configuration II with probability T is denoted by $\mathcal{F}(T)$. The notation used here is that of the above-mentioned paper whenever possible. For values of $\mathcal{E}(S)$ and $\mathcal{F}(T)$, and for method of solution refer to RM-791.

A Generalization

As described above, each one of the signals σ_1, σ_2, σ_3 has probability 1/3. It may be desirable to consider the problem in which the relative probability of each signal may vary. For instance, a detecting instrument (like radar), may point to a region around a point of probability higher than the neighboring points. Or vice versa, to a circle of points of high probability around a center of relatively low probability.

Therefore consider a case in which a probability a, $0 \le a \le 1$ is assigned to σ_2 (E in the middle point), and equal probabilities

$\frac{1-a}{2}$ to σ_1 and σ_2. Then $\mathcal{E}(S)$ and $\mathcal{F}(T)$ can be evaluated in terms of the parameter a.

It is sometimes desirable although not done here, to compare pure strategies with the optimal mixed strategy obtained here. For that purpose it is convenient to allow P to remain stationary if he so chooses, contrary to the rules of the game in RM-791. It turns out that this does not alter the optimal strategies since one is found in which this choice of P is precluded. Whenever a method of reasoning (or a proof) is a mere repetition of a method (or proof) in RM-791, it will be omitted, or simply sketched. Only when a modification is required or a simplification possible, will details be given.

Expressions for the Payoff Functions.

Configuration I.

The mixed strategies for P are given by the following table:

	1	2
σ_1	1	0
σ_2	$p_2(S)$	$1 - p_2(S)$
σ_3	$p_3(S)$	$1 - p_3(S)$
σ_4	0	1
σ_5	0	1

TABLE I

The subscripts on the left hand side from 1 through 5 indicate each of the signals shown in Figure 1. The numbers 1, 2 on the top indicate P's moving 1 or 2 to the right. Here P has nothing to gain by standing still, so this alternative is precluded. The entries in the boxes indicate the probability of P's moving 1 or 2 points given each signal. P determines these probabilities.

It is shown in RM-791 that the mixed strategies for E can be given in terms of the numbers A, B, C; the probabilities of E's finishing at the end of the game (the little game) at points 2, 3, 4 correspondingly.

Then in the same way as is done in RM-791, the payoff function is computed to be:

$$
(1) \qquad \phi = 1 + \frac{1-a}{2}(A+C)\mathcal{F}\left(\frac{C}{A+C}\right) + \left(a\,A + \frac{1+a}{2}C\right)\mathcal{E}(0)
+ p_2\left(\frac{1-a}{2}B - aA\right)\mathcal{E}(0)
+ p_3\left[-\frac{1-a}{2}(A+C)\mathcal{F}\left(\frac{C}{A+C}\right) + \left(aB + \frac{1-a}{2}C\right)\mathcal{E}\left(\frac{C}{B+C}\right)\right]
$$

Configuration II

The strategies for E are given by the following table:

	-1	0	1
2	0	$1 - q_1$	q_1
0	q_{-1}	$1 - q_{-1}$	0

TABLE II

The numbers 2, 0 on the left indicate the position of E at the beginning of the game (Figure 2). The numbers -1, 0, 1 indicate one move to the left, standing still, one move to the right respectively. The numbers in the boxes indicate the probabilities assigned to each combination (mixed strategy).

The strategies for P are given in the table below:

	-2	-1	0	1	2
σ_1	1	0	0	0	0
σ_2	$1 - p_2$	p_2	0	0	0
σ_3	$1 - p_3$	p_3	0	0	0
σ_4	0	$1 - p_4 - p'_4$	p'_4	p_4	0
σ_5	0	0	0	p_5	$1 - p_5$
σ_6	0	0	0	p_6	$1 - p_6$
σ_7	0	0	0	0	1

TABLE III

The entries on the left hand indicate the signals in Figure 2. The numbers across the top have an explanation similar to those on Table II. The entries in the boxes have a similar explanation.

Notice first of all that σ_2 and σ_3 are not equivalent any longer*, since they occur with different probabilities. The same can be said of σ_5 and σ_6. Also notice that 0 on top indicates that P is allowed to stand still, but only under σ_4 does it seem advisable to do so. His probability of standing still under a given mixed strategy is denoted by p'_4.

In a manner similar to that employed for ϕ, the payoff function for configuration II is found to be:

---

* As in RM-791.

$$(2)\qquad \psi = 1 + \mathcal{E}(0)\Big[(1-T)\Big\{\frac{1+a}{2}\, q_{-1}p_2 + \frac{1-a}{2}(1-p_2-q_{-1}) + \frac{1+a}{2}\, q_{-1}q_3 + a(1-p_3-q_{-1})\Big\} + T\Big\{\frac{1+a}{2}\, q_1p_5 + a(1-q_1-p_5) + \frac{1+a}{2}\, q_1\, p_6 + \frac{1-a}{2}\,(1-q_1-p_6)\Big\}\Big]$$

$$+ \frac{1-a}{2}\, p'_4\, \mathcal{F}(T)\Big[(1-T)\,(1-q_{-1}) + T(1-q_1)\Big]$$

$$+ \frac{1-a}{2}\,\mathcal{E}(1)\Big[(1-T)\, p_4(1-q_{-1}) + T(1-p_4-p'_4)(1-q_1)\Big].$$

Optimal strategies and values of $\mathcal{E}(S)$ and $\mathcal{F}(T)$ are obtained by solving the functional equations:

$$(3)\qquad \mathcal{E}(S) = \max_{\substack{A,B,C\\ A+B+C=1\\ C\leq S}}\ \min_{p_2,p_3}\ \phi = \min_{p_2,\, p_3}\ \max_{\substack{A,B,C\\ A+B+C=1\\ C\leq S}}\ \phi$$

$$(4)\qquad \mathcal{F}(T) = \max_{q_{-1},q_1}\ \min_{p_2,p_3,p_4,p'_4,\, p_5,\, p_6}\ \psi$$

$$= \min_{p_2,p_3,\, p_4,\, p'_4,p_5,p_6}\ \max_{q_{-1},q_1}\ \psi$$

The last part of each equation (max min = min max) represents an assumption. Solutions of equations (3) and (4), together with strategies which give rise to those solutions represent a solution to the game.

The Value of $\mathcal{E}(0)$

Theorem 1. The value of $\mathcal{E}(0)$ and the corresponding strategies are given as follows:

$a > 1/3$ $\qquad\qquad \mathcal{E}(0) = \dfrac{2}{1+a}$,

with strategies: A and B arbitrary in the set of values defined by $A + B = 1$, $\dfrac{1-a}{1+a} \leq A \leq \dfrac{2a}{1+a}$, $C = 0$, $p_2 = 1$, $p_3 = 0$.

$a = 1/3$ (cf. RM-791, p. 8)

$\mathcal{E}(0) = 3/2$, with strategies $A = 1/2$, $B = 1/2$, $C = 0$, p_2 and p_3 arbitrary among the values $p_2 + p_3 = 1$.

$a < 1/3$

$\mathcal{E}(0) = \frac{1}{1 - a}$, with strategies: A and B arbitrary among the values $A + B = 1$, $\frac{2a}{1+a} \leq A \leq \frac{1-a}{1+a}$, $p_2 = 0$, $p_3 = 1$.

Remark: The method of proof for the three cases mentioned is essentially the same trial and error procedure. Therefore the proof will only be given for $a > 1/3$. Furthermore a complete proof is quite lengthy, so it will only be shown that for $a > 1/3$, $\mathcal{E}(0) = \frac{2}{1+a}$.

Proof: The problem is to evaluate $\mathcal{E}(S)$ for $S = 0$. In this case $C = 0$ and $B = 1 - A$. Also $\mathcal{F}\left(\frac{A}{A+C}\right) = \mathcal{F}(0) = \mathcal{E}\left(\frac{C}{B+C}\right) = \mathcal{E}(0)$. Substituting these values in (1) and simplifying we obtain

(8) $$\emptyset = 1 + \mathcal{E}(0)\mathcal{L}, \quad \text{where}$$

(9) $$\mathcal{L} = \frac{1+a}{2}A + p_2\left(\frac{1-a}{2} - \frac{1+a}{2}A\right) + p_3\left(a - \frac{1+a}{2}A\right),$$

and from (3) we have

(10) $$\mathcal{E}(0) = 1 + \mathcal{E}(0) \max_{A} \min_{p_2, p_3} \mathcal{L}.$$

We show that $\max_{A} \min_{p_2,p_3} \mathcal{L} = \frac{1-a}{2}$. This establishes the value of $\mathcal{E}(0)$, since $\mathcal{E}(0) = 1 + \mathcal{E}(0)\frac{1-a}{2}$, and then solving for $\mathcal{E}(0)$ we obtain

(11) $$\mathcal{E}(0) = \frac{2}{1+a}.$$

Now let $a > 1/3$. This is equivalent to $\frac{1-a}{2} < a$ and $\frac{2a}{1+a} < \frac{1-a}{1+a}$.

