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AN INVESTIGATION OF THE EXACT SOLUTIONS OF THE LINEARIZED EQUATIONS FOR THE FLOW PAST CONICAL BODIES

Part III.

Supersonic Flow Past an Elliptic Cone At an Angle of Attack

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PREFACE

This is the fifth of a series of reports submitted under this title. In order to make clear to the reader the purpose of this publication, it is perhaps not superfluous to review the previous reports.

In April, 1947, the authors submitted Part I, Supersonic Flow Past an Arrow Wing at an Angle of Attack.

In this report, the general linearized method was explained and the following essentially new features were introduced: (a) The irrotational character of the flow was taken into account through formulae dubbed "Weierstrass formulae" and a new complex potential was introduced. (b) Angles of attack or yaw were shown to be amenable to quantitative consideration by means of the Lorentz transformations, well known in relativity theory. (c) The introduction of hyperbolic stereographic parameters made the methods of conformal mapping applicable. (d) The theory of (a), (b) and (c) was then applied to the case of a delta wing. It was found that an infinity of solutions were compatible with the conditions of the problem, and that uniqueness could only be attained by imposing a further condition, that of finiteness of lift.

In July, 1947, the authors submitted Part II, Supersonic Flow Past an Elliptic Cone at Zero Angle of Attack.

In this report, the fundamental differential equation was shown to be separable in terms of curvilinear coordinates, one of whose families of surfaces is an elliptic cone. The solution was expressed in terms of a Fourier series for the downstream velocity component w whose components were obtained as solutions of an infinite system of linear equations.

In February, 1948, Bumblebee Report No. 75 was published. It contained the material of Parts I and II, much unified and simplified, especially by the use of the transformation theory of elliptic functions.

In May, 1948, a report, CM471, entitled: "Supersonic Flow Past a Delta Wing at Angles of Attack and Yaw" was submitted by the present authors. In it the delta wing, when arbitrarily inclined with respect to the windstream, was treated by the method of the previous publications. However, the treatment given there excludes the case of a trailing edge. The treatment of an angle of yaw great enough to produce a trailing edge and therefore a vortex sheet is reserved to a future report.

In the present report, the methods developed are applied to the elliptic cone at an angle of attack. This is possible by fusing the method of Part II with the idea of the Lorentz transformation. The approach used here is the three-dimensional one of Part II. It is obvious that for an elliptic cone there exists no principle difference between an angle of attack and one of yaw. However, in this report the cone is always supposed to be rotated around an axis parallel to one of the major axes of the cross sectional ellipse. The extension of the theory to encompasss the case where simultaneously an angle of attack and of yaw exist is obvious. Since it offers no new difficulty other than one of greater complexity, it has been left aside here.

The authors wish, at this time, to express their indebtedness to the following co-workers: Homer W. Schamp, Jr. who laid the groundwork of the calculations and Roderick E. Reid who did the bulk of the numerical work. It was also of considerable help to us to be able to use a network calculation machine owned by the Dow Chemical Company of Midland, Michigan. The courtesy of Dr. Jason Saunderson of that company was greatly appreciated.

AN INVESTIGATION OF THE EXACT SOLUTIONS OF THE LINEARIZED EQUATIONS FOR THE FLOW PAST CONICAL BODIES

PART III. SUPERSONIC FLOW PAST AN ELLIPTIC CONE AT AN ANGLE OF ATTACK

PART I: DETERMINATION OF THE MATHEMATICAL CONE PARAMETERS FOR A GIVEN FLOW FIELD

1. Determination of the Elliptic Function Parameters k and y_o for a Given Cone at Angle of Attack Zero

The elliptic cone has, when referred to a coordinate system whose Z-axis is its axis of symmetry, the equation:

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} - Z^2 = 0 \qquad A > B \tag{1.1}$$

On the other hand, the hydrodynamical equations demand the parametrization of space by a triply infinite system of surfaces, as was described in Part I, Sections 2 and 4, and in B.B. Rep. p. 58. The surfaces that concern us at this moment are the surfaces y = const. , which are elliptic cones around the Z-axis:

$$\frac{\beta^2 X^2}{k^2 nd^2(y,k')} + \frac{\beta^2 Y^2}{k^2 sd^2(y,k')} - Z^2 = 0 \tag{1.2}$$

In this equation β is, as always, related to the Mach number by

$$\beta^2 = M^2 - 1 \,,$$

k is the modulus of the elliptic functions, and $k' = \sqrt{1 - k^2}$. By varying y and keeping k constant, a one-parameter family of cones is obtained, such that for $y = 0$ the delta wing formed by the two straight lines

$$\frac{\beta X}{k} = \pm Z \tag{1.3}$$

results; while as y increases toward K' (the complete elliptic integral belonging to the modulus k), the cones become gradually rounder and finally merge with the circular Mach cone:

$$\beta^2 (X^2 + Y^2) - Z^2 = 0 \tag{1.4}$$

From (1.3) it is clear that as k is varied, a different one-parameter family (namely one belonging to a different delta wing) is generated. It is clear that for a given Mach number and a given material cone characterized by A and B the elliptic parameters k and y_o can be determined; they are found by comparing (1.1) and (1.2) through the relations:

$$A = \frac{k}{\beta} \, nd(y_o, k') \qquad B = \frac{k}{\beta} \, sd(y_o, k') \tag{1.5}$$

For practical purposes the inverse of these is needed, since one desires to find k and y_o for a cone with <u>given</u> A and B, and a <u>given</u> β. By using the identities between the elliptic functions one can eliminate y_o to obtain:

$$k^2 = \frac{\frac{1}{B^2} - \frac{1}{A^2}}{\frac{1}{A^2}\left(\frac{1}{\beta^2 B^2} - 1\right)} \, , \qquad k'^2 = 1 - k^2 . \tag{1.6}$$

In view of the inequalities

$$A > B \, , \quad \beta A < 1 \, ,$$

it follows from the foregoing expressions that the modulus is restricted in accordance with the inequality:

$$0 < k^2 < 1 \, .$$

Having determined the modulus, one merely has to consult the tables[1] to obtain y_o from:

$$\mathrm{sn}\,(y_o, k') = \frac{B}{A} \, . \tag{1.7}$$

Equations (1.6) and (1.7) enable one to find the variables k and y_o as functions of the experimental quantities A, B, and β.

2. Behavior of a Cone under Euclidean Rotation; Various Types of Contact with Mach Cone

When the cone is at an angle of attack α its equation is no longer (1.1) but somewhat more complicated. The coordinate system X, Y, Z is rotated with respect to another coordinate system $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$. While mathematically there

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[1] Die ellipischen Funktionen von Jacobi, by L. M. Milne - Thompson, Berlin, 1931. See also: Smithsonian Elliptic Function Tables by G. W. and R. M. Spenceley, Washington, 1947.

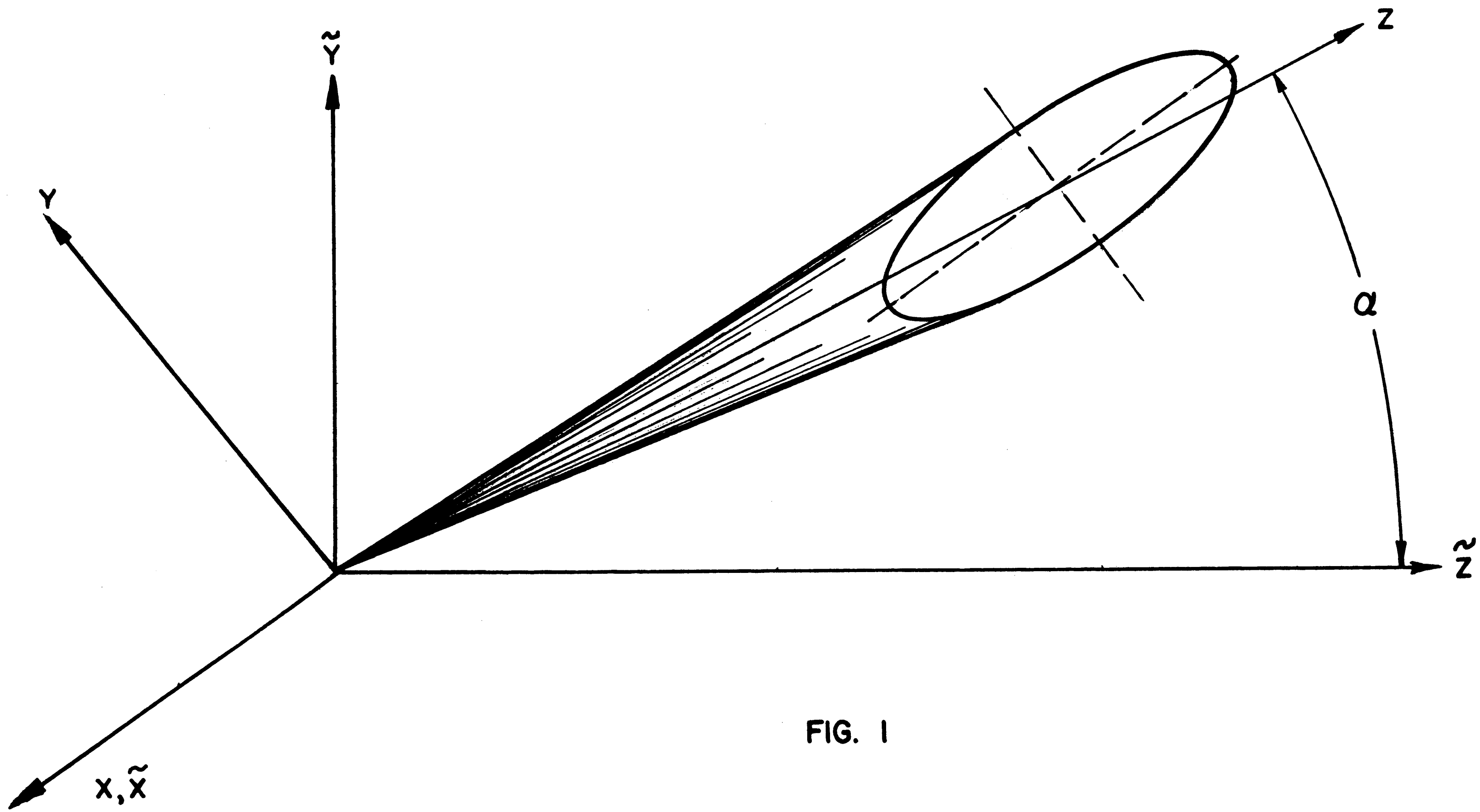


FIG. 1

is no restriction whatever on the magnitude of α we are, for physical reasons, only interested in angles of attack which will nevertheless confine the material cone to the interior of the Mach cone

$$\tilde{X}^2 + \tilde{Y}^2 - \frac{1}{\beta^2}\tilde{Z}^2 = 0. \tag{1.9}$$

For reasons which will become clear in the following sections, it is of some interest to investigate briefly the limiting case when the material cone is tangent to the Mach cone. According as the material cone is close in shape to the delta wing ($B \ll A$) or, on the other hand, close in shape to a circular cone ($B \approx A$) it is clear that the two cones will touch in two symmetrically arranged rays or in one ray in the $\tilde{Y}$, $\tilde{Z}$ plane. The following Figure 2 illustrates these situations by depicting simply the intersection of both cones with the plane $\tilde{Z} = 1$. The Mach cone is shown as a circle of radius $1/\beta$; and three different material cones, of identical A but increasing B are shown, first in the center for $\alpha = 0$, then in contact with the Mach cone. The two-point contact (two-ray contact in space) is shown as case 1; case 3 is the case where the ellipse and the circle touch in the point $\tilde{X} = 0$, $\tilde{Y} = 1/\beta$.

Now from the point of view of differential geometry the circle and the ellipse have, in all these contacts, a common tangent. So in case 1 there are two points of second order osculation and in case 3 one point of second order osculation. This consideration shows that there is a transition case between these others, here called case 2, at which the circle and the ellipse not only share the tangent, but also have a common radius of curvature (third order osculation).

The quantitative relations are therefore understood if we merely consider the radius of curvature which the elliptic section has at its upper

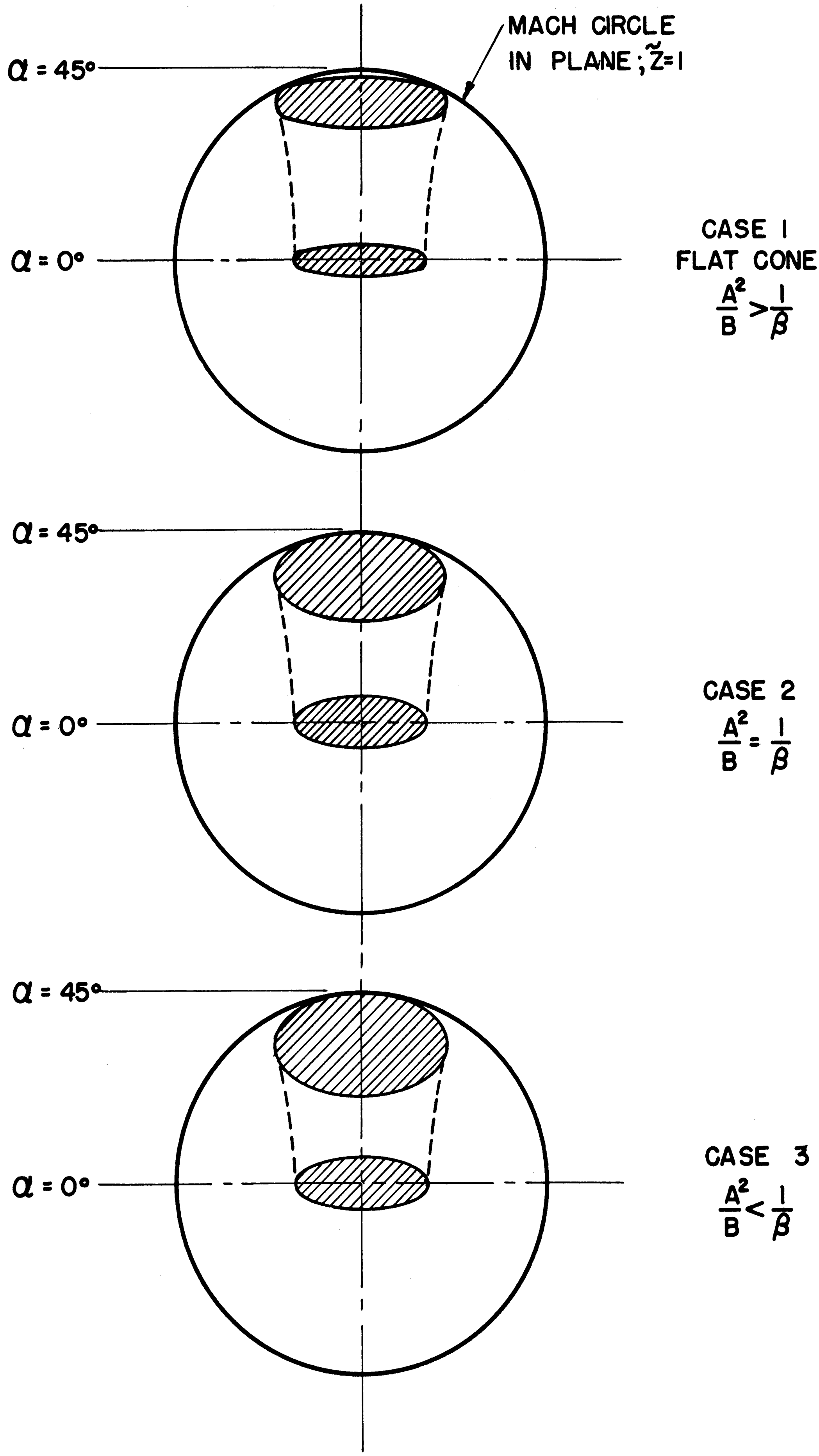

MACH CIRCLE
IN PLANE; $\tilde{Z}=1$
$\alpha = 45°$
$\alpha = 0°$
CASE 1
FLAT CONE
$\frac{A^2}{B} > \frac{1}{\beta}$
$\alpha = 45°$
$\alpha = 0°$
CASE 2
$\frac{A^2}{B} = \frac{1}{\beta}$
$\alpha = 45°$
$\alpha = 0°$
CASE 3
$\frac{A^2}{B} < \frac{1}{\beta}$

FIG. 2

vertex at the moment of contact. Calling this radius ρ we have case 1, 2, or 3, according as

$$\rho \gtreqless \frac{1}{\beta} . \tag{1.10}$$

At this moment we need the following little theorem: The radius of curvature of the intersection ellipse (formed by an elliptic cone with the plane $\tilde{Z} = 1$) is independent of α.

Proof: The radius of curvature $X = 0$, $Y = B$ (point P_0 of Figure 3) of the ellipse:

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1$$

is known to be:

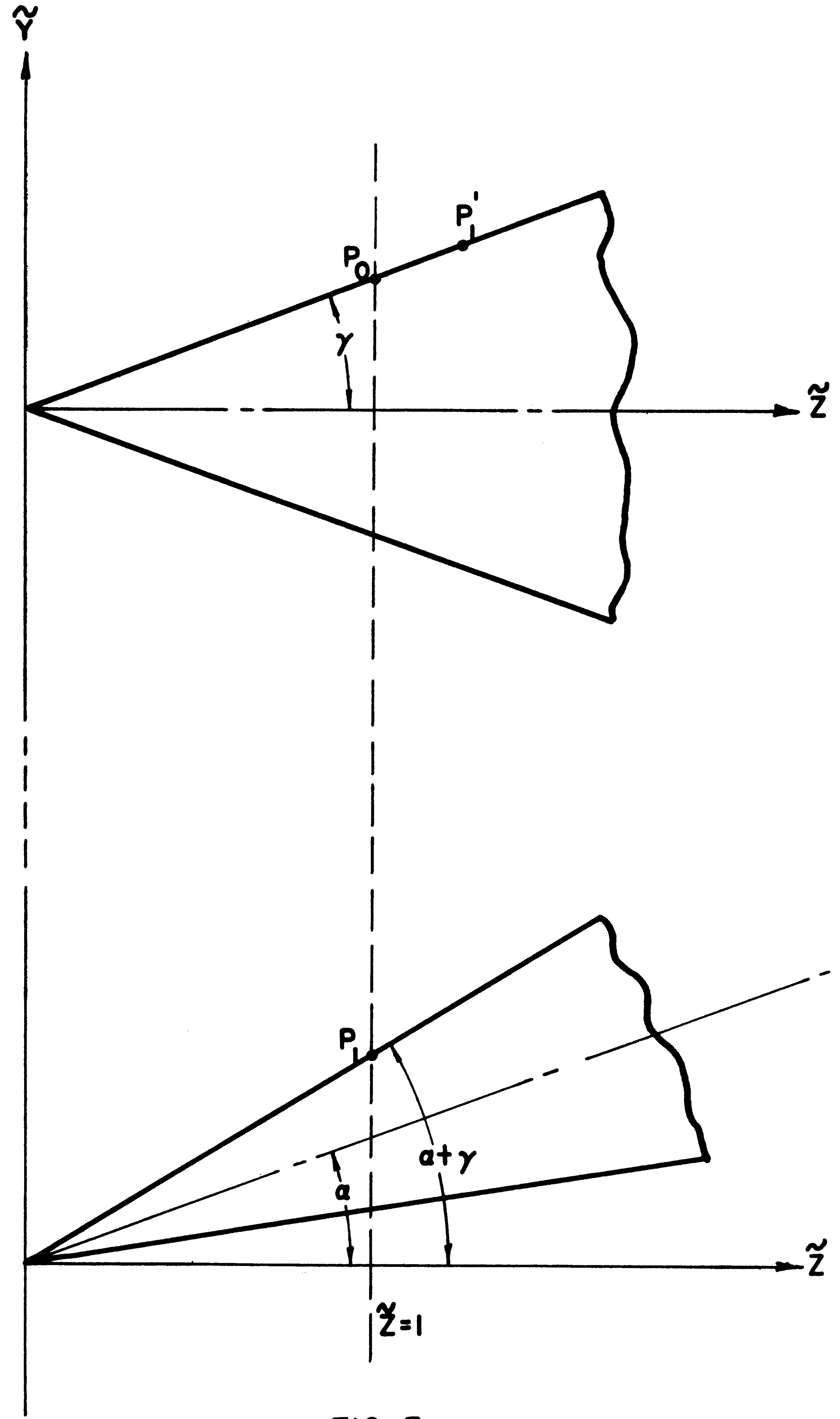
$$\rho_0 = \frac{A^2}{B} .$$

To calculate the curvature of the section with $\tilde{Z} = 1$ after tilting we use Meusnier's theorem which says that the radius ρ of a section making an angle ϕ with the normal is connected with the radius of curvature of the normal section R by

$$\rho = R \cos \phi .$$

After elevating the cone by α, the point P_1 has the distance

$$d = \frac{1}{\cos(\alpha + \gamma)}$$



$\tilde{Y}$
P_1'
P_0
γ
$\tilde{Z}$
P_1
$\alpha+\gamma$
α
$\tilde{Z}$
$\tilde{Z}=1$

FIG. 3

from the origin. Before elevating the point P_1 lies beyond the plane $\tilde{Z} = 1$ and is marked by P_1' in the upper half of Figure 3. Its Z coordinate is given by

$$Z_{P_1'} = \frac{\cos\gamma}{\cos(\alpha+\gamma)} .$$

Hence, the radius of curvature of the section with this $Z = Z_{P_1'}$ plane will be slightly larger, namely:

$$\rho_1 = \frac{A^2}{B}\,\frac{\cos\gamma}{\cos(\alpha+\gamma)} .$$

Now we apply Meusnier's theorem twice. The curvature of the normal section at P_o is:

$$R_{P_o} = \frac{\rho_o}{\cos\gamma} ,$$

γ being the angle between the normal at P_o and the line $\tilde{Z} = 1$; then the curvature of the normal section at P_1' is obtained by proportionate magnification:

$$R_{P_1'} = R_{P_o} Z_{P_1'} = \frac{\rho_o}{\cos(\alpha+\gamma)} .$$

Finally, by Meusnier's theorem again we find for the radius of curvature of the section at P_1:

$$\rho_1 = R_{P_1'} \cos(\alpha+\gamma) = \rho_o .$$

This proves the theorem:

It is therefore possible to put each flow with a given cone (A,B) and a given Mach number (M) into one of the following three cases:

Case 1: "Flat Cone": $\dfrac{A^2}{B} > \dfrac{1}{\beta}$ 2-point contact, 2nd order osculation each.

Case 2: Transition Case: $\dfrac{A^2}{B} = \dfrac{1}{\beta}$ 1-point contact, 3rd order osculation.

Case 3: "Round Cone": $\dfrac{A^2}{B} < \dfrac{1}{\beta}$ 1-point contact, 2nd order osculation. (1.11)

3. Behavior under Euclidean Rotation, Continued; Limiting Values of Angles of Attack

The equation of the tilted cone results, upon combining (1.1) and (1.8):

$$\frac{\tilde{X}^2}{A^2} + \left(\frac{\cos^2\alpha}{B^2} - \sin^2\alpha\right)\tilde{Y}^2 + \left(\frac{\sin^2\alpha}{B^2} - \cos^2\alpha\right)\tilde{Z}^2 - \left(1+\frac{1}{B^2}\right)\sin 2\alpha\,\tilde{Y}\tilde{Z} = 0. \tag{1.12}$$

To find the limiting values of α, it is necessary to determine the intersection of the ellipse

$$\frac{\tilde{X}^2}{A^2} + \left(\frac{\cos^2\alpha}{B^2} - \sin^2\alpha\right)\tilde{Y}^2 + \left(\frac{\sin^2\alpha}{B^2} - \cos^2\alpha\right) - \left(1+\frac{1}{B^2}\right)\sin 2\alpha\,\tilde{Y} = 0$$

with the Mach circle

$$\tilde{X}^2 + \tilde{Y}^2 = \frac{1}{\beta^2} .$$

The ordinates of the (in general, four) intersection points are given by:

$$\frac{1}{u}\left(-v \pm \sqrt{v^2 - uw}\right) ,$$

where
$$\mathcal{U} = \frac{\cos^2\alpha}{B^2} - \sin^2\alpha - \frac{1}{A^2},$$

$$\mathcal{V} = -\left(1 + \frac{1}{B^2}\right)\sin\alpha\cos\alpha,$$

$$\mathcal{W} = \frac{\sin^2\alpha}{B^2} - \cos^2\alpha + \frac{1}{\beta^2 A^2}.$$

For tangency, the case in which we are interested, the radicand $v^2 - uw$ must vanish. Calling the angle for which tangency is attained α_t, we obtain

$$\sin^2\alpha_t = \frac{\left(\frac{1}{\beta^2 A^2} - 1\right)\left(\frac{A^2}{B^2} - 1\right)}{\left(1 + \frac{1}{\beta^2}\right)\left(1 + \frac{1}{B^2}\right)}. \tag{1.13}$$

This is clearly the limiting angle in case 1.

Turning now to case 3, we see that the limiting angle, now to be called α_τ, is given by (Figure 4)

$$\alpha_\tau = \mu - \gamma.$$

Since
$$\sin^2\mu = \frac{1}{1+\beta^2},$$

and
$$\sin^2\gamma = \frac{1}{1 + \frac{1}{B^2}},$$

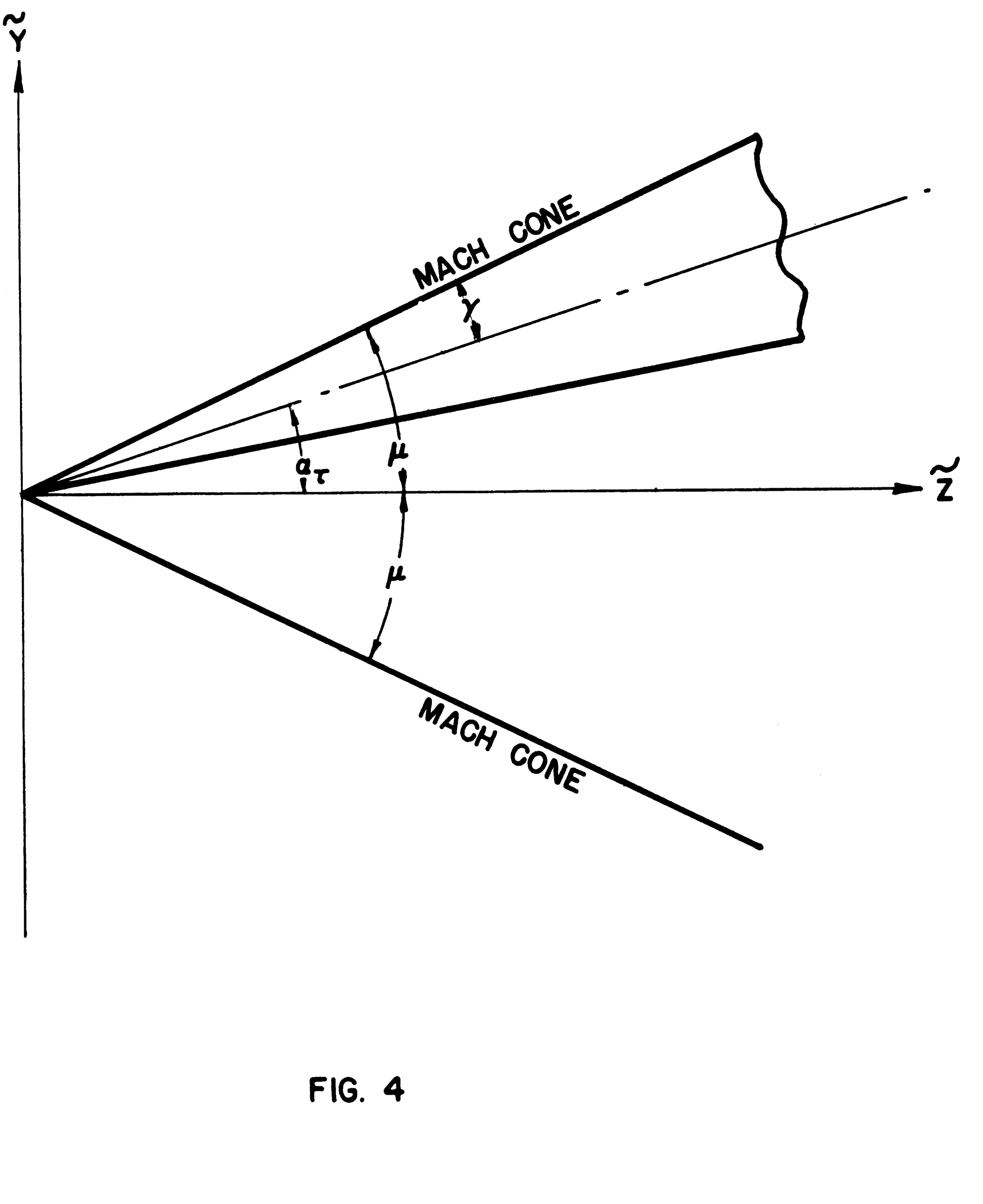


FIG. 4

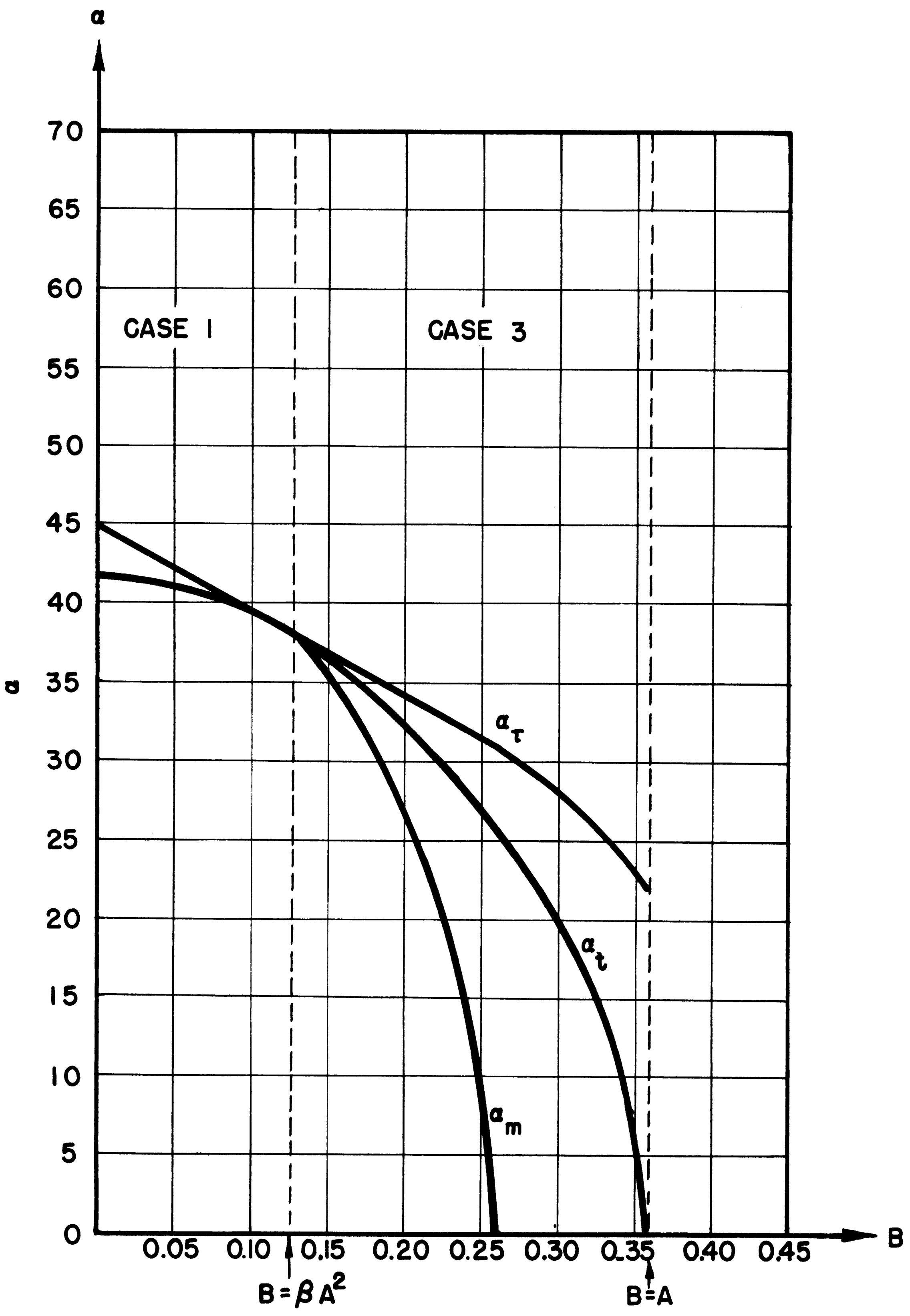

α
70
65
60
55
50
45
40
35
30
25
20
15
10
5
0
α
CASE 1
CASE 3
α_τ
α_t
α_m
0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45
B
$B = \beta A^2$
$B = A$

FIG. 5

we obtain

$$\sin^2\alpha_\tau = \frac{(1-\beta B)^2}{(1+\beta^2)(1+B^2)}, \tag{1.14}$$

an expression which is independent of A. For case 2, i.e.,

$$\frac{A^2}{B} = \frac{1}{\beta} \quad ; \quad \alpha_t = \alpha_\tau .$$

It is of interest to consider the difference of the two angles. One obtains from (1.13) and (1.14)

$$\sin^2\alpha_\tau - \sin^2\alpha_t = \frac{\left(\frac{B}{A} - \beta A\right)^2}{(1+\beta^2)(1+B^2)} \geq 0,$$

the equality sign holding for case 2. In Figure 5 the two angles α_t and α_τ are plotted for fixed A and β as functions of the variable B which increases from zero (arrow wing) to A (circular cone). From this figure it is evident that for a "flat" cone ($A^2/B > 1/\beta$) the physical range of angles is from zero to α_t, while for a "round" cone ($A^2/B < 1/\beta$) that range is from $\alpha = 0$, past α_t, to α_τ, at which latter value contact is finally achieved.

