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ASYMPTOTIC THEORY OF DIFFRACTION

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ABSTRACT

Given a smooth, convex conducting body of revolution with a plane electromagnetic wave propagating in the direction of the axis of revolution, the problem considered is that of finding an expression, valid for small values of wavelength, which describes the currents in the vicinity of the caustic in the shaded region of the surface.

The problem is formulated in terms of an integral equation obtainable from a three-dimensional Green's function. The integration with respect to the azimuthal variable is carried out by two different schemes and the results discussed in relation to one another. The remaining integration, which is over a geodesic path, defines an integral equation which possesses a singular kernel. This singular equation is then studied in conjunction with a bounded kernel.

The body of revolution under consideration to this point is then specialized to the case of the sphere in order to compare the theory with known results, and some of the physical implications of the theory are discussed.

FOREWORD

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CHAPTER I

INTRODUCTION

1.1 Brief Survey

If an electromagnetic wave exists in space and then an obstacle is introduced into that space, changes occur in the electromagnetic field. In the study of this phenomenon it has been found that if a complete description of the currents induced on the obstacle by the original or incident electromagnetic wave can be obtained, then the induced currents may be used as a source distribution which generates a scattered field. The field resulting from the incident field and the obstacle may then be completely described in terms of the incident field and the scattered field. In this sense it may be remarked that the problem of properly describing the currents on the surface of the obstacle is of fundamental importance and may be designated as the diffraction problem.

The problem of adequately describing the surface currents, even in the case of an obstacle for which an exact solution to the problem can be found, is very difficult. Many restrictive assumptions must be made before any detailed study may be undertaken. Therefore, the following set of assumptions is not to be considered as exhaustive, but is to be interpreted as limiting the class of problems to be studied in order to establish a starting point.

Consider the case where a plane electromagnetic wave impinges upon a smooth (i.e. no edges), convex, conducting body of finite volume. Further require that the dimensions of the body be large when compared to the wavelength of the incident wave. The finite volume restriction may be removed in the case where one wishes to consider the two dimensional theory of diffraction.

Under these assumptions it is found that a shadow is formed, and that even for the case of the large sphere, knowing the exact solution is of little value because of the slow convergence of the Mie series.

The first effective solution of this shadow problem was completed by Watson (1918, 1919) when he was able to transform the series solution into a residue series which converged rapidly. Much later Fock (1946) and Franz (1954) were able to generalize the theory to include convex bodies other than the circle and the sphere although a number of questions in connection with the residue series were left unanswered. In this regard Goodrich and Kazarinoff (1963) discussed convergence of the residue series and Ursell (1968) concerned himself with the problem of exponentially large terms appearing in the application of the Watson transform for a cross-section other than the circle. Goodrich (1959) extended the work of Fock and was able to penetrate more deeply into the shadow zone with the applicability of the solution.

Franz and Depperman (1952, 1954) examined both the problem of the circle and the sphere by making use of the integral equation of diffraction theory as set forth by Maue (1949) and by decomposing the problem into a geometric optics wave and a creeping wave. From this stage of development of the theory until the present time many different authors have coped with several aspects of the diffraction problem. Because the viewpoint of this work is the integral equation approach to the diffraction phenomena, the remainder of this short introduction will be confined to work which deals with this approach.

Cullen (1958) began with the same integral equation that Fock used in one of his papers and was able to solve the equation directly. Hong (1966) also studied the surface waves for the scalar and vector incident waves.

In all of the work mentioned to this point it is implicit in the analysis that the results are not valid if the creeping waves converge. If the creeping waves do converge to a point, a combination of incoming and outgoing waves may be continued to a solution which remains finite at this point caustic. This process was investigated by Franz and Depperman (op cit), Goodrich, Harrison, Kleinman and Senior (1961) and is also discussed by Hönl, Maue and Westpfahl (1961).

The purpose of this research is to investigate the surface waves in the neighborhood of the caustic and to investigate the feasibility of attempting to find terms other than the leading term in this neighborhood.

1.2 Preliminary Remarks

By using the standard Green's function techniques, the exact integral equation which relates the surface currents to the incident field may be derived. If the current and the magnetic field quantities are represented by the complex vectors $\bar{J}$ and $\bar{H}$ respectively, the resulting equation is (Hönl et al):

$$\frac{1}{2}\bar{J}(\bar{r}) + \oint_S \bar{n} \times \left\{\bar{J}(\bar{r}) \times \nabla G(\bar{r} - \bar{r}')\right\} dS' = \bar{n} \times \bar{H}(\bar{r}) \tag{1.2.1}$$

where $\bar{r}$, $\bar{r}'$ are vectors measuring the distance from the origin to the observation and source points respectively, $\bar{n}$ is the outward unit vector normal to the surface, and the integration is to be carried out over the entire surface. Making use of the three dimensional Green's function, one may write

$$\nabla G(\bar{r}-\bar{r}') = \frac{(\bar{r}-\bar{r}')(ik|\bar{r}-\bar{r}'| - 1)}{4\pi|\bar{r}-\bar{r}'|^3} e^{ik|\bar{r}-\bar{r}'|} \quad . \tag{1.2.2}$$

It is also to be noted that in this work the time factor $e^{-i\omega t}$ has been suppressed. This is as far as one can proceed without becoming more specific as to the nature of the surface under consideration.

The purpose of the next chapter is then two-fold, firstly to adequately describe the surface and secondly to rewrite the problem in terms of that description. The reduction to geodesic coordinates (Chapter II) and the stationary phase argument (Chapter III) follow the work of Hong with minor exceptions.

CHAPTER II

THE DIFFRACTING SURFACE

2.1 The Geodesic Coordinate System

For a suitably smooth surface the length element can always be reduced to geodesic coordinates such that

$$ds^2 = du^2 + G\,(u,v)\;dv^2 \quad . \tag{2.1.1}$$

Further if one picks a point on the surface and considers all of the geodesic lines emanating from that point (these are the u-lines or the v = constant lines) and constructs the orthogonal trajectories of the geodesic lines, this defines a geodesic polar coordinate system and it is then allowable to write

$$\sqrt{G\,(0,v)} = 0, \quad \left[\frac{\partial}{\partial u}\sqrt{G\,(u,v)}\right]_{u=0} = 1 \quad . \tag{2.1.2}$$

$\sqrt{G}$ may be expanded into a Taylor series in terms of u, and if this is done the result is

$$\sqrt{G\,(u,v)} = u - \frac{1}{6}\,K\,(0,v)\,u^3 + R\,(u,v) \tag{2.1.3}$$

where $K\,(0,v)$ is the Gaussian curvature evaluated at the origin of the coordinate system, and $R\,(u,v)$ is of order $n > 3$ in u.

If $\sqrt{G\,(u,v)} = \sqrt{G(u)}$ (i.e. the torsion of the geodesic is zero), the geodesic curve is limited to a plane curve, and hence the surfaces under consideration must be limited to surfaces of revolution. This results in the simplification that if $\bar{r}$ is a position vector then

$$\frac{\partial \bar{r}}{\partial v} = \bar{b}\,. \tag{2.1.4}$$

That is to say that the binormal vector is tangent to the surface if the surface is a surface of revolution.

Also,

$$\frac{\partial \bar{r}}{\partial u} = \bar{t} \ . \tag{2.1.5}$$

The higher order derivatives of $\bar{r}$ can then be given in terms of the derivatives of $\bar{t}$, $\bar{n}$, and $\bar{b}$ where $\bar{t}$, $\bar{n}$, and $\frac{\bar{b}}{\sqrt{G}}$ are the unit tangent, normal, and binormal vectors respectively. The derivatives are given by the Gauss-Weingarten equations:

$$\begin{aligned} \frac{\partial \bar{t}}{\partial u} &= -\kappa_g \bar{n} \ , \\ \frac{\partial \bar{t}}{\partial v} &= \frac{\partial \bar{b}}{\partial u} = \kappa_{tt} \bar{b} \ , \\ \frac{\partial \bar{b}}{\partial v} &= -G\left[\kappa_{tt} \bar{t} + \kappa_{tn} \bar{n}\right] , \\ \frac{\partial \bar{n}}{\partial u} &= \kappa_g \bar{t} \ , \\ \frac{\partial \bar{n}}{\partial v} &= \kappa_{tn} \bar{b} \end{aligned} \tag{2.1.6}$$

where κ_g is the curvature of the geodesic, κ_{tt} and κ_{tn} are the tangential and normal components of the curvature of the u = constant curves.

The Gaussian (or total) curvature is given by

$$K = \kappa_g \kappa_{tn} = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2} \ , \tag{2.1.7}$$

and

$$\kappa_{tt} = \frac{1}{2G} \frac{\partial G}{\partial u} \ . \tag{2.1.8}$$

In addition to these equations there are conditions that must be imposed in order to satisfactorily define the mixed partial derivatives.

Among these conditions are the general results known as the Codazi condition which are given by:

$$\frac{\partial\kappa_{tn}}{\partial u} = \kappa_{tt}(\kappa_g - \kappa_{tn}) \quad ,$$

$$\frac{\partial\kappa_g}{\partial v} = 0 \quad . \tag{2.1.9}$$

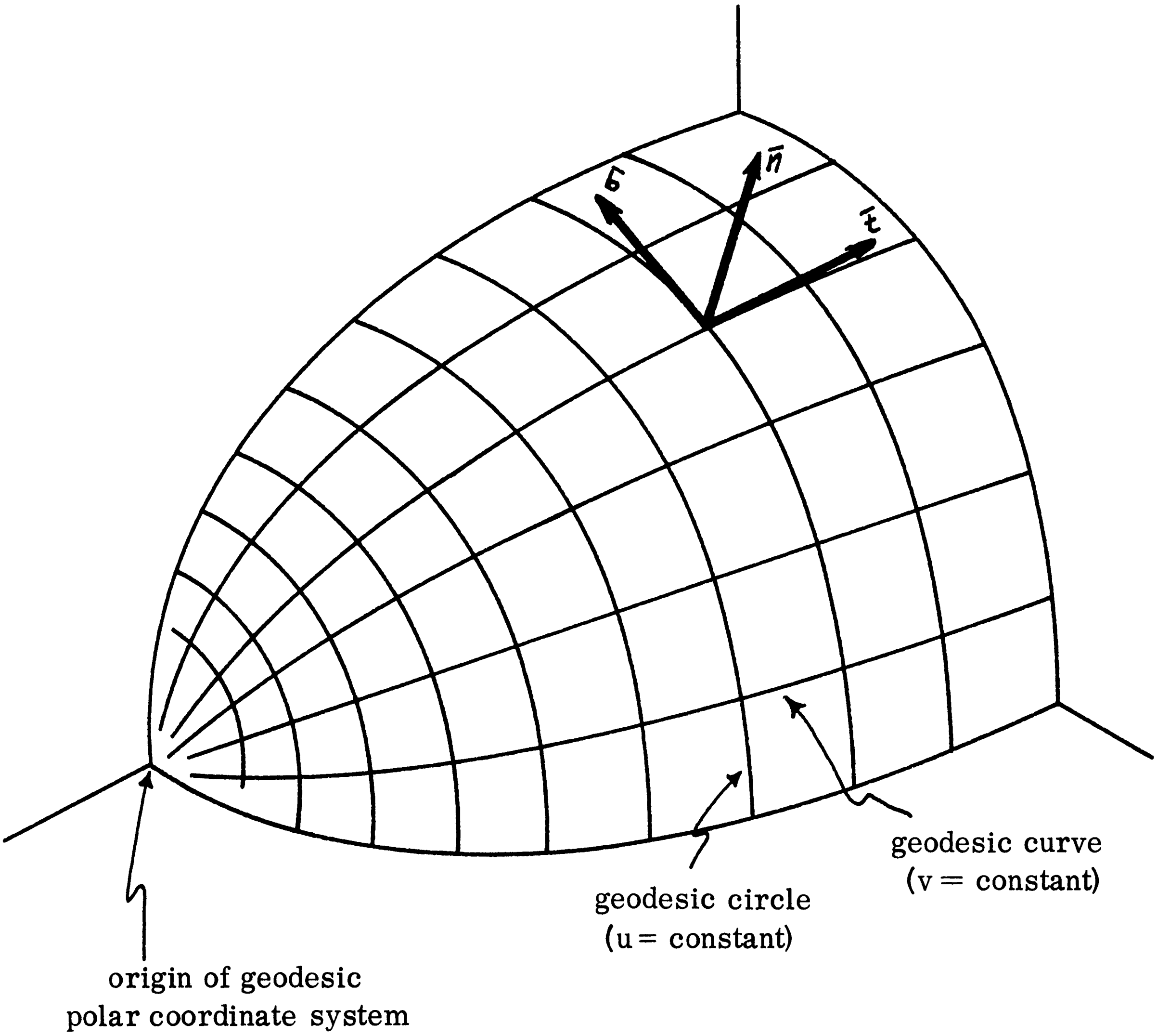

FIG. 1: SECTION OF A SURFACE OF REVOLUTION ILLUSTRATING THE NORMAL, TANGENT, AND BINORMAL VECTORS.

2.2 The Distance Between Two Points on the Surface

The position vector used in the last section is sufficient to define the geodesic coordinates associated with an arbitrary smooth body, but in the work which follows it is necessary to define the distance between two points on the surface in terms of the geodesic coordinates. In order to accomplish this, define $\bar{R}$ by:

$$\bar{R} = \bar{r}(u', v') - \bar{r}(u, v) \ , \tag{2.2.1}$$

and make use of (2.1.4), (2.1.5) and (2.1.6) in writing a Taylor series expansion for $\bar{R}$ about the point (u, v). The result of this expansion is

$$\bar{R} = R_t \bar{t} + R_b \bar{b} + R_n \bar{n} \tag{2.2.2}$$

where the components are given by:

$$R_t = (u'-u) - \frac{1}{6}\kappa_g^2 (u'-u)^3 - \frac{1}{8}\kappa_g \dot{\kappa}_g (u'-u)^4 - \frac{1}{120}(-4\kappa_g \ddot{\kappa}_g - 3\dot{\kappa}_g^2 + \kappa_g^4) \cdot (u'-u)^5 + \ldots - \frac{1}{2} G\kappa_{tt}(v'-v)^2 - \frac{1}{2} G\kappa_{tt}^2 (u'-u)(v'-v)^2 + \{\text{even terms in } (v'-v)\} \tag{2.2.3}$$

$$\frac{R_b}{\sqrt{G}} = (v'-v) + \kappa_{tt}(u'-u)(v'-v) - \frac{1}{2}\kappa_g \kappa_{tn}(u'-u)^2(v'-v) - \frac{1}{6} G\kappa_t^2 (v'-v)^3 + \{\text{odd terms in } (v'-v)\} \tag{2.2.4}$$

$$R_n = -\frac{1}{2}\kappa_g (u'-u)^2 - \frac{1}{6}\dot{\kappa}_g (u'-u)^3 + \ldots + \left[-\frac{1}{2} G\kappa_{tn}(v'-v)^2 - \frac{1}{2} G\kappa_{tt}\kappa_{tn}(u'-u)(v'-v)^2 + \{\text{even terms in } (v'-v)\}\right] . \tag{2.2.5}$$

In the above equations (and in the following) the metric and various curvatures appearing without argument are to be interpreted as the value of the function at the point (u, v). The dots appearing above some functions signify the derivative of that function with respect to u.