(12) $$\max_{A} \min_{p_2,p_3} \mathcal{L} \geq \min_{p_2,p_3}\left[\frac{1+a}{2}A + p_2\left(\frac{1-a}{2} - \frac{1+a}{2}A\right) + p_3\left(a - \frac{1+a}{2}A\right)\right]$$

where on the right hand side, A is chose to lie in the interval $\frac{1-a}{1+a} \leq A \leq \frac{2a}{1+a}$. But in this interval the coefficient of p_2 is ≤ 0 and that of p_3 is ≥ 0. So the values $p_2 = 1$, $p_3 = 0$ are minimizing values and the expression on the right of (12) becomes equal to $\frac{1-a}{2}$.

On the other hand:

$$(13) \qquad \max_{A} \min_{p_2, p_3} \mathcal{L} \leq \max_{A} \left[\frac{1+a}{2} A + \frac{1-a}{2} - \frac{1+a}{2} A \right]$$

(setting $p_2 = 1$, $p_3 = 0$) or

$$\max_{A} \min_{p_2, p_3} \mathcal{L} \leq \frac{1-a}{2}.$$

This completes the proof.

<u>The Value of</u> $\mathcal{F}(T)$

In the proof of the following theorem it will be assumed that

1) $\mathcal{F}(T) - T\,\mathcal{E}(1) \geq 0$. The negation of this assumption leads to a contradiction. For a proof of this see the appendix.

2) $\mathcal{F}(T) = \mathcal{F}(1-T) = \mathcal{F}(\min(T, 1-T))$. Since $\min(T, 1-T) \leq \frac{1}{2}$, the value of the function need only to be given for $T \leq \frac{1}{2}$. This assumption is evident from its interpretation in terms of configuration.

<u>Theorem 2</u>. The value of $\mathcal{F}(T)$ for $T \leq \frac{1}{2}$, and a set of optimal strategies are given as follows.

For $a \geq 1/3$:

$$(14) \qquad \mathcal{F}(T) = \frac{2}{1+a} + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}$$ and optimal strategies are

$p_2 = 0$, $p_3 = 1$, $p_6 = 0$, $p_5 = 1 + \frac{1-a}{2}\mathcal{E}(1)$, $p_4 = p'_4 = 0$, $q_{-1} = q_1 = \frac{2a}{1+a}$.

For a $\leq 1/3$:

(15) $$\mathcal{F}(T) = \frac{1}{1-a} + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}$$ and optimal strategies are

$p_2 = 1$, $p_3 = 0$, $p_4 = p'_4 = 0$, $p_5 = \frac{(1-a)^2}{1+a}\,\mathcal{E}(1)$, $p_6 = 1$, $q_{-1} = q_1 = \frac{2a}{1+a}$.

Only part of the proof of this theorem will be given here. We will only refer to the case a $\leq 1/3$, since the other case follows exactly the same method.

The proof again consists in showing the two inequalities

(16) $$\mathcal{F}(T) \leq \frac{1}{1-a} + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}$$

(17) $$\mathcal{F}(T) \geq \frac{1}{1-a} + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}.$$

The proof of (16) is almost a step by step repetition of the proof of the corresponding inequality for a = 1/3 given in RM-791, Theorem 2, P. 12. Therefore it will not be given here. The proof of (17) goes as follows:

Substituting the values $q_{-1} = q_1 = \frac{2a}{1+a}$, we obtain:

(18) $$\mathcal{F}(T) = \max_{q_{-1},q_1}\ \min_{p_2,p_3,p_4,p'_4,p_5,p_6} \psi$$

$$\geq \min_{p_2,p_3,p_4,p'_4,p_5,p_6} \Big\{ (1-T)\,\mathcal{E}(0)\left(a - \frac{1-a}{2}\right)p_2$$

$$+ T\,\mathcal{E}(0)\left(a - \frac{1-a}{2}\right)p_6 + (1-2T)\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}\,p_4$$

$$+ \frac{(1-a)^2}{2(1+a)}\Big(\mathcal{F}(T) - T\,\mathcal{E}(1)\Big)p'_4 + 1$$

$$+ \frac{1-a}{2}\,\mathcal{E}(0) + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1-a)} \Big\}.$$

Now $a - \frac{1-a}{2} \leq 0$ for a $\leq 1/3$ so that the coefficients of p_2 and p_6 are nonpositive, and therefore the minimizing values of p_2 and p_6 are $p_2 = p_6 = 1$.

Again, the coefficient of p_4 is nonnegative since $T \leq \frac{1}{2}$. Also the coefficient of p'_4 is nonnegative since $\mathcal{F}(T) - T\,\mathcal{E}(1)$ was assumed to be nonnegative. This fact will be shown in the details in the appendix. Therefore the minimizing values of p_4 and p'_4 are $p_4 = p'_4 = 0$. Thus we have:

$$\begin{aligned}
(19) \qquad \mathcal{F}(T) &\geq (1 - T)\,\mathcal{E}(0)\left(a - \frac{1-a}{2}\right) + T\,\mathcal{E}(0)\left(a - \frac{1-a}{2}\right) \\
&\quad + 1 + \frac{1-a}{2}\,\mathcal{E}(0) + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)} \\
&= \mathcal{E}(0)\left(a - \frac{1-a}{2}\right) + \mathcal{E}(0)\,\frac{1-a}{2} + 1 + \\
&\quad T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)} = 1 + a\,\mathcal{E}(0) + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)} \\
&= \frac{1}{1-a} + T\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}
\end{aligned}$$

which was to be shown.

Here as in the previous proof the values pertaining to optimal strategies seem to come out of the air. Therefore for the sake of completeness, I would like at this moment to indicate a procedure for computing them.* This is merely an outline, and no proofs are given.

The procedure is to assume that the optimal values of q_{-1} and q_1 are interior to the interval of definition of these variables. Therefore for these values we have $\dfrac{\partial \psi}{\partial q_{-1}} = \dfrac{\partial \psi}{\partial q_1} = 0$. This gives two relations binding $p_2, p_3, p_4, p'_4, p_5, p_6$. From these relations we can find those variables among $p_2, p_3, p_4, p'_4, p_5, p_6$ achieving their optimal values at points interior to their intervals of definition. By setting the partial derivatives of ψ with respect to these variables equal to zero, we compute q_1 and q_{-1}. From all relations involved, the values of $p_2, p_3, p_4, p'_4, p_5, p_6$ can be obtained.

* Suggested by R. M. Thrall.

Lemma 1. $\mathcal{E}(1) \le \frac{4}{1+a}$.

Proof. Setting $p_2 = p_3 = 0$ in (1) and noting that for S = 1, C can range over the whole interval $0 \le C \le 1$ we have

(20) $$\mathcal{E}(1) \le \max_{\substack{A,B,C \\ A+B+C=1 \\ C \le 1}} \left\{ 1 + \frac{1-a}{2}(A + C)\,\mathcal{F}\left(\frac{C}{A+C}\right) + \left(aA + \frac{1+a}{2}C\right)\mathcal{E}(0) \right\}$$

Now it can be verified from Theorem 1 that $\mathcal{E}(0) \le \frac{2}{1+a}$ for all a, and from Theorem 2 that $\mathcal{F}(T) \le 2 + \frac{1-a}{2}\mathcal{E}(1)$ for all a and T.

Substituting these estimates in (20) :

(21) $$\mathcal{E}(1) \le \max_{\substack{A,C \\ C \le 1 \\ A+C=1}} \left\{ 1 + \frac{1-a}{2}(A + C)\left[2 + \frac{1-a}{2}\mathcal{E}(1)\right] + \left(aA + \frac{1+a}{2}C\right)\frac{2}{1+a} \right\}.$$

The maximum occurs at A = 0, C = 1. So:

(22) $$\mathcal{E}(1) \le 1 + (1-a) + \left(\frac{1-a}{2}\right)^2 \mathcal{E}(1) + 1 = 3 - a + \frac{(1-a)^2}{4}\mathcal{E}(1),$$

(23) $$\frac{(1+a)(3-a)}{4}\,\mathcal{E}(1) \le 3 - a\ ,$$

(24) $$\mathcal{E}(1) \le \frac{4}{1+a}$$ Q. E. D.

Lemma 2.

$$\frac{1 + \frac{3+a}{4(1-a)}}{1 - \frac{(1-a)^3}{8(1+a)}} \le \frac{2(1+a)}{(1-a)^3}$$

Proof: Combining the two inequalities: $\frac{2-a}{2} \leq 1$, $\frac{(1-a)^2}{1+a} \leq 1$, the following relation is obtained: $\frac{(1-a)^2(2-a)}{2(1+a)} \leq 1$. From this

$$\frac{5(1-a)^3}{8(1+a)} + \frac{(3+a)(1-a)^2}{8(1+a)} = \frac{(1-a)^2(2-a)}{2(1+a)} \leq 1, \text{ or}$$

$$\frac{(1-a)^3}{2(1+a)} + \frac{(3+a)(1-a)^2}{8(1+a)} \leq 1 - \frac{(1-a)^3}{8(1+a)}, \text{ and finally:}$$

$$\frac{1 + \frac{3+a}{4(1-a)}}{1 - \frac{(1-a)^3}{8(1+a)}} \leq \frac{2(1+a)}{(1-a)^3}$$

Theorem 3. The value of $\mathcal{E}(1)$ and corresponding strategies are given as follows: $A = B = 0$, $C = 1$, $p_2 = p_3 = 0$.

$$\mathcal{E}(1) = 1 + \mathcal{E}(0) \tag{25}$$

Remark: The proof of this theorem for $a \geq 1/3$ is a paraphrase of the corresponding proof in RM-791, with lemma 1 here used in place of lemma 4 there. However most of the proof for the case $a \leq 1/3$, although following a similar pattern, is essentially different. This proof will be given here.