It is justified to ask the question: What <u>does</u> happen to a "round" cone as it passes the position $\alpha = \alpha_t$? The answer to this can best be given if one remembers that by means of the formulae:

$$\beta(\tilde{X} + i\tilde{Y}) = R\,\frac{\xi + i\eta}{\lambda}, \quad \tilde{Z} = R\,\frac{1-\lambda}{\lambda}, \tag{1.15}$$
$$R^2 = \tilde{Z}^2 - \beta^2(\tilde{X}^2 + \tilde{Y}^2), \quad 2\lambda = 1 - \xi^2 - \eta^2,$$

(B.B Rept. 75, p. 11, formulae 1.10) the elliptic cones are projected into the ξ, η plane where they form plane cyclids. (See Figure 4 of the same report.) When an elliptic cone is tilted, its projection becomes more like that of a circular cone, as can, in a quick way, be understood by just thinking of the intersection with the plane $\tilde{Z} = 1$. It may be shown that the stereographic projection of a "round" cone when just at angle α_t degenerates into a circle[1], although the proof has been suppressed here. When tilting further to an $\alpha > \alpha_t$ the stereographic projection is now a cyclid whose two (internal) foci lie, one beneath the other, on the η-axis. For a circular cone α_t equals zero; hence when tilting we are always in the range $\alpha_t < \alpha < \alpha_\tau$. It is obvious that in this case the cyclids will have their foci on the η-axis.

4. Behavior of the Cone under Lorentz Transformations

The special case of the Lorentz transformation of concern here is

$$\begin{aligned} \beta X &= \beta \tilde{X}, & \beta \tilde{X} &= \beta X, \\ \beta Y &= \beta \tilde{Y} ch\tilde{\alpha} - \tilde{Z} sh\tilde{\alpha}, & \beta \tilde{Y} &= \beta Y ch\tilde{\alpha} + Z sh\tilde{\alpha}, \\ Z &= -\beta \tilde{Y} sh\tilde{\alpha} + \tilde{Z} ch\tilde{\alpha}, & \tilde{Z} &= \beta Y sh\tilde{\alpha} + Z ch\tilde{\alpha}. \end{aligned} \tag{1.16}$$

The quantity $\tilde{\alpha}$ now takes the place of the Euclidean angle α of (1.8) and instead of preserving the square of the radius vector, now the following relation holds:

$$Z^2 - \beta^2 (X^2 + Y^2) = \tilde{Z}^2 - \beta^2 (\tilde{X}^2 + \tilde{Y}^2).$$

Thus not the radius vector, but the quantity R of (1.15) is preserved. But

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[1]However, it is not true that for $\alpha = \alpha_t$ the intersection of the cone with the plane $\tilde{Z} = 1$ is a circle. This occurs at a different angle.

since the equation R = const. is an hyperboloid with the Mach cone (1.9) as its asymptote, a Lorentz transformation shifts points along these hyperboloids but never from one hyperboloid to another.

As $\tilde{\alpha}$ changes from 0 to ∞, a point Q in the $\tilde{Y}$, $\tilde{Z}$ plane with coordinates $\tilde{Z} = \tilde{Z}_o$, $\tilde{Y} = 0$ moves along the hyperbola $\tilde{Z}^2 - \beta^2 \tilde{Y}^2 = \tilde{Z}_o^2$. From Figure 6 it is seen that an elliptic cone OPQR is transformed into another inclined cone OP'Q'R', but into one of different principal axis; and the points on it are shifted outward.

In Section 1 of Part I we stated the connection formulae between the quantities (A,B) for a cone at zero angle of attack and the parameters (k,y_o). From what has just been said it is evident that the corresponding formulae connecting the quantities (A,B,α) for a cone at a non-zero angle of attack and the parameters $(k,y_o,\tilde{\alpha})$ are much more complicated.

Upon transforming the cone equation (1.2) with (1.16), the following equation results:

$$\frac{\beta^2 \tilde{X}^2}{k^2 nd^2(y_o,k')} + \left(\frac{ch^2\tilde{\alpha}}{k^2 sd^2(y_o,k')} - sh^2\tilde{\alpha}\right)\tilde{Y}^2 + \left(\frac{sh^2\tilde{\alpha}}{k^2 sd^2(y_o,k')} - ch^2\tilde{\alpha}\right)\tilde{Z}^2$$
$$- \frac{cd^2(y_o,k')}{k^2} sh\, 2\tilde{\alpha}\, \beta \tilde{Y} \tilde{Z} = 0. \qquad (1.17)$$

The particular y surface which is the material cone has been denoted by y_o. Since all elliptic functions from now on are modulo k', this symbol shall be suppressed. In (1.17) we have a second form of the equation of the inclined elliptic cone; while the first form (1.12) is in accordance with the geometry of the experiment, the second one is required by the differential equation of the problem. By the comparison of the coefficients of (1.12) and (1.17) the relations follow:

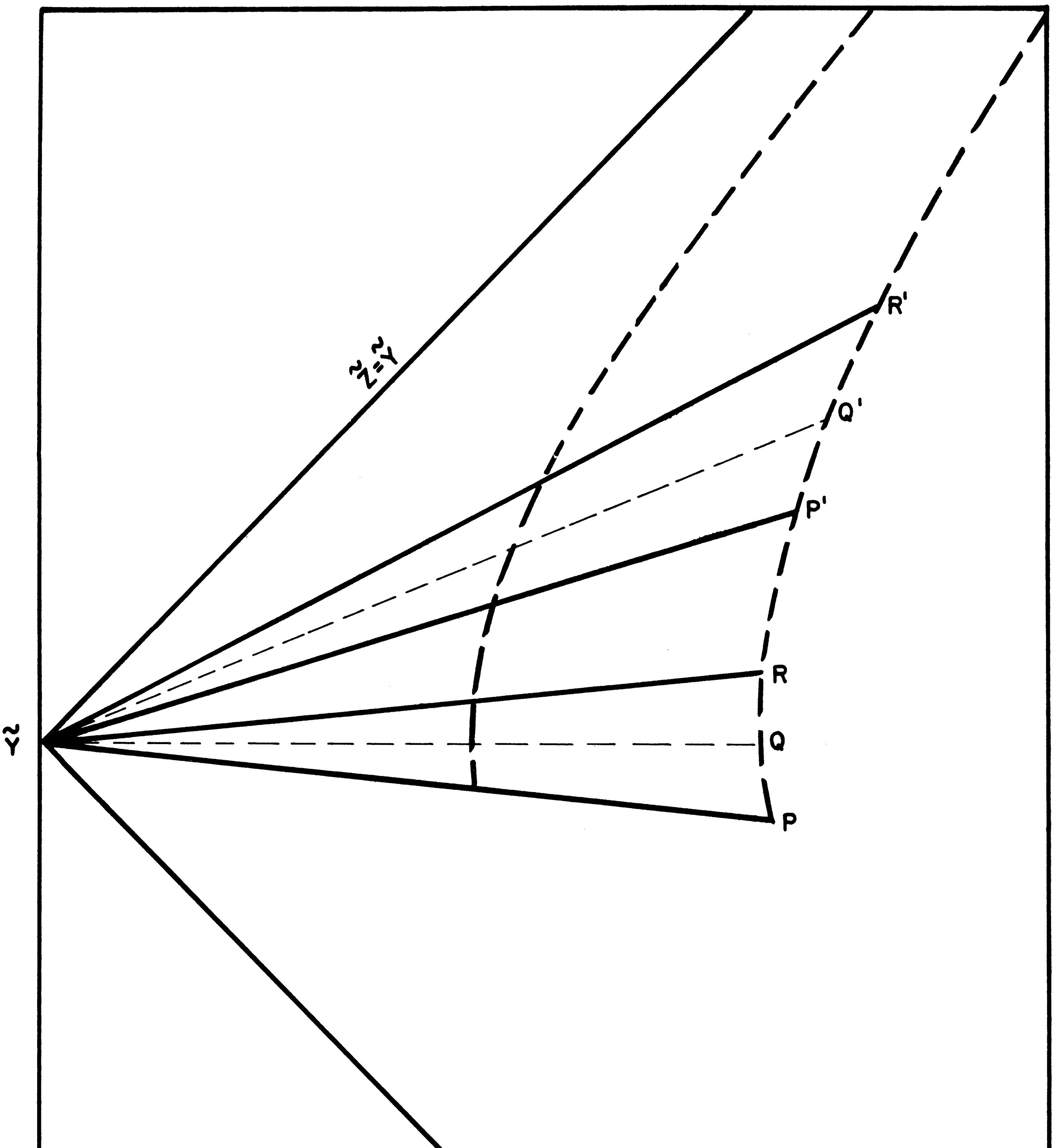

FIG. 6

$$
\begin{aligned}
&1)\quad A^2\left(\frac{\cos^2\alpha}{B^2} - \sin^2\alpha\right) = \frac{ch^2\tilde{\alpha}}{sn^2 y_0} - k^2 nd^2 y_0\, sh^2\tilde{\alpha}\,,\\
&2)\quad \beta^2 A^2\left(\frac{\sin^2\alpha}{B^2} - \cos^2\alpha\right) = \frac{sh^2\tilde{\alpha}}{sn^2 y_0} - k^2 nd^2 y_0\, ch^2\tilde{\alpha}\,;\\
&3)\quad \beta A^2\left(1 + \frac{1}{B^2}\right)\sin 2\alpha = cs^2 y_0\; nd^2 y_0\; sh\, 2\tilde{\alpha}\,.
\end{aligned}
\tag{1.18}
$$

Our task is now to solve these, so as to get the mathematical parameters k, y_0 and $\tilde{\alpha}$ as functions of A, B, α, and β, which are the physical parameters.

We form 1 + 2; these give, after some easy transformations:

$$
cs^2 y_0\; nd^2 y_0\; ch\, 2\tilde{\alpha} = A^2 \mathcal{L}(B, \beta, \alpha)\,,
$$

with

$$
\mathcal{L} = \frac{1}{B^2} - \beta^2 - (1-\beta^2)\left(1 + \frac{1}{B^2}\right)\sin^2\alpha\,. \tag{1.19}
$$

The above gives, together with 3, by squaring and subtracting:

$$
cs^2 y_0\; nd^2 y_0 = \frac{1}{sn^2 y_0} - \frac{k^2}{dn^2 y_0} = A^2\sqrt{\mathcal{L}^2 - \beta^2\left(1 + \frac{1}{B^2}\right)^2 \sin^2 2\alpha}\;. \tag{1.20}
$$

Two identical forms of the left-hand side have been written down. Relation (1.19) is free of $\tilde{\alpha}$. Another one results by forming 1 - 2:

$$
\frac{1}{sn^2 y_0} + \frac{k^2}{dn^2 y_0} = A^2 \mathcal{M}(B, \beta, \alpha)\,, \tag{1.21}
$$

with

$$
\mathcal{M} = \frac{1}{B^2} + \beta^2 - (1+\beta^2)\left(1 + \frac{1}{B^2}\right)\sin^2\alpha\,.
$$

The radicand of (1.20) may be written

$$\mathcal{L}^2 - \beta^2\left(1+\frac{1}{B^2}\right)^2 \sin^2 2\overset{\circ}{\alpha} = \mathcal{M}^2 - \left(\frac{2\beta}{B}\right)^2. \qquad (1.22)$$

From (1.20) and (1.21) the following are obtained:

$$\mathrm{sn}^2 y_0 = \frac{1}{\mathcal{U}}, \quad \mathrm{dn}^2 y_0 = \frac{k^2}{\mathcal{V}}, \qquad (1.23)$$

where the abbreviations U and V stand for:

$$\mathcal{U} = \frac{A^2}{2}\left(\mathcal{M} + \sqrt{\mathcal{M}^2 - \left(\frac{2\beta}{B}\right)^2}\right),$$

$$\mathcal{V} = \frac{A^2}{2}\left(\mathcal{M} - \sqrt{\mathcal{M}^2 - \left(\frac{2\beta}{B}\right)^2}\right). \qquad (1.24)$$

From (1.23) it is now easy to eliminate y_0, with the result

$$k^2 = \mathcal{V}\,\frac{(\mathcal{U}-1)}{(\mathcal{U}-\mathcal{V})}, \qquad (1.25)$$

for which one may write, somewhat more explicitly:

$$k^2 = \frac{1}{2}\left(\frac{\mathcal{M}}{\sqrt{\mathcal{M}^2 - \left(\frac{2\beta}{B}\right)^2}} - 1\right)\left(\frac{A^2}{2}\left(\mathcal{M} + \sqrt{\mathcal{M}^2 - \left(\frac{2\beta}{B}\right)^2}\right) - 1\right). \qquad (1.25)$$

Now we know k^2 and y_0. To get $\tilde{\alpha}$, one may divide (1.18_3) by (1.20)

$$\operatorname{sh} 2\tilde{\alpha} = \beta\left(1 + \frac{1}{B^2}\right)\frac{\sin 2\alpha}{\sqrt{\mathfrak{M}^2 - \left(\frac{2\beta}{B}\right)^2}} . \tag{1.26}$$

Formulae (1.23), (1.25), and (1.26) solve the problem of finding the mathematical cone variables k, y_o, and $\tilde{\alpha}$ as functions of the physical cone, and wind-stream variables A, B, α, and β.

5. Remarks on the Dependence of k^2 upon α (1.25); Discussion of Small Values of α

As (1.25) and especially (1.27) show, the $k^2(\alpha)$ curve starts from a value k_o^2 at $\alpha = 0$ with a horizontal tangent. We shall show that otherwise the curves look totally different, according as cases 1, 2, or 3 obtain.

1. $k^2(\alpha)$ rises monotonically for case 1, but has a maximum for case 3. Proof:

$$\frac{d k^2}{d(\sin^2\alpha)} = (1+\beta^2)\left(1 + \frac{1}{B^2}\right)\frac{uv}{(u-v)^2}\,\frac{(u+v-2)}{\sqrt{\mathfrak{M}^2 - \left(\frac{2\beta}{B}\right)^2}} ,$$

where the notation of the previous section is employed. Hence for a maximum[1]

$$A^2 \mathfrak{M} = 2 ,$$

and calling the angle of attack for which this maximum is reached α_m, we have

$$\sin^2\alpha_m = \frac{\frac{1}{B^2} + \beta^2 + \frac{2}{A^2}}{(1+\beta^2)\left(1 + \frac{1}{B^2}\right)} .$$

[1]It is a maximum, since the derivative is always positive for small α.

The maximum has no significance for case 1, since for $\alpha = \alpha_m$

$$\mathcal{U}, \mathcal{V} = 1 \pm \sqrt{1 - \left(\frac{\beta A^2}{B}\right)^2} .$$

So the maximum will not be real except in case 3. Now in this latter case the total angle range goes from 0 past α_t to α_τ. Since, with the help of (1.13),

$$\sin^2\alpha_t - \sin^2\alpha_m = \frac{\frac{1}{A^2}\left[1 - \left(\frac{\beta A^2}{B}\right)^2\right]}{(1+\beta^2)\left(1+\frac{1}{B^2}\right)} ,$$

we see that

$$0 < \alpha_m < \alpha_t < \alpha_\tau .$$

2. In case 1, k^2 attains the value 1 for $\alpha = \alpha_t$. Proof: The square root occurring in (1.24) to (1.26) is in this case

$$\sqrt{\mathfrak{M}^2 - \left(\frac{2\beta}{B}\right)^2} = \pm \frac{1}{A^2}\left[1 - \left(\frac{\beta A^2}{B}\right)^2\right] .$$

Further:

$$\mathcal{U} = \left(\frac{\beta A^2}{B}\right)^2 , \quad \mathcal{V} = 1 ,$$

and

$$k^2 = 1 .$$

3. In case 3, k^2 vanishes for $\alpha = \alpha_t$. Now

$$\mathcal{U} = 1 \ , \qquad \mathcal{V} = \left(\frac{\beta A^2}{B}\right)^2 ,$$

$$k^2(\alpha_t) = 0 \ .$$

This is in accordance with the geometric statement made on Page 15 that the cyclids, which are the stereographic projections of our cones, degenerate at this moment into circles. For $\alpha > \alpha_t$, k^2 is therefore negative.

4. In case 3, k^2 is negatively infinite for $\alpha = \alpha_\tau$. Proof:

$$\mathfrak{M}(\alpha_\tau) = \frac{2\beta}{B} \ ,$$

$$\mathcal{U} = \mathcal{V} = \frac{\beta A^2}{B} < 1 \ .$$

Hence

$$k^2(\alpha_\tau) = -\infty \ .$$

5. In case 2, $\beta A^2/B = 1$, k^2 has the value 1/2 when $\alpha_t = \alpha_\tau$; this limiting value is approached with a vertical tangent. The proof is lengthy, but straightforward.

In conclusion, we record the expressions for k, y_o and $\tilde{\alpha}$ for small values of the angle of attack α.

$$\mathcal{U} = \frac{A^2}{B^2}\left(1 - \frac{(1+\beta^2)(1+\frac{1}{B^2})}{\frac{1}{B^2}-\beta^2}\alpha^2\right),$$

$$\mathcal{V} = A^2\beta^2\left(1 + \frac{(1+\beta^2)(1+\frac{1}{B^2})}{\frac{1}{B^2}-\beta^2}\alpha^2\right),$$

$$sh\,2\tilde{\alpha} = \frac{2\beta\left(1+\frac{1}{B^2}\right)}{\frac{1}{B^2}-\beta^2}\alpha + \frac{2\beta\left(1+\frac{1}{B^2}\right)}{\frac{1}{B^2}-\beta^2}\left\{\frac{\left(\frac{1}{B^2}+\beta^2\right)\left(1+\beta^2\right)\left(1+\frac{1}{B^2}\right)}{\left(\frac{1}{B^2}-\beta^2\right)^2} - \frac{2}{3}\right\}\alpha^3,$$

$$k^2 = \frac{\beta^2\left(\frac{A^2}{B^2}-1\right)}{\frac{1}{B^2}-\beta^2}\left\{1 + \frac{(1+\beta^2)(1+\frac{1}{B^2})}{\frac{1}{B^2}-\beta^2}\,\frac{\left[\left(\frac{A^2}{B^2}-1\right)\left(\frac{1}{B^2}+\beta^2\right)-\left(\frac{1}{B^2}-\beta^2\right)\right]}{\left(\frac{1}{B^2}-\beta^2\right)\left(\frac{A^2}{B^2}-1\right)}\alpha^2\right\}, \tag{1.27}$$

$$sn(y_0,k') = \frac{1}{\sqrt{\mathcal{U}}} = \frac{B}{A}\left\{1 + \frac{1}{2}\,\frac{(1+\beta^2)(1+\frac{1}{B^2})}{\frac{1}{B^2}-\beta^2}\alpha^2\right\}.$$

In (3) of Equation (1.27) it is convenient to compute k' first and then to obtain y_0 by means of (2). If the parameter y_0 is replaced by the parameter $sn(y_0,k')$, the use of the Jacobi Zeta-function can be avoided in the analysis that follows.

PART II: SOLUTION OF THE PROBLEM IN THE FORM OF AN INFINITE SYSTEM OF LINEAR EQUATIONS. TWO APPROXIMATE METHODS

1. Integral Formulae

In this section we derive the integral formulae for the components of the velocity of the flow past the cone at an angle of attack. These formulae correspond to those which in the earlier reports were referred to as the Weierstrass formulae[1]. We begin with the conditions of irrotationality of the flow, namely

$$\frac{\partial w}{\partial \tilde{Y}} - \frac{\partial v}{\partial \tilde{Z}} = 0 \quad ; \quad \frac{\partial u}{\partial \tilde{Z}} - \frac{\partial w}{\partial \tilde{X}} = 0 \quad ; \quad \frac{\partial v}{\partial \tilde{X}} - \frac{\partial u}{\partial \tilde{Y}} = 0$$

where, it should be recalled, the $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ coordinate system is in the given flow space with the $\tilde{Z}$-axis in the direction of the flow, and with respect to which the cone has the (Lorentzian) angle of attack $\tilde{\alpha}$. By means of the Lorentz transformations (1.16) these three equations are readily expressed in terms of the coordinate system X, Y, Z that is aligned with respect to the cone, and in terms of which the cone has the simple equation (1.2). They are, respectively,

$$\begin{aligned} \frac{\partial w}{\partial Y}\cosh\tilde{\alpha} - \frac{\partial w}{\partial Z}\beta\sinh\tilde{\alpha} + \frac{\partial v}{\partial Y}\frac{\sinh\tilde{\alpha}}{\beta} - \frac{\partial v}{\partial Z}\cosh\tilde{\alpha} &= 0, \\ -\frac{\partial u}{\partial Y}\frac{\sinh\tilde{\alpha}}{\beta} + \frac{\partial u}{\partial Z}\cosh\tilde{\alpha} - \frac{\partial w}{\partial X} &= 0, \\ \frac{\partial v}{\partial X} - \frac{\partial u}{\partial Y}\cosh\tilde{\alpha} + \frac{\partial u}{\partial Z}\beta\sinh\tilde{\alpha} &= 0. \end{aligned} \tag{2.1}$$

---

[1] See, for example, Part II, Supersonic Flow Past an Elliptic Cone at Zero Angle of Attack, p. 16; also Bumblebee Report No. 75, pp. 60 and 61.

Next we introduce the complex variable z = x + iy, where, as in the earlier reports[1], x and y are related to the coordinates X, Y, and Z by the equations

$$\beta X = \frac{k}{k'} R\, cn(x,k)\, cn(iy,k) = \frac{k}{k'} R\, cn(x,k)\, nc(y,k'),$$

$$\beta Y = -ikR\, sn(x,k)\, sn(iy,k) = kR\, sn(x,k)\, sc(y,k'),$$

$$Z = \frac{1}{k'} R\, dn(x,k)\, dn(iy,k) - \frac{1}{k'} R\, dn(x,k)\, dc(y,k'). \qquad (2.2)$$

Since it can be shown that the components of velocity u, v, and w are harmonic functions of the variables x and y, we may define the functions U(z), V(z), and W(z) of the complex variable z by setting[2]

$$U(z) = u + i\bar{u}\,; \quad V(z) = v + i\bar{v}\,; \quad W(z) = w + i\bar{w}\,;$$

where $\bar{u}$, $\bar{v}$, and $\bar{w}$ are the real harmonic functions conjugate to u, v, and w, respectively. Then, making use of the Cauchy-Riemann equations, we may write

$$\frac{\partial w}{\partial Y} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial Y} - \frac{\partial \bar{w}}{\partial x}\frac{\partial y}{\partial Y} = \mathcal{R}\left(\frac{dW}{dz}\frac{\partial z}{\partial Y}\right),$$

where $\mathcal{R}(\ )$ denotes the real part of the quantity within the parentheses. In general

$$\frac{\partial u_i}{\partial X_j} = \mathcal{R}\left(\frac{dU_i}{dz}\frac{\partial z}{\partial X_j}\right),$$

---

[1]See Part II, loc. cit., pp. 4 and 10. Note that the sign of y is reversed from that chosen in Bumblebee Report No. 75, p. 57, et seq.

