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AN INVESTIGATION OF THE EXACT SOLUTIONS OF THE LINEARIZED EQUATIONS FOR THE FLOW PAST CONICAL BODIES

BY

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PREFACE

The problem of the steady flow of a gas over bodies of conical shape moving in a uniform stream at supersonic speeds has received considerable attention in the literature. Among the first successful treatments of this problem was that given by v. Kármán and Moore[1)] [1] for the case of slender bodies of revolution in which the assumptions essential to the linear theory of Glauert [2] and Prandtl [3] were made. This work was immediately followed by the more general treatment of the cone of revolution by Taylor and Maccoll [4] . Later Busemann [5], [6] gave an extended analytical treatment of the problem of the flow over a body with axial symmetry.

In the original investigation of this problem by Busemann [7], it is assumed that the flow over the conical body is irrotational and such that the characteristic quantities of the flow, namely, velocity, pressure and density, are constant along rays issuing from the vertex of the cone. This assumption was retained in his later treatment of the problem and was also adopted by Taylor and Maccoll in their work. Busemann referred to a region in which the flow has this property as a conical flow field (the

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1) Numbers in brackets refer to the bibliography.

common point of the family of rays along which the velocity is constant is called the "center" of the conical field), and the term has been in common usage since[2]. In 1943 Busemann [8] formulated a very general method of treating the linearized flow problem of Glauert and Prandtl for a conical body under the assumption that the flow past the body is conical. This method has recently received widespread attention. Notable among the applications which have been made of this method for the investigation of the general properties of supersonic flow and for the treatment of special types of bodies are those of Stewart [9], Gurevich [10], Lagerstrom [11], and Hayes [12].

The basis of the present treatment of the equations for the linearized flow past a conical body is the method suggested in the work of Busemann [8]. The purpose, however, is to develop a method of determining the solution of these equations which satisfies the precise boundary conditions along the surface of the body and thus to obtain a flow for which the surface is an actual stream surface. This is in contrast with the previous treatments in which the actual conditions on the boundary are replaced by the approximate "linearized" conditions. The work is presented in three parts. Part I is devoted to the theory of conical flow and the formulation of an analytical treatment of the general problem. Parts II and III are devoted respectively to the determination of the conical flow past a plane arrow-shaped wing with angle of attack and zero yaw, and an elliptic cone with zero angle of attack by means of the method developed in the first part.

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2) It should be remarked that the term conical flow does not imply that the velocity of the flow at a point of the field is in the direction of the ray passing through this point and the center of the flow.

In the general treatment of Part I, it is shown that a supersonic conical flow in space can be completely described in terms of functions depending on a single complex variable. In particular, the components of the velocity of the flow are determined by a single analytic function of a complex variable which in this theory plays a role of comparable importance to that played by the complex velocity potential in the theory of the steady, irrotational motion of an incompressible, ideal fluid. Hence, when the general problem of conical flow is formulated in this way its analytical treatment naturally lends itself to applications of the well-established theory of functions of a complex variable, especially to applications of the techniques utilizing conformal mapping.

In the treatment of the arrow wing in Part II, it is shown that the boundary conditions are not sufficient to determine the conical flow past the wing. This fact was previously pointed out by Stewart [9] in his solution of the corresponding problem with the simpler linearized boundary conditions. In order to determine the order of the infinite singularities which the velocity of the flow is permitted to have at the edges of the wing, he supplemented the boundary conditions by requiring the lift coefficient of the wing to be finite. It is shown in Part II that the addition of this supplementary condition suffices to determine the flow uniquely, and hence, it is shown that among all conical flows for which the wing is a stream surface, there is a unique one for which the coefficient of lift is finite.

The work in Part III on the conical flow past an elliptic cone indicates that the incompleteness of the boundary conditions in the case of the arrow wing springs from the singularities of the surface at the wing edges. For it is shown in this part that the boundary conditions are sufficient to determine a unique conical flow past an elliptic cone without edge having a finite velocity at every point of the field.

PART I. THE GENERAL PROBLEM OF THE SUPERSONIC CONICAL FLOW

1.1 Statement of the General Problem

Consider the flow over a slender conical body placed in a uniform stream moving with supersonic speed, the vertex of the body pointing in the direction of the oncoming stream. Let the origin of the rectangular system of X, Y, and Z coordinates be taken at this point with the Z-axis in the direction of the uniform flow. Let w_∞ denote the velocity of the uniform stream, and let u, v, and w denote the components of the additional velocity; the components of the total velocity of the disturbed flow at any point are therefore u, v, and $w + w_\infty$. Since the flow is irrotational, the additional velocity is expressible as a gradient of a velocity potential $\Phi(X, Y, Z)$:

$$u = \frac{\partial \Phi}{\partial X}, \qquad v = \frac{\partial \Phi}{\partial Y}, \qquad w = \frac{\partial \Phi}{\partial Z}, \tag{1.1}$$

where, in accordance with the assumptions of the linear theory, the potential Φ satisfies the linear differential equation:[3)]

$$\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} - \beta^2 \frac{\partial^2 \Phi}{\partial Z^2} = 0, \tag{1.2}$$

in which $\beta = (M^2 - 1)^{1/2}$ and M is the constant Mach number of the uniform stream.

The characteristics of the differential equation (1.2) are the conical surfaces of revolution with axes parallel to the Z-axis (i.e., parallel to the direction of the flow of the uniform stream) and with semi-vertex angle equal to the Mach angle μ defined by the equation:

$$\mu = \sin^{-1}\left(\frac{1}{M}\right) = \cot^{-1} \beta . \tag{1.3}$$

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3) cf. Glauert [2] and Prandtl [3]. Also R. Sauer, Theoretische E in fuhrung in die Gas dynamik. Springer, Berlin, 1943. Reprinted by Edwards Bros., Ann Arbor, Michigan, 1945, p.24.

The downstream halves of these cones are significant physically since each envelopes the domain which is influenced by a small disturbance originating at its vertex. These semi-cones[4] are therefore identified with the well-known Mach cones of sonic disturbance in a stream of uniform supersonic velocity.

If it is assumed that the conical body is entirely enclosed within the Mach cone attached to the vertex of the conical body, then, in accordance with the linear theory, the Mach cone will constitute a boundary between the region of the constant state of the flow ahead of the body and the region of the disturbed flow adjacent it. The problem is, therefore, to determine the distribution of the velocity of the flow in the region between the two surfaces subject to the two conditions: (a) that conical body be a stream surface and (b) that the transition from the constant state of flow to the disturbed flow take place across the characteristic or Mach cone. The first of these conditions implies that at each point of the body surface the component of the total velocity in the direction of the normal at the point is zero[5]. Thus if $S(X, Y, Z) = 0$ represents the equation of body surface, this condition becomes

$$u \frac{\partial S}{\partial X} + v \frac{\partial S}{\partial Y} + (w_\infty + w) \frac{\partial S}{\partial Z} = 0 \qquad (1.4)$$

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4) For convenience, the semi-conical surfaces generated by a half-line, or ray, emanating from a point will henceforth be referred to as a cone.

5) In conformity with the observed properties of the motion of a fluid, it must also be required that the velocity of the fluid be continuous at each regular point of the surface of the obstacle, that is, at each point at which the surface has a continuous normal vector.

or, by equation (1.1),

$$\frac{\partial \Phi}{\partial X}\frac{\partial S}{\partial X} + \frac{\partial \Phi}{\partial Y}\frac{\partial S}{\partial Y} + (w_\infty + \frac{\partial \Phi}{\partial Z})\frac{\partial S}{\partial Z} = 0 . \qquad (1.5)$$

The second of the conditions implies that the velocity is continuous across the Mach cone[6], and therefore that the components u, v, and w of the additional velocity satisfy the following equations on this Mach cone[7]:

$$u = v = w = 0, \qquad (1.6)$$

or, by equation (1.1),

$$\frac{\partial \Phi}{\partial X} = \frac{\partial \Phi}{\partial Y} = \frac{\partial \Phi}{\partial Z} = 0 . \qquad (1.7)$$

Therefore, the problem of determining the (linearized) flow past the conical obstacle may be formulated as (Problem I), that of determining the function Φ (X, Y, Z) which satisfies the differential equation (1.2) in the region between the body and the Mach cone and the boundary conditions (1.4) and (1.7) at points on these two surfaces, respectively.

The functions u, v, and w corresponding to a solution Φ of the problem formulated above will also satisfy differential equations of the type (1.2), as can easily be verified by differentiating equation (1.2) with respect to either X, Y, or Z and making use of the corresponding equations in (1.1). Therefore, in the region between the Mach cone and the surface of the conical obstacle u, v, and w satisfy the three equations

$$\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} - \beta^2 \frac{\partial^2 u}{\partial Z^2} = 0, \text{ etc.}, \qquad (1.8)$$

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6) See, for example, Sauer: loc. cit., pp. 110 and 132.

7) It can be shown that for a conical (linearized) flow the condition (1.6) is equivalent to merely requiring that the component of the total velocity tangent to the Mach cone be continuous on passing through this surface. In this connection, it should be observed that in the two-dimensional flow over a wedge the corresponding condition requiring the continuity of the velocity component tangent to the Mach lines leads to the familiar discontinuity of the velocity vector across these lines.

where in the last two equations, u is replaced by v and w, respectively. However, the three functions derived in this manner from equation (1.1) are not entirely independent of one another, but in addition, satisfy the following equations in the region between the Mach cone and the body

$$\frac{\partial w}{\partial Y} - \frac{\partial v}{\partial Z} = 0, \qquad \frac{\partial u}{\partial Z} - \frac{\partial w}{\partial X} = 0, \qquad \frac{\partial v}{\partial X} - \frac{\partial u}{\partial Y} = 0. \quad (1.9)$$

These are simply the conditions of irrotationality of the velocity field defined by the functions u, v, and w. It is also evident that the functions u, v, and w corresponding to the solution Φ of Problem I will satisfy the boundary conditions in equations (1.4) and (1.6) on the surface of the body and on the Mach cone, respectively.

Following Busemann[8], it is observed that the equations (1.4), (1.6), (1.8), and (1.9) as well as the equations for the surfaces[9] of the conical body and the Mach cone, are not changed by the operation of replacing the variables X, Y, and Z by tX, tY, and tZ, respectively, where t is an arbitrary positive constant. Therefore, if the functions u(X, Y, Z), v(X, Y, Z), and w(X, Y, Z) satisfy the differential equations (1.8) and (1.9) and the boundary conditions (1.4) and (1.6), then the functions u(tX, tY, tZ), v(tX, tY, tZ), and w(tX, tY, tZ) do likewise. This corresponds to the statement that an arbitrary magnification (or contraction) of the field of flow compatible with the surface of the obstacle is again a field of flow compatible with this surface. This observation lead Busemann to conjecture[10] that the velocity of the flow is constant along rays

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8) cf. Busemann [7] .

9) In general, the equation of a cone with vertex at the origin is homogeneous in the variables X, Y, and Z.

10) The truth of this conjecture would immediately follow from the property that the solution of Problem I is unique, except for an additive constant, provided that the problem as it has been formulated enjoyed this property. However, the latter is not true in general, since it is shown that in the case of the arrow wing there are infinitely many functions $\Phi(x,y,z)$ satisfying the conditions of the problem. Nevertheless, these functions are all homogeneous of degree one.

emanating from the origin, or vertex of the conical obstacle, and consequently that the flow is conical. This hypothesis is also adopted in the remainder of this report. Hence, only those solutions $\Phi(X, Y, Z)$ of Problem I will be sought for which the corresponding flow is conical, i.e., for which the components u, v, and w of the additional velocity in equation (1.1) are homogeneous functions of degree zero. Apart from additive constants, the admissible solutions $\Phi(X, Y, Z)$ are therefore homogeneous functions of degree one. In this connection, it should be remarked that by virtue of this property of homogeneity, equation (1.1) is readily solvable for the function Φ. Thus, making use of Euler's relation for homogeneous functions and equation (1.1), it is evident that Φ can be expressed in the form:

$$\Phi = X \frac{\partial \Phi}{\partial X} + Y \frac{\partial \Phi}{\partial Y} + Z \frac{\partial \Phi}{\partial Z} + \text{const} = uX + vY + wZ + \text{const}. \tag{1.1a}$$

Since the differential equation (1.8) is obtained from (1.2) by a process of differentiation, the differential equation (1.8) when expressed in terms of Φ is of higher order than (1.2). It is therefore not necessarily true that the function Φ which is determined by the converse procedure from the solutions of the differential equations (1.8) and (1.9) will satisfy equation (1.2) and thus represent the potential of a linear flow. However, if in addition, the solutions u, v, and w of the system of equations in (1.8) and (1.9) are required to be homogeneous of degree zero, that is, define a conical flow field, then it is readily shown that the corresponding function $\Phi(x, y, z)$ obtained from them by means of equation (1.1), or more simply, by (1.1a), is a solution of equation (1.2). For, on substituting (1.1a) in (1.2), making use of (1.9) and the homogeneity of the functions u, v, and w, the left hand member of (1.2), which is here denoted by $L\{\Phi\}$, becomes

$$L\{\Phi\} \equiv \frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} - \beta^2 \frac{\partial^2 \Phi}{\partial Z^2} = \frac{\partial u}{\partial X} + \frac{\partial v}{\partial Y} - \beta^2 \frac{\partial w}{\partial Z} .$$

Since u, v, and w are of degree zero, the quantity $L\{\Phi\}$ is either identically zero, whence equation (1.2) is satisfied, or is homogeneous of degree -1. In the latter event, it follows from Euler's relation that

$$X \frac{\partial L}{\partial X} + Y \frac{\partial L}{\partial Y} + Z \frac{\partial L}{\partial Z} = -L.$$

However, by equations (1.9) and (1.8) each of the partial derivatives in the left hand member of this equation vanishes identically. Consequently, L also vanishes identically. Hence, the function Φ defined in this way satisfies the differential equation (1.2). It is evident that Φ will also satisfy the boundary conditions (1.5) and (1.7) if the functions u, v, and w satisfy the corresponding conditions (1.4) and (1.6). It therefore follows that in order to determine the admissible solutions Φ of Problem I, it is sufficient to seek solutions u, v, and w of the differential equations (1.8) and (1.9) which are homogeneous of degree zero and which satisfy the boundary conditions (1.4) and (1.6).

In summarizing the results of the preceding paragraphs, it can be stated that the problem of determining the conical flow past a conical body which is contained within the Mach cone attached to its vertex is equivalent to either: (Problem I) determining the single function Φ satisfying the differential equation (1.2) in the region between the body and the Mach cone and the boundary conditions (1.5) and (1.7) on these surfaces, respectively; or (Problem II) determining the three functions u, v, and w which are homogeneous of degree zero in the variables X, Y, Z, and which satisfy the differential equations (1.8) and (1.9) between the body and Mach cone and the boundary conditions (1.4) and (1.6) on these surfaces, respectively. It will be shown in the following section that when considered in the latter form, the problem does in general admit of a very elegant formulation in terms of functions of a complex variable.

1.2 Transformation of the Equation for the General Conical Flow

Since the functions u, v, and w are constant along rays emanating from the origin, the values of these functions along any such ray depend only on two independent quantities specifying the direction of the ray. The procedure is therefore to make an appropriate coordinate transformation so as to reduce the differential equations and boundary conditions defining u, v, and w to a differential system depending on two independent variables which can be treated by an established method of integration.

Consider the mapping from the flow space to the space of the parameters ξ, η, and R, with $R > 0$, by means of the coordinate transformation[11])

$$X = R\,\frac{\xi}{\beta\lambda}, \qquad Y = R\,\frac{\eta}{\beta\lambda}, \qquad Z = R\,\frac{1-\lambda}{\lambda}, \tag{1.10}$$

where

$$\lambda = \frac{1}{2}\,(1 - \xi^2 - \eta^2) \quad \text{and} \quad R = \sqrt{Z^2 - \beta^2(X^2+Y^2)}\;. \tag{1.10a}$$

It is evident that these equations set up a one-to-one correspondence between

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11) This transformation is analogous to the transformation from the (X,Y,Z)-coordinates to the stereographic parameters (ξ, η, R), where $R=(X^2+Y^2+Z^2)^{1/2}$ and ξ, η are the coordinates of the points obtained by means of a stereographic projection of the point (X/R,Y/R,Z/R) of the unit sphere with center at the origin onto the (ξ, η)-plane, taking (0,0,-1) as the center of the projection. In equation (1.10) the unit sphere is replaced by the hyperboloid $Z^2-\beta^2(X^2+Y^2)=1$.

It should be noted that the second of equations (1.10a) effectively specifies a particular branch of the inverse of the transformation defined by equations (1.10). Equations (1.10) with λ defined as in (1.10a) assign to each interior point (X,Y,Z) of the region of definition two distinct points of the (ξ, η, R)-space, one interior to and the other exterior to the cylinder $\xi^2+ \eta^2=1$.

the points of the region bounded by the Mach cone, i.e., the region defined by the relations

$$Z > 0, \qquad Z^2 - \beta^2(X^2 + Y^2) > 0,$$

and the points of the (ξ, η, R)-space $(R > 0)$ bounded by the circular cylinder with unit radius and axis along the R-axis. This correspondence is such that images of cones in the (X, Y, Z)-space with vertices at the origin and contained entirely within the Mach cone are cylinders in the (ξ, η, R)-space lying entirely within the unit circular cylinder; the Mach cone itself corresponds to this unit circular cylinder as a limiting case. Equations (1.10) and (1.10a) therefore define a one-to-one mapping of the region between the conical body and the Mach cone in the flow space into the region of the (ξ, η, R)-space $(R > 0)$ bounded externally by the unit circular cylinder and internally by a parallel cylinder corresponding to the conical body.

It follows from the properties of the transformation that rays through the origin of the flow space along which the functions u, v, and w are constant are mapped into lines which are parallel to the R-axis. The corresponding "flow" defined in the image space by these functions is therefore constant along lines parallel to the R-axis. As a consequence, the "flow" in the image space is two-dimensional in character since the functions u, v, and w have the same values at corresponding points in all planes parallel to the (ξ, η)-plane and are therefore everywhere defined by their values over any one of these planes. The two-dimensional domain into which the flow pattern is projected by the transformation defined by equations (1.10) and(1.10a) consists of a doubly-connected region (see Figure 1) of the (ξ, η)-plane bounded externally by the unit circle with center at the origin, and internally by a simple closed curve whose shape is determined by the shape and orientation of the conical obstacle.