The distance between two points on the diffracting surface may also be written

$$R = \left| \bar{r} - \bar{r}' \right|$$

$$= \left[R_z^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi' - \phi) \right]^{1/2}$$

$$= \left[R_o^2 + 4\,\rho\rho' \sin^2 \frac{\phi' - \phi}{2} \right]^{1/2} \tag{2.2.6}$$

where R_z is the component of the distance which is parallel to the axis of rotation of the diffracting surface, (ρ', ϕ') and (ρ, ϕ) are the polar coordinates (referred to the axis of symmetry) of the source and observation points respectively, and R_o is the distance between source and observation points when $\phi' = \phi$.

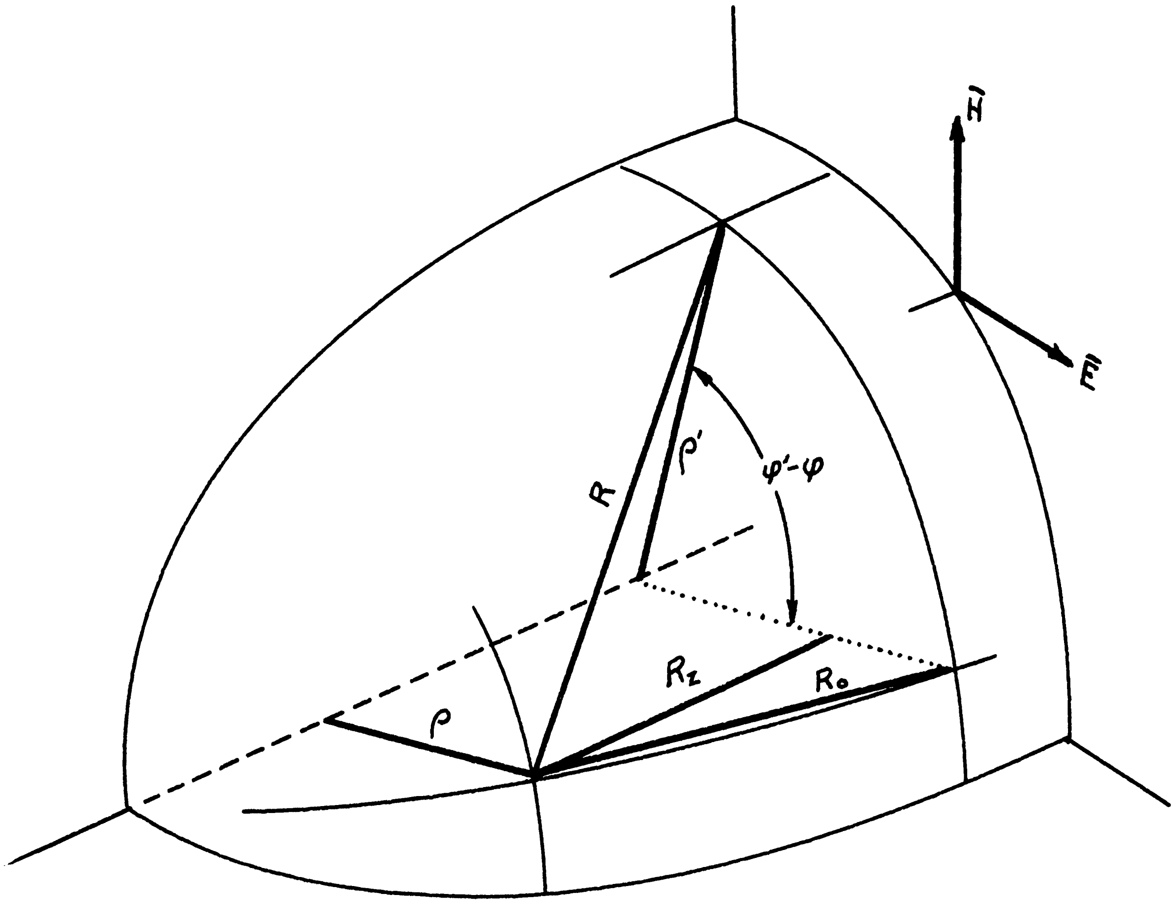

FIG. 2: ILLUSTRATION OF THE ALTERNATE DISTANCE PARAMETERS AND THE INCIDENT (VECTOR) PLANE WAVE.

2.3 Formulation of the Problem

The nonhomogeneity appearing in (1.2.1) contains an incident magnetic field vector which may be expressed by

$$\bar{H}(u,v) = \left[\sin\phi\, \bar{n}(\beta,v) - \cos\phi\, \bar{b}(\beta,v)\right] e^{-ik\bar{t}(\beta,v)\cdot\bar{r}(u,v)} \qquad (2.3.1)$$

where β is the u value assigned to the shadow boundary, and ϕ is the polarization angle of the incident field (refer to Fig. 2). It is convenient to normalize the radius of the circular cross-section of the body of revolution to the maximum radius which occurs; namely the radius at the shadow boundary. Under this assumption

$$\bar{H}(u,v) = \left[\sin v\, \bar{n}(\beta,v) - \cos v\, \bar{b}(\beta,v)\right] e^{-ik\bar{t}(\beta,v)\cdot\bar{r}(u,v)} \qquad (2.3.2)$$

This incident magnetic field may then be operated upon to produce the incident surface wave which will explicitly contain the factors $\sin v$ and $\cos v$ as listed below in (2.3.4).

This is as far as the present discussion of the incident field will go in this report, namely to observe the appearance of the factors $\sin v$ and $\cos v$ which will motivate some of the assumptions in the work to follow. For an asymptotic estimate of the incident field and its application to the penumbra region refer to the work of Hong (1966).

If now $\bar{J}(\bar{r})$ is written in the form

$$\bar{J}(\bar{r}) = \theta(\bar{r})\,\bar{t}(\bar{r}) + \phi(\bar{r})\,\bar{b}(\bar{r}), \qquad (2.3.3)$$

and substituted into the integral equation, the result is of the general form

$$\frac{1}{2}\left[\theta(\bar{r})\,\bar{t}(\bar{r}) + \phi(\bar{r})\,\bar{b}(\bar{r})\right] = \sin v\, J_b(\bar{r})\bar{b}(\bar{r}) - \cos v\, J_t(\bar{r})\,\bar{t}(\bar{r}) -$$

$$- \iint ds' \left[\theta(\bar{r}')\,\bar{t}(\bar{r}') + \phi(\bar{r}')\,\bar{b}(\bar{r}')\right] \bar{n}\cdot\nabla G(\bar{r}-\bar{r}') \,. \qquad (2.3.4)$$

Thus it is seen that if the unit vectors do not change rapidly as a function of $\bar{r}$ and if the main contribution to the integral comes from a neighborhood of the point (u, v) (i.e. $\bar{r} \simeq \bar{r}'$), then each of the component currents must satisfy the integral equation which also arises when an acoustically hard surface is subjected to a plane acoustic wave. For this reason the first problem to be studied will be the scalar problem under the assumption, at least for the present, that the only differences which show up in these problems are the conditions which must be satisfied at the shadow boundary.

By using the notation of the last section and (1.2.2), the scalar equation of interest may be written

$$\psi(u,v) = 2\psi_{inc}(\beta, v)\begin{pmatrix}1\\ \sin v\\ \cos v\end{pmatrix} - \frac{1}{2\pi}\iint dv'\,du'\,\sqrt{G(u')}\,\psi(u',v')$$

$$\cdot\ \frac{1-ikR}{R^3}\,\bar{n}(u',v')\cdot\bar{R}\,e^{ikR} \quad . \tag{2.3.5}$$

where

$$\bar{R} = \bar{r}(u',v') - \bar{r}(u,v) \quad . \tag{2.3.6}$$

The notation $\begin{pmatrix}1\\ \sin v\\ \cos v\end{pmatrix}$ means that any one of the three quantities may multiply the other factor. The "one" multiplier refers to the scalar problem and the two remaining factors indicate which component of the vector problem is under consideration.

Because the particular problem of interest in this paper is the description of the currents deep in the shadow region, it is assumed that the incident wave term in the integral equation will be deleted. That is to say that the main concern of this research will be the study of the homogeneous integral equation. This approach was taken by Franz and Depperman (1952) in their study of creeping waves.

In Cullen's paper it is stated that the homogeneous equation has no non-trivial solution because it would represent a free oscillation of current which would radiate energy and damp out.

Hönl et al (1961) consider the homogeneous integral equation approach to be a valid approximation (in the optical limit) as long as the integration is carried out only over the shaded region of the body, and as long as values are chosen for the constants appearing in the solutions so that a smooth transition is obtained as one moves into and through the penumbra region.

Goodrich (1969) is presently working on research that would indicate that it is possible that the concept of using the homogeneous equation and confining the integration to be over the shadow region only may be exact in the case of the circle and the sphere.

With these thoughts in mind the only touchstone used in this research will be the results obtained for the currents in the vicinity of the caustic.

CHAPTER III

REDUCTION OF THE TWO-FOLD EQUATION

3.1 Observations Concerning the Point of Stationary Phase

First consider the exponential factor contained in the integrand of (2.3.5) and notice that

$$\frac{\partial R}{\partial v'} = \frac{\partial}{\partial v'}\sqrt{\bar{R}\cdot\bar{R}} = \frac{1}{R}\left[R_t \frac{\partial R_t}{\partial v'} + R_b \frac{\partial R_b}{\partial v'} + R_n \frac{d R_n}{d v'}\right]$$

$$= \frac{G(v'-v)}{R}\left[1+\kappa_{tt}(u'-u)+\ldots\right] . \qquad (3.1.1)$$

This means that the integrand contains a point of stationary phase at the point $v' = v$. (assuming $\psi(u,v)$ does not vary rapidly as a function of v).

In the following work designate the difference $\bar{r}(u',v) - \bar{r}(u,v)$ by the symbol $\bar{R}_o$. If a stationary phase analysis is carried out for the integration with respect to v', the equation that remains will contain the factor $\exp(ikR_o)$. Notice that R_o may also be expressed by

$$R_o = \left[R_{to}^2 + R_{no}^2\right]^{1/2}$$

$$= |u'-u|\left(1 - \frac{\kappa_g^2}{12}(u'-u)^2 - \frac{\kappa_g\dot{\kappa}_g}{12}(u'-u)^3 + \ldots\right)^{1/2}$$

$$= |u'-u|\left(1 - \frac{\kappa_g^2}{24}(u'-u)^2 - \frac{\kappa_g\dot{\kappa}_g}{24}(u'-u)^3 + \ldots\right) \qquad (3.1.2)$$

This result indicates that the integration may be partitioned according as $u' < u$ or $u < u'$. Also it may be noticed that if the leading term in the above expansion for R_o is removed, the resulting expansion, namely

$$|u'-u|\left(-\frac{\kappa_g^2}{24}(u'-u)^2 - \frac{\kappa_g\kappa_g}{24}(u'-u)^3 + \ldots\right)$$

possesses a point of stationary phase at $u' = u$. This observation in turn indicates that the unknown should perhaps be written as the sum of an incoming and an outgoing surface wave. If this is substituted into the integral equation, the exponential terms that appear are given by

$$\exp\left\{ik\left(u - \frac{\kappa_g^2}{24}(u-u')^3\right)\right\}, \quad \exp\left\{ik\left(-u+2u' - \frac{\kappa_g^2}{24}(u'-u)^3\right)\right\},$$

$$\exp\left\{ik\left(u - 2u' - \frac{\kappa_g^2}{24}(u-u')^3\right)\right\}, \quad \text{and} \quad \exp\left\{ik\left(-u - \frac{\kappa_g^2}{24}(u'-u)^3\right)\right\}$$

where the first term corresponds to the outgoing wave for $u' < u$, the second one corresponds to the outgoing wave for $u' > u$, the third one corresponds to the incoming wave for $u' < u$, and the fourth and last one corresponds to the incoming wave when $u' > u$. The first and fourth exponentials listed above contain a point of stationary phase when $u' = u$. If, however, either the second or third exponentials have points of stationary phase, they must occur when R_o is a decreasing function of its argument. This is not possible if the variable of integration is restricted to the shadow region of a convex body. Therefore both of these integrations may be asymptotically neglected if the difference of u and u' is large enough that $k\,|u' - u| >> 1$. The maximum contribution to these these two integrals will then also appear in the neighborhood of $u' = u$. For these reasons the kernel of the equation may be approximated by its behavior near the point $u' = u$ in all of the work to follow.

The last observation of this section is to point out that neglecting the integrations which do not possess a point of stationary phase is tantamount to assuming that there is no coupling between incoming and outgoing waves for the high frequency case.