[2]See Part II, loc. cit., p. 13, also Bumblebee Report, No. 75, p. 60.

where X_j represents either X, Y, or Z; u_i represents u, v, or w; and U_i the corresponding function U, V, or W. Making use of these relations, the Equations (2.1) may be written in the form

$$\mathcal{R}\left\{ \left(\frac{\partial z}{\partial Y}\frac{sh\tilde{\alpha}}{\beta} - \frac{\partial z}{\partial Z} ch\tilde{\alpha}\right)\frac{dV}{dz} + \left(\frac{\partial z}{\partial Y} ch\tilde{\alpha} - \frac{\partial z}{\partial Z} sh\tilde{\alpha}\right)\frac{dW}{dz}\right\} = 0,$$

$$\mathcal{R}\left\{\left(-\frac{\partial z}{\partial Y}\frac{sh\tilde{\alpha}}{\beta} + \frac{\partial z}{\partial Z} ch\tilde{\alpha}\right)\frac{dU}{dz} - \frac{\partial z}{\partial X}\frac{dW}{dz}\right\} = 0,$$

$$\mathcal{R}\left\{\left(-\frac{\partial z}{\partial Y} ch\tilde{\alpha} + \frac{\partial z}{\partial Z}\beta sh\tilde{\alpha}\right)\frac{dU}{dz} + \frac{\partial z}{\partial X}\frac{dV}{dz}\right\} = 0.$$

By solving this antisymmetric set for U', V', and W' the following is obtained:

$$U' = \frac{\partial z}{\partial X}\mathcal{F}(z),$$

$$V' = \left(\frac{\partial z}{\partial Y} ch\tilde{\alpha} - \frac{\partial z}{\partial Z}\beta sh\tilde{\alpha}\right)\mathcal{F}(z), \tag{2.3}$$

$$W' = \left(-\frac{\partial z}{\partial Y}\frac{sh\tilde{\alpha}}{\beta} + \frac{\partial z}{\partial Z} ch\tilde{\alpha}\right)\mathcal{F}(z),$$

where $\mathcal{F}(z)$ is an arbitrary analytic function of z. The formulae for the derivative of z with respect to X, Y, and Z were derived on Page 15 of the Part II report and will be reproduced here:

$$\frac{\partial z}{\partial X} = -\frac{\beta}{kk'R}\, ds(z,k) = \frac{i\beta}{k'R}\, cn(z',k),$$

$$\frac{\partial z}{\partial Y} = \frac{i\beta}{kR}\, ns(z,k) = -\frac{i\beta}{R}\, sn(z',k),$$

$$\frac{\partial z}{\partial Z} = -\frac{1}{k'R}\, cs(z',k) = \frac{i}{k'R}\, dn(z',k).$$

z = x + i y PLANE

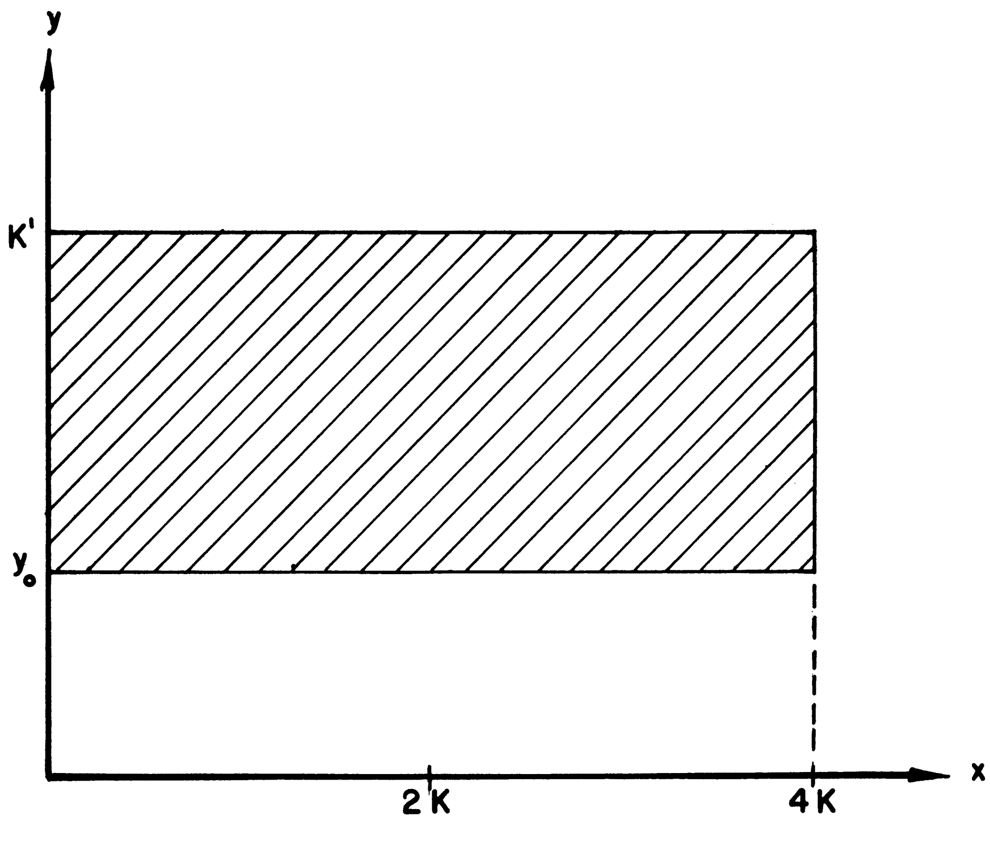

FIG. 7

The second form arises upon substituting

$$z' = iK' - z. \tag{2.4}$$

Upon substituting into (2.3), the velocity component can be written as indefinite integrals,

$$U = \frac{i\beta}{k'} \int sn\, z' \mathcal{F}(z')\, dz',$$

$$V = i\beta \int \left(-sn\, z' ch\, \tilde{\alpha} - \frac{1}{k'} dn\, z' sh\, \tilde{\alpha}\right) \mathcal{F}(z')\, dz', \tag{2.5}$$

$$W = i \int \left(sn\, z' sh\, \tilde{\alpha} + \frac{1}{k'} dn\, z' ch\, \tilde{\alpha}\right) \mathcal{F}(z')\, dz'.$$

These formulae consitute the generalizations to angles of attack of formulae (37), Page 16 of the Part II report. The function $\mathcal{F}(z')$ is determined by the conditions that the components u, v, and w satisfy on the Mach cone and the surface of the conical body. These conditions are discussed in the next section.

The correspondence between the plane of the complex variable $z = x + iy$ and the physical space of the flow is made clear in Figure 7. The shaded rectangular region

$$0 < x < 4K \quad ; \quad y_0 < y < K'$$

is the map of the region between the tilted material cone (1.17) and the Mach cone (1.9). The side $y = y_0$ corresponds to the surface of the material cone, and the side $y = K'$ to the Mach cone. These surfaces are described once as a point in the z-plane moves along the respective side of the rectangle with x ranging from 0 to $4K$. As has already been shown, the eccentricity of the material cones depends in part upon the ordinate y_0. Thus, the smaller the value of y_0, the flatter the cone becomes. In the limit when $y_0 = 0$, the rectangle

$$0 < x < 4K \quad , \quad 0 < y < K' ,$$

corresponds to the space between the Mach cone and a delta wing at an angle of attack.

2. The Boundary Conditions

In accordance with the linear theory of supersonic flow[1], the flow past the conical body lying entirely within its Mach cone is such that: (a) the transition from the constant state of the flow ahead of the body to the disturbed flow around the body takes place across the Mach cone, and (b) the body surface is a stream surface. Consequently, the function F(z') of the Equations (2.5) defining an arbitrary conical flow field must be determined so that: (a) on the surface of the Mach cone (1.9)

$$u = v = w = 0 , \tag{2.6}$$

and (b) on the surface of the material cone (1.17), whose equation may for convenience be written

$$\tilde{S}(\tilde{X}, \tilde{Y}, \tilde{Z}) = 0 ,$$

$$u \frac{\partial \tilde{S}}{\partial \tilde{X}} + v \frac{\partial \tilde{S}}{\partial \tilde{Y}} + (w + w_\infty) \frac{\partial \tilde{S}}{\partial \tilde{Z}} = 0 . \tag{2.7}$$

The derivatives in Equation (2.7) are readily expressed in terms of the coordinates x and y of the z-plane with the aid of the Lorentz transformations (1.16). We may write

[1]See Bumblebee Report No. 75, Section 1.1.

$$\frac{\partial \tilde{S}}{\partial \tilde{X}} = \frac{\partial S}{\partial X},$$

$$\frac{\partial \tilde{S}}{\partial \tilde{Y}} = \frac{\partial S}{\partial Y}\, ch\tilde{\alpha} - \frac{\partial S}{\partial Z}\, \beta sh\tilde{\alpha},$$

$$\frac{\partial \tilde{S}}{\partial \tilde{Z}} = -\frac{\partial S}{\partial Y}\, \frac{sh\tilde{\alpha}}{\beta} + \frac{\partial S}{\partial Z}\, ch\tilde{\alpha},$$

where S(X,Y,Z) = 0 is Equation (1.2) in terms of the rotated system of coordinates X, Y, and Z. After performing the derivatives of the left-hand member of Equation (1.2), making use of Equations (2.2), and substituting the resulting expressions for the derivatives of S(X,Y,Z) in Equation (2.7), we obtain the result

$$g_1(\tilde{\alpha}, y_0, x)\, u + g_2(\tilde{\alpha}, y_0, x)\, v + g_3(\tilde{\alpha}, y_0, x)\,(w + w_\infty) = 0, \tag{2.8}$$

where

$$g_1 = \frac{\beta}{k'}\, sn\, y_0\, dn\, y_0\, cn\, x,$$

$$g_2 = \beta\left(dn\, y_0\, sn\, x\, ch\tilde{\alpha} + \frac{k}{k'}\, sn\, y_0\, dn\, x\, sh\tilde{\alpha}\right),$$

$$g_3 = -\left(dn\, y_0\, sn\, x\, sh\tilde{\alpha} + \frac{k}{k'}\, sn\, y_0\, dn\, x\, ch\tilde{\alpha}\right).$$

The problem of the conical flow past the cone at an arbitrary angle of attack may therefore be stated as that of determining the function F(z') so that the functions u, v, and w given by the real parts, respectively of the three integrals in (2.5) satisfy the condition (2.6) on the side $y = K'$ of the rectangle (Figure 7) in the Z-plane, and the condition (2.8) on the side $y = y_0$.

The boundary condition (2.8) has the same encumbering property as in the case of a zero angle of attack: its coefficients are functions of x^1. In order to satisfy this condition, it has been found necessary to express $\mathfrak{F}(z')$ as well as the coefficients g_i in terms of their Fourier series with respect to the variable x. As a consequence, the problem leads to the treatment of an infinite set of linear equations for the coefficients of the expansion of $\mathfrak{F}(z')$.

In the earlier work[1] on the cone for the case $\alpha = 0$, the boundary condition on the side $y = y_o$ was taken in the form obtainable from (2.8) by dividing both members of this equation by the coefficient $g_3(\alpha, y_o, x)$. In the present case the quotients g_1/g_3 and g_2/g_3 formed in this way are finite only as long as

$$\operatorname{th} \tilde{\alpha} < k \operatorname{sd}(y_o, k').$$

In other words, the function $g_3(\alpha, y_o, x)$ does not vanish as long as the angle of attack of the cone is less than the vertical flare angle (see Figure 8). But when, as the angle of attack α increases, the lower side of the cone becomes horizontal, $g_3(\alpha, y_o, x)$ vanishes for $x = 3K$. For larger values of α, $g_3(\alpha, y_o, x)$ vanishes twice in the interval $2K < x < 4K$. Hence, for these values of the angle of attack α, the quotients g_1/g_3 and g_2/g_3 are not developable as Fourier series. To avoid this difficulty the boundary condition on the side $y = y_o$ is taken in the form (2.8).

3. Fourier Series for the Function (z')

It will be seen that the following hypothesis as to form for $\mathfrak{F}(z')$ will give velocity components of the right symmetry:

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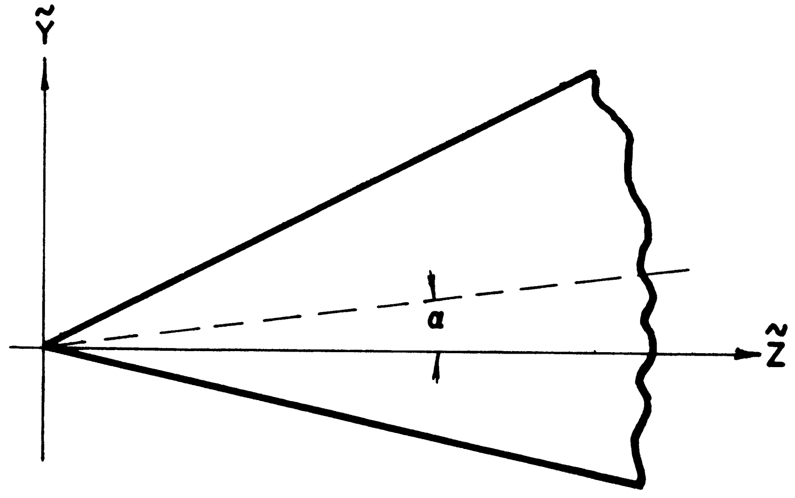
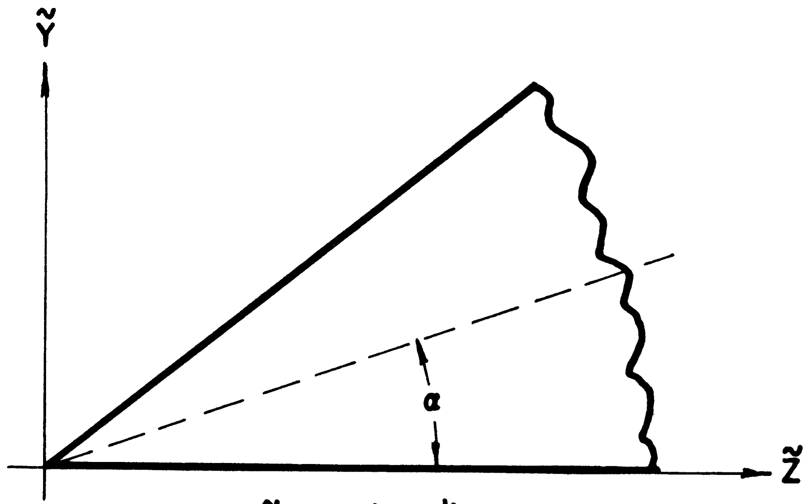
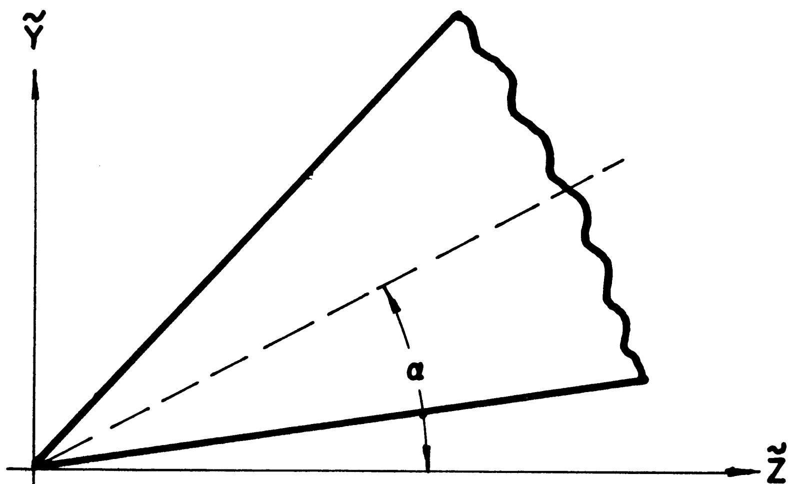
[1] See Bumblebee Report No. 75, p. 62.



$\text{th}\ \tilde{\alpha} < \text{ksd}\,(y_0, k')$



$\text{th}\ \tilde{\alpha} = \text{ksd}\,(y_0, k')$



$\text{th}\ \tilde{\alpha} > \text{ksd}\,(y_0, k')$

FIG. 8

$$\mathfrak{F}(z') = \sum_n a_n e^{i\frac{n\pi z'}{K}} + i\sum_n b_n e^{i\frac{2n+1}{2}\frac{\pi z'}{K}},$$

with

$$a_n = a_{-n},$$

$$b_n = -b_{-n-1}, \qquad (2.9)$$

$$a_n, b_n \quad \text{(both real)}.$$

$\sum$ signs for which no limits of summation are written down are tacitly assumed to imply summation from $-\infty$ to $+\infty$. The first sum in (2.9) along gives u, v, and w series of form (41) of the Part II report and is therefore sufficient for no angle of attack. Hence one now expects a_n to depend on even, and b_n on odd, powers of α.

This expression for $\mathfrak{F}(z')$ is now introduced into (2.5) to obtain the velocities. We need also the Fourier series for the elliptic functions occurring in the integrands. These familiar series we write as follows:

$$dn(x,k) = \sum_n \delta_n e^{i\frac{n\pi x}{K}},$$

$$cn(x,k) = \sum_n \gamma_n e^{i\frac{2n+1}{2}\frac{\pi x}{K}},$$

$$sn(x,k) = i\sum_n \sigma_n e^{i\frac{2n+1}{2}\frac{\pi x}{K}},$$

with:

$$\delta_n = \frac{\pi}{2K} \frac{1}{ch \frac{n\pi K'}{K}} = \delta_{-n} ,$$

$$\gamma_n = \frac{\pi}{2Kk} \frac{1}{ch \frac{2n+1}{2} \frac{\pi K'}{K}} = \gamma_{-n-1} ,$$

$$\sigma_n = -\frac{\pi}{2Kk} \frac{1}{sh \frac{2n+1}{2} \frac{\pi K'}{K}} = -\sigma_{-n-1} , \tag{2.10}$$

The last two occurred previously in our work, but were differently abbreviated then (Part II report, (43), Bumblebee Report No. 75, (3.33)) as β_n and α_n. The relations connecting them are:

$$\gamma_n = i\, k' (-)^n \beta_n ,$$

$$\sigma_n = (-)^{n+1} \alpha_n ,$$

4. Fourier Series for U, V, W, and their Real Parts

Substitution of (2.9) into (2.5$_1$) gives

$$\left(\frac{i\beta}{k'}\right)^{-1} U' = \sum_m \gamma_m e^{i \frac{2m+1}{2} \frac{\pi z'}{K}} \left\{ \sum_n a_n e^{i \frac{n\pi z'}{K}} + i \sum_n b_n e^{i \frac{2n+1}{2} \frac{\pi z'}{K}} \right\}$$

$$= \sum_m \sum_n \gamma_m a_n e^{i \frac{2(m+n)+1}{2} \frac{\pi z'}{K}} + i \sum_m \sum_n \gamma_m b_n e^{i (m+n+1) \frac{\pi z'}{K}}$$

In the first term re-name:

$$m + n = \mu \quad , \quad n = \nu \; ,$$

and in the second term:

$$m + n + 1 = \mu \quad , \quad n = \nu \; .$$

Then

$$\left(\frac{i\beta}{k'}\right)^{-1} U' = \sum_{\mu}\sum_{\nu} \gamma_{\mu-\nu}\, a_{\nu}\, e^{i\frac{2\mu+1}{2}\frac{\pi z'}{K}} + i\sum_{\mu}\sum_{\nu} \gamma_{\mu-\nu-1}\, b_{\nu}\, e^{i\mu\frac{\pi z'}{K}} ,$$

and by integration:

$$U = \sum_{\mu} A_{\mu}\, e^{i\frac{2\mu+1}{2}\frac{\pi z'}{K}} + i\sum_{\mu}{}' \alpha_{\mu}\, e^{i\mu\frac{\pi z'}{K}} ,$$

with:

$$A_{\mu} = \frac{\beta K}{\pi k'}\,\frac{2}{2\mu+1} \sum_{\nu} \gamma_{\mu-\nu}\, a_{\nu} ,$$

$$\alpha_{\mu} = \frac{\beta K}{\pi k'}\,\frac{1}{\mu} \sum_{\nu} \gamma_{\mu-\nu-1}\, b_{\nu} . \qquad (2.11)$$

The accent on the second sum sign signifies, as from now on, that the term with $\mu = 0$ is to be omitted. A term proportionate to z' seems to arise when integrating, but its coefficient $\sum_{\nu} b_{\nu}\gamma_{\nu}$ vanishes.

The results for V and W will be given only, since the procedure is analogous.

$$V = i \sum_{\mu} B_{\mu} e^{i \frac{2\mu+1}{2} \frac{\pi z'}{K}} + \sum_{\mu}{}' \mathcal{B}_{\mu} e^{i \mu \frac{\pi z'}{K}} + i \mathcal{B}_0 z',$$

$$B_{\mu} = -\beta \frac{K}{\pi} \frac{2}{2\mu+1} \sum_{\nu} \left(ch\tilde{\alpha}\, \sigma_{\mu-\nu} a_{\nu} + \frac{sh\tilde{\alpha}}{k'} \delta_{\mu-\nu} b_{\nu} \right),$$

with:

$$\mathcal{B}_{\mu} = -\beta \frac{K}{\pi} \frac{1}{\mu} \sum_{\nu} \left(\frac{sh\tilde{\alpha}}{k'} \delta_{\mu-\nu} a_{\nu} - ch\tilde{\alpha}\, \sigma_{\mu-\nu-1} b_{\nu} \right),$$

$$\mathcal{B}_0 = -\beta \sum_{\nu} \left(\frac{sh\tilde{\alpha}}{k'} \delta_{\nu} a_{\nu} + ch\tilde{\alpha}\, \sigma_{\nu} b_{\nu} \right). \quad (2.12)$$

$$W = i \sum_{\mu} C_{\mu} e^{i \frac{2\mu+1}{2} \frac{\pi z'}{K}} + \sum_{\mu}{}' \mathcal{L}_{\mu} e^{i \mu \frac{\pi z'}{K}} + i \mathcal{L}_0 z',$$

$$C_{\mu} = \frac{K}{\pi} \frac{2}{2\mu+1} \sum_{\nu} \left(sh\tilde{\alpha}\, \sigma_{\mu-\nu} a_{\nu} + \frac{ch\tilde{\alpha}}{k'} \delta_{\mu-\nu} b_{\nu} \right),$$

with:

$$\mathcal{L}_{\mu} = \frac{K}{\pi} \frac{1}{\mu} \sum_{\nu} \left(\frac{ch\tilde{\alpha}}{k'} \delta_{\mu-\nu} a_{\nu} - sh\tilde{\alpha}\, \sigma_{\mu-\nu-1} b_{\nu} \right),$$

$$\mathcal{L}_0 = \sum_{\nu} \left(\frac{ch\tilde{\alpha}}{k'} \delta_{\nu} a_{\nu} + sh\tilde{\alpha}\, \sigma_{\nu} b_{\nu} \right).$$

(2.13)

An additive constant has been suppressed in each of the complex velocities.

Next we have to write the real parts u, v, and w as exponential series in x on the straight line

$$z' = iK' - iy_0 - x \,. \tag{2.14}$$

The method to be used for this end exploits in an essential manner the symmetry or antisymmetry of the coefficients, which is, of course, a consequence of (2.9) and (2.10). It can be proved that in the series free from i, the series coefficients are antisymmetric, while in those having i as factor, the coefficients are symmetric. This will be shown on only one term, taken from the first series in (2.11). Let us consider

$$A_\mu = \frac{\beta K}{\pi k'} \frac{2}{2\mu+1} \sum_\nu \gamma_{\mu-\nu}\, a_\nu \,.$$

Then:

$$A_{-\mu-1} = -\frac{\beta K}{\pi k'} \frac{2}{2\mu+1} \sum_\nu \gamma_{-\mu-\nu-1}\, a_\nu$$

$$= -\frac{\beta K}{\pi k'} \frac{2}{2\mu+1} \sum_\nu \gamma_{\mu+\nu}\, a_\nu \,.$$

Now we replace ν by $-\nu'$ and use the symmetry of a_ν

$$A_{-\mu-1} = -\frac{\beta K}{\pi k'} \frac{2}{2\mu+1} \sum_{\nu'} \gamma_{\mu-\nu'}\, a_{\nu'} = -A_\mu \,. \tag{2.15}$$

Similarly for other terms. To get u, add to (2.11) its complex conjugate, while taking into account (2.14):

$$2u = \sum_\mu A_\mu e^{-\frac{2\mu+1}{2}\eta}(e^{i\cdots} + e^{-i\cdots}) + i\sum_\mu{}' \mathfrak{a}_\mu e^{-\mu\eta}(e^{-i\cdots} - e^{i\cdots}).$$

The abbreviation

$$\eta = \frac{\pi}{K}(K' - y_0) \tag{2.16}$$

is advantageous. The imaginary exponents are not written out. Now, in the first sum, replace μ by $-\mu - 1$, and in the third, by $-\mu$, with the result that:

$$u = -\sum_\mu A_\mu \operatorname{sh}\frac{2\mu+1}{2}\eta\, e^{i\frac{2\mu+1}{2}\frac{\pi x}{K}} + i\sum_\mu{}' \mathfrak{a}_\mu \operatorname{sh}\mu\eta\, e^{i\mu\frac{\pi x}{K}}. \tag{2.17}$$

Similarly:

$$v = i\sum_\mu B_\mu \operatorname{sh}\frac{2\mu+1}{2}\eta\, e^{i\frac{2\mu+1}{2}\frac{\pi x}{K}} - \sum_\mu{}' \mathfrak{b}_\mu \operatorname{sh}\mu\eta\, e^{i\mu\frac{\pi x}{K}} \tag{2.18}$$

$$- \mathfrak{b}_0 (K' - y_0),$$

$$w = i\sum_\mu C_\mu \operatorname{sh}\frac{2\mu+1}{2}\eta\, e^{i\frac{2\mu+1}{2}\frac{\pi x}{K}} - \sum_\mu{}' \tilde{\mathfrak{c}}_\mu \operatorname{sh}\mu\eta\, e^{i\mu\frac{\pi x}{K}} \tag{2.19}$$

$$- \tilde{\mathfrak{c}}_0 (K' - y_0).$$

These expressions are only complex in appearance. It should be noted that the velocity components all vanish for $y_0 = K'$, i.e., on the Mach cone, as they should according to (2.6). This fact serves as a-posteriori justification of our suppression of additive constants in the integration which led to (2.11), (2.12), and (2.13). Such additive constants, if added, would have to be adjusted at this moment so as to satisfy the Mach cone condition (2.6).

5. Introduction into the Boundary Condition

The last series, together with (2.10), will now be substituted into the boundary condition (2.8). In view of the lengthiness of the procedure, each term will be taken up separately.

The first term:

$$g_1 u = \frac{\beta}{k'}\,\mathrm{sn}\,y_0\,\mathrm{dn}\,y_0 \sum_m \gamma_m\, e^{i\frac{2m+1}{2}\frac{\pi x}{K}} \left\{ -\sum_n A_n\,\mathrm{sh}\frac{2n+1}{2}\eta\, e^{i\frac{2n+1}{2}\frac{\pi x}{K}} + i\sum_n{}' \alpha_n\,\mathrm{sh}\,n\eta\, e^{i\frac{n\pi x}{K}} \right\}.$$

In the first series product indices will be renamed as follows:

$$m+n+1 = \lambda \quad , \quad n = \mu .$$

In the second product:

$$m+n = \lambda \quad , \quad n = \mu .$$

Hence:

$$g_1 u = \sum_\lambda (1)_\lambda\, e^{i\frac{\lambda\pi x}{K}} + i\sum_\lambda [1]_\lambda\, e^{i\frac{2\lambda+1}{2}\frac{\pi x}{K}} ,$$

with:
$$(1)_\lambda = -\frac{\beta}{k'}\, \mathrm{sn}\, y_0\, \mathrm{dn}\, y_0 \sum_\mu \gamma_{\mu-\lambda} A_\mu\, \mathrm{sh}\, \frac{2\mu+1}{2}\eta ,$$

$$[1]_\lambda = \frac{\beta}{k'}\, \mathrm{sn}\, y_0\, \mathrm{dn}\, y_0 {\sum_\mu}' \gamma_{\mu-\lambda-1} \mathcal{O}_\mu\, \mathrm{sh}\,\mu\eta . \tag{2.20a}$$

The second term:
$$g_2 v = \beta\left(\mathrm{dn}\, y_0\, \mathrm{ch}\,\tilde{\alpha} \sum_m \sigma_m e^{i\frac{2m+1}{2}\frac{\pi x}{K}} + \frac{k}{k'}\, \mathrm{sn}\, y_0\, \mathrm{sh}\,\tilde{\alpha} \sum_m \delta_m e^{i\frac{m\pi x}{K}}\right)$$
$$\times\left\{ i\sum_n B_n\, \mathrm{sh}\, \frac{2n+1}{2}\eta\, e^{i\frac{2n+1}{2}\frac{\pi x}{K}} - {\sum_n}' \mathcal{L}_n\, \mathrm{sh}\, n\eta\, e^{i\frac{n\pi x}{K}} - \mathcal{L}_0 (K'-y_0)\right\}.$$

Renaming similarly, one obtains:
$$g_2 v = \sum_\lambda (2)_\lambda e^{i\frac{\lambda\pi x}{K}} + i\sum_\lambda [2]_\lambda e^{i\frac{2\lambda+1}{2}\frac{\pi x}{K}} ,$$

with:
$$\begin{aligned}(2)_\lambda = {} & \beta\, \mathrm{dn}\, y_0\, \mathrm{ch}\,\tilde{\alpha} \sum_\mu \sigma_{\mu-\lambda} B_\mu\, \mathrm{sh}\, \frac{2\mu+1}{2}\eta \\ & - \frac{\beta k}{k'}\, \mathrm{sn}\, y_0\, \mathrm{sh}\,\tilde{\alpha} {\sum_\mu}' \delta_{\mu-\lambda} \mathcal{L}_\mu\, \mathrm{sh}\,\mu\eta \\ & - \frac{\beta k}{k'}\, \mathrm{sn}\, y_0\, \mathrm{sh}\,\tilde{\alpha}\, \mathcal{L}_0\, \delta_\lambda (K'-y_0),\end{aligned}$$

$$\begin{aligned}[2]_\lambda = {} & \beta\, \mathrm{dn}\, y_0\, \mathrm{ch}\,\tilde{\alpha} {\sum_\mu}' \sigma_{\mu-\lambda-1} \mathcal{L}_\mu\, \mathrm{sh}\,\mu\eta \\ & + \beta\frac{k}{k'}\, \mathrm{sn}\, y_0\, \mathrm{sh}\,\tilde{\alpha} \sum_\mu \delta_{\mu-\lambda} B_\mu\, \mathrm{sh}\, \frac{2\mu+1}{2}\eta \\ & - \beta\, \mathrm{dn}\, y_0\, \mathrm{ch}\,\tilde{\alpha}\, \mathcal{L}_0\, \sigma_\lambda (K'-y_0).\end{aligned} \tag{2.20b}$$

The third term:

$$g_3(w+w_\infty) = -\left(i\, dn y_0\, sh\tilde{\alpha} \sum_m \sigma_m e^{i\frac{2m+1}{2}\frac{\pi x}{K}} + \frac{k}{k'} sn y_0\, ch\tilde{\alpha} \sum_m \delta_m e^{i\frac{m\pi x}{K}} \right)$$

$$\times \left\{ i \sum_n C_n\, sh\frac{2n+1}{2}\eta\, e^{i\frac{2n+1}{2}\frac{\pi x}{K}} - \sum_n{}' \mathcal{L}_n\, sh\, n\eta\, e^{i\frac{n\pi x}{K}} - \mathcal{L}_0(K'-y_0) + w_\infty \right\}.$$

This becomes:

$$g_3(w+w_\infty) = \sum_\lambda (3)_\lambda\, e^{i\frac{\lambda\pi x}{K}} + i \sum_\lambda [3]_\lambda\, e^{i\frac{2\lambda+1}{2}\frac{\pi x}{K}},$$

with:

$$\begin{aligned}(3)_\lambda = &- dn y_0\, sh\tilde{\alpha} \sum_\mu \sigma_{\mu-\lambda} C_\mu\, sh\frac{2\mu+1}{2}\eta \\ &+ \frac{k}{k'} sn y_0\, ch\tilde{\alpha} \left[\mathcal{L}_0(K'-y_0) - w_\infty\right]\delta_\lambda \\ &+ \frac{k}{k'} sn y_0\, ch\tilde{\alpha} \sum_\mu{}' \delta_{\mu-\lambda} \mathcal{L}_\mu\, sh\mu\eta\,, \\ [3]_\lambda = &- dn y_0\, sh\tilde{\alpha} \sum_\mu{}' \sigma_{\mu-\lambda-1} \mathcal{L}_\mu\, sh\mu\eta \\ &- \frac{k}{k'} sn y_0\, ch\tilde{\alpha} \sum_\mu \delta_{\mu-\lambda} C_\mu\, sh\frac{2\mu+1}{2}\eta \\ &+ dn y_0\, sh\tilde{\alpha} \left[\mathcal{L}_0(K'-y_0) - w_\infty\right]\sigma_\lambda\,. \end{aligned} \tag{2.20c}$$

Thus the whole boundary condition is seen to be expressible as the sum of two exponential series. The first of these has patently real coefficients, and proceeds according to even powers of exp. $i(\pi x/2K)$ while the second has imaginary coefficients and proceeds according to odd powers of the same exponential. It can be shown, using the numerous symmetry statements made earlier, that the coefficients of the former series are symmetric in λ, while the coefficients of the latter are antisymmetric. This will be carried out here only on $(1)_\lambda$ and $[1]_\lambda$ of (2.20a).

$$(1)_\lambda = \cdots \sum_\mu \gamma_{\mu-\lambda} A_\mu \, sh \frac{2\mu+1}{2} \eta .$$

We have for $(1)_{-\lambda}$, while at the same time we replace the dummy index μ by $-\mu - 1$:

$$(1)_{-\lambda} = \cdots \sum_\mu \gamma_{\lambda-\mu-1} A_{-\mu-1} \, sh\left(- \frac{2\mu+1}{2} \eta\right).$$

But according to (2.10) and (2.15)

$$\gamma_n = \gamma_{-n-1} \quad , \qquad A_n = - A_{-n-1} ,$$

hence $$(1)_{-\lambda} = \cdots \sum_\mu \gamma_{\mu-\lambda} A_\mu \, sh \frac{2\mu+1}{2} \eta = (1)_\lambda .$$

Likewise $$[1]_{-\lambda-1} = \cdots \sum_\mu{}' \gamma_{\mu+\lambda} \mathcal{C}_\mu \, sh \mu \eta$$

$$= \cdots \sum_{\mu'}{}' \gamma_{\lambda-\mu'} \mathcal{C}_{-\mu'} \, sh(-\mu' \eta),$$

after putting $\mu' = -\mu$. Since

$$\gamma_{\lambda-\mu'} = \gamma_{\mu'-\lambda-1} \quad , \qquad \mathcal{C}_{-\mu'} = \mathcal{C}_{\mu'}$$

the result is

$$[1]_{-\lambda-1} = -[1]_{\lambda} .$$

For the other coefficients the result is analogous in that all three coefficients in round brackets are symmetric and all those in square brackets antisymmetric. Hence by combining positive and negative indices the terms of the real symmetric series can be put into the form of a cosine series and the terms of imaginary antisymmetric series into that of a sine series with real coefficients. Therefore, it follows that:

$$\left(C\right)_{\lambda} \equiv (1)_{\lambda} + (2)_{\lambda} + (3)_{\lambda} = 0 ,$$

$$\left[S\right]_{\lambda} \equiv [1]_{\lambda} + [2]_{\lambda} + [3]_{\lambda} = 0 , \tag{2.21}$$

and this is to be regarded as a system of infinitely many linear equations for the series coefficients a_n and b_n of (2.9).