Proof: (for $a \leq 1/3$)

We set $\mathcal{E}(0) = \frac{1}{1-a}$ in ϕ.

i) $\mathcal{E}(1) \geq 1 + \frac{1}{1-a} = 1 + \mathcal{E}(0)$.

Substitute the values $A = B = 0$, $C = 1$ in (1), and the following results:

$$\mathcal{E}(1) \geq \min_{p_2, p_3} \left[1 + \frac{1-a}{2}\mathcal{F}(1) + \frac{1+a}{2}\mathcal{E}(0) + p_3\left(-\frac{1-a}{2}\mathcal{F}(1) + \frac{1-a}{2}\mathcal{E}(1) \right) \right] \tag{26}$$

$$= \min_{p_2, p_3} \left[1 + \frac{1}{1-a}\left(\frac{1-a}{2} + \frac{1+a}{2}\right) + \frac{1-a}{2} p_3\left(-\frac{1}{1-a} + \mathcal{E}(1)\right)\right]$$

since $\mathcal{F}(1) = \mathcal{F}(0) = \mathcal{E}(0)$. The minimizing value of p_3 is either 0 or 1 according as the quantity $-\frac{1}{1-a} + \mathcal{E}(1)$ is positive or negative (p_3 is arbitrary if the parenthesis is 0).

If $-\frac{1}{1-a} + \mathcal{E}(1) < 0$ then $\mathcal{E}(1) < \frac{1}{1-a}$, and the minimizing value of $p_3 = 1$. Then

(27) $$\mathcal{E}(1) \geq 1 + \frac{1}{1-a} - \frac{1}{2} + \frac{1-a}{2}\mathcal{E}(1) > \frac{1}{1-a} ,$$

a contradiction. Therefore $\mathcal{E}(1) \geq \frac{1}{1-a}$ or $-\frac{1}{1-a} + \mathcal{E}(1) \geq 0$ and $p_3 = 0$ is always a minimizing value. Then

$$\mathcal{E}(1) \geq 1 + \frac{1}{1-a} = 1 + \mathcal{E}(0)$$

ii) $\mathcal{E}(1) \leq 1 + \frac{1}{1-a}$.

Set $p_2 = p_3 = 0$ in (1), to obtain

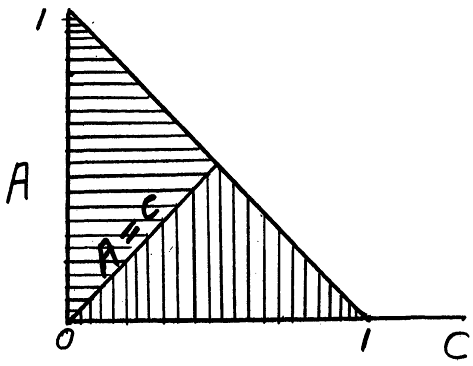
(28) $$\mathcal{E}(1) \leq \max_{A,B,C} \left[1 + \frac{1-a}{2}(A + C)\mathcal{F}\left(\frac{C}{A+C}\right) + \left(aA + \frac{1+a}{2}C\right)\frac{1}{1-a}\right]$$

$$= \max \left[1 + \left(\frac{1-a}{2}\mathcal{F}\left(\frac{C}{A+C}\right) + \frac{a}{1-a}\right)(A + C) + \frac{C}{2}\right]$$

$$= \max_{A+C \leq 1} \beta(C, A)$$

We propose to evaluate $\max \beta(C, A)$. The set $A + C \leq 1$ can be represented as in Figure 3, by the area enclosed by the line $A + C = 1$ and the coordinate C-axis, and A-axis. Consider the following two subsets:

(1) Those points for which $C < A$, lying above the line $A = C$, indicated by the horizontal lines. (2) the complementary set for which $A \leq C$, below and including the line $A = C$, indicated by the verticals.



et (C, A) lie on set (1). Then $C < A$. Using the value of Theorem 2 we have

$$(29) \qquad \beta(C, A) = 1 + (A + C)\left[\frac{1-a}{2}\left(\frac{1}{1-a} + \frac{C}{A+C}\,\mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}\right) + \frac{a}{1-a}\right] + \frac{C}{2}$$

$$= 1 + (A + C)\left(\frac{1}{2} + \frac{a}{1-a}\right) + \left[\frac{1}{2} + \frac{(1-a)^3}{4(1+a)}\,\mathcal{E}(1)\right] C$$

It is evident from (29) that the coefficient of C is greater than that of A. Therefore a maximum cannot occur on set (1), since for any pair (C_o, A_o) such that $C_o + A_o \leq 1$ and $C_o < A_o$, we can (properly) increase the value of $\beta(C_o, A_o)$ by increasing slightly C_o and decreasing A_o, without violating the above inequalities.

We conclude then that any maximizing pair (C, A) of β must lie on set (2). In this set $A \leq C$ and :

(30) $$\beta(C, A) = 1 + (A + C)\left[\frac{1-a}{2}\left(\frac{1}{1-a} + \frac{A}{A+C}\mathcal{E}(1)\frac{(1-a)^2}{2(1+a)}\right) + \frac{a}{1-a}\right] + \frac{C}{2}$$

$$= 1 + \left(\frac{1}{2} + \mathcal{E}(1)\frac{(1-a)^3}{4(1+a)} + \frac{a}{1-a}\right)A + \left(\frac{1}{2} + \frac{a}{1-a} + \frac{1}{2}\right)C$$

$$= 1 + k_1 A + k_2 C .$$

Here again the argument must be divided into two parts according as

a) $k_1 > k_2$, or b) $k_1 \leq k_2$.

a) Assume $k_1 > k_2$. Then from this follows

(31) $$\mathcal{E}(1)\frac{(1-a)^3}{4(1+a)} > \frac{1}{2} \quad \text{or}$$

(32) $$\mathcal{E}(1) > \frac{2(1+a)}{(1-a)^3}$$

But then the maximizing pair of $\beta(C, A)$ is $(\frac{1}{2}, \frac{1}{2})$ and then:

(33) $$\mathcal{E}(1) \leq 1 + \tfrac{1}{2}\left[\tfrac{1}{2} + \mathcal{E}(1)\frac{(1-a)^3}{4(1+a)} + \frac{a}{1-a} + \frac{1}{2} + \frac{a}{1-a} + \frac{1}{2}\right]$$

$$= 1 + \frac{3+a}{4(1-a)} + \mathcal{E}(1)\frac{(1-a)^3}{8(1+a)}$$

from which finally results:

(34) $$(1) \qquad \frac{1 + \dfrac{3+a}{4(1-a)}}{1 - \dfrac{(1-a)^3}{8(1+a)}}$$

but this by lemma 2, contradicts (32).

So we conclude $k_1 \leq k_2$, from which it follows that the maximizing pair is (1, 0) and

(35) $$\max \beta(C, A) = 2 + \frac{a}{1-a} = 1 + \frac{1}{1-a}, \text{ and } \mathcal{E}(1) \leq 1 + \frac{1}{1-a} .$$

With $\mathcal{E}(0)$ and $\mathcal{E}(1)$ evaluated, all the unknown values in the expression for $\mathcal{F}(T)$ are obtained, and therefore the function $\mathcal{F}(T)$ is known. Furthermore, with this function known too, all the unknown symbols in the expression for $\emptyset$ in (1) may be replaced to obtain a function of A, B, C, p_2, p_3, linear in any of these variables. A summary of these values follows.

Summary of Result Obtained and General Remarks.

All values and strategies obtained to this point are collected in Table IV. For all functional values, the values of a are split into two overlapping sets $a \leq 1/3$, $a \geq 1/3$, and this is the way that the computations were performed except for the case of $\mathcal{E}(0)$, where the case $a = 1/3$ is treated separately. This is not shown in Table IV, but a glance at Theorem 1 will show that there is some arbitrariness in the choice of strategies, even though the value of $\mathcal{E}(0) = 3/2$ conforms to the two general formulas given, $\frac{1}{1 - a}$ and $\frac{2}{1 + a}$.

It is also opportune to observe that the point of view of the proof of Theorem 1 is to start with the function $\emptyset$ with S = 0 and to set out to find in a direct way the optimal value of this function and all the sets of values of A, B, C, p_2, p_3 that provide this optimal value. In this sense theorem 1 completely solves the problem. This approach is in contrast with that of Theorems 2 and 3. There specific strategies are shown to yield the optimal value of the payoff function, without a claim that those are all the optimal strategies (they are probably not). Also in the last two theorems, no indication is given of how these strategies are obtained. As a matter of fact the strategies presented are generalizations (with the parameter a worked into them) of strategies given in corresponding theorems in RM-791.