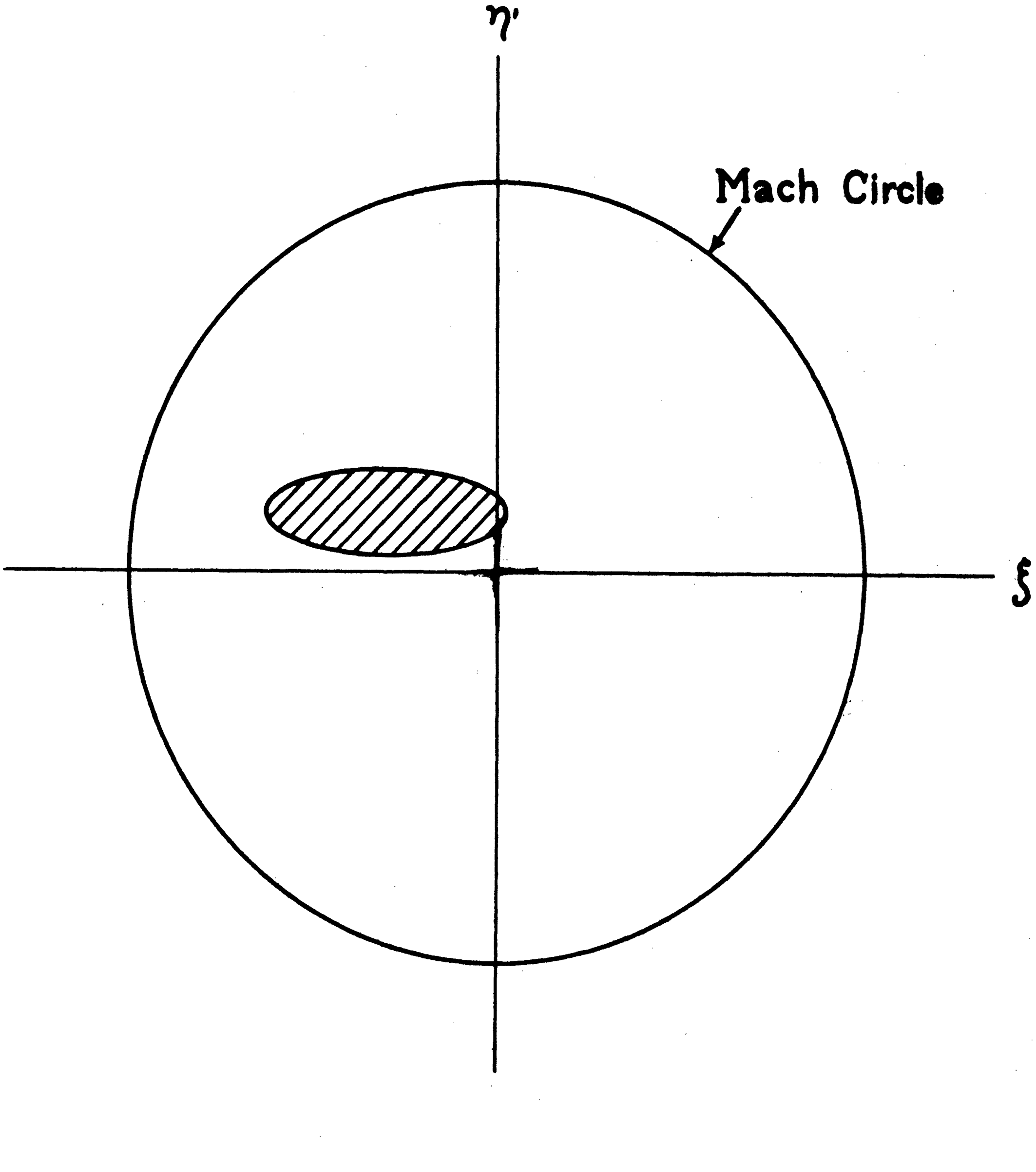

Fig. 1. ζ-plane

On making the substitutions of equation (1.10) in the differential equation (1.2), it becomes

$$\Delta \Phi = \lambda^{-2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \Phi}{\partial R}\right), \tag{1.11}$$

where

$$\Delta \Phi = \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} .$$

If, in particular, it is required that the function Φ (X, Y, Z) be homogeneous of degree n in the variables X, Y, and Z, it follows from equation (1.10) that

$$R \frac{\partial \Phi}{\partial R} = n \Phi$$

and, hence, by equation (1.11) that

$$\Delta \Phi = \frac{n(n+1)}{\lambda^2} \Phi .$$

Therefore, the differential equation in the variables ξ and η which determines the admissible solutions Φ of Problem I representing the potentials of a conical flow is obtained by setting n = 1, namely,

$$\Delta \Phi = \frac{2}{\lambda^2} \Phi. \tag{1.12}$$

This equation holds throughout the doubly-connected region of the (ξ, η)-plane. On the other hand, the admissible solutions u, v, and w of Problem II representing the components of the additional velocity of the conical flow are homogeneous of degree zero. Hence, these functions satisfy the two-dimensional Laplace equations

$$\Delta u = \Delta v = \Delta w = 0$$

throughout the doubly-connected region of the (ξ, η)-plane. In other words, an irrotational conical flow is defined by a triplet of harmonic functions of the variables ξ and η which in addition satisfy equation (1.9).

If the harmonic functions u, v, and w are considered as the real parts respectively of the three analytic functions U(ζ), V(ζ), and W(ζ)

of the complex variable $\zeta = \xi + i\eta$, the set of functions satisfying equation (1.9) can be expressed in an especially elegant form. For, by equation (1.10), it follows that[12)]

$$\frac{\partial}{\partial X} = \frac{\beta}{R} \mathcal{R}\left\{ \frac{1}{2}(1 + \zeta^2) \frac{d}{d\zeta} \right\},$$

with similar expressions for the derivatives with respect to Y and Z. Hence, equation (1.9) can be placed in the form:

$$\begin{aligned} &\mathcal{R}\left\{ \frac{dV}{d\zeta} + \frac{i\beta}{2}(1 - \zeta^2) \frac{dW}{d\zeta} \right\} = 0, \\ &\mathcal{R}\left\{ \frac{dU}{d\zeta} + \frac{\beta}{2}(1 + \zeta^2) \frac{dW}{d\zeta} \right\} = 0, \\ &\mathcal{R}\left\{ \frac{i\beta}{2}(1 - \zeta^2)\frac{dU}{d\zeta} - \frac{\beta}{2}(1 + \zeta^2)\frac{dV}{d\zeta} \right\} = 0. \end{aligned} \tag{1.13}$$

These equations hold within the doubly-connected region of the ζ-plane. However, the vanishing of the real part of an analytic function in any portion of its region of definition implies that the imaginary part and, consequently, the function itself must be constant throughout this region. Since the determinant of the coefficients of $dU/d\zeta$, $dV/d\zeta$, and $dW/d\zeta$ in the foregoing equations is identically zero, the quantity contained in each brace is, in fact, equal to zero. This is evidently true if, and only if, the analytic functions U, V, and W are given by means of the integral formulae:

$$\begin{aligned} U(\zeta) &= \frac{\beta}{2}\int (1 + \zeta^2)\, F(\zeta)\, d\zeta, \\ V(\zeta) &= i\frac{\beta}{2}\int (1 - \zeta^2)\, F(\zeta)\, d\zeta, \\ W(\zeta) &= -\int \zeta\, F(\zeta)\, d\zeta, \end{aligned} \tag{1.14}$$

12) The symbol $\mathcal{R}\{\ \}$ represents the real part of the complex quantity within the brace.

where $F(\zeta)$ is an arbitrary analytic function of the complex variable ζ.[13)]

Finally, the boundary conditions (1.4) and (1.6) yield a pair of conditions satisfied by the functions u, v, and w on the inner and outer boundaries, respectively, of the doubly-connected region of the plane. Thus, since the function S(X, Y, Z) is homogeneous in the variables X, Y, and Z, equation (1.4) will take the following form on making the substitutions (1.10):

$$Au + Bv + C(w_{\infty} + w) = 0, \tag{1.15}$$

where the coefficients A, B, and C are in general functions of ξ and η. This condition is satisfied on the inner boundary corresponding to the surface of the conical body. On the other hand, it follows from (1.6) that the functions u, v, and w vanish on the unit circle forming the outer boundary, i.e.,

$$u = v = w = 0 \quad \text{for} \quad |\zeta| = 1. \tag{1.16}$$

Thus, beginning with Problem II of the preceding section and reformulating the differential equations and boundary conditions in terms of the variables ξ and η, or the complex variable $\zeta = \xi + i\eta$, it is shown in this section that real parts u, v, and w of the complex functions $U(\zeta)$, $V(\zeta)$, and $W(\zeta)$, respectively, of (1.14) which satisfy the conditions

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13) The analogy between the formulae (1.14) and those representing the solution of the minimal surface problem as formulated by Weierstrass is self-evident. The latter involves the determination of a triplet of analytic functions $f_n(\zeta)$ (n=1,2,3) satisfying the condition $\sum_n [f_n'(\zeta)]^2 = 0$. The real parts of the analytic function $f_n(\zeta)$ are the coordinate functions defining the minimal surface. See, for example, Courant-Hilbert, Methoden der Mathematischen Physik, Vol II., Berlin, 1937, Chap. III, § 2. It follows from (1.14) that the triplet of functions $U(\zeta)$, $V(\zeta)$, and $W(\zeta)$ satisfy an analogous condition: $[U'(\zeta)]^2 + [V'(\zeta)]^2 - \beta^2 [W'(\zeta)]^2 = 0$. However, this single condition is not sufficient to insure the complete satisfaction of the three equations (1.13), two of which are independent, corresponding to the conditions of irrotationality (1.9). As a consequence of this, it will be seen that the integral formulae in (1.14) are not invariant in form under conformal transformations of the doubly-connected region.

(1.15) and (1.16) determine an irrotational, conical flow over the conical body. The (linearized) conical flow problem is therefore equivalent to that of determining a single function $F(\zeta)$ which is analytic in the double-connected region of the ζ-plane and such that the real parts u, v, and w of the complex functions defined by the formulas (1.14) satisfy the boundary conditions (1.15) and (1.16) on the inner and outer boundary, respectively, of this region. This form of the problem has the obvious advantage that it is susceptible to treatment by the methods developed in the theory of functions of a complex variable for solving boundary value problems by means of conformal mapping.

It should be remarked that the function $F(\zeta)$ plays a role in the present theory of irrotational, conical flow, which is analogous to and as important as that which is played by the complex potential in the theory of the irrotational motion of an incompressible fluid. The functions u, v, and w corresponding to any arbitrary analytic function $F(\zeta)$ determine a flow with the conical property. However, it is readily verified that these functions will not in general vanish over the surface of the Mach cone in accordance (1.6) unless the $F(\zeta)$ can be represented at points of an annular neighborhood of the unit circle by a Laurent series of the form

$$F(\zeta) = \frac{1}{\zeta^2}\left\{ a_o + \sum_{n=1}^{\infty} \left(a_n \zeta^n + \frac{\bar{a}_n}{\zeta^n}\right)\right\},$$

where the coefficient a_o is real, and the coefficient $\bar{a}_n$ is the complex conjugate of the coefficient a_n. For the simple case

$$F(\zeta) = \frac{a_o}{\zeta^2},$$

the formulae (1.14) yield the familiar Kármán-Moore solution for the flow over a circular cone.

It should also be remarked at this point that in general the function $F(\zeta)$ is necessarily continuous on the unit circle. This follows, in accordance with the general theorems on the properties of an analytic function, from the fact that the real part of the analytic function $W(\zeta)$ is constant along this circle. For, as an immediate consequence of the latter, the derivative $W'(\zeta)$ exists and is continuous on the unit circle and, consequently, by the last of equation (1.14), $F(\zeta)$ is continuous on that circle.

1.3 Application of the Lorentz Group of Rotations

It is evident that the differential equation (1.2) remains unchanged in form under the following linear transformation from the points (X, Y, Z) to the points (X_1, Y_1, Z_1) of the space:

$$\begin{aligned} \beta X_1 &= \beta a_{11}X + \beta a_{12}Y + a_{13}Z, \\ \beta Y_1 &= \beta a_{21}X + \beta a_{22}Y + a_{23}Z, \\ Z_1 &= \beta a_{31}X + \beta a_{32}Y + a_{33}Z, \end{aligned} \qquad (1.17)$$

where

$$a_{11}^2 + a_{12}^2 - a_{13}^2 = a_{21}^2 + a_{22}^2 - a_{23}^2 = - a_{31}^2 - a_{32}^2 + a_{33}^2 = 1,$$

and

$$a_{i1}a_{j1} + a_{i2}a_{j2} - a_{i3}a_{j3} = 0 \qquad \text{for } i \neq j .$$

Therefore, any solution of the differential equation (1.2) is transformed by the linear substitutions (1.17) into a solution of the same differential equation in terms of the coordinates (X_1, Y_1, Z_1). It is easily verified that the Mach cone is also invariant under this same transformation of the space. Hence this transformation has the property that it maps any given linear, supersonic flow field bounded by a given Mach cone into another such field bounded by the same Mach cone[14).

In particular, consider the two distinct subgroups of the group of transformation defined by (1.17):

$$\text{"Pitch":}\quad \begin{aligned} \beta X_1 &= \beta X \\ \beta Y_1 &= \beta Y \cosh \tilde{\alpha} - Z \sinh \tilde{\alpha} \\ Z_1 &= -\beta Y \sinh \tilde{\alpha} + Z \cosh \tilde{\alpha} \end{aligned} \tag{1.18}$$

$$\text{"Yaw":}\quad \begin{aligned} \beta X_1 &= \beta X \cosh \tilde{\psi} - Z \sinh \tilde{\psi} \\ \beta Y_1 &= \beta Y \\ Z_1 &= -\beta X \sinh \tilde{\psi} + Z \cosh \tilde{\psi} \,. \end{aligned} \tag{1.19}$$

These two families of transformations constitute the so-called Lorentz groups of non-Euclidean rotations of space about the X- and Y-axis, respectively, leaving the Mach cone unaltered; $\tilde{\alpha}$ and $\tilde{\psi}$ denote the non-Euclidean "angles of rotation" in pitch and yaw, respectively. It is readily verified that the actual Euclidean angle of rotation α in pitch and the non-Euclidean angle α are related by the equation

$$\beta \tan \alpha = \tanh \tilde{\alpha}, \tag{1.20}$$

with a similar relation for the angles in yaw. In particular, the position of a line making an angle γ with the YZ-plane and contained in a plane which makes an angle α with the XZ-plane (for example, an edge of the arrow wing in Figure 2) can be specified by means of the corresponding non-Euclidean angles of rotation $\tilde{\alpha}$ and $\tilde{\gamma}$ about the X- and Y-axis, provided that

$$\beta \tan \alpha = \tanh \tilde{\alpha} \qquad \text{and} \qquad \beta \tan \gamma \sec \alpha = \tanh \tilde{\gamma} \;\; \operatorname{sech} \tilde{\alpha} \,. \tag{1.21}$$

---

14) Since the transformation defined by (1.17) is in general not conformal, the form of the boundary conditions (1.5) is not preserved under the transformation.

The transformations corresponding to (1.18) and (1.19) in the (ξ, η)-plane are especially simple. It is readily shown that these transformations of the region of the flow bounded by the Mach cone onto itself correspond to two distinct subgroups of linear fractional transformations which map the interior of the unit circle conformally onto itself. Thus, by making the substitutions (1.10), it is seen that (1.18) and (1.19) correspond respectively to the following transformations of the ζ-plane onto the ζ_1-plane:

$$\zeta_1 = \frac{\zeta - ib}{1 + ib\zeta}, \tag{1.22}$$

where

$$b = \tanh \frac{\tilde{\alpha}}{2}, \tag{1.22a}$$

and

$$\zeta_1 = \frac{\zeta - a}{1 - a\zeta}, \tag{1.23}$$

where

$$a = \tanh \frac{\tilde{\psi}}{2}. \tag{1.23a}$$

The first of these maps the interior of the unit circle in the ζ-plane onto the region bounded by the unit circle in the ζ_1-plane in such a manner that points of the η-axis correspond to points of the η_1-axis, whereas the mapping effected by the second carries the points of the ξ-axis into the points of the ξ_1-axis,

PART II. SUPERSONIC FLOW PAST AN ARROW WING AT AN ANGLE OF ATTACK

2.1 Formulation of the Problem

The general theory which is presented in Part I is here applied to the problem of the flow around an arrow wing formed by a sector of a plane inclined at an angle of α with respect to the uniform flow and at zero yaw. The flare angle; i.e., the angle of the sector, is taken as 2γ (see Figure 2). It is assumed throughout that the angle of attack α and the flare angle 2γ are so restricted that the wing is contained entirely within the Mach cone of its vertex. This case may be regarded as the limiting case of the flow around an elliptic cone whose normal cross section is an ellipse with vanishing minor axes.