6. Closer Inspection of the System of Linear Equations

We put (2.11) into (2.20a), (2.12) into (2.20b), and (2.13) into (2.20c). The equation $\left(\mathbf{C}\right)_{\lambda} = 0$ takes the form

$$-\frac{\beta^2 K}{\pi k'^2}\, \mathrm{sn}\, y_0\, \mathrm{dn}\, y_0 \sum_\nu \sum_\mu \gamma_{\mu-\lambda}\gamma_{\mu-\nu} \frac{\mathrm{sh}\frac{2\mu+1}{2}\eta}{\frac{2\mu+1}{2}} a_\nu$$

$$-(\beta^2 \mathrm{ch}^2\tilde{\alpha} + \mathrm{sh}^2\tilde{\alpha})\frac{K}{\pi}\, \mathrm{dn}\, y_0 \sum_\nu \sum_\mu \sigma_{\mu-\lambda}\sigma_{\mu-\nu} \frac{\mathrm{sh}\frac{2\mu+1}{2}\eta}{\frac{2\mu+1}{2}} a_\nu$$

$$-(1+\beta^2)\,\mathrm{sh}\,\tilde{\alpha}\,\mathrm{ch}\,\tilde{\alpha}\frac{K}{\pi k'}\, \mathrm{dn}\, y_0 \sum_\nu \sum_\mu \sigma_{\mu-\lambda}\delta_{\mu-\nu} \frac{\mathrm{sh}\frac{2\mu+1}{2}\eta}{\frac{2\mu+1}{2}} b_\nu$$

$$+(\beta^2 \mathrm{sh}^2\tilde{\alpha} + \mathrm{ch}^2\tilde{\alpha})\frac{Kk}{\pi k'^2}\, \mathrm{sn}\, y_0 \sum_\nu \sum_\mu{}' \delta_{\mu-\lambda}\delta_{\mu-\nu} \frac{\mathrm{sh}\,\mu\eta}{\mu} a_\nu$$

$$-(1+\beta^2)\,\mathrm{sh}\,\tilde{\alpha}\,\mathrm{ch}\,\tilde{\alpha}\frac{Kk}{\pi k'}\, \mathrm{sn}\, y_0 \sum_\nu \sum_\mu{}' \delta_{\mu-\lambda}\sigma_{\mu-\nu-1} \frac{\mathrm{sh}\,\mu\eta}{\mu} b_\nu$$

$$+(\beta^2 \mathrm{sh}^2\tilde{\alpha} + \mathrm{ch}^2\tilde{\alpha})\frac{k}{k'^2}\, \mathrm{sn}\, y_0\, (K'-y_0)\,\delta_\lambda \sum_\nu \delta_\nu a_\nu$$

$$+(1+\beta^2)\,\mathrm{sh}\,\tilde{\alpha}\,\mathrm{ch}\,\tilde{\alpha}\frac{k}{k'}\, \mathrm{sn}\, y_0\, (K'-y_0)\,\delta_\lambda \sum_\nu \sigma_\nu b_\nu$$
$$= \frac{k}{k'}\, \mathrm{sn}\, y_0\, \mathrm{ch}\,\tilde{\alpha}\, \delta_\lambda\, w_\infty .$$

The introduction of the following abbreviations is advantageous:

$$t_\mu = \frac{\mathrm{sh}\frac{2\mu+1}{2}\eta}{\frac{2\mu+1}{2}} = t_{-\mu-1}\,,$$
$$\tau_\mu = \frac{\mathrm{sh}\,\mu\eta}{\mu} = \tau_{-\mu}\,, \tag{2.22}$$

$$l_{11} = \beta^2 \mathrm{ch}^2\tilde{\alpha} + \mathrm{sh}^2\tilde{\alpha}\,,$$
$$l_{22} = \beta^2 \mathrm{sh}^2\tilde{\alpha} + \mathrm{ch}^2\tilde{\alpha}\,, \tag{2.23}$$
$$l_{12} = (1+\beta^2)\,\mathrm{sh}\,\tilde{\alpha}\,\mathrm{ch}\,\tilde{\alpha}\,,$$

The l_{ij} are the transformation coefficients which also appear in another connection.[1] The final form of the first equation of (2.21) becomes

$$
\begin{aligned}
&\frac{K}{\pi}\sum_{\nu}\Big\{-\frac{\beta^2}{k'^2}\,\mathrm{sn}y_0\,\mathrm{dn}y_0\sum_{\mu}\gamma_{\mu-\lambda}\,\gamma_{\mu-\nu}\,t_\mu \\
&\qquad -l_{11}\,\mathrm{dn}y_0\sum_{\mu}\sigma_{\mu-\lambda}\,\sigma_{\mu-\nu}\,t_\mu \\
&\qquad +\frac{k}{k'^2}\,\mathrm{sn}y_0\,l_{22}\Big(\sum_{\mu}{}'\,\delta_{\mu-\lambda}\,\delta_{\mu-\nu}\,\tau_\mu+\delta_\lambda\,\delta_\nu\,\eta\Big)\Big\}a_\nu \\
&-\frac{K}{\pi k'}\,l_{12}\sum_{\nu}\Big\{\mathrm{dn}y_0\sum_{\mu}\sigma_{\mu-\lambda}\,\delta_{\mu-\nu}\,t_\mu \\
&\qquad +k\,\mathrm{sn}y_0\Big(\sum_{\mu}{}'\,\delta_{\mu-\lambda}\,\sigma_{\mu-\nu-1}\,\tau_\mu-\delta_\lambda\,\sigma_\nu\,\eta\Big)\Big\}b_\nu \\
&=\frac{k}{k'}\,\mathrm{sn}y_0\,\mathrm{ch}\tilde{\alpha}\,\delta_\lambda\,w_\infty\,.
\end{aligned}
\tag{2.24}
$$

We now turn to $[S]_\lambda = 0$ which becomes in the same manner

[1] R.F.C. Bartels and O. Laporte, in Proceedings of the Conformal Mapping Conference held by the Inst. of Numerical Analysis, N.C.L.A., June, 1949.

$$\frac{\beta^2 K}{\pi k'^2} \operatorname{sn} y_0 \operatorname{dn} y_0 \sum_{\nu} \sum_{\mu}{}' \gamma_{\mu-\lambda-1}\, \gamma_{\mu-\nu-1} \frac{\operatorname{sh} \mu\eta}{\mu}\, b_\nu$$

$$+\left(\beta^2 \operatorname{ch}^2 \tilde{\alpha} + \operatorname{sh}^2 \tilde{\alpha}\right) \frac{K}{\pi} \operatorname{dn} y_0 \sum_{\nu} \sum_{\mu}{}' \sigma_{\mu-\lambda-1}\, \sigma_{\mu-\nu-1} \frac{\operatorname{sh} \mu\eta}{\mu}\, b_\nu$$

$$-\left(1+\beta^2\right) \operatorname{sh} \tilde{\alpha} \operatorname{ch} \tilde{\alpha} \frac{K}{\pi k'} \operatorname{dn} y_0 \sum_{\nu} \sum_{\mu}{}' \sigma_{\mu-\lambda-1}\, \delta_{\mu-\nu} \frac{\operatorname{sh} \mu\eta}{\mu}\, a_\nu$$

$$-\left(\beta^2 \operatorname{sh}^2 \tilde{\alpha} + \operatorname{ch}^2 \tilde{\alpha}\right) \frac{K k}{\pi k'^2} \operatorname{sn} y_0 \sum_{\nu} \sum_{\mu} \delta_{\mu-\lambda}\, \delta_{\mu-\nu} \frac{\operatorname{sh} \frac{2\mu+1}{2}\eta}{\frac{2\mu+1}{2}}\, b_\nu$$

$$-\left(1+\beta^2\right) \operatorname{sh} \tilde{\alpha} \operatorname{ch} \tilde{\alpha} \frac{K k}{\pi k'} \operatorname{sn} y_0 \sum_{\nu} \sum_{\mu} \delta_{\mu-\lambda}\, \sigma_{\mu-\nu} \frac{\operatorname{sh} \frac{2\mu+1}{2}\eta}{\frac{2\mu+1}{2}}\, a_\nu$$

$$+\left(\beta^2 \operatorname{ch}^2 \tilde{\alpha} + \operatorname{sh}^2 \tilde{\alpha}\right) \operatorname{dn} y_0 \left(K' - y_0\right) \sigma_\lambda \sum_{\nu} \sigma_\nu\, b_\nu$$

$$+\left(1+\beta^2\right) \operatorname{sh} \tilde{\alpha} \operatorname{ch} \tilde{\alpha} \frac{1}{k'} \operatorname{dn} y_0 \left(K' - y_0\right) \sigma_\lambda \sum_{\nu} \delta_\nu\, a_\nu$$

$$= \operatorname{sh} \tilde{\alpha}\, \sigma_\lambda \operatorname{dn} y_0\, w_\infty .$$

This also simplifies using formulae (2.22) and (2.23)

$$\frac{K}{\pi}\sum_{\nu}\Big\{\frac{\beta^2}{k'^2}\,\mathrm{sn}y_0\,\mathrm{dn}y_0\sum_{\mu}{}'\gamma_{\mu-\lambda-1}\gamma_{\mu-\nu-1}\,\tau_\mu$$

$$-\frac{k}{k'^2}\,\mathrm{sn}y_0\,l_{22}\sum_{\mu}\delta_{\mu-\lambda}\,\delta_{\mu-\nu}\,t_\mu$$

$$+\mathrm{dn}y_0\,l_{11}\Big(\sum_{\mu}{}'\sigma_{\mu-\lambda-1}\,\sigma_{\mu-\nu-1}\,\tau_\mu+\sigma_\lambda\sigma_\nu\,\eta\Big)\Big\}b_\nu \tag{2.25}$$

$$-\frac{K}{\pi k'}\,l_{12}\sum_{\nu}\Big\{k\,\mathrm{sn}y_0\sum_{\mu}\delta_{\mu-\lambda}\,\sigma_{\mu-\nu}\,t_\mu$$

$$+\Big(\sum_{\mu}{}'\sigma_{\mu-\lambda-1}\,\delta_{\mu-\nu}\,\tau_\mu-\sigma_\lambda\,\delta_\nu\,\eta\Big)\Big\}a_\nu$$

$$=\mathrm{dn}y_0\,\mathrm{sh}\tilde{\alpha}\,\sigma_\lambda\,w_\infty\,.$$

7. The Eight Types of Infinite Series; Discussion of Small α

Already in the earlier work, for $\alpha = 0$, two types of infinite series, then called $S_{\lambda\mu}$ and $\Sigma_{\lambda\mu}$, occurred in an essential fashion within the coefficient matrix of the linear equation system. The considerably greater complication of the angle of attack makes itself felt in the appearance of six more, but very similar, types of series. Inasmuch as a systematic notation is necessary, the symbols $S_{\lambda\mu}$ and $\Sigma_{\lambda\mu}$ have been discarded and the following table of definitions has been decided upon.[1]

$$
\begin{aligned}
{}_1P_{\lambda\nu} &= \sum_\mu \gamma_{\mu-\lambda}\,\gamma_{\mu-\nu}\, t_\mu\,, & {}_4P_{\lambda\nu} &= \sum_\mu{}' \gamma_{\mu-\lambda-1}\,\gamma_{\mu-\nu-1}\,\tau_\mu\,,\\
{}_2P_{\lambda\nu} &= \sum_\mu \sigma_{\mu-\lambda}\,\sigma_{\mu-\nu}\, t_\mu\,, & {}_5P_{\lambda\nu} &= \sum_\mu{}' \sigma_{\mu-\lambda-1}\,\sigma_{\mu-\nu-1}\,\tau_\mu\,,\\
{}_3P_{\lambda\nu} &= \sum_\mu{}' \delta_{\mu-\lambda}\,\delta_{\mu-\nu}\,\tau_\mu\,, & {}_6P_{\lambda\nu} &= \sum_\mu \delta_{\mu-\lambda}\,\delta_{\mu-\nu}\, t_\mu\,,
\end{aligned}
\qquad (2.26)
$$

$$
{}_7P_{\lambda\nu} = \sum_\mu \sigma_{\mu-\lambda}\,\delta_{\mu-\nu}\, t_\mu\,,
$$

$$
{}_8P_{\lambda\nu} = \sum_\mu{}' \delta_{\mu-\lambda}\,\sigma_{\mu-\nu-1}\,\tau_\mu\,.
$$

[1] This table does not exhaust the possibilities for further, related series of the unsymmetric kind; four more unsymmetric series will appear when angles of yaw are considered as well. These further series, incidentally, show the same interesting properties as ${}_1P$ to ${}_8P$ that are to be discussed in Section 8.

As is seen, ${}_1\mathfrak{p}$ to ${}_6\mathfrak{p}$ are symmetric, ${}_7\mathfrak{p}$ and ${}_8\mathfrak{p}$ are unsymmetric, in λ and ν. They are the only series occurring both in (2.24) and (2.25). Looking at the round brackets of these equations, one sees that a zeroth term appears for ${}_3\mathfrak{p}$, ${}_5\mathfrak{p}$ and ${}_8\mathfrak{p}$, which is in each case the limit of the general term of the series for $\mu \to 0$. With this understanding, the accent on the sum signs of the $\mathcal{T}_\mu$ series could have been avoided.[1] All eight series converge rapidly for any possible value of y_0, even including $y_0 = 0$. (Connection between y_0 and η is $\eta = \frac{\pi}{K}(K' - y_0)$.)

Using these definitions, (2.24) and (2.25) can be written:

$$\frac{K}{\pi}\sum_\nu \left\{ -\frac{\beta^2}{k'^2} \operatorname{sn} y_0 \operatorname{dn} y_0 \, {}_1\mathfrak{p}_{\lambda\nu} - \operatorname{dn} y_0 \, l_{11} \, {}_2\mathfrak{p}_{\lambda\nu} + \frac{k}{k'^2} \operatorname{sn} y_0 \, l_{22} \left({}_3\mathfrak{p}_{\lambda\nu} + \delta_\lambda \delta_\nu \eta \right) \right\} a_\nu$$
$$- \frac{K}{\pi k'} l_{12} \sum_\nu \left\{ \operatorname{dn} y_0 \, {}_7\mathfrak{p}_{\lambda\nu} + k \operatorname{sn} y_0 \left({}_8\mathfrak{p}_{\lambda\nu} - \delta_\lambda \sigma_\nu \eta \right) \right\} b_\nu \qquad (2.27_1)$$
$$= \frac{k}{k'} \operatorname{sn} y_0 \operatorname{ch} \tilde{\alpha} \, \delta_\lambda w_\infty ,$$

$$\frac{K}{\pi}\sum_\nu \left\{ \frac{\beta^2}{k'^2} \operatorname{sn} y_0 \operatorname{dn} y_0 \, {}_4\mathfrak{p}_{\lambda\nu} - \frac{k}{k'^2} \operatorname{sn} y_0 \, l_{22} \, {}_6\mathfrak{p}_{\lambda\nu} + \operatorname{dn} y_0 \, l_{11} \left({}_5\mathfrak{p}_{\lambda\nu} + \sigma_\lambda \sigma_\nu \eta \right) \right\} b_\nu$$
$$- \frac{K}{\pi k'} l_{12} \sum_\nu \left\{ k \operatorname{sn} y_0 \, {}_7\mathfrak{p}_{\nu\lambda} + \operatorname{dn} y_0 \left({}_8\mathfrak{p}_{\nu\lambda} - \sigma_\lambda \delta_\nu \eta \right) \right\} a_\nu \qquad (2.27_2)$$
$$= \operatorname{dn} y_0 \operatorname{sh} \tilde{\alpha} \, \sigma_\lambda w_\infty .$$

---

[1]The contribution of the zeroth term of ${}_4\mathfrak{p}$ vanishes after summation with respect to ν.

With these last two equations the problem is solved to the same degree to which the problem of the elliptic cone without angle of attack was solved in earlier reports of this series.

For the present, it is instructive to consider the problem for small angles of attack. It seems reasonable to attempt a successive approximation scheme in powers of sh $\tilde{\alpha}$ of this sort:

$$a_\nu = {}_0a_\nu + \operatorname{sh}^2\tilde{\alpha}\,{}_2a_\nu + \cdots ,$$

$$b_\nu = \operatorname{sh}\tilde{\alpha}\,{}_1b_\nu + \cdots . \tag{2.28}$$

The first set alone contributes to the zeroth order:

$$\begin{aligned} &\frac{K}{\pi}\sum_\nu \Big\{ -\frac{\beta^2}{k'^2}\operatorname{sn}y_0 \operatorname{dn}y_0\,{}_1p_{\lambda\nu} - \beta^2 \operatorname{dn}y_0\,{}_2p_{\lambda\nu} \\ &\quad + \frac{k}{k'^2}\operatorname{sn}y_0\left({}_3p_{\lambda\nu} + \delta_\lambda\delta_\nu\eta\right)\Big\}\,{}_0a_\nu \\ &= \frac{k}{k'}\operatorname{sn}y_0\,\delta_\lambda\,w_\infty . \end{aligned} \tag{2.29}$$

This equation solves the problem for $\alpha = 0$ and is therefore the equivalent of our earlier results. That no exact agreement in form exists is due to the slight modification in method as discussed in Section 2 of Part II.

The first order correction is obtained from the second equation as follows:

$$\frac{K}{\pi}\sum_{\nu}\left\{\frac{\beta^2}{k'^2}\, sn y_0\, dn y_0\, {}_4P_{\lambda\nu} - \frac{k}{k'^2}\, sn y_0\, {}_6P_{\lambda\nu} + \beta^2 dn y_0\left({}_5P_{\lambda\nu} + \sigma_\lambda \sigma_\nu \eta\right)\right\}{}_1b_\nu \tag{2.30}$$

$$= dn y_0\, \sigma_\lambda\, w_\infty + \frac{K}{\pi k'}(1+\beta^2)\sum_{\nu}\left\{k\, sn y_0\, {}_7P_{\nu\lambda} + dn y_0\left({}_8P_{\nu\lambda} - \sigma_\lambda \delta_\nu \eta\right)\right\}{}_0a_\nu .$$

In this equation system the right-hand side contains terms originally on the left; but their position is justified by the fact that for a_ν now ${}_0a_\nu$ must be used. A second order correction would again arise from (2.25_2).

8. Further Discussion of the Eight Infinite Series (2.24)

Whether the approximate system (2.29), (2.30) is solved, or the rigorous system (2.27_1), (2.27_2), the series ${}_1P$ to ${}_8P$ of (2.26) should be known for a reasonable range of their arguments. However, the arguments are so numerous as to make tabulation difficult. At one time it was proposed that the series be calculated at the Dahlgren Laboratory by means of the Mark II calculating machine, but the idea was abandoned because of the considerable cost.

Some time has been devoted to attempts of summing the series, i.e., of expressing them in other analytical forms, not involving infinite sums. It has been found, using an extension of the method described in Appendix B of the Part II report (or Bumblebee Report No. 75, Appendix) that all series can be written as indefinite integrals with respect to η whose integrands are products of one elliptic function with several hyperbolic functions. It is known that indefinite integrals of this type cannot be reduced in closed form. Hence, the best that can be derived from this is graphical integration, but it is doubtful whether this process is faster than the direct summation of the rather rapidly converging series themselves.

We spoke above of the large number of arguments of the series. Apart from depending upon the continuous variables y_0 and k (and therefore upon the angle of attack) they are matrices as far as the indices λ and ν are concerned. In order to calculate n equations of (2.25_1) and (2.25_2), $4n^2$ matrix elements of ${}_1P_{\lambda\nu}$ have to be calculated. It is therefore encouraging to be able to report a method which will materially reduce the labor of calculating. Instead of having to calculate an (infinite) square array, it will be shown to be sufficient to calculate two diagonals.

For the transformation of the ${}_1P_{\lambda\nu}$ which is being considered, it is necessary to use the following three elementary identities: let a and b be any two real quantities; then:

$$(\mathring{i}_1) \qquad \frac{1}{\operatorname{ch} a \operatorname{ch} b} = \frac{\operatorname{th} a - \operatorname{th} b}{\operatorname{sh}(a-b)} \, ,$$

$$(\mathring{i}_2) \qquad \frac{1}{\operatorname{sh} a \operatorname{sh} b} \qquad \frac{\operatorname{coth} b - \operatorname{coth} a}{\operatorname{sh}(a-b)} \, ,$$

$$(\mathring{i}_3) \qquad \frac{1}{\operatorname{sh} a \operatorname{ch} b} \qquad \frac{\operatorname{coth} a - \operatorname{th} b}{\operatorname{ch}(a-b)} \, .$$

As an example only, ${}_1P_{\lambda\nu}$ will be considered. Using the abbreviation

$$\frac{\pi K'}{2K} = \zeta \, ,$$

put $a = [2(\mu-\lambda)+1]\zeta$; $b = [2(\mu-\nu)+1]\zeta$

into identity (i_1):

$$\gamma_{\mu-\lambda}\,\gamma_{\mu-\nu} = \left(\frac{\pi}{2Kk}\right)^2 \frac{1}{sh\,2(\nu-\lambda)\zeta}\left\{th\,[2(\mu-\lambda)+1]\zeta - th\,[2(\mu-\nu)+1]\zeta\right\}.$$

Hence ${}_1p_{\lambda\nu}$ may be written:

$$sh\,2(\nu-\lambda)\zeta\;{}_1p_{\lambda\nu} = T_\lambda^* - T_\nu^*,$$

with

$$T_\lambda^* = \left(\frac{\pi}{2Kk}\right)^2 \sum_\mu th\,[2(\mu-\lambda)+1]\zeta\; t_\mu . \tag{2.31}$$

The obvious and considerable advantage is now that the series, ${}_1p_{\lambda\nu}$ after multiplication by sh $2(\nu-\lambda)\zeta$ can be decomposed into the difference of two series, each of which depends only upon one index. This decomposition is exactly of the kind used in spectroscopy, where the frequencies of the spectrum lines of an atom or molecule can be written as differences of "spectral terms" or of "energy levels."

However, the decomposition is at present only formal, because the series T_λ^* obviously diverges for real η . (Hence the last two equations have to be taken with a mathematical grain of salt). The situation is nevertheless easily remediable in any of two ways. Either one subtracts from T_λ^* a series independent of λ, but of the same degree of divergence, thereby leaving $T_\lambda^* - T_\nu^*$ unaffected. Or, by replacing μ by $-\mu - 1$, one has an alternative:

$$T_\lambda^* = -\left(\frac{\pi}{2Kk}\right)^2 \sum_\mu \mathrm{th}\,[2(\mu+\lambda)+1]\zeta \; t_\mu$$

$$= - T_{-\lambda}^* \,.$$

Hence one has a definition which is free from any taint:[1]

$$T_\lambda = \frac{1}{2}\left(\frac{\pi}{2Kk}\right)^2 \sum_\mu \left\{\mathrm{th}[2(\mu-\lambda)+1]\zeta - \mathrm{th}\,[2(\mu+\lambda)+1]\zeta\right\} t_\mu$$

or, using (i_1)

$$T_\lambda = -\frac{1}{2}\mathrm{sh}\,4\lambda\zeta \left(\frac{\pi}{2Kk}\right)^2 \sum_\mu \frac{1}{\mathrm{ch}[2(\mu-\lambda)+1]\zeta\,\mathrm{ch}[2(\mu+\lambda)+1]\zeta}\, t_\mu$$

or $$T_\lambda = -\frac{1}{2}\mathrm{sh}\,4\lambda\zeta \;{}_1P_{\lambda,-\lambda} \,. \tag{2.32}$$

One can even avoid the T_λ symbol altogether and write:

$$\mathrm{sh}\,2(\lambda-\nu)\zeta \;{}_1P_{\lambda\nu} = \frac{1}{2}\mathrm{sh}\,4\lambda\zeta \;{}_1P_{\lambda,-\lambda} - \frac{1}{2}\mathrm{sh}\,4\nu\zeta \;{}_1P_{\nu,-\nu} \,. \tag{2.33}$$

This method can be applied to all eight series through application of identity

---

[1]From now on, the taint having been removed, we shall also remove the asterisk from the T_λ symbol.

