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# AN EFFICIENT FORMULATION OF ROBOT ARM DYNAMICS FOR CONTROL ANALYSIS AND MANIPULATOR DESIGN

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## TABLE OF CONTENTS

## ABSTRACT

This paper presents the development of the generalized d'Alembert equations of motion for application to robot manipulators with rotary joints. These equations, when applied to a robot arm, result in an efficient and explicit set of closed form second order nonlinear differential equations with vector cross product terms. They give well "structured" equations of motion suitable for state-space control analysis. The interaction and coupling reaction forces/torques between the neighboring joints of a manipulator can be easily identified as coming from the *translational* and *rotational* effects of the links. With this information, either a simplified dynamic model can be developed or an appropriate controller can be designed to compensate the nonlinear effects. Application to obtaining simplified dynamic model is discussed together with the computational complexities of the dynamic coefficients in the generalized d'Alembert equations of motion. The dynamic equations of the first three links of a PUMA robot are worked out to illustrate the method.


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## 1. Introduction

A priori information needed for manipulator control analysis is a set of closed form differential equations describing the dynamic behavior of the manipulator. Various approaches are available to formulate the robot arm dynamics, such as Lagrange-Euler (L-E) [1-5], Newton-Euler (N-E) [7-12], and recursive Lagrangian (R-L) [13], though mainly two approaches are used by most researchers - the Lagrange-Euler and the Newton-Euler formulations. The L-E equations of motion maintain a well structured form, but computationally it is very difficult to utilize for real time control purposes unless the equations of motion are simplified. The N-E formulation results in a very efficient set of recursive equations, but they are very difficult to use for deriving state-space control laws. This paper presents the generalized d'Alembert (G-D) equations of motion which give well "structured" equations suitable for control analysis. In addition to having faster computation time for computing the dynamic coefficients than the L-E equations of motion, the G-D equations of motion explicitly indicate the contributions of the *translational* and *rotational* effects of the links. Such information is useful for designing a controller in state-space or in obtaining an appropriate approximate model of the manipulator which simplifies the design of the controller.

Assuming rigid body motion, the Lagrange-Euler equations of motion, excluding the gear friction and backlash, are a set of second order coupled nonlinear differential equations. In general, the necessary generalized torque $\tau_i$ for joint $i$ to drive the $i^{th}$ link of the robot arm can be written as [1-5],

$$\sum_{k=i}^{n} \sum_{j=1}^{k} Tr\left\{ \frac{\partial \mathbf{T}_0^k}{\partial \theta_j} \mathbf{J}_k \left( \frac{\partial \mathbf{T}_0^k}{\partial \theta_i} \right)^T \right\} \ddot{\theta}_j + \sum_{r=i}^{n} \sum_{j=1}^{r} \sum_{k=1}^{r} Tr\left\{ \frac{\partial^2 \mathbf{T}_0^r}{\partial \theta_j \partial \theta_k} \mathbf{J}_r \left( \frac{\partial \mathbf{T}_0^r}{\partial \theta_i} \right)^T \right\} \dot{\theta}_j \dot{\theta}_k$$
$$- \sum_{j=i}^{n} m_j \mathbf{g} \left\{ \frac{\partial \mathbf{T}_0^j}{\partial \theta_i} \right\} \bar{\mathbf{r}}_j = \tau_i \quad ; \text{ for } i = 1,2, \cdots, n \tag{1}$$

These equations can be expressed in matrix form explicitly,

$$\sum_{k=1}^{n} C_{ik} \ddot{\theta}_k + \sum_{k=1}^{n} \sum_{m=1}^{n} C_{ikm} \dot{\theta}_k \dot{\theta}_m + C_i = \tau_i \quad ; \text{ for } i = 1,2, \cdots, n \tag{2}$$

where

$$C_{ik} = \sum_{j=max(i,k)}^{n} Tr\left\{ \frac{\partial \mathbf{T}_0^j}{\partial \theta_k} \mathbf{J}_j \left( \frac{\partial \mathbf{T}_0^j}{\partial \theta_i} \right)^T \right\} \quad ; \text{ for } i,k = 1,2, \cdots, n \tag{3}$$

$$C_{ikm} = \sum_{j=max(i,k,m)}^{n} Tr\left\{ \frac{\partial^2 \mathbf{T}_0^j}{\partial\theta_k \partial\theta_m} \mathbf{J}_j \left( \frac{\partial \mathbf{T}_0^j}{\partial\theta_i} \right)^T \right\} \quad ; \text{ for } i,k,m = 1,2, \cdots, n \tag{4}$$

$$C_i = \sum_{j=i}^{n} \left[ -m_j \mathbf{g} \left\{ \frac{\partial \mathbf{T}_0^j}{\partial\theta_i} \right\} \bar{\mathbf{r}}_j \right] \quad ; \text{ for } i = 1,2, \cdots, n \tag{5}$$

and $\mathbf{T}_0^i$ is a $4\times4$ homogeneous link transformation matrix which relates the spatial relationship between the $i^{th}$ and the base coordinate frames, $\mathbf{J}_i$ is the inertial matrix of link $i$ about the $i^{th}$ coordinate frame, $\bar{\mathbf{r}}_i$ is the position of the center of mass of link $i$ with respect to the $i^{th}$ coordinate system, $\mathbf{g}$ is the gravity row vector $=$ $(g_x, g_y, g_z, 0)$ and $|\mathbf{g}| = 9.8062 m/s^2$, the superscript $T$ on vectors and matrices indicates the transpose operation, and $n$ is the number of degrees-of-freedom of the robot arm.

The above dynamic equations of motion are highly nonlinear and consist of inertia loading, coupling Coriolis and centrifugal reaction forces between joints, and gravity loading effects. Furthermore, these interaction torques/forces depend on the manipulator's physical parameters, instantaneous joint configuration, and the load it is carrying. To improve the speed of computation, simplified sets of equations have been proposed by many investigators [2-4, 6]. In general, these "approximate" models simplify the underlying physics by neglecting second order terms such as the Coriolis and centrifugal reaction terms. However, at high arm speeds the neglected terms become significant, making the accurate position control of the robot arm more difficult [16].

As an alternative to deriving more efficient equations of motion, Newton-Euler equations of motion were developed [7-12]. The derivation was based mainly on the "moving coordinate systems" and d'Alembert's principle. The resulting Newton-Euler equations of motion, excluding the gear friction and backlash, are a set of compact forward and backward recursive equations. This set of recursive equations can be applied to the robot links sequentially. The forward recursion propagates kinematics information (such as angular velocities, angular accelerations, linear accelerations, and linear acceleration about the center of mass of each link) from the base reference frame (inertial frame) to the end-effector. The backward recursion propagates the forces exerted on each link from the end-effector of the manipulator to the base reference frame and the applied joint torques are computed from these forces. Because of the nature of the formulation and the method of systematically computing the joint torques, computations are much simpler. The most significant of this formulation is that the computation time of the applied torques is found linearly proportional to the number of joints of the robot arm and independent of the robot arm configuration. This enables the implementation of simple real-time control algorithm for a robot arm in the joint-variable space.

The inefficiency of the equations of motion as formulated by the L-E method comes mainly from the $4 \times 4$ homogeneous matrices describing the kinematic chain [14-15], while the efficiency of the N-E formulation can be seen from the vector formulation and its recursive nature. Turney et al. [14] explicitly verified that one can obtain the L-E equations of motion from the N-E equations, while Silver [15] showed the equivalence of the L-E and the N-E equations of motion through tensor analysis. To further improve the computation time of the Lagrangian formulation, Hollerbach [13] exploited the recursive nature of the Lagrangian formulations. However, the recursive equations destroy the "structure" of the dynamic model which is quite useful in providing insight for the controller design. For state-space control analysis, one would like to obtain an explicit set of closed form differential equations that describe the dynamic behavior of a manipulator and the interaction and coupling reaction forces in the equations can be easily identified so that an appropriate controller can be designed to compensate their effects. To obtain such an efficient set of equations of motion, Huston [12] employed Kane's dynamical equations to develop an algorithmic approach for deriving the equations of motion suitable for computer implementation. Another approach of obtaining an efficient set of closed form equations of motion is based on the generalized d'Alembert principle. Chace [17] successfully used the technique for multi-freedom mechanical parts. This paper extends Chace's work to develop the generalized d'Alembert equations of motion (G-D) for mechanical manipulators with rotary joints and shows the equations of motion explicitly in vector-matrix form. The method uses Euler transformation matrices ( or rotation matrices ) and relative position vectors between joints to increase the computational efficiency. The G-D equations of motion of a three-link PUMA[1] robot arm are then worked out showing the simplicity of the method.

For a rather "loose" comparison, the computation of the applied joint torques from the G-D equations of motion is of order $O(n^3)$, while the L-E equations of motion is of order $O(n^4)$, and the N-E equations of motion is of order $O(n)$, where $n$ is the number of degrees of freedom of the robot arm.

## 2. The Generalized d'Alembert Equations of Motion

Computationally, the L-E equations of motion are inefficient due to the $4 \times 4$ homogeneous matrix manipulations, while the efficiency of the N-E formulation can be seen from the vector formulation and its recursive nature. In order to obtain an efficient set of closed form equations of motion, one can

PUMA[1] is a trademark of Unimation Inc.

utilize the relative position vector and rotation matrix representation to describe each link's kinematics information, obtain the kinetic and potential energies of the robot arm to form the Lagrangian function and apply the Lagrange-Euler formulation to obtain the equations of motion.

With reference to Figure 1, assuming that the links of the robot arm are rigid bodies, the angular velocity $\omega_s$ of link $s$ with respect to the base coordinate frame can be expressed as a sum of the relative angular velocities from the lower joints,

$$\omega_s = \sum_{j=1}^{s} \dot{\theta}_j \mathbf{z}_{j-1} \tag{6}$$

where $\mathbf{z}_{j-1}$ is the axis of rotation of joint $j$ with reference to the base coordinate frame. Premultiplying the above angular velocity by the rotation matrix $\mathbf{R}_s^0$ changes its reference to the $s^{th}$ link coordinate frame,

$$\mathbf{R}_s^0 \omega_s = \sum_{j=1}^{s} \dot{\theta}_j \mathbf{R}_s^0 \mathbf{z}_{j-1} \tag{7}$$

Let $\bar{\mathbf{r}}_s$ be the position vector to the center of mass of link $s$ from the base coordinate frame. This position vector can be expressed as,

$$\bar{\mathbf{r}}_s = \sum_{j=1}^{s-1} \mathbf{p}_j^* + \bar{\mathbf{c}}_s = \mathbf{p}_{s-1} + \bar{\mathbf{c}}_s \tag{8}$$

where $\mathbf{p}_j^*$ is the position vector from the origin of link $j-1$ coordinate frame to the origin of link $j$ coordinate frame with respect to the base coordinate system, $\mathbf{p}_{s-1}$ is the position vector from the base coordinate frame to the origin of link $s-1$ coordinate frame, and $\bar{\mathbf{c}}_s$ is the position vector of the center of mass of link $s$ from the $(s-1)^{th}$ coordinate frame with reference to the base coordinate frame.

Using Eqs. 6-8, the linear velocity of link $s$, $\mathbf{v}_s$, with respect to the base coordinate frame can be computed as a sum of the linear velocities from the lower links,

$$\mathbf{v}_s = \sum_{k=1}^{s-1} \left\{ \left( \sum_{j=1}^{k} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \mathbf{p}_k^* \right\} + \left( \sum_{j=1}^{s} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \bar{\mathbf{c}}_s \tag{9}$$

The kinetic energy of link $s$ $(1 \leq s \leq n)$ with mass $m_s$ can be expressed as the summation of the kinetic energies due to the translational and rotational effects at its center of mass [18]:

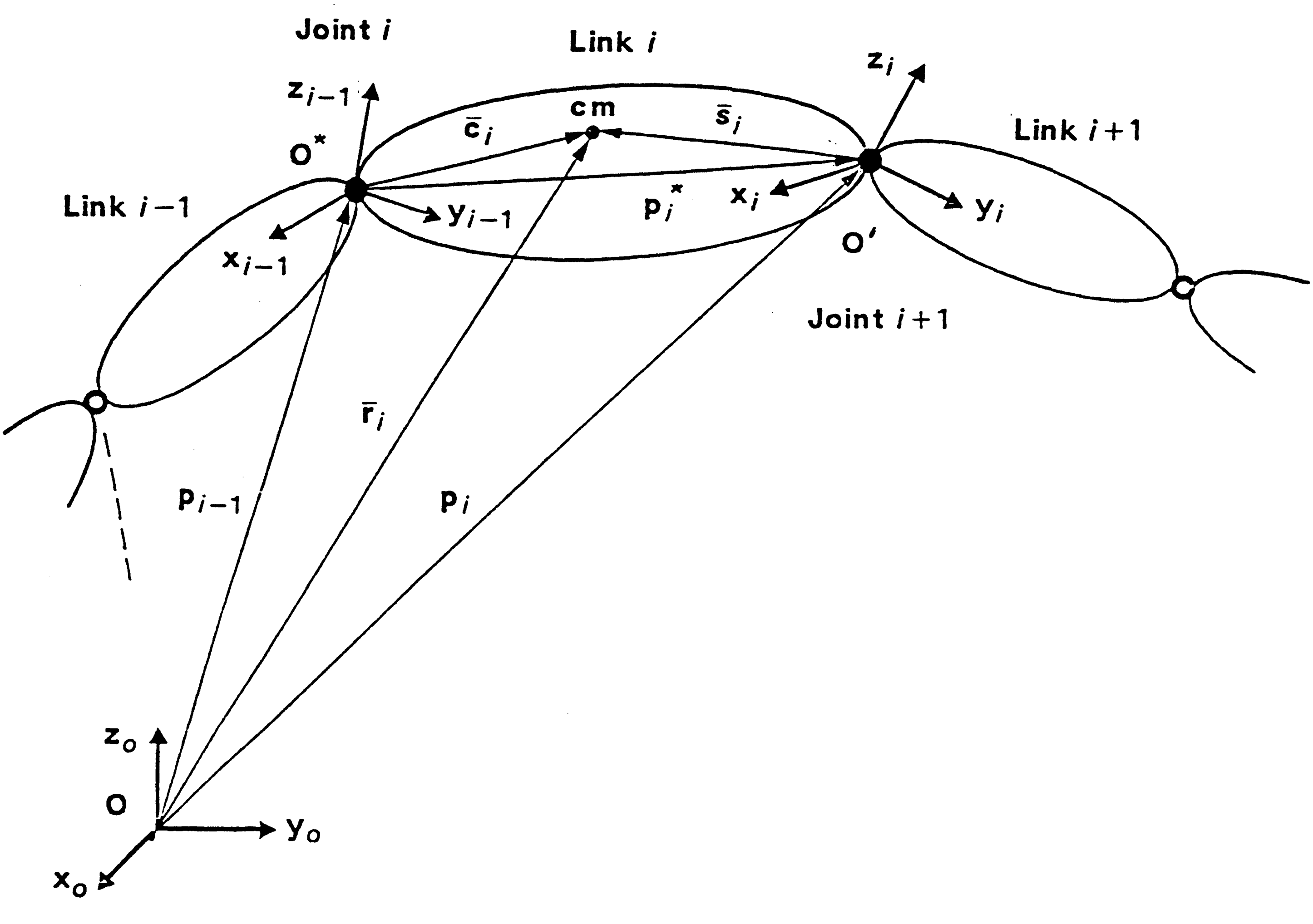



**Figure 1 Vector Definition for the G-D Equations**

$$K_s = (K_s)_{tran} + (K_s)_{rot} = \frac{1}{2}\, m_s(\mathbf{v}_s \cdot \mathbf{v}_s) + \frac{1}{2}\, (\mathbf{R}_s^0 \omega_s)^T \mathbf{I}_s\, (\mathbf{R}_s^0 \omega_s) \tag{10}$$

where $\mathbf{I}_s$ is the inertia tensor matrix of link $s$ about its center of mass expressed in the $s^{th}$ coordinate system.

For ease of discussion and derivation, the equations of motion due to the translational, rotational, and gravitational effects of the links will be considered and treated separately. Applying the Lagrange-Euler formulation to the above translational kinetic energy of link $s$ with respect to the generalized coordinate $\theta_i$ ($s \geq i$), we have

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial (K_s)_{tran}}{\partial \dot{\theta}_i}\right] - \frac{\partial (K_s)_{tran}}{\partial \theta_i} &= \frac{d}{dt}\left[m_s\, \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i}\right] - m_s\, \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \theta_i} \\ &= m_s \dot{\mathbf{v}}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i} + m_s \mathbf{v}_s \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i}\right) - m_s\, \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \theta_i} \end{aligned} \tag{11}$$

where

$$\frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i} = \mathbf{z}_{i-1} \times \left(\mathbf{p}_i^* + \mathbf{p}_{i+1}^* + \cdots + \mathbf{p}_{s-1}^* + \bar{\mathbf{c}}_s\right) = \mathbf{z}_{i-1} \times \left(\,\bar{\mathbf{r}}_s - \mathbf{p}_{i-1}\right) \quad ; \quad s \geq i \tag{12}$$

Using the following identities,

$$\frac{d}{dt}\left(\frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i}\right) = \frac{\partial \dot{\mathbf{v}}_s}{\partial \dot{\theta}_i} \quad \text{and} \quad \frac{\partial \dot{\mathbf{v}}_s}{\partial \dot{\theta}_i} = \frac{\partial \mathbf{v}_s}{\partial \theta_i} \tag{13}$$

Eq. 11 becomes

$$\frac{d}{dt}\left[\frac{\partial (K_s)_{tran}}{\partial \dot{\theta}_i}\right] - \frac{\partial (K_s)_{tran}}{\partial \theta_i} = m_s \dot{\mathbf{v}}_s \cdot \left\{\mathbf{z}_{i-1} \times (\,\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})\right\} \tag{14}$$

Summing all the links from $i$ to $n$ gives us the reaction torques due to the translational effect of all the links,

$$\frac{d}{dt}\left[\frac{\partial (K.E.)_{tran}}{\partial \dot{\theta}_i}\right] - \frac{\partial (K.E.)_{tran}}{\partial \theta_i} = \sum_{s=i}^{n}\left[\frac{d}{dt}\left(\frac{\partial (K_s)_{tran}}{\partial \dot{\theta}_i}\right) - \frac{\partial (K_s)_{tran}}{\partial \theta_i}\right] \tag{15}$$

$$= \sum_{s=i}^{n} m_s \dot{\mathbf{v}}_s \cdot \left\{\mathbf{z}_{i-1} \times (\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})\right\}$$

where, the acceleration of link $s$ is

$$\dot{\mathbf{v}}_s = \frac{d}{dt}\left\{\sum_{k=1}^{s-1}\left[\left(\sum_{j=1}^{k}\dot{\theta}_j \mathbf{z}_{j-1}\right) \times \mathbf{p}_k^*\right] + \left(\sum_{j=1}^{s}\dot{\theta}_j \mathbf{z}_{j-1}\right) \times \bar{\mathbf{c}}_s\right\}$$

$$= \sum_{k=1}^{s-1}\left[\left(\sum_{j=1}^{k}\ddot{\theta}_j \mathbf{z}_{j-1}\right) \times \mathbf{p}_k^* + \left\{\left(\sum_{j=1}^{k}\dot{\theta}_j \mathbf{z}_{j-1}\right) \times \left[\left(\sum_{j=1}^{k}\dot{\theta}_j \mathbf{z}_{j-1}\right) \times \mathbf{p}_k^*\right]\right\}\right] \tag{16}$$

$$+ \left\{\left(\sum_{j=1}^{s}\ddot{\theta}_j \mathbf{z}_{j-1}\right) \times \bar{\mathbf{c}}_s\right\} + \left\{\left(\sum_{j=1}^{s}\dot{\theta}_j \mathbf{z}_{j-1}\right) \times \left[\left(\sum_{j=1}^{s}\dot{\theta}_j \mathbf{z}_{j-1}\right) \times \bar{\mathbf{c}}_s\right]\right\}$$

$$+ \sum_{k=2}^{s-1}\left\{\sum_{p=2}^{k}\left[\left(\left(\sum_{q=1}^{p-1}\dot{\theta}_q \mathbf{z}_{q-1}\right) \times \dot{\theta}_p \mathbf{z}_{p-1}\right) \times \mathbf{p}_k^*\right]\right\} + \left\{\sum_{p=2}^{s}\left[\left(\left(\sum_{q=1}^{p-1}\dot{\theta}_q \mathbf{z}_{q-1}\right) \times \dot{\theta}_p \mathbf{z}_{p-1}\right) \times \bar{\mathbf{c}}_s\right]\right\}$$

Next, the kinetic energy due to the rotational effect of link $s$ is:

$$(K_s)_{rot} = \frac{1}{2}\left(\mathbf{R}_s^0 \omega_s\right)^T \mathbf{I}_s \left(\mathbf{R}_s^0 \omega_s\right) = \frac{1}{2}\left\{\sum_{j=1}^{s}\dot{\theta}_j \mathbf{R}_s^0 \mathbf{z}_{j-1}\right\}^T \mathbf{I}_s \left\{\sum_{j=1}^{s}\dot{\theta}_j \mathbf{R}_s^0 \mathbf{z}_{j-1}\right\} \tag{17}$$

where $\mathbf{I}_s$ is the inertia matrix about the center of mass of link $s$ with respect to the base coordinate frame. Since

$$\frac{\partial (K_s)_{rot}}{\partial \dot{\theta}_i} = \left[\mathbf{R}_s^0 \mathbf{z}_{i-1}\right]^T \mathbf{I}_s \left[\sum_{j=1}^{s}\dot{\theta}_j \mathbf{R}_s^0 \mathbf{z}_{j-1}\right], \quad ; \; s \geq i \tag{18}$$

$$\frac{\partial}{\partial \theta_i}\left(\mathbf{R}_s^0 \mathbf{z}_{j-1}\right) = \mathbf{R}_s^0 \mathbf{z}_{j-1} \times \mathbf{R}_s^0 \mathbf{z}_{i-1} \quad ; \; i \geq j \tag{19}$$

and

$$\frac{d}{dt}\left(\mathbf{R}_s^0 \mathbf{z}_{i-1}\right) = \sum_{j=i}^{s}\left(\frac{\partial}{\partial \theta_j}\mathbf{R}_s^0 \mathbf{z}_{i-1}\right)\frac{d\theta_j}{dt} = \mathbf{R}_s^0 \mathbf{z}_{i-1} \times \left(\sum_{j=i}^{s}\dot{\theta}_j \mathbf{R}_s^0 \mathbf{z}_{j-1}\right) \tag{20}$$

then the time derivative of Eq. 18 is

$$\frac{d}{dt}\left\{\frac{\partial(K_s)_{rot}}{\partial\theta_i}\right\} = \left[\frac{d}{dt}\mathbf{R}_s^0\mathbf{z}_{i-1}\right]^T\mathbf{I}_s\left[\sum_{j=1}^{s}\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right]$$

$$+\left[\mathbf{R}_s^0\mathbf{z}_{i-1}\right]^T\mathbf{I}_s\left[\sum_{j=1}^{s}\ddot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right]+\left[\mathbf{R}_s^0\mathbf{z}_{i-1}\right]^T\mathbf{I}_s\left[\sum_{j=1}^{s}\dot{\theta}_j\left(\frac{d}{dt}\mathbf{R}_s^0\mathbf{z}_{j-1}\right)\right] \tag{21}$$

$$=\left[\mathbf{R}_s^0\mathbf{z}_{i-1}\times\sum_{j=i}^{s}\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right]^T\mathbf{I}_s\left[\sum_{j=1}^{s}\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right]$$

$$+\left[\mathbf{R}_s^0\mathbf{z}_{i-1}\right]^T\mathbf{I}_s\left[\sum_{j=1}^{s}\ddot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right]+\left[\mathbf{R}_s^0\mathbf{z}_{i-1}\right]^T\mathbf{I}_s\left[\sum_{j=1}^{s}\left(\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\times\sum_{k=j+1}^{s}\dot{\theta}_k\mathbf{R}_s^0\mathbf{z}_{k-1}\right)\right]$$

Next using Eq. 19, we can find the partial derivative of $(K_s)_{rot}$ with respect to the generalized coordinate $\theta_i$ $(s \geq i)$,

$$\frac{\partial(K_s)_{rot}}{\partial\theta_i} = \left\{\left(\sum_{j=1}^{i}\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right)\times\mathbf{R}_s^0\mathbf{z}_{i-1}\right\}^T\mathbf{I}_s\left\{\sum_{j=1}^{s}\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right\} \tag{22}$$

Subtracting Eq. 22 from Eq. 21 and summing all the links from $i$ to $n$ gives us the reaction torques due to the rotational effects of all the links,

$$\frac{d}{dt}\left(\frac{\partial(K.E.)_{rot}}{\partial\dot{\theta}_i}\right)-\frac{\partial(K.E.)_{rot}}{\partial\theta_i}=\sum_{s=i}^{n}\left[\frac{d}{dt}\left(\frac{\partial(K_s)_{rot}}{\partial\dot{\theta}_i}\right)-\frac{\partial(K_s)_{rot}}{\partial\theta_i}\right]$$

$$=\sum_{s=i}^{n}\left[\left(\mathbf{R}_s^0\mathbf{z}_{i-1}\right)^T\mathbf{I}_s\left(\sum_{j=1}^{s}\ddot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right)+\left(\mathbf{R}_s^0\mathbf{z}_{i-1}\right)^T\mathbf{I}_s\left\{\sum_{j=1}^{s}\left(\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\times\left(\sum_{k=j+1}^{s}\dot{\theta}_k\mathbf{R}_s^0\mathbf{z}_{k-1}\right)\right)\right\}\right. \tag{23}$$

$$\left.+\left\{\mathbf{R}_s^0\mathbf{z}_{i-1}\times\left(\sum_{k=1}^{s}\dot{\theta}_k\mathbf{R}_s^0\mathbf{z}_{k-1}\right)\right\}^T\mathbf{I}_s\left(\sum_{j=1}^{s}\dot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right)\right] \quad ;\ i=1,2,\cdots,n$$