Let the vertex of the wing be placed at the origin and let the equation of its plane be $Y = Z \tan\alpha$. Making the substitutions, equation (1.10), this becomes, with the aid of (1.21)

$$\xi^2 + \eta^2 - 2 \coth\tilde{\alpha}\, \eta + 1 = 0 \; . \qquad (2.1)$$

The arrow wing characterized by the angles α and γ, or the corresponding quantities $\tilde{\alpha}$ and $\tilde{\gamma}$ of equation (1.21), is therefore projected into the ζ - plane as an arc of the circle determined by equation (2.1). The edges of the wing correspond to the end points $(\pm\xi_E \; \eta_E)$ of this arc (see Figure 3a). This circle and the unit circle $|\zeta| = 1$ intersect orthogonally. Consequently, its position is completely fixed by its intercept b with the η -axis and, therefore, by the angle of attack of the wing. For by equation (2.1) or, more simply, by (1.22a), it follows that

$$b = \tanh \frac{\tilde{\alpha}}{2} \; . \qquad (2.2)$$

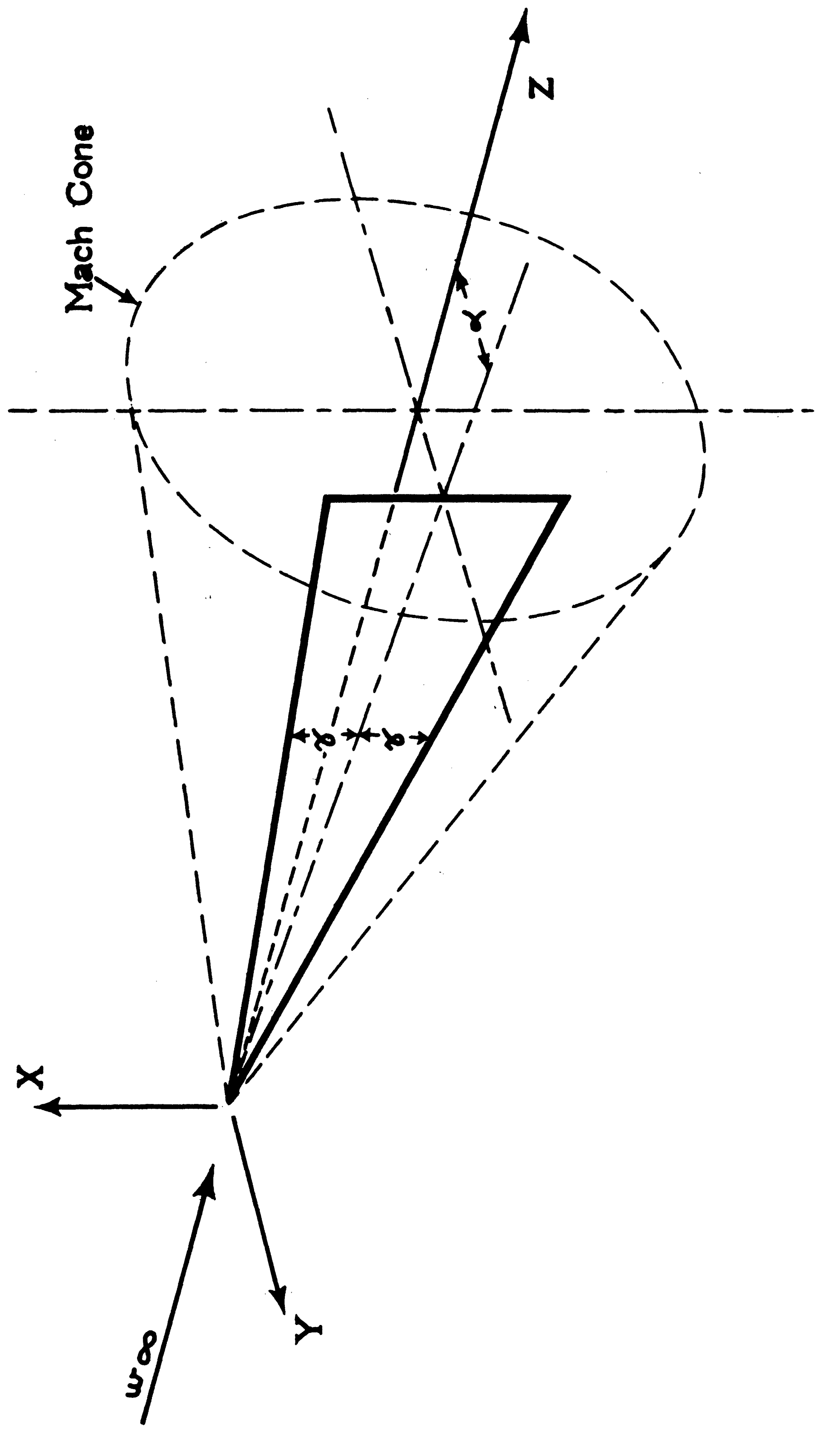

Fig. 2. Arrow wing at an angle of attack

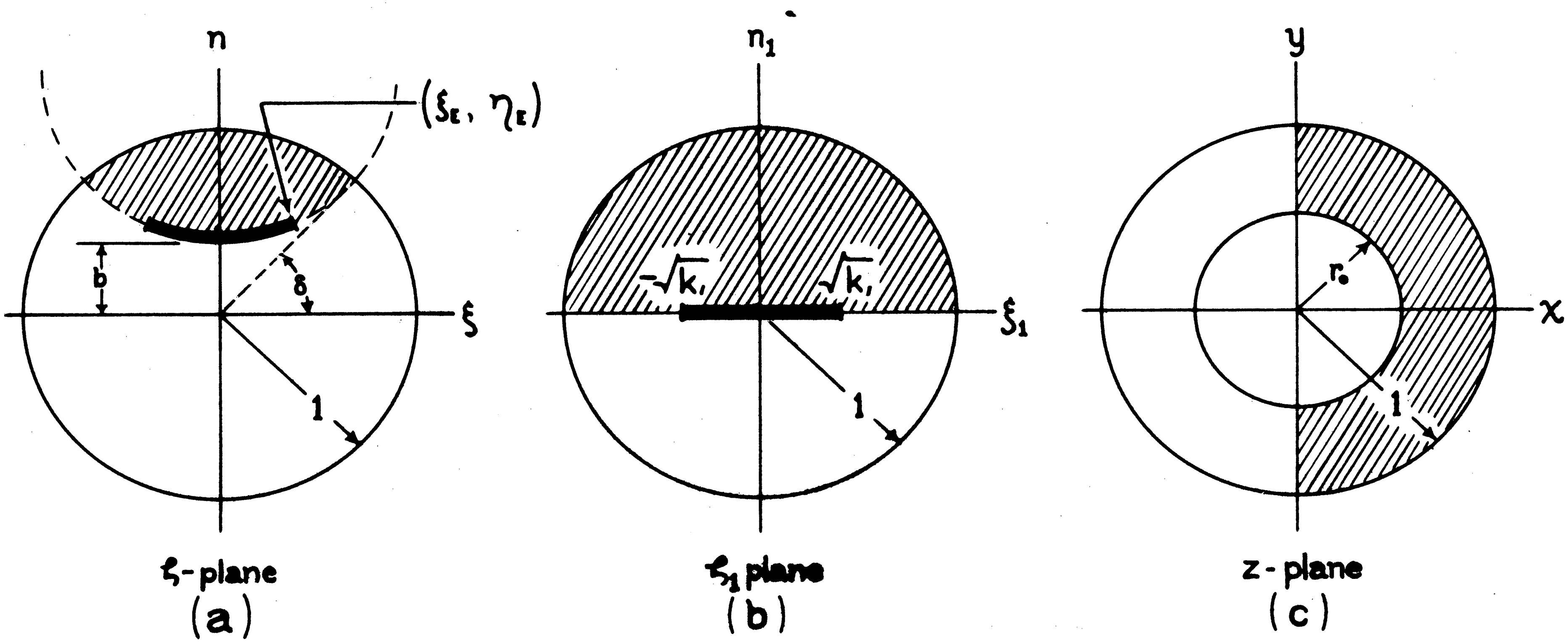

Fig. 3. Correspondence between the ζ-plane, ζ_1-plane, and z-plane.

Hence, in accordance with Section 1.2, the problem of the flow over the arrow wing is that of determining a function $F(\zeta)$ which is analytic in the doubly-connected region of the ζ-plane bounded externally by the unit circle and "internally" by the circular arc $(\pm\xi_E, \eta_E)$ and such that the real parts u, v, and w of the functions $U(\zeta)$, $V(\zeta)$ and $W(\zeta)$, respectively, in equation (1.14) are single valued and satisfy the following boundary conditions corresponding to equations (1.15) and (1.16) respectively:

$$\begin{aligned} &(1) \text{ On the circular arc:} && v - w\tan\alpha = w_\infty \tan\alpha \,, \\ &(2) \text{ On the unit circle } |\zeta| = 1: && u = v = w = 0 \,. \end{aligned} \tag{2.3}$$

The solution of the problem just formulated is accomplished by solving an equivalent problem for an annular region bounded by two concentric circles obtained by the method of conformal mapping. In more detail, let the relation

$$\zeta = g(z) \tag{2.4}$$

map the doubly-connected region of the ζ-plane conformally onto the annular region $r_0 \leq |z| \leq 1$ in the plane of the complex variable $z = x + iy$ (see Figure 3c) in such a manner that the points of the unit circle $|\zeta| = 1$ correspond to points on the unit circle $|z| = 1$ forming the exterior boundary of the annulus, and the points of the (two-sided) circular arc $(\pm\xi_E, \eta_E)$ correspond to points of the circle $|z| = r_0$ forming interior boundary of the annulus. Then the functions $U(z)$, $V(z)$, and $W(z)$ obtained from the functions $U(\zeta)$, $V(\zeta)$ and $W(\zeta)$, respectively, by the change of variable in (2.4) are analytic in the annulus and their real parts u, v, and w, respectively, satisfy the following conditions on the boundaries of the annulus:

$$(1)\ \text{On the circle } |z| = r_0: \quad v - w\tan\alpha = w_\infty \tan\alpha, \qquad (2.5)$$

$$(2)\ \text{On the circle } |z| = 1: \quad u = v = w = 0.$$

Moreover, the integral formulae in (1.14) when expressed in terms of z become:

$$U(z) = \frac{\beta}{2}\int \left[1 + g^2(z)\right] G(z)\, dz,$$

$$V(z) = \frac{i\beta}{2}\int \left[1 - g^2(z)\right] G(z)\, dz, \qquad (2.6)$$

$$W(z) = -\int G(z)\, dz,$$

where $G(z)$ is an analytic function of z in the annulus and is related to the function $F(\zeta)$ of equation (1.14) by the equation

$$F(\zeta) = \frac{G(z)}{g'(z)}. \qquad (2.7)$$

The equivalent problem for the annular region is, therefore, that of determining a function $G(z)$ which is analytic in the annulus and such that the real parts u, v, and w of the analytic functions $U(z)$, $V(z)$, and $W(z)$, respectively, defined by the integral formulae (2.6) satisfy the conditions (2.5) on the boundary of this region. It is important to note that the function $G(z)$ is necessarily continuous on the circle $|z| = 1$. This follows from the continuity of the function $F(\zeta)$ on the unit circle of the ζ-plane and the fact that the derivative $g'(z)$ of the mapping function is continuous for $|z| = 1$.

A further modification of the statement of the problem is found desirable. Since the functions $V(z)$ and $W(z)$ are analytic in the annulus and have single valued real parts in this region, the function

$$H(z) = V(z) - W(z)\tan\alpha \qquad (2.8)$$

also enjoys the same properties. By equation (2.5), the real part of this function is constant on the boundary of the annulus, in fact:

$$
\begin{aligned}
&(1)\ \text{On } |z| = r_0, \quad \mathcal{R}\left\{H(z)\right\} = w_\infty \tan\alpha, \\
&(2)\ \text{On } |z| = 1, \quad \mathcal{R}\left\{H(z)\right\} = 0.
\end{aligned}
\tag{2.9}
$$

Conversely, if $H(z)$ satisfies the first of these conditions, the functions $U(z)$, $V(z)$ and $W(z)$ will obviously satisfy the first of the conditions (2.5). It is readily shown that the converse is also true for the second boundary condition. In other words, the boundary conditions (2.9) are completely equivalent to those of (2.5). Moreover, by equations (2.6) and (2.8), $H(z)$ is expressed in terms of $G(z)$ by the equation:

$$H(z) = \int L(z)\ G(z)\ dz\,, \tag{2.10}$$

where

$$L(z) = -\frac{i}{2}\left\{\beta\left[g^2(z) - 1\right] + 2i\ g(z)\ \tan\alpha\right\}. \tag{2.11}$$

Thus, in view of the preceding observations, the problem of the conical flow over the arrow wing consists of determining the function $G(z)$ which is analytic in the annular region $r_0 < |z| < 1$ and continuous on the circle $|z| = 1$, and which is such that the real part of the function $H(z)$ defined by equation (2.10) is single valued and satisfies the condition (2.9) on the boundary of the annulus. The functions $U(z)$, $V(z)$ and $W(z)$ representing the components of the "complex velocity" of the flow are obtained from the solution of this problem by means of equation (2.6). It is necessary, of course, to select the appropriate constants of integration in the integral formulae of equation (2.6) in order that the components u, v and w of the real velocity vanish on the circle $|z| = 1$.

2.2 Conformal Mapping of the (ξ, η)-Domain onto the Annulus

The conformal mapping of the doubly-connected region of the ζ-plane (Figure 3a) bounded externally by the unit circle and internally by the circular arc $(\pm\xi_E, \eta_E)$ onto the annulus is effected in two steps. The first step is accomplished by the linear fractional transformation defined by equation (1.22) or by the equation

$$\zeta = \frac{\zeta_1 + ib}{1 - ib\zeta_1} , \qquad (2.12)$$

where b is the intercept of the circular arc with the η-axis and is given by (2.2). This relation maps the region of the ζ-plane bounded by the unit circle onto the region bounded by the unit circle in the plane of the complex variable $\zeta_1 = \xi_1 + i\eta_1$ in such a manner that the points of the circle in the ζ-plane defined by equation (2.1) correspond to the points of the real axis in the ζ_1-plane. The edge points $(\pm\xi_E, \eta_E)$ are thereby mapped into two symmetrically situated points $(\pm\xi_{1E}, 0)$ on the real ξ_1-axis (see Figure 3b).

In Section 1.3, it has been pointed out that the mapping effected by (2.12) is equivalent to the non-Euclidean rotation of the flow space about the X-axis, defined by equation (1.18), where $\tilde{\alpha}$ is given by (1.22a), namely

$$b = \tanh \frac{\tilde{\alpha}}{2} .$$

It is evident that as a result of applying (1.18), the plane $Y = Z \tan\alpha$ containing the wing is "rotated" into the plane $Y_1 = 0$. It is evident further that the edges of the wing, which in the plane $Y = Z \tan\alpha$ form an angle 2γ (see Figure 2), are "rotated" into the lines

$$\beta X_1 = \pm Z_1 \tanh\tilde{\gamma}, \qquad (2.13)$$

where, in accordance with the second of equation (1.21),

$$\beta \tan\gamma \sec\alpha = \tanh\tilde{\gamma} \operatorname{sech} \tilde{\alpha}.$$

The wing edges (2.13) in the (X_1, Y_1, Z_1) space correspond to the edge points $(\pm\xi_{1_E}, 0)$ on the real ξ_1-axis in the ζ_1-plane. Since this correspondence is established by means of equation (1.10), it follows from these and equation (2.13) that

$$\xi_{1_E} = \tanh \frac{\tilde{\gamma}}{2} . \tag{2.14}$$

The circular region $|\zeta_1| \leqq 1$ in the ζ_1-plane with the symmetrically placed slit on the real axis is then mapped onto the annular region $r_o \leqq |z| \leqq 1$ in the plane of the complex variable $z = x + iy$ by means of the Jacobi elliptic function

$$\zeta_1 = -\sqrt{k_1} \operatorname{sn} \left(\frac{2K_1}{\pi i} \log z - \frac{iK_1'}{2}; k_1\right), \tag{2.15}$$

where $\frac{2K_1}{\pi i} \log z - \frac{iK_1'}{2}$ is the argument, and k_1 is the modulus of this function; K_1 and K_1' are the two complete elliptic integrals of the first kind belonging to k and the complementary modulus k_1', respectively. (The transition from the ζ-plane to the z-plane via the ζ_1-plane is shown in Figure 3.) The radius r_o of the interior circular boundary of the annulus is determined by the equation

$$r_o = e^{-\frac{\pi K_1'}{4K_1}} \tag{2.16}$$

The points of this circle are mapped into the slit extending from $\sqrt{k_1}$ to $-\sqrt{k_1}$ along the real axis in the ζ_1-plane in such a manner that the points $z = \pm r_o i$ correspond respectively to the points $\zeta_1 = \mp\sqrt{k_1}$. This is immediately evident on setting $z = r_o e^{i\theta}$ in (2.15). For then

$$\zeta_1 = -\sqrt{k_1}\ \mathrm{sn}\ (\frac{2K_1}{\pi}\theta;\ k_1),$$

whence it follows[15] that ζ_1 is real for all values of θ (real) and, in particular, that for $\theta = -\frac{\pi}{2}$, 0, and $\frac{\pi}{2}$, the values of ζ_1 are $\sqrt{k_1}$, 0, and $-\sqrt{k_1}$, respectively. Hence, the desired mapping function is completely defined on setting

$$\sqrt{k_1} = \zeta_{1_E} = \tanh \frac{\tilde{\gamma}}{2}\ . \tag{2.17}$$

It is readily verified that by equation (2.15) the points of the unit circles in the ζ_1-plane and the z-plane correspond to one another and that, in particular, the points $\zeta_1 = \pm 1$ are mapped into the points $z = \pm i$, respectively. For, on setting $z = e^{i\theta}$, equation (2.15) becomes

$$\zeta_1 = -\sqrt{k_1}\ \mathrm{sn}\ (\frac{2K_1}{\pi}\theta - \frac{iK_1'}{2};\ k_1)\ ,$$

whence for $\theta = \pm\frac{\pi}{2}$ the values of ζ_1 are ∓ 1, respectively. Also[15]

$$\bar{\zeta}_1 = -\sqrt{k_1}\ \mathrm{sn}\ (\frac{2K_1}{\pi}\theta + \frac{iK_1'}{2};\ k_1) = -\frac{1}{\sqrt{k_1}\ \mathrm{sn}\ (\frac{2K_1}{\pi}\theta - \frac{iK_1'}{2};\ k_1)}\ ,$$

where $\bar{\zeta}_1$ represents the conjugate of the complex number ζ_1. Therefore, for points $z = e^{i\theta}$ on the unit circle about the origin in the z-plane,

$$|\zeta_1|^2 = \zeta_1\bar{\zeta}_1 = 1\ ,$$

whence the corresponding point in the ζ_1-plane also lies on the unit circle about the origin.