(i_1) to ${}_1\mathfrak{p}$, ${}_3\mathfrak{p}$, ${}_4\mathfrak{p}$, ${}_6\mathfrak{p}$; of (i_2) to ${}_2\mathfrak{p}$, ${}_5\mathfrak{p}$; and of (i_3) to ${}_7\mathfrak{p}$, ${}_8\mathfrak{p}$. The following table summarizes the results:

(1) $$2\,\mathrm{sh}\,2(\lambda-\nu)\zeta\;{}_1P_{\lambda\nu} = \mathrm{sh}\,4\lambda\zeta\;{}_1P_{\lambda,-\lambda} - \mathrm{sh}\,4\nu\zeta\;{}_1P_{\nu,-\nu}\,,$$

(2) $$2\,\mathrm{sh}\,2(\lambda-\nu)\zeta\;{}_2P_{\lambda\nu} = \mathrm{sh}\,4\lambda\zeta\;{}_2P_{\lambda,-\lambda} - \mathrm{sh}\,4\nu\zeta\;{}_2P_{\nu,-\nu}\,,$$

(3) $$2\,\mathrm{sh}\,2(\lambda-\nu)\zeta\;{}_3P_{\lambda\nu} = \mathrm{sh}\,4\lambda\zeta\;{}_3P_{\lambda,-\lambda} - \mathrm{sh}\,4\lambda\zeta\;{}_3P_{\nu,-\nu}\,,$$

(4) $$2\,\mathrm{sh}\,2(\lambda-\nu)\zeta\;{}_4P_{\lambda\nu} = \mathrm{sh}\,2(2\lambda+1)\zeta\;{}_4P_{\lambda,-\lambda-1} - \mathrm{sh}\,2(2\nu+1)\zeta\;{}_4P_{\nu,-\nu-1}\,,$$

(5) $$2\,\mathrm{sh}\,2(\lambda-\nu)\zeta\;{}_5P_{\lambda\nu} = \mathrm{sh}\,2(2\lambda+1)\zeta\;{}_5P_{\lambda,-\lambda-1} - \mathrm{sh}\,2(2\nu+1)\zeta\;{}_5P_{\nu,-\nu-1}\,,$$

(6) $$2\,\mathrm{sh}\,2(\lambda-\nu)\zeta\;{}_6P_{\lambda\nu} = \mathrm{sh}\,2(2\lambda+1)\zeta\;{}_6P_{\lambda,-\lambda-1} - \mathrm{sh}\,2(2\nu+1)\zeta\;{}_6P_{\nu,-\nu-1}\,,$$

(7) $$2\,\mathrm{ch}\left[2(\nu-\lambda)+1\right]\zeta\;{}_7P_{\lambda\nu} = -k\,\mathrm{sh}\,4\lambda\zeta\;{}_2P_{\lambda,-\lambda} - \frac{1}{k}\,\mathrm{sh}\,2(2\nu+1)\zeta\;{}_6P_{\nu,-\nu-1}\,,$$

(8) $$2\,\mathrm{ch}\left[2(\lambda-\nu)+1\right]\zeta\;{}_8P_{\lambda\nu} = -\frac{1}{k}\,\mathrm{sh}\,4\lambda\zeta\;{}_3P_{\lambda,-\lambda} - k\,\mathrm{sh}\,2(2\nu+1)\zeta\;{}_5P_{\nu,-\nu-1}\,.$$

Several facts are worth remarking about these decomposition formulae. While ${}_1\mathfrak{p}$ to ${}_6\mathfrak{p}$ employ the same type of expression both for λ and for ν, the formulae for ${}_7\mathfrak{p}$ and ${}_8\mathfrak{p}$ are unsymmetric. However, no new series are needed in the decomposition formulae for ${}_7\mathfrak{p}$ and ${}_8\mathfrak{p}$; one sees that for the eight series ${}_i\mathfrak{p}$ only six "term sequences" need be computed, viz:

$$ {}_1p_{\lambda,-\lambda} \quad , \quad {}_2p_{\lambda,-\lambda} \quad , \quad {}_3p_{\lambda,-\lambda} \quad , $$

$$ {}_4p_{\lambda,-\lambda-1} \quad , \quad {}_5p_{\lambda,-\lambda-1} \quad , \quad {}_6p_{\lambda,-\lambda-1} \quad , $$

(with ${}_7p$ and ${}_8p$ playing the roles of "inter-system combinations"). Even among the above six series one can notice the relationship of ${}_3p$ and ${}_4p$ and ${}_1p$ and ${}_6p$ when admitting half-integer indices.

It is to be noted that the first six of the above identities become nugatory for $\lambda = \nu$. Nor does it help to let ν approach λ for only the original definition of ${}_ip_{\lambda\lambda}$ $(i = 1, 2, 3, 4, 5, 6)$ as may directly be got from (2.24) results. But for ${}_7p$ and ${}_8p$ this is not so; their diagonal elements can perfectly well be obtained from their decomposition formulae $(2.33_{7,8})$.

We now turn to a discussion of the computational simplifications effected by the decomposition. The formulae show clearly the advantage of working with auxiliary, antisymmetric matrices defined by

$$ {}_i\pi_{\lambda\nu} = \sinh 2(\lambda-\nu)\zeta \; {}_ip_{\lambda\nu} , \qquad i = 1, \cdots, 6 \tag{2.35} $$

such that

$$ {}_i\pi_{\lambda\nu} = {}_i\pi_{\lambda,-\lambda} - {}_i\pi_{\nu,-\nu} . \tag{2.36} $$

Hence, for instance

$$ {}_{i}^{\circ}\pi_{\lambda,\nu+1} - {}_{i}^{\circ}\pi_{\lambda\nu} = {}_{i}^{\circ}\pi_{\nu+1,\nu} , $$

or a function of ν only. This result confirms that in order to know a ${}_{i}^{\circ}\pi$ matrix, one has only to calculate the values ${}_{i}^{\circ}\pi$ assumes along any straightline sequence of lattice points. All others can be found by addition and subtraction. Figure 9 illustrates this. The curly brackets at the right-hand margin labelled $\pi_{1,2}$, $\pi_{0,1}$, $\pi_{1,0}$, etc., are meant to indicate that the difference of any pair of elements above one another is constant for that pair of rows and is equal to the particular π indicated at the right. Similarly, correspondingly situated elements in neighboring columns have the constant difference indicated at the bottom. One is reminded of the regularities between the frequencies of the spectrum lines after they are arranged into "multipletts." From this behavior it becomes evident that one merely needs to calculate, for instance, the elements $\pi_{1,1}$, $\pi_{2,2}$, $\pi_{3,3}$, . . . in order to obtain all other π_{λ} by the application of the "Rydberg-Ritz Combination Principle." After that, one obtains, by division with sh $2(\lambda - \nu)\zeta$, all $\rho_{\lambda\nu}$ except the diagonal elements $\rho_{\lambda,\lambda}$. These latter have to be found by direct calculation using the original definition.

9. Avoidance of Negative Summation Indices by Symmetrization

The linear equation system (2.27_1), (2.27_2) will once more be briefly considered. It should be realized that the summation indices λ, μ and ν may assume all values from $-\infty$ to $+\infty$, but that for numerical calculation one often prefers to have positive indices only. However, from the point of view of brevity of writing this restriction is not advantageous, since the term, the

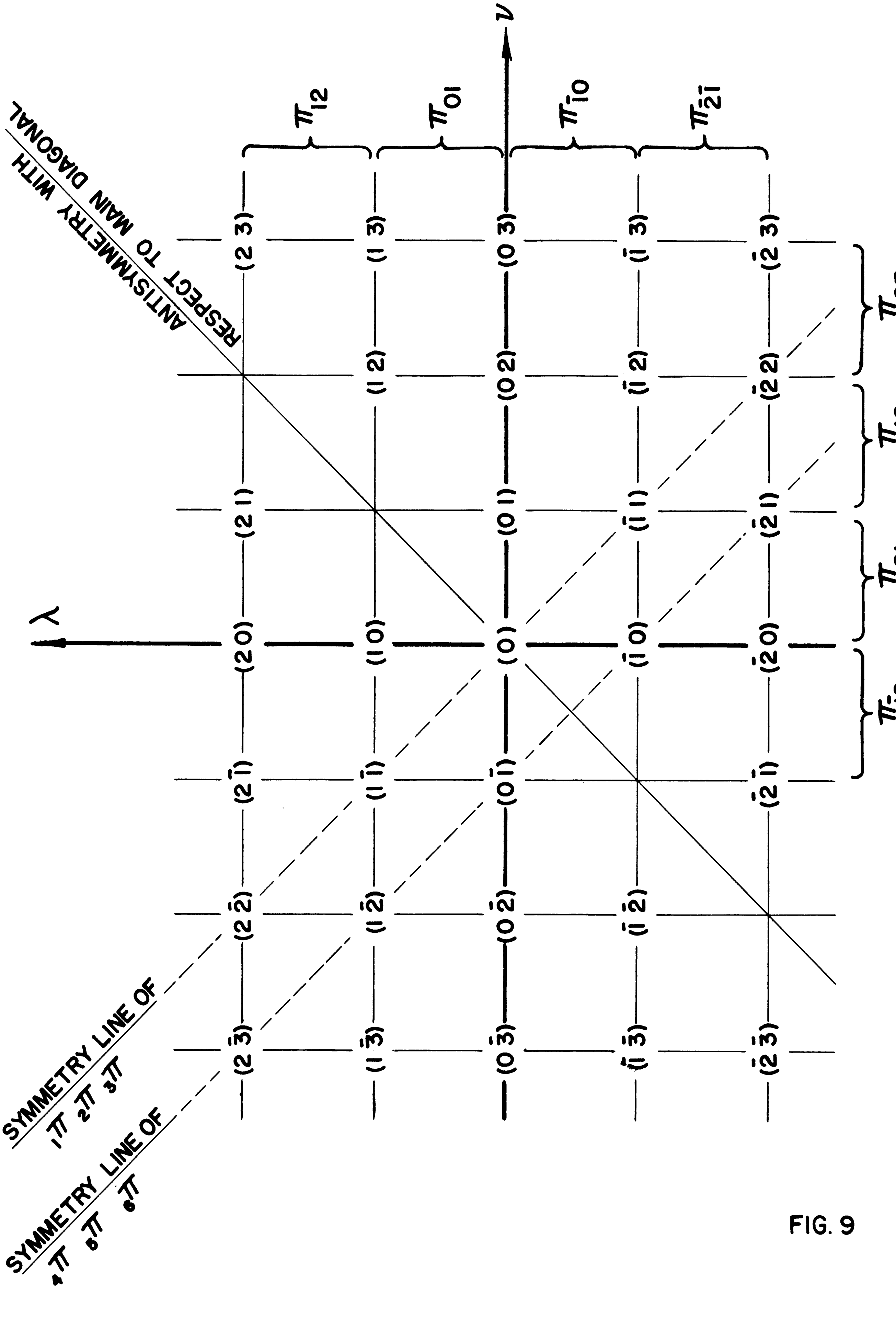

ANTISYMMETRY WITH RESPECT TO MAIN DIAGONAL
SYMMETRY LINE OF ${}_1\pi$ ${}_2\pi$ ${}_3\pi$
SYMMETRY LINE OF ${}_4\pi$ ${}_5\pi$ ${}_6\pi$
ν
γ
π_{12}
π_{01}
$\pi_{\bar{1}0}$
$\pi_{\bar{2}\bar{1}}$
π_{23}
π_{12}
π_{01}
$\pi_{\bar{1}0}$

FIG. 9

index of which is equal to zero, frequently has to be written down separately. The effect of this avoidance of negative indices on the eight series ${}_{i}p_{\lambda\nu}$ is that linear combinations have to be introduced as follows:

$$p^{1,2,3}_{\lambda\nu} = {}_{1,2,3}p_{\lambda\nu} + {}_{1,2,3}p_{\lambda,-\nu} + {}_{1,2,3}p_{-\lambda,\nu} + {}_{1,2,3}p_{-\lambda,-\nu} ,$$

$$p^{4,5,6}_{\lambda\nu} = {}_{4,5,6}p_{\lambda\nu} + {}_{4,5,6}p_{-\lambda-1,-\nu-1} - {}_{4,5,6}p_{\lambda,-\nu-1} - {}_{4,5,6}p_{-\lambda-1,\nu} , \quad (2.36) .$$

$$p^{7,8}_{\lambda\nu} = {}_{7,8}p_{\lambda\nu} + {}_{7,8}p_{-\lambda,\nu} - {}_{7,8}p_{\lambda,-\nu-1} - {}_{7,8}p_{-\lambda,-\nu-1} .$$

The resulting series,which are now distinguished by superscripts rather than as their constituents by subscripts on the left, need only be recorded for positive values of their indices λ and ν . However, since the Rydberg-Ritz type of addition scheme holds only for the ${}_{i}\pi_{\lambda,\nu}$ which are intimately connected with the ${}_{i}p_{\lambda,\nu}$ the latter series must always be calculated first.

10. A Further Approximative Method: Pointwise Fulfillment of the Boundary Condition

In Section 7 we developed an approximate method which presupposes that the angle of attack is small. The following developments are free from this assumption, but instead are based upon the idea of approximately fulfilling the boundary conditions.

As was seen in Section 2, it is the unfortunate fact that the boundary condition on the material cone is endowed with coefficients yet depending on the variable x which leads, for the determination of the Fourier coefficients a_n, b_n , to the infinite system of linear equations. Since experience (especially for zero α) has shown that for all but the slimmest cones the

Fourier series will converge rather rapidly, it seems attractive to seek an approximate solution in the form of a finite Fourier series for $\mathfrak{F}(z')$, or in case of zero α, for the velocity component w. The finite set of the coefficients of these series can then be determined by fulfilling the boundary condition at a finite number of points or rather rays, $\mathcal{X}_i$, along the surface of the material cone. This method, which is fashioned after the Glauert-Lotz methods of calculating spanwise lift distributions, is of course not restricted to elliptic cones, but can be used for more general cones. It seems natural to space the points $\mathcal{X}_i$ along the circumference of the cone more closely in such regions where rapid variations of the velocities can be expected.

Suppose that, for an elliptic case with $\alpha = 0$ the following complex potential is assumed

$$\mathfrak{F}(z') = a_0 + 2\, a_1 \cos \frac{\pi z'}{K} .$$

Then we shall fulfill the boundary condition at the four vertical rays $\mathcal{X} = 0$, 2K and $\mathcal{X} = \pm K$, which gives, due to symmetry, two linear equations for a_0 and a_1. But in the case of a nonvanishing angle of attack, we enter the boundary condition with

$$\mathfrak{F}(z') = a_0 + 2\, a_1 \cos \frac{\pi z'}{K} - 2\, b_0 \sin \frac{\pi z'}{2K} .$$

However, the same values of $\mathcal{X}$ afford the determination of the three coefficients since, due to lower symmetry, the points $\mathcal{X} = K$ and 3K now give different results. Similarly, when operating with a_0, a_1, a_2 for $\alpha = 0$, one should put $\mathcal{X} = 0$, K/2, and K, thereby fulfilling the boundary condition along eight rays, while for $\alpha \neq 0$ the coefficients of a series with a_0, a_1, a_2, b_0, b_1 can be found at $\mathfrak{X} = 0, \pm K/2, \pm K$, which satisfies the boundary condition at the same eight rays. The following discussion is for an $\mathfrak{F}(z')$ of the above form, the coefficient being found at $\mathfrak{X} = 0, \pm K$. At $\mathfrak{X} = 0$ the boundary condition consists of the following three terms:

$$g_1 u = -\beta \, \mathrm{sn} y_0 \, \mathrm{dn} y_0 \sum_\mu A_\mu \, \mathrm{sh} \frac{2\mu+1}{2} \eta$$

$$= -\frac{\beta^2 K}{\pi k'} \mathrm{sn} y_0 \, \mathrm{dn} y_0 \left(\sum_\mu \gamma_\mu t_\mu a_0 + 2 \sum_\mu \gamma_{\mu+1} t_\mu a_1 \right),$$

where the abbreviations introduced in (2.22) are used.

The second and the last term are at $\mathfrak{X} = 0$:

$$g_2 v = -\beta k \, \mathrm{sn} y_0 \, \mathrm{sh} \tilde{\alpha} \left\{ \sum_\mu{}' \mathcal{L}_\mu \, \mathrm{sh} \mu\eta + \mathcal{L}_0 (K' - y_0) \right\}$$

$$= \beta^2 k \, \mathrm{sn} y_0 \, \mathrm{sh} \tilde{\alpha} \left\{ \frac{K}{\pi} \sum_\mu{}' \left[\frac{\mathrm{sh} \tilde{\alpha}}{k'} \left(\delta_\mu \tau_\mu a_0 + 2 \delta_{\mu+1} \tau_\mu a_1 \right) \right. \right.$$

$$\left. + 2 \, \mathrm{ch} \tilde{\alpha} \, \sigma_\mu \tau_\mu b_0 \right]$$

$$\left. + \left[\frac{\mathrm{sh} \tilde{\alpha}}{k'} \left(\delta_0 a_0 + 2 \delta_1 a_1 \right) + 2 \, \mathrm{ch} \tilde{\alpha} \, \sigma_0 b_0 \right] (K' - y_0) \right\},$$

$$g_3(w+w_\infty) = -k\, sn\, y_0\, ch\tilde{\alpha}\left[-\sum_\mu{}' \mathcal{L}_\mu\, sh\,\mu\eta - \mathcal{L}_0(K'-y_0) + w_\infty\right]$$

$$= k\, sn\, y_0\, ch\tilde{\alpha}\left\{\frac{K}{\pi}\sum_\mu{}'\left[\frac{ch\tilde{\alpha}}{k'}(\delta_\mu \tau_\mu a_0 + 2\delta_{\mu+1}\tau_\mu a_1)\right.\right.$$

$$\left. + 2\, sh\tilde{\alpha}\, \sigma_\mu \tau_\mu b_0\right]$$

$$+ \left[\frac{ch\tilde{\alpha}}{k'}(\delta_0 a_0 + 2\delta_1 a_1) + 2\, sh\tilde{\alpha}\, \sigma_0 b_0\right](K'-y_0)$$

$$\left. + w_\infty \right\}$$

Proceeding similarly for $x = \pm K$, the following three expressions in which all upper signs or all lower signs should be taken:

$$g_1 u = 0,$$

$$g_2 v = \beta k' \left(\pm \,\mathrm{dn}\, y_0 \,\mathrm{ch}\,\tilde{\alpha} + k \,\mathrm{sn}\, y_0 \,\mathrm{sh}\,\tilde{\alpha} \right) \left[\mp \sum_{\mu} (-)^{\mu} B_{\mu} \,\mathrm{sh}\, \frac{2\mu+1}{2}\eta \right.$$

$$\left. - \sum_{\mu}{}' (-)^{\mu} \mathcal{L}_{\mu} \,\mathrm{sh}\,\mu\eta - \mathcal{L}_0 (K' - y_0) \right]$$

$$= \beta k' \left(\pm \,\mathrm{dn}\, y_0 \,\mathrm{ch}\,\tilde{\alpha} + k \,\mathrm{sn}\, y_0 \,\mathrm{sh}\,\tilde{\alpha} \right) \left\{ \pm \frac{\beta K}{\pi} \sum_{\mu} \left[\right. \right.$$

$$\left. \mathrm{ch}\,\tilde{\alpha} \left(\sigma_{\mu} a_0 + 2\sigma_{\mu+1} a_1 \right) + 2 \frac{\mathrm{sh}\,\tilde{\alpha}}{k'} \delta_{\mu} b_0 \right] (-)^{\mu} t_{\mu}$$

$$+ \frac{\beta K}{\pi} \sum_{\mu}{}' \left[\frac{\mathrm{sh}\,\tilde{\alpha}}{k'} \left(\delta_{\mu} a_0 + 2\delta_{\mu+1} a_1 \right) + 2 \,\mathrm{ch}\,\tilde{\alpha}\, \sigma_{\mu} b_0 \right] (-)^{\mu} \tau_{\mu}$$

$$\left. + \beta \left[\frac{\mathrm{sh}\,\tilde{\alpha}}{k'} \left(\delta_0 a_0 + 2\delta_1 a_1 \right) + 2 \,\mathrm{ch}\,\tilde{\alpha}\, \sigma_0 b_0 \right] (K' - y_0) \right\},$$

$$g_3 (w + w_\infty) = -k' \left(\pm \,\mathrm{dn}\, y_0 \,\mathrm{sh}\,\tilde{\alpha} + k \,\mathrm{sn}\, y_0 \,\mathrm{ch}\,\tilde{\alpha} \right) \left[\mp \sum_{\mu} (-)^{\mu} C_{\mu} \right.$$

$$\left. \mathrm{sh}\, \frac{2\mu+1}{2}\eta - \sum_{\mu}{}' (-)^{\mu} \mathcal{L}_{\mu} \,\mathrm{sh}\,\mu\eta - \mathcal{L}_0 (K' - y_0) + w_\infty \right]$$

$$= -k' \left(\pm \,\mathrm{dn}\, y_0 \,\mathrm{sh}\,\tilde{\alpha} + k \,\mathrm{sn}\, y_0 \,\mathrm{ch}\,\tilde{\alpha} \right) \left\{ \mp \frac{K}{\pi} \sum_{\mu} \left[\right. \right.$$

$$\left. \mathrm{sh}\,\tilde{\alpha} \left(\sigma_{\mu} a_0 + 2\sigma_{\mu+1} a_1 \right) + 2 \frac{\mathrm{ch}\,\tilde{\alpha}}{k'} \delta_{\mu} b_0 \right] (-)^{\mu} t_{\mu}$$

$$- \frac{K}{\pi} \sum_{\mu}{}' \left[\frac{\mathrm{ch}\,\tilde{\alpha}}{k'} \left(\delta_{\mu} a_0 + 2\delta_{\mu+1} a_1 \right) + 2 \,\mathrm{sh}\,\tilde{\alpha}\, \sigma_{\mu} b_0 \right] (-)^{\mu} \tau_{\mu}$$

$$\left. - \left[\frac{\mathrm{ch}\,\tilde{\alpha}}{k'} \left(\delta_0 a_0 + 2\delta_1 a_1 \right) + 2 \,\mathrm{sh}\,\tilde{\alpha}\, \sigma_0 b_0 \right] (K' - y_0) + w_\infty \right\}$$

The last six (or counting the ambiguous signs doubly, the last nine) formulae furnish the desired three equations for a_o, a_1, and b_o. Since in the absence of an angle of attack, and therefore of b_o, the boundary conditions at $\pm$ K are identical, one has the right to put the terms endowed with the $\pm$ signs equal to zero separately. In this way the following set of equations is obtained, of which the last one is obtained by subtracting the boundary conditions at $\pm$ K from one another.

$$\mathcal{H}_{00}\, a_0 + \mathcal{H}_{01}\, a_1 + \mathcal{L}_{00}\, b_0 = R_0 ,$$

$$\mathcal{H}_{10}\, a_0 + \mathcal{H}_{11}\, a_1 + \mathcal{L}_{10}\, b_0 = R_1 ,$$

$$\mathcal{H}_{20}\, a_0 + \mathcal{H}_{21}\, a_1 + \mathcal{L}_{20}\, b_0 = R_2 .$$

The meaning of the twelve coefficients is, employing the abbreviations (2.23):

$$\mathcal{H}_{00} = -\frac{\beta^2 K}{\pi k'}\, dn y_0 \sum_\mu \gamma_\mu t_\mu + \frac{Kk}{\pi k'}\, l_{11}\left(\sum_\mu{}' \delta_\mu \tau_\mu + \delta_0 \eta\right),$$

$$\mathcal{H}_{01} = -2\frac{\beta^2 K}{\pi k'}\, dn y_0 \sum_\mu \gamma_{\mu+1} t_\mu + \frac{2Kk}{\pi k'}\, l_{11}\left(\sum_\mu{}' \delta_{\mu+1} \tau_\mu + \delta_1 \eta\right),$$

$$\mathcal{L}_{00} = 2\frac{Kk}{\pi}\, l_{12}\left(\sum_\mu{}' \sigma_\mu \tau_\mu + \sigma_0 \eta\right),$$

$$R_0 = k\, ch\tilde{\alpha}\, w_\infty ,$$

$$\mathcal{H}_{10} = \frac{K}{\pi}\left\{k' \operatorname{dn} y_0\, l_{22} \sum_\mu (-)^\mu \sigma_\mu t_\mu + k \operatorname{sn} y_0\, l_{11}\left(\sum_\mu{}' (-)^\mu \delta_\mu \tau_\mu + \delta_0 \eta\right)\right\},$$

$$\mathcal{H}_{11} = \frac{2K}{\pi}\left\{k' \operatorname{dn} y_0\, l_{22} \sum_\mu (-)^\mu \sigma_{\mu+1} t_\mu + k \operatorname{sn} y_0\, l_{11}\left(\sum_\mu{}' (-)^\mu \delta_{\mu+1} \tau_\mu + \delta_1 \eta\right)\right\},$$

$$\mathcal{L}_{10} = \frac{2K}{\pi} l_{12}\left\{\operatorname{dn} y_0 \sum_\mu (-)^\mu \sigma_\mu t_\mu + k k' \operatorname{sn} y_0\left(\sum_\mu{}' (-)^\mu \sigma_\mu \tau_\mu + \sigma_0 \eta\right)\right\},$$

$$R_1 = k k' \operatorname{sn} y_0 \operatorname{ch} \tilde{\alpha}\, w_\infty ,$$

$$\mathcal{H}_{20} = \frac{K}{\pi} l_{12}\left\{k k' \operatorname{sn} y_0 \sum_\mu (-)^\mu \sigma_\mu t_\mu + \operatorname{dn} y_0\left(\sum_\mu{}' (-)^\mu \delta_\mu \tau_\mu + \delta_0 \eta\right)\right\},$$

$$\mathcal{H}_{21} = \frac{2K}{\pi} l_{12}\left\{k k' \operatorname{sn} y_0 \sum_\mu (-)^\mu \sigma_{\mu+1} t_\mu + \operatorname{dn} y_0\left(\sum_\mu{}' (-)^\mu \delta_{\mu+1} \tau_\mu + \delta_1 \eta\right)\right\},$$

$$\mathcal{L}_{20} = \frac{2K}{\pi}\left\{k \operatorname{sn} y_0\, l_{11} \sum_\mu (-)^\mu \delta_\mu t_\mu + k' \operatorname{dn} y_0\, l_{22}\left(\sum_\mu{}' (-)^\mu \sigma_\mu \tau_\mu + \sigma_0 \eta\right)\right\}$$

$$R_2 = k' \operatorname{dn} y_0 \operatorname{sh} \tilde{\alpha}\, w_\infty .$$

At first cursory glance it might not appear to be such a simplification to determine a_o, a_1, and b_o from three equations whose coefficients are themselves infinite series. However, one can see immediately that these latter series are of much simpler character than the previous eight series ${}_1p_{\lambda\nu}$, . . . ${}_8p_{\lambda\nu}$. They involve only a product of two hyperbolic functions rather than of three, and hence can be summed numerically quite quickly. If expressions in closed form are desired, one can show the sums occurring in:

$$\mathcal{H}_{00}, \mathcal{H}_{10}, \mathcal{H}_{20}; \mathcal{L}_{00}, \mathcal{L}_{10}, \mathcal{L}_{20},$$

can all be written as indefinite integrals of elliptic functions and can therefore be expressed in terms of inverse trigonometric functions of elliptic functions.[1] The sums occurring in the residual coefficients, that is to say in

$$\mathcal{H}_{01}, \mathcal{H}_{11}, \mathcal{H}_{21},$$

are expressible in terms of indefinite integrals of products of elliptic and hyperbolic functions. It is, however, believed that the rapidly converging original series lend themselves better to a numerical approach.

For zero angle of attack the system reduces to the first two equations, but without the third column:

$$\mathcal{H}_{00}\, a_0 + \mathcal{H}_{01}\, a_1 = R_0,$$

$$\mathcal{H}_{10}\, a_0 + \mathcal{H}_{11}\, a_1 = R_1,$$

[1]See Whittaker-Watson

Through combination of the present approximation method with that of Section 7 for small angles of attack, a still greater simplification can be achieved. As was pointed out in that section, it is proper to regard b_o as a quantity of order α; hence the terms $\mathcal{L}_{00}\, b_o$ and $\mathcal{L}_{10}\, b_o$ are of second order. Thus, up to terms of second order in α the coefficients a_o and a_1 retain the values they have for zero α. Finally b_o is obtained from the last equation when using these values of a_o and a_1.