The potential energy of the robot arm equals to the sum of the potential energies of each link,

$$P.E. = \sum_{s=1}^{n}P_s \tag{24}$$

where $P_s$ is the potential energy of link $s$ and is found to be

$$P_s = -\mathbf{g} \cdot m_s\bar{\mathbf{r}}_s = -\mathbf{g} \cdot m_s\left(\mathbf{p}_{i-1} + \mathbf{p}_i^* + \cdots + \bar{\mathbf{c}}_s\right) \tag{25}$$

where $\mathbf{g} = (g_x, g_y, g_z)^T$ and $|\mathbf{g}| = 9.8062 m/s^2$. Applying the Lagrange-Euler formulation to the potential energy of link $s$ with respect to the generalized coordinate $\theta_i$ $(s \geq i)$, we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial(P_s)}{\partial\dot{\theta}_i}\right) - \frac{\partial(P_s)}{\partial\theta_i} &= -\frac{\partial(P_s)}{\partial\theta_i} = \mathbf{g} \cdot m_s \frac{\partial\left(\mathbf{p}_{i-1} + \mathbf{p}_i^* + \cdots + \bar{\mathbf{c}}_s\right)}{\partial\theta_i} \\ &= \mathbf{g} \cdot m_s \frac{\partial\left(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1}\right)}{\partial\theta_i} = \mathbf{g} \cdot m_s\left\{\mathbf{z}_{i-1} \times \left(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1}\right)\right\} \end{aligned} \tag{26}$$

where $\mathbf{p}_{i-1}$ is not a function of $\theta_i$. Summing all the links from $i$ to $n$ gives us the reaction torques due to the gravity effects of all the links,

$$\frac{d}{dt}\left(\frac{\partial(P.E.)}{\partial\dot{\theta}_i}\right) - \frac{\partial(P.E.)}{\partial\theta_i} = -\sum_{s=i}^{n}\frac{\partial(P_s)}{\partial\theta_i} = \sum_{s=i}^{n}\mathbf{g} \cdot m_s\left\{\mathbf{z}_{i-1} \times \left(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1}\right)\right\} \tag{27}$$

The summation of Eqs. 15, 23, and 27 is equal to the generalized applied torque exerted at joint $i$ to drive link $i$,

$$\tau_i = \left[\frac{d}{dt}\left(\frac{\partial(K.E.)_{tran}}{\partial\dot{\theta}_i}\right) - \frac{\partial(K.E.)_{tran}}{\partial\theta_i}\right] + \left[\frac{d}{dt}\left(\frac{\partial(K.E.)_{rot}}{\partial\dot{\theta}_i}\right) - \frac{\partial(K.E.)_{rot}}{\partial\theta_i}\right] + \frac{\partial(P.E.)}{\partial\theta_i}$$

$$= \sum_{s=i}^{n}\left[m_s\left\{\left(\sum_{k=1}^{s-1}\left(\sum_{j=1}^{k}\ddot{\theta}_j\mathbf{z}_{j-1}\right) \times \mathbf{p}_k^*\right) + \left(\left(\sum_{j=1}^{s}\ddot{\theta}_j\mathbf{z}_{j-1}\right) \times \bar{\mathbf{c}}_s\right)\right\} \cdot \left\{\mathbf{z}_{i-1} \times \left(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1}\right)\right\}\right]$$

$$+ \sum_{s=i}^{n}\left[\left(\mathbf{R}_s^0\mathbf{z}_{i-1}\right)^T\mathbf{I}_s\left(\sum_{j=1}^{s}\ddot{\theta}_j\mathbf{R}_s^0\mathbf{z}_{j-1}\right)\right] + \sum_{s=i}^{n}\left[m_s\left[\sum_{k=1}^{s-1}\left\{\left(\left(\sum_{p=1}^{k}\dot{\theta}_p\mathbf{z}_{p-1}\right) \times \left(\left(\sum_{q=1}^{k}\dot{\theta}_q\mathbf{z}_{q-1}\right) \times \mathbf{p}_k^*\right)\right)\right.\right.\right.$$

$$\left.\left.\left. + \left\{\sum_{p=2}^{k}\left(\left(\sum_{q=1}^{p-1}\dot{\theta}_q\mathbf{z}_{q-1}\right) \times \dot{\theta}_p\mathbf{z}_{p-1}\right) \times \mathbf{p}_k^*\right\}\right\}\right] \cdot \left\{\mathbf{z}_{i-1} \times \left(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1}\right)\right\}\right]$$

$$+ \sum_{s=i}^{n} \left[ m_s \left\{ \left( \left( \sum_{p=1}^{s} \dot{\theta}_p \mathbf{z}_{p-1} \right) \times \left( \left( \sum_{q=1}^{s} \dot{\theta}_q \mathbf{z}_{q-1} \right) \times \bar{\mathbf{c}}_s \right) \right\} \right. \tag{28}$$

$$+ \left\{ \sum_{p=2}^{s} \left( \left( \sum_{q=1}^{p-1} \dot{\theta}_q \mathbf{z}_{q-1} \right) \times \dot{\theta}_p \mathbf{z}_{p-1} \right) \times \bar{\mathbf{c}}_s \right\} \right\} \cdot \left\{ \mathbf{z}_{i-1} \times ( \bar{\mathbf{r}}_s - \mathbf{p}_{i-1}) \right\} \Bigg]$$

$$+ \sum_{s=i}^{n} \left[ (\mathbf{R}_s^0 \mathbf{z}_{i-1})^T \mathbf{I}_s \left\{ \sum_{j=1}^{s} \left( \dot{\theta}_j \mathbf{R}_s^0 \mathbf{z}_{j-1} \times \left( \sum_{k=j+1}^{s} \dot{\theta}_k \mathbf{R}_s^0 \mathbf{z}_{k-1} \right) \right) \right\} \right.$$

$$+ \left\{ \mathbf{R}_s^0 \mathbf{z}_{i-1} \times \left( \sum_{p=1}^{s} \dot{\theta}_p \mathbf{R}_s^0 \mathbf{z}_{p-1} \right) \right\}^T \mathbf{I}_s \left( \sum_{q=1}^{s} \dot{\theta}_q \mathbf{R}_s^0 \mathbf{z}_{q-1} \right) \Bigg]$$

$$- \mathbf{g} \cdot \left\{ \mathbf{z}_{i-1} \times \left( \sum_{j=i}^{n} m_j ( \bar{\mathbf{r}}_j - \mathbf{p}_{i-1}) \right) \right\} \quad ; \text{ for } \quad i = 1, 2, \cdots, n$$

The above equation can be re-written in a more "structured" form as ( for $i = 1, 2, \cdots, n$ ):

$$\sum_{j=1}^{n} D_{ij} \ddot{\theta}_j + H_i^{tran}(\theta, \dot{\theta}) + H_i^{rot}(\theta, \dot{\theta}) + G_i = \tau_i \tag{29}$$

where, for $i = 1, 2, \cdots, n$,

$$D_{ij} = D_{ij}^{rot} + D_{ij}^{tran} = \sum_{s=j}^{n} \left\{ (\mathbf{R}_s^0 \mathbf{z}_{i-1})^T \mathbf{I}_s (\mathbf{R}_s^0 \mathbf{z}_{j-1}) \right\}$$

$$+ \sum_{s=j}^{n} \left[ m_s \left\{ \mathbf{z}_{j-1} \times \left( \sum_{k=j}^{s-1} \mathbf{p}_k^* + \bar{\mathbf{c}}_s \right) \right\} \cdot \left\{ \mathbf{z}_{i-1} \times ( \bar{\mathbf{r}}_s - \mathbf{p}_{i-1}) \right\} \right] \quad ; \quad i \leq j \tag{30}$$

$$= \sum_{s=j}^{n} \left\{ (\mathbf{R}_s^0 \mathbf{z}_{i-1})^T \mathbf{I}_s (\mathbf{R}_s^0 \mathbf{z}_{j-1}) \right\} + \sum_{s=j}^{n} \left[ m_s \left\{ \mathbf{z}_{j-1} \times ( \bar{\mathbf{r}}_s - \mathbf{p}_{j-1}) \right\} \cdot \left\{ \mathbf{z}_{i-1} \times ( \bar{\mathbf{r}}_s - \mathbf{p}_{i-1}) \right\} \right] \quad ; \quad i \leq j$$

$$H_i^{tran}(\theta, \dot{\theta}) = \sum_{s=i}^{n} \left[ m_s \left[ \sum_{k=1}^{s-1} \left\{ \left( \left( \sum_{p=1}^{k} \dot{\theta}_p \mathbf{z}_{p-1} \right) \times \left( \left( \sum_{q=1}^{k} \dot{\theta}_q \mathbf{z}_{q-1} \right) \times \mathbf{p}_k^* \right) \right\} \right. \right.$$

$$+\left\{\sum_{p=2}^{k}\left[\left[\sum_{q=1}^{p-1}\dot\theta_q \mathbf{z}_{q-1}\right] \times \dot\theta_p \mathbf{z}_{p-1}\right] \times \mathbf{p}_k^*\right\}\Bigg] \cdot \left\{\mathbf{z}_{i-1} \times (\,\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})\right\}\Bigg] \tag{31}$$

$$+\sum_{s=i}^{n}\left[m_s\left\{\left\{\left[\sum_{p=1}^{s}\dot\theta_p \mathbf{z}_{p-1}\right] \times \left[\left[\sum_{q=1}^{s}\dot\theta_q \mathbf{z}_{q-1}\right] \times \bar{\mathbf{c}}_s\right]\right\}\right.\right.$$

$$\left.+\left\{\sum_{p=2}^{s}\left[\left[\sum_{q=1}^{p-1}\dot\theta_q \mathbf{z}_{q-1}\right] \times \dot\theta_p \mathbf{z}_{p-1}\right] \times \bar{\mathbf{c}}_s\right\}\right\} \cdot \left\{\mathbf{z}_{i-1} \times (\,\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})\right\}\Bigg]$$

$$H_i^{rot}(\theta,\dot\theta) = \sum_{s=i}^{n}\Bigg[\left(\mathbf{R}_s^0 \mathbf{z}_{i-1}\right)^T \mathbf{I}_s\left\{\sum_{j=1}^{s}\left[\dot\theta_j \mathbf{R}_s^0 \mathbf{z}_{j-1} \times \left(\sum_{k=j+1}^{s}\dot\theta_k \mathbf{R}_s^0 \mathbf{z}_{k-1}\right)\right]\right\}$$
$$+\left\{\mathbf{R}_s^0 \mathbf{z}_{i-1} \times \left(\sum_{p=1}^{s}\dot\theta_p \mathbf{R}_s^0 \mathbf{z}_{p-1}\right)\right\}^T \mathbf{I}_s\left(\sum_{q=1}^{s}\dot\theta_q \mathbf{R}_s^0 \mathbf{z}_{q-1}\right)\Bigg] \tag{32}$$

$$G_i = -\mathbf{g} \cdot \left[\mathbf{z}_{i-1} \times \sum_{j=i}^{n} m_j(\,\bar{\mathbf{r}}_j - \mathbf{p}_{i-1})\right] \tag{33}$$

The dynamic coefficients $D_{ij}$ and $G_i$ are functions of both the joint variables and inertial parameters of the manipulator, while the $H_i^{tran}$ and $H_i^{rot}$ are functions of the joint variables, the joint velocities and inertial parameters of the manipulator. These coefficients have the following physical interpretations:

(1) The elements of the $D_{ij}$ matrix are related to the link inertias of the manipulator. Eq. 30 reveals the acceleration effects of joint $j$ acting on joint $i$ where the driving torque $\tau_i$ acts. The first term of Eq. 30 indicates the inertial effects of moving link $j$ on joint $i$ due to the *rotational* motion of link $j$, and vice versa. If $i = j$, it is the effective inertias felt at joint $i$ due to the rotational motion of link $i$; while if $i \neq j$, it is the pseudo products of inertia of link $j$ felt at joint $i$ due to the rotational motion of link $j$. The second term has the same physical meaning except it is due to the *translational* motion of link $j$ acting on joint $i$.

(2) The $H_i^{tran}(\theta,\dot\theta)$ is related to the velocities of the joint variables. Eq. 31 represents the combined centrifugal and Coriolis reaction torques felt at joint $i$ due to the velocities of joints $p$ and $q$ resulted from the *translational*

motion of links $p$ and $q$. The first and third terms of Eq. 31 constitute the centrifugal and Coriolis reaction forces from all the links below link $s$ and link $s$ respectively in the kinematic chain due to the translational motion of the links. If $p = q$, then it represents the centrifugal reaction forces felt at joint $i$. If $p \neq q$, then it indicates the Coriolis forces acting on joint $i$. The second and fourth terms of Eq. 31 indicate the Coriolis reaction forces contributed from the links below link $s$ and link $s$ respectively due to the translational motion of the links.