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15) cf. E. T. Whittaker and G. N. Watson, Modern Analysis, 4th Ed., London, 1935, pp. 493 et seq.

By combining equations (2.8) and (2.11), the function $g(z)$ in equation (2.4) which accomplishes the desired mapping of the doubly connected region in the ζ-plane onto the annulus in the z-plane is, therefore, given by the equation

$$\zeta = g(z) = \frac{ib - \sqrt{k_1}\ \mathrm{sn}\ (z_1;k_1)}{1 + ib\sqrt{k_1}\,\mathrm{sn}\ (z_1;k_1)} , \tag{2.18}$$

where

$$z_1 = \frac{2K_1}{\pi i} \log z - \frac{iK_1'}{2} . \tag{2.19}$$

It should be observed that the circles concentric to the circular boundaries of the annulus are mapped by the transformation of equation (2.15) into a set of algebraic curves of the fourth degree in the ζ_1-plane called cyclids. Similarly, the rays θ = constant in the z-plane are mapped into cyclids which together with the first set form a net of orthogonal curve families. As shown in Figure 4, each cyclid consists of two closed oval branches surrounding the four foci $\pm\sqrt{k_1}$, $\pm 1/\sqrt{k_1}$; the oval branches encircling the line segment between $+\sqrt{k_1}$ and $-\sqrt{k_1}$ and lying within the unit circle $|\zeta_1| = 1$ correspond under the transformation (2.15) to the circles concentric to the boundaries of the annulus and having radii between the values 1 and r_0. The cyclids and their three dimensional generalizations, the cyclidic surfaces, have been studied in great detail by Darboux, by Holzmüller [13], by Klein and Bôcher [14]. They are important in the present investigation insofar as they constitute the appropriate system of curvilinear coordinates for the domain inside of the Mach cone when the obstacle is either a plane arrow wing or, as in the case treated in Part III, an elliptic cone.

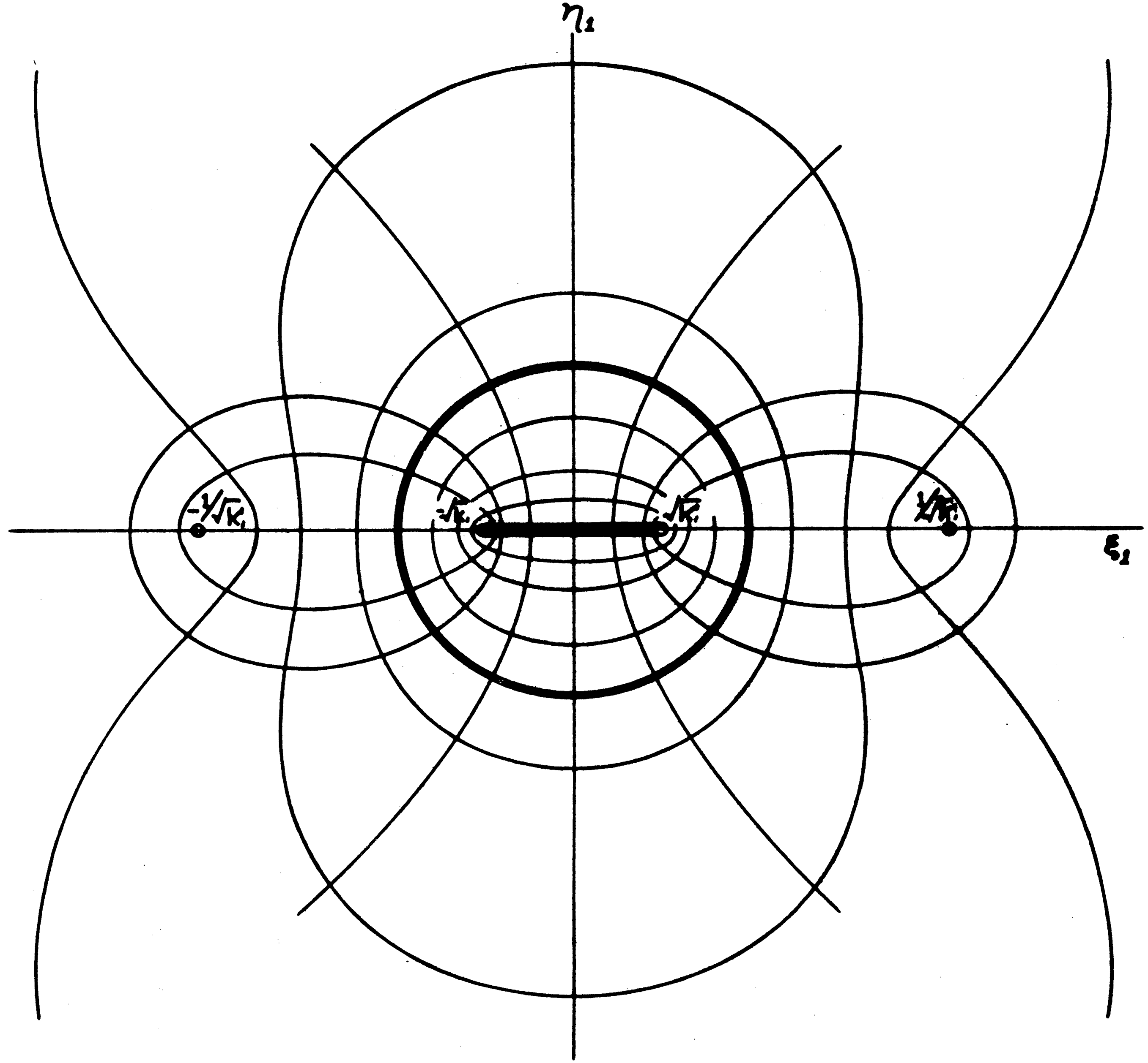

Fig. 4. Orthogonal net of cyclids in the ζ_1-plane.

2.3 Determination of the Properties of the Function G(z)

It is shown in this section that the function G(z) representing the solution of the problem formulated in Section 2.1 for the conical flow over the arrow wing is expressible in terms of an elliptic function. The derivation depends on the special form that the boundary conditions (1.4) and (1.6) take when expressed as the conditions satisfied by the function H(z) in equation (2.9). Most important, it is shown that these conditions are not sufficient in the case of the arrow wing to specify completely the solution G(z), but must be supplemented by an additional condition such as the finiteness of the total normal force coefficient of the wing. It is shown in Part III that for the case of a body in the shape of an elliptic cone without sharp edges no supplementary condition is necessary. The boundary conditions (1.4) and (1.6) are in this case sufficient to determine the flow over the cone completely.

Since the function H(z) is analytic and bounded within the annulus $r_0 < |z| < 1$ (except possibly in the neighborhood of the points corresponding to the wing edges; namely, $z = \pm r_0 i$) and its real part is constant on the circular boundaries, it follows, in accordance with the general properties of an analytic function, that the derivative H'(z) exists and is continuous at all points of these circles (except possibly at the edge points $z = \pm r_0 i$). Therefore, the function h(z) is defined by the equation

$$h(z) = z\,H'(z), \tag{2.20}$$

or by the equation

$$H(z) = w_\infty \tan\alpha + \int_{r_0}^{z} h(\nu)\,\frac{d\nu}{\nu}\,, \tag{2.21}$$

where the constant of integration is dictated by the first of the boundary conditions (2.9). Then the function $h(z)$ is continuous on the boundary of the annulus, except possibly at the points $z = \pm r_o i$, and, by the second of equations (2.9),

$$\int_{r_o}^{1} h(x)\,\frac{dx}{x} = -w_\infty \tan\alpha, \tag{2.22}$$

Consequently, along the circle $|z| = 1$, where $z = e^{i\theta}$,

$$\mathcal{R}\left\{H(z)\right\} = \mathcal{R}\left\{i\int_0^\theta h(e^{i\varphi})\,d\varphi\right\};$$

whereas, on the circle $|z| = r_o$, where $z = r_o e^{i\theta}$,

$$\mathcal{R}\left\{H(z)\right\} - w_\infty \tan\alpha = \mathcal{R}\left\{i\int_0^\theta h(r_o e^{i\varphi})\,d\varphi\right\}.$$

Since the boundary conditions (2.9) require that these quantities vanish identically, it follows that the imaginary part of the function $h(z)$ is zero at all points on the circular boundaries of the annulus, and therefore that the values of the function $h(z)$ are real on these circles.

Zeros of the function $h(z)$ are readily found by writing it in the form

$$h(z) = z\,L(z)\,G(z) \tag{2.23}$$

with the aid of equations (2.10), (2.11) and (2.20). The function $L(z)$ when expressed in terms of ζ by means of the relation (2.4) is a quadratic function of ζ; namely,

$$L = -\frac{i}{2}\left\{\beta(\zeta^2 - 1) + 2i\zeta\tan\alpha\right\}. \tag{2.24}$$

The two simple zeros of this function, namely,

$$\zeta = \pm\sqrt{1 - \frac{\tan^2\alpha}{\beta^2}} - i\,\frac{\tan\alpha}{\beta}$$

are evidently distinct and lie on the unit circle in the ζ-plane provided that

$$\tan\alpha < \beta = \tan(\pi/2 - \mu), \tag{2.25}$$

where μ is the Mach angle defined in equation (1.3). Consequently, for angles of attack α consistent with this condition, the function $L(z)$ possesses two simple zeros on the circle $|z| = 1$ in the z-plane. It is evident that these points are symmetrical with respect to the real axis in this plane, say at the points $z = c, \bar{c}$ (see Figure 3). Since the function $G(z)$ is continuous on the circle $|z| = 1$, it follows that the function $h(z)$ possesses zeros of at least the first order at the points $z = c, \bar{c}$ on the circle $|z| = 1$.

In addition to the limitation imposed on the magnitude of the angle of attack α by the inequality (2.25); namely, that $\alpha < \pi/2 - \mu$, it is also necessary that $\alpha < \mu$ in order that the arrow wing be contained entirely within the Mach cone. It is evident that the second inequality automatically implies the first for large values of M (small μ). However, the first inequality restricts the range of values of the angle of attack for which the subsequent results are applicable when the Mach number of the flow is near unity (i.e., when μ is near $\pi/2$). It is readily verified that both of these inequalities are satisfied simultaneously if the following inequality is satisfied:

$$\frac{\sin 2\alpha}{\sin 2\mu} < 1. \tag{2.26}$$

Consider now the analytic continuation of the function $h(z)$ throughout the entire z-plane. Since $h(z)$ is real and continuous on the circular boundaries of the annulus (except possibly at the points $z = \pm r_0 i$), it follows from the principle of reflection[16] that the extended function $h(z)$ is analytic in the whole of the z-plane (except possibly at the points $z = \pm r_0 i$ and their successive reflections with respect to the circles bounding the annulus), real on the circles (see Figure 5),

$$|z| = r_0^n \quad , (n = 0, \pm 1, \pm 2, \ldots),$$

and such that for any point $z = \tau$ within the annulus

$$\ldots = \overline{h(\tau_1)} = h(\tau) = h(\tau_{-2}) = \ldots = h(\tau_{-2n}) = \ldots, \quad (2.27)$$

where τ_1 is the reflected image[17] of τ with respect to the circle $|z| = r_0$; τ_{-2} is the reflected image of τ_1 with respect to the circle $|z| = 1$; τ_{-4} is the reflected image of τ_1 with respect to the circle $|z| = r_0^{-1}$; and, in general, τ_{-2n} is the reflected image of τ_1 with respect to the circle $|z| = r_0^{-n+1}$. Therefore

$$\ldots = \tau_1 = \frac{r_0^2}{\tau} = \frac{1}{\tau_{-2}} = \frac{r_0^{-2}}{\tau_{-4}} = \ldots = \frac{r_0^{-2n+2}}{\tau_{-2n}} = \ldots,$$

and

$$\tau_{-2n} = r_0^{-2n} \tau .$$

---

16) See, for example, E. J. Townsend, Functions of a Complex Variable, Holt, New York, 1915, p. 255.

17) The point z" is said to be the reflected image of z' with respect to the circle $|z| = a$ if $z' \overline{z''} = a^2$.

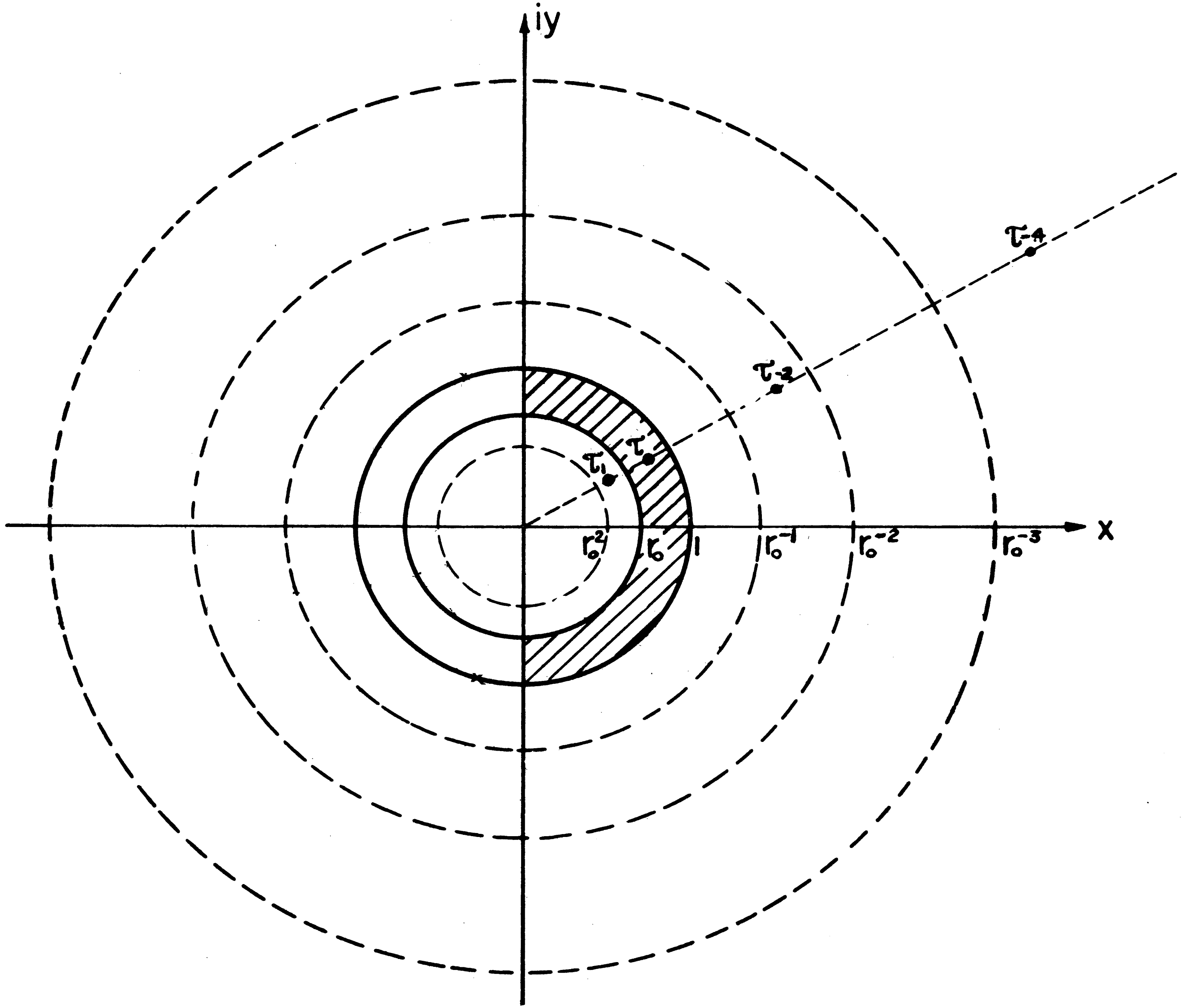

Figure 5. z-plane. Analytic Continuation of h(Z).