TABLE IV

Summary of Results Obtained

Symbol	Value and Strategies			
	$a \leq 1/3$		$a \geq 1/3$	
	Value	Strategies	Value	Strategies
$\mathcal{E}(0)$	$\frac{1}{1-a}$	$A + B = 1$ $\frac{2a}{1+a} \leq A \leq \frac{1-a}{1+a}$ $p_2 = 0, \quad p_3 = 1$	$\frac{2}{1+a}$	$A + B = 1$ $\frac{1-a}{1+a} \leq A \leq \frac{2a}{1+a}$ $p_2 = 1, \; p_3 = 0$
$\mathcal{E}(1)$	$\frac{2-a}{1-a}$	$A = B = 0 \quad C = 1$ $p_2 = p_3 = 0$	$\frac{3+a}{1+a}$	$A = B = 0 \quad C = 1$ $p_2 = p_3 = 0$
$\mathcal{F}(T)$	$\frac{1}{1-a} + \left[\frac{(2-a)(1-a)}{2(1+a)} \min(T, 1-T)\right]$	$p_2 = 1, \; p_3 = p_4 = p'_4 = 0$ $p_5 = \frac{(2-a)(1-a)}{1+a}; \; p_6 = 1$ $q_{-1} = q_1 = \frac{2a}{1+a}$	$\frac{2}{1+a} + \left[\frac{(1-a)^2(3+a)}{2(1+a)^2} \min(T, 1-T)\right]$	$p_2 = 0, \; p_3 = 1, \; p_4 = p'_4 = 0, \; p_6 = 0$ $p_5 = 1 + \frac{(1-a)(3+a)}{2(1+a)}$ $q_{-1} = q_1 = \frac{2a}{1+a}$

As already stated, $\mathcal{E}(1) = 1 + \mathcal{E}(0)$, for all a. From this and from an inspection of Table IV or Theorem 2, for that matter, the following general expression for $\mathcal{F}(T)$ is obtained.

$$\mathcal{F}(T) = \mathcal{E}(0) + \mathcal{E}(1)\,\frac{(1-a)^2}{2(1+a)}\min(T, 1-T) \tag{36}$$

$$= \mathcal{E}(0) + (1 + \mathcal{E}(0))\,\frac{(1-a)^2}{2(1+a)}\min(T, 1-T),$$

for all a.

We can now give an expression for ϕ in terms of all the known functional values. This we obtain by substituting the value of $\mathcal{F}(T)$ given in (36), in (1). We obtain

$$\begin{aligned}\phi = {} & 1 + \frac{1+a}{2}\mathcal{E}(0)(A+C) + \mathcal{E}(1)\frac{(1-a)^3}{4(1+a)}\min(A, C) \\ & + \frac{1-a}{2}\mathcal{E}(0)\,C + p_2\left(\frac{1-a}{2}B - aA\right)\mathcal{E}(0) \\ & + p_3\left[-\frac{1-a}{2}\mathcal{E}(0)(A+C) - \frac{(1-a)^3}{4(1+a)}\mathcal{E}(1)\min(A, C)\right. \\ & \left. + \left(aB + \frac{1-a}{2}C\right)\mathcal{E}\left(\frac{C}{B+C}\right)\right]\end{aligned} \tag{37}$$

This expression can be condensed by using what will be referred to in the rest of the paper as the λ-notation, as follows:

$$\begin{aligned}\phi = {} & 1 + \lambda_1(A+C) + \lambda_2\min(A, C) + \lambda_3 C + p_2(\lambda_3 B - \lambda_4 A) \\ & + p_3\left[-\lambda_3(A+C) - \lambda_2\min(A, C) + (\lambda_5 B + \lambda_6 C)\,\mathcal{E}\left(\frac{C}{B+C}\right)\right]\end{aligned} \tag{38}$$

where the values of the λ's are given by Table V. The expressions of (37) and (38) are valid for all values of a. This makes them extremely useful in the treatment of $\mathcal{E}(S)$ below, since the necessary computations do not depend, as they have until now, on whether a is greater than or less than 1/3.

TABLE V

Values of the λ's.

Symbol	Value: Formula	Value: $a \le 1/3$	Value: $a \ge 1/3$
λ_1	$\frac{1+a}{2}\,\mathcal{E}(0)$	$\frac{1+a}{2(1-a)}$	1
λ_2	$\frac{(1-a)^3}{4(1+a)}\,\mathcal{E}(1)$	$\frac{(1-a)^2\,(2-a)}{4(1+a)}$	$\frac{(1-a)\;(3+a)}{4(1+a)^2}$
λ_3	$\frac{1-a}{2}\,\mathcal{E}(0)$	$\frac{1}{2}$	$\frac{1-a}{1+a}$
λ_4	$a\,\mathcal{E}(0)$	$\frac{a}{1-a}$	$\frac{2a}{1+a}$
λ_5	a	a	a
λ_6	$\frac{1-a}{2}$	$\frac{1-a}{2}$	$\frac{1-a}{a}$

The last few paragraphs represent an effort to remove the split that exists at a = 1/3 in the treatment of $\mathcal{F}$(T), and associated values. Some general expressions were obtained, but still the proofs and computations had to be separated into two cases. However I cannot help feeling that it must be possible to bring forth some more general considerations which will bring the two cases together, but this is all I can say at the moment.

Notice that the split begins with the values and strategies of $\mathcal{E}$(0), which exerts such an influence on the remaining computations, that the split in values of $\mathcal{F}$(T) and $\mathcal{E}$(1) are entirely accountable in terms of the difference in $\mathcal{E}$(0). Not so regarding strategies. In the case of $\mathcal{E}$(1) there is no split as can be seen from Table IV. The same table shows however that strategies vary from the case of $a \leq 1/3$ to $a \geq 1/3$ for Configuration II in spite of the fact that the expression of $\mathcal{F}$(T) in terms of $\mathcal{E}$(0) is the same in both cases.

One final remark before we plunge into the computation of $\mathcal{E}$(S). It was pointed out at the outset that P will be allowed to stand still, at variance with RM-791. This is represented by p'_4 in Configuration II. From Theorem 3, it turns out that there is an optimal strategy for which $p'_4 = 0$, which constitutes some evidence that Isaacs' prescription that P stand still constitutes no essential restriction. Notice, however, that Theorem 3 does <u>not</u> show that $p'_4 = 0$ for <u>all</u> optimal strategies.

<u>Determination of</u> $\mathcal{E}$(S).

In this section I have extended (as far as has been possible) the computations appearing in RM-791 for $\mathcal{E}$(S) to the general case treated here. Very few things are proved, since proofs for them do not exist to the best of my knowledge. It is merely shown that if the computations for a = 1/3 are repeated for an arbitrary a the given results are obtained. Therefore, the formal approach of the previous sections will be abandoned. A few computations and several remarks will be given instead of lemmas and theorems.

The fact that $a = 1/3$ is an exception will become still more evident, since it turns out that $\mathcal{E}(S)$ is not composed of linear pieces for the general case as it is for $a = 1/3$.

We first transform $\emptyset$ from (38) into a more suitable form for the purpose of computation. We first substitute the value of B as $1 - A - C$ and obtain:

$$(39) \qquad \emptyset = 1 + \lambda_1(A + C) + \lambda_2 \min(A, C) + \lambda_3 C + p_2\Big[\lambda_3(1 - C) - (\lambda_3 + \lambda_4)A\Big] + p_3\Big\{ - \lambda_3 (A + C) - \lambda_2 \min(A, C) + \Big[\lambda_5(1 - A) + (\lambda_6 - \lambda_5) C\Big] \mathcal{E}\Big(\frac{C}{B + C}\Big) \Big\}$$

Then:

$$(40) \qquad \mathcal{E}(S) = \max_{0 \le C \le S} \mathcal{M}(C) \quad \text{where}$$

$$(41) \qquad \mathcal{M}(C) = 1 + (\lambda_1 + \lambda_3) C + \max_{0 \le A \le 1-C} \theta(A, C)$$

and

$$(42) \qquad \theta(A, C) = \lambda_1 A + \lambda_2 \min(A, C) + \min_{p_2, p_3}\Big\{ p_2 \Big[\lambda_3(1 - C) - (\lambda_3 + \lambda_4)A\Big] + p_3 \lambda(A, C)\Big\}$$

and finally

$$(43) \qquad \lambda(A, C) = -\lambda_3(A + C) - \lambda_2 \min(A, C) + \Big[\lambda_5(1 - A) + (\lambda_6 - \lambda_5) C\Big] \mathcal{E}\Big(\frac{C}{B + C}\Big).$$

The first step is computing $\mathcal{M}(C)$. This is done following the same steps described in Isaacs' paper.

Consider the number $C_1 = \dfrac{\lambda_2 + \lambda_3}{\lambda_2 + \lambda_6 \mathcal{E}^{(1)}}$. It can be shown that this number is $\geq \frac{1}{2}$ by substituting the values of the λ's from the second column of Table V, and the formula $\mathcal{E}(1) = 1 + \mathcal{E}(0)$, and noticing that $\mathcal{E}(0) \geq 1$.

Then for C in the interval $C_1 \leq C \leq 1$ the value of $\mathcal{M}(C)$ is given by:

(44) $$\mathcal{M}(C) = 1 + \lambda_1 + \lambda_2 - \lambda_4 + (\lambda_3 + \lambda_4 - \lambda_2)\,C$$

and the strategies are:

$A = 1 - C, \qquad p_3 = 0, \qquad p_2 = 1.$

The proof of this will only be sketched, since it again represents a repetition of the proof for a = 1/3, and since it rests on a principle which has been used over and over in this report.