(3) The $H_i^{rot}(\theta,\dot{\theta})$ is also related to the velocities of the joint variables. Similar to the $H_i^{tran}(\theta,\dot{\theta})$, Eq. 32 reveals the combined centrifugal and Coriolis reaction torques felt at joint $i$ due to the velocities of joints $p$ and $q$ resulted from the *rotational* motion of links $p$ and $q$. The first term of Eq. 32 indicates purely the Coriolis reaction forces of joints $p$ and $q$ acting on joint $i$ due to the rotational motion of the links. The second term is the combined centrifugal and Coriolis reaction forces acting on joint $i$. If $p = q$, then it indicates the centrifugal reaction forces felt at joint $i$, while if $p \neq q$, then it represents the Coriolis forces acting on joint $i$ due to the rotational motion of the links.

(4) The coefficient $G_i$ represents the gravity effects acting on joint $i$ from the links above joint $i$

Since these coefficients are used quite often in designing a feedback controller for the manipulator, it would be useful to evaluate the computational complexities of these coefficients in Eqs. 30-33. An example of using these coefficients in designing a feedback controller is the nonlinear decoupled control [19]. At first sight, Eqs. 30-33 seem to require a large amount of computations. However, most of the cross product terms can be computed very fast. As an indication of their computational complexities, a block diagram explicitly showing the procedure in calculating these coefficients for every set point in the trajectory in terms of multiplication and addition operations is shown in Figure 2. Table 1 summaries the computational complexities of the L-E, N-E, and G-D equations of motion in terms of required mathematical operations per trajectory set point.

### 3. Applications to Obtaining Simplified Dynamic Model

The main objective of developing the G-D equations of motion (Eqs. 29-33) is to facilitate the design of suitable controller for the manipulator in state-space or in obtaining approximate dynamic model which simplifies the design of the controller. Similar to the L-E equations of motion ( Eqs. 2-5 ), the G-D equations of motion are explicitly expressed in vector-matrix form and all the interaction and coupling reaction forces that are present in a manipulator can be easily identified in Eqs. 30-33. Furthermore the elements in the $D_{ij}$ matrix, the $H_i^{tran}$, the $H_i^{rot}$, and the $G_i$ vectors can be clearly identified as coming from the

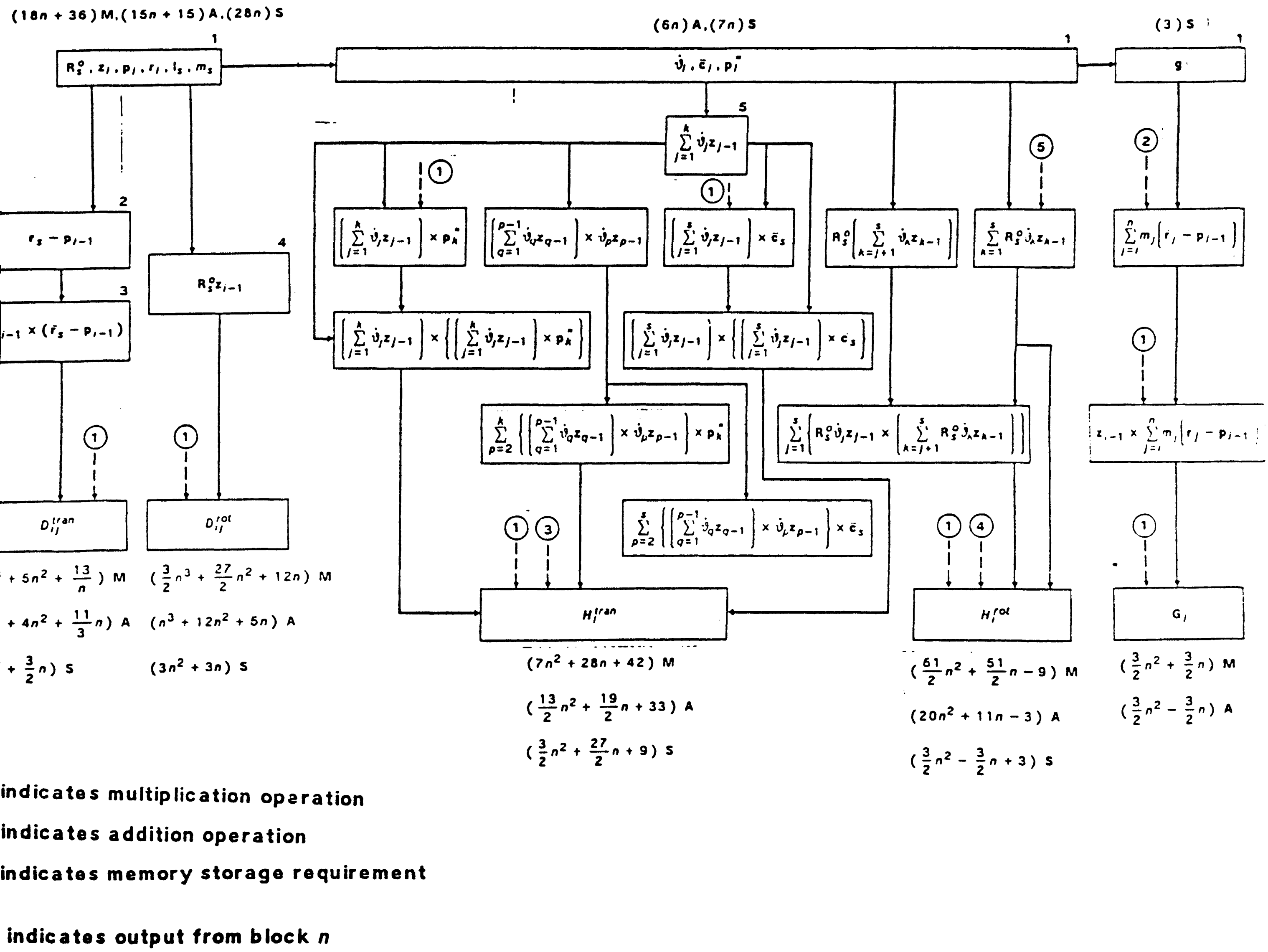



**Figure 2 Computational Procedure for $D_{ij}$, $H_i^{tran}$, $H_i^{rot}$, and $G_i$**

| Approach | Lagrange-Euler | Newton-Euler | Generalized d'Alembert |
|---|---|---|---|
| Multiplications | $\frac{128}{3}n^4 + \frac{512}{3}n^3 + \frac{739}{3}n^2 + \frac{160}{3}n$ | $132n$ | $\frac{13}{6}n^3 + \frac{105}{2}n^2 + \frac{268}{3}n + 69$ |
| Additions | $\frac{98}{3}n^4 + \frac{781}{6}n^3 + \frac{559}{3}n^2 + \frac{245}{6}n$ | $111n - 4$ | $\frac{4}{3}n^3 + 44n^2 + \frac{146}{3}n + 45$ |
| Kinematics Representation | 4×4 Homogeneous Matrices | Rotation Matrices and Position Vectors | Rotation Matrices and Position Vectors |
| Equations of Motion | Closed-form Differential Equations | Recursive Equations | Closed-form Differential Equations |

where $n$ = number of degrees-of-freedom of the robot arm

**Table 1 Comparison of Robot Arm Dynamics Computational Complexities**

translational and the rotational motion of the links. This greatly aids the construction of simplified dynamic model for control purpose. For example, for a PUMA or T3[2] robot arm, the elements of the $D_{ij}$ matrix come from the translational and rotational effects of the links. These effects depend on the joint variables and the inertial parameters of the manipulator. For the first three joints $(\theta_1, \theta_2, \theta_3)$, because of their usually long link length for maximum reach and long distance traveled between the initial position and final position, the effects of translational motion will dominate the rotational motion. In contrast to the first three joints, the rotational effects will dominate for the last three joints. Hence, one can simplify the computation of the $D_{ij}$ matrix by considering only the translational effects for the first three joints and the rotational effects for the last three joints. Similarly, one can evaluate the contribution of $H_i^{tran}$ and $H_i^{rot}$ and eliminate their computations if they are insignificant. The resulting simplified model retains all the major interaction and coupling reaction forces at a reduced computation time and greatly aids the design of an appropriate control law for controlling the robot arm.

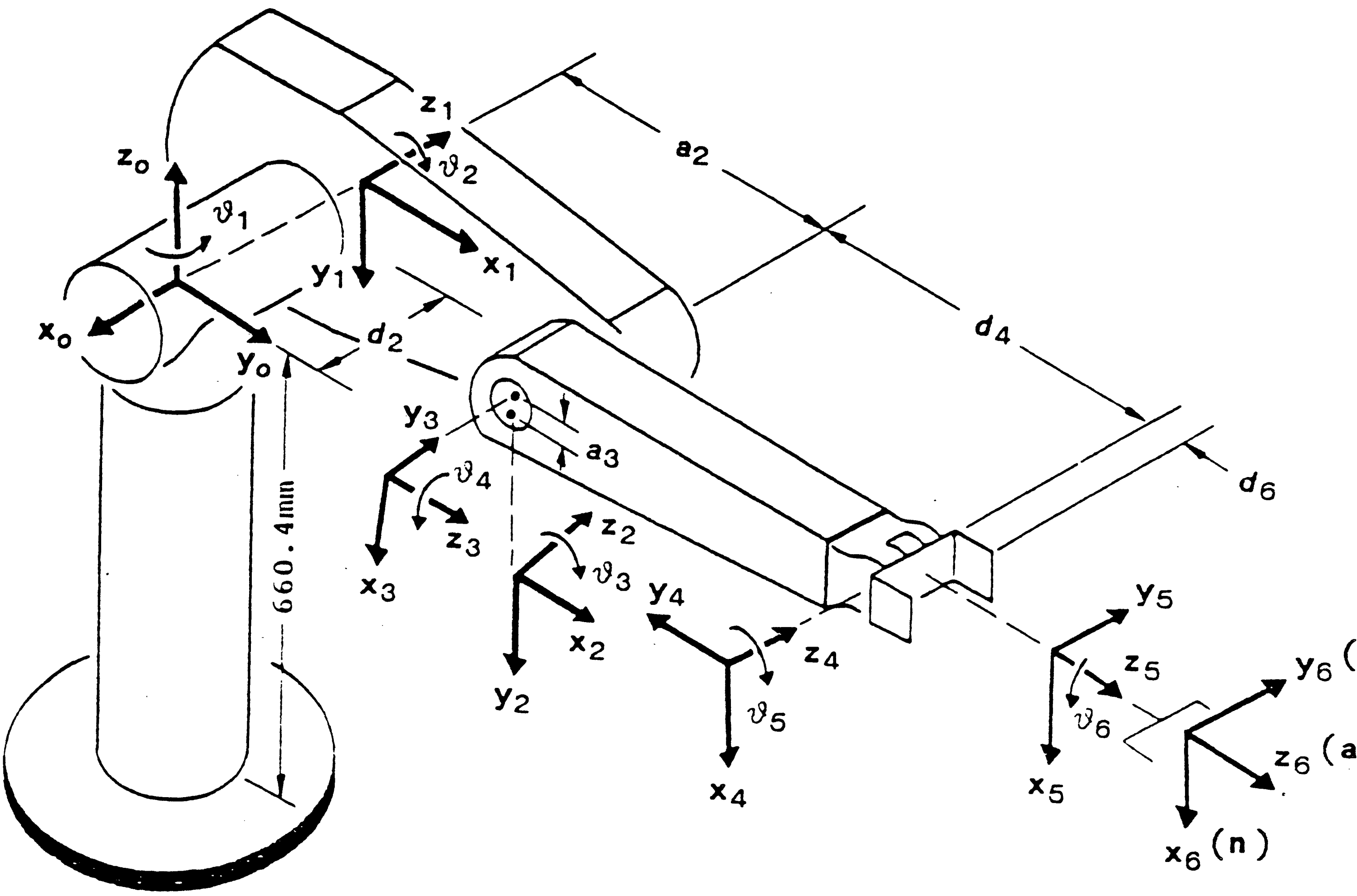
Current industrial robots are designed kinematically to reach any given point in their specific workspace without considering the efficiency of the dynamics and control strategies. Since physical link parameters play an important role in determinating its sphere of influence and the magnitude of the interaction and coupling reaction forces between joints, it is important to choose these parameter values properly so that the dynamic coefficients used in the controller can be computed rapidly to obtain the necessary applied torques to the joint actuators. Eqs. 30-33 can be used extensively to verify the effects of the links' length, the link inertia, and location of the center of mass on the magnitude of the interaction and coupling reaction forces between joints. Based on Eqs. 30-33, one can design a robot arm with a simplified dynamic model for control purpose.