PART III NUMERICAL ASPECTS OF THE PROBLEM:

1. Selection of Appropriate Cones for Study

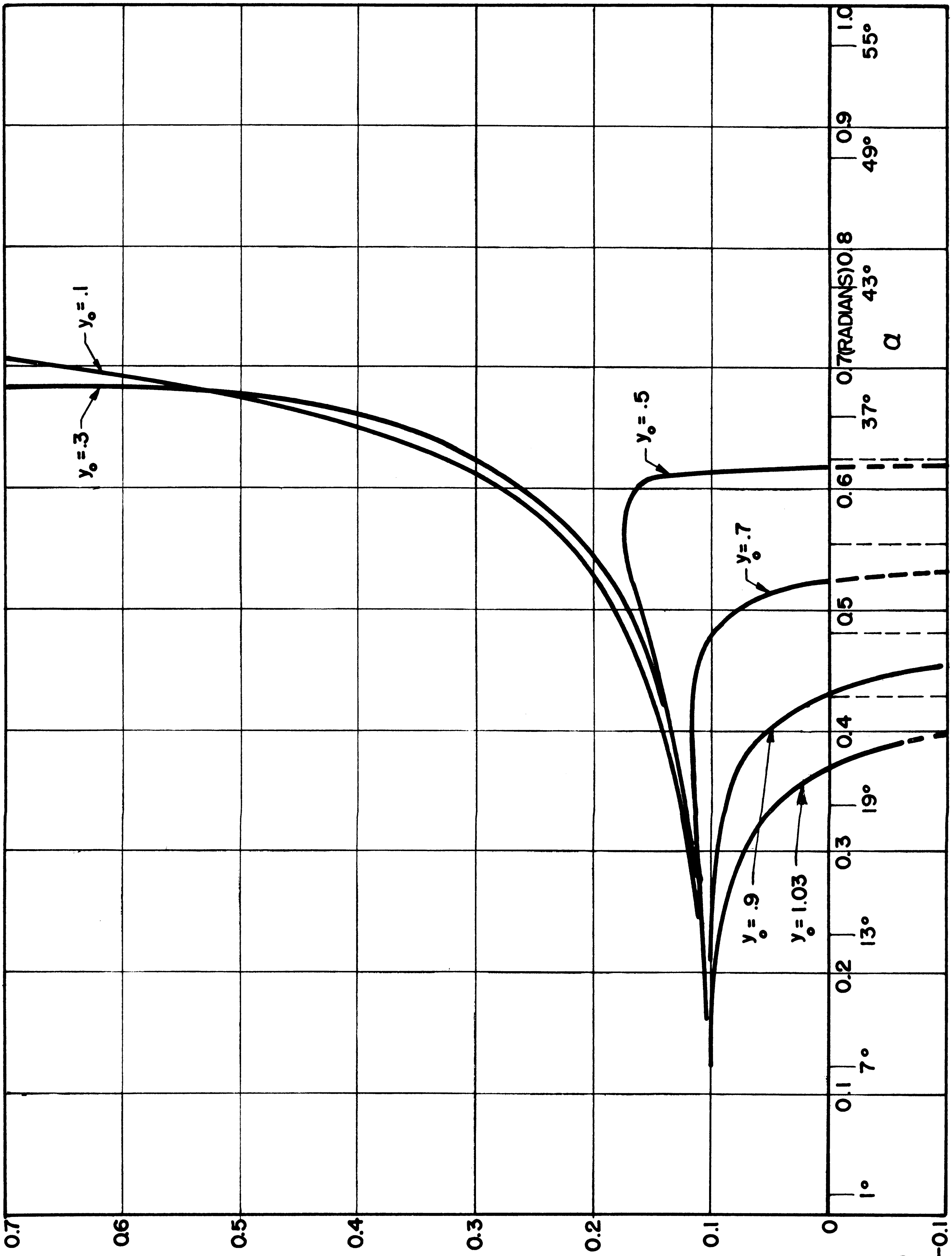
The problem of computing the flow past an elliptic cone at an angle of attack has been shown to resolve itself into the solving of two infinite systems of inhomogeneous linear equations for the Fourier coefficients a_n, b_n of $\mathcal{F}(z')$. A survey was made to determine those physical cones (characterized by A, B in $\tilde{Z} = 1$ plane and by α) which, corresponding to definite mathematical cones (characterized by k^2, y_o, $\tilde{\alpha}$) are most convenient for the problem of solving these linear equations numerically. Formulae have been developed above (Section 4, of Part I) which show the dependence of k^2, y_o, $\tilde{\alpha}$ upon A, B, α at fixed Mach number M. The first part of the survey consists of calculations and plots, exhibiting k^2, y_o, $\tilde{\alpha}$ as functions of α for a variety of physical cones i.e. (A, B) - values. (See Figures 10 - 12). Each curve is labelled with a y_o and a k^2 value which are the values of these constants at zero angle of attack. Now at $\alpha = 0$ the following relations are valid:

$$y_o = K' - \frac{K}{\pi}\eta$$

$$A = \frac{k}{\beta}\, nd(y_o, k')$$

$$\frac{B}{A} = sn(y_o, k')$$

Using these formulae, one can calculate the physical cones represented by each of the curves in Figures 10 - 12. In Figure 10 all curves start at $k^2_{\alpha=0} = 0.1$, and each curve (corresponding to a different (A, B) - value) is labelled with some value of $y_{o\,\alpha=0}$. The diagram shows the distinction between cones of



$y_o = .1$
$y_o = .3$
$y_o = .5$
$y_o = .7$
$y_o = .9$
$y_o = 1.03$
α (RADIANS)
0.1
0.2
0.3
0.4
0.5
0.61
0.7
0.8
0.9
1.0
1°
7°
13°
19°
37°
43°
49°
55°
K'
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1

FIG. 10

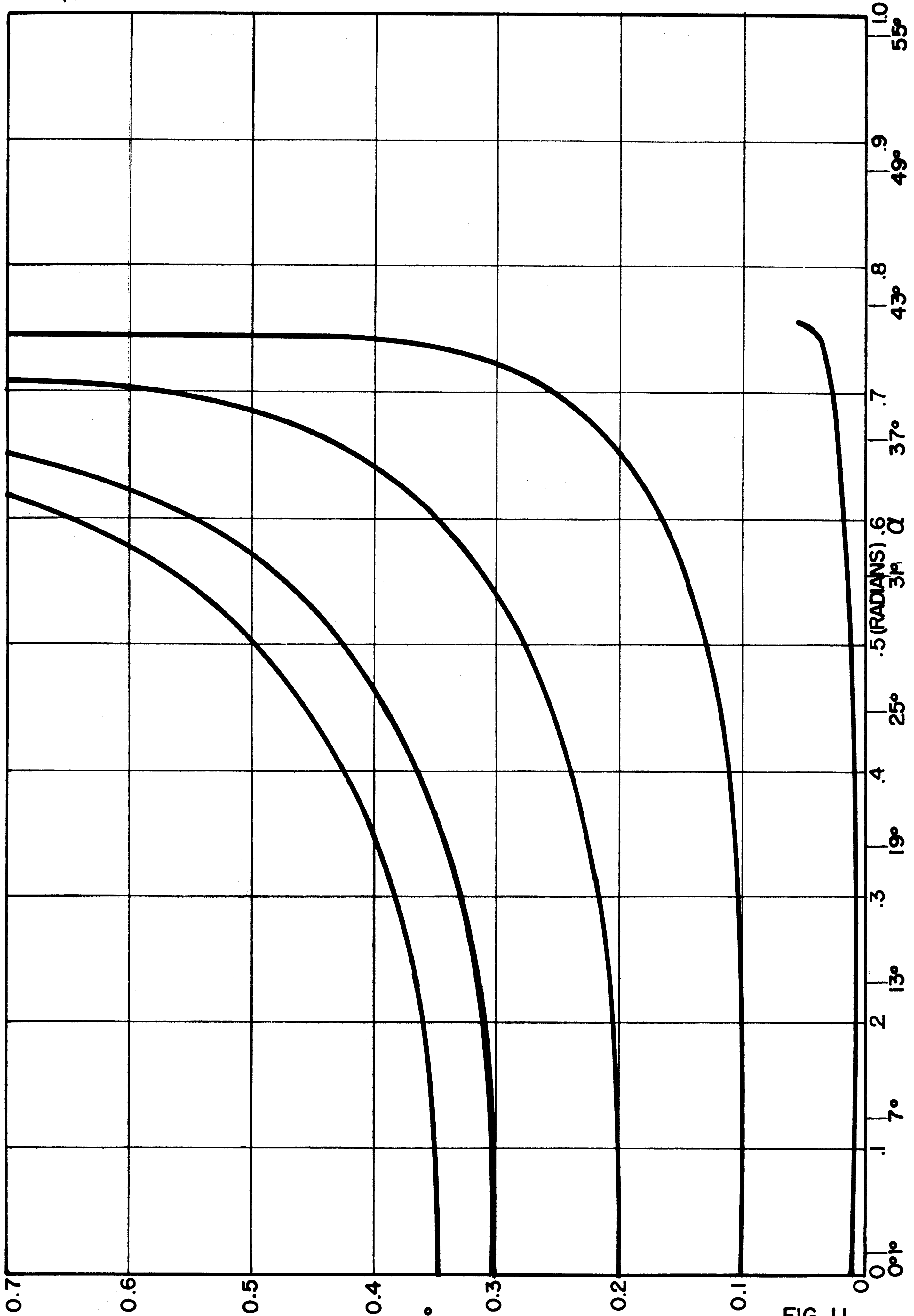

0
.1
.2
.3
.4
.5 (RADIANS) .6 α
.7
.8
.9
1.0
0°
7°
13°
19°
25°
31°
37°
43°
49°
55°
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7

FIG. 11

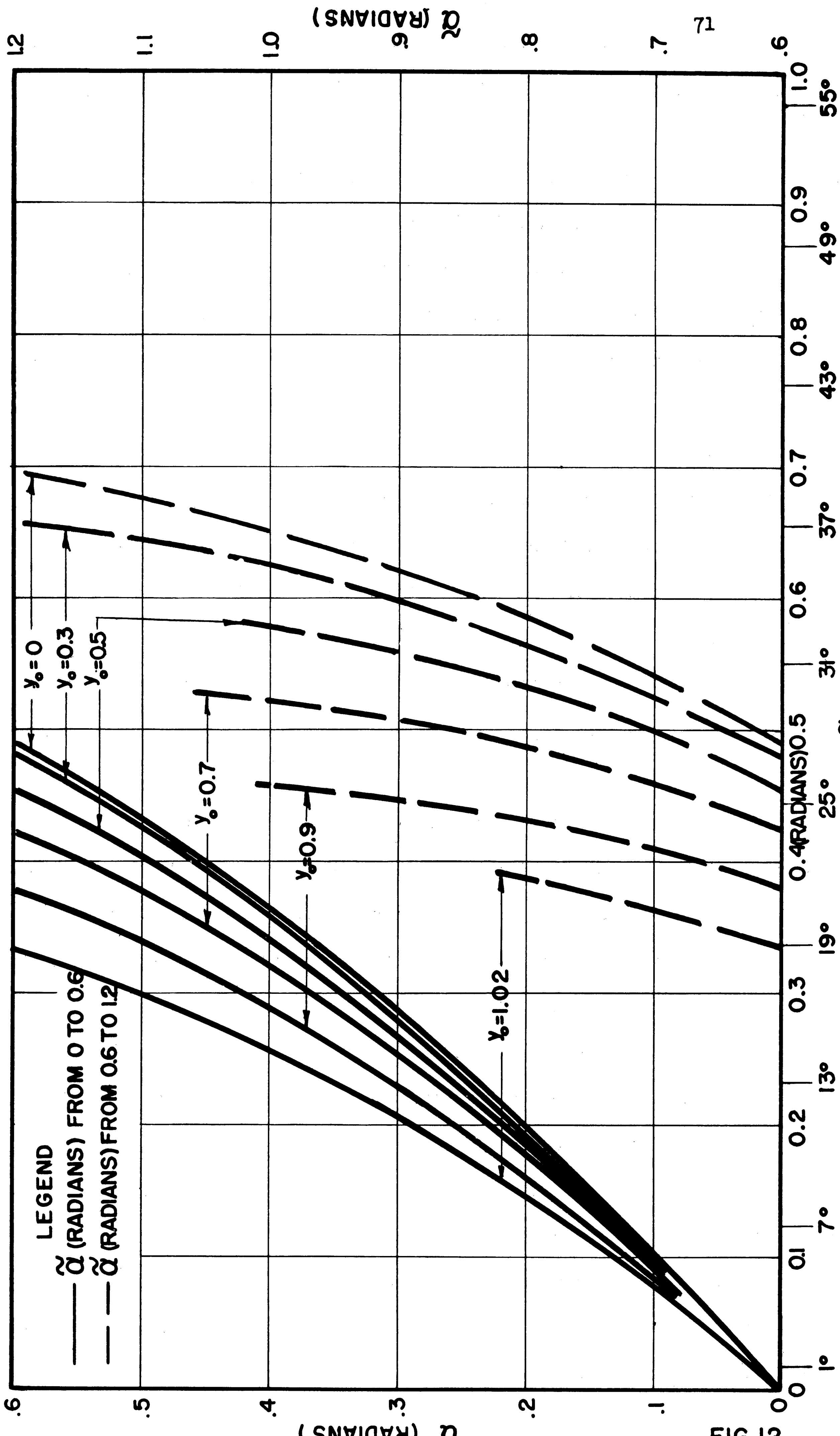
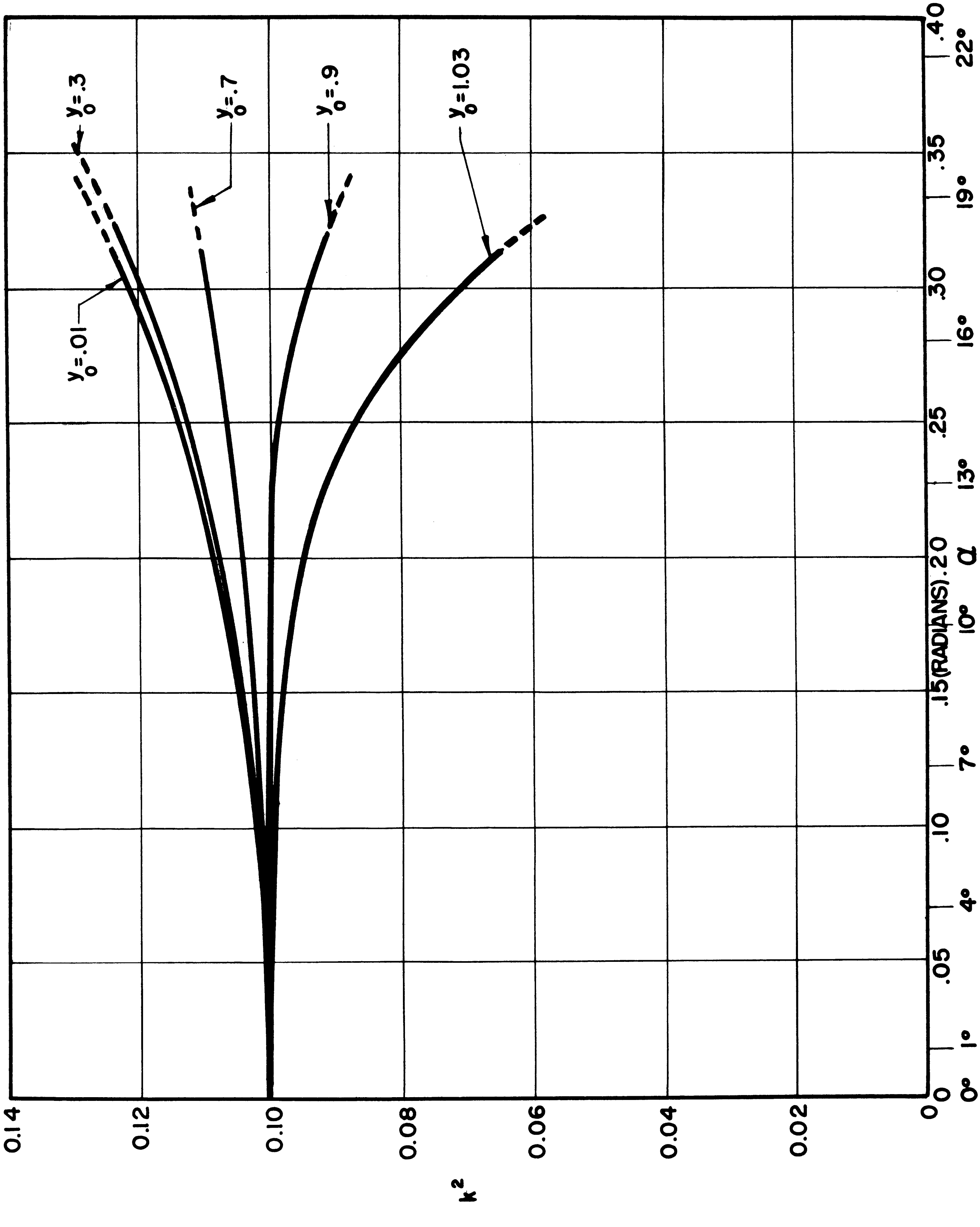

LEGEND
— $\tilde{\alpha}$ (RADIANS) FROM 0 TO 0.6
— — $\tilde{\alpha}$ (RADIANS) FROM 0.6 TO 1.2
$y_o = 0$
$y_o = 0.3$
$y_o = 0.5$
$y_o = 0.7$
$y_o = 0.9$
$y_o = 1.02$
$\tilde{\alpha}$ (RADIANS)
α (RADIANS)
α

FIG. 12

case 1 and case 3 (see Section 2, of Part I) inasmuch as the former rise monotonically while the latter exhibit the maxima discussed in Section 3, of Part I. In Figure 11 y_o is plotted as a function of α for four cones characterized by their $k^2_{\alpha=0}$ and $y_{o_{\alpha=0}}$ - values at $\alpha = 0$, (viz. $k^2 = 0.1$, $y_o = 0.1, 0.2, 0.3, 0.345$). All these curves are for cones of case 1 ($A^2/B > 1/\beta$). Figure 12 shows $\tilde{\alpha}$ as function of α for a variety of cones characterized by $k^2_{\alpha=0} = 0.1$ and various $y_{o_{\alpha=0}}$ values. Cones of cases 1, 2, or 3 do not give rise to essentially different curves as was the case previously for $k^2(\alpha)$. In Figures 13, 14 k^2 and y_o are drawn in the range of more immediate interest $0 \leq \alpha \leq 12°$. The Lorentz angle $\tilde{\alpha}$ enters the linear equation schemes in a rather harmless manner so that one need not concern himself with $\tilde{\alpha}$ as a function of α when selecting appropriate cones. It is evident from the above Figures 13, 14 that the parameters k^2, y_o do not depend strongly on α in the range $0 \leq \alpha \leq 10°$.

Now, there are several criteria which aid one in selecting the most convenient (A, B) - values. First of all, it must be realized that the singularities along the edges of a delta wing (characterized by $y_o = 0$) make it impossible to expand $\mathfrak{F}(z')$ in a Fourier series in the immediate neighborhood of $y_o = 0$. Thus, one expects the linear equation schemes to break down when y_o is near zero in the sense that more and more a_n, b_n are required to describe $\mathfrak{F}(z')$ with a prescribed accuracy. Secondly, one must select an (A, B) - value at some low M-value such that at all higher M - values the inequality $A\beta < 1$ is maintained (see Section 1, of Part I). Now from the three formulae given above one can construct a set of level-curves of constant η in the plane (A, B/A). Figure 15 shows an example of such a curve for the particular value at $\eta = 4$, $M = \sqrt{2}$ for which $k^2 = 0.1$. There is one point on this curve which has been chosen as an appropriate cone; it has (A, B) = (0.352; 0.163) and



$y_o = .3$
$y_o = .7$
$y_o = .9$
$y_o = 1.03$
$y_o = .01$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0
k^2
0
.05
.10
.15 (RADIANS) .20 α
.25
.30
.35
.40
0°
1°
4°
7°
10°
13°
16°
19°
22°

FIG. 13

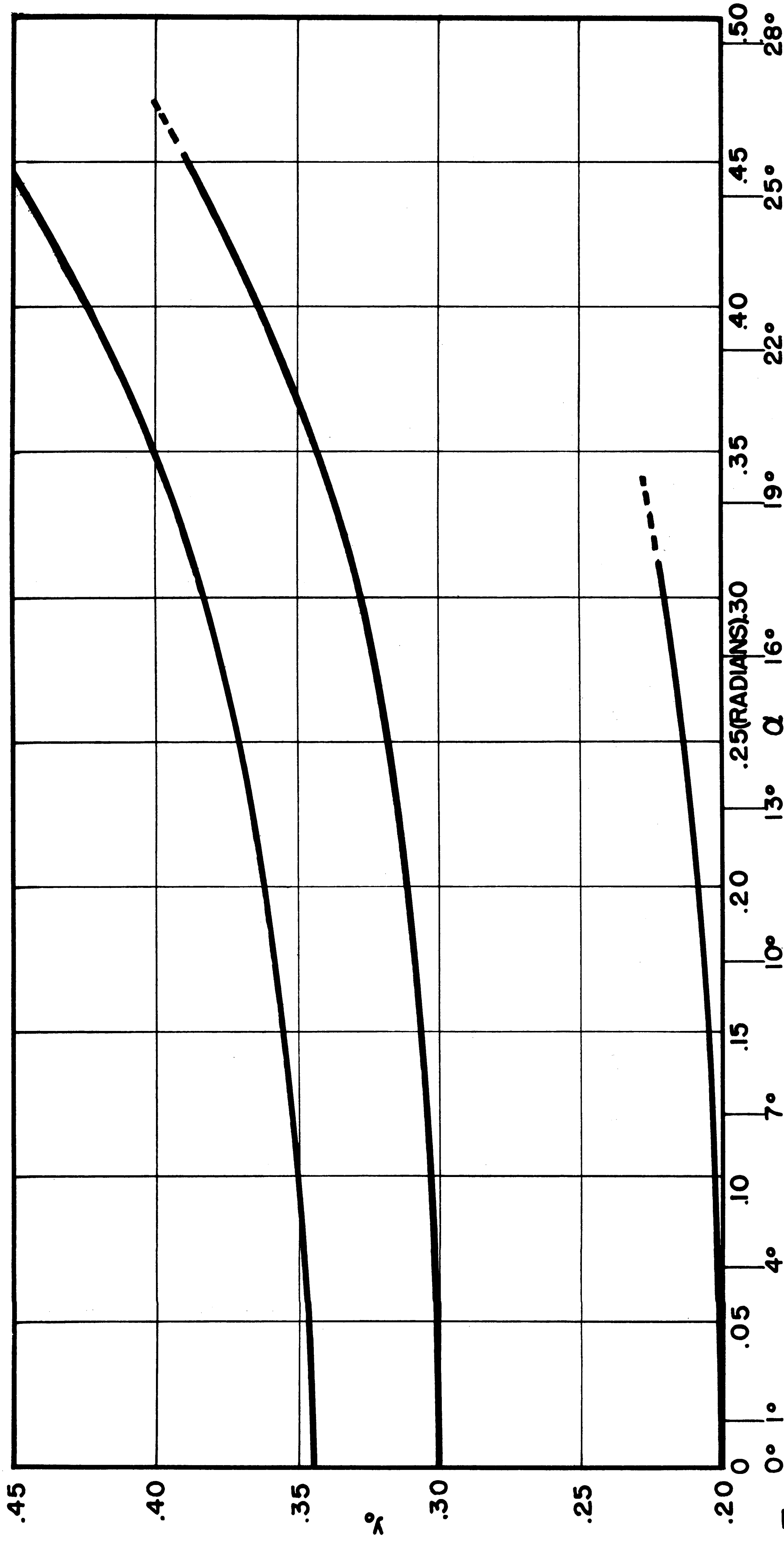

.45
.40
.35
.30
.25
.20
γ_0
0
.05
.10
.15
.20
.25 (RADIANS) α
.30
.35
.40
.45
.50
0°
1°
4°
7°
10°
13°
16°
19°
22°
25°
28°

FIG.14

will be used below. Also there is another point very close to this point having $\eta = 4.00084$ and (A, B) = (0.35587, 0.17207) which is to be used below in solving the linear equation schemes at $\alpha = 0$ and $\alpha = 10°$. The inequalities in Figure 15 merely indicate tendencies toward poor or good convergence and are not meant as absolute criteria. One can summarize the behavior of η as follows ($\eta = 4$ will be used for this illustration): It is desirable that $\eta \leq 4$ and A small ($A\beta < 1$) this means that k^2 is to be small. If A is decreased then so is k^2 which, however, increases η. For a given (A, β), η is smaller for a large B/A ratio but one also desires $B/A \ll 1$ so that $A^2/B > 1/\beta$ i.e. if flat cones are desired (see Figure 2). These statements concern themselves with $\alpha = 0$. For $\alpha \neq 0$ and for larger M one can obtain (see Section 4, of Part I) smaller values of η for given (A, B/A) however this is fairly insensitive.

Of course, once an (A, B) - value is decided upon at some low Mach number, e.g., $M = \sqrt{2}$, then this cone is to be used at all other M-values. The effect of keeping (A, B) constant at all Mach numbers is to cause rather large changes in the values of k^2 and y_o at $\alpha = 0$. Figures 16 - 20 show this effect for five M - values for the cone decided upon in Figure 15. Sketches are shown of the relative sizes of the fixed elliptic cone and the Mach cone in the $\tilde{Z} = 1$ plane. The curves show that k^2, y_o depend strongly on M (or β) and rather weakly upon α.

2. Calculation of a_n at $\alpha = 0$ Using Exact Linear Equations.

In Report II, page 28, a cone with ($A = 0.4748$, $B = 0.3734$; $k^2 = 0.1$, $\eta = 3$) was used to calculate the Fourier coefficients C_λ of w (z) at $M = \sqrt{2}$. As shown above (Section 2, of Part II) the method used there is not convenient

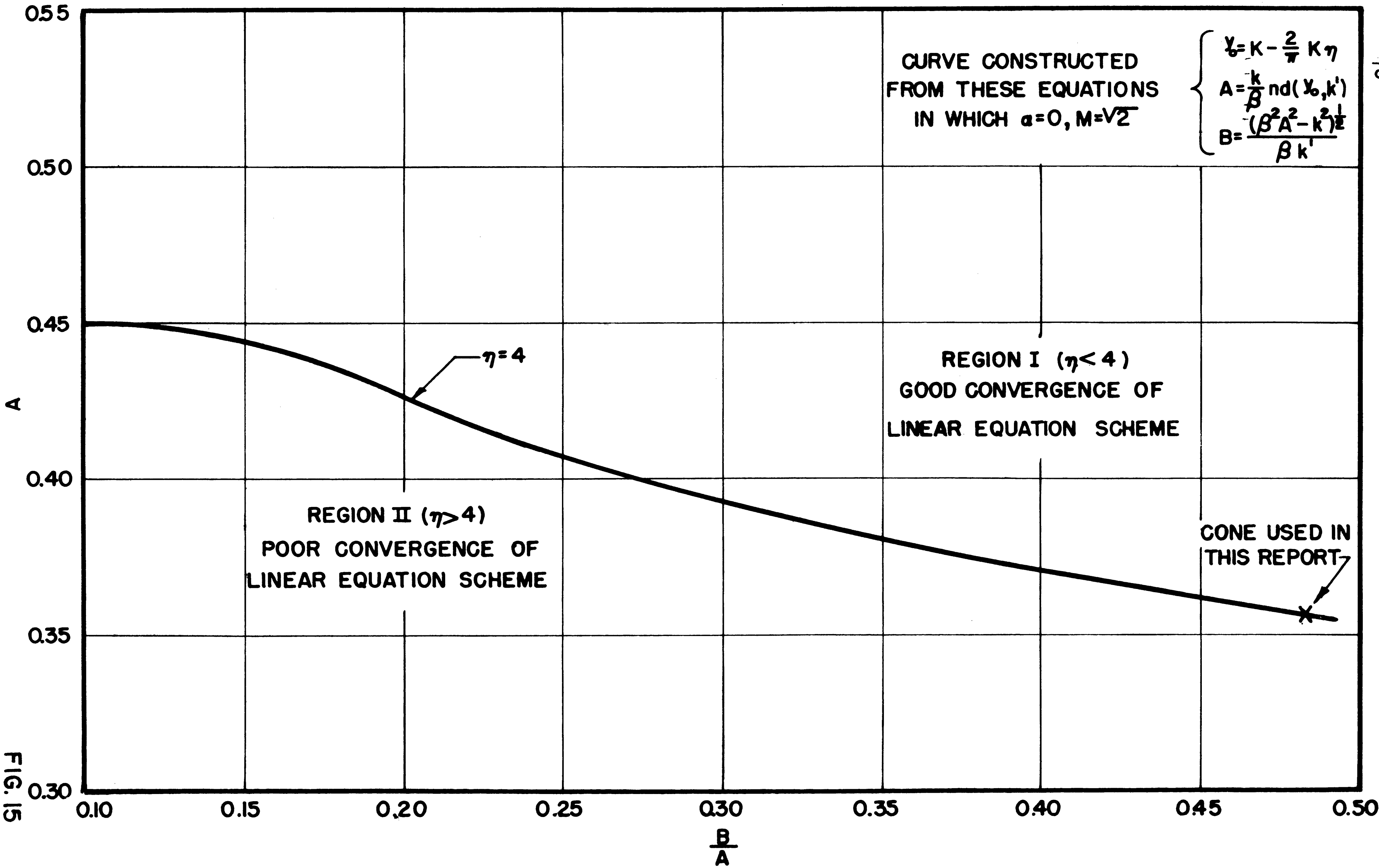
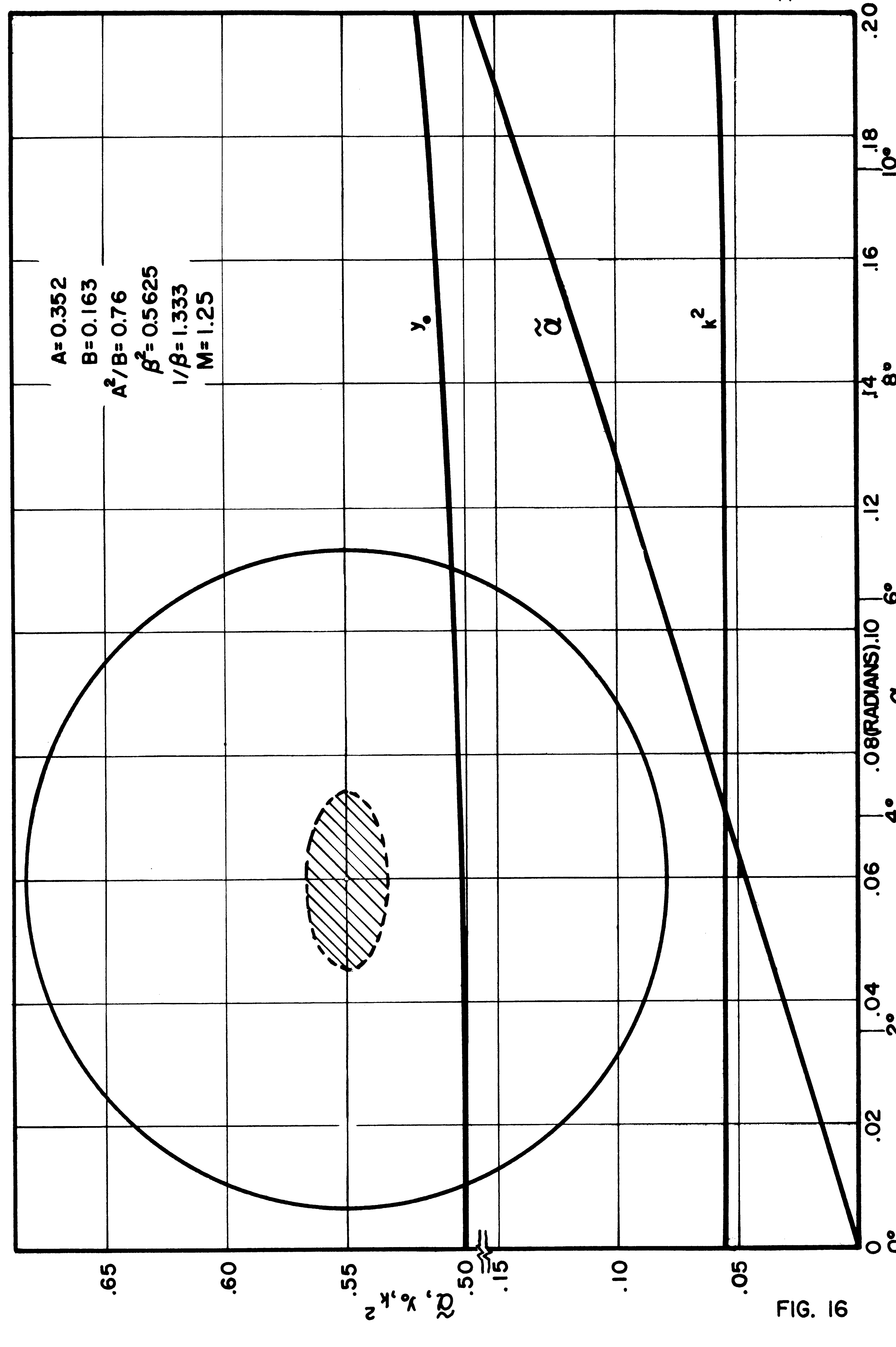