By setting $p_2 = 1$, $p_3 = 0$ in (42), we see that the maximum for this particular set of values of p_2 and p_3 occurs at A = 1 - C and the value obtained in (42) is

(45) $$(\lambda_1 + \lambda_2 - \lambda_4)\,(1 - C)$$

This is obtained using the fact that $C \geq \frac{1}{2}$ which is equivalent to $1 - C \leq C$. So we see that $\max \theta(A, C) \geq$ this value.

On the other hand substituting A = 1 - C in (42) and (43), we see in (43) the $\mathcal{E}\left(\dfrac{C}{1 - A}\right)$ becomes $\mathcal{E}(1)$. Then for C in the above interval, $\lambda(1 - C_1, C) \geq 0$, and the minimizing p_3 in (42) is 0. On the other hand, the coefficient of $p_2 \leq 0$ and $p_2 = 1$ is the minimizing value, and so the value obtained is (45), which then must be equal to max $\theta(A,C)$. Substituting this value in (41) we obtain the value (44).

To perform the evaluation of $\mu(C)$ in the next interval, applying the method described by Isaacs, it is necessary to make the following assumptions:

1) $\mu(C)$ is increasing everywhere and therefore $\mathcal{E}(S) = \mu(S)$.

2) For every value F in the interval $C_1 \leq F \leq 1$, there is a pair of values A and C such that $F = \frac{C}{1 - A}$, $A \leq C$, and

$$\mu(C) = 1 + (\lambda_1 + \lambda_3) C + \theta(A, C).$$

It might be possible to prove the validity of these assumptions by a more detailed analysis of $\mu(C)$, but I know of no way of doing it. They are not shown to my satisfaction in RM-791.

However, by using them we can show that there is a 1-1 correspondence between the values of F and those of C defined in assumption 2) and that while F describes the interval $C_1 \leq F \leq 1$, C describes an interval $C_2 \leq C \leq C_1$ and it is possible to compute C_2. Also the A discussed in assumption 2) can be computed as a function of C.

In what remains of this section, it will be assumed that $a \geq 1/3$. The whole argument that follows is valid for an arbitrary a, except for the assumption that expression (51) ≤ 0. This inequality is valid for $a \geq 1/3$, but not for an arbitrary a, and the main conclusion depends on this assumption. So the assumption that $a \geq 1/3$ may be replaces by the assumption that the expression (51) ≤ 0.

Consider a pair of values A, C for which $C_1 \leq \frac{C}{1 - A} \leq 1$. Then by assumption 1), $\lambda(A, C)$ assumes the form:

(46) $$\lambda(A, C) = -\lambda_3(A + C) - \lambda_2 \min (A, C) + \left[\lambda_5(1 - A) + (\lambda_6 - \lambda_5) C\right] \left[(1 + \lambda_1 + \lambda_2 - \lambda_4) + (\lambda_3 + \lambda_4 - \lambda_2) \frac{C}{1 - A}\right]$$

Consider a fixed value of C and an arbitrary value of A satisfying the inequality

(47) $$C_1 \leq \frac{C}{1 - A} \leq 1 \quad .$$

If for that value, $\lambda(A, C) > 0$, then it is clear from (42) that the minimizing value of p_3 is 0, and then $\theta(A, C)$ is an increasing function of A. This is clear if $p_2 = 0$ (minimizing value), since then $\theta(A, C) = \lambda_1 A + \lambda_2 \min (A, C)$ an increasing function of A. If the minimizing p_2 is 1, then $\theta(A, C) = (\lambda_1 - \lambda_3 - \lambda_4)A + \min (A, C) = \min (A, C)$, an increasing function of A.

On the other hand, if $\lambda(A, C) \leq 0$ for the prescribed values, then the minimizing p_3 is 1 and the terms involving A are:

(48) $$(\lambda_1 - \lambda_3 - \lambda_5 (1 + \lambda_1 + \lambda_2 - \lambda_4)) A - p_2 (\lambda_3 + \lambda_4) A + (\lambda_6 - \lambda_5) (\lambda_3 + \lambda_4 - \lambda_2) \frac{C^2}{1 - A}$$

or

(49) $$-p_2(\lambda_3 + \lambda_4) A + \left[\lambda_1 - \lambda_3 - \lambda_5(1 + \lambda_1 + \lambda_2 - \lambda_4)\right] A + (\lambda_6 - \lambda_5) (\lambda_3 + \lambda_4 - \lambda_2) \frac{C^2}{1 - A}$$

Plainly the first term is a decreasing function of A. It is my purpose to show that the last two terms form a decreasing function of A, and therefore $\theta(A, C)$ is decreasing.

Taking the derivative with respect to A of these two terms we get

$$(50) \qquad \lambda_1 - \lambda_3 - \lambda_5 (1 + \lambda_1 + \lambda_2 - \lambda_4) + (\lambda_6 - \lambda_5)(\lambda_3 + \lambda_4 - \lambda_2) \frac{C^2}{(1-A)^2}$$

$$\leq (\lambda_1 - \lambda_3) - \lambda_5 (1 + \lambda_1 + \lambda_2 - \lambda_4) + (\lambda_6 - \lambda_5)(\lambda_3 + \lambda_4 - \lambda_2).$$

This is so because $\frac{C}{1 - A} \leq 1$.

The right hand member becomes

$$(51) \qquad \lambda_1 + \lambda_6(\lambda_3 + \lambda_4 - \lambda_2) \quad - \lambda_3 - \lambda_5(1 + \lambda_1 + \lambda_3) .$$

It can be shown (See appendix) that (51) is ≤ 0. Then, this shows that (49) is a decreasing function of A, and so is $\theta(A, C)$.

To summarize if $\lambda(A, C) \geq 0$, $\theta(A, C)$ is increasing with A, and if $\lambda(A, C) \leq 0$, $\theta(A, C)$ is decreasing with A. It follows that if $\lambda(A, C)$ vanishes at all for values of A, for which $A \leq C$, $C_1 \leq \frac{C}{1 - A} \leq 1$, (assuming C fixed), then any such value maximizes $\theta(A, C)$ and $\mu(C) = 1 + (\lambda_1 + \lambda_3)C + \theta(A, C)$, for that value of A. Our next task is determining for what values of C, the corresponding values of A exist, and to compute them. We start by setting (46) equal to zero.

To simplify the computations let us introduce the following notations:

(52) a) $\mu_1 = 1 + \lambda_1 + \lambda_2 - \lambda_4$

b) $\mu_2 = \lambda_3 + \lambda_4 - \lambda_2$

c) $\mu_3 = \lambda_6 - \lambda_5$

d) $\mu_4 = \lambda_2 + \lambda_3$.

Then with this notation, (46) equated to zero becomes

(53) $$-\lambda_3(A + C) - \lambda_2 A + \left[\lambda_5(1 - A) + \mu_3 C\right]\left[\mu_1 + \mu_2 \frac{C}{1 - A}\right] = 0.$$

Also the fact that $A \leq C$ is used. Multiplying and regrouping we get

(54) $$-\mu_4 A + (\mu_1 \mu_3 + \lambda_5 \mu_2 - \lambda_3) C + \lambda_5 \mu_1 (1 - A) + \mu_2 \mu_3 \frac{C^2}{1 - A} = 0.$$

This is clearly a quadratic equation in both A and C. At this point a change of variables will simplify matters. We replace the variables A, C by $\frac{C}{1 - A}$, C. So let

(55) $$F = \frac{C}{1 - A}.$$

With this substitution (54) becomes

(56) $$-\mu_4(F - C) + (\mu_1 \mu_3 + \lambda_5 \mu_2 - \lambda_3) CF + \lambda_5 \mu_1 C + \mu_2 \mu_3 CF^2 = 0.$$

which is evidently linear in C. Accordingly, we can solve for C:

(57) $$C = \frac{\mu_4 F}{\mu_2 \mu_3 F^2 + (\mu_1 \mu_3 + \lambda_5 \mu_2 - \lambda_3) F + \mu_4 + \lambda_5 \mu_1}$$

Finally we introduce a notation to replace the λ - notation and the μ -notation. It will be referred to η -notation.

(58) a) $\eta_1 = \mu_4 = \lambda_2 + \lambda_3$

b) $\eta_2 = \mu_2 \mu_3 = (\lambda_6 - \lambda_5)(\lambda_3 + \lambda_4 - \lambda_2)$

$$\text{c)}\quad \eta_3 = \mu_1 \mu_3 + \lambda_5 \mu_2 - \lambda_3$$

$$= (\lambda_6 - \lambda_5)(1 + \lambda_1 + \lambda_2 - \lambda_4)$$

$$+ \lambda_5(\lambda_3 + \lambda_4 - \lambda_2) - \lambda_3$$

$$\text{d)}\quad \eta_4 = \mu_4 + \lambda_5 \mu_1$$

$$= \lambda_2 + \lambda_3 + \lambda_5(1 + \lambda_1 + \lambda_2 - \lambda_4) .$$

Then (57) becomes

(59) $$C = \frac{\eta_1 F}{\eta_2 F^2 + \eta_3 F + \eta_4}$$

This gives C as a function of F in the interval $C_1 \leq F \leq 1$. From (55) we obtain A as a function of F, thus:

(60) $$A = 1 - \frac{\eta_1}{\eta_2 F^2 + \eta_3 F + \eta_4}$$

It would be desirable at this point to show that $A \leq C$. This is impossible, since it can be shown that there are values of a for which this is not so. To see this, notice that the η's are algebraic expressions in a. So for a fixed F, A, C are continuous functions of a. Now for a = 1, by substituting from Table V and from (58) we get:

(61) $$C = 0, \qquad A = 1, \qquad A > C.$$

And although we exclude a = 1, there must be values of a close enough to 1 for which $A > C$ holds. So we only deal with those values of a for which $A \leq C$. What those values are however, I have not been able to determine.