Hence, since $z = \tau$ is any arbitrary point of the annulus, it follows from the last equation and equation (2.27) that for all values of z

$$h(z) = h(r_o^{-2n} z), \quad n = 0, \pm 1, \pm 2, \ldots \quad . \tag{2.28}$$

On the other hand, since $h(z)$ is continuous in the annulus, it is necessarily a periodic function of $\arg(z)$ with the period 2π. Consequently, the function $h(z)$ as well as its analytic extension also satisfies the following relation for all values of z:

$$h(z) = h(ze^{2m\pi i}), \quad m = 0, \pm 1, \pm 2, \ldots \quad . \tag{2.29}$$

As in equation (2.19), set

$$z_1 = \frac{2K_1}{\pi i} \log z - i\frac{K_1'}{2}$$

and define the function $h_1(z_1)$ of the complex variable $z_1 = x_1 + iy_1$, in terms of the extended function $h(z)$ by the relation

$$h_1(z_1) = \frac{\pi}{2K_1} h(z) \; . \tag{2.30}$$

Then, except possibly at the points corresponding to the edge points and at the points derived from these by successive reflections with respect to the circular boundaries of the annulus, the function $h_1(z_1)$ is analytic throughout the entire z_1-plane. Moreover, as a consequence of equations (2.16), (2.28) and (2.29) it follows that

$$h_1(z_1 + niK_1') = h_1(z_1) \text{ and } h_1(z_1 + 4mK_1) = h_1(z_1), \quad (m, n = 0, \pm 1, \pm 2, \ldots). \tag{2.31}$$

Hence, $h_1(z_1)$ is a doubly periodic function of z_1 with the periods $4K_1$

and iK_1', and consequently is completely defined by its values in the period rectangle (see Figure 6. The shaded region in this figure corresponds to the shaded regions in Figure 3.):

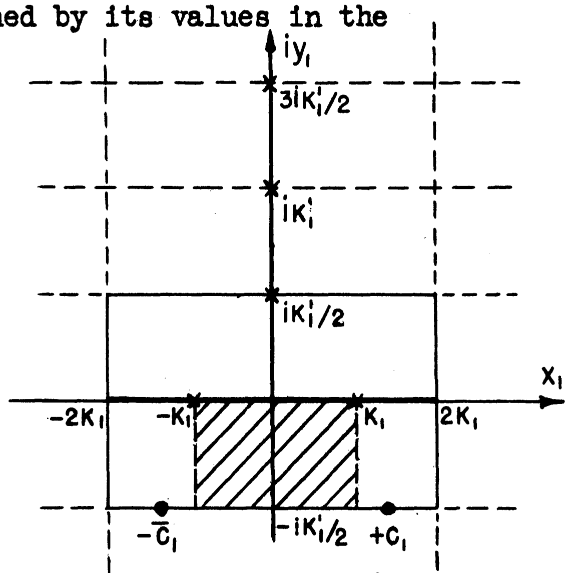

FIG. 6 z_1 - PLANE

$$-2K_1 \leqq x_1 < 2K_1$$

$$-\frac{K_1'}{2} \leqq y_1 < \frac{K_1'}{2}.$$

It should be noted that the lower half of this rectangle is mapped onto the annulus $r_0 \leqq |z| \leqq 1$ in the z-plane by the transformation (2.19) in such a manner that the Mach circle $|z| = 1$ corresponds to the side $y_1 = -\frac{K_1'}{2}$ and the circle $|z| = r_0$ corresponds to the real axis $y_1 = 0$; in particular, the wing edge points $z = \pm r_0 i$ are mapped into the points $z_1 = \pm K_1$, respectively. It should also be observed that as a consequence of equation (2.27)

$$h_1(\bar{z}_1) = \overline{h_1(z_1)}. \tag{2.32}$$

Since the function $h_1(z_1)$ is real at points on the real axis, this relation is also a direct consequence of the principle of reflection. Thus by means of equations (2.31) and (2.32), the function $h_1(z_1)$ is completely defined at all points of the z_1-plane in terms of its values in the half of the period rectangle which is mapped into the annulus in the z-plane. Furthermore, the function $h_1(z_1)$ vanishes at the two points, say $z_1 = c_1$ and $-\bar{c}_1$, on the side $y_1 = -\frac{K_1'}{2}$ corresponding to the two zeros $z = c$ and $\bar{c}$, respectively, of the function $h(z)$ on the circle $|z| = 1$ in the z-plane.

It is well known[18] that a doubly periodic function is either everywhere constant or possesses infinite singularities in each period parallelogram. Since $h_1(z_1)$ has zeros at $z_1 = c_1$ and $-\overline{c_1}$, the first alternative would imply that $h_1(z_1)$ is everywhere equal to zero. As a consequence, $H(z)$ would, in accordance with equations (2.20) and (2.30) be everywhere constant. The latter situation is inconsistent with the boundary conditions (2.9) satisfied by $H(z)$. Therefore $h_1(z_1)$ cannot remain bounded in a period rectangle - nor, indeed, in the half-period rectangle corresponding to the annulus - but must possess certain infinite singularities (poles) in the rectangle. Hence, $h_1(z_1)$ is an elliptic function of the variable z_1. As such, it is completely determined, apart from a constant multiplier, by the locations and orders of its zeros and poles in a period rectangle - and, consequently, by the locations and orders of its zeros and poles in the half-period rectangles corresponding to the annulus in the z-plane and, consequently, to the original domain of disturbance around the wing in the flow space.

Since the velocity components u, v, and w derivable from $h_1(z_1)$ by means of equations (2.6), (2.23) and (2.30) are continuous throughout the domain in the flow space, including the points of the Mach cone, and at all points of the arrow wing not lying along an edge[19], it follows that the poles of the elliptic function $h_1(z_1)$ are restricted to lie at the points of the z_1-plane corresponding to the wing edges; that is, at the points $z_1 = \pm K_1$. However, neither the order of these poles nor the order of the zeros of $h_1(z_1)$ at the points $z_1 = c_1$ and $-\overline{c_1}$

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18) Whittaker and Watson, loc. cit. Liouville's theorem, p. 431

19) See footnote 5, page 6.

are prescribed by the boundary conditions (2.9). Nor, for that matter, are the disposition and order of other possible zeros[20] of $h_1(z_1)$ prescribed by these boundary conditions. It is shown in the following paragraph that as a consequence of the boundary conditions (2.9) the residues of the poles of $h_1(z_1)$ are necessarily equal to zero, but, other than this, no further limitations are placed on the orders of the poles by these conditions. Therefore, the function $h_1(z_1)$ is not uniquely determined by the boundary conditions (2.9). Consequently the conical flow past the arrow wing is not completely determined by the boundary conditions formulated in equations (1.4) and (1.6) for the general flow problem from which the equations in (2.9) are derived. On the other hand, it is shown in Section 2.6 that the normal force exerted on a finite portion of the wing is finite only if the order of the poles of $h_1(z_1)$ is at most equal to two[21]. Since the residue of each pole must be zero, the order of the poles is then exactly two. The elliptic function $h_1(z_1)$ is thereby uniquely determined by the addition of this supplementary condition. As a consequence of equation (2.23), the function $G(z)$ representing the solution of the problem is then also uniquely determined. Hence, among all the conical flows over the arrow wing satisfying the boundary conditions in equations (1.4) and (1.6), there is a unique one for which the normal force coefficient of the wing is finite. The particular functions $h_1(z_1)$ and $G(z)$ which determine the flow with finite lift coefficient are given in the following section.

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20) It is known that the sum of the orders of the zeros in a period parallelogram is equal to the sum of the orders of the poles. Whittaker and Watson, loc. cit., p. 432.

21) As a consequence of this restriction, the possibility that the singularities at the wing edges are essential singular points is precluded.

In order to prove that the residue of each of the poles of the elliptic function $h_1(z_1)$ is zero, let this function be represented in the neighborhood of a pole, say $z_1 = K_1$, in the form

$$h_1(z_1) = \frac{a_m}{(z_1-K_1)^m} + \frac{a_{m-1}}{(z_1-K_1)^{m-1}} + \cdots + \frac{a_1}{(z_1-K_1)} + h_o(z_1),$$

where the coefficients a_m, a_{m-1}, . . ., a_1, are constants, and $h_o(z_1)$ is a continuous function of z_1 in the neighborhood of $z_1 = K_1$. Since the values of $h_1(z_1)$ are real at all points of the real axis in the z_1-plane, the coefficients a_m, a_{m-1}, . . ., a_1, are real and the values of the function $h_o(z_1)$ are also real at points of the real axis. Therefore, by equations (2.19), (2.21) and (2.30), the values of the function $H(z)$ at points in the vicinity of the point $z = ir_o$ in the z-plane corresponding to a wing edge are given by an expression of the form:

$$H(z) = w_\infty \tan\alpha + i\int_0^{z_1} h_1(\nu_1)\, d\nu_1 ,$$

$$= w_\infty \tan\alpha - \frac{i\, a_m}{(m-1)(z_1-K_1)^{m-1}} - \frac{i\, a_{m-1}}{(m-2)(z_1-K_1)^{m-2}}$$

$$- \cdots + ia_1 \log(z_1-K_1) + i\int_0^{z_1} h_o(\nu_1)\, d\nu_1$$

$$= w_\infty \tan\alpha + ia_1 \log(z_1-K_1) + iH_o(z) .$$

where $z_1 = 2K_1/\pi i \log z - iK_1'/2$ and $H_o(z)$ is a function whose values are real at points on the circle $|z| = r_o$. Hence, at points on the circle $|z| = r_o$ where $z = r_o e^{i\theta}$ in the vicinity of the edge point $z = r_o i$, the values of the real part of the function $H(z)$ are given by

$$R\left\{H(z)\right\} = w_\infty \tan\alpha + \mathcal{R}\left\{ia_1 \log(\theta - \pi/2)\right\}.$$

Since the second term in this expression experiences a finite jump of magnitude $\pm a_1\pi$ on passing through the edge point $z_1 = r_0 i$ where $\theta = \pi/2$, the function $H(z)$ cannot satisfy the second of the conditions in equation (2.9) unless $a_1 = 0$; that is, unless the residue of the pole of $h_1(z_1)$ at the point $z_1 = K_1$ is zero. A similar argument shows that the residue of the pole at $z_1 = -K_1$ is also zero. Hence, the residues of the poles of $h_1(z_1)$ at the points $z = \pm K_1$ are zero and, consequently, these poles must be at least of the second order.

2.4 Determination of the function $G(z)$

In the preceding section, it is shown that the function $G(z)$ is expressible in terms of an elliptic function $h_1(z_1)$ of the variable $z_1 = 2K_1/\pi i \log z - iK_1'/2$ having the following properties:

(a) Periods $4K_1$ and iK_1'.

(b) Poles with zero residue at $z_1 = \pm K_1$.

(c) Zeros at the points $z_1 = c_1$ and $-\overline{c_1}$ (see Figure 6) at which the function L in equation (2.24) vanishes.

(d) Real for real values of z_1.

Functions possessing these properties are readily constructed with the aid of the Jacobi elliptic functions. In this section the function $G(z)$ corresponding to the particular elliptic function $h_1(z_1)$ having poles of second order is derived. For, as shown in Section 2.6, the lift coefficient of the wing is not finite for a conical flow corresponding to a function with poles of order greater than two at the points corresponding to the wing edges.

Consider the elliptic function[22)]

$$T(z_1) = dc(z_1;k_1) + k_1 cd(z_1;k_1) \tag{2.33}$$

This function has the periods $4K_1$ and iK_1'. It has poles of first order at $z_1 = \pm K_1$ and is real for real values of z_1. In fact, this function is real on all four sides of the period rectangle (see Figure 6):

$$x_1 = \pm 2K_1 \text{ and } y_1 = \pm K_1'/2$$

and is such that

$$T(-\overline{z_1}) = T(\overline{z_1}) = \overline{T(z_1)}$$

Therefore the derivative

$$T'(z_1) = \frac{d}{dz_1} T(z_1)$$

has the same periods and has second order poles with zero residues at $z_1 = \pm K_1$. Moreover, the derivative is also real for real values of z_1. On the other hand, the function $T^2(z_1)$ also possesses these same properties. Hence it follows from the general properties of elliptic functions that the most general doubly periodic function with the periods $4K_1$, iK_1', having second order poles with zero residues at $z_1 = \pm K_1$, and which is real for real values of z_1 is a real multiple of the function

$$T^2(z_1) + A'T'(z_1) - A \quad ,$$

where A and A' are real constants. However, this function will vanish at the two symmetrically placed zeros $z_1 = c_1$ and $-\overline{c_1}$ only if

$$A' = 0.$$

22) Here use is made of the conventional notation for the quotients of the Jacobi elliptic functions; e.g., $dc(z_1;k_1) = dn(z_1;k_1)/cn(z_1;k_1)$,etc. It should be remarked that the procedure at this point can be somewhat simplified by making use of Gauss' transformation introduced in Section 3.2.

Hence, the function $h_1(z_1)$ possessing the appropriate properties (a) to (d) listed above and whose poles are of the second order is defined by the equation[23)]

$$h_1(z_1) = B\left[T^2(z_1) - A\right] , \tag{2.34}$$

where A is the real constant defined by the equation

$$A = T^2(c_1) = T^2(-\overline{c_1}) , \tag{2.35}$$

and where B is a real constant. The constant factor B is determined by the boundary conditions (2.9). Thus, by substituting (2.34) in equations (2.22) and (2.30), it is found that the constant B has the value given by the equation

$$B = 2\,\frac{w_\infty \tan\alpha}{AK_1' - 2(k_1K_1' + E_1')} , \tag{2.36}$$

where E_1' is the complete elliptic integral of the second kind corresponding to the complementary modulus k_1'[24)].

The value of the constant A in equation (2.35) depends on the position of the zeros of the function $L(z)$ in the z_1-plane; that is, it depends on the location of the points designated $z_1 = c_1$ and $-\overline{c_1}$ (see Figure 6). Writing equation (2.24) in the form

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23) It is immediately evident that arbitrarily many functions satisfying the properties (a) to (d) can be constructed by adding a real constant to an appropriate linear combination (containing real constants) of the functions $T^2(z_1)$, $T'(z_1)$, their first, and higher derivatives; the constants being adjusted so that the function possesses the desired zeros. Each such function gives rise to a conical flow over the arrow wing satisfying the boundary conditions (1.4) and (1.6). However, except for multiples of the function in (2.34), all those functions have poles of higher order than the second and therefore yield flows for which the coefficient of lift is infinite.

24) cf. Whittaker and Watson, loc. cit., pp. 517 et seq.

$$L = -\frac{i}{2}\,\frac{\beta^2(\zeta^2-1)^2 + 4\zeta^2\tan^2\alpha}{\beta(\zeta^2-1) - 2i\zeta\tan\alpha}\ ,$$

and making the substitutions defined in equation (2.18), one obtains, after considerable manipulation, the following expression for the function L in terms of the variable z_1:

$$L = 2ib\csc 2\alpha\,\frac{(1-k_1)^2 + 4k_1p^2}{(1+k_1)\,(1-k_1)^2}$$

$$\frac{\left[T^2(z_1) - A_1\right]\ \mathrm{cn}z_1\mathrm{dn}z_1}{\left[1 - ib\sqrt{k_1}\ \mathrm{sn}\ z_1\right]^2\left[T(z_1) - 2i\sqrt{k_1}(1+k_1)p\ \mathrm{sc}\ z_1\mathrm{nd}z_1\right]}\ , \tag{2.37}$$

where

$$p = \frac{\sin 2\alpha}{\sin 2\mu} \tag{2.38}$$

and A_1 is a real constant defined by the equation

$$A_1 = \frac{4k_1(1+k_1)^2\,p^2}{(1-k_1)^2 + 4k_1p^2}\ .$$

It is immediately evident from this that the zeros $z_1 = c_1$ and $-\bar{c}_1$ of the function L in the z_1-plane are among the roots of the equation

$$T^2(z_1) - A_1 = 0\ ,$$

whence

$$T^2(c_1) = T^2(-\bar{c}_1) = A_1\ ,$$

Hence, on comparing this with equation (2.35), it follows that the constant A in equation (2.34) is given by the equation:

$$A = A_1 = \frac{4k_1(1+k_1)^2p^2}{(1-k_1)^2 + 4k_1p^2} \tag{2.39}$$

Finally, the function G(z) representing the particular solution of the problem of the conical flow over the arrow wing for which the normal force coefficient is finite is obtained from equations (2.23), (2.30), (2.35), (2.37) and (2.39); namely,

$$G(z) = -\, i \, \frac{A\, B(1-k_1)^2 K_1}{4bk_1(1+k_1)\; p^2 \pi} \;\sin 2\alpha$$

$$\frac{(1-ib\sqrt{k_1}\,\mathrm{sn}\; z_1)^2 \left[T(z_1)-2i\sqrt{k_1}(1+k_1)p\;\mathrm{sc}\; z_1 \mathrm{nd} z_1\right]}{z\;\mathrm{cn} z_1\;\mathrm{dn} z_1} \;, \qquad (2.40)$$

where $z_1 = 2K_1/\pi i \log z = iK_1'/2$ and the constants b, B, p, and A are given respectively by equations (1.22a), (2.36), (2.38) and (2.39).

With the aid of equations (2.21) and (2.40), the function H(z) when expressed in terms of the variable z_1 becomes

$$H(z_1) = w_\infty \tan\alpha + iB\Big\{ \left[\mathrm{dc}(z_1;k_1) + k^2 \mathrm{cd}(z_1;k_1)\right] \mathrm{sn}(z_1;k_1)$$

$$-2E(z_1;k_1) + \left[2(1+k_1) - A\right] z_1 \Big\} \;, \qquad (2.41)$$

where $E(z_1;k_1)$ is the fundamental elliptic integral of the second kind[25].

2.5 Determination of the Function W(z) and the Values of the Velocity Component w along the Wing Surface

In accordance with the last of equations (2.6), the component W(z) of the complex velocity is given by the integral

$$W = -\int_1^z \cancel{g(\nu)}\; G(\nu)\; d\nu \;,$$

where the choice of the lower limit is consistent with the condition that

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25) cf. Whittaker and Watson, loc. cit., p. 517.

the real part of function $W(z)$ vanish along the circle $|z| = 1$; it should be remarked that in order to satisfy the latter requirement it is sufficient to select any point on the circle $|z| = 1$ as the initial point of the path of integration. Making use of the expression obtained for $G(z)$ in equation (2.40), the component $W(z)$ can be written in the form:

$$W(z) = \left\{ N\left[dc(z_1;k_1) - k_1 cd(z_1;k_1)\right] - PH(z) + Q\cdot\log z\right\} \sin 2\alpha , \tag{2.42}$$

where $z_1 = 2K_1/\pi i \log z - iK_1'/2$ and the real constants N, P and Q are defined by the equations

$$N = B\,\frac{\sqrt{k_1}(1-k_1)(1-q^2)p}{q^2\left[(1-k_1)^2+4k_1p^2\right]} = \frac{B}{2}\,\frac{p(1-q^2)\,\sinh\tilde{\gamma}}{q^2(1+p^2\,\sinh^2\gamma)} ,$$

$$P = 1/2\,\frac{(1-k_1)^2q^2+4k_1p^2}{q^2\left[(1-k_1)^2+4k_1p^2\right]} = \frac{1}{2}\,\frac{q^2+p^2\sinh^2\tilde{\gamma}}{q^2(1+p^2\sinh^2\tilde{\gamma})} ,$$

$$Q = \frac{1-k_1}{\sqrt{k_1}}\,\frac{A\,K_1N}{p\pi} = \frac{A\,BK_1\,(1-q^2)}{\pi q^2(1+p^2\sinh^2\tilde{\gamma})} ,$$

in which

$$q = \frac{\sin\alpha}{\sin\mu} . \tag{2.43}$$

The constants B, p and A are defined in equations (2.36), (2.38) and (2.39), respectively.