## 4. Equations of Motion for a Three-link PUMA Robot

Consider the first three links of a PUMA robot arm shown in Figure 3. We would like to derive the generalized d'Alembert equations of motion for it. $m_i$, $i = 1,2,3$, represent its respect link mass, and the link inertia tensor matrices are

$$\mathbf{I}_i = \begin{bmatrix} I_{ixx} & 0 & 0 \\ 0 & I_{iyy} & 0 \\ 0 & 0 & I_{izz} \end{bmatrix} \quad ; \quad i = 1,2,3.$$

T3[2] is a trademark of Cincinnati Malcron




| PUMA Robot Link Coordinate Parameters | | | | | |
|---|---|---|---|---|---|
| Joint $i$ | $\vartheta_i$ | $\alpha_i$ | $a_i$ | $d_i$ | Range |
| 1 | 90 | -90 | 0 | 0 | -160 to +160 |
| 2 | 0 | 0 | 431.8 mm | 149.09 mm | -225 to +45 |
| 3 | 90 | 90 | - 20.32 mm | 0 | -45 to +225 |
| 4 | 0 | -90 | 0 | 433.07 mm | -110 to +170 |
| 5 | 0 | 90 | 0 | 0 | -100 to +100 |
| 6 | 0 | 0 | 0 | 56.25 mm | -266 to +266 |

**Figure 3 Link Coordinate Frames for a PUMA Robot Arm**

The homogeneous transformation matrices, $A_{i-1}^{i}$ $i = 1,2,3$, are found to be

$$A_0^1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \; A_1^2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \; A_2^3 = \begin{bmatrix} C_3 & 0 & S_3 & a_3 C_3 \\ S_3 & 0 & -C_3 & a_3 S_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Various rotation matrices used in deriving the equations of motion are

$$R_0^2 = R_0^1 \cdot R_1^2 = \begin{bmatrix} C_1 C_2 & -C_1 S_2 & -S_1 \\ S_1 C_2 & -S_1 S_2 & C_1 \\ -S_2 & -C_2 & 0 \end{bmatrix}$$

$$R_0^3 = R_0^2 \cdot R_2^3 = \begin{bmatrix} C_1 C_{23} & -S_1 & C_1 S_{23} \\ S_1 C_{23} & C_1 & S_1 S_{23} \\ -S_{23} & 0 & C_{23} \end{bmatrix}$$

$$R_1^3 = R_1^2 \cdot R_2^3 = \begin{bmatrix} C_{23} & 0 & S_{23} \\ S_{23} & 0 & -C_{23} \\ 0 & 1 & 0 \end{bmatrix}$$

where $C_i = \cos\theta_i$ , $S_i = \sin\theta_i$ , $C_{ij} = \cos(\theta_i + \theta_j)$ , $S_{ij} = \sin(\theta_i + \theta_j)$.
The physical parameters of the manipulator such as $p_i^*$ , $\bar{c}_i$ , $\bar{r}_i$ , and $p_i$ are

$$p_1^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \; p_2^* = R_0^1 \begin{bmatrix} a_2 C_2 \\ a_2 S_2 \\ 0 \end{bmatrix} = \begin{bmatrix} a_2 C_1 C_2 \\ a_2 S_1 C_2 \\ -a_2 S_2 \end{bmatrix} = \begin{bmatrix} p_{2x}^* \\ p_{2y}^* \\ p_{2z}^* \end{bmatrix}$$

$$p_3^* = R_0^2 \begin{bmatrix} a_3 C_3 \\ a_3 S_3 \\ 0 \end{bmatrix} = \begin{bmatrix} a_3 C_1 C_{23} \\ a_3 S_1 C_{23} \\ -a_3 S_{23} \end{bmatrix} = \begin{bmatrix} p_{3x}^* \\ p_{3y}^* \\ p_{3z}^* \end{bmatrix}$$

$$\mathbf{p}_0 = \mathbf{p}_1 = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad , \quad \mathbf{p}_2 = \mathbf{p}_1 + \mathbf{p}_2^* = \begin{bmatrix}a_2 C_1 C_2\\ a_2 S_1 C_2\\ -a_2 S_2\end{bmatrix} = \begin{bmatrix}p_{2x}\\p_{2y}\\p_{2z}\end{bmatrix}$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \mathbf{p}_3^* = \begin{bmatrix}a_2 C_1 C_2 + a_3 C_1 C_{23}\\ a_2 S_1 C_2 + a_3 S_1 C_{23}\\ -a_2 S_2 - a_3 S_{23}\end{bmatrix} = \begin{bmatrix}p_{3x}\\p_{3y}\\p_{3z}\end{bmatrix}$$

$$\bar{\mathbf{r}}_1 = \begin{bmatrix}\bar{r}_{1x}\\ \bar{r}_{1y}\\ \bar{r}_{1z}\end{bmatrix} \ , \ \bar{\mathbf{r}}_2 = \begin{bmatrix}\bar{r}_{2x}\\ \bar{r}_{2y}\\ \bar{r}_{2z}\end{bmatrix} \ , \ \bar{\mathbf{r}}_3 = \begin{bmatrix}\bar{r}_{3x}\\ \bar{r}_{3y}\\ \bar{r}_{3z}\end{bmatrix} \ , \ \bar{\mathbf{c}}_1 = \begin{bmatrix}\bar{c}_{1x}\\ \bar{c}_{1y}\\ \bar{c}_{1z}\end{bmatrix} \ , \ \bar{\mathbf{c}}_2 = \begin{bmatrix}\bar{c}_{2x}\\ \bar{c}_{2y}\\ \bar{c}_{2z}\end{bmatrix} \ , \ \bar{\mathbf{c}}_3 = \begin{bmatrix}\bar{c}_{3x}\\ \bar{c}_{3y}\\ \bar{c}_{3z}\end{bmatrix}$$

The following equivalences are used in the rest of the equations that follow.

$$\mathbf{p}_2 = \mathbf{p}_2^* \ , \ \mathbf{p}_3 = \mathbf{p}_2 + \mathbf{p}_3^* \ , \ \bar{\mathbf{r}}_1 = \bar{\mathbf{c}}_1 \ , \ \bar{\mathbf{r}}_2 = \bar{\mathbf{c}}_2$$

$$\bar{\mathbf{r}}_i = \mathbf{p}_{i-1} + \bar{\mathbf{c}}_i \quad , \quad i = 1, 2, 3.$$

The joint axes of motion with respect to the base coordinate frame are,

$$\mathbf{z}_0 = \begin{bmatrix}0\\0\\1\end{bmatrix} \ , \ \mathbf{z}_1 = \mathbf{R}_0^1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-S_1\\ C_1\\ 0\end{bmatrix} \ , \ \mathbf{z}_2 = \mathbf{R}_0^2\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-S_1\\ C_1\\ 0\end{bmatrix} \ , \ \mathbf{z}_3 = \mathbf{R}_0^3\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}C_1 S_{23}\\ S_1 S_{23}\\ C_{23}\end{bmatrix}$$

$$\mathbf{R}_1^0\mathbf{z}_0 = \begin{bmatrix}0\\-1\\0\end{bmatrix} \ , \ \mathbf{R}_2^0\mathbf{z}_0 = \begin{bmatrix}-S_2\\-C_2\\0\end{bmatrix} \ , \ \mathbf{R}_3^0\mathbf{z}_0 = \begin{bmatrix}-S_{23}\\0\\C_{23}\end{bmatrix}$$

$$\mathbf{R}_1^0\mathbf{z}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad , \quad \mathbf{R}_2^0\mathbf{z}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad , \quad \mathbf{R}_3^0\mathbf{z}_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$\mathbf{R}_1^0\mathbf{z}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad , \quad \mathbf{R}_2^0\mathbf{z}_2 = \begin{bmatrix}0\\0\\1\end{bmatrix} \quad , \quad \mathbf{R}_3^0\mathbf{z}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Using the $D_{ij}$ equation in Eq. 30, we derive the elements of the $\mathbf{D}(\theta)$ matrix.

$$D_{33} = (\mathbf{R}_3^0\mathbf{z}_2)^T\mathbf{I}_3(\mathbf{R}_3^0\mathbf{z}_2) + m_3\left\{\mathbf{z}_2\times(\bar{\mathbf{r}}_3 - \mathbf{p}_2)\right\} \cdot \left\{\mathbf{z}_2\times(\bar{\mathbf{r}}_3 - \mathbf{p}_2)\right\}$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix}^T \mathbf{I}_3 \begin{bmatrix}0\\1\\0\end{bmatrix} + m_3\left\{\begin{bmatrix}-S_1\\C_1\\0\end{bmatrix} \times \begin{bmatrix}\bar{c}_{3x}\\\bar{c}_{3y}\\\bar{c}_{3z}\end{bmatrix}\right\} \cdot \left\{\begin{bmatrix}-S_1\\C_1\\0\end{bmatrix} \times \begin{bmatrix}\bar{c}_{3x}\\\bar{c}_{3y}\\\bar{c}_{3z}\end{bmatrix}\right\}$$

$$= I_{3yy} + m_3\left\{\bar{c}_{3z}^2 + (S_1\bar{c}_{3y} + C_1\bar{c}_{3x})^2\right\}$$

$$D_{22} = (\mathbf{R}_2^0\mathbf{z}_1)^T\mathbf{I}_2(\mathbf{R}_2^0\mathbf{z}_1) + (\mathbf{R}_3^0\mathbf{z}_1)^T\mathbf{I}_3(\mathbf{R}_3^0\mathbf{z}_1) + m_2\left\{\mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1)\right\} \cdot \left\{\mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1)\right\}$$

$$+ m_3\left\{\mathbf{z}_1 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_1)\right\} \cdot \left\{\mathbf{z}_1 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_1)\right\}$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix}^T \mathbf{I}_2 \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}^T \mathbf{I}_3 \begin{bmatrix}0\\1\\0\end{bmatrix} + m_2\left\{\begin{bmatrix}-S_1\\C_1\\0\end{bmatrix} \times \begin{bmatrix}\bar{c}_{2x}\\\bar{c}_{2y}\\\bar{c}_{2z}\end{bmatrix}\right\} \cdot \left\{\begin{bmatrix}-S_1\\C_1\\0\end{bmatrix} \times \begin{bmatrix}\bar{c}_{2x}\\\bar{c}_{2y}\\\bar{c}_{2z}\end{bmatrix}\right\}$$

$$+ m_3\left\{\begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \bar{r}_{3x} \\ \bar{r}_{3y} \\ \bar{r}_{3z} \end{bmatrix}\right\} \cdot \left\{\begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \bar{r}_{3x} \\ \bar{r}_{3y} \\ \bar{r}_{3z} \end{bmatrix}\right\}$$

$$= I_{2zz} + I_{3yy} + m_2\left\{\bar{c}_{2z}^2 + (S_1\bar{c}_{2y} + C_1\bar{c}_{2x})^2\right\} + m_3\left\{\bar{r}_{3z}^2 + (S_1\bar{r}_{3y} + C_1\bar{r}_{3x})^2\right\}$$

$$D_{23} = (\mathbf{R}_3^0\mathbf{z}_1)^T\mathbf{I}_3(\mathbf{R}_3^0\mathbf{z}_2) + m_3\left\{\mathbf{z}_2 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_2)\right\} \cdot \left\{\mathbf{z}_1 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_1)\right\}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \mathbf{I}_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + m_3\left\{\begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \bar{c}_{3x} \\ \bar{c}_{3y} \\ \bar{c}_{3z} \end{bmatrix}\right\} \cdot \left\{\begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \bar{r}_{3x} \\ \bar{r}_{3y} \\ \bar{r}_{3z} \end{bmatrix}\right\}$$

$$= I_{3yy} + m_3\left\{\bar{c}_{3z}\bar{r}_{3z} + (S_1\bar{c}_{3y} + C_1\bar{c}_{3x})\ (S_1\bar{r}_{3y} + C_1\bar{r}_{3x})\right\}$$

$$D_{13} = (\mathbf{R}_3^0\mathbf{z}_0)^T\, \mathbf{I}_3\, (\mathbf{R}_3^0\mathbf{z}_2) + m_3\left\{\mathbf{z}_2 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_2)\right\} \cdot \left\{\mathbf{z}_0 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_0)\right\}$$

$$= \begin{bmatrix} -S_{23} \\ 0 \\ C_{23} \end{bmatrix}^T \mathbf{I}_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + m_3\begin{bmatrix} C_1\bar{c}_{3z} \\ S_1\bar{c}_{3z} \\ -S_1\bar{c}_{3y} - C_1\bar{c}_{3x} \end{bmatrix} \cdot \begin{bmatrix} -\bar{r}_{3y} \\ \bar{r}_{3x} \\ 0 \end{bmatrix}$$

$$= m_3(-C_1\bar{c}_{3z}\bar{r}_{3y} + S_1\bar{c}_{3z}\bar{r}_{3x})$$

$$D_{12} = (\mathbf{R}_2^0\mathbf{z}_0)^T\mathbf{I}_2(\mathbf{R}_2^0\mathbf{z}_1) + (\mathbf{R}_3^0\mathbf{z}_0)^T\mathbf{I}_3(\mathbf{R}_3^0\mathbf{z}_1) + m_2\left\{\mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1)\right\} \cdot \left\{\mathbf{z}_0 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_0)\right\}$$

$$+ m_3\left\{\mathbf{z}_1 \times (\overline{\mathbf{r}}_3 - \mathbf{p}_1)\right\} \cdot \left\{\mathbf{z}_0 \times (\overline{\mathbf{r}}_3 - \mathbf{p}_0)\right\}$$