CURVE CONSTRUCTED FROM THESE EQUATIONS IN WHICH $a=0, M=\sqrt{2}$
$\gamma_0 = K - \frac{2}{\pi} K \eta$
$A = \frac{k}{\beta} \mathrm{nd}(\gamma_0, k')$
$B = \frac{(\beta^2 A^2 - k^2)^{\frac{1}{2}}}{\beta k'}$
$\eta = 4$
REGION I ($\eta < 4$) GOOD CONVERGENCE OF LINEAR EQUATION SCHEME
REGION II ($\eta > 4$) POOR CONVERGENCE OF LINEAR EQUATION SCHEME
CONE USED IN THIS REPORT
A
0.55
0.50
0.45
0.40
0.35
0.30
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
0.50
$\frac{B}{A}$

FIG. 15



A = 0.352
B = 0.163
A^2/B = 0.76
β^2 = 0.5625
$1/\beta$ = 1.333
M = 1.25
y_0
$\tilde{\alpha}$
k^2
$\tilde{\alpha}$, y_0, k^2
.65
.60
.55
.50
.15
.10
.05
0
.02
.04
.06
.08 (RADIANS) .10
.12
.14
.16
.18
.20
α
0°
2°
4°
6°
8°
10°

FIG. 16

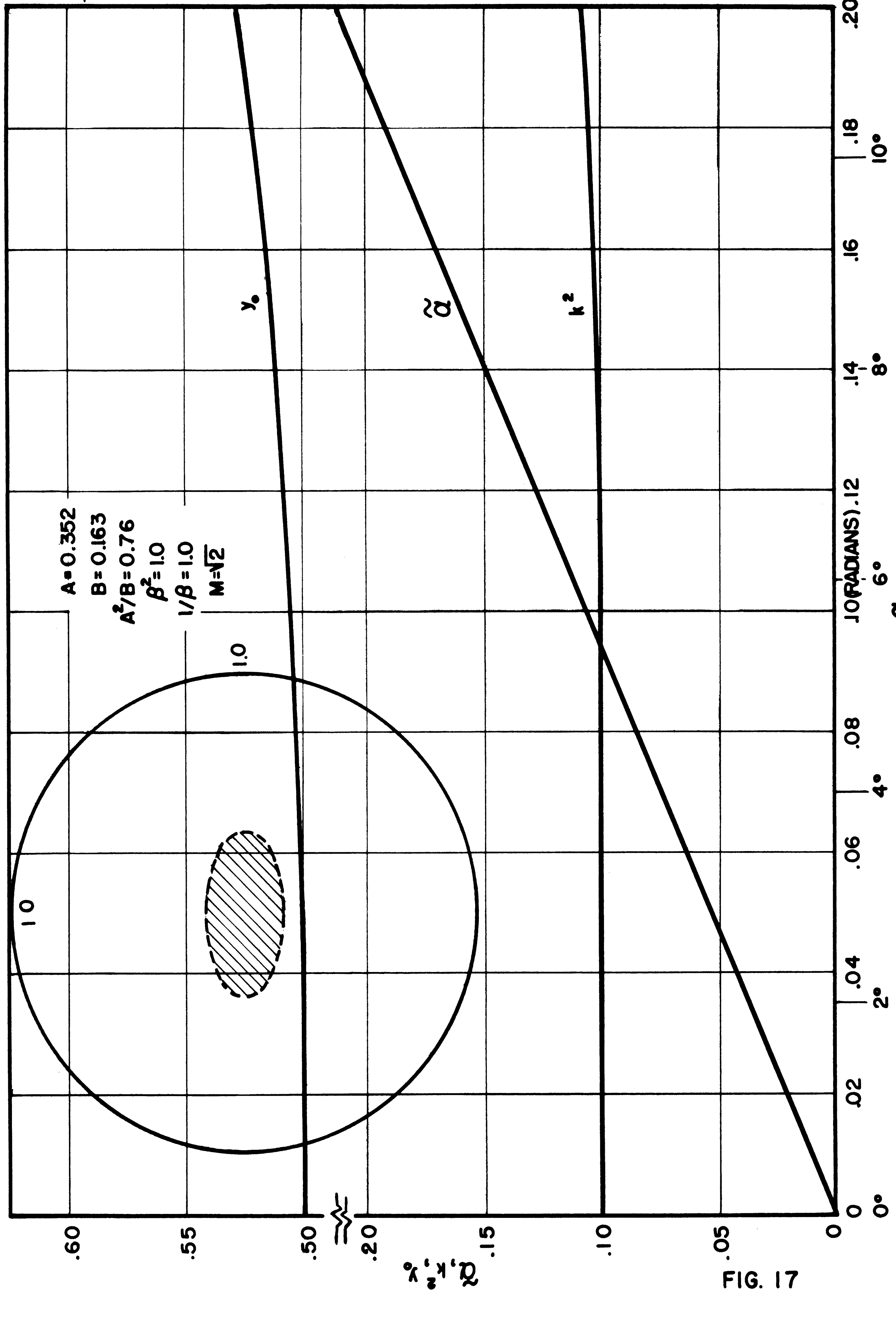
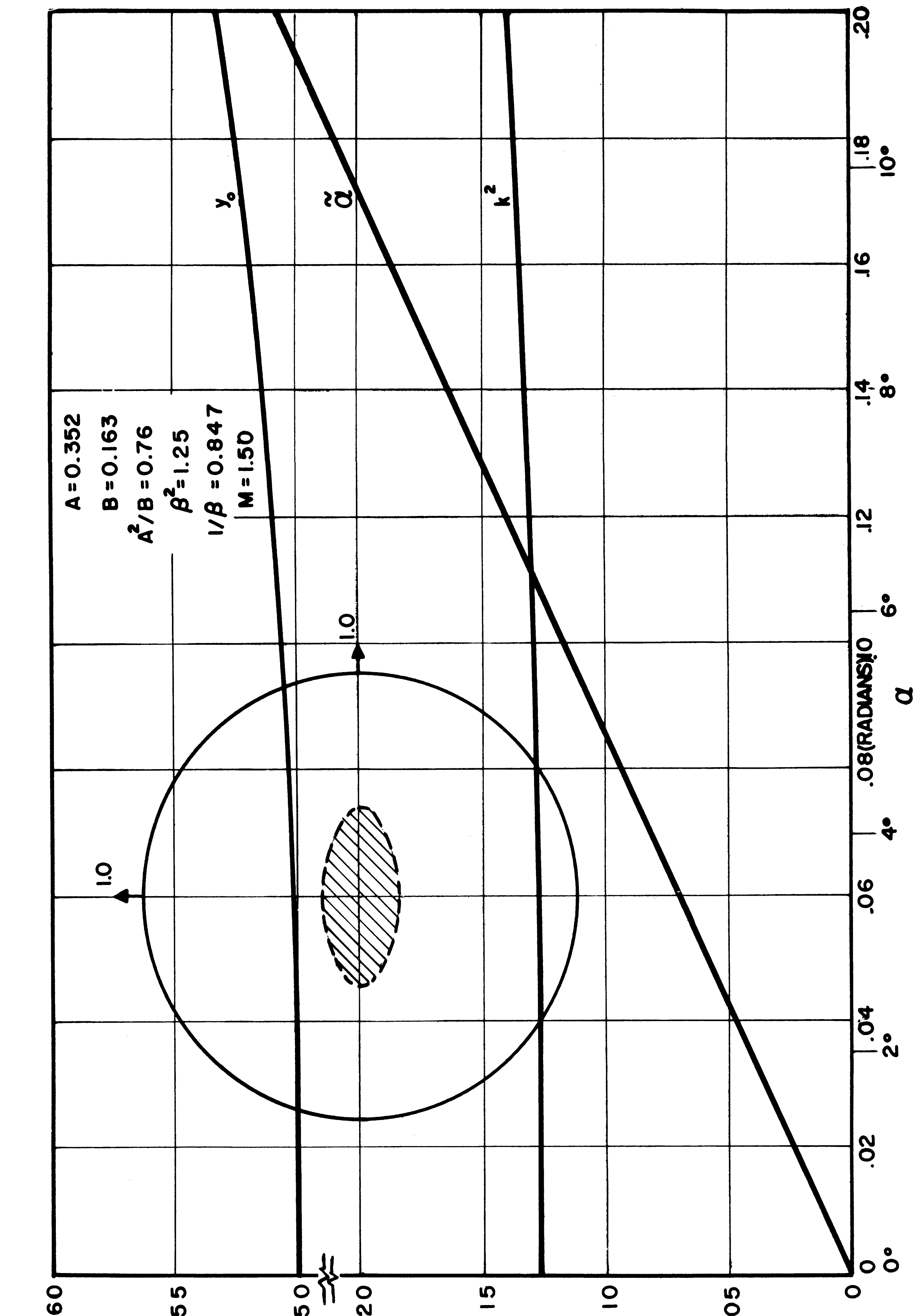
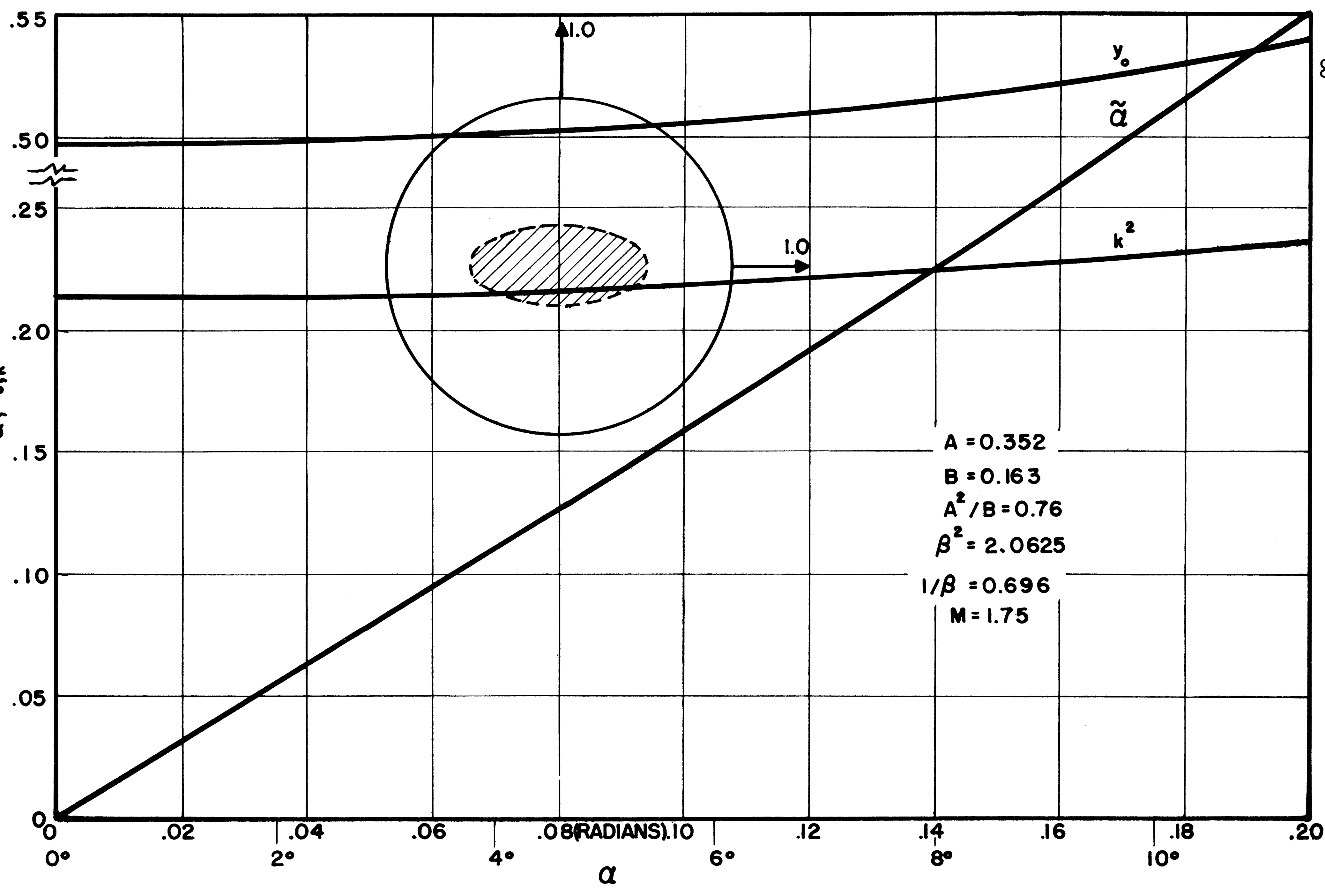

A = 0.352
B = 0.163
$A^2/B = 0.76$
$\beta^2 = 1.0$
$1/\beta = 1.0$
$M = \sqrt{2}$
1.0
1.0
y_0
$\tilde{\alpha}$
k^2
$\tilde{\alpha}, k^2, y_0$
.60
.55
.50
.20
.15
.10
.05
0
0
.02
.04
.06
.08
.10 (RADIANS)
.12
.14
.16
.18
.20
0°
2°
4°
6°
8°
10°
α

FIG. 17



A = 0.352
B = 0.163
A²/B = 0.76
β² = 1.25
1/β = 0.847
M = 1.50
y₀
α̃
k²
1.0
1.0
0 .02 .04 .06 .08 (RADIANS) .10 .12 .14 .16 .18 .20
0° 2° 4° 6° 8° 10°
α
0 .05 .10 .15 .20 .50 .55 .60
α̃, y₀, k²

FIG. 18



.55
.50
.25
.20
.15
.10
.05
0
0
.02
.04
.06
.08
.10
.12
.14
.16
.18
.20
0°
2°
4°
6°
8°
10°
(RADIANS)
α
$\tilde{\alpha}, y_o, k^2$
y_o
$\tilde{\alpha}$
k^2
1.0
1.0
A = 0.352
B = 0.163
A^2/B = 0.76
β^2 = 2.0625
$1/\beta$ = 0.696
M = 1.75

FIG. 19

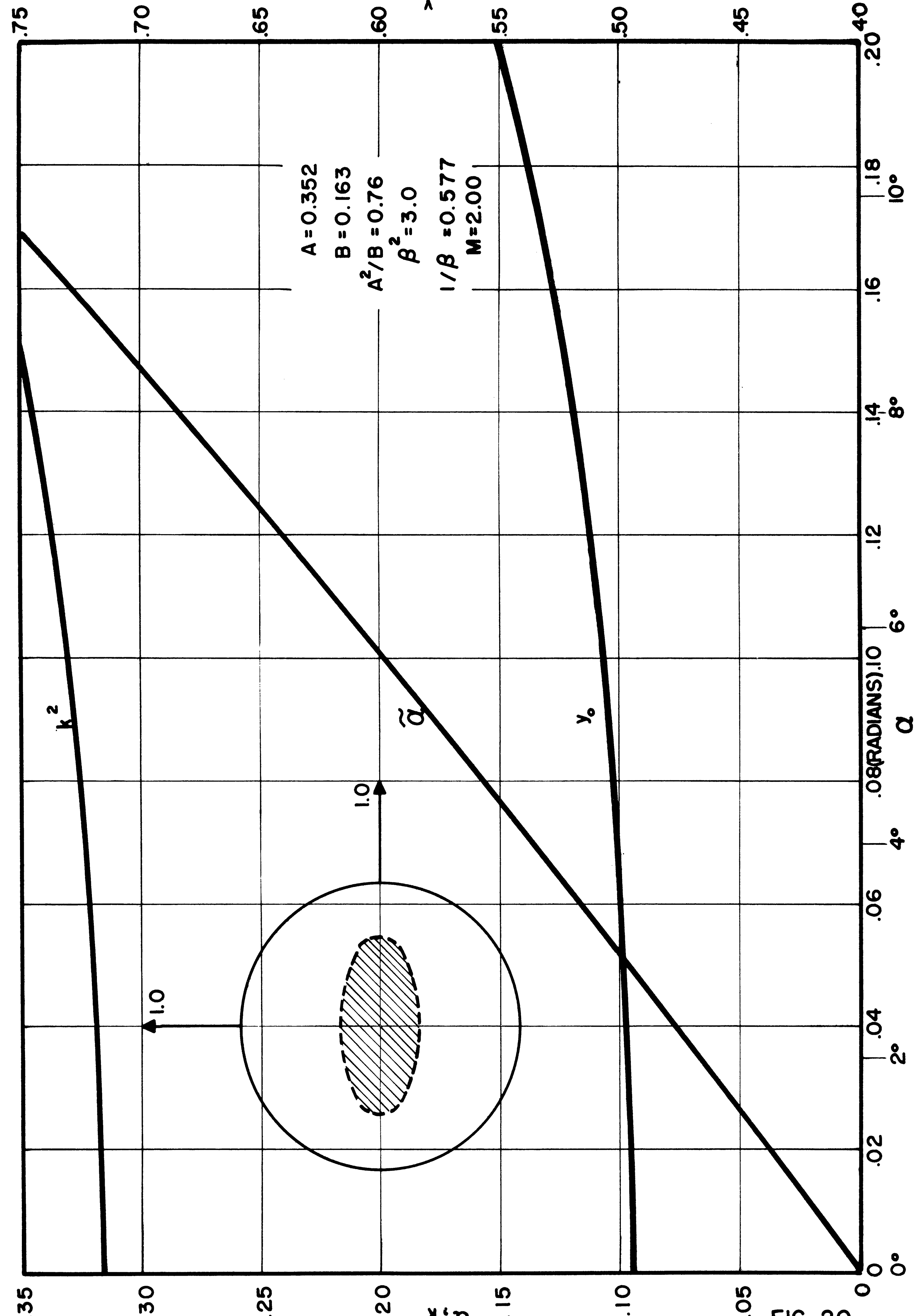

A = 0.352
B = 0.163
$A^2/B = 0.76$
$\beta^2 = 3.0$
$1/\beta = 0.577$
M = 2.00
k^2
$\tilde{\alpha}$
y_o
1.0
1.0
α (RADIANS)
$\tilde{\alpha}, k^2$
y_o
0°
2°
4°
6°
8°
10°
0
.02
.04
.06
.08
.10
.12
.14
.16
.18
.20
.05
.10
.15
.20
.25
.30
.35
.40
.45
.50
.55
.60
.65
.70
.75

FIG. 20

DATA FOR FIGURES 17 - 20

$A = 0.352$, $B = 0.163$

$M = \sqrt{2}$ $\beta^2 = 1$

α	$\tilde{\alpha}$	k^2	y_0	$\eta/2$
0°	0	0.1000	0.49923	2.049
1	0.0184	0.1000	0.49942	2.049
2	0.369	0.1002	0.49997	2.048
3	0.0555	0.1005	0.50092	2.047
4	0.0740	0.1008	0.50201	1.993
5	0.0925	0.1013	0.50373	1.991
6	0.1115	0.1018	0.50582	1.989
7	0.1305	0.1025	0.50832	1.986
8	0.1500	0.1032	0.51128	1.984
9	0.1690	0.1042	0.51472	1.980
10	0.1890	0.1052	0.51862	1.976
11	0.2090	0.1064	0.52302	1.972
12	0.2295	0.1076	0.52793	1.967

$M = 1.5$ $\beta^2 = 1.25$

α	$\tilde{\alpha}$	k^2	y_0	$\eta/2$
0°	0	0.1258	0.49851	1.907
1	0.0195	0.1259	0.49872	1.907
2	0.0386	0.1261	0.49935	1.906
3	0.0585	0.1265	0.50042	1.904
4	0.0785	0.1270	0.50192	1.899
5	0.0980	0.1277	0.50387	1.894
6	0.118	0.1285	0.50627	1.889
7	0.138	0.1295	0.50914	1.882
8	0.159	0.1306	0.51250	1.874
9	0.179	0.1320	0.51637	1.866
10	0.200	0.1335	0.52081	1.855
11	0.221	0.1352	0.52562	1.844
12	0.243	0.1371	0.53127	1.832

$M = 1.75$ $\beta^2 = 2.0625$

α	$\tilde{\alpha}$	k^2	y_0	$\eta/2$
0°	0	0.2124	0.49680	1.634
1	0.0273	0.2126	0.49710	1.634
2	0.0546	0.2131	0.49796	1.631
3	0.0820	0.2140	0.49917	1.628
4	0.1097	0.2153	0.50124	1.623
5	0.1376	0.2170	0.50395	1.616
6	0.1660	0.2191	0.50731	1.608
7	0.1948	0.2216	0.51136	1.595
8	0.2242	0.2246	0.51612	1.586
9	0.2543	0.2282	0.52143	1.573
10	0.2853	0.2322	0.52788	1.557
11	0.3170	0.2368	0.53528	1.540
12	0.3500	0.2421	0.54383	1.520

$M = 2$ $\beta^2 = 3$

α	$\tilde{\alpha}$	k^2	y_0	$\eta/2$
0°	0	0.3173	0.4946	1.418
1	0.0338	0.3177	0.4950	1.417
2	0.0675	0.3189	0.4962	1.414
3	0.1017	0.3208	0.4981	1.408
4	0.1362	0.3237	0.5005	1.401
5	0.1714	0.3273	0.5042	1.392
6	0.2073	0.3320	0.5087	1.383
7	0.2436	0.3377	0.5139	1.366
8	0.2816	0.3445	0.5205	1.348
9	0.3210	0.3526	0.5284	1.328
10	0.3620	0.3623	0.5314	1.310
11	0.4047	0.3736	0.5474	1.278
12	0.4505	0.3872	0.5601	1.247

for angles of attack so it was decided to repeat the calculation of these C_λ using the method of this report, as a method of checking previous results. The appropriate linear equations for the a_n are obtained from formulae (2.27_1) (2.27_2) by setting $\alpha = 0$. They are:

$$-\frac{K}{4\pi k'^2}\,\mathrm{sn}\,y_0\,\mathrm{dn}\,y_0 \sum_\mu \overset{1}{P}_{\lambda\mu} a_\mu - \frac{K}{4\pi}\,\mathrm{dn}\,y_0 \sum_\mu \overset{2}{P}_{\lambda\mu} a_\mu$$
$$+\frac{Kk}{4\pi k'^2}\,\mathrm{sn}\,y_0 \sum_\mu \overset{3}{P}_{\lambda\mu} a_\mu + \frac{k}{k'^2}\,\mathrm{sn}\,y_0\,(K'-y_0)\delta_\lambda \sum_\mu \delta_\mu a_\mu = \frac{k}{k'}\,\mathrm{sn}\,y_0\,\delta_\lambda \cdot w_\infty$$

Using the above values of k^2, y_0, η the three series $\overset{1}{P}_{\lambda\nu}$, $\overset{2}{P}_{\lambda\nu}$, $\overset{3}{P}_{\lambda\nu}$ were computed. Tables are presented of the necessary coefficients σ_μ, γ^μ, δ_μ used in computing these series. The resulting series are also tabulated below. Numerically the linear equations have the following form:

$$-1.223a_0 + 1.0658(-1)a_1 + 4.0132(-3)a_2 + 2.1581(-4)a_3 = 3.2473(-1)W_\infty$$
$$5.3289(-2)a_0 - 5.2677a_1 + 2.5158(-1)a_2 + 1.604(-2)a_3 = 4.2762(-3)W_\infty$$
$$2.0072(-3)a_0 - 2.5158a_1 - 6.2557(+1)a_2 + 2.1842\,a_3 = 2.8159(-5)W_\infty$$
$$1.0790(-4)a_0 + 1.6038(-2)a_1 + 2.1842a_2 - 9.0001(+2)a_3 = 1.840(-7)W_\infty$$

The solutions are:

$$a_0 = -2.6597(-1)W_\infty \qquad a_2 = -1.4991(-5)W_\infty$$
$$a_1 = -3.5071(-3)W_\infty \qquad a_3 = -1.5053(-7)W_\infty$$

From the definition (2.13) one has the relations:

$$C_\lambda = -\frac{2K}{\pi k'}\,\frac{1}{\lambda} \sum_\mu \delta_{\lambda-\mu} a_\mu \qquad \lambda > 0$$

$$C_0 = -\frac{1}{k'} \sum_\mu \delta_\mu a_\mu \qquad \lambda = 0$$

TABLE OF COEFFICIENTS

$\eta = 3 \qquad \zeta = 2.51151$

n	σ_n	γ_n	δ_n	t_n	τ_n
10	-7.66021 (-23)	7.66021 (-23)	2.98516 (-22)	2.28064 (12)	5.34325 (11)
9	-1.16332 (-20)	1.16332 (-20)	4.53346 (-20)	1.25499 (11)	2.95582 (10)
8	-1.76669 (-18)	1.76669 (-18)	6.88479 (-18)	6.98329 (9)	1.65558 (9)
7	-2.68301 (-16)	2.68301 (-16)	1.04567 (-15)	3.94035 (8)	9.42014 (7)
6	-4.07501 (-14)	4.07501 (-14)	1.58802 (-13)	2.26360 (7)	5.47167 (6)
5	-6.18853 (-12)	6.18853 (-12)	2.41167 (-11)	1.33188 (6)	3.26902 (5)
4	-9.39829 (-10)	9.39829 (-10)	3.66751 (-9)	8.10462 (4)	2.03444 (4)
3	-1.42728 (-7)	1.42728 (-7)	5.56211 (-7)	5.18794 (3)	1.35051 (3)
2	-2.16777 (-5)	2.16777 (-5)	8.44781 (-5)	3.61608 (2)	1.00857 (2)
1	-3.29206 (-3)	3.29206 (-3)	1.28285 (-2)	3.00020 (1)	1.00179 (1)
0	-5.03275 (-1)	4.96686 (-1)	9.74176 (-1)	4.25856 (0)	3.
-1	+5.03275 (-1)	4.96686 (-1)	1.28285 (-2)	4.25856 (0)	1.00179 (1)
-2	+3.29206 (-3)	3.29206 (-3)	8.44781 (-5)	3.00020 (1)	1.00857 (2)
-3	+2.16777 (-5)	2.16777 (-5)	5.56211 (-7)	3.61608 (2)	1.35051 (3)
-4	+1.42728 (-7)	1.42728 (-7)	3.66251 (-9)	5.18794 (3)	2.03444 (4)
-5	+9.29829 (-10)	9.39829 (-10)	2.41167 (-11)	8.10462 (4)	3.26902 (5)
-6	+6.18853 (-12)	6.18853 (-12)	1.58802 (-13)	1.33188 (6)	5.47167 (6)
-7	+4.07501 (-14)	4.07501 (-14)	1.04567 (-15)	2.26360 (7)	9.42014 (7)
-8	+2.68301 (-16)	2.68301 (-16)	6.88479 (-18)	3.94035 (8)	1.65558 (9)
-9	+1.76669 (-18)	1.76669 (-18)	4.53346 (-20)	6.98329 (9)	2.95582 (10)
-10	+1.16332 (-20)	1.16332 (-20)	2.98516 (-22)	1.25499 (11)	5.34325 (11)

$_1p_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3	
3	2.64515(-9)	2.17990(-7)	3.08216(-5)	4.63277(-3)	6.96345(-1)	9.77495(1)	1369.94	
2	2.17990(-7)	5.19937(-6)	4.42766(-4)	5.99619(-2)	7.99937	96.6653	9.77495(1)	
1	3.08216(-5)	4.42766(-4)	1.45729(-2)	1.10662	8.45594	7.99937	6.96345(-1)	
0	4.63277(-3)	5.99619(-2)	1.10662	2.10180	1.10662	5.99619(-2)	4.63277(-3)	
-1	6.96345(-1)	7.99937	8.45594	1.10662	1.45729(-2)	4.42766(-4)	3.08216(-5)	
-2	9.77495(1)	96.6653	7.99937	5.99619(-2)	4.42766(-4)	5.19937(-6)	2.17990(-7)	
-3	1369.94	9.77495(+1)	6.96345(-1)	4.63277(-3)	3.08216(-5)	2.17990(-7)	2.64515(-9)	

$p'_{\lambda\nu}$

λ \ ν	0	1	2	3
0	8.40720	4.42648	0.239848	0.0185311
1	4.42648	16.9410	15.9996	1.39275
2	0.239848	15.9996	193.331	195.499
3	0.0185311	1.39275	195.499	2739.88

$A = 0.475$
$B = 0.373$
$M = \sqrt{2}$
$\eta = 3$
$k^2 = 0.1$
$\zeta = 2.51151$

$_2p_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3	
3	-3.92240(-9)	-1.72808(-7)	-2.42193(-5)	-3.94123(-3)	-5.91871(-1)	-82.8913	1406.50	
2	-1.72808(-7)	-4.57572(-6)	-3.87422(-4)	-5.27693(-2)	-6.99183	99.2455	-82.8913	
1	-2.42193(-5)	-3.87422(-4)	-1.28017(-2)	-1.02186	8.68168	-6.99183	-5.91871(-1)	
0	-3.94123(-3)	-5.27693(-2)	-1.02186	2.15791	-1.02186	-5.27693(-2)	-3.94123(-3)	
-1	-5.91871(-1)	-6.99183	8.68168	-1.02186	-1.28017(-2)	-3.87422(-4)	-2.42193(-5)	
-2	-82.8913	99.2455	-6.99183	-5.27693(-2)	-3.87422(-4)	-4.57572(-6)	-1.72808(-7)	
-3	1406.50	-82.8913	-5.91871(-1)	-3.94123(-3)	-2.42193(-5)	-1.72808(-7)	-3.92240(-9)	