The expression for the real part of $W(z)$; i.e., the component w of the additional velocity, is particularly simple at points on the circle $|z| = r_o \equiv e^{-\pi K_1'/4K_1}$ corresponding to the wing surface. For these points $\mathcal{R}\left\{H(z)\right\} = w_\infty \tan\alpha$. Therefore, for $z = r_o e^{i\theta}$

$$w = -C\,\frac{w_\infty \sin 2\alpha}{\beta}\left[dc\left(\frac{2K_1}{\pi}\,\theta;k_1\right) - k_1 cd\left(\frac{2K_1}{\pi}\,\theta;k_1\right)\right] - Dw_\infty \sin^2\alpha\,, \quad (2.44)$$

where

$$C = \frac{(1-q^2)\,\sinh\tilde{\gamma}}{(1 + p^2\sinh^2\gamma)\left[2(k_1K_1' + E_1') - AK_1'\right]}\,,$$

$$(2.45)$$

$$D = \frac{AK_1'q^{-2} - 4P(k_1K_1' + E_1')}{AK_1' - 2(k_1K_1' + E_1')}\,.$$

Making use of the relation in equation (2.15) between points of the ζ_1-plane and the z-plane, the values of w at points along the segment of the real ξ_1-axis corresponding to the surface of the wing are obtainable from equation (2.44) in the form

$$[w]_{\eta_1=0+} = -\sqrt{k_1}(1-k_1)\;C\,\frac{w_\infty \sin 2\alpha}{\beta}\;\frac{1+\xi_1^2}{\sqrt{(k_1-\xi_1^2)(1-k_1\xi_1^2)}} - Dw_\infty \sin^2\alpha\,,$$

$$(2.46)$$

$$[w]_{\eta_1=0-} = +\sqrt{k_1}(1-k_1)\;C\,\frac{w_\infty \sin 2\alpha}{\beta}\;\frac{1+\xi_1^2}{\sqrt{(k_1-\xi_1^2)(1-k_1\xi_1^2)}} - Dw_\infty \sin^2\alpha\,.$$

The first of these expressions represents the values of w along the upper side of the segment of the real axis (corresponding to the "lower" surface of the wing), and the second, the values of w on the lower side of this segment (corresponding to the "upper" surface of the wing).

2.6 Calculation of the Normal Force Coefficient C_n for the Arrow Wing

In the linearized theory of the flow of gases at supersonic speeds, the difference Δp between the pressure at any point in the stream and the pressure throughout the undisturbed region of the flow ahead of the leading points of disturbance is given by[26)]

$$\Delta p = -\rho_\infty w_\infty w$$

where ρ_∞ is the density of the gas in the undisturbed region. Therefore, the total <u>normal force</u> applied to the triangular tip of the arrow wing (see Figure 7) is given, within the limits of accuracy attainable by the linear theory, by the integral

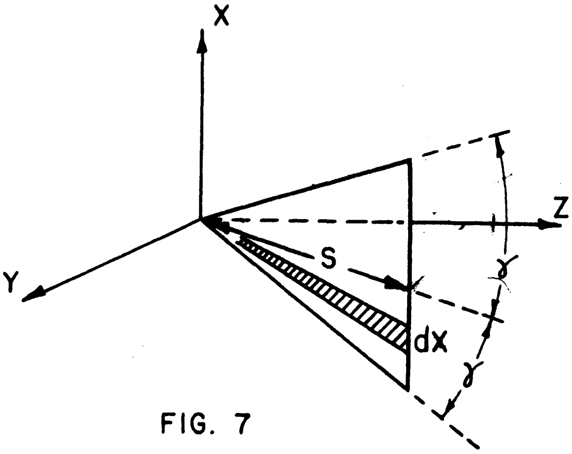

FIG. 7

$$-1/2\ \rho_\infty w_\infty s \int_{-s\tan\gamma}^{s\tan\gamma} (w_2 - w_1)\, dX ,$$

where w_2 and w_1 are the values of the velocity component w on the positive and negative faces (relative to the sense of the Y-coordinate), respectively, and where s is the altitude of the triangular tip. The nondimensional normal force coefficient C_n obtained by dividing this expression by the stagnation pressure $1/2\ \rho_\infty w_\infty{}^2$ and the area $s^2 \tan\gamma$ of the portion of the wing surface under consideration is given by

$$C_n = -\frac{\cot\gamma}{s\ w_\infty} \int_{-s\ \tan\gamma}^{s\ \tan\gamma} (w_2 - w_1)\, dX .$$

The relation between the X-coordinate of a point on the surface of the wing along the line Y = constant = s sinα, Z = constant = s cosα, and the ξ_1-coordinate of the corresponding point on the segment of the real axis in the ζ_1-plane can be obtained in the following form from equations (1.10) and (2.12).

$$X = 2s\ \sin\alpha\ \operatorname{csch}\tilde{\alpha}\ \frac{\xi_1}{1 + \xi_1^2}$$

26) Sauer, loc. cit., p. 23

The normal force coefficient can be written in terms of an integral over the segment of the real ξ_1-axis between the points $-\sqrt{k_1}$ and $+\sqrt{k_1}$ as follows:

$$C_n = -2\,\frac{\coth\tilde{\gamma}}{W_\infty}\int_{-\sqrt{k_1}}^{\sqrt{k_1}} \Delta w\,\frac{1-\xi_1^2}{(1+\xi_1^2)^2}\,d\xi_1\;, \tag{2.47}$$

where, making use of the notation employed in equation (2.46),

$$\Delta w = \left[w\right]_{\eta_1=0+} - \left[w\right]_{\eta_1=0-}\;. \tag{2.48}$$

In accordance with equations (2.15) and (2.19), the relation between the ξ_1-coordinate of a point on the segment of the real axis in the ζ_1-plane corresponding to the wing surface and the x_1-coordinate of the corresponding point on the real axis in the z_1-plane (see Figure 6) is given by the equation

$$\xi_1 = -\sqrt{k_1}\;\mathrm{sn}\,(x_1;k_1).$$

Therefore, the expression for the normal force coefficient of the arrow wing can also be written in terms of an integral over the segment of the real axis between $x_1 = -2K_1$ and $x_1 = +2K_1$ as follows:

$$C_n = -2\sqrt{k_1}\,\frac{\coth\tilde{\gamma}}{W_\infty}\int_{-2K_1}^{2K_1} \Delta w\;\mathrm{cn}(x_1;k_1)\mathrm{dn}(x_1;k_1)\,\frac{1-k_1\mathrm{sn}^2(x_1;k_1)}{\left[1+k_1\mathrm{sn}^2(x_1;k_1)\right]^2}\,dx_1.$$

It is shown in Section 2.3 that the component W of the complex velocity has singularities of the nature of poles at the points $z_1 = \pm K_1$ corresponding to the wing edges. Therefore, the velocity component w representing the real part of this function which appears in the integrand of the integral defining the normal force coefficient C_n becomes infinite at the two points on the path of integration corresponding to $x_1 = \pm K_1$. However, the Jacobi elliptic function $\mathrm{cn}(\nu;k_1)$ has simple zeros at $\nu = \pm K_1$. Consequently, the normal force coefficient C_n is finite if and only if the poles of the function W(z) at the edges of the wing are at most of the first order. Since in accordance with equations (2.6) and (2.23)

$$h(z) = -W'(z)\frac{zL(z)}{g(z)} ,$$

it follows that <u>the normal force coefficient</u> C_n <u>for the arrow wing is finite if and only if the poles of the function</u> h(z), <u>and consequently also those of the elliptic function</u> $h_1(z_1)$, <u>are at most of the second order</u>. The only function $h_1(z_1)$ which satisfies this condition is shown in Section 2.3 to be of the form given by equation (2.34), and the corresponding component W(z) of the complex velocity with the simple poles is given by equation (2.42).

The value of the normal force coefficient corresponding to the function W(z) obtained in Section 2.3 is readily obtained with the aid of the formula in equation (2.47). Thus, making use of the values of the velocity component w along the upper and lower side of the segment of the real axis in the ζ_1-plane corresponding to the wing surface given by equation (2.46), the increment Δw in the velocity across the wing surface is given by the equation

$$\begin{aligned} \Delta w &= [w]_{\eta_1=0+} - [w]_{\eta_1=0-} \\ &= -2\sqrt{k_1}(1-k_1)\, C\, \frac{w_\infty \sin 2\alpha}{\beta} \cdot \frac{1+\xi_1^2}{\sqrt{(k_1-\xi_1^2)(1-k_1\xi_1^2)}} . \end{aligned} \tag{2.49}$$

Making use of the properties of symmetry of the integrand of the integral in equation (2.47) and introducing the variable x_1 by means of the relation

$$\xi_1 = -\sqrt{k_1}\ \mathrm{sn}(x_1;k_1),$$

the expression for the coefficient C_n can be written in the form[27]

$$C_n = 8\sqrt{k_1}(1-k_1)\ C\ \frac{\coth\tilde{\gamma}\ \sin 2\alpha}{\beta}\int_0^{K_1} \frac{1-k_1\mathrm{sn}^2(x_1;k_1)}{1+k_1\mathrm{sn}^2(x_1;k_1)}\ dx_1 .$$

Since

$$\int_0^{K_1} \frac{1-k_1\mathrm{sn}^2(x_1;k_1)}{1+k_1\mathrm{sn}^2(x_1;k_1)}\ dx_1 = \frac{\pi}{2(1+k_1)}\ ,$$

the normal force coefficient C_n is easily placed in the form given in the following equation

$$C_n = +4\pi C'\ \sin 2\alpha\ , \tag{2.50}$$

where

$$C' = \frac{(1-q^2)\ \tanh\tilde{\gamma}/2}{(1+p^2\sinh^2\tilde{\gamma})\ 2(k_1K_1'+E') - AK_1'}\ . \tag{2.51}$$

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27) See footnote 22 regarding possible simplifications by the introduction of a new variable by means of Gauss' transformation considered in Section 3.2.

PART III. SUPERSONIC FLOW PAST AN ELLIPTIC CONE AT ZERO ANGLE OF ATTACK

3.1 Statement of the Problem.

In this part of the report there is constructed an exact solution of the linearized conical flow equations for the case of the flow past a body in the shape of an elliptic cone whose axis of symmetry coincides with the direction of the undisturbed stream velocity w_∞ ahead of the body. Following the general theory in the first part of the report, the conical body is assumed to be contained entirely within the Mach cone attached to its vertex. This problem is a natural generalization of the problem of Karman and Moore [1] for the flow past a circular cone at zero angle of attack and therefore amply deserves the attention given it here. In addition, the problem of the flow past the elliptic cone has many points in common with the flow past the arrow wing treated in Part II since the latter surface may be regarded simply as the limiting case of an elliptic cone. This connection between the two problems has already been referred to in Section 2.2.

In accordance with Section 1.2 the problem is referrred to a doubly-connected region of the ζ -plane (see Figure 1) into which the region of the XYZ-space between the material cone and the Mach cone is projected by the transformation (1.10). The problem in this form is to determine the function $F(\zeta)$ such that the components of velocity u, v, andw forming the real part of the complex functions $U(\zeta)$, $V(\zeta)$, and $W(\zeta)$ defined by the "Weierstrass integral formulae" in (1.14) satisfy the boundary conditions (1.15) and 1.16) on the boundaries of the region in the ζ -plane. In this case, the inner boundary of the region correponding to the elliptic cone is an oval curve belonging to the family of cyclids referred to in Section 2.2. This family of curves also includes as a number the outer circular boundary of the region.

In the treatment of the problem of the elliptic cone presented here, use is made of the system of curvilinear coordinates in the ζ -plane which employs as a set of coordinate lines the family of cyclids to which the inner boundary of the doubly-connected region belongs. In other words, by a method of conformal mapping, this region of the ζ -plane is mapped periodically into a strip in another plane in such a manner that the family of cyclids corresponds to parallel lines in this plane. It is shown that the system of curvilinear coordinates in the ζ -plane corresponds to a system of non-orthogonal elliptic cone coordinates in the XYZ-space. One of the families of coordinate surfaces is the set of elliptic cones which is projected by the transformation (1.10) into the family of cyclids in the ζ -plane and which contains the material cone as a member.

3.2 The Elliptic Cone Coordinates Appropriate to the Problem.

It was asserted in Section 2.2 that an elliptic cone in the XYZ-space is projected by the transformation (1.10) into a cyclid in the ζ -plane which surrounds the foci $\pm\sqrt{k_1}$ and is contained within the unit circle $|\zeta| = 1$[28]. Several of these cyclids are indicated in Figure 4. It was also indicated in that section that the relation

$$\zeta = -\sqrt{k_1}\ \mathrm{sn}(z_1;k_1) \qquad (3.1)$$

maps the doubly-connected region of the ζ -plane within the unit circle into a rectangle in the z_1-plane (see Figure 6) in such a manner that the cyclids of Figure 4 correspond to the lines parallel to the real axis in the plane of the complex variable $z_1 = x_1 + iy_1$. It is shown in this section that the lines x_1 = const. and y_1 = const. correspond to two families

[28] Since the elliptic cone under consideration has zero angle of attack, the distinction between the ζ and ζ_1 planes of Part II is not necessary here. Therefore the ζ_1-plane of Part II will be denoted in this part simply as the ζ -plane.

of surfaces in XYZ-space which together with the surfaces R = const. defined in equations (1.10a) form a triple system of coordinate surfaces, the material cone belonging to the family of elliptic cones corresponding to the lines x_1 = const.

Separating the real and imaginary parts of the complex quantities in equation (3.1) by making use of the addition theorem for the Jacobi elliptic function $sn(z_1, k_1)$,[29] the following equations are obtained:

$$\xi = -\sqrt{k_1}\frac{snx_1 cniy_1 dniy_1}{1-k_1^2 sn^2x_1 sn^2iy_1},$$

$$\eta = i\sqrt{k_1}\frac{sniy_1 cnx_1 dnx_1}{1-k_1^2 sn^2x_1 sn^2iy_1},$$

and, after slight changes, also the equation

$$\xi^2+\eta^2 = \frac{sn^2x_1 - sn^2iy_1}{1-k_1^2 sn^2x_1 sn^2iy_1}.$$

By substituting these equations into equations (1.10) and (1.10a) the following relationships between the Cartesian coordinates X, Y and Z of the flow space and the parameters x_1, y_1, and R are obtained

$$\beta X = -2R\sqrt{k_1}\,\frac{snx_1}{1-k_1 sn^2x_1}\cdot\frac{cniy_1 dniy_1}{1+k_1 sn^2iy_1},$$

$$\beta Y = 2iR\sqrt{k_1}\frac{cnx_1 dnx_1}{1-k_1 sn^2x_1}\cdot\frac{sniy_1}{1+k_1 sn^2iy_1}, \qquad (3.2)$$

$$Z = R\,\frac{1+k_1 sn^2x_1}{1-k_1 sn^2x_1}\cdot\frac{1-k_1 sn^2iy_1}{1+k_1 sn^2iy_1}.$$

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[29] Cf. Whittaker and Watson, loc. cit. p. 494.

Inspection reveals that each coordinate is a product of factors depending separately on x_1, y_1, and R. In particular, each of the six factors which depend upon the parameters x_1 and y_1 may be considerably simplified by means of Gauss' transformation of the Jacobi elliptic functions. [30] In such a transformation the ratio K_1'/K_1 of the fractional periods is doubled. Consequently, the modulus k_1 of the elliptic functions is also changed, and the independent variable undergoes a change in scale. The following table gives a complete review of the changes:

$$z_1' = (1 + k_1)z_1, \tag{3.3}$$

$$k = \frac{2\sqrt{k_1}}{1 + k_1} \quad \text{or} \quad \frac{1}{k} = \frac{1}{2}\left(\frac{1}{\sqrt{k_1}} + \sqrt{k_1}\right), \tag{3.4}$$

$$1 + k' = \frac{2}{1 + k_1} ;$$

$$K = (1 + k_1)K_1, \quad K' = \frac{1}{2}(1 + k_1)K_1' ,$$

$$\frac{K'}{K} = 2\frac{K_1'}{K_1} ; \tag{3.5}$$

$$\mathrm{sn}(z_1';k) = (1 + k_1)\frac{\mathrm{sn}(z_1;k_1)}{1 + k_1\mathrm{sn}^2(z_1;k_1)} ,$$

$$\mathrm{cn}(z_1';k) = \frac{\mathrm{cn}(z_1;k_1)\mathrm{dn}(z_1;k_1)}{1 + k_1\mathrm{sn}^2(z_1;k_1)} , \tag{3.6}$$

$$\mathrm{dn}(z_1';k) = \frac{1 - k_1\mathrm{sn}^2(z_1;k_1)}{1 + k_1\mathrm{sn}^2(z_1;k_1)} .$$

In each relation the quantities on the right-hand side are in terms of the parameters employed in the development in Part II while the new variables,

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[30] The formulae for Gauss' transformation of the Jacobi elliptic functions can be found in R. Fricke, Die Elliptischen Funktionen, Leipzig (1922), Vol. 2, p293.

moduli and periods introduced by Gauss' transformation are on the left-hand side.

Making use of the relations (3.3) to (3.6), the expressions for the Cartesian coordinates in (3.2) become when expressed in terms of the complex variable $z_1' = x_1' + iy_1'$ as follows:

$$\begin{aligned} \beta X &= -kR\,\mathrm{sd}(x_1';k)\ \mathrm{cn}(iy_1';k), \\ \beta Y &= ikR\,\mathrm{cd}(x_1';k)\ \mathrm{sn}(iy_1';k), \\ Z &= R\,\mathrm{nd}(x_1';k)\ \mathrm{dn}(iy_1';k). \end{aligned} \tag{3.7}$$

Finally, it is convenient to introduce the complex variable [31] $z = x + iy$ by means of the relation

$$z_1' = z - K\ . \tag{3.8}$$

The expressions for the Cartesian coordinates (3.7) become when expressed in forms of z as follows:

$$\begin{aligned} \beta X &= \frac{k}{k'} R\,\mathrm{cn}(x;k)\ \mathrm{cn}(iy;k), \\ \beta Y &= ikR\,\mathrm{sn}(x;k)\ \mathrm{sn}(iy;k), \\ Z &= \frac{1}{k'} R\ \mathrm{dn}(x;k)\ \mathrm{dn}(iy;k) \end{aligned} \tag{3.9}$$

Making use of equations (3.1, (3.8), and the last of equations (3.6), the transformation from the doubly-connected region in the ζ -plane to the plane of the complex variable z is found to be given by the relation:

$$\zeta = \frac{\mathrm{dn}(z;k) - k'}{k\ \mathrm{cn}(z;k)}\ . \tag{3.10}$$

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[31] The complex variable z introduced at this point should not be identified with the variable z used in Sections 2.1 to 2.4 on the solution of the problem for the arrow wing. No confusion should result because of its use in the two senses for the two separate treatments.