$$= \begin{bmatrix} -S_2 \\ -C_2 \\ 0 \end{bmatrix}^T \mathbf{I}_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -S_{23} \\ 0 \\ C_{23} \end{bmatrix}^T \mathbf{I}_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + m_2 \begin{bmatrix} C_1\overline{r}_{3z} \\ S_1\overline{r}_{3z} \\ -S_1\overline{r}_{3y} - C_1\overline{r}_{3x} \end{bmatrix} \cdot \begin{bmatrix} -\overline{r}_{2y} \\ \overline{r}_{2x} \\ 0 \end{bmatrix} + m_3 \begin{bmatrix} C_1\overline{r}_{3z} \\ S_1\overline{r}_{3z} \\ -S_1\overline{r}_{3y} - C_1\overline{r}_{3x} \end{bmatrix} \cdot \begin{bmatrix} -\overline{r}_{3y} \\ \overline{r}_{3x} \\ 0 \end{bmatrix}$$

$$= m_2(-C_1\overline{r}_{3z}\overline{r}_{2y} + S_1\overline{r}_{3z}\overline{r}_{2x}) + m_3(-C_1\overline{r}_{3z}\overline{r}_{3y} + S_1\overline{r}_{3z}\overline{r}_{3x})$$

$$D_{11} = (\mathbf{R}_1^0\mathbf{z}_0)^T\mathbf{I}_1(\mathbf{R}_1^0\mathbf{z}_0) + (\mathbf{R}_2^0\mathbf{z}_0)^T\mathbf{I}_2(\mathbf{R}_2^0\mathbf{z}_0) + (\mathbf{R}_3^0\mathbf{z}_0)^T\mathbf{I}_3(\mathbf{R}_3^0\mathbf{z}_0)$$

$$+ m_1\left\{\mathbf{z}_0 \times (\overline{\mathbf{r}}_1 - \mathbf{p}_0)\right\} \cdot \left\{\mathbf{z}_0 \times (\overline{\mathbf{r}}_1 - \mathbf{p}_0)\right\} + m_2\left\{\mathbf{z}_0 \times (\overline{\mathbf{r}}_2 - \mathbf{p}_0)\right\} \cdot \left\{\mathbf{z}_0 \times (\overline{\mathbf{r}}_2 - \mathbf{p}_0)\right\}$$

$$+ m_3\left\{\mathbf{z}_0 \times (\overline{\mathbf{r}}_3 - \mathbf{p}_0)\right\} \cdot \left\{\mathbf{z}_0 \times (\overline{\mathbf{r}}_3 - \mathbf{p}_0)\right\}$$

$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}^T \mathbf{I}_1 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -S_2 \\ -C_2 \\ 0 \end{bmatrix}^T \mathbf{I}_2 \begin{bmatrix} -S_2 \\ -C_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -S_{23} \\ 0 \\ C_{23} \end{bmatrix}^T \mathbf{I}_3 \begin{bmatrix} -S_{23} \\ 0 \\ C_{23} \end{bmatrix}$$

$$+ m_1(\overline{r}_{1y}^2 + \overline{r}_{1x}^2) + m_2(\overline{r}_{2y}^2 + \overline{r}_{2x}^2) + m_3(\overline{r}_{3y}^2 + \overline{r}_{3x}^2)$$

$$= I_{1yy} + S_2^2 I_{2xx} + C_2^2 I_{2yy} + S_{23}^2 I_{3xx} + C_{23}^2 I_{3zz} + m_1(\overline{r}_{1y}^2 + \overline{r}_{1x}^2) + m_2(\overline{r}_{2y}^2 + \overline{r}_{2x}^2) + m_3(\overline{r}_{3y}^2 + \overline{r}_{3x}^2)$$

To derive the components of $H_i^{tran}(\theta, \dot{\theta})$ and $H_i^{rot}(\theta, \dot{\theta})$, we use Eqs. 31 and 32 respectively,

$$H_3^{tran} = m_3\Big[\dot\theta_1\mathbf{z}_0 \times (\dot\theta_1\mathbf{z}_0 \times \mathbf{p}_1^*) + (\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1) \times \Big\{(\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1) \times \mathbf{p}_2^*\Big\}$$

$$+ (\dot\theta_1\mathbf{z}_0 \times \dot\theta_2\mathbf{z}_1) \times \mathbf{p}_2^*\Big] \cdot \Big\{\mathbf{z}_2 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_2)\Big\}$$

$$+ m_3\Big[(\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1 + \dot\theta_3\mathbf{z}_2) \times \Big\{(\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1 + \dot\theta_3\mathbf{z}_2) \times \bar{\mathbf{c}}_3\Big\}$$

$$+ \Big\{(\dot\theta_1\mathbf{z}_0 \times \dot\theta_2\mathbf{z}_1) + (\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1) \times \dot\theta_3\mathbf{z}_2\Big\} \times \bar{\mathbf{c}}_3\Big] \cdot \Big\{\mathbf{z}_2 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_2)\Big\}$$

$$= m_3\left[\left\{\begin{bmatrix} S_1\dot\theta_2 \\ C_1\dot\theta_2 \\ \dot\theta_1 \end{bmatrix} \times \left\{\begin{bmatrix} -S_1\dot\theta_2 \\ C_1\dot\theta_2 \\ \dot\theta_1 \end{bmatrix} \times \begin{bmatrix} p_{2x}^* \\ p_{2y}^* \\ p_{2z}^* \end{bmatrix}\right\} + \begin{bmatrix} -C_1\dot\theta_1\dot\theta_2 \\ -S_1\dot\theta_1\dot\theta_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} p_{2x}^* \\ p_{2y}^* \\ p_{2z}^* \end{bmatrix}\right\}\right] \cdot \begin{bmatrix} C_1\bar{c}_{3x} \\ S_1\bar{c}_{3x} \\ -S_1\bar{c}_{3y} - C_1\bar{c}_{3z} \end{bmatrix}$$

$$+ m_3\left[\begin{bmatrix} -S_1(\dot\theta_2 + \dot\theta_3) \\ C_1(\dot\theta_2 + \dot\theta_3) \\ \dot\theta_1 \end{bmatrix} \times \left\{\begin{bmatrix} -S_1(\dot\theta_2 + \dot\theta_3) \\ C_1(\dot\theta_2 + \dot\theta_3) \\ \dot\theta_1 \end{bmatrix} \times \begin{bmatrix} \bar{c}_{3x} \\ \bar{c}_{3y} \\ \bar{c}_{3z} \end{bmatrix}\right\} + \left\{\begin{bmatrix} -C_1\dot\theta_1\dot\theta_2 \\ -S_1\dot\theta_1\dot\theta_2 \\ 0 \end{bmatrix}\right.\right.$$

$$\left.\left. + \begin{bmatrix} -S_1\dot\theta_2 \\ C_1\dot\theta_2 \\ \dot\theta_1 \end{bmatrix} \times \begin{bmatrix} -S_1\dot\theta_3 \\ C_1\dot\theta_3 \\ 0 \end{bmatrix}\right\} \times \begin{bmatrix} \bar{c}_{3x} \\ \bar{c}_{3y} \\ \bar{c}_{3z} \end{bmatrix}\right] \cdot \begin{bmatrix} C_1\bar{c}_{3x} \\ S_1\bar{c}_{3x} \\ -S_1\bar{c}_{3y} - C_1\bar{c}_{3z} \end{bmatrix}$$

$$= m_3\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \cdot \begin{bmatrix} C_1\bar{c}_{3x} \\ S_1\bar{c}_{3x} \\ -S_1\bar{c}_{3y} - C_1\bar{c}_{3z} \end{bmatrix}$$

where

$$\alpha = -(p^*_{2x} + \bar{c}_{3x})\dot{\theta}_1^2 - (S_1 C_1 p^*_{2y} + C_1 p^*_{2x} + S_1 C_1 \bar{c}_{3y} + C_1 \bar{c}_{3x})\dot{\theta}_2^2$$
$$- (S_1 C_1 \bar{c}_{3y} + C_1^2 \bar{c}_{3x})\dot{\theta}_3^2 - (2 S_1 p^*_{2x} + 2 S_1 \bar{c}_{3x})\dot{\theta}_1 \dot{\theta}_2$$
$$- (2 S_1 C_1 \bar{c}_{3y} + 2 C_1^2 \bar{c}_{3x})\dot{\theta}_2 \dot{\theta}_3 - (2 S_1 \bar{c}_{3z})\dot{\theta}_1 \dot{\theta}_3$$

$$\beta = -(p^*_{2y} + \bar{c}_{3y})\dot{\theta}_1^2 - (S_1 C_1 p^*_{2x} + S_1^2 p^*_{2y} + S_1 C_1 \bar{c}_{3x} + S_1^2 \bar{c}_{3y})\dot{\theta}_2^2$$
$$- (S_1 C_1 \bar{c}_{3x} + S_1^2 \bar{c}_{3y})\dot{\theta}_3^2 + 2(C_1 p^*_{2z} + C_1 \bar{c}_{3z})\dot{\theta}_1 \dot{\theta}_2$$
$$- (2 S_1^2 \bar{c}_{3y} + 2 S_1 C_1 \bar{c}_{3x})\dot{\theta}_2 \dot{\theta}_3 + (2 C_1 \bar{c}_{3z})\dot{\theta}_1 \dot{\theta}_3$$

$$\gamma = -(p^*_{2z} + \bar{c}_{3z})\dot{\theta}_2^2 - \bar{c}_{3z}\dot{\theta}_3^2 + (C_1 p^*_{2y} - C_1 p^*_{2z})\dot{\theta}_1 \dot{\theta}_2 - 2\bar{c}_{3z}\dot{\theta}_2 \dot{\theta}_3$$

After some simple manipulations, we obtain

$$H_3^{tran} = m_3 \Bigg[ \Big\{ C_1 \bar{c}_{3z}(-p^*_{2x} - \bar{c}_{3x}) + S_1 \bar{c}_{3z}(-p^*_{2y} - \bar{c}_{3y}) \Big\} \dot{\theta}_1^2$$

$$+ \Big\{ -S_1 \bar{c}_{3z} p^*_{2y} - C_1 \bar{c}_{3z} p^*_{2x} + (S_1 \bar{c}_{3y} + C_1 \bar{c}_{3x}) p^*_{2z} \Big\} \dot{\theta}_2^2 + \Big\{ (-S_1 \bar{c}_{3y} - C_1 \bar{c}_{3x})(C_1 p^*_{2y} - C_1 p^*_{2z}) \Big\} \dot{\theta}_1 \dot{\theta}_2 \Bigg]$$

From Eq. 32, we have

$$H_3^{rot} = (\mathbf{R}_3^0 \mathbf{z}_2)^T \mathbf{I}_3 \Big\{ \dot{\theta}_1 \mathbf{R}_3^0 \mathbf{z}_0 \times (\dot{\theta}_2 \mathbf{R}_3^0 \mathbf{z}_1 + \dot{\theta}_3 \mathbf{R}_3^0 \mathbf{z}_2) + \dot{\theta}_2 \mathbf{R}_3^0 \mathbf{z}_1 \times \dot{\theta}_3 \mathbf{R}_3^0 \mathbf{z}_2 \Big\}$$

$$+ \Big\{ \mathbf{R}_3^0 \mathbf{z}_2 \times (\dot{\theta}_1 \mathbf{R}_3^0 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{R}_3^0 \mathbf{z}_1 + \dot{\theta}_3 \mathbf{R}_3^0 \mathbf{z}_2) \Big\}^T \mathbf{I}_3 \Big\{ \dot{\theta}_1 \mathbf{R}_3^0 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{R}_3^0 \mathbf{z}_1 + \dot{\theta}_3 \mathbf{R}_3^0 \mathbf{z}_2 \Big\}$$

$$= \begin{bmatrix}0\\1\\0\end{bmatrix}^T \mathbf{I}_3 \left\{ \begin{bmatrix} -S_{23}\dot{\theta}_1 \\ 0 \\ C_{23}\dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ \dot{\theta}_2 + \dot{\theta}_3 \\ 0 \end{bmatrix} \right\} + \left\{ \begin{bmatrix}0\\1\\0\end{bmatrix} \times \begin{bmatrix} -S_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \\ C_{23}\dot{\theta}_1 \end{bmatrix} \right\}^T \mathbf{I}_3 \begin{bmatrix} -S_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \\ C_{23}\dot{\theta}_1 \end{bmatrix}$$

$$= -I_{3xx}C_{23}S_{23}\dot{\theta}_1^2 + I_{3zz}C_{23}S_{23}\dot{\theta}_1^2 = C_{23}S_{23}(I_{3zz} - I_{3xx})\dot{\theta}_1^2$$

Hence,

$$H_3 = H_3^{tran} + H_3^{rot} = \left\{ C_{23}S_{23}(I_{3zz} - I_{3xx}) - m_3 C_1 \bar{c}_{3z}(p_{2x}^* + \bar{c}_{3x}) - m_3 S_1 \bar{c}_{3z}(p_{2y}^* + \bar{c}_{3y}) \right\} \dot{\theta}_1^2$$

$$+ \left\{ p_{2z}^*(S_1\bar{c}_{3y} + C_1\bar{c}_{3x}) - S_1\bar{c}_{3z}p_{2y}^* - C_1\bar{c}_{3z}p_{2x}^* \right\} \dot{\theta}_2^2 + \left\{ (S_1\bar{c}_{3y} + C_1\bar{c}_{3x})(C_1 p_{2z}^* - C_1 p_{2y}^*) \right\} \dot{\theta}_1 \dot{\theta}_2$$