3.2 Development of the Asymptotic Integral Equation

In this section an asymptotic treatment of the v' integration appearing in (2.3.5) will be carried out. To accomplish this recall that if

$$z = f(\xi) = z_o + \sum_{n=1}^{\infty} A_n \xi^n , \; A_1 \neq 0 \qquad (3.2.1)$$

then

$$\xi(z) = \xi_o + \sum_{n=1}^{\infty} B_n z^n \qquad (3.2.2)$$

where B_n is defined, by way of residue theory and Cauchy's integral formula, by the following equation

$$nB_n = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{d\xi^{n-1}} \left[\frac{\xi}{f(\xi)} \right]^n \right\}_{\xi=0} . \qquad (3.2.3)$$

With this in mind examine the exponential factor and let

$$\frac{w^2}{(v'-v)^2} = \frac{R-R_o}{(v'-v)^2} = \sum_{n=0}^{\infty} A_n (v'-v)^n \; , \; A_o \neq 0 \; . \qquad (3.2.4)$$

If $f(\xi)$ is then chosen to be $\sqrt{\frac{R-R_o}{v'-v}}$, then

$$\frac{\xi}{f(\xi)} = \left[\sum_{n=0}^{\infty} A_n (v'-v)^{n-1} \right]^{-1/2} . \qquad (3.2.5)$$

Therefore, for the problem under consideration, the series of interest is given by

$$\frac{d(v'-v)}{dw} = \sum_{n=1}^{\infty} nB_n w^{n-1} \qquad (3.2.6)$$

where nB_n is calculated by placing (3.2.5) into (3.2.3) which yields the result

$$nB_n = \frac{1}{A_o^{n/2}\,(n-1)!}\left\{\frac{d^{n-1}}{d\xi^{n-1}}\left[\sum_{p=0}^{\infty}\frac{A_p}{A_o}\,\xi^{p-1}\right]^{-n/2}\right\}_{\xi=0} . \qquad (3.2.7)$$

For reference the first three B_n are listed:

$$A_o^{1/2}\,B_1 = 1$$

$$2A_o\,B_2 = -\frac{A_1}{A_o} \qquad (3.2.8)$$

$$6A_o^{3/2}\,B_3 = \frac{15}{4}\,\frac{A_1^2}{A_o^2} - 3\,\frac{A_2}{A_o} .$$

Next consider the factor

$$\psi(u',v')\,\frac{1-ikR}{R^3}\,\bar{n}\,(u',v')\cdot\bar{R} = \sum_{n=0}^{\infty} C_n\,(v'-v)^n . \qquad (3.2.9)$$

Since this factor is to be evaluated in the neighborhood of $u' = u$, the approximation $\bar{n}\,(u',v) \simeq -\bar{n}\,(u,v)$ may be used to simplify the results. The first few C_n are also listed for reference.

$$C_o = \psi(u',v)\,\frac{1-ikR_o}{R_o^3}\,\bar{n}\,(u',v)\cdot\bar{R}_o \simeq \psi(u',v)\,\frac{1-ikR_o}{2R_o}\,\kappa_g$$

$$C_1 = \frac{\partial\psi(u',v)}{\partial v}\,\frac{1-ikR_o}{2R_o}\,\kappa_g \qquad (3.2.10)$$

$$2C_2 \simeq \left[\frac{\partial^2\psi(u',v')}{\partial v'^2}\,\frac{1-ikR}{2R}\,\kappa_g + \psi(u',v')\,\frac{\partial}{\partial R}\,\frac{1-ikR}{R^3}\,\frac{\partial^2 R}{\partial v'^2}\,\bar{n}\cdot\bar{R} + \right.$$

$$\left. + \;\psi(u',v')\,\frac{1-ikR}{R^3}\,\frac{\partial 2}{\partial v'^2}\,(\bar{n}\cdot\bar{R})\right]_{v'=v} .$$

By making use of all of this information, a series in powers of w may be written as follows

$$\frac{d(v'-v)}{dw}\,\psi(u',v')\,\frac{1-ikR}{R^3}\,\bar{n}(u',v')\cdot\bar{R} = \sum_{n=0}^{\infty} a_n w^n . \tag{3.2.11}$$

The a_n may be evaluated from the series representations of the factors in a straightforward manner. The results for the first few a_n are listed (in terms of B_n and C_n) below in (3.2.14).

Under the substitution $R = R_o + w^2$ and the general result (3.2.11) the integral equation (2.3.5) becomes

$$\psi(u,0) = -\frac{1}{2\pi}\int du'\sqrt{G(u')}\, e^{ikR_o}\, a_o \int \sum_{n=0}^{\infty} \frac{a_n}{a_o}\, w^n e^{ikw^2}\, dw . \tag{3.2.12}$$

Under the transformation $ikw^2 = -t$

$$a_o \int \sum_{n=0}^{\infty} \frac{a_n}{a_o}\, w^n e^{ikw^2}\, dw \rightarrow \frac{a_o}{2}\sqrt{\frac{i}{k}} \sum_{n=0}^{\infty} \frac{a_n}{a_o}\, \frac{e^{in\frac{\pi}{4}}}{k^{\frac{n}{2}}} \int_0^{\infty} t^{\frac{n-1}{2}}\, e^{-t}\, dt$$

where $t = 0$ corresponds to the point of stationary phase and the stationary phase theory along with the theory of contour integration allow the integration to be carried out over the path indicated.

Using the fact that the above integration with respect to t defines the Gamma function of argument $\frac{n-1}{2} + 1$, (3.2.12) is transformed as follows:

$$\psi(u,0) = -\frac{1}{4}\sqrt{\frac{i}{\pi k}}\int du'\sqrt{G(u')}\, e^{ikR_o}\, a_o \sum_{n=0}^{\infty} \frac{a_n \Gamma\left(\frac{n-1}{2}+1\right) e^{in\frac{\pi}{4}}}{a_o \Gamma\left(\frac{1}{2}\right) k^{\frac{n}{2}}} . \tag{3.2.13}$$

Finally using (3.2.11), (3.2.6), and (3.2.9), (3.2.13) may be written in terms of B_n and C_n as follows:

$$\psi(u,0) = -\frac{1}{4}\sqrt{\frac{i}{\pi k}}\int du'\,\sqrt{G(u')}\;C_oB_1\,e^{ikR_o}\left[1+\left(2\,\frac{B_2}{B_1}+\frac{B_1C_1}{C_o}\right)\sqrt{\frac{i}{\pi k}}\,+\right.$$

$$\left.+\frac{1}{2}\left(3\,\frac{B_3}{B_1}+3\,\frac{B_2C_1}{C_o}+\frac{C_2B_1^2}{C_o}\right)\frac{e^{i\frac{\pi}{2}}}{k}+O\left(k^{-\frac{3}{2}}\right)\right]. \qquad (3.2.14)$$

This derivation is purely formal and until one becomes more precise about the limits of the v' integration and the corresponding limits of the w and t integrations the terms other than the leading term are apt to be overshadowed by the error introduced by arbitrarily extending the limits of integration. For the purpose of this report the first term is all that is required and therefore no analysis other than this will be undertaken. The only purpose of the section following this one is to complete, in some sense, the formalism established and it will be referred to only once in the work to follow.

3.3 Decomposition of the Integral Equation

In the case where the axially incident field is given to be scalar field it is recognized that for a body of revolution the surface field should not be a function of v. Therefore, in (3.2.10) all terms involving derivatives of $\psi(u,v)$ with respect to v may be dropped. It is then noticed that a factor $\psi(u,0)$ is common to each of the C_n listed in (3.2.10). In order to introduce the unknown explicitly, make the substitution $a_o = \psi(u)\,a_o'$ in (3.2.13) and $C_o = \psi(u)\,C_o'$ in (3.2.14).

The equation finally attains the general form

$$\psi(u) = \frac{\lambda}{\sqrt{k}}\int du'\,\psi(u')\,A(u,u')\left[1+\frac{B(u,u')}{\sqrt{k}}+\frac{C(u,u')}{k}+\dots\right]. \qquad (3.3.1)$$

where λ is taken to be a complex constant.

From this representation it is easily seen that if $\psi(u)$ is assumed to be of the form

$$\psi = \psi_o + \frac{\psi_1}{\sqrt{k}} + \frac{\psi_2}{k} + \ldots \tag{3.3.2}$$

that the following identification can be made:

$$\psi_o(u) = \frac{\lambda}{\sqrt{k}} \int du' \, \psi_o(u') \, A(u,u')$$

$$\psi_1(u) = \frac{\lambda}{\sqrt{k}} \left[\int du' \psi_1(u') \, A(u,u') + \int du' \psi_o(u') \, A(u,u') \, B(u,u') \right]$$

$$\psi_2(u) = \frac{\lambda}{\sqrt{k}} \left[\int du' \psi_2(u') \, A(u,u') + \int du' \psi_1(u') \, A(u,u') \, B(u,u') + \int du' \psi_o(u') \, A(u,u') \, C(u,u') \right]$$

$$\vdots \tag{3.3.3}$$

It is immediately noticed that if the homogeneous equation contained in the set (3.3.3) possesses only the trivial solution, that (3.3.1) necessarily possesses only the trivial solution. If (3.3.1) were cast as a Fredholm equation, it is very likely that there would be no solution for the conditions on the non-homogeneous terms which would be required by the alternative theorem would not normally be satisfied. If the problem were cast as a Volterra equation, and the homogeneous equation possesses a non-trivial solution, the procedure would be straightforward in that each of the non-homogeneous equations must be solved, and then the solution(s) of the homogeneous equation must be appended to that solution.

3.4 An Equation Defining the Outgoing Surface Wave

As indicated in 3.1 it will be assumed that no coupling exists between the incoming and the outgoing surface waves. The equation that governs the outgoing surface wave may then be written

$$\psi(u) = -\frac{1}{4}\sqrt{\frac{i}{\pi k}}\int_0^u du' \sqrt{G(u')}\,\psi(u')\,C_o'\,B_1\,e^{ikR_o} . \qquad (3.4.1)$$

The product $\psi(u')\,C_o'$ is given directly by (3.2.10); however, further work is needed to approximate the factor B_1.

First note that (3.2.8) indicates that

$$B_1 = \left[\frac{1}{2}\left.\frac{\partial^2 R}{\partial v^2}\right|_{v=0}\right]^{1/2} . \qquad (3.4.2)$$

By following the procedure established in (3.1.1) and applying (2.2.3), (2.2.4), and (2.2.5) an approximation may be obtained for B_1 for values of u' near u. The following procedure is an outline of the application of this plan.

$$\left.\frac{\partial^2 R}{\partial v^2}\right|_{v=0} = \frac{1}{R_o}\left[\left(\frac{\partial R_b}{\partial v}\right)^2 + R_t\frac{\partial^2 R_t}{\partial v^2} + R_n\frac{\partial^2 R_n}{\partial v^2}\right]_{v=0}$$

$$\simeq \frac{G}{R_o}\left[\left\{1+\kappa_{tt}(u'-u) - \frac{1}{2}\kappa_g\kappa_{tn}(u'-u)^2\right\}^2 - \right.$$

$$\left. - \left\{1+\kappa_{tt}(u'-u)\right\}\bar{R}_o\cdot\left\{\kappa_{tt}\,\bar{t} + \kappa_{tn}\,\bar{n}\right\}\right]$$

$$\simeq \frac{\sqrt{G(u)}}{R_o}\left(\sqrt{G(u)}\left[1+\kappa_{tt}(u'-u) - \frac{1}{2}\kappa_g\kappa_{tn}(u'-u)^2\right]\right) . \qquad (3.4.3)$$

In light of (2.1.7) and (2.1.8) it is noticed that the quantity contained in the parentheses is nothing more than the first few terms of the series expansion of $\sqrt{G(u')}$ about the point u. Therefore (3.4.3) may be approximated by

$$B_1 \simeq \sqrt{2R_o}\,\left[G(u')\,G(u)\right]^{-\frac{1}{4}} . \qquad (3.4.4)$$

If these approximations are substituted into (3.4.1) the integral equation becomes

$$\psi(u) = -\frac{1}{4}\sqrt{\frac{2i}{\pi k}}\,\kappa_g \int_0^u du' \left[\frac{G(u')}{G(u)}\right]^{\frac{1}{4}} \psi(u') \frac{1-ikR_o}{\sqrt{R_o}} e^{ikR_o} . \quad (3.4.5)$$

This equation may be written

$$\psi(u) = -\sqrt{\frac{i}{8\pi\kappa}}\,\kappa_g \int_0^u d\tau\, \frac{1-ik\tau}{\sqrt{\tau}}\, e^{ik\tau - ik\frac{\kappa_g^2}{24}\tau^3}\, \psi(u-\tau) \quad (3.4.6)$$

under the assumption that $\psi(u)$ contains a factor e^{iku} (i.e. that it is outgoing) and the transformation $u - u' = \tau$. It is to be observed that the solution of (3.4.6) must be multiplied by $G^{-1/4}(u)$ in order to be a solution of (3.4.5).

3.5 An Equation Defining the Incoming Surface Wave

In the development of (3.4.6) no approximations concerning the kernel were made that would be changed by considering $u' > u$. The only differences which must be accounted for are the region of integration, and the approximation of R_o according to (3.1.2). The equation of interest may then be written directly, and is

$$\psi(u) = -\sqrt{\frac{i}{8\pi k}}\,\kappa_g \int_0^{\beta-u} d\tau\, \frac{1-ik\tau}{\sqrt{\tau}}\, e^{ik\tau - ik\frac{\kappa_g^2}{24}\tau^3}\, \psi(u+\tau) \quad (3.5.1)$$

under the assumption that $\psi(u)$ contains a factor e^{-iku} and the transformation $\tau = u' - u$. Again, the solution of (3.5.1) must be multiplied by $G^{-1/4}(u)$ to be a solution of the original equation which represents the surface field. The value $u = \beta$ represents the shadow boundary.

In both equations it is seen that the point of stationary phase may be associated with a "source" and/or "sink" point, and that the field at an observation point depends upon the distance (along the surface) from the "source" point to the extent that the integration is to take place over that distance interval. The upper limits of integration may, in some cases, be extended under the assumption that this will not appreciably change the result of the integration. This is, for instance, the approach taken by Hönl et al (1961).

Designate by $\psi^{(1,2)}(u)$ the solutions of (3.4.6) and (3.5.1) respectively. Then by associating the upper sign with the one, and the lower sign with the two, both equations may be summarized by

$$\psi^{(1,2)}(u) = -\sqrt{\frac{i}{8\pi k}}\ \kappa_g \int_0^{\infty} d\tau \frac{1-ik\tau}{\sqrt{\tau}}\ e^{ik\tau - ik\frac{\kappa_g^2}{24}\tau^3}\ \psi^{(1,2)}(u \mp \tau). \tag{3.5.2}$$

3.6 Remarks Concerning the Vector Problem

If only the first term of (3.2.14) is considered and it is assumed that the unknown may be written in terms of a product of a function of v only with a function of u only, it is evident that any choice for a function of v will satisfy the requirements in the shadow region. If it is assumed that the first term of (3.2.14) applies up to the shadow boundary the nonhomogeneous term from (2.3.5) may be used to limit the class of acceptable choices. Again this assumption is of a formal nature and to justify its use one should supply an answer to the question: "In what regions of the surface is it valid to apply saddle point integration techniques?"

If the nonhomogeneous term in (2.3.5) is assumed to have a factor cos v it is readily apparent that if the unknown is assumed to possess a factor of cos v, then this factor is common to all terms. That is to say that it is an acceptable choice for the function of v in the solution. Similar remarks apply for the case where the nonhomogeneity contains a factor sin v.

The generalization formed by taking a linear combination of sin v and cos v does not readily yield any information from either (2.3.4) or (2.3.5). In fact when viewed in these equations any attempt at generalization does not seem appropriate or necessary.

The conclusions are that each of the components of the vector problem, when viewed as functions of u, must satisfy the scalar equation. However, to be properly identified, they must be multiplied by the appropriate polarization factor, cos v or sin v, before being interpreted as the currents induced by an incident electromagnetic plane wave. In light of this discussion the factors $\sin v$ and/or $\cos v$ may be dropped from the discussion and reintroduced at will without changing any results.