It is immediately evident from this relation that the points of the unit circle $|\zeta| = 1$ in the ζ-plane correspond to points of the line y = -K' in the z-plane between x = o and x = 4K; the mapping of the unit circle onto the line y =-K' is repeated periodically in segments of length 4K. Also the points of the (two sided) slit along the real axis in the ζ-plane between $\zeta = +\sqrt{k_1}$ and $-\sqrt{k_1}$ correspond to the points of the real axis in the z-plane between x = o and x = 4K. Hence, the doubly connected-region of the ζ-plane bounded on the exterior by the unit circle and on the interior by the slit along the real axis between $\zeta = \pm\sqrt{k_1}$ is mapped into the following rectangle in the z-plane by means of the relation (3.10):

$$o < x < 4K \ , \ -K' < y < o. \tag{3.11}$$

The formulae (3.9) can evidently be considered as relations between the Cartesian coordinates (X,Y,Z) of a point in the flow space and a system of curvilinear coordinates (x,y,R). The geometric significance of this point of view becomes clear when the surfaces for which either x,y, or R is constant are determined. Thus, by eliminating successively the pairs x and R, y and R, and x and y from equations (3.9) the following three equations are obtained:

$$\frac{\beta^2 X^2}{k^2 nc^2(y;k')} + \frac{\beta^2 Y^2}{k^2 sc^2(y;k')} - \frac{Z^2}{dc^2(y;k')} = 0 \ , \tag{3.12}$$

$$-\frac{\beta^2 X^2}{k^2 cn^2(x;k)} + \frac{\beta^2 Y^2}{k^2 sn^2(x;k)} + \frac{Z^2}{dn^2(x;k)} = 0 \ , \tag{3.13}$$

$$Z^2 - \beta^2(X^2 + Y^2) = R^2 \ . \tag{3.14}$$

The first of these equations represents a family of elliptic cones around the Z-axis. The major axes of the cross sections of these cones by planes perpendicular to the Z-axis are horizontal (i.e., parallel to the X-axis). The second equation determines a family of elliptic cones around the X-axis, while the third represents a family of hyperboloids of revolution about the Z-axis. These three families of surfaces constitute the coordinate surfaces for the system of coordinates (x, y, R). These surfaces are not orthogonal in the ordinary Euclidean sense. Therein lies their greatest disadvantage. However, their advantage lies in the fact that the differential equation (1.2) becomes separable when expressed in terms of the variables of x, y, and R.

Consider, in particular, the family of elliptic cones (3.12) corresponding to the lines y = const. in the z-plane. As the value of y approaches -K', the corresponding cone approaches the Mach cone, namely,

$$\beta^2(X^2 + Y^2) - Z^2 = 0 \ . \tag{3.15}$$

On the other hand, as the value of y approaches zero, the corresponding cone approaches the arrow wing formed by a sector in the XZ-plane. Therefore, the cones (3.12) cooresponding to the parallel lines in the z-plane between the lines y = 0 and y =-K' sweep out the region of XYZ-space between the Mach cone and the plane sector. Let the surface of the symmetrically placed conical obstacle in the flow space be given by the equation

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} - Z^2 = 0 \ , \tag{3.16}$$

where $a > b$ and, since the body is entirely contained within the Mach cone,

$$\beta b < \beta a < 1 \ . \tag{3.17}$$

This cone is included in the family of elliptic cones (3.12) and corresponds to a particular value of y, say $y = -y_o$, provided that k' and y_o are defined by the equations

$$k'^2 = \frac{1 - \beta^2 a^2}{1 - \beta^2 b^2} \ ,$$
$$\operatorname{sn}(y_o; k') = + \frac{b}{a} \ . \tag{3.18}$$

It is evident from the preceding discussion that the XYZ-space between the Mach cone and the material cone is mapped into the rectangle

$$o \leqq x \leqq 4K, \qquad -y_o \leqq y \leqq o, \qquad (y_o \leqq K'), \tag{3.19}$$

in the z-plane by means of the coordinate transformation in (3.19) provided that k' and y_o are defined by (3.18). This mapping is effected in such a manner that the Mach cone corresponds to the side y = o and the material cone, to the side $y = -y_o$. As a consequence, with these values of k' and y_o, the transformation (3.10) effects a mapping of the doubly-connected region in the ζ -plane bounded by the unit circle and the oval-shaped cyclid corresponding to the material cone onto the rectangle (3.19) in the z-plane, the unit circle and the cyclid corresponding respectively to the sides $y = -K'$ and $y = -y_o$.

3.3 Formulation of the Problem in Terms of the Elliptic Cone Coordinates.

The functions $U(\zeta)$, $V(\zeta)$, and $W(\zeta)$ of the complex variable which are defined in Section 1.2 and whose real parts are the components of the velocity u, v, and w, respectively, are readily expressed in terms of the variable z. Thus, by substituting in the formulae (1.14) the value of ζ given by equation (3.10), the corresponding "Weierstrass integral formulae" for the functions U, V, and W in terms of the variable z become[32]

$$U = -\frac{\beta}{kk'} \int dsz\, G(z)\, dz,$$

$$V = -\frac{i\beta}{k} \int nsz\, G(z)\, dz, \tag{3.20}$$

$$W = \frac{1}{k'} \int csz\, G(z) dz,$$

[32]The function G(z) defined here should not be confused with that defined in Section 2.1 in connection with the separate problem of the arrow wing.

where $G(z)$ is an analytic function of z in the rectangle (3.19) of the z-plane and is related to the function $F(\zeta)$ of equations (1.14) by the equation

$$F(\zeta) = G(z)\left(\frac{d\zeta}{dz}\right)^{-2} . \tag{3.21}$$

It is also convenient to express the functions U, V, and W in terms of the complex variable $z' = x' + iy'$ defined by the equation

$$z' = +z + iK'. \tag{3.22}$$

When expressed in terms of this variable equations (3.20) become

$$\begin{aligned} U &= - i \frac{\beta}{k'} \int \mathrm{cn}z' \, G_1(z') \, dz' , \\ V &= - i\beta \int \mathrm{sn}z' \, G_1(z') \, dz', \\ W &= \frac{i}{k'} \int \mathrm{dn}z' \, G_1(z') \, dz' , \end{aligned} \tag{3.23}$$

where the relation between the function $G_1(z')$ and the function $F(\zeta)$ in equations (1.14) is given by the equation

$$F(\zeta) = G_1(z') \left(\frac{d\zeta}{dz'}\right)^{-2} . \tag{3.24}$$

Actually these relations are used in the form:

$$\begin{aligned} \frac{dU}{dz'} &= - \beta \, \mathrm{cd}z' \, \frac{dW}{dz'} , \\ \frac{dV}{dz'} &= - \beta k' \, \mathrm{sd}z' \, \frac{dW}{dz'} . \end{aligned} \tag{3.25}$$

The boundary conditions (1.4) pertaining to the material cone are easily expressed in the form which the components of velocity u, v, and w must satisfy on the side $y = y_o$ of the rectangle in the z-plane. The function denoted by $S(X, Y, Z)$ in equation (1.4) is in this case understood to represent the left-hand member of equation (3.16) or the left-hand member of equation (3.12) in which $y = -y_o$. Thus, making use of the relations (3.9) and setting $y = -y_o$, the boundary condition (1.4) assumes the form:

$$\frac{\beta}{k} \operatorname{dn}(y_o;k') \operatorname{cd}(x;k)u + \frac{\beta k'}{k} \operatorname{ds}(y_o;k') \operatorname{sd}(x;k)v - w = w_\infty . \qquad (3.26)$$

This condition is a linear expression in the functions u, v, and w. However, the presence of the variable x in the coefficients of this expression gives rise to considerable complications in the procedure for determining the functions u, v, and w. There is some advantage in the fact that the functions cd(x;k) and sd(x;k) which are present in the coefficients of this expression are also present in the relations (3.25).

The boundary condition (1.6) pertaining to the Mach cone becomes in the z-plane simply the condition that the functions u, v, and w vanish along the side $y = 0$ of the rectangle in the z-plane. Thus, for $y = 0$

$$u = v = w = 0 . \qquad (3.27)$$

As a consequence of the foregoing discussion, it follows that the problem of determining the conical flow past an arbitrary cone represented by equation (3.16) corresponds to determing the analytic functions U(z), V(z), and W(z) which satisfy the relations (3.25) throughout the rectangle (3.19) in the z-plane and whose real parts u, v, and w, respectively, satisfy the boundary conditions (3.26) and (3.27) on the sides $y = -y_o$ and $y = 0$, respectively, of this rectangle. The procedure of determining the solution of the problem in this form is to express the functions U(z), V(z), and W(z) in terms of Fourier series in the complex variable z with period 4K, and then to evaluate the coefficients by means of the differential relations (3.25) and the boundary conditions (3.26) and (3.27).

3.4 Special Fourier Series Employed in the Solution.

In view of the fact that the elliptic cone $y = -y_o$ does not have an angle of attack, the velocity components w, u, v possess the following symmetry properties:

	w	u	v
x = o	symmetric	symmetric	antisymmetric
x = K	symmetric	antisymmetric	symmetric

This is evident when one remembers that x = 0 means Y = 0 or the plane of the major axes, and x = K means X = 0 or the plane of the minor axes of the ellipses Z = const.

Therefore let the functions u, v, and w be expressed in the following form:

$$w = C_o(K' + y) + \sum_1^\infty C_n sh\left[\frac{n\pi}{K}(K' + y)\right] \cos\frac{n\pi x}{K},$$

$$u = \sum_1^\infty A_n sh\left[\frac{(2n+1)\pi}{2K}(K' + y)\right] \cos\frac{(2n+1)\pi x}{2K}, \tag{3.28}$$

$$v = \sum_1^\infty B_n sh\left[\frac{(2n+1)\pi}{2K}(K' + y)\right] \sin\frac{(2n+1)\pi x}{2K}.$$

Since the complex velocities W, U, V are needed in (3.25) they will be recorded here:

$$W = -iC_o z' - i\sum_1^\infty C_n \sin\frac{n\pi z'}{K}$$

$$U = -i\sum_1^\infty A_n \sin\frac{(2n+1)\pi z'}{K}, \tag{3.29}$$

$$V = -i\sum_1^\infty B_n \cos\frac{(2n+1)\pi z'}{K}.$$

It will be found convenient to write all these Fourier series as series of exponential functions proceeding from $-\infty$ to $+\infty$. For (3.29) one gets:

$$W = -iC_o z' - 1/2 \sum_{-\infty}^{+\infty}{}' C_n e^{i\frac{n\pi z'}{K}} \; ; \; C_{-n} = -C_n \; ,$$
$$U = \quad - 1/2 \sum_{-\infty}^{+\infty} A_n e^{i\frac{2n+1}{2}\frac{\pi}{K} z'} \; ; \; A_{-n-1} = -A_n \; , \qquad (3.30)$$
$$V = \quad - i/2 \sum_{-\infty}^{+\infty} B_n e^{i\frac{2n+1}{2}\frac{\pi}{K} z'} \; ; \; B_{-n-1} = +B_n \; .$$

The accent on the sum sign of the first formula indicates that the term with $n = o$ is absent. The three series (3.28) may be written

$$w = \sum_{-\infty}^{+\infty} N_n e^{i\frac{n\pi x}{K}} \; ,$$
$$u = \sum_{-\infty}^{+\infty} L_n e^{i\frac{2n+1}{2}\frac{\pi x}{K}} \qquad (3.31)$$
$$v = \sum_{-\infty}^{+\infty} M_n e^{i\frac{2n+1}{2}\frac{\pi x}{K}} \; ,$$

with

$$N_n = 1/2 \, C_n \mathrm{sh} \frac{n\pi}{K} (K' + y) = + N_{-n} \; ,$$
$$N_o = C_o (K' + y_o) \; ,$$
$$L_n = 1/2 \, A_n \gamma_n = + L_{-n-1} \; , \qquad (3.32)$$
$$M_n = 1/2i \, B_n \gamma_n = - M_{-n-1} \; ,$$
$$\gamma_n = \mathrm{sh} \frac{2n+1}{2} \frac{\pi}{K} (K' + y_o) \; .$$

The relations (3.25), as well as the boundary condition (3.26) call for the Fourier development of the elliptic functions cd and sd or x or z' which are in the text books.[33] When written as series of expotentials, they are:

$$\mathrm{cdz}' = \sum_{-\infty}^{+\infty} \alpha_n e^{i(2n+1)\frac{\pi}{2}\frac{z'}{K}} ,$$

$$\mathrm{sdz}' = \sum_{-\infty}^{+\infty} \beta_n e^{i(2n+1)\frac{\pi}{2}\frac{z'}{K}} , \qquad (3.33)$$

with

$$\alpha_n = \frac{\pi}{2kK} \frac{(-1)^n}{\mathrm{sh}(2n+1)\frac{\pi}{2}\frac{K'}{K}} ,$$

$$\beta_n = \frac{\pi}{2ikk'K} \frac{(-1)^n}{\mathrm{ch}(2n+1)\frac{\pi}{2}\frac{K'}{K}} .$$

3.5 Solution in Terms of Fourier Series.

The relations (3.25) make it possible to express the series coefficients A_n and B_n in terms of C_n. We introduce the derivatives with the respect to z' of the series (3.30) and also the series (3.33) into (3.25) and obtain:

$$- \frac{i\pi}{4K} \sum_n (2n+1) A_n e^{i\frac{2n+1}{2}\frac{\pi}{K}z'}$$

$$= \beta \sum_n \alpha_n e^{i\frac{2n+1}{2}\frac{\pi}{K}z'} \left(-iC_o - \frac{i\pi}{2K} \sum_m mC_m e^{i\frac{M\pi z'}{K}}\right) .$$

In the series product we rename indices as follows:

$$n + m = n'; \; m = \lambda; \; n = n' - \lambda \quad . \qquad (3.34)$$

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[33] Cf. Whittaker and Watson, loc. cit., p. 511.. The formula for the Fourier series of cnu on this page contains a typographical error: The exponent of q in the denominator should be 2n + 1 instead of 2n - 1.

Then

$$\sum_n (2n + 1) A_n e^{i\frac{2n+1}{2}\frac{\pi}{K} z'}$$

$$= \frac{4K}{\pi} \beta(C_o \sum_n \alpha_n e^{i\frac{2n+1}{2}\frac{\pi}{K} z'} + \frac{\pi}{2K} \sum_{n'} \sum_\lambda \alpha_{n'-\lambda} \lambda C_\lambda e^{i\frac{2n'+1}{2}\frac{\pi}{K} z'}) .$$

Therefore

$$A_n = \frac{4K\beta}{\pi} \frac{1}{2n+1} (C_o \alpha_n + \frac{\pi}{2K} \sum_\lambda \alpha_{n-\lambda} \lambda C_\lambda) . \tag{3.35}$$

Similarly:

$$B_n = \frac{4iKk'\beta}{\pi} \frac{1}{2n+1} (C_o \beta_n + \frac{\pi}{2K} \sum_\lambda \beta_{n-\lambda} \lambda C_\lambda) . \tag{3.36}$$

We shall now treat the boundary condition (3.26) in the same way. Using the series (3.31) for the real velocities and the series (3.33) (but for the real argument x) we get:

$$\frac{\beta}{k} dny_o \sum_n \alpha_n e^{i\frac{2n+1}{2}\frac{\pi x}{K}} \sum_m e^{i\frac{2m+1}{2}\frac{\pi x}{K}} L_m$$

$$+ \frac{\beta k'}{k} dsy_o \quad \beta_n e^{i\frac{2n+1}{2}\frac{\pi x}{K}} \quad M_m e^{i\frac{2m+1}{2}\frac{\pi x}{K}}$$

$$- \quad N_n e^{i\frac{n\pi x}{K}} = w$$

Now we rename indices as follows:

$$m + n + 1 = n', \; m = \lambda, \; n = n' - \lambda - 1 ,$$

whereupon the last formula becomes:

$$\frac{\beta}{k} dny_o \sum_{n'} \sum_\lambda \alpha_{n'-\lambda-1} L_\lambda e^{i\frac{n'\pi x}{K}}$$

$$+ \frac{\beta k'}{k} dsy_o \sum_{n'} \sum_\lambda \beta_{n'-\lambda-1} M_\lambda e^{i\frac{n'\pi x}{K}} - \sum_n N_n e^{i\frac{n\pi x}{K}} = w_\infty$$

By comparing coefficients the following equations result:

$$\frac{\beta}{k}\, dny_o \sum_{\lambda} \alpha_{\lambda-n} L_\lambda - \frac{\beta k'}{k}\, dsy_o \sum_{\lambda} \beta_{\lambda-n} M_\lambda - N_n = w_\infty \delta_{n,o} \tag{3.37}$$

In this system it is sufficient to regard n as going from 0 to ∞ , because all the expotential series are actually cosine series.