The next task is to show that as F describes the interval $C_1 \leq F \leq 1$, C describes some interval. For this call the function to the right of (59) L(F). We first show $\frac{dL}{dF} > 0$, and then evaluate L at the end points. We first consider the function $\eta_2 F^2 + \eta_3 F + \eta_4 = F(\eta_2 F + \eta_3) + \eta_4$. If $\eta_2 F + \eta_3 \geq 0$, then the function $\geq \eta_4 = \lambda_2 + \lambda_3 + \lambda_5(1 + \lambda_1 + \lambda_2 - \lambda_4)$. It is easy to see from Table V that this expression is positive for values of $a \neq 1$. Again if $\eta_2 F + \eta_3 < 0$ then the function $\geq \eta_2 F + \eta_3 + \eta_4$. If $\eta_2 \geq 0$ then this in turn is $\geq \eta_3 + \eta_4$. Again by substituting in the values of η_3 and η_4 we can see that $\eta_2 F^2 + \eta_3 F + \eta_4 > 0$. Finally if $\eta_2 < 0$, then $\eta_2 F^2 + \eta_3 F + \eta_4 \geq \eta_2 + \eta_3 + \eta_4 = \lambda_2 + \lambda_6(1 + \lambda_1 + \lambda_3) > 0$ for $a \neq 1$. Summarizing then, $\eta_2 F^2 + \eta_3 F + \eta_4 > 0$ for all F and all $a \neq 1$. Therefore L(F) is well defined, and therefore the corresponding values of C and A are also well defined.

We now show $\frac{dL}{dF} > 0$.

(62) $$\frac{dL}{dF} = \frac{\eta_4 - \eta_2 F^2}{[\eta_2 F^2 + \eta_3 F + \eta_4]^2}$$

The denominator is positive as we have shown. If $\eta_2 < 0$ then the numerator is $> \eta_4$, and we saw η_4 to be positive. On the other hand if $\eta_2 \geq 0$ then the numerator $\geq \eta_4 - \eta_2$. Now:

(63) $$\eta_4 - \eta_2 = \lambda_2 + \lambda_3 + \lambda_5(1 + \lambda_1 + \lambda_3) - \lambda_6(\lambda_3 + \lambda_4) + \lambda_6 \lambda_2.$$

It is evident that to show that $\lambda_4 - \lambda_2 > 0$, it suffices to show that $\lambda_3 - \lambda_6(\lambda_3 + \lambda_4) > 0$.

$$(64) \qquad \lambda_3 - \lambda_6(\lambda_3 + \lambda_4) = \left[\frac{1-a}{2} - \frac{1-a}{2}\,\frac{1+a}{2}\right]\mathcal{E}(0)$$

$$> \left[\frac{1-a}{2} - \frac{1-a}{2}\right]\mathcal{E}(0) = 0$$

for $a < 1$. Therefore $\frac{dL}{dF} > 0$.

Let

$$(65) \qquad C_2 = L(C_1) = \frac{\eta_1 C_1}{\eta_2 C_1^2 + \eta_3 C_1 + \eta_4}$$

Let us examine L(1) more closely:

$$(66) \qquad L(1) = \frac{\eta_1}{\eta_2 + \eta_3 + \eta_4}$$

We have already seen that the denominator of (66) equals $\lambda_2 + \lambda_6(1 + \lambda_1 + \lambda_2)$. From Table V we see that $1 + \lambda_1 + \lambda_3 = \mathcal{E}(1)$. So:

$$(67) \qquad L(1) = \frac{\lambda_2 + \lambda_3}{\lambda_2 + \lambda_6\,\mathcal{E}(1)} = C_1.$$

Thus we see that as F describes the interval $C_1 \leq F \leq 1$, C describes the interval $C_2 \leq C \leq C_1$, and the maximizing values of A are given by (60). It remains to compute $\mu(C)$ and this will give us the value of $\mathcal{E}(S)$ in this interval, since we are assuming that $\mathcal{E}(S) = \mu(S)$. We evaluate $\mu(C)$ by turning back to (41) and (42). There are two cases to distinguish: (a) $A > \frac{\lambda_3}{\lambda_3 + \lambda_4}(1 - C)$ and (b) $A \leq \frac{\lambda_3}{\lambda_3 + \lambda_4}(1 - C)$. In the first case we see from (42) that the minimizing value of $p_2 = 1$, and

$$(68) \qquad \mu(C) = \Omega(F) = 1 + \lambda_3 + \lambda_1 C(F) + (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4)A(F)$$

where A(F) and C(F) are given by (60) and (59) respectively.

In the second case $\mathcal{M}(C)$ is given

(69) $$\mathcal{M}(C) = \Omega(F) = 1 + (\lambda_1 + \lambda_2)A(F) + (\lambda_1 + \lambda_3)C(F)$$

Both cases (a) and (b) occur for suitable values of a. A precise determination of these values has not been possible. It is even possible for the same value of a to have both cases, depending on the value of F. Thus (68) and (69) give the value of $\Omega(F)$ in the interval $C_1 \leq F \leq 1$ or what is the same, $\mathcal{M}(C)$ in the interval $C_2 \leq C \leq C_1$. We have thus succeeded in evaluating $\mathcal{M}(C)$ and therefore $\mathcal{E}(S)$ in the interval $[C_2, 1]$.

To summarize, $\mathcal{E}(S)$ in the closed interval from C_2 to 1 is obtained by setting $\mathcal{E}(S) = \mathcal{M}(S)$ and is given in the interval $[C_2, C_1]$ in case (a) by (68), and in case (b) by (69).

Supplement: Some Proposed 2-dimensional Extensions of Isaacs' Game.

The following is a brief account of several extensions to the plane of the pursuit game of RM-791, proposed in Project M720-1. It represents partly my own work, but mostly that of persons whose names are attached to the specific games. The brief attention given to these games was mostly of a descriptive non-quantitive nature.

The first attempts, proposed by R. M. Thrall, consisted in subdividing the plane by a system of congruent regular polygons, and having E move from one polygon to an adjacent or remain stationary, and P move from one polygon to an adjacent polygon, to one adjacent to an adjacent polygon, or remain stationary. A signal would consist of a polygon and those adjacent to it, with a set of numbers indicating the conditional probability that E be at each one of them, given that E is at one of them.

Of these the case of triangles, squares and hexagons were considered specifically. The square and hexagonal variations were abandoned almost immediately because they appeared to be hopelessly complicated. The game with triangles was considered in slightly more detail. It is to be played on an infinite arrangement, a portion of which appears in Figure 4. Here E and P are to move across the sides of the triangles, E across one side, P across two at most. This is somewhat unrealistic. For example, notice that triangle 38 in Figure 4 is "next" to 39. Yet, neither E nor P can get from 38 to 39. This perhaps could be remedied by prescribing that the players may also move through vertices as well as across sides. This however has not been considered. Again, from Figure 4, assuming E is at 38, the set of signals may be represented in Figure 5 below:

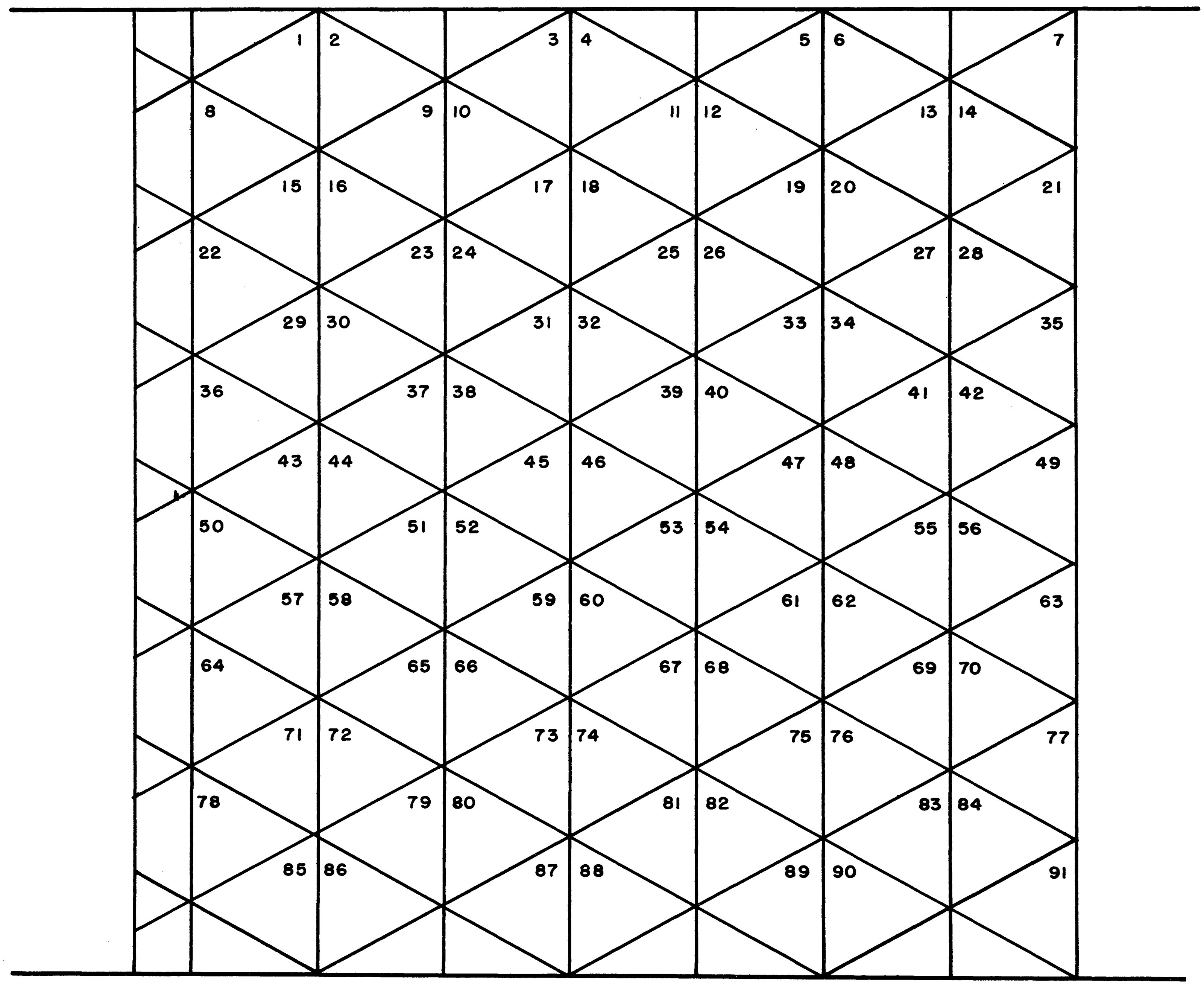

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60
61
62
63
64
65
66
67
68
69
70
71
72
73
74
75
76
77
78
79
80
81
82
83
84
85
86
87
88
89
90
91

FIG. 4

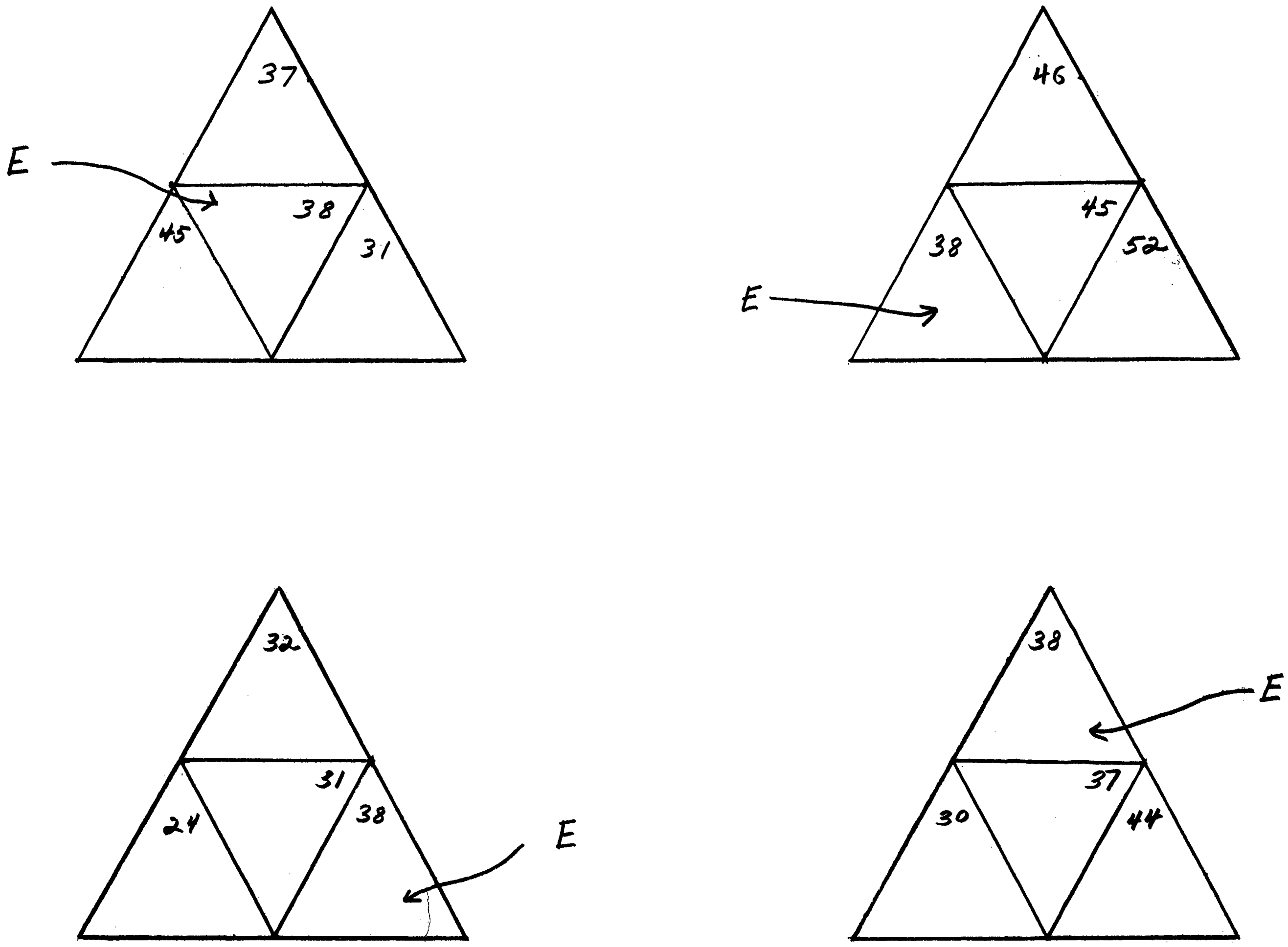


Figure 5

If we are to search for a closed set of configurations, we find, contrary to the one-dimensional game, that there is no one such set which is evidently better than any other. In fact there are many closed sets of configurations, and no way of telling beforehand which one is best. But the union of closed sets is closed, therefore there is a maximal closed set of configurations. It is with this set that we must work. An example of a closed set of configurations (which is not maximal) is given in Figure 6.

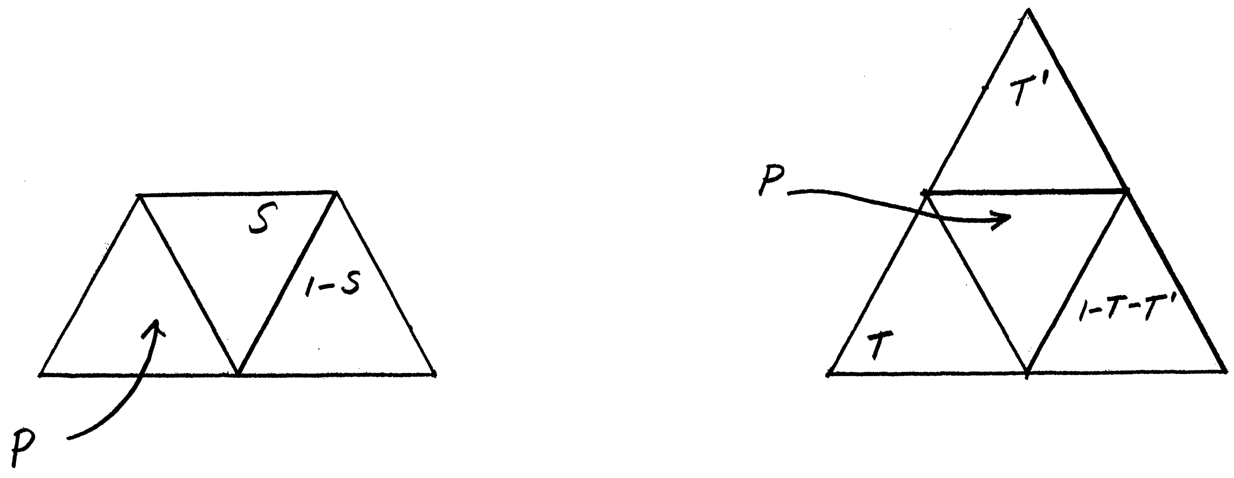


Figure 6 has an explanation similar to that of Figures 1 and 2. The arrow indicates the position of P; S, 1 - S, T, T', 1 - T - T' indicate the probability of E being at the corresponding triangles.

The following game was suggested by W. Hoffman of Project M720-1, as perhaps a simpler generalization. It is actually a one-dimensional arrangement in two dimensions. The board consists of an infinite rectangular lattice, a portion of which is illustrated in Figure 7.