CHAPTER IV

AN ALTERNATE REDUCTION OF THE TWO-FOLD EQUATION

4.1 Alternate Form of the Kernel

By making use of the alternate form for describing the distance between two points on the surface, the integration of the three-dimensional Green's function, e^{ikR}/R, may be carried out as follows:

$$\int_{\phi-\pi}^{\phi+\pi} d\phi' \frac{e^{ikR}}{R} = \int_{\phi-\pi}^{\phi+\pi} d\phi' \frac{e^{ik\sqrt{R_o^2 + 4\rho\rho' \sin^2 \frac{\phi'-\phi}{2}}}}{\sqrt{R_o^2 + 4\rho\rho' \sin^2 \frac{\phi'-\phi}{2}}}$$

$$= \frac{1}{\sqrt{\rho\rho'}} \int_{-1}^{1} \frac{dz}{\sqrt{1-z^2}} \frac{e^{i2k\sqrt{\rho\rho'}\sqrt{\frac{R_o^2}{4\rho\rho'} + z^2}}}{\sqrt{\frac{R_o^2}{4\rho\rho'} + z^2}}$$

$$= \frac{1}{\sqrt{\rho\rho'}} \int_{-1^+}^{1^-} dz \left(1 + \frac{z^2}{2} + \frac{3}{8} z^4 + \ldots\right) \frac{e^{i2k\sqrt{\rho\rho'}\sqrt{\frac{R_o^2}{4\rho\rho'} + z^2}}}{\sqrt{\frac{R_o^2}{4\rho\rho'} + z^2}} . \qquad (4.1.1)$$

Consider for a moment the following integration by parts:

$$\int_a^b dz \frac{z^{2n} e^{ik\sqrt{x^2+z^2}}}{\sqrt{x^2+z^2}} = \frac{z^{2n-1}}{ik} e^{ik\sqrt{x^2+z^2}} \Bigg|_a^b -$$

$$- \frac{2n-1}{ik} \int_a^b dz \; z^{2(n-1)} e^{ik\sqrt{x^2+z^2}} . \qquad (4.1.2)$$

If this result is applied to the preceeding equation with $a = -1^+$, $b = 1^-$, and $n > 0$ it is seen that, from the second term on, the integration is $O(k^{-1})$. If the limits of the first integration are extended to include all of the real axis, the error introduced is twice the above integration by parts with $a = 1$, $b \to \infty$ and $n = 0$ and is therefore also $O(k^{-1})$.

The same results could have been obtained in the original form of the integral by recognizing the point of stationary phase at $\phi' = \phi$ (the points $-\pi, \pi$ are not candidates as points of stationary phase because they do not lie in the open interval under consideration), replacing $\sin\frac{\phi'-\phi}{2}$ by the first term in its series expansion and extending the limits of integration.

Then, using the equation (Magnus and Oberhettinger, 1949, p. 27):

$$H_o^{(1)}(kx) = \frac{-i}{\pi}\int_{-\infty}^{\infty} dt\, \frac{e^{ik\sqrt{x^2+t^2}}}{\sqrt{x^2+t^2}} \quad (k, x \text{ real and positive}) \qquad (4.1.3)$$

it is seen that

$$\int_{\phi-\pi}^{\phi+\pi} d\phi'\, \frac{e^{ikr}}{R} = \frac{1}{\sqrt{\rho\rho'}}\left[i\pi\, H_o^{(1)}(kR_o) + O(k^{-1})\right] . \qquad (4.1.4)$$

The three dimensional Green's function for the Neumann boundary condition may be written in the operator notation

$$\frac{1-ikR}{R}\, e^{ikR} = \left(1 - k\frac{d}{dk}\right)\frac{e^{ikR}}{R} . \qquad (4.1.5)$$

If, in the Neumann problem, the integration with respect to the azimuthal variable and the above operator are interchanged the result is

$$\left(1-k\frac{d}{dk}\right)\int_{\phi-\pi}^{\phi+\pi} d\phi'\, \frac{e^{ikR}}{R} = \left(1-k\frac{d}{dk}\right)\frac{i\pi}{\sqrt{\rho\rho'}}\left[H_o^{(1)}(kR_o) + O(k^{-1})\right]$$

$$= \frac{i\pi}{\sqrt{\rho\rho'}}\left[kR_o H_1^{(1)}(kR_o) + O(1)\right] . \qquad (4.1.6)$$

By forgetting for a moment the factor multiplying the brackets of this equation it is noticed that

$$kR_o H_1^{(1)}(kR_o) \sim \sqrt{\frac{2kR_o}{\pi}}\, e^{i\left(kR_o - \frac{3\pi}{4}\right)} \qquad (4.1.7)$$

for large values of the product kR_o. If this is compared with the corresponding part of the kernel in (3.4.5), assuming in that equation that $kR_o >> 1$, it is observed that the functional form in both representations is the same.

4.2 An Error Analysis

A question may be asked at this point. An assumption that the unknown contained an exponential factor moves the point of stationary phase from some non-zero point (the point $u'-u = (2n+1)\pi$ in the case of the sphere) to the point $u'-u = 0$. At this point $R_o = 0$. What error is introduced into the problem by using the asymptotic representation of the Hankel function?

Consider, for a moment, the integration

$$\int_0^\infty kR_o H_1^{(1)}(kR_o)\, \psi(u')\, du' = \int_0^{u-\frac{N}{k}} kR_o H_1^{(1)}(kR_o)\, \psi(u')\, du' \; +$$

$$+ \int_{u-\frac{N}{k}}^{u+\frac{N}{k}} kR_o H_1^{(1)}(kR_o)\, \psi(u')\, du' \; +$$

$$+ \int_{u+\frac{N}{k}}^{\infty} kR_o H_1^{(1)}(kR_o)\, \psi(u')\, du' \qquad (4.2.1)$$

where it is to be assumed that N is large enough that it is reasonable to replace $kR_o H_1^{(1)}(kR_o)$ by its asymptotic representation in the first and third integrations on the right-hand side of (4.2.1). The second integration may be bounded as follows

$$\left| \int_{u-\frac{N}{k}}^{u+\frac{N}{k}} kR_o H_1^{(1)}(kR_o)\, \psi(u')\, du' \right| \leq \frac{\Psi}{k} \left| \int_{u-\frac{N}{k}}^{u+\frac{N}{k}} k(u-u')\, H_1^{(1)}\big(k(u-u')\big)\, dku' \right|$$

$$= \frac{\Psi}{k} \left| \int_{-N}^{N} z \frac{d}{dz} H_o^{(1)}(z)\, dz \right| \qquad (4.2.2)$$

where it has been assumed that $\psi(u)$ is bounded and that Ψ is the maximum value of $\psi(u)$ in the interval under consideration. Ψ may include k to some power, in which case (4.2.2) simply demonstrates that the contribution of the middle integral is of the order of the solution generated plus one in inverse powers of k. Therefore, to this order the solutions generated by the asymptotic approximation of the Hankel functions are valid, and as $k \to \infty$ may be written as integrations from 0 to u and from u to ∞ as recorded previously in Chapter Three.

4.3 Development of the Alternate Equations

The kernel of the equation, namely $kR_o H_1^{(1)}(kR_o)$, has only one extreme value when the kernel is assumed to be a function of a real (non-negative) variable and this occurs when $R_o = 0$. Because this point is also the stationary phase point when considering the shadow region (away from the caustic) it seems reasonable to assume that the major contribution to the integration will come from a neighborhood of this extreme value, and that the contribution gained from allowing u' to stray away from this neighborhood will not appreciably alter the results. This reasoning again allows the kernel of the equation to be approximated by its behavior near this extreme value which in turn requires that $u' \simeq u$. Thus it is again assumed that

$$\bar{n}(u', v) \simeq - \bar{n}(u, v) \qquad (4.3.1)$$

and consequently

$$\frac{\bar{n}(u', v)}{R^2} \cdot \bar{R} \simeq \frac{\kappa_g}{2} \; . \qquad (4.3.2)$$

Incorporating this information into the problem, one obtains the integral equation

$$\psi(u) = -\frac{i\kappa_g}{4}\int du' \sqrt{\frac{G(u')}{\rho\rho'}}\; kR_o\, H_1^{(1)}(kR_o)\,\psi(u'). \qquad (4.3.3)$$

From the geometry of the problem it is evident that the polar coordinate distances must be of the general form

$$\rho(u) = r(u)\sqrt{G(u)}$$
$$\rho(u') = r(u')\sqrt{G(u')} \qquad (4.3.4)$$

if it is now assumed that the origin of the coordinate system is somewhere along the axis of rotation of the body under consideration. Fortunately, even though (4.3.4) would require further information concerning r(u) and r(u'), the geometric mean of the two polar coordinate distances may be easily related to the geodesic coordinates of the surface.

The desired result may be derived from the series which defines the components of two $\bar{R}$ spaces along with the Taylor series expansion for $\sqrt{G(u')}$, as was done when evaluating $\frac{\partial^2 R}{\partial v^2}$. When doing this it is found that

$$R^2 \simeq R_o^2 + v^2\sqrt{G(u)\,G(u')}\left[1+\kappa_{tt}(u'-u) - \frac{\kappa_g\kappa_{tn}}{2}(u'-u)^2 - \bar{R}_o\cdot\left\{\kappa_{tt}\bar{t} + \kappa_{tn}\bar{n}\right\}\right] + \frac{v^4}{4}\,G(u)\,G(u')\left\{\kappa_{tt}^2 + \kappa_{tn}^2\right\}. \qquad (4.3.5)$$

In the last term of this expression $\kappa_{tn} \xrightarrow[u\to 0]{} 1$, $\kappa_{tt} \xrightarrow[u\to 0]{} u^{-1}$, and $\sqrt{G(u)} \xrightarrow[u\to 0]{} u$. Therefore, this term behaves functionally as $v^4\sqrt{G(u')}$ and is, consequently, bounded for small u. Further if v^4 is neglected in comparison with v^2 (recall that the original kernel of the equation possesses a point of stationary phase at $v = 0$), this equation may be compared directly with R^2 as calculated from the R found in (4.1.1) and the identification

$$\rho\rho' \simeq \sqrt{G(u)\,G(u')}\left[1+\frac{1}{6}\left(\kappa_g^2\,\kappa_{tt}+\dot{\kappa}_g\,\kappa_{tn}\right)(u'-u)^3\right]$$

$$= \begin{cases} \sqrt{G(u)\,G(u')}\,, & u>\epsilon>0. \\ \sqrt{G(u)\,G(u')}\left(\frac{u'}{u}\right)\frac{\kappa_g^2}{6}(u'-u)^2\left(1-\frac{u}{u'}\right), & u\to 0. \end{cases} \qquad (4.3.6)$$

is then apparent.

For small values of u'-u it can be shown from (2.2.3) and (2.2.5) that

$$R_o \simeq 2\ell\left|\sin\frac{u'-u}{2\ell}\right| \,.\, \simeq |u'-u|\,. \qquad (4.3.7)$$

where ℓ has units of length.

Because the form of R_o is precisely the form that one obtains for the sphere problem it would appear reasonable to assign the convention that when u' goes through $2\pi\ell$ length units the perimeter of the obstacle (around the v = constant section) will have been traversed once. This, along with the ϕ' integration of section 4.1 implies that integration with respect to u' from zero to $\pi\ell$ is sufficient to have defined one integration over the entire surface. By covering the entire surface with the integration it is tacitly assumed that the creeping waves exist in the illuminated region as well as in the shaded region. This will be discussed further later in the paper.

One further simplification to be used is to assume that the unknown is a function of ku instead of the u that has been written thus far.

Using these thoughts and assuming for the moment that u is bounded away from zero, one notices that the equation takes the form

$$\psi(ku) = -\frac{i\kappa_g}{4}\int_0^{\pi\ell} du'\left[\frac{G(u')}{G(u)}\right]^{1/4} k\,|u'-u|\,H_1^{(1)}(k\,|u'-u|)\,\psi(ku'). \qquad (4.3.8)$$

The factor $\left[\frac{G(u')}{G(u)}\right]^{1/4}$ may be removed from consideration as long as the answer obtained is multiplied by $G^{-1/4}(u)$ as was done in the preceding work. If $H_1^{(1)}(k\,|u'-u|)$ is replaced by the first term of its asymptotic expansion (for large values of $k|u'-u|$) it is noticed that the equation obtained is precisely the same equation that generated the equations for the incoming and outgoing surface waves of Chapter Three if it is assumed that the term $(kR_0)^{-1/2}$ may be neglected when compared to $(kR_0)^{1/2}$ in (3.4.5). This equation, under these conditions has been solved (refer to the work of Hong, 1966), and these results will be assumed to be known.

If it is no longer required that u be bounded away from zero the second part of (4.3.6) applies. The first thing to come into discussion is the last factor of this expression. As long as u' is greater than u this factor may be immediately neglected. In order to integrate over the entire surface one could add to the integral from u to π, the integral with limits of 2π and $2\pi+u$. From the symmetry of the problem the latter integral will be the same as an integration from 0 to u. Therefore the last factor will be neglected entirely for all values of u'.

Thus this equation is given by

$$\psi(ku) = -\frac{i\sqrt{6}}{4}k\int_0^{\pi\ell} du'\left[\frac{G(u')}{G(u)}\right]^{1/4}\sqrt{\frac{u}{u'}}\;H_1^{(1)}(k|u'-u|)\,\psi(ku')\,, \tag{4.3.9}$$

and it is seen that the factor to be removed from this equation contains an additional factor of $\sqrt{\frac{u}{u'}}$. The factor $\left[\frac{G(u')}{G(u)}\right]^{1/4}\sqrt{\frac{u}{u'}}$ may be deleted from this equation with the understanding that any solutions of the "deleted" equation must be multiplied by $(u)^{1/2}G^{-1/4}(u)$ before they may be interpreted as solutions of (4.3.9).

Because the last two integral equations are intended to specify the same unknown in different regions of the surface it is not unreasonable to attempt some connection between the two. The solution of (4.3.8) was found (in the work by Hong) by finding a solution in the penumbra region and using this as a vehicle for launching the creeping waves into shadow region. These launched waves are then supported by (4.3.8).

In this connection it should be noted that, as long as one stays away from the caustic, the equation for the surface currents, when written in terms of the geodesic polar coordinate system, is exactly the same as the equation, when one stays away from the shadow boundary, written in terms of the geodesic coordinate system generated by the shadow boundary and the direction of the incident field at each point of the shadow boundary. With these thoughts in mind the "connection" between these equations will simply be that the solution in the shadow region will be the sum of the solution generated by Hong and the solution generated by (4.3.9).

It is possible, however, that a solution of (4.3.8) could be a factor in the solution of (4.3.9). If this is so, this factor may be removed in the same manner as the other factors have been removed and need not be considered until the final step when it must be introduced as a multiplier.

CHAPTER V

THE SOLUTION IN THE VICINITY OF THE CAUSTIC

5.1 Observations Concerning Possible Solutions

The starting point of this discussion will be the equation obtainable from (4.3.9) after removing all of the known common factors. This equation may be written

$$\frac{4i}{k\sqrt{6}}\,\psi(ku) = \int_0^u du'\, H_1^{(1)}(k\{u-u'\})\,\psi(ku') + \int_u^{\pi\ell} du'\, H_1^{(1)}(k\{u'-u\})\,\psi(ku') \; . \tag{5.1.1}$$

At this stage of the development of the problem it is of some benefit to consider the behavior of the equation and its solution for negative values of u. It is apparent from the physical problem that if the incident plane wave is a scalar wave, one would not expect any difference in the induced surface wave if one were merely to consider different initial geodesic paths. That is to say that the surface wave should be independent of v and in particular the solution of the problem should be an even function of its argument (assuming of course that the removed factors, the convergence factor, $G^{-1/4}(u)$, in all cases and a distance factor from the equation valid in the neighborhood of the caustic, will not introduce any phase change and therefore only their magnitudes are of importance for $u < 0$).