The last step is to express the L_λ and M_λ in terms of the C_λ by means of (3.32), (3.35), and (3.36). We introduce the following abbreviations:

$$s_{\lambda\mu} = \sum_{\nu=-\infty}^{\infty} \frac{\alpha_{\nu-\lambda}\, \alpha_{\nu+\lambda}}{2\nu+1} \gamma_\nu \ ,$$

$$\sigma_{\lambda\mu} = \sum_{\nu=-\infty}^{\infty} \frac{\beta_{\nu-\lambda}\, \beta_{\nu+\lambda}}{2\nu+1} \gamma_\nu \ . \tag{3.38}$$

The result of the substitution is for n = o:

$$\left\{ \frac{2\beta^2 K dny_o}{k\pi} s_{oo} - \frac{2\beta^2 k'^2 K dsy_o}{k\pi} \sigma_{oo} - (K' + y_o) \right\} C_o$$

$$+ \sum_{\mu} \left\{ \frac{\beta^2}{k} dny_o\, \mu s_{o\mu} - \frac{\beta^2 k'^2}{k} dsy_o\, \mu\, \sigma_{o\mu} \right\} C_\mu = w_\infty \ , \tag{3.39}$$

and for n > o:

$$\left\{ \frac{2\beta^2 K}{k\pi} dny_o\, s_{no} - \frac{2\beta^2 k'^2 K}{k\pi} dsy_o\, \sigma_{no} \right\} C_o$$

$$+ \sum_{\mu} \left\{ \frac{\beta^2}{k} dny_o\, \mu\, s_{n\mu} - \frac{\beta^2 k'^2}{k} dsy_o\, \mu\, \sigma_{n\mu} \right\} C_\mu \tag{3.40}$$

$$- 1/2\ sh \frac{n\pi}{K} (K' + y_o)\, C_n = 0 \ .$$

The sums with respect to μ which at present run over all positive and negative values may be written as sums from zero to infinity. With

$$S_{\lambda,\mu} = s_{\lambda,\mu} + s_{\lambda,-\mu} \ ,$$

$$\Sigma_{\lambda,\mu} = \sigma_{\lambda,\mu} + \sigma_{\lambda,-\mu} \ , \tag{3.41}$$

the equation system becomes

$$\left\{\frac{\beta^2 K}{\pi k}\ dny_o\ S_{oo} - \frac{\beta^2 k'^2 K}{\pi k}\ dsy_o \Sigma_{oo}\ -(K' + y_o)\right\} C_o$$
$$+\sum_{\mu=1}^{\infty}\left\{\frac{\beta^2}{k}\ dny_o\,\mu\, S_{o\mu} - \frac{\beta^2 k'^2}{k}\ dsy_o\,\mu\,\Sigma_{o\mu}\right\} C_\mu = w_\infty ,$$
$$\left\{\frac{\beta^2 K}{\pi k}\ dny_o\ S_{no} - \frac{\beta^2 k'^2 K}{\pi k}\ dsy_o\ \Sigma_{no}\right\} C_o \qquad (3.42)$$
$$+\sum_{\mu=1}^{\infty}\left\{\frac{\beta^2}{k}\ dny_o\ \mu\ S_{n\mu} - \frac{\beta^2 k'^2}{k}\ dsy_o\ \mu\ \Sigma_{n\mu}\right\} C_\mu$$
$$-\ 1/2\ \text{sh}\ \frac{n\pi}{K}\ (K' + y_o)\ C_n = 0\ .$$

For this it is more convenient to write

$$\sum_{\mu=0}^{\infty} a_{n\mu}\, C_\mu = w_\infty\, \delta_{no} \qquad (3.43)$$

by introducing

$$a_{oo} = \frac{\beta^2 K}{\pi k}\ dny_o S_{oo} - \frac{\beta^2 k'^2 K}{\pi k}\ dsy_o \Sigma_{oo} - (K' + y_o)\ ,$$
$$a_{o\mu} = \frac{\beta^2}{k}\ dny_o\ \mu\, S_{o\mu} - \frac{\beta^2 k'^2}{k}\ dsy_o\ \mu\, \Sigma_{o\mu}\ ,$$
$$a_{no} = \frac{\beta^2 K}{\pi k}\ dny_o\ S_{no} - \frac{\beta^2 k'^2 K}{\pi k}\ dsy_o\ \Sigma_{no}\ , \qquad (3.44)$$
$$a_n = \frac{\beta^2}{k}\ dny_o \mu S_{n\mu} - \frac{\beta^2 k'^2}{k}\ dsy_o\ \mu \Sigma_{n\mu}$$
$$-\ 1/2\ \text{sh}\ \frac{n\pi}{K}\ (K' + y_o)\ .$$

The infinite series $S_{n\mu}$ and $\Sigma_{n\mu}$ will be dealt with in Appendix .

The solution of (3.44) proceeds as follows: After having calculated the coefficients $a_{n\mu}$ for an elliptic cone of given eccentricity (k) and flare (y_o) subjected to a windstream of given speed (w_∞ ,β) the infinite linear equation system (3.43) must be solved for the coefficients C_μ. These coefficients are then substituted into the first equation (3.44). The component w of the additional velocity -- the only one which is important for the calcuation of the drag -- is then completely known. Should the components u and v also be desired, the Fourier coefficients A_n and B_n may be found from formulae (3.35) and (3.36).

3.6 Numerical Example.

In order to test the practicability of the solution a numerical example was calculated. Unfortunately time and help were lacking to test, through calculation of several examples, the fastness of the convergence as a function of the parameters such as Mach number, eccentricity, flare angle, etc. As soon as values for these parameters have been decided upon, the coefficients of the infinite linear equation system (3.43) which are given through (3.44) and (3.38) can be calculated.

Since the coefficients $a_{n\mu}$ increase rapidly with increasing indices, it is convenient to divide the n^{th} equation by its diagonal coefficient a_{nn}. The thus normalized equation system is

$$\sum_{\mu} a^{N}_{n\mu} \quad C_{\mu} \quad = w_{\infty} \quad a^{-1}_{oo} \quad \delta_{no} \quad , \tag{3.45}$$

where

$$a^{N}_{n\mu} = a_{n\mu}/a_{nn} \quad .$$

Finally it was found convenient to symmetrize the matrix by multiplying (3.45) by a^{N}_{nk} and summing with respect to n. In this way one gets:

$$A_{k\mu} C_{\mu} \quad = R_k \quad , \tag{3.96}$$

where

$$R_k = w_{\infty} \quad a^{-1}_{oo} \, a_{ok} \quad ,$$

and

$$A_k = \sum_{n} a^{N}_{nk} \, a^{N}_{n\mu} \tag{3.47}$$

is evidently symmetric[34]. The solution of the equation system (3.46) can be effected in several ways. First a successive-approximation scheme fashioned after the Liouville-Neumann method for solving integral equations[35] was employed. However, the convergence was found to be slow and the fluctuation, especially for the higher C_μ, to be considerable. Therefore, the following method was employed[36]: In the first three equations (k = o, 1, 2) all terms with $\mu > 2$ were neglected and C_o as well as C_1 computed. Then the equations with k = 2, 3, 4 were taken, and terms with $\mu > 4$ were neglected. In the terms with $\mu = o$ and 1 the just calculated C_o and C_1 were used. From these equations C_2 and C_3 were obtained. The calculation was continued in this fashion but since C_3 was already of order 10^{-4} of C_o the accuracy attained was regarded as sufficient. As a check the so determined values of C_μ were re-substituted into (3.45) and the residues, or differences of the left sides minus the right sides were calculated.

TABLE I

n	Residues of Equation (3.45)
o	.00002
1	-.00004
2	-.000004
3	.00001

Table of residues, i.e., of the differences of the left hand sides of (3.45) using the calculated values (3.51) minus the right hand sides.

[34] While a symmetric matrix is advantageous one has to accept the disadvantage that now every equation of (3.43) has a right hand side. There is little doubt that the solution scheme to be described immediately could have been applied to (3.45) directly.

[35] Cf. Whittaker and Watson, loc. cit., p. 221.

[36] This method is due to Mr. H. Schamp who carried out the numerical calculations.

Since these residues were found to be satisfactorily small, the thus obtained C_μ values were regarded as final.

In the actual numerical example the following values were used:

$$\beta = 1, \; w_\infty = \sqrt{2}c, \; M = \sqrt{2} \; . \qquad (3.48)$$

By taking

$$k = .316227; \; k^2 = .1, \; k' = .948681; \; k'^2 = .9, \qquad (3.49)$$
$$K = 1.6124; \; K' = 2.5781 \; ,$$

a one-parameter family of elliptic cones is selected which contains cones approximating the (circular) Mach cone down to an arrow wing of zero thickness for which

$$\operatorname{tg}\gamma = .316227; \; \gamma = 17°33'$$

γ being the Euclidean semi-flare angle (see Section 2.1). Finally we choose one particular cone of this family as material cone by taking

$$y_0 = 1.0383; \; \frac{\pi}{2}\,\frac{K' - y_0}{K} = 1.5 \; . \qquad (3.50)$$

Its flare is characterized by the ray along the top vertex

$$\beta X = k \operatorname{sd} y_0 . Z$$

which is now[37]

$$X = .3733Z \; ,$$

and by the ray along the side

$$\beta X = k \operatorname{sd} y_0 . Z$$

which is now

$$X = .4748Z$$

[37] Elliptic functions were computed by means of the tables: L. M. Milne-Thomson, Die Elliptischen Funktionen von Jacobi, Berlin, 1931.

The two semi-flare angles which belong to these rays are 20°28' and 25°24' respectively. The coefficients of the linear equation system (3.43) or of the systems (3.45)(3.46) derived therefrom are given in Tables II and III. The solution by means of the method just described is:

$$\begin{aligned} C_0 &= .27322 \,, \\ C_1 &= .0073450 \,, \\ C_2 &= .000051826 \,, \\ C_3 &= .000012955 \,. \end{aligned} \qquad (3.51)$$

Each of these coefficients must yet be multiplied by w_∞ or $\sqrt{2}C$. These, then, are the Fourier coefficients of the velocity component w of the first formula (3.28).

Since C_1 is only about .04 of C_o a good approximation would be to put $w = C_o\,(K' + y)$ and to calculate u and v from (3.25). This is not very surprising since the elliptic cone chosen for the present numerical test does not possess a very marked eccentricity. It is to be expected that for cones more closely approximating the arrow wing the convergence will be less rapid.

TABLE II

	$a^N_{n,\mu}$						
n \ μ =	0	1	2	3	4	5	Right Hand Side
0	1	$-2.20512 \cdot 10^{-1}$	$2.68682 \cdot 10^{-2}$	$-3.341 \cdot 10^{-3}$	$5.21 \cdot 10^{-4}$	$-5.68 \cdot 10^{-5}$	$.27159 \cdot W_\infty$
1	$-2.69455 \cdot 10^{-2}$	1	$-4.44729 \cdot 10^{-1}$	$5.9928 \cdot 10^{-2}$	$-8.0069 \cdot 10^{-3}$	$1.026 \cdot 10^{-3}$	0
2	$6.9167 \cdot 10^{-5}$	$-9.369 \cdot 10^{-3}$	1	$-3.56056 \cdot 10^{-1}$	$4.98228 \cdot 10^{-2}$	$-5.01606 \cdot 10^{-3}$	0
3	$-2.657 \cdot 10^{-7}$	$3.9003 \cdot 10^{-5}$	$-1.09996 \cdot 10^{-2}$	1	$-3.28693 \cdot 10^{-1}$	$4.60298 \cdot 10^{-2}$	0
4	$1.49 \cdot 10^{-9}$	$-1.8717 \cdot 10^{-7}$	$5.52823 \cdot 10^{-5}$	$-1.18037 \cdot 10^{-2}$	1	$-3.15110 \cdot 10^{-1}$	0
5	$-6.31 \cdot 10^{-12}$	$9.319 \cdot 10^{-10}$	$-2.16336 \cdot 10^{-7}$	$6.42697 \cdot 10^{-5}$	$-1.22498 \cdot 10^{-2}$	1	0
6					$6.99563 \cdot 10^{-5}$	$-1.25221 \cdot 10^{-2}$	0

Table of the matrix $a^N_{n,\mu}$ and of the right hand sides of (3.45).

TABLE III

	$A_{k\mu}$						
= k	$\mu = 0$	1	2	3	4	5	Right Hand Side Divided by w_∞
0	1.00073	- .247459	.0389208	- .004980	.000740	- .0000848	- .271591
1		1.04871	-.46258	.064040	-.008601	.001087	- .059889
2			1.19862	- .393798	.057068	- .0059978	.0072972
3				1.13052	-.35867	.051661	-.00090739
4					1.11070	-.34275	.00014150
5						1.10160	-.000015424

Table of the symmetrized matrix $A_{k\mu}$ and of the right hand sides of (3.46).

APPENDIX TO PART III

Alternative Expressions for $S_{\lambda\mu}$

The series $S_{\lambda\mu}$ and $\Sigma_{\lambda\mu}$ are of prime importance for the calculation of the coefficient matrix $a_{n\mu}$ of (3.43). For rapid calculation and for high values of the indices it is sometimes convenient to use formulae different from (3.38). The following developments are carried out on the series $S_{\lambda\mu}$ only. Completely analogous formulae - suppressed here may be derived for $\Sigma_{\lambda\mu}$. We may write $S_{\lambda\mu}$ using (3.33):

$$S_{\lambda\mu} = \frac{1}{2}\sum_{-\infty}^{+\infty}{}_{\nu}(\alpha_{\nu-\lambda} + \alpha_{\nu+\lambda})(\alpha_{\nu-\mu} + \alpha_{\nu+\mu}) \frac{\gamma_\nu}{2\nu+1}, \qquad (3.52)$$

with

$$\gamma_\nu = \text{sh } (2\nu+1)\eta \;,$$

where η is used as abbreviation for $(\pi/2K)(K' + y_0)$. Let, in this appendix, a prime stand for the derivative with respect to η.

Then

$$\begin{aligned} S'_{\lambda\mu} &= \frac{1}{2}\sum (\alpha_{\nu-\lambda} + \alpha_{\nu+\lambda})(\alpha_{\nu-\mu} + \alpha_{\nu+\mu})\, e^{(2\nu+1)\eta} \\ &= \frac{1}{2} e^{2\lambda\eta} \sum \alpha_{\nu-\lambda} (\alpha_{\nu-\mu} + \alpha_{\nu+\mu})\, e^{[2(\nu-\lambda)+1]\eta} \\ &+ \frac{1}{2} e^{-2\lambda\eta} \sum \alpha_{\nu+\lambda} (\alpha_{\nu-\mu} + \alpha_{\nu+\mu})\, e^{[2(\nu+\lambda)+1]\eta} \\ &= \frac{1}{2} e^{2\lambda\eta} \sum \alpha_\sigma (\alpha_{\sigma+\lambda-\mu} + \alpha_{\sigma+\lambda+\mu})\, e^{(2\sigma+1)\eta} \\ &+ \frac{1}{2} e^{-2\lambda\eta} \sum \alpha_\sigma (\alpha_{\sigma-\lambda-\mu} + \alpha_{\sigma-\lambda+\mu})\, e^{(2\sigma+1)\eta} \end{aligned}$$

Let us use the abbreviation

$$\begin{aligned} T_K &= \sum \alpha_{\sigma-K}\,\alpha_\sigma\, e^{(2\sigma+1)\eta} \\ &= e^{2K\eta} \sum \alpha_{\sigma-K}\,\alpha_\sigma\, e^{-(2\sigma+1)\eta} \;. \end{aligned}$$

Therefore

$$T_K\,(1 - e^{-2K\eta}) = 2\sum \alpha_{\sigma-K}\,\alpha_{\sigma}\,\mathrm{ch}(2\sigma+1)\eta$$

$$= S'_{K,0}$$

Going back to the more general S'_λ we have

$$S'_{\lambda\mu} = \frac{1}{2}\,e^{2\lambda\eta}\,(T_{\mu-\lambda} + T_{-\mu-\lambda})$$

$$+\frac{1}{2}\,e^{-2\lambda\eta}\,(T_{\lambda+\mu} + T_{\lambda-\mu})\,,$$

for which one may write:

$$S'_{\lambda,\mu} = \frac{1}{2}\left\{\frac{\mathrm{ch}(\mu+\lambda)}{\mathrm{ch}(\mu-\lambda)}\,S'_{\mu-\lambda,0} + \frac{\mathrm{ch}(\mu-\lambda)}{\mathrm{ch}(\mu+\lambda)}\,S'_{\mu+\lambda,0}\right\} \qquad (3.53)$$

This identity shows that it is merely necessary to have a formula for $S'_{n,0}$. Such a formula will now be derived. From (3.33) it is seen that

$$\mathrm{cd}(z+iy_1) = \sum \alpha_n\,e^{-(2n+1)\frac{\pi}{2}\frac{y_1}{K}}\,e^{i(2n+1)\frac{\pi}{2}\frac{z}{K}}$$

where y_1 is some real parameter small enough not to destroy convergence. Multiplying this series and (3.33) we get:

$$\mathrm{cd}z\cdot\mathrm{cd}(z+iy_1) = \sum_\lambda\sum_\mu \alpha_{\mu-\lambda}\,\alpha_\mu\,e^{-(2\mu+1)\frac{\pi}{2}\frac{y_1}{K}}e^{i\lambda\pi\frac{z}{K}}\,,$$

and therefore

$$\mathrm{cd}z\quad \mathrm{cd}(z+iy_1) + \mathrm{cd}(z-iy_1)\Big\} =$$

$$= \sum_\lambda\sum_\mu (\alpha_{\mu-\lambda}\,\alpha_\mu + \alpha_{\mu+\lambda}\,\alpha_\mu)\,\mathrm{ch}(2\mu+1)\frac{y_1}{K}\,e^{i\frac{\lambda\pi z}{K}}$$

$$= \sum_\lambda S'_{\lambda,0}\,e^{i\frac{\lambda\pi z}{K}} \qquad (3.54)$$

The function on the left is therefore a generating function for the $S'_{\lambda,0}$. To obtain an expression for $S'_{\lambda,0}$ in closed form we merely have to develop the elliptic function on the left in a Fourier series. This is done by the standard procedure[38] of integrating around the period rectangle and taking residues at the poles $z = K + iK' \pm iy_1$. The result is

$$S'_{\lambda,0} = (-1)^{\lambda} \frac{\pi}{Kk^2} \, cs(y_1;k') \, \frac{sh \frac{\lambda\pi y_1}{K}}{sh \frac{\lambda\pi K'}{K}} \quad (3.55)$$

From this $S_{\lambda,0}$ may be obtained by integration with respect to y. From (3.55) and (3.53) one may readily extrapolate the S sums for large values of the indices.

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38) cf. Whittaker and Watson, loc. cit., p. 510.

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