Similarly, using Eqs. 31 and 32, we obtain

$$H_2^{trans} = m_2 \left\{ \dot{\theta}_1 \mathbf{z}_0 \times (\dot{\theta}_1 \mathbf{z}_0 \times \mathbf{p}_1^*) \right\} \cdot \left\{ \mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1) \right\}$$

$$+ m_3 \Bigg[ \left\{ \dot{\theta}_1 \mathbf{z}_0 \times (\dot{\theta}_1 \mathbf{z}_0 \times \mathbf{p}_1^*) + \left\{ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1) \times \left\{ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1) \times \mathbf{p}_2^* \right\} \right\} \right.$$

$$\left. + (\dot{\theta}_1 \mathbf{z}_0 \times \dot{\theta}_2 \mathbf{z}_1) \times \mathbf{p}_2^* \Bigg] \cdot \left\{ \mathbf{z}_1 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_1) \right\}$$

$$+ m_2 \left[ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1) \times \left\{ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2 \right\} + (\dot{\theta}_1 \mathbf{z}_0 \times \dot{\theta}_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2 \right] \cdot \left\{ \mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1) \right\}$$

$$+\ m_3\Bigg[(\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1 + \dot\theta_3\mathbf{z}_2) \times \Big\{(\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1 + \dot\theta_3\mathbf{z}_2) \times \bar{\mathbf{c}}_3\Big\}$$

$$+ \Big\{(\dot\theta_1\mathbf{z}_0 \times \dot\theta_2\mathbf{z}_1) + (\dot\theta_1\mathbf{z}_0 + \dot\theta_2\mathbf{z}_1) \times \dot\theta_3\mathbf{z}_2\Big\} \times \bar{\mathbf{c}}_3\Bigg] \cdot \Big\{\mathbf{z}_1 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_1)\Big\}$$

$$= m_3\begin{bmatrix} -S_1C_1\dot\theta_2^2 p_{2y}^{\bullet} - C_1^2\dot\theta_2^2 p_{2x}^{\bullet} - 2S_1\dot\theta_1\dot\theta_2 p_{2z}^{\bullet} - \dot\theta_1^2 p_{2x}^{\bullet} \\ -S_1^2 p_{2y}^{\bullet}\dot\theta_2^2 - S_1C_1\dot\theta_2^2 p_{2x}^{\bullet} + 2C_1\dot\theta_1\dot\theta_2 p_{2z}^{\bullet} - \dot\theta_1^2 p_{2y}^{\bullet} \\ -S_1^2\dot\theta_2^2 p_{2z}^{\bullet} - C_1^2\dot\theta_2^2 p_{2x}^{\bullet} + C_1\dot\theta_1\dot\theta_2 p_{2y}^{\bullet} - C_1\dot\theta_1\dot\theta_2 p_{2z}^{\bullet} \end{bmatrix} \cdot \begin{bmatrix} C_1\bar{r}_{3z} \\ S_1\bar{r}_{3z} \\ -S_1\bar{r}_{3y} - C_1\bar{r}_{3x} \end{bmatrix}$$

$$+\ m_2\begin{bmatrix} -S_1C_1\dot\theta_2^2\bar{c}_{2y} - C_1^2\dot\theta_2^2\bar{c}_{2x} - 2S_1\bar{c}_{2z}\dot\theta_1\dot\theta_2 - \bar{c}_{2x}\dot\theta_1^2 \\ -S_1^2\dot\theta_2^2\bar{c}_{2y} - S_1C_1\dot\theta_2^2\bar{c}_{2x} + 2C_1\bar{c}_{2z}\dot\theta_1\dot\theta_2 - \dot\theta_1^2\bar{c}_{2y} \\ -S_1^2\bar{c}_{2z}\dot\theta_2^2 - C_1\bar{c}_{2z}\dot\theta_2^2 + C_1\bar{c}_{2y}\dot\theta_1\dot\theta_2 - C_1\bar{c}_{2z}\dot\theta_1\dot\theta_2 \end{bmatrix} \cdot \begin{bmatrix} C_1\bar{c}_{2z} \\ S_1\bar{c}_{2z} \\ -S_1\bar{c}_{2y} - C_1\bar{c}_{2x} \end{bmatrix}$$

$$+\ m_3\begin{bmatrix} -S_1C_1(\dot\theta_2 + \dot\theta_3)^2\bar{c}_{3y} - C_1^2(\dot\theta_2 + \dot\theta_3)^2\bar{c}_{3x} - 2S_1\bar{c}_{3z}\dot\theta_1(\dot\theta_2 + \dot\theta_3) - \bar{c}_{3x}\dot\theta_1^2 \\ -S_1^2(\dot\theta_2 + \dot\theta_3)\bar{c}_{3y} - S_1C_1(\dot\theta_2 + \dot\theta_3)^2\bar{c}_{3x} + 2C_1\bar{c}_{3z}\dot\theta_1(\dot\theta_2 + \dot\theta_3) - \dot\theta_1^2\bar{c}_{3y} \\ -S_1^2\bar{c}_{3z}(\dot\theta_2 + \dot\theta_3)^2 - C_1^2\bar{c}_{3z}(\dot\theta_2 + \dot\theta_3)^2 + C_1\bar{c}_{3y}\dot\theta_1(\dot\theta_2 + \dot\theta_3) - C_1\bar{c}_{3z}\dot\theta_1(\dot\theta_2 + \dot\theta_3) \end{bmatrix} \cdot \begin{bmatrix} C_1\bar{r}_{3z} \\ S_1\bar{r}_{3z} \\ -S_1\bar{r}_{3y} - C_1\bar{r}_{3x} \end{bmatrix}$$

$$= m_2\Big[(-C_1\bar{c}_{2x}\bar{c}_{2z} - S_1\bar{c}_{2y}\bar{c}_{2z})\dot\theta_1^2 + (S_1\bar{c}_{2y} + C_1\bar{c}_{2x})(C_1\bar{c}_{2z} - C_1\bar{c}_{2y})\dot\theta_1\dot\theta_2\Big]$$

$$+\ m_3\Big[(-C_1 p_{2x}^{\bullet}\bar{r}_{3z} - S_1 p_{2y}^{\bullet}\bar{r}_{3z} - C_1\bar{c}_{3x}\bar{r}_{3z} - S_1\bar{c}_{3y}\bar{r}_{3z})\dot\theta_1^2$$

$$+ (S_1 p_{2z}^{\bullet}\bar{r}_{3y} + C_1 p_{2z}^{\bullet}\bar{r}_{3x} - S_1 p_{2y}^{\bullet}\bar{r}_{3z} - C_1 p_{2x}^{\bullet}\bar{r}_{3z} + S_1\bar{c}_{3z}\bar{r}_{3y} + C_1\bar{c}_{3z}\bar{r}_{3x} - S_1\bar{c}_{3y}\bar{r}_{3z} - C_1\bar{c}_{3x}\bar{r}_{3z})\dot\theta_2^2$$

$$+ (S_1\bar{c}_{3z}\bar{r}_{3y} + C_1\bar{c}_{3z}\bar{r}_{3x} - S_1\bar{c}_{3y}\bar{r}_{3z} - C_1\bar{c}_{3x}\bar{r}_{3z})\dot\theta_3^2 + (C_1S_1 p_{2z}^{\bullet}\bar{r}_{3y} + C_1^2 p_{2z}^{\bullet}\bar{r}_{3x} - C_1S_1 p_{2y}^{\bullet}\bar{r}_{3y} - C_1^2 p_{2y}^{\bullet}\bar{r}_{3x}$$

$$+ C_1S_1\bar{c}_{3z}\bar{r}_{3y} + C_1^2\bar{c}_{3z}\bar{r}_{3x} - C_1S_1\bar{c}_{3y}\bar{r}_{3y} - C_1^2\bar{c}_{3y}\bar{r}_{3x})\dot{\theta}_1\dot{\theta}_2 + 2(S_1\bar{c}_{3z}\bar{r}_{3y} + C_1\bar{c}_{3z}\bar{r}_{3x} - S_1\bar{c}_{3y}\bar{r}_{3z} - C_1\bar{c}$$

$$+ (C_1S_1\bar{c}_{3z}\bar{r}_{3y} + C_1^2\bar{c}_{3z}\bar{r}_{3x} - C_1S_1\bar{c}_{3y}\bar{r}_{3y} - C_1^2\bar{c}_{3y}\bar{r}_{3x})\dot{\theta}_1\dot{\theta}_3\Big]$$

$$H_2^{rot} = (\mathbf{R}_2^0\mathbf{z}_1)^T\mathbf{I}_2(\dot{\theta}_1\mathbf{R}_2^0\mathbf{z}_0 \times \dot{\theta}_2\mathbf{R}_2^0\mathbf{z}_1)$$

$$+ (\mathbf{R}_3^0\mathbf{z}_1)^T\mathbf{I}_3\Big\{\dot{\theta}_1\mathbf{R}_3^0\mathbf{z}_0 \times (\dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 + \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2) + \dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 \times \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2\Big\}$$

$$+ \Big\{\mathbf{R}_2^0\mathbf{z}_1 \times (\dot{\theta}_1\mathbf{R}_2^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_2^0\mathbf{z}_1)\Big\}^T \mathbf{I}_2(\dot{\theta}_1\mathbf{R}_2^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_2^0\mathbf{z}_1)$$

$$+ \Big\{\mathbf{R}_3^0\mathbf{z}_1 \times (\dot{\theta}_1\mathbf{R}_3^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 + \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2)\Big\}^T \mathbf{I}_3(\dot{\theta}_1\mathbf{R}_3^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 + \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T \mathbf{I}_2 \begin{bmatrix} -C_2\dot{\theta}_1\dot{\theta}_2 \\ S_2\dot{\theta}_1\dot{\theta}_2 \\ 0 \end{bmatrix} + 0 + \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -S_2\dot{\theta}_1 \\ -C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \right\}^T \mathbf{I}_2 \begin{bmatrix} -S_2\dot{\theta}_1 \\ -C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + C_{23}S_{23}(I_{3zz} - I_{3xx})\dot{\theta}_1^2$$

$$= \Big\{S_2C_2(I_{2yy} - I_{2xx}) + C_{23}S_{23}(I_{3zz} - I_{3xx})\Big\}\dot{\theta}_1^2$$

$$H_1^{trans} = m_2\Big[\dot{\theta}_1\mathbf{z}_0 \times (\dot{\theta}_1\mathbf{z}_0 \times \mathbf{p}_1^*)\Big] \cdot \Big\{\mathbf{z}_0 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_0)\Big\}$$

$$+ m_3\Big[\Big\{\dot{\theta}_1\mathbf{z}_0 \times (\dot{\theta}_1\mathbf{z}_0 \times \mathbf{p}_1^*) + \Big\{(\dot{\theta}_1\mathbf{z}_0 + \dot{\theta}_2\mathbf{z}_1) \times \Big\{(\dot{\theta}_1\mathbf{z}_0 + \dot{\theta}_2\mathbf{z}_1) \times \mathbf{p}_2^*\Big\}\Big\}$$

$$+ (\dot{\theta}_1 \mathbf{z}_0 \times \dot{\theta}_2 \mathbf{z}_1) \times \mathbf{p}_2^{\bullet} \Big] \cdot \Big\{ \mathbf{z}_0 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_0) \Big\} + m_1 \Big\{ \dot{\theta}_1 \mathbf{z}_0 \times (\dot{\theta}_1 \mathbf{z}_0 \times \bar{\mathbf{c}}_1) \Big\} \cdot \Big\{ \mathbf{z}_0 \times (\bar{\mathbf{r}}_1 - \mathbf{p}_0) \Big\}$$

$$+ m_2 \Big[ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1) \times \Big\{ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2 \Big\} + (\dot{\theta}_1 \mathbf{z}_0 \times \dot{\theta}_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2 \Big] \cdot \Big\{ \mathbf{z}_0 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_0) \Big\}$$

$$+ m_3 \Big[ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1 + \dot{\theta}_3 \mathbf{z}_2) \times \Big\{ (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_1 \mathbf{z}_1 + \dot{\theta}_3 \mathbf{z}_2) \times \bar{\mathbf{c}}_3 \Big\}$$

$$+ \Big\{ (\dot{\theta}_1 \mathbf{z}_0 \times \dot{\theta}_2 \mathbf{z}_1) + (\dot{\theta}_1 \mathbf{z}_0 + \dot{\theta}_2 \mathbf{z}_1) \times \dot{\theta}_3 \mathbf{z}_2 \Big\} \times \bar{\mathbf{c}}_3 \Big] \cdot \Big\{ \mathbf{z}_0 \times (\bar{\mathbf{r}}_3 - \mathbf{p}_0) \Big\}$$