For the case in which the incident wave is a vector plane wave, each of the components, when considered as a function of u only, must also be an even function of its argument. This will first be demonstrated for the binormally directed component.

If one considers the incident magnetic field on the illuminated portion of the surface, but close to the shadow boundary, it is apparent that at diametrically opposite points on the surface the tangential components are negatives of one another, as is indicated in Fig. 3a. By letting ϵ denote a small

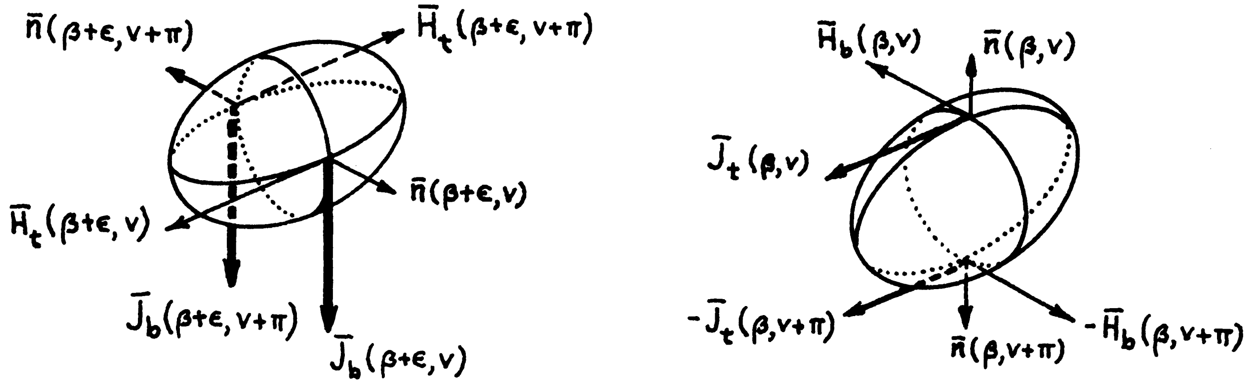

Figure 3a   Figure 3b

FIG. 3: RELATIONSHIP BETWEEN THE POLARIZATION AND INDUCED CURRENT.

positive quantity (to insure being slightly on the illuminated side of the shadow boundary) the following equations may be seen to hold:

$$\begin{aligned} \bar{n}\,(\beta+\epsilon,\ v+\pi) &\simeq -\bar{n}\,(\beta+\epsilon,\ v) \\ \bar{t}\,(\beta+\epsilon,\ v+\pi) &\simeq \bar{t}\,(\beta+\epsilon,\ v) \\ \bar{b}\,(\beta+\epsilon,\ v+\pi) &\simeq -\bar{b}\,(\beta+\epsilon,\ v)\ . \end{aligned} \qquad (5.1.2)$$

Therefore, the vector describing the b component may be written

$$\begin{aligned} \bar{\psi}\,(-k\{\beta+\epsilon\},\ v) &= \bar{\psi}_b\,(k\{\beta+\epsilon\},\ v+\pi) \\ &= \sin(v+\pi)\ \psi_b\,(k\{\beta+\epsilon\})\ \bar{n}\,(\beta+\epsilon,\ v+\pi) \times \bar{t}\,(\beta+\epsilon,\ v+\pi) \\ &= -\sin(v+\pi)\ \psi_b(k\{\beta+\epsilon\})\ \bar{b}\,(\beta+\epsilon,\ v)\ . \end{aligned} \qquad (5.1.3)$$

while for the diametrically opposite point,

$$\bar{\psi}_b (k\{\beta+\epsilon\}, v) = \sin v \psi_b (k\{\beta+\epsilon\})\bar{n}(\beta+\epsilon, v) \times [-\bar{t}(\beta+\epsilon, v)]$$

$$= -\sin v \psi_b (k\{\beta+\epsilon\})\, \bar{b}(\beta+\epsilon, v) \qquad (5.1.4)$$

Upon comparing (5.1.3) and (5.1.4) it is seen that the "total" b component of the surface field which is launched into the shadow region is an odd function of its argument. If one considers only $\psi_b (k\{\beta+\epsilon\})$ (i.e. do not consider the sign change introduced by $\sin(v+\pi)$), it is seen that it is an even function of its argument. Because this function is even when entering the penumbra region, it is reasonable to expect it to be an even function throughout the shadow region and, indeed, over the entire surface.

In a similar fashion the same results may be drawn for the t-directed current component by recognizing that the incident b-directed magnetic field does not change directions at diametrically opposed points. Refer to Fig. 3b.

In line with these thoughts, integrate in the opposite direction (or if preferred integrate from $-\pi\ell$ to 0 which, due to the symmetry of the problem will yield the same result as the integration given in (5.1.1)) and evaluate $\psi(-ku)$. This results in

$$\frac{4i}{k\sqrt{6}}\psi(-ku) = -\int_0^{-u} du' H_1^{(1)}(-k\{u+u'\})\psi(ku') - \int_{-u}^{-\pi\ell} du' H_1^{(1)}(k\{u'+u\})\psi(ku').$$

$$= \int_0^{u} du' H_1^{(2)}(k\{u-u'\})\psi(-ku') + \int_u^{\pi\ell} du' H_1^{(2)}(k\{u'-u\})\psi(-ku').$$

(5.1.5)

If it is assumed that $\psi(-ku) = \psi(ku)$ and (5.1.5) is added to (5.1.1), the result is a third equation given by

$$\frac{4i}{k\sqrt{6}} \psi(ku) = \int_0^u du' \, J_1(k\{u-u'\}) \, \psi(ku') + \int_u^{\pi\ell} du' \, J_1(k\{u'-u\}) \, \psi(ku') . \tag{5.1.6}$$

A fourth equation could be generated by taking the difference between (5.1.5) and (5.1.1), but this equation, which involves $Y_1(ku)$, retains the singularity which appears in (5.1.1) and (5.1.5) and destroys the incoming, outgoing surface wave interpretation while offering no new insight into the problem. This equation will therefore not be considered.

It is worth noticing at this point that if the kernel of (5.1.5) is considered to be $H_1^{(2)}(k|u-u'|)$, the transposed kernel is given by

$$\begin{aligned} H_1^{(2)*}(k|u'-u|) &= H_1^{(1)}(k|u'-u|^*) \\ &= H_1^{(1)}(k|u'-u|). \end{aligned} \tag{5.1.7}$$

Therefore, the integral operators defined for these two equations are the adjoints of one another; but, because of the singularities of the kernels, nothing may be inferred about the possibility of solutions which are members of a particular class of functions.

On the other hand, the combination of the two adjoint problems, (5.1.6), yields a Fredholm equation with a square integrable, symmetric kernel. However the multiplier appearing on the left-hand side of (5.1.6) is complex and is therefore certainly not a possible candidate for an eigenvalue. This in turn implies that if any non-trivial solutions to this problem do in fact exist, they are not members of the class of integrable functions.

5.2 The Volterra Equation

Define the variables z and ξ by the equations

$$z = ku, \quad \xi = ku' \tag{5.2.1}$$

and rewrite (5.1.6). This results in the equation

$$\psi(z) = -\frac{i\sqrt{6}}{4}\left[\int_0^z d\xi\, J_1(z-\xi)\,\psi(\xi) + \int_z^{k\pi\ell} d\xi\, J_1(\xi-z)\,\psi(\xi)\right] . \tag{5.2.2}$$

Suppose now that the solution can be represented by a series of Bessel functions for values of $z > 0$. The second term may then be examined as follows:

$$\left|\int_z^{k\pi\ell} d\xi J_1(\xi-z)J_n(\xi)\right| = \left|\int_z^{k\pi\ell} d\xi J_0'(\xi-z)J_n(\xi)\right|$$

$$= \left|-J_0(k\pi\ell-z)J_n(k\pi\ell)+J_n(z)+\int_z^{k\pi\ell} J_0(\xi-z)J_n'(\xi)d\xi\right|$$

$$\leq J_n(k\pi\ell)\left|1-J_0(k\pi\ell-z)\right| . \tag{5.2.3}$$

Since, for large k, the inequality (5.2.3) is $O(k^{-1/2})$ this integral may be neglected with respect to the first integration provided the solution of the remaining Volterra equation meets the above assumption.

The Volterra equation that remains possesses no integrable solutions but may possess singular solutions (refer to Mikhlin, 1960). In anticipation of this, consider extending the discussion to include generalized functions. In particular, introduce the Dirac δ-function into the kernel of the integral equation by defining the operator L by the following:

$$L[\psi] = \int_0^z d\xi \left[2\delta(z-\xi)+\frac{i\sqrt{6}}{4} J_1(z-\xi)\right]\psi(\xi). \tag{5.2.4}$$

In this terminology the problem under discussion is to find a sequence of (generalized) functions, t_α, such that

$$L\left[t_\alpha\right] = 0 \qquad \text{almost everywhere,} \qquad (5.2.5)$$

with the added restriction that the sequence should contain at least one function which approaches a finite, non-zero limit as its argument approaches zero. The last restriction is necessary, even in the case where membership to t_α is severly restricted, because there is no uniqueness theorem for (5.2.5) once one allows nonintegrable solutions. Physically, of course, it is known that there is a finite non-zero current at the caustic.

In view of the fact that one physically expects a smooth solution in the shadow region with the possible exception of the caustic, the generalized function terms appearing in the sequence will also be required to be smooth with the exception of the caustic. This means that the generalized functions appearing here will be restricted to being the Dirac δ-function and its derivatives.

5.3 The Generation of Solutions

In the following work use will be made of the following convolution

$$\int_0^z d\xi \, J_u(z-\xi) \, J_v(\xi) = 2 \sum_{n=0}^{\infty} (-1)^n J_{u+v+2n+1}(z) \; . \qquad (5.3.1)$$

Also the convention will be used that

$$\int_0^z d\xi \, \delta(\xi) \, f(\xi) = \frac{1}{2} f(0) \; . \qquad (5.3.2)$$

By using these equations and the definition of L, the solution generated by $\delta(z)$ may be evaluated as follows:

$$L\left[\delta(z)\right] = 2\delta(z) + \frac{i\sqrt{6}}{8} J_1(z)$$

$$L\left[\frac{i\sqrt{6}}{8} J_1(z)\right] = \frac{i\sqrt{6}}{8} J_1(z) + \left(\frac{i\sqrt{6}}{8}\right)^2 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+3}(z)$$

$$\vdots$$

indicating that

$$L\left[\delta(z) - \frac{i\sqrt{6}}{8} J_1(z) + \left(\frac{i\sqrt{6}}{8}\right)^2 2\sum_{n=0}^{\infty}(-1) J_{2n+3}(z) - \ldots\right] = 2\delta(z) \qquad (5.3.3)$$

While this series does represent a formal solution of (5.2.5) it is not an acceptable choice because it does not contain a function which remains finite and non-zero when $z = 0$.

If the series generated by $\delta'(z)$ is evaluated in the same fashion (recalling the general formula $\int_0^z d\xi\,\delta^{(n)}(\xi)\, f(z) = \frac{(-1)^n}{2} f(0)$, and using the recursion formula $2J_1'(z) = J_o(z) - J_1(z)$) the comparable steps are given by:

$$L\left[\delta'(z)\right] = -\frac{i\sqrt{6}}{8} J_1'(z)$$

$$L\left[\frac{i\sqrt{6}}{8} J_1'(z)\right] = \frac{i\sqrt{6}}{8} J_1'(z) + \left(\frac{i\sqrt{6}}{8}\right)^2 \left[J_2(z) - 2\sum_{n=0}^{\infty}(-1)^n J_{2n+4}\right]$$

$$\vdots$$

and therefore

$$L\left[\delta'(z) + \frac{i\sqrt{6}}{8} J_1'(z) - \left(\frac{i\sqrt{6}}{8}\right)^2 \left[J_2(z) - 2\sum_{n=0}^{\infty}(-1)^n J_{2n+4}\right] + \right.$$

$$\left. + \ldots\right] \equiv 0. \qquad (5.3.4)$$

Thus it is seen that if z is assumed to be close enough to zero so that z^2 may be neglected, an acceptable generalized function, i.e. one which meets all of the requirements set forth above, to use as a solution is

$$t_\alpha = \delta(z) + \frac{i\sqrt{6}}{8} J_1'(z). \qquad (5.3.5)$$

For the reason that (5.3.4) turned out to be identically zero it may be stated that the series t_α not only satisfies (5.2.5), but also satisfies the more stringent problem

$$L\left[t_\alpha\right] = 0.$$

in the sense of distributions.

That is to say that with the definition

$$< f, g > = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx \qquad (5.3.6)$$

for the scalar product of two (real) functions, the sequence, t_α, satisfies the equation

$$< L\left[t_\alpha\right], \phi > = < t_\alpha, \phi > , \qquad (5.3.7)$$

where the symbol ϕ in this equation is used to denote the class of test functions which are infinitely continuously differentiable and vanish outside of some bounded set.

The series indicated in (5.3.4) is actually the least singular solution of (5.2.5) in the sense that $\delta^{(2n+1)}(z)$ may be used to generate a suitable solution for any value of n. This solution may also be generated by taking the appropriate number of derivatives of the least singular solution. Thus a more general solution would be a linear combination of the least singular solution and its derivatives. However, in view of the recursive relationships for Bessel functions, it is not to be expected to gain any further information by considering the derivatives of the least singular solutions.

The solution generated thus far appears to be applicable to the scalar case and to the t-directed current component in the vector case. From the formalism established it is very difficult to say anything about the b-directed current component because the current does not pass through the caustic in a discontinuous fashion, but travels around the caustic in smaller and smaller circles as u becomes smaller. One obvious approach is to write the convolution in the other sense, i.e. define the operator L^* by the equation

$$L^*\left[\psi\right] = \int_0^z d\xi \left[2\delta(\xi) + \frac{i\sqrt{6}}{4} J_1(\xi)\right] \psi(z-\xi). \tag{5.3.8}$$

Now when the approximation $\psi(z) = \delta(z)$ is made on the right side of the equation it must be noticed that the two dimensional δ-function is no longer being evaluated at the origin of the rectangular coordinate system used to form the first solution. The coordinate system has been transformed, and in evaluating the transformed δ-function, the Jacobian of the coordinate transformation must be used (refer to Friedman, 1956). Since the area of the caustic is locally a spherical surface (as evidenced by the disappearance of the curvature variable in the equations valid near the caustic) it is assumed that locally the Jacobian appears as $\sqrt{G(u)}$. Thus in the region of the caustic (but not exactly at the caustic), the factor of the volume δ-function in which we are interested is given by

$$\frac{\delta(z-\xi)}{\sqrt{G(z)}} \simeq \frac{\delta(z-\xi)}{z} \quad . \tag{5.3.9}$$

This then is the generating function which will be used on the right-hand side of (5.3.8). Symbolically, in the operator notation, let $L^*\left[\frac{\delta(z)}{z}\right]$ be defined to be $1/z\, L^*\left[\delta(z)\right]$. Therefore the first attempt, which was subsequently discarded, may be put into use by simply dividing it by z since

$$2\,\frac{J_1(z)}{z} \xrightarrow[z \to 0]{} 1. \tag{5.3.10}$$

It is to be noted that this solution does satisfy (5.2.5) but, because the δ-function does not "add out" as it did in the previous case, it does not satisfy the more restrictive problem that the previous solution did.