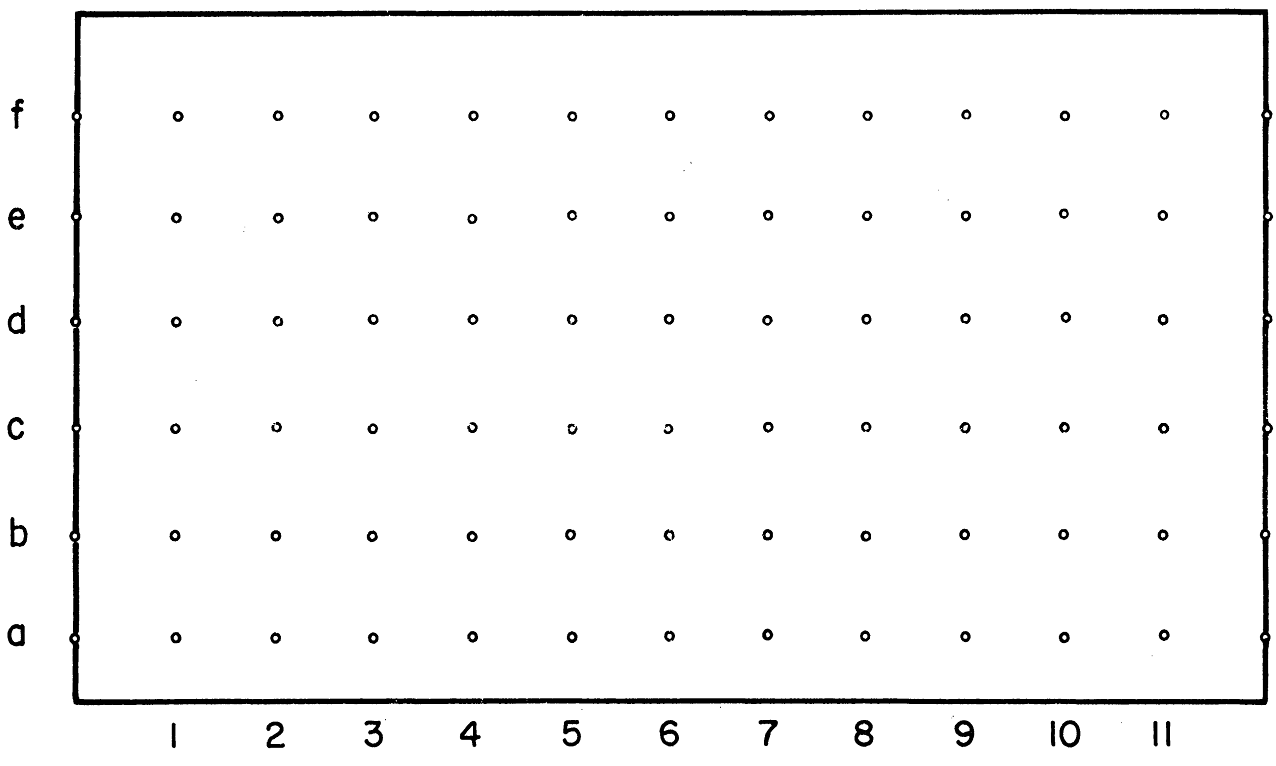


FIG. 7

Here two points can be defined to be adjacent if they belong to a square, with sides parallel to the edges of the figure, whose four sides contain (altogether) only four points. Then, with this definition of adjacent we can give the usual rules for the movement of P and E from point to point. The signal given for this game is such that it indicates both the position of E and the direction of movement. From Figure 7, it can be seen that E can get to any point from four different directions (along four lines). Each direction indicates a set of three signals. So there are twelve possible signals, although once the direction is determined, only three of them are possible. The signals may best be described as follows: Given E to be at some point, and the line along which it moves to it. The points on this line (one of which is the point at which E is) form a lineal set of the kind used in the main portion of this report and in RM-791. Then the choice of signals is that prescribed on that line by the one-dimensional game. If E remains stationary, the direction is assumed to be that of the previous move. To make the game completely unambiguous, an initial direction may be determined by a chance device. An example will serve to illustrate and clarify this. In Figure 7, suppose E has arrived at point (d, 6) from (c,5). Then the possible signals are:

$$\sigma_1 : \left[(b, 4), (c, 5), (d, 6)\right]$$

$$\sigma_2 : \left[(c, 5), (d, 6), (e, 7)\right]$$

$$\sigma_3 : \left[(d, 6), (e, 7), (f, 8)\right] .$$

If on the other hand, E had arrived at (d, 6) from (e, 6), then the signals would have been

$$\sigma_1 : [(b, 6), (c, 6), (d, 6)]$$

$$\sigma_2 : [(c, 6), (d, 6), (e, 6)]$$

$$\sigma_3 : [(d, 6), (e, 6), (f, 6)] .$$

It would appear that this scheme allows E a great deal of mobility. It turns out that this mobility lasts only as long as P is not close enough for capture. As soon as E changes direction it reveals the position from which such a change occurred (the intersection of the line of the previous signal with that of the present one); and that two of the three choices for the present signal reveal his position unambiguously. This property will probably make a solution simpler. It may even be possible to give the solution in terms of solutions to one-dimensional games. This possibility has not been investigated.

Regarding closed sets of configurations, it turns out, as should be expected, that configurations I and II of the one-dimensional game, form a closed set here too. But there are others, and something like a maximal set may be necessary.

<u>Proof of the inequality</u> $\mathcal{F}(T) - \mathcal{E}(1)\, T \geq 0$.

This proof is done as a supplement to the proof of theorem 2. So it will only be done for $a \leq 1/3$. Also according to previous remarks. it can be assumed that $T \leq \frac{1}{2}$.

Suppose $\mathcal{F}(T) - \mathcal{E}(1)\, T < 0$. Then, going back to the proof of Theorem 2, and to (18) we see that in that case, the minimizing value of p_4' is 1. Then we have:

(70) $$\mathcal{F}(T) \geq 1 + \frac{1-a}{2}\mathcal{E}(0) + \frac{(1-a)^2}{2(1+a)}\mathcal{F}(T),$$

(71) $$\mathcal{F}(T) \geq \frac{3}{2}\left(\frac{1}{1 - \frac{(1-a)^2}{2(1+a)}}\right)$$

This gives a lower bound for $\mathcal{F}(T)$, and (16) obtained in the first part of the proof of Theorem 2 gives an upper bound. Turning now to (1) we find, using (16) that:

(72) $$\mathcal{E}(1) \leq \max_{A,C}\left\{1 + \frac{1-a}{2}(A+C)\left[\frac{1}{1-a} + \mathcal{E}(1)\frac{(1-a)^2}{2(1+a)}\frac{\min(A,C)}{A+C}\right] + \left(aA + \frac{1+a}{2}C\right)\mathcal{E}(0)\right\} \leq 1 + \frac{1-a}{2}\left[\frac{1}{1-a} + \mathcal{E}(1)\frac{(1-a)^2}{2(1+a)}\left(\frac{1}{2}\right)\right] + \left(a + \frac{1+a}{2}\right)\frac{1}{1-a}$$

$$\leq \frac{3}{2} + \frac{(1-a)^2}{4(1+a)}\mathcal{E}(1) + \frac{1+3a}{2(1-a)}$$

Finally:

(73)
$$\mathcal{E}(1) \leq \frac{\frac{2}{1-a}}{1 - \frac{(1-a)^2}{4(1+a)}}$$

It is quite clear that

(74)
$$\frac{1}{2}\,\frac{\frac{2}{1-a}}{1-\frac{(1-a)^2}{4(1+a)}} = \frac{1}{1-a}\,\frac{1}{1-\frac{(1-a)^2}{4(1+a)}} < \frac{3}{2}\left(\frac{1}{1-\frac{(1-a)^2}{2(1+a)}}\right).$$

Therefore

(75)
$$T\,\mathcal{E}(1) \leq \tfrac{1}{2}\,\mathcal{E}(1) \leq \frac{1}{1-a}\left(\frac{1}{1-\frac{(1-a)^2}{4(1+a)}}\right) < \frac{3}{2}\left(\frac{1}{1-\frac{(1-a)^2}{2(1+a)}}\right) \leq \mathcal{F}(T)$$

a contradiction. This establishes

(76)
$$\mathcal{F}(T) - T\,\mathcal{E}(1) \geq 0. \quad \text{Q. e. d.}$$

A similar proof holds for $a \geq 1/3$.

<u>Proof of the inequality</u>

$$\lambda_1 + \lambda_6(\lambda_3 + \lambda_4 - \lambda_2) - \lambda_3 - \lambda_5(1 + \lambda_1 + \lambda_3) \leq 0$$

This inequality which involves (51), arose in connection with the evaluation of $\mathcal{E}(S)$ in the interval $C_2 \leq C \leq C_1$. The proof is valid only for $a \geq 1/3$. Values of a can be found close enough to 0 for which the inequality runs in the opposite direction.

By substituting the values of the λ's from Table V into (51) and simplifying, the inequality becomes

(77)
$$\frac{1-a}{2}\left[\frac{1+a}{2}\,\mathcal{E}(0) - \frac{(1-a)^3}{4(1+a)}\,\mathcal{E}(1)\right] - a \leq 0.$$

Now let $a \geq 1/3$. The left hand side becomes

$$\frac{1-a}{2}\left[1 - \frac{(1+a)^3(3+a)}{4(1+a)^2}\right] - a \leq \frac{1-a}{2} - a \leq 0$$

which proves the inequality.