$$= m_3 \begin{bmatrix} - S_1 C_1 \dot{\theta}_2^2 p_{2y}^{\bullet} - C_1^2 \dot{\theta}_2^2 p_{2x}^{\bullet} - 2 S_1 \dot{\theta}_1 \dot{\theta}_2 p_{2x}^{\bullet} - \dot{\theta}_1^2 p_{2x}^{\bullet} \\ - S_1^2 p_{2y}^{\bullet} \dot{\theta}_2^2 - S_1 C_1 \dot{\theta}_2^2 p_{2x}^{\bullet} + 2 C_1 \dot{\theta}_1 \dot{\theta}_2 p_{2x}^{\bullet} - \dot{\theta}_1^2 p_{2y}^{\bullet} \\ A \end{bmatrix} \cdot \begin{bmatrix} -\bar{r}_{3y} \\ \bar{r}_{3x} \\ 0 \end{bmatrix} + m_1 \begin{bmatrix} -\bar{c}_{1x} \dot{\theta}_1^2 \\ -\bar{c}_{1y} \dot{\theta}_1^2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\bar{c}_{1y} \\ \bar{c}_{1x} \\ 0 \end{bmatrix}$$

$$+ m_2 \begin{bmatrix} - S_1 C_1 \dot{\theta}_2^2 \bar{c}_{2y} - C_1^2 \dot{\theta}_2^2 \bar{c}_{2x} - 2 S_1 \bar{c}_{2x} \dot{\theta}_1 \dot{\theta}_2 - \bar{c}_{2x} \dot{\theta}_1^2 \\ - S_1^2 \bar{c}_{2y} \dot{\theta}_2^2 - S_1 C_1 \dot{\theta}_2^2 \bar{c}_{2x} + 2 C_1 \dot{\theta}_1 \dot{\theta}_2 \bar{c}_{2x} - \dot{\theta}_1^2 \bar{c}_{2y} \\ B \end{bmatrix} \cdot \begin{bmatrix} -\bar{r}_{2y} \\ \bar{r}_{2x} \\ 0 \end{bmatrix}$$

$$+ m_3 \begin{bmatrix} - S_1 C_1 (\dot{\theta}_2 + \dot{\theta}_3)^2 \bar{c}_{3y} - C_1^2 (\dot{\theta}_2 + \dot{\theta}_3)^2 \bar{c}_{3x} - 2 S_1 \bar{c}_{3x} \dot{\theta}_1 (\dot{\theta}_2 + \dot{\theta}_3) - \bar{c}_{3x} \dot{\theta}_1^2 \\ - S_1^2 (\dot{\theta}_2 + \dot{\theta}_3)^2 \bar{c}_{3y} - S_1 C_1 (\dot{\theta}_2 + \dot{\theta}_3)^2 \bar{c}_{3x} + 2 C_1 \bar{c}_{3x} \dot{\theta}_1 (\dot{\theta}_2 + \dot{\theta}_3) - \bar{c}_{3y} \dot{\theta}_1^2 \\ C \end{bmatrix} \cdot \begin{bmatrix} -\bar{r}_{3y} \\ \bar{r}_{3x} \\ 0 \end{bmatrix}$$

$$= m_2 \Big[ (\bar{c}_{2x} \bar{r}_{2y} - \bar{c}_{2y} \bar{r}_{2x}) \dot{\theta}_1^2 + (S_1 C_1 \bar{c}_{2y} \bar{r}_{2y} + C_1^2 \bar{c}_{2x} \bar{r}_{2y} - S_1^2 \bar{c}_{2y} \bar{r}_{2x} - S_1 C_1 \bar{c}_{2x} \bar{r}_{2x}) \dot{\theta}_2^2$$

$$+ 2(S_1\bar{c}_{2z}\bar{r}_{2y} + C_1\bar{c}_{2z}\bar{r}_{2x})\dot{\theta}_1\dot{\theta}_2\Big] + m_3\Big[(\dot{p}_{2x}\bar{r}_{3y} - \dot{p}_{2y}\bar{r}_{3x} + \bar{c}_{3x}\bar{r}_{3y} - \bar{c}_{3y}\bar{r}_{3x})\dot{\theta}_1^2$$

$$+ (S_1C_1\dot{p}_{2y}\bar{r}_{3y} + C_1^2\dot{p}_{2x}\bar{r}_{3y} - S_1^2\dot{p}_{2y}\bar{r}_{3x} - S_1C_1\dot{p}_{2x}\bar{r}_{3x} + S_1C_1\bar{c}_{3y}\bar{r}_{3y} + C_1^2\bar{c}_{3x}\bar{r}_{3y}$$

$$- S_1^2\bar{c}_{3y}\bar{r}_{3x} - S_1C_1\bar{c}_{3x}\bar{r}_{3x})\dot{\theta}_2^2 + (S_1C_1\bar{c}_{3y}\bar{r}_{3y} + C_1^2\bar{c}_{3x}\bar{r}_{3y} - S_1^2\bar{c}_{3y}\bar{r}_{3x} - S_1C_1\bar{c}_{3x}\bar{r}_{3x})\dot{\theta}_3^2$$

$$+ 2(S_1\dot{p}_{2z}\bar{r}_{3y} + C_1\dot{p}_{2z}\bar{r}_{3x} + S_1\bar{c}_{3z}\bar{r}_{3y} + C_1\bar{c}_{3z}\bar{r}_{3x})\dot{\theta}_1\dot{\theta}_2$$

$$+ 2(S_1C_1\bar{c}_{3y}\bar{r}_{3y} + C_1^2\bar{c}_{3x}\bar{r}_{3y} - S_1^2\bar{c}_{3y}\bar{r}_{3x} - S_1C_2\bar{c}_{3x}\bar{r}_{3x})\dot{\theta}_2\dot{\theta}_3 + 2(S_1\bar{c}_{3z}\bar{r}_{3y} + C_1\bar{c}_{3z}\bar{r}_{3x})\dot{\theta}_1\dot{\theta}_3\Big]$$

where $A, B$, and $C$ need not to be computed.

$$H_1^{rot} = (\mathbf{R}_1^0\mathbf{z}_0 \times \dot{\theta}_1\mathbf{R}_1^0\mathbf{z}_0)^T\mathbf{I}_1\,(\dot{\theta}_1\mathbf{R}_1^0\mathbf{z}_0) + (\mathbf{R}_2^0\mathbf{z}_0)^T\mathbf{I}_2\,(\dot{\theta}_1\mathbf{R}_2^0\mathbf{z}_0 \times \dot{\theta}_2\mathbf{R}_2^0\mathbf{z}_1)$$

$$+ \left\{\mathbf{R}_2^0\mathbf{z}_0 \times (\dot{\theta}_1\mathbf{R}_2^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_2^0\mathbf{z}_1)\right\}^T\mathbf{I}_2\,(\dot{\theta}_1\mathbf{R}_2^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_2^0\mathbf{z}_1)$$

$$+ (\mathbf{R}_3^0\mathbf{z}_0)^T\mathbf{I}_3\left\{\dot{\theta}_1\mathbf{R}_3^0\mathbf{z}_0 \times (\dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 + \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2) + \dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 \times \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2\right\}$$

$$+ \left\{\mathbf{R}_3^0\mathbf{z}_0 \times (\dot{\theta}_1\mathbf{R}_3^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 + \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2)\right\}^T\mathbf{I}_3(\dot{\theta}_1\mathbf{R}_3^0\mathbf{z}_0 + \dot{\theta}_2\mathbf{R}_3^0\mathbf{z}_1 + \dot{\theta}_3\mathbf{R}_3^0\mathbf{z}_2)$$

$$= \begin{bmatrix} -S_2 \\ -C_2 \\ 0 \end{bmatrix}^T \mathbf{I}_2 \begin{bmatrix} -C_2\dot{\theta}_1\dot{\theta}_2 \\ S_2\dot{\theta}_1\dot{\theta}_2 \\ 0 \end{bmatrix} + \left\{\begin{bmatrix} -S_2 \\ -C_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -S_2\dot{\theta}_1 \\ -C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}\right\}^T \mathbf{I}_2 \begin{bmatrix} -S_2\dot{\theta}_1 \\ -C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}\Bigg\}$$

$$+\left\{\begin{bmatrix}-S_{23}\\0\\C_{23}\end{bmatrix}\times\begin{bmatrix}-S_{23}\dot{\theta}_1\\0.\\C_{23}\dot{\theta}_1\end{bmatrix}\times\begin{bmatrix}0\\\dot{\theta}_2+\dot{\theta}_3\\0\end{bmatrix}\right\}+\left\{\begin{bmatrix}-S_{23}\\0\\C_{23}\end{bmatrix}\times\begin{bmatrix}-S_{23}\dot{\theta}_1\\\dot{\theta}_2+\dot{\theta}_3\\C_{23}\dot{\theta}_1\end{bmatrix}\right\}^T\mathbf{I}_3\begin{bmatrix}-S_{23}\dot{\theta}_1\\\dot{\theta}_2+\dot{\theta}_3\\C_{23}\dot{\theta}_1\end{bmatrix}$$

$$= 2(S_2C_2I_{2xx} - S_2C_2I_{2yy})\dot{\theta}_1\dot{\theta}_2 + 2(S_{23}C_{23}I_{3xx} - S_{23}C_{23}I_{3zz})\dot{\theta}_1\dot{\theta}_2 + 2(S_{23}C_{23}I_{3xx} - S_{23}C_{23}I_{3zz})\dot{\theta}_1\dot{\theta}_3$$

To derive the elements of the gravity vector $\mathbf{G}(\theta)$, we use Eq. 33,

$$G_3 = -\mathbf{g}\cdot\left[\mathbf{z}_2\times m_3(\bar{\mathbf{r}}_3-\mathbf{p}_2)\right] = -\begin{bmatrix}0\\0\\g\end{bmatrix}\cdot\left[\begin{bmatrix}-S_1\\C_1\\0\end{bmatrix}\times\begin{bmatrix}m_3\bar{c}_{3x}\\m_3\bar{c}_{3y}\\m_3\bar{c}_{3z}\end{bmatrix}\right]$$

$$= m_3gS_1\bar{c}_{3y} + m_3gC_1\bar{c}_{3x}$$

$$G_2 = -\mathbf{g}\cdot\left[\mathbf{z}_1\times\left\{m_2(\bar{\mathbf{r}}_2-\mathbf{p}_1)+m_3(\bar{\mathbf{r}}_3-\mathbf{p}_1)\right\}\right] = \begin{bmatrix}0\\0\\g\end{bmatrix}\cdot\left[\begin{bmatrix}-S_1\\C_1\\0\end{bmatrix}\times\begin{bmatrix}m_2\bar{r}_{2x}+m_3\bar{r}_{3x}\\m_2\bar{r}_{2y}+m_3\bar{r}_{3y}\\m_2\bar{r}_{2z}+m_3\bar{r}_{3z}\end{bmatrix}\right]$$

$$= m_2g(\bar{r}_{2y}S_1 + \bar{r}_{2x}C_1) + m_3g(\bar{r}_{3y}S_1 + \bar{r}_{3x}C_1)$$

$$G_1 = -\mathbf{g}\cdot\left[\mathbf{z}_0\times\begin{bmatrix}m_1\bar{r}_{1x}+m_2\bar{r}_{2x}+m_3\bar{r}_{3x}\\m_1\bar{r}_{1y}+m_2\bar{r}_{2y}+m_3\bar{r}_{3y}\\m_1\bar{r}_{1z}+m_2\bar{r}_{2z}+m_3\bar{r}_{3z}\end{bmatrix}\right] = -\begin{bmatrix}0\\0\\g\end{bmatrix}\cdot\left[\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}m_1\bar{r}_{1x}+m_2\bar{r}_{2x}+m_3\bar{r}_{3x}\\m_1\bar{r}_{1y}+m_2\bar{r}_{2y}+m_3\bar{r}_{3y}\\m_1\bar{r}_{1z}+m_2\bar{r}_{2z}+m_3\bar{r}_{3z}\end{bmatrix}\right] = 0$$

Thus the equations of motion for the first three links of a PUMA robot arm are:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{12} & D_{22} & D_{23} \\ D_{13} & D_{23} & D_{33} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} H_1^{tran} \\ H_2^{tran} \\ H_3^{tran} \end{bmatrix} + \begin{bmatrix} H_1^{rot} \\ H_2^{rot} \\ H_3^{rot} \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix}$$

## 5. Conclusion

Three different formulations for robot arm dynamics have been presented and discussed. The L-E equations of motion can be put in a well structured form, but computationally it is very difficult to utilize for real time control purposes unless the equations of motion are simplified. The N-E formulation results in a very efficient set of recursive equations, but they are very difficult to use for deriving advanced control laws. The G-D equations of motion give well "structured" equations at the expense of higher computations. In addition to having faster computation time than the L-E equations of motion, the G-D equations of motion explicitly indicate the contributions of the translational and rotational effects of the links. Such information is useful for control analysis in obtaining an appropriate approximate model of the manipulator. To briefly summarize the results, an user is able to choose between a formulation which is highly structured but computationally inefficient ( L-E ), a formulation which has efficient computations at the expense of the "structure" of the equations of motion ( N-E ), and a formulation which retains the "structure" of the problem with only a moderate computing penalty ( G-D ).

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