5.4 Decomposition of the Solution

The work of the last section possesses several deficiencies which will now be pointed out and steps will be taken to rectify them to some extent. One major difficulty which may not be arbitrarily removed is the problem of uniqueness. Therefore in the work to follow the initial source function which is used to generate the solutions will be assumed to be of unit magnitude and all results will therefore be normalized to this extent. The allowable source functions are to be restricted to those allowed in the last section under the conditions that each one must generate at least one term which is finite and non-zero at the caustic.

In the development of (5.1.6) it was assumed, from physical reasoning, that the "total solution" of the problem must be an even function of its argument. In attempting to decompose the total solution into two parts which may be interpreted as incoming and outgoing surface waves, it may not be inferred that each of these waves is represented by an even function. That is to say that (5.1.1) and (5.1.5) may in fact possess solutions which contain terms which would have added to zero when (5.1.6) was formed.

With these thoughts in mind it may be of benefit to outline precisely the intended purpose of this section. Firstly the study of the singular equations under the influence of the allowable source functions will be undertaken. Secondly, the results of this study will then be compared to the results of the last section as far as functional behavior is concerned. Part of the purpose of this comparison is to discover what part the terms neglected in the last section by assuming $z \simeq 0$ play in the description of the solution.

It is assumed, in analogy with the work of the last section, that (5.1.1) and (5.1.5) may be approximated by the Volterra equations

$$\psi(z) = -\frac{i\sqrt{6}}{4}\int_0^z d\xi\, H_1^{(1,2)}(z-\xi)\,\psi(\xi) \quad . \tag{5.4.1}$$

This equation may also be written in terms of the Dirac δ-function as was done in the last section. That equation may in turn be written as a contour integral where the contour, C, encloses the interval $[0, z]$. Since the solution consists of a distribution, as well as classical functions, the contour integration will provide an analytic representation of the solution at all points in the complex plane which lie in the complement of the support of the distribution. That is to say that the analytic representation of the classical part of the solution will coincide with its restriction to the closed interval, but the representation of the δ-function is not valid where the δ-functions are non-zero.

Since the analytic (Cauchy) representation of the Dirac δ-function is given by (refer to Bremermann, 1965, p. 60)

$$\hat{\delta}(z) = \frac{1}{2\pi i} < \delta(t), (z-t)^{-1} > = -\frac{1}{2\pi i z} \quad , \tag{5.4.2}$$

the problem now under consideration is expressed by

$$\frac{1}{2} \oint_C dt \left[\frac{-1}{\pi i (z-t)} + \frac{i\sqrt{6}}{4} H_1^{(1,2)} (z-t) \right] \hat{\psi}(t) = 0 \tag{5.4.3}$$

where the carat appearing above the unknown is to remind one that the analytic representation of the solution is now being sought. It may be noticed that since the Dirac δ-function is an even function, the sign appearing with its representation in the first term of the kernel is rather arbitrary and the procedure adopted here is to just follow what is formally indicated. Secondly, in view of the fact that the constant multiplier is not determinable, the direction of integration is arbitrary and for this work the counterclockwise path was chosen.

For the sake of convenience make the substitution

$$A = \frac{i\sqrt{6}}{4} \tag{5.4.4}$$

in the work to follow.

By using the information of the last section the first trial solution for this equation will be

$$\hat{\psi}_o(z) = - \frac{1}{2\pi iz} \quad . \tag{5.4.5}$$

Before one can calculate the function generated by this choice one must examine the series defining the function $Y_1(z)$ and make the following observation:

$$z H_1^{(1,2)}(z) = z\left[J_1(z) \pm i\, Y_1(z)\right] \xrightarrow[z \to 0]{} \pm \frac{2}{\pi i} \tag{5.4.6}$$

where the upper sign is associated with the superscript 1 on the Hankel function, and the lower sign is associated with the superscript 2 on the Hankel functions.

Then, when source function (5.4.2) is placed into the contour integration the calculation will proceed as follows:

$$- \frac{1}{2} \oint_C dt \left[\frac{-1}{\pi i(z-t)} + A H_1^{(1,2)}(z-t)\right] \frac{1}{2\pi it}$$

$$= - \frac{1}{2}\left[\frac{1}{\pi i}\left\{\frac{1}{z} - \frac{1}{z}\right\} + A H_1^{(1,2)}(z) - A\left(\pm \frac{2}{\pi iz}\right)\right]$$

$$= - A\left[\frac{H_1^{(1,2)}(z)}{2} \mp \frac{1}{\pi iz}\right] \quad . \tag{5.4.7}$$

The next approximation is obtained by iterating this term. Namely:

$$- \frac{A}{2} \oint_C dt \left[\frac{-1}{\pi i(z-t)} + A H_1^{(1,2)}(z-t) \right] \left[\frac{H_1^{(1,2)}(t)}{2} \mp \frac{1}{\pi i t} \right]$$

$$= \pm A \oint_C dt \left[\frac{-1}{\pi i(z-t)} + A H_1^{(1,2)}(z-t) \right] \frac{1}{2\pi i t}$$

$$- \frac{A}{2} \oint_C dt \left[\frac{-1}{\pi i(z-t)} + A H_1^{(1,2)}(z-t) \right] \frac{H_1^{(1,2)}(t)}{2}$$

$$= \mp 2A \left[-A \left\{ \frac{H_1^{(1,2)}(z)}{2} \mp \frac{1}{i\pi z} \right\} \right] - \frac{A}{2} \left[H_1^{(1,2)}(z) \mp \frac{2}{\pi i z} + 0 \right]$$

$$= A \left[\mp 2A - 1 \right] \left[\frac{H_1^{(1,2)}(z)}{2} \mp \frac{1}{\pi i z} \right] \qquad (5.4.8)$$

From these results it is immediately noticed that by simply multiplying (5.4.5) by $\pm$ 2A and adding that to (5.4.5) one generates precisely (5.4.8). Therefore one solution of (5.4.3) is given by

$$\hat{\psi}(z) = (1 \pm 2A) \left(\frac{1}{2\pi i z} \right) - A \left[\frac{H_1^{(1,2)}(z)}{2} \mp \frac{1}{\pi i z} \right] . \qquad (5.4.9)$$

It should be recalled that in order to qualify as a surface wave these solutions should be divided by z. If this division is carried out and the two solutions are added several interesting details become manifest. Firstly, most of the source terms that appear add to zero. The only one remaining is in fact the one to be used in the generation of the other solution of interest. Secondly, except for a factor of two which will be accounted for later, the surface behavior appears precisely in the fashion predicted in Section 5.3. Thirdly, it is noticed that all of the terms generated in 5.3, which were subsequently ignored, are conspicuously absent in the finite surface wave which is generated from the singular equations.

If now the generating function is assumed to be $(2\pi i z^2)^{-1}$, the analytic representation of $\delta'(z)$, the procedure follows the pattern established above. In order to implement this recall that

$$\frac{d^n}{da^n} f(a) = \frac{n!}{2\pi i} \oint \frac{f(t)\, dt}{(z-a)^{n+1}} \quad . \tag{5.4.10}$$

One further computation is needed; namely

$$z^2 \frac{d}{dz} H_1^{(1,2)}(z) \Big|_{z=0} = - \frac{z^2}{2} H_2^{(1,2)}(z) \Big|_{z=0} = \pm \frac{2}{\pi i} \quad . \tag{5.4.11}$$

When this information is incorporated into the contour integration under the assumed source function the calculation proceeds as follows:

$$\frac{1}{2} \oint dt \left[\frac{-1}{i\pi(z-t)} + A H_1^{(1,2)}(z-t) \right] \frac{1}{2\pi i t^2}$$

$$= \frac{1}{2} \left[\frac{1}{i\pi} \left\{ \frac{1}{z^2} + \frac{1}{z^2} \right\} + A H_1'^{(1,2)}(z) \mp A \frac{2}{\pi i z^2} \right]$$

$$= A \left[\frac{H_1'^{(1,2)}(z)}{2} \mp \frac{1}{\pi i z^2} \right] + \frac{1}{i\pi z^2} \quad . \tag{5.4.12}$$

If the first term of this outcome is placed into the contour integral it generates the next approximation which is

$$A \; [1 \mp 2A] \left[\frac{H_1'^{(1,2)}(z)}{2} \mp \frac{1}{\pi i z^2} \right] \mp \frac{2A}{i\pi z^2} \tag{5.4.13}$$

From this work it is evident that if the approximation

$$\hat{\psi}(z) = - [1 \mp 2A] \frac{1}{2\pi i z^2} + A \left[\frac{H_1'^{(1,2)}(z)}{2} \mp \frac{1}{i\pi z^2} \right] \tag{5.4.14}$$

is placed into the contour integral it produces a remainder term

$$- \frac{1}{\pi i z^2} \quad . \tag{5.4.15}$$

An interesting reciprocity has become evident at this stage of the development of the problem. In treating the problem in the context of a generalized function one exact solution was found and one solution left a remainder term at the caustic. After transforming the problem to one of analytic representation these roles were reversed to the extent that the exact solution transformed to one with a remainder term and the one with the term left at the caustic transformed to an exact solution. It is to be noted that the remainder term in the transformed solution corresponds to a derivative of the Dirac δ-function with support at the caustic in the generalized solution.

If the two solutions (5.4.14) are added, the same remarks may be made concerning the sum as were made in the previous case. One further point of interest is to notice that by applying the same formalism to obtain the final results it is evident that the phase of the solutions representing the two current components generated at the caustic differ by 180°.

The only remaining question is the factor of two appearing in this section which was not predicted in the preceding work. In the original Volterra equation the standard practice of multiplying by one-half was used because the endpoints of the interval coincided with the support of Dirac δ-functions. Physically, this may be interpreted as considering one-half of the source to be within the interval and the other half to be outside of the interval. In defining the path of the contour integration, one could have passed through the endpoints of the interval and would have accomplished a division by two yielding the same result.

5.5 Further Considerations and Refinements

In the formalism of the last section it was found that the source function required was multiplied by a complex constant in both cases. Also it is to be noted that when the singular solutions are combined to yield a bounded solution, in each case the bounded function approaches a limit of one-half (excluding the complex constant A) as the argument of the function approaches zero. The first step of this section will be to renormalize the problem by multiplying the solutions of Section 5.4 by the factor

$$\frac{2}{1 \pm 2A} \quad . \tag{5.5.1}$$

which is equivalent to assuming that a source of magnitude two and a phase angle zero is what will arbitrarily be used to generate the surface field in the neighborhood of the caustic (for the b-directed component).

Under the influence of the source of magnitude two the singular expressions for the b-directed current components are given by

$$\psi_b(z) = \frac{-A}{1 \pm 2A} \frac{H_1^{(1,2)}(z)}{z} \tag{5.5.2}$$

for $z \neq 0$.

For the scalar problem and the t-directed current component the corresponding equation is

$$\psi_t(z) = \frac{A}{1 \pm 2A} \frac{d}{dz} H_1^{(1,2)}(z) \ . \tag{5.5.3}$$

The denominator appearing in both of these equations is

$$\frac{1}{1 \pm i\frac{\sqrt{6}}{2}} \simeq 0.632\, e^{\mp i 50^\circ 46'} \tag{5.5.4}$$

which means that (5.5.2) may be written

$$- \frac{i\sqrt{6}}{4} (0.632)\, e^{\pm i 50^\circ 46'} \frac{H_1^{(1,2)}(z)}{z}$$

$$= 0.3873\, e^{-i(90^\circ \mp 50^\circ 46')} \frac{H_1^{(1,2)}(z)}{z} \quad . \qquad (5.5.5)$$

If these two solutions are added, the incoming and outgoing surface waves combine to yield the finite solution

$$\psi_b(z) = -2(0.3873) \sin 39^\circ 14' \; 2 \frac{J_1(z)}{z} = -(0.4899)\; 2 \frac{J_1(z)}{z} \qquad (5.5.6)$$

The same procedure applied to (5.5.2) produces the finite combination of incoming and outgoing surface waves given by

$$\psi_t(z) = (0.4899)\; 2\; J_1'(z) \quad . \qquad (5.5.7)$$

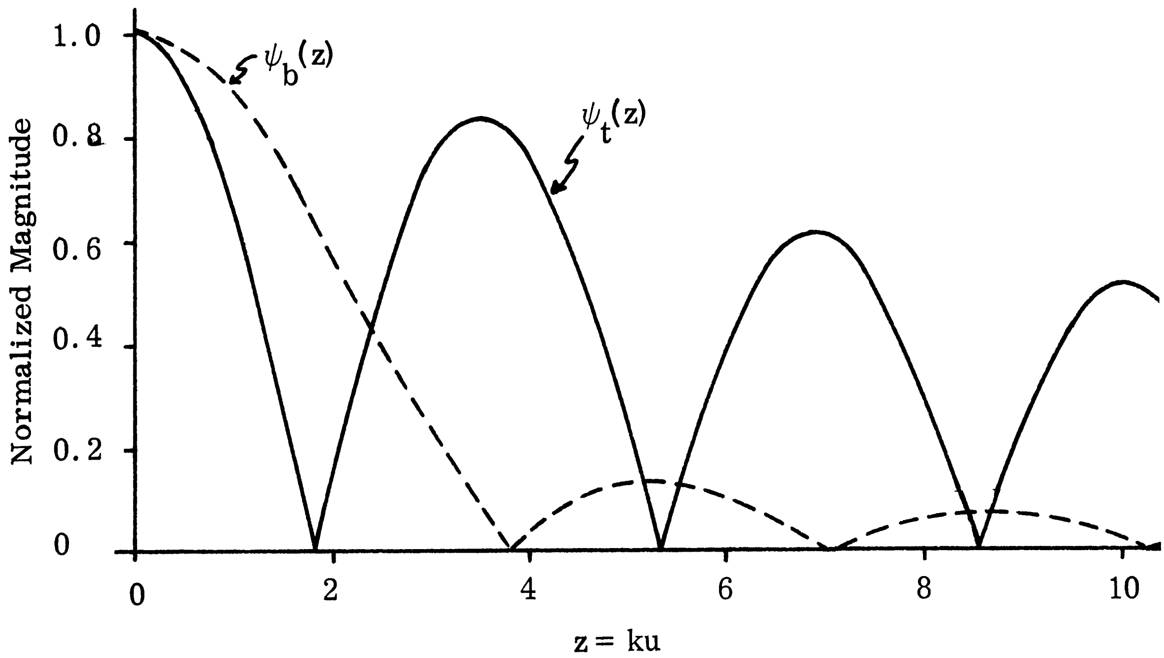

FIG. 4: MAGNITUDE OF CURRENTS GENERATED AT THE CAUSTIC.