$p^2_{\lambda\nu}$

λ \ ν	0	1	2	3
0	8.63164	-4.08744	-2.11077(-1)	-1.57649(-2)
1	-4.08744	17.3378	-13.9844	-1.18379
2	-2.11077(-1)	-13.9844	198.491	-165.783
3	-1.57649(-2)	-1.18379	-165.783	2813.00

$_3p_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3	
3	4.87002(-10)	3.98230(-8)	5.65182(-6)	8.52885(-4)	1.28712(-1)	18.1596	1285.03	
2	3.98230(-8)	8.63500(-7)	7.10005(-5)	9.95954(-3)	1.38701	95.9391	18.596	
1	5.65182(-6)	7.1005(-5)	1.65032(-3)	1.25316(-1)	9.52379	1.38701	1.28712(-1)	
0	8.52885(-4)	9.95852(-3)	1.25316(-1)	3.29874(-3)	1.25316(-1)	9.95854(-3)	8.52885(-4)	
-1	1.28712(-1)	1.38701	9.52379	1.25316(-1)	1.65032(-3)	7.10005(-5)	5.65182(-6)	
-2	18.1596	95.9391	1.38701	9.95854(-3)	7.10005(-5)	8.63500(-7)	3.98230(-8)	
-3	1285.03	18.1596	1.28712(-1)	8.52885(-4)	5.65182(-6)	3.98230(-8)	4.87002(-10)	

$p^3_{\lambda\nu}$

λ \ ν	0	1	2	3
0	1.31950(-2)	5.01264(-1)	3.98342(-2)	3.41154(-3)
1	5.01264(-1)	19.0509	2.77416	0.257435
2	3.98342(-2)	2.77544	191.878	36.3192
3	3.41154(-3)	0.257435	36.3192	2570.06

TABLE OF COEFFICIENTS

$\zeta = 2.51151$, $\eta = 4$

n	σ_n	γ_n	δ_n	t_n	τ_n
10	-7.66021 (-23)	7.66021 (-23)	2.98516 (-22)	8.28224 (16)	1.17693 (16)
9	-1.16332 (-20)	1.16332 (-20)	4.53346 (-20)	1.67663 (15)	2.39513 (14)
8	-1.76669 (-18)	1.76669 (-18)	6.88479 (-18)	3.43213 (13)	4.93519 (12)
7	-2.68301 (-16)	2.68301 (-16)	1.04567 (-15)	7.12433 (11)	1.03304 (11)
6	-4.07501 (-14)	4.07501 (-14)	1.58802 (-13)	1.50562 (10)	2.20743 (9)
5	-6.18853 (-12)	6.18853 (-12)	2.41167 (-11)	3.25901 (8)	4.85165 (7)
4	-9.39829 (-10)	9.39829 (-10)	3.66751 (-9)	7.29555 (6)	1.11076 (6)
3	-1.42728 (-7)	1.42728 (-7)	5.56211 (-7)	1.71800 (5)	2.54592 (4)
2	-2.16777 (-5)	2.16777 (-5)	8.44781 (-5)	4.40528 (3)	7.45240 (2)
1	-3.29206 (-3)	3.29206 (-3)	1.28285 (-2)	1.34475 (2)	1.36450 (1)
·0	-5.03275 (-1)	4.96686 (-1)	9.74176 (-1)	7.25372 (0)	4.
-1	+5.03275 (-1)	4.96686 (-1)	1.28285 (-2)	7.25372 (0)	1.36450 (1)
-2	+3.29206 (-3)	3.29206 (-3)	8.44781 (-5)	1.34475 (2)	7.45240 (2)
-3	+2.16777 (-5)	2.16777 (-5)	5.56211 (-7)	4.40528 (3)	2.54592 (4)
-4	+1.42728 (-7)	1.42728 (-7)	3.66251 (-9)	1.71800 (5)	1.11076 (6)
-5	+9.39829 (-10)	9.39829 (-10)	2.41167 (-11)	7.29555 (6)	4.85165 (7)
-6	+6.18853 (-12)	6.18853 (-12)	1.58802 (-13)	3.25901 (8)	2.20743 (9)
-7	+4.07501 (-14)	4.07501 (-14)	1.04567 (-15)	1.50562 (10)	1.03304 (11)
-8	+2.68301 (-16)	2.68301 (-16)	6.88479 (-18)	7.12433 (11)	4.93519 (12)
-9	+1.76669 (-18)	1.76669 (-18)	4.53346 (-20)	3.43213 (13)	2.39513 (14)
-10	+1.16332 (-20)	1.16332 (-20)	2.98516 (-22)	1.67663 (15)	1.17693 (16)

${}_{1}p_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3
3	3.49260(-8)	2.73196(-6)	4.03361(-4)	6.11694(-2)	9.27712	1368.46	43548.3
2	2.73196(-6)	2.42212(-5)	1.92679(-3)	0.279334	40.3989	1121.81	1368.46
1	4.03361(-4)	1.92679(-3)	2.66213(-2)	2.02154	35.0118	40.3989	9.27712
0	6.11694(-2)	0.279334	2.02154	3.58185	2.02154	0.279334	6.11694(-2)
-1	9.27712	40.3989	35.0118	2.02154	2.66213(-2)	1.92679(-3)	4.03361(-4)
-2	1368.46	1121.81	40.3989	0.279334	1.92679(-3)	2.42212(-5)	2.73196(-6)
-3	43548.3	1368.46	9.27712	6.11694(-2)	4.03361(-4)	2.73196(-6)	3.49260(-8)

$p'_{\lambda\nu}$

λ \ ν	0	1	2	3
0	14.3274	8.08616	1.11734	0.244678
1	8.08616	70.0768	80.8017	18.5550
2	1.11734	80.8017	2243.62	2736.92
3	0.244678	18.5550	2736.92	87096.6

$A = 0.356$
$B = 0.172$
$M = \sqrt{2}$
$\eta = 4.00084$
$k^2 = 0.1$
$\zeta = 2.51192$

$_2p_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3
3	-2.12665(-8)	-1.66768(-6)	-2.45633(-4)	-3.72337(-2)	-5.64455	-830.463	44709.5
2	-1.66768(-5)	-1.61797(-5)	-1.29804(-3)	-0.186597	-26.7347	1151.71	-830.463
1	-245633(-4)	-1.29804(-3)	-2.10978(-2)	-1.60212	35.9458	-26.7347	-5.64455
0	-3.72337(-2)	-0.186597	-1.60212	3.67745	-1.60212	-0.186597	-3.72337(-2)
-1	-5.64455	-26.7347	35.9458	-1.60212	-2.10996(-2)	-1.29804(-3)	-2.45633(-4)
-2	-830.463	1151.71	-26.7347	-0.186597	-1.29804(-3)	-1.61797(-5)	-1.66768(-6)
-3	44709.5	-830.463	-5.64455	-3.72337(-2)	-2.45633(-4)	-1.66768(-6)	-2.12665(-8)

$p^2_{\lambda\nu}$

λ \ ν	0	1	2	3
0	14.7098	-6.40848	-0.746388	-0.148935
1	-6.40848	71.8476	-53.4720	-11.2896
2	-0.746388	-53.4720	2303.42	-1660.93
3	-0.148935	-11.2896	-1660.93	89419.0

$_2\pi_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3
3	-130460.	-67364.9	-65334.7	65213.0	-65091.3	-63061.2	0
2	-67364.9	-4303.56	-2273.45	-2151.78	-2030.10	0	63061.2
1	-65334.7	-2273.45	-243.314	-121.657	0	2030.10	65091.3
0	65213.0	-2151.78	-121.657	0	121.657	2151.78	65213.0
-1	65091.3	2030.10	0	121.657	243.314	2273.45	65334.7
-2	63061.2	0	2030.10	2151.78	2273.45	4303.56	67364.9
-3	0	63061.2	65091.3	65213.0	65334.7	67364.9	130460.

${}_3p_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3
3	8.37660(-9)	6.54264(-7)	9.65879(-5)	1.46697(-2)	2.22692	328.676	24344.5
2	6.54011(-7)	5.52843(-6)	4.19924(-4)	6.37583(-2)	9.51117	711.447	328.676
1	9.66453(-5)	4.27219(-4)	2.25663(-3)	0.171349	13.0722	9.51117	2.22692
0	1.46697(-2)	6.37583(-2)	0.171349	4.50177(-3)	0.171349	6.37583(-2)	1.46697(-2)
-1	2.22692	9.51117	13.0722	0.171349	2.25663(-3)	4.27219(-4)	9.66453(-5)
-2	328.676	711.447	9.51117	6.37583(-2)	4.19924(-4)	5.52843(-6)	6.54011(-7)
-3	24344.5	328.677	2.22692	1.46697(-2)	9.65979(-5)	6.54264(-7)	8.37660(-9)

$p^3_{\lambda\nu}$

λ \ ν	0	1	2	3
0	1.80071(-2)	0.685396	0.255033	5.86788(-2)
1	0.685396	26.1489	19.0232	4.45403
2	0.255033	19.0232	1422.89	657.352
3	5.86788(-2)	4.45403	657.352	48689.0

$_4P_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3
3	1.05985(-7)	1.56715(-5)	2.37867(-3)	0.361236	54.8348	8101.81	280828.
2	8.65372(-7)	6.70267(-5)	9.98019(-3)	1.51551	225.563	6476.60	8101.81
1	6.70267(-5)	3.99651(-4)	3.04976(-2)	4.58656	187.488	225.563	54.8348
0	9.98019(-3)	3.04965(-2)	4.47289(-2)	3.37442	4.58656	1.51551	0.361236
-1	1.51551	4.58656	3.37442	4.47289(-2)	3.04965(-2)	9.98019(-3)	2.37867(-3)
-2	225.563	187.488	4.58656	3.04976(-2)	3.99651(-4)	6.70267(-5)	1.56715(-5)
-3	6476.60	225.563	1.51551	9.98019(-3)	6.70267(-5)	8.65372(-7)	1.05985(-7)

$P^4_{\lambda\nu}$

λ \ ν	0	1	2	3
0	6.65938	9.11213	3.01106	0.627715
1	9.11213	374.975	451.126	109.670
2	3.01106	451.126	12953.2	16203.6
3	0.627715	109.670	16203.6	561656.

$_5p_{\lambda\nu}$

λ \ ν	-3	-2	-1	0	1	2	3	4
4				-5.86655(-2)	-8.90913	-1352.17	-20074.	12593800.
3	-6.06670(-8)	-8.93790(-6)	-1.35674(-3)	-0.206044	-31.2796	-4603.74	288315.	-200740.
2	-5.58909(-7)	-4.30675(-5)	-6.44420(-3)	-0.979048	-146.462	6649.27	-4603.74	-1352.17
1	-4.30675(-5)	-1.90509(-4)	-1.43171(-2)	-2.21954	192.492	-146.462	-31.2796	-8.90913
0	-6.44420(-3)	-1.43171(-2)	4.53210(-2)	3.46432	-2.21954	-0.979048	-0.206044	-5.86655(-2)
-1	-0.979048	-2.21954	3.46432	4.53210(-2)	-1.43171(-2)	-6.44420(-3)	-1.35674(-3)	
-2	-146.462	192.492	-2.21954	-1.43171(-2)	-1.90509(-4)	-4.30675(-5)	-8.93790(-6)	
-3	6649.27	-146.462	-0.979048	-644420(-3)	-4.30675(-5)	-5.58909(-7)	-6.06670(-8)	

$p^5_{\lambda\nu}$

λ \ ν	0	1	2	3
0	6.83800	-4.41045	-1.94521	-0.409375
1	-4.41045	386.984	-292.924	-62.5592
2	-1.94521	-292.924	13298.5	-9207.48
3	-0.409375	-62.5592	-9207.48	576630.

$_6p_{\lambda\nu}$

λ \ ν	-4	-3	-2	-1	0	1	2	3
3		2.88104(-8)	4.26831(-6)	6.47708(-4)	9.83568(-2)	14.9269	2209.92	164245.
2	2.88104(-8)	2.21013(-7)	1.73152(-5)	2.55259(-3)	0.38649	56.9182	4209.04	2209.92
1	4.26831(-6)	1.73152(-5)	1.61872(-4)	1.28906(-2)	1.77600	128.346	56.9182	14.9269
0	647709(-4)	2.55259(-3)	1.28906(-2)	0.181594	6.90728	1.77600	0.386495	9.83566(-2)
-1	9.83568(-2)	0.386495	1.77600	6.90728	0.181594	1.28906(-2)	2.55259(-3)	6.47709(-4)
-2	14.9269	56.9182	128.346	1.77600	1.28906(-2)	1.61872(-4)	1.73152(-5)	4.26831(-6)
-3	2209.92	4209.04	56.9182	0.386495	2.55259(-3)	1.73152(-5)	2.21013(-7)	2.88104(-8)
-4	164245.	2209.92	14.9269	9.83568(-2)	6.47709(-4)	4.26831(-6)	2.88104(-8)	

$p^6_{\lambda\nu}$

λ \ ν	0	I	2	3
0	13.4514	3.52622	0.767885	0.195418
1	3.52622	256.692	113.836	29.8538
2	0.767885	113.836	8418.08	4419.84
3	0.195418	29.8538	4419.84	328490.

${}_6\pi_{\lambda\nu}$

λ \ ν	-4	-3	-2	-1	0	1	2	3
3	344548.	176738.	172416.	172281	172267	172132	167810	0
2	176738.	8927.67	4605.59	4470.73	4456.94	4322.08	0	-167810
1	172416.	4605.59	283.511	148.650	134.861	0	-4322.08	-172132.
0	172281.	4470.73	148.650	13.7893	0	-134.861	-4456.94	-172267.
-1	172267.	4456.94	134.861	0	-13.7893	-148.650	-4470.73	-172281.
-2	172132	4322.08	0	-134.861	148.650	-283.511	-4605.59	-172416.
-3	167810.	0	-4322.08	-4456.94	-4470.73	-4605.59	-8927.67	-176738.
-4	0	-167810.	-172132.	-172267.	-172281.	-172416	-176738.	-344548.

$_{7}p_{\lambda\nu}$

λ \ ν	-4	-3	-2	-1	0	1	2	3
3	7.47898(-9)	6.97785(-8)	6.42729(-6)	9.56335(-4)	+0.144947	+21.5553	+1048.59	-84510.1
2	1.09574(-6)	4.51405(-6)	5.22949(-5)	4.94183(-3)	+0.703862	+37.4410	-2166.21	-581.439
1	1.66211(-4)	6.55791(-4)	3.42495(-3)	6.44070(-2)	+2.68722	-66.0697	-15.0437	-3.83319
0	2.52400(-2)	9.93309(-2)	0.479018	3.51516	-3.51516	-0.479018	-9.93309(-2)	-2.52400(-2)
-1	3.83319	15.0437	66.0697	-2.68722	-6.44070(-2)	-3.42495(-3)	-6.55791(-4)	-1.66211(-4)
-2	581.439	2166.21	-37.4410	-0.703862	-4.94183(-3)	-5.22949(-5)	-4.51405(-6)	-1.09574(-6)
-3	84510.1	-1048.59	-21.5553	-0.144947	-9.56335(-4)	-6.42729(-6)	-6.97785(-8)	-7.47898(-9)

$p^{7}_{\lambda\nu}$

λ \ ν	0	1	2	3
0	-14.0606	-1.91607	-0.397324	-0.100960
1	5.24563	-132.146	-30.0887	-7.66671
2	1.39784	74.8820	-4332.42	-1162.88
3	0.287981	43.1106	2097.18	-169.020

$_7\pi_{\lambda\nu}$

λ \ ν	-4	-3	-2	-1	0	1	2	3
3	565407.	34743.4	21075.8	20649.3	20605.7	20179.2	6511.53	-524152.
2	545460.	14796.4	1128.72	702.254	658.648	232.180	-13435.5	-544099.
1	544818.	14154.4	486.743	60.2743	16.6683	-409.800	-14077.5	-544741.
0	544780.	14116.0	448.272	21.8030	-21.8030	-448.272	-14116.0	-544780
-1	544741.	14077.5	409.800	-16.6683	-60.2743	-486.743	-14154.4	-544818.
-2	544099.	13435.5	-232.180	-658.648	-702.254	-1128.72	-14796.4	-545460.
-3	524154.	-6511.53	-20179.2	-20605.7	-20649.3	-21075.8	-34743.4	-565407.

$_8p_{\lambda\nu}$

λ \ ν	-4	-3	-2	-1	0	1	2	3
3	-2.58407(-9)	-1.70389(-7)	-2.48034(-5)	-3.76433(-3)	-0.571736	-86.7693	-12524.3	5297.25
2	-2.33888(-7)	-1.79838(-6)	-9.78031(-5)	-1.63570(-2)	-2.48521	-366.364	200.695	119.447
1	-3.48227(-5)	-1.67298(-4)	-6.60727(-4)	-4.33863(-2)	-6.72137	+1.87129	3.77077	0.842664
0	-5.28650(-3)	-2.53205(-2)	-5.63684(-2)	8.77256(-2)	-8.77256(-2)	5.63684(-2)	2.53205(-2)	5.28650(-3)
-1	-0.842664	-3.77077	-1.87129	6.72137	4.33863(-2)	6.60727(-4)	1.67298(-4)	3.48227(-5)
-2	-119.447	-200.695	366.364	2.48521	1.63570(-2)	9.78031(-5)	1.79838(-6)	2.33888(-7)
-3	-5297.25	12524.3	86.7693	0.571736	3.76433(-3)	2.48034(-5)	1.70389(-7)	2.58407(-9)

$p^8_{\lambda\nu}$

λ \ ν	0	1	2	3
0	-0.354902	0.225474	0.101282	0.021146
1	-13.3560	3.74390	7.54187	1.68540
2	-4.93771	-732.728	401.390	238.894
3	-1.13594	-173.539	-25048.6	10594.5

These C_λ are given below along with old values in Report II:

	C_0	C_1	C_2	C_3
Old Method of Report II	2.7322(-1)	7.384(-3)	5(-5)	4.5(-8)
New Method	2.7321(-1)	7.389(-3)	4.4(-5)	7.6(-8)

3. Computation of the Eight Series

A particular cone (A, B) = (0.35587, 0.17207) at $M = \sqrt{2}$ was decided upon to illustrate the method of solving the linear equation systems at angles of attack. The mathematical parameters have the following values at $\alpha = 0$

$$k^2 = 0.1$$

$$y_o = 0.525068 \qquad \rho = 1 \cdot \frac{A^2}{B} = 0.7368 \quad \text{(Case 3)}$$

$$\eta = 4.00084$$

One could plot k^2, y_o, $\tilde{\alpha}$ as functions of α in the range $0 \leq \alpha \leq 12°$ say; however, it has already been shown (Section 1) for a cone very similar to the present cone that these parameters (k^2, y_o) are only weakly dependent upon α. In this report these parameters will be assumed constant in range $0 \leq \alpha \leq 10°$. The eight $\overset{\circ}{P}{}^{i}_{\lambda\nu}$ have been computed using the auxiliary ${}_{\overset{\circ}{i}}\pi_{\lambda\nu}$ described in this report page 51. The coefficients σ_μ, γ_μ, δ_μ necessary for computing these series are tabulated below. Also a few of the ${}_{\overset{\circ}{i}}\pi_{\lambda\nu}$ are given to exhibit the symmetry properties discussed in Section 8, of Part II.

4. Solution of Linear Equations at $\alpha = 0$ for the Cone in Section 3.

The first three series $P^{i}_{\lambda\nu}$ $\dot{i} = 1, 2, 3$ of Section 3 were used to solve the linear equations for the a_n at $\alpha = 0$, $M = \sqrt{2}$. The appropriate linear equations are (see Section 2 above)

$$-4.63724a_o + 1.05295a_1 + 9.20406(-2)a_2 + 1.35384(-2)a_3 = 3.24725(-1)\ W_\infty$$

$$5.26476(-1)a_o - 4.93185(+1)a_1 + 6.46239a_2 + 1.02473a_3 = 4.27616(-3)\ W_\infty$$

$$4.60202(-2)a_o + 6.46239a_1 - 1.52767(+3)a_2 + 1.49145(+2)a_3 = 2.81593(-5)\ W_\infty$$

$$6.76917(-3)a_o + 1.02473a_1 + 1.49145(+2)a_2 - 5.98945(+4)a_3 = 1.85403(-7)\ W_\infty$$

The solutions are:

$a_o = -7.022(-2)\ W_\infty$ $\quad$ $a_2 = -5.67(-6)\ W_\infty$

$a_1 = -8.37(-4)\ W_\infty$ $\quad$ $a_3 = -3.14(-8)\ W_\infty$

The Fourier coefficients of W(z) viz. C_λ are obtained from the relations given in Section 2. The results are given below:

$C_o = 7.21(-2)\ W_\infty$ $\quad$ $C_2 = 1.2(-5)\ W_\infty$

$C_1 = 1.86(-3)\ W_\infty$ $\quad$ $C_3 = 7\ \ (-8)\ W_\infty$

These are to be compared below (Section 5) with the corresponding Fourier coefficients of component w for $\alpha = 10°$. The following results had been obtained earlier using the method of Report II.

$C_o = 7.28(-2)\ W_\infty$ $\quad$ $C_2 = 1.3(-5)\ W_\infty$

$C_1 = 1.97(-3)\ W_\infty$ $\quad$ $C_3 = 9\ \ (-8)\ W_\infty$

5. Exact Solution of the Linear Equation Systems at $\alpha = 10°$.

As stated in Section 3 the eight $\overset{i}{p}_{\lambda\nu}$ calculated with the values of k^2, y_o at $\alpha = 0$ are to be used now to set up the linear equations at α=10°, using however the exact value of $\tilde{\alpha}$. A spot check reveals that the matrix elements of the $\overset{i}{p}_{\lambda\nu}$ would change by about 10 percent if the latter were computed with the correct values of k^2,y_o at 10°. For purposes of illustration this is of

NORMALIZED EXACT LINEAR EQUATIONS FOR a_ν AND b_ν :

λ	a_0	a_1	a_2	a_3	b_0	b_1	b_2	b_3	RHS
0	1	-0.266533	-2.48655(-2)	-3.96730(-3)	-0.514855	-7.53162(-2)	-1.53113(-2)	-3.92300(-3)	-6.87928(-2)
1	-1.24058(-2)	1	-0.163359	-2.79663(-2)	1.11323(-2)	-0.493442	-0.108037	-2.76807(-2)	-8.43304(-5)
2	-3.73955(-5)	-5.27825(-3)	1	-0.131939	1.59196(-4)	-6.19719(-3)	-0.516901	-0.136009	-1.79430(-8)
3	-1.52140(-7)	-2.30417(-5)	-3.36437(-3)	1	2.86444(-7)	4.10244(-5)	-6.83261(-3)	-0.516909	-3.012561(-12)
0	-5.77541(-2)	0.281286	0.106966	2.47430(-2)	1	5.07243(-2)	-1.73145(-2)	-3.43932(-3)	-2.07198(-2)
1	1.62077(-5)	2.22537(-2)	0.442841	0.102198	1.37013(-3)	1	-7.28758(-2)	-3.13213(-3)	-3.67671(-6)
2	-2.97687(-7)	-4.23401(-5)	6.14357(-3)	0.441430	-1.35934(-5)	-2.11814(-3)	1	-1.17079(-2)	-7.03684(-10)
3	-7.86885(-10)	-1.52519(-7)	-1.61758(-5)	7.67411(-3)	-6.27466(-8)	-2.11549(-6)	-2.72068(-4)	1	-1.07665(-13)

little consequence since in a more elaborate numerical investigation the $\overset{i}{P}_{\lambda\nu}$ will surely be calculated as a function of α. The linear equations are given below with the solutions a_n, b_n and the corresponding Fourier coefficients of W (z): C_λ and $\tilde{\mathcal{L}}_\lambda$. The results are given with three significant figures because of the use of the approximate values of k^2, y_o at $\alpha = 10°$. Similarly, when using the eight $\overset{\circ}{\overset{i}{P}}_{\lambda\nu}$ of section 3 one uses only the first three significant figures.

The solutions of the linear equations are:

$$a_o = -8.20\ (-2)\ W_\infty \qquad b_o = -2.52\ (-2)\ W_\infty$$

$$a_1 = -7.96\ (-4)\ W_\infty \qquad b_1 = +5.12\ (-5)\ W_\infty$$

$$a_2 = -2.98\ (-6)\ W_\infty \qquad b_2 = -2.1\ (-7)\ W_\infty$$

$$a_3 = -2\ (-8)\ W_\infty \qquad b_3 = +2\ (-8)\ W_\infty$$

The Fourier coefficients of W (z) are obtained from Section 4, Part II, Formula (2.13).

Using the solutions a_n, b_n one finds:

$$C_o = 1.21\ (-2)\ W_\infty \qquad \tilde{\mathcal{L}}_o = -8.05\ (-2)\ W_\infty$$

$$C_1 = 9.42\ (-3)\ W_\infty \qquad \tilde{\mathcal{L}}_1 = -3.53\ (-3)\ W_\infty$$

$$C_2 = 2.00\ (-3)\ W_\infty \qquad \tilde{\mathcal{L}}_2 = -1.05\ (-5)\ W_\infty$$

It is evident that the b_n are of the same order of magnitude as the a_n at $\alpha = 10°$, $M = \sqrt{2}$. The a_n and b_n behave differently in regard to sign, as indicated.

6. Approximate Solution for Small Angles of Attack

As discussed in Section 7, of Part II when α is small the first set of linear equations just gives the usual system for at $\alpha = 0$ as in Section 4 while the other set gives a system of inhomogeneous linear equations in which the b_n are the unknowns and the a_n (these were obtained in Section 4) are known.

For the same cone used in Sections 4 and 5 these linear equations are

$$\sum_{\mu}\left\{0.120276\, p^{(4)}_{\lambda\mu} + 0.22386\, p^{(5)}_{\lambda\mu} - 0.0427712\, p^{(6)}_{\lambda\mu} + 3.58176\,\sigma_{\lambda}\sigma_{\mu}\right\}{}_1b_{\mu}$$

$$= 1.74463\,\sigma_{\lambda} + \sum_{\mu}\left\{0.471938\, p^{(8)}_{\mu\lambda} + 0.081153\, p^{(7)}_{\mu\lambda} - 7.55101\,\sigma_{\lambda}\sigma_{\mu}\right\}{}_0a_{\mu}$$

Using the ${}_0a_{\mu}$ of Section 4 this set of linear equations for ${}_1b_{\mu}$ can be written in the following normalized form:

$${}_1b_o - 1.3605(-2)\,{}_1b_1 - 3.9833(-2)\,{}_1b_2 - 9.1985(-3)\,{}_1b_3 = -1.9453(-1)W_{\infty}$$

$$-3.01229(-4)\,{}_1b_o + {}_1b_1 - -1.34520(-1)\,{}_1b_2 - 1.73786(-2)\,{}_1b_3 = 6.2666(-5)W_{\infty}$$

$$-2.5414(-5)\,{}_1b_o - 3.8763(-3)\,{}_1b_1 + {}_1b_2 - 7.21751(-2)\,{}_1b_3 = -9.1834(-9)W_{\infty}$$

$$-1.3419(-7)\,{}_1b_o - 1.14504(-5)\,{}_1b_1 - 1.65029(-3)\,{}_1b_2 + {}_1b_3 = -3.1002(-10)W_{\infty}$$

The solutions are:

$${}_1b_o = -2.284(-1)W_{\infty}$$

$${}_1b_1 = 3.99\,(-6)W_{\infty}$$

$${}_1b_2 = -5.84(-6)W_{\infty}$$

The actual b_{μ} are equal to $\delta h\tilde{\alpha}\,{}_1b_{\mu}$; this indicates that for small angles only ${}_1b_o$ is required along with the a_{μ}.

7. Approximate Solution Using Method of Finite Fourier Series

As shown in Section 10, of Part II, one can set up an approximate scheme in which $\mathfrak{F}(z')$ is represented by a finite Fourier series and in which the boundary conditions are satisfied at a selected set of points. As a simple example of the type of numerical work involved in using this method, the case of $\alpha = 0$, $M = \sqrt{2}$ will be resolved here and compared with results of Section 4. One satisfies the boundary conditions at $\mathcal{X} = 0$, $\mathcal{X} = K$; this yields two linear equations for a_o and a_1 which are

$$\mathcal{H}_{00}\, a_0 + \mathcal{H}_{01}\, a_1 = R_0$$

$$\mathcal{H}_{10}\, a_0 + \mathcal{H}_{11}\, a_1 = R_1$$

where the coefficients have been defined in Section 10, Part II. Substituting appropriate constants and summing the series one obtains the following set of linear equations

$$3.929\, a_o + 77.87\, a_1 = -\, 0.3162\, W_\infty$$

$$6.242\, a_o - 110.89\, a_1 = -\, 0.3162\, W_\infty$$

which yields the following values

$$a_o = -6.48\ (-2)\ W_\infty$$

$$a_1 = -7.93\ (-4)\ W_\infty$$

these are to be compared with the exact values of Section 4, viz:

$$a_o = -7.02\ (-2)\ W_\infty$$

$$a_1 = -8.37\ (-4)\ W_\infty$$

The solutions agree roughly to within ten percent. As more points are taken on the cone; say $\mathcal{X} = K/2$, $K/4$, etc., the accuracy of this method will become much better. It should be emphasized that this calculation requires only several hours to complete and the addition of more linear equations when the boundary conditions are satisfied at other points will not introduce much more computational labor.