If one studies the set (3.3.3) it becomes apparent that a solution of the first equation is a term of a solution of each of the remaining equations in the set. This means that although the work accomplished thus far is only a first order approximation, it may be considerably sharpened by forming a geometric series each term of which contains the solution of the homogeneous equation and the common ratio being $k^{-1/2}$. Because all of the rest of the multipliers which must appear in the solution go to one as z goes to zero, this hypothesis may be quickly checked by simply summing the series, multiplying it by 0.4899, and comparing it with the exact value at the caustic. The "exact" value for $k = 20$ was read from a graph on page 566 of Honl et al (1961) (also to be found in King and Wu, 1959), while the remaining values were taken from Ducmanis and Liepa (1965).

The quantity k under discussion here is the dimensionless product of the phase constant and a characteristic length of the body under consideration. This length was chosen to be the normalized radius at the shadow boundary (assuming the caustic to caustic length $= 2a \geq 2$; if not, choose the length to be a). In any case, the dimension of each k in the work to follow is clear from context.

TABLE 1: MAGNITUDE OF THE CURRENT AT THE CAUSTIC ASYMPTOTIC PREDICTION VS. EXACT VALUE

k	$0.4899 \dfrac{1}{1-\frac{1}{\sqrt{k}}}$	Exact Value
2	1.67	1.41
3	1.16	1.27
4	0.98	1.18
6	0.83	1.04
8	0.76	0.94
10	0.72	0.86
20	0.63	0.61

In order to check the expressions derived thus far, the body under consideration will be specialized to the case of the sphere. When making this specialization it is to be noted that the following three substitutions may be made, if desired, in order to cast the solution in terms of the standard notation of the sphere problem. One may replace k by ka (this is a result of the definition of the length unit in the geodesic coordinate system); $\sqrt{G(u)}$ may be replaced by $\sin\theta$; and u may be replaced by $\pi-\theta$ everywhere else it appears.

In order to attempt a complete comparison, use will be made of the following current component expressions (for details concerning these expressions see, for example, Goodrich (1959)):

$$f(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{i\xi t}}{w(t)}\, dt\,, \tag{5.5.8}$$

and

$$g(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{i\xi t}}{w'(t)}\, dt \tag{5.5.9}$$

where

$$\xi = \left(\frac{k}{2}\right)^{1/3} u \tag{5.5.10}$$

and $w(t)$ is the Airy integral defined by

$$w(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} \exp\left(iz - \frac{z^3}{3}\right) dz \quad . \tag{5.5.11}$$

In the construction of the curves to follow, use was made of the National Bureau of Standards tables (1964) and the tabulation of $f(\xi)$ and $g(\xi)$ found in Logan (1959). Since all of the calculations made in order to construct the graphs were made by hand, no attempt was made to establish the number of significant figures, nor are any tabulated values summarized here. The value of k chosen for the calculations is $k = 10$.

The first attempt at comparing consisted of combining the currents launched at the shadow boundary with the currents generated at the caustic. While the results for the b-directed current were remarkably good (refer to Fig. 5), the t-directed current did not fare so well. However it was discovered that if the magnitude of $J_1'(ku)$ was used, the magnitude of the current improves in some regions (refer to Fig. 6).

As mentioned on page 30, it is possible that the solution found in the region of the shadow boundary will be a factor in the solution which is valid at the caustic. Since a magnetic field vector at the caustic may be assumed to be the source for both currents, and because it would be tangent to the surface (for both components), it is most likely that the same factor appears in both solutions. With this in mind $g(\xi)$ was normalized, and denoted by $\bar{g}(\xi)$, and all of the assumptions concerning both the solution at the caustic and the solution at the shadow boundary are incorporated into the following descriptions of the current components:

$$\psi_b(ku) = e^{ik(\beta-u)} \left[\left(\frac{k}{2}\right)^{-1/3} f(\tilde{\beta}-\xi) - 0.4899 \frac{\sqrt{ku}}{\sqrt{k-1}} \frac{\bar{g}(\xi)}{G^{1/4}(u)} \frac{2J_1(ku)}{ku} \right] \tag{5.5.12}$$

and

$$\psi_t(ku) = e^{ik(\beta-u)} \left[g(\tilde{\beta}-\xi) + 0.4899 \frac{\sqrt{ku}}{\sqrt{k-1}} \frac{\bar{g}(\xi)}{G^{1/4}(u)} 2J_1'(ku) \right]. \tag{5.5.13}$$

where $\tilde{\beta} = \left(\frac{k}{2}\right)^{1/3} \beta$.

Figures 7 and 8 illustrate the results obtained from (5.5.12) and (5.5.13). It is noticed that $\psi_t(ku)$ again suffers bady (insofar as magnitude is concerned) at alternate peaks of the solution. If it assumed that the phase variation is due entirely to $\bar{g}(\xi)$ (i.e. the factor $\bar{g}(\xi)$ is insensitive to the phase changes of the current generated at the caustic) $\frac{J_1(ku)}{ku}$ and $J_1'(ku)$

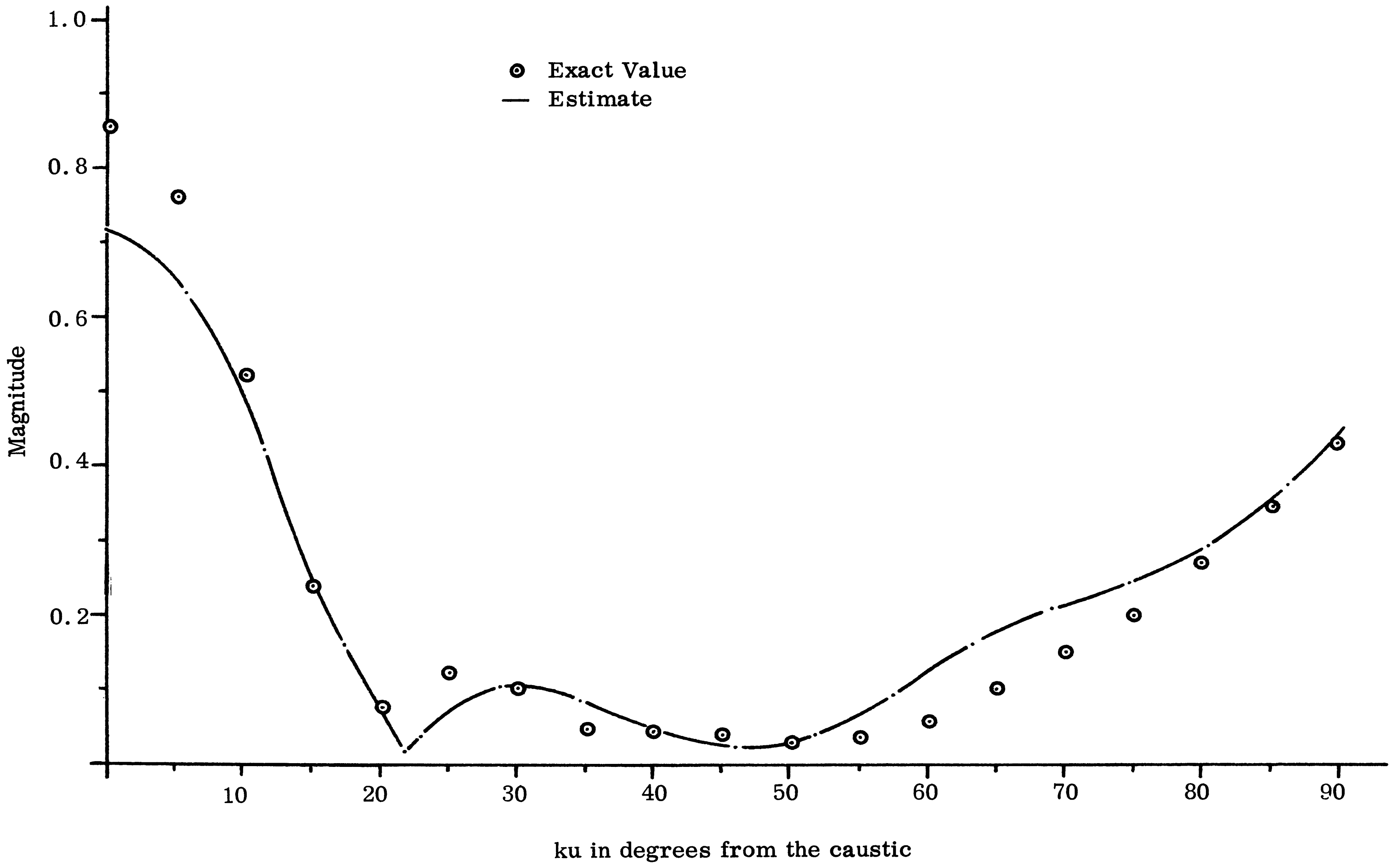

FIG. 5: MAGNITUDE OF $\psi_b(ku)$ IF THE CREEPING WAVE IS ADDED TO THE WAVE GENERATED AT THE CAUSTIC

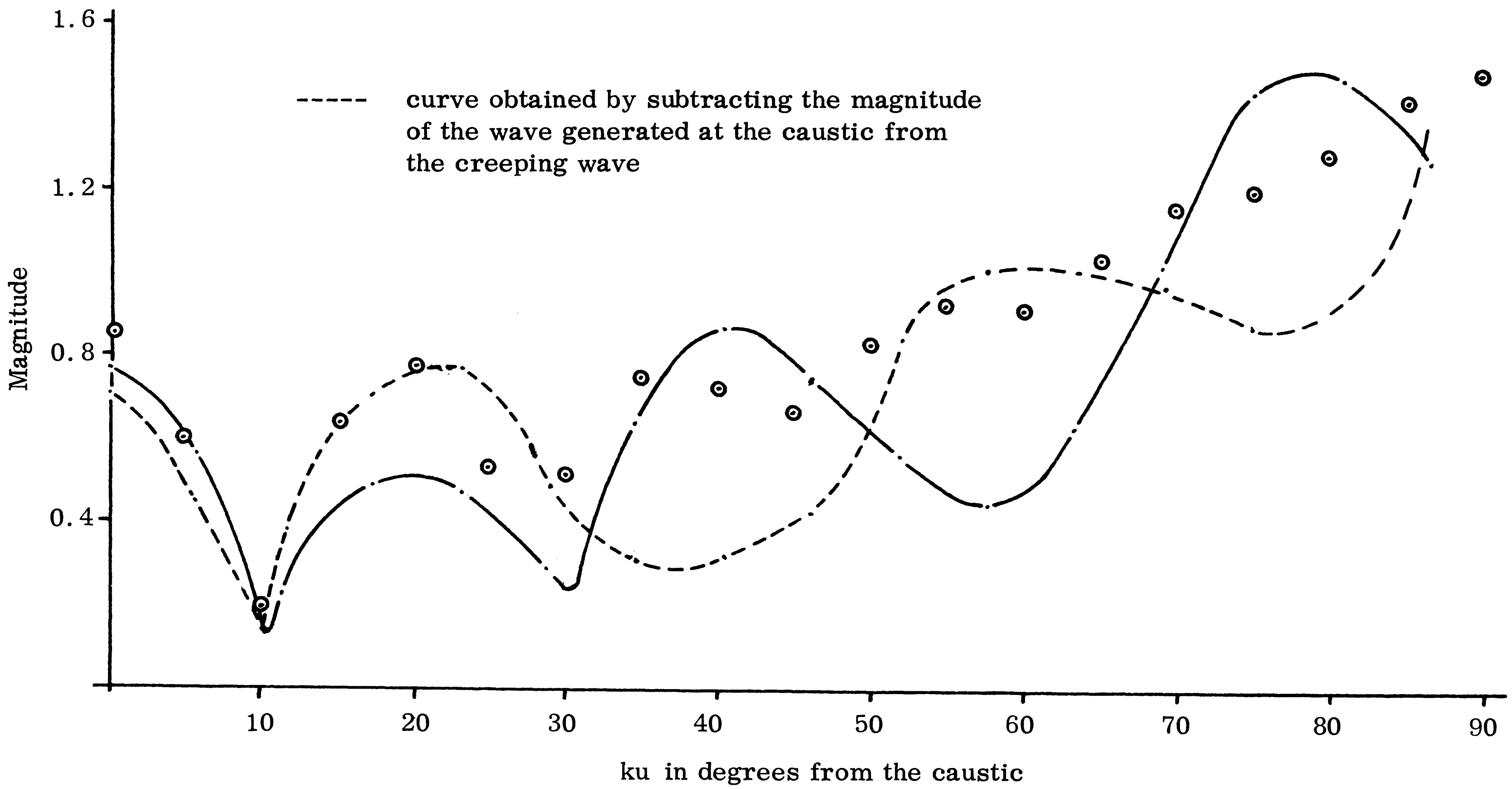

FIG. 6: MAGNITUDE OF ψ_t(ku) IF THE CREEPING WAVE IS ADDED TO THE WAVE GENERATED AT THE CAUSTIC.

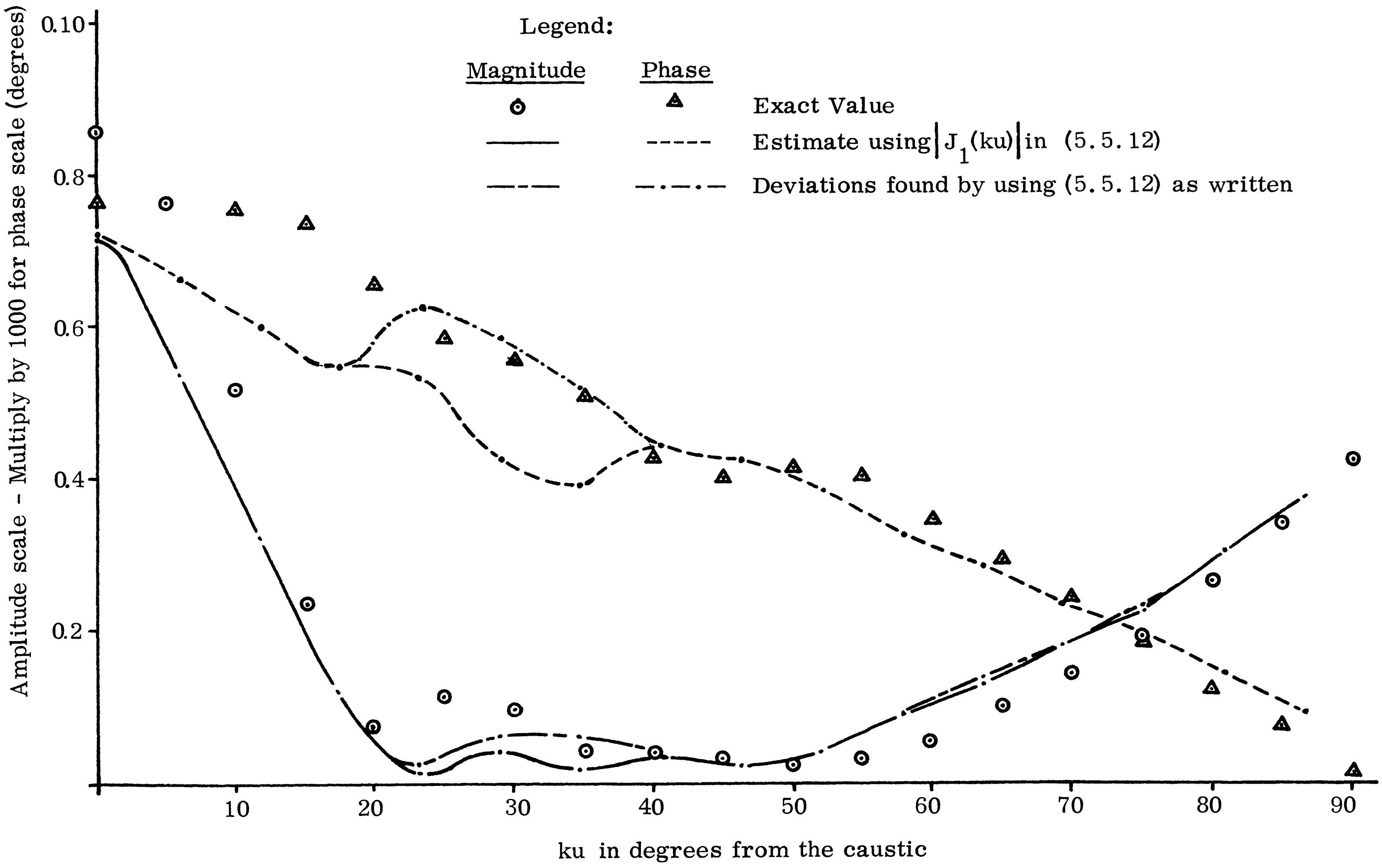

FIG. 7: AMPLITUDE AND PHASE VARIATION FOR $\psi_b(ku)$.

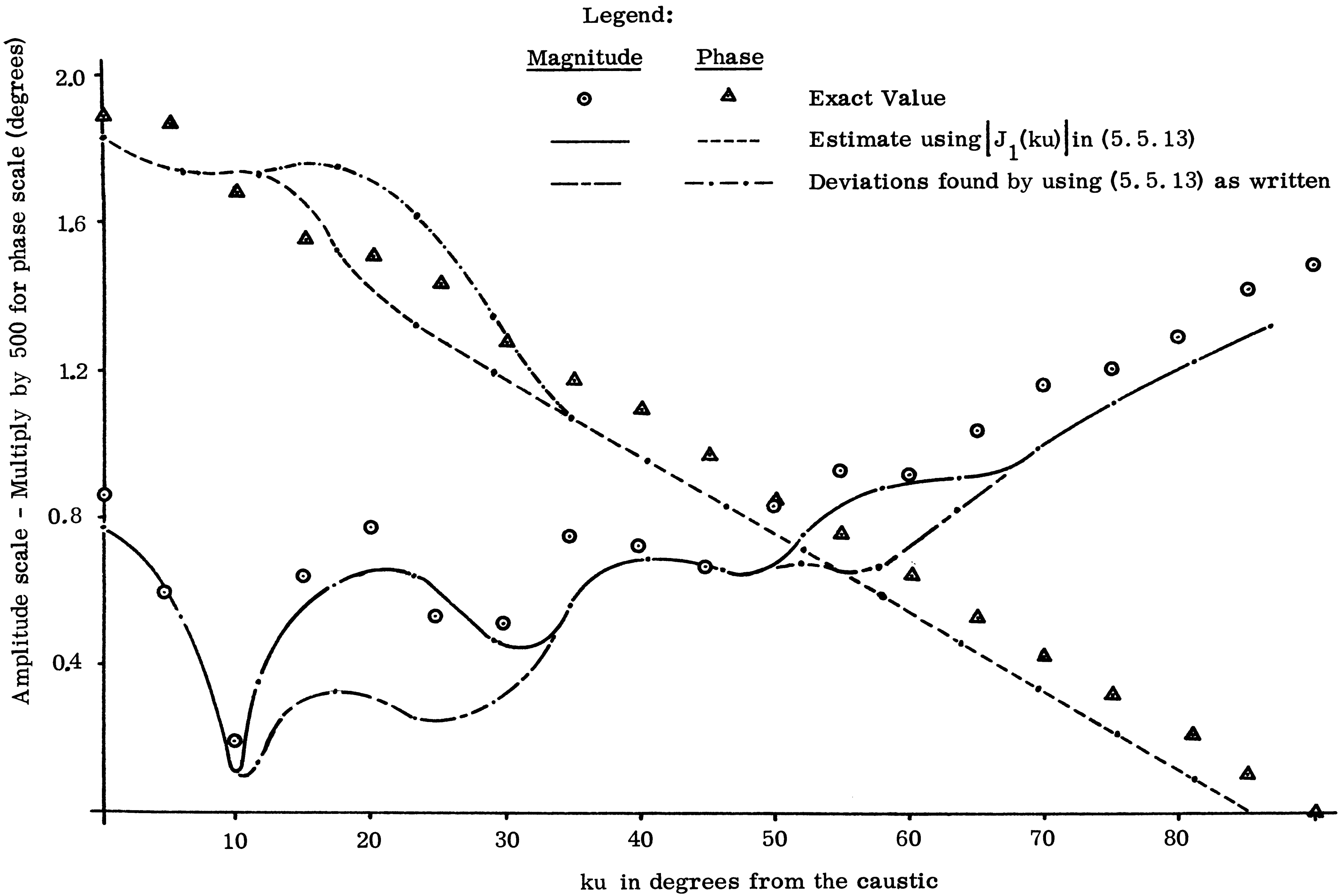

FIG. 8: AMPLITUDE AND PHASE VARIATION FOR $\psi_t(ku)$

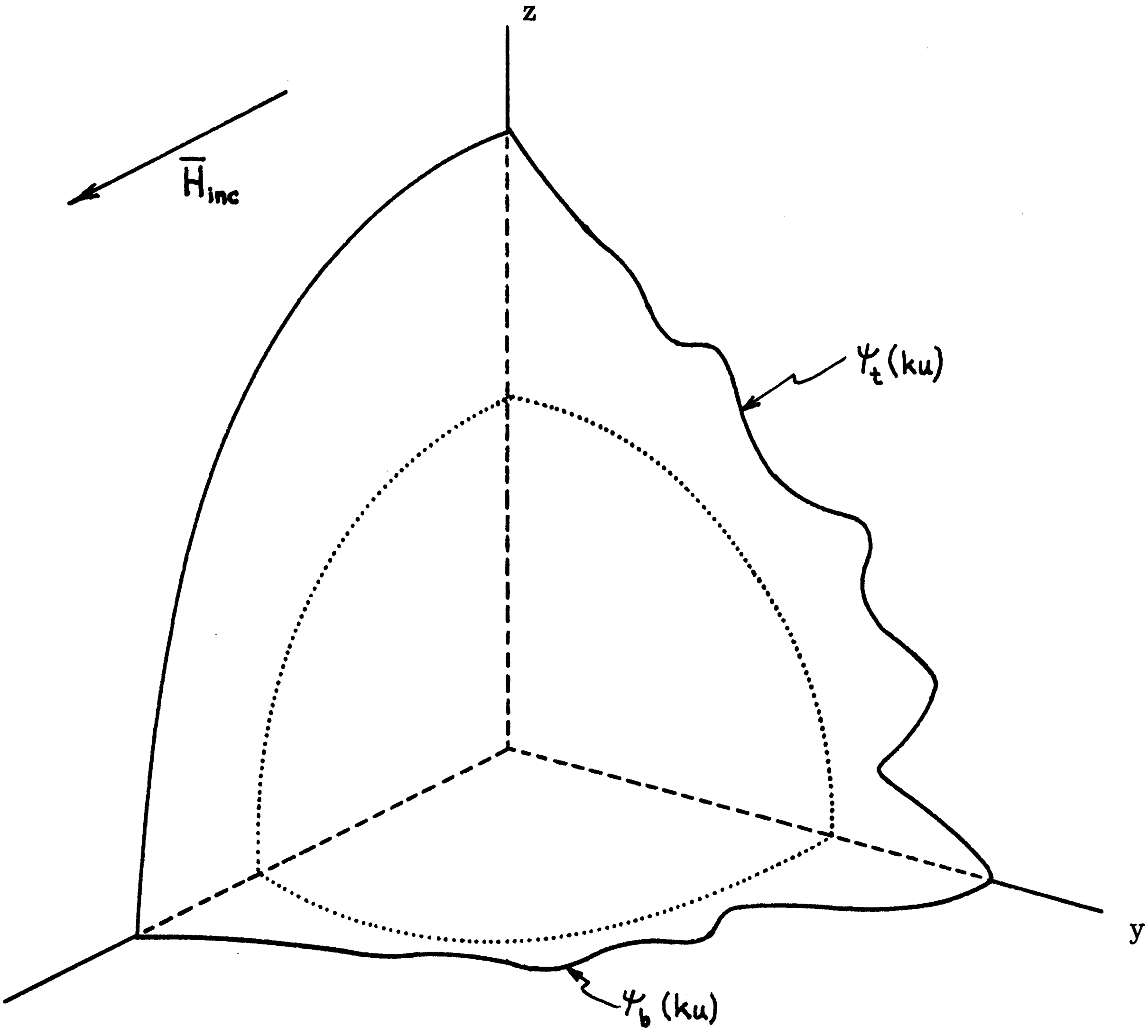

FIG. 9: AN ILLUSTRATION OF THE EXACT CURRENT MAGNITUDES. The surface of the sphere is the zero magnitude reference.

may be replaced by their respective magnitudes. The results of this assumption are also found in Figs. 7 and 8, and appear to be more applicable than the previous results.

It appears to this observer that the latter description of the currents is the one to be preferred in that the general trend seems to be more in line with what one would expect from an analysis such as this.

The points labeled as exact in Figs. 5, 6, 7 and 8 were taken from Ducmanis and Liepa (1965).

CHAPTER VI

SUMMARY AND CONCLUSIONS

6.1 Physical Interpretations

In writing the integral equation which describes the surface field, it was found that the only curvature term which appears in the equation for the leading term in the expansion is the curvature of the geodesic, κ_g. Furthermore when the equation was specialized to the region of the caustic it was found that the curvature of the geodesic divided out. Because a value of $\kappa_g = 1$ would represent a sphere, this is interpreted as meaning that the region of the caustic will appear locally to be a sphere for all bodies of revolution (provided of course that the radius of such a sphere is large enough to apply the asymptotic formalism).

From previous work it is apparent that the penumbra region behaves as a generator to the extent that non-zero currents in that region launch creeping waves into the shadow region. As long as one remains away from the caustic region the behavior of the creeping waves is influenced almost entirely by the conditions in the penumbra region. This work points out that the caustic may be thought of as a source/sink, generating outgoing waves and absorbing incoming waves, and that the behavior in the region of the caustic is influenced almost entirely by the source/sink concept. This interpretation was anticipated in a paper by Kazarinoff and Senior (1962).

Further the results of this work would indicate that the source/sink at the caustic may be considered as a point antenna residing on the surface at the caustic. As such, it would be expected that any deleted neighborhood of the caustic lies in the shadow region of the surface with respect to this source and would therefore be governed by the slowly varying shadow region solution (determined only by the polarization of the source). As it turns out the currents generated by the caustic and modulated by the slowly varying solution will creep around the surface and will converge at the caustic in the illuminated region. For the reason that the creeping waves have travelled so far and have all but disappeared, the source in the illuminated region will not be as strongly excited as the source in the shaded region was, and its effect will be more difficult to observe in view of the fact that it is masked by the current induced by the incident wave. This statement must be presumed because of mathematical difficulties in decoupling incoming and outgoing waves asymptotically for oblate spheroids for example (because of the argument concerning $R(\tau)$ being a decreasing function of τ), although for spheres and prolate spheroids this argument holds and the results should follow immediately once one discovers the strength of the source.

6.2 Comparison with Previous Results

If one were to study equation (4.3.9) and decide that, except for the convergence factor, everything should be expressed as a product of k and u, one would be tempted to multiply and divide by $\sqrt{k}$. By doing this it would be recognized that the solution must now contain a factor of $\sqrt{ku}$ instead of the factor of $\sqrt{u}$ as assumed in this work.

After doing this, one would observe that if the solutions obtained in Chapter 5 were asymptotically expanded, the results at the caustic would remain finite instead of going to zero as they presently do. (This serendipity comes about because the approach in this work generated a uniformly asymptotic kernel for the singular equations.) This writer believes that this is precisely the problem encountered in attempting to continue the solution known to be valid away from the caustic to a solution valid at the caustic and that the appearance of $\sqrt{k}$ in such a continuation is in error. This problem is also discussed in Kazarinoff and Senior (1962).

6.3 Comments Concerning Further Study

In view of all of the places where terms of some order of k have been neglected in this work , it does not seem feasible to go back and study these terms in order to generate higher order terms, especially in view of the fact that there is no tight bound on the error introduced by using the convolution of the kernel and the unknown to describe the unknown.

It appears that the next areas of advancement should be: 1) to study more carefully the representation of creeping waves on bodies other than the circle and the sphere under the following consideration. The representation of the creeping waves is in a sense complete for these geometries because the surface field, regardless of how many times it has wound around the body, is representable by the series $\sum e^{\lambda_j u}$. This representation does not appear to apply to the case where the cross section is not circular, and in fact should probably be replaced by a representation of the order of $\sum f_j(u)\, e^{\lambda_j u}$ where the $f_j(u)$ are periodic functions in some sense (i.e. periodic, almost periodic, or mean periodic). 2) Work must be done in the case where the distance cannot be expressed as the difference of two surface variables, and thus the convolution results, as applied in this paper, are no longer of any value. Such a case would be the problem of diffraction by an ellipsoid, and would be further complicated by the fact that the geodesic torsion will not be zero.

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13. ABSTRACT

Given a smooth, convex conducting body of revolution with a plane electromagnetic wave propagating in the direction of the axis of revolution, the problem considered is that of finding an expression, valid for small values of wavelength, which describes the currents in the vicinity of the caustic in the shaded region of the surface.

The problem is formulated in terms of an integral equation obtainable from a three-dimensional Green's function. The integration with respect to the azimuthal variable is carried out by two different schemes and the results discussed in relation to one another. The remaining integration, which is over a geodesic path, defines an integral equation which possesses a singular kernel. This singular equation is then studied in conjunction with a bounded kernel.

The body of revolution under consideration to this point is then specialized to the case of the sphere in order to compare the theory with known results, and some of the physical implications of the theory are discussed.

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