

The (s,S) Policy for the Production/Inventory
System with Compound Poisson Demands

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Technical Report 88-2
April 1988

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Abstract

A production/inventory system is considered for items produced by a single production facility. The demand for the item is assumed to arrive according to a compound Poisson process. The processing time required to produce an item is assumed to follow an arbitrary distribution. An (s,S) policy is considered in which production stops at the instant that the inventory level is raised to S and production begins again at an inspection point when the inventory level is observed to have dropped to or below s for the first time. The time interval between two successive inspection points during a non-production period is a random variable which follows an arbitrary distribution. Under a cost structure which includes a set-up cost, a linear holding cost and a linear backorder cost, an expression for the expected cost per unit time for given control values is obtained, and using a convex property of the cost function, a procedure to find the optimal (s,S) policy is presented. The continuous review (s,S) policy with compound Poisson demands is also discussed.

1. Introduction

There has been a considerable amount of work on the (s,S) inventory control policy. Most of these studies treat inventory systems in which as many items as needed can be replenished all at once. While this is a realistic assumption in many cases, in some real manufacturing systems, the production facility can only produce one item at a time, and hence inventory can only be replenished on an item-by-item basis. Motivated by this fact, in this paper we analyze production/inventory systems where inventory is replenished on an item-by-item basis.

In the inventory system considered here a single production facility produces a single type of item. The demand for the item is assumed to follow a compound Poisson process: demand arrives according to a Poisson process with rate λ and the size of each demand, X , is a random variable following an arbitrary discrete distribution. The processing time for producing (replenishing) an item is assumed to be an independent and identically distributed random variable U which follows an arbitrary distribution. For stability of the system, we assume that $\lambda E(X)E(U) < 1$. If a demand occurs, it is supplied directly from the inventory, if there is a sufficient inventory of items available. If the inventory is not available, the demand is backordered. When the "on hand inventory" is not sufficient to meet the demand, only the unsatisfied portion of the demand is backordered. The relevant costs which are to be traded off in this paper include a set-up cost, a holding cost and a backorder cost. We assume that inventory holding costs are incurred linearly over time with respect to the inventory level; backorder costs are incurred linearly over time with respect to the backorder level; a set-up cost is incurred each time the production facility is turned on. The objective is to find a production/inventory policy to minimize the expected cost per unit time in the long run.

Since (s,S) policies are known to be effective in a variety of inventory situations[2,10,11], the operating policy considered in this paper is an (s,S) policy. The characteristics of the (s,S) policy considered in this paper will be described below.

As soon as the inventory level is raised to S , the production facility is turned off and a non-production period begins. During the non-production period, the inventory level is inspected only at certain points in time (inspection points). The time interval between two successive inspections during the non-production period is a random variable following an arbitrary distribution. The non-production period lasts until an inspection point is encountered at which the inventory level is observed to have first reached or dropped below s . At this point, the production facility is turned on and a production period begins. During the production period, inventory is replenished on an item-by-item basis. If the inventory level reaches S , the production period ends and the next non-production period begins, which initiates another cycle in the (s,S) policy system (refer to figure 1).

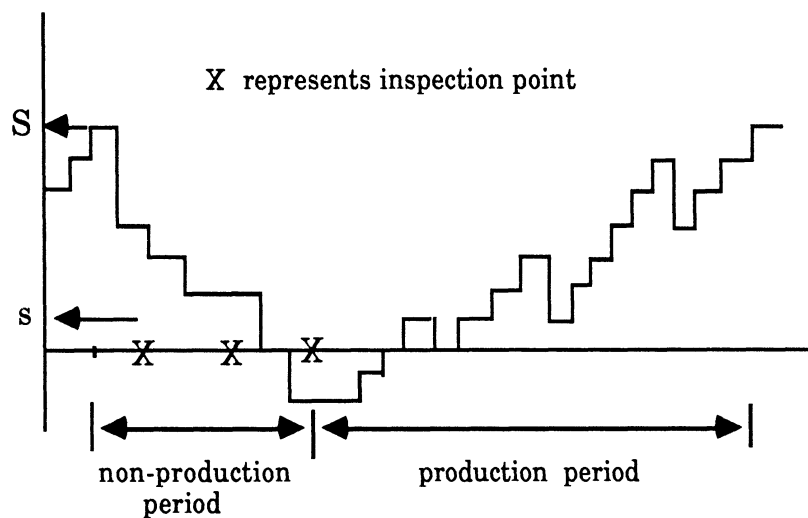


Fig. 1. The (s,S) Policy with Compound Poisson Demands and Irregular Inspection Intervals

The random inspection interval policy in this paper would be used, for example, in situations where the production facility is used for some secondary work during a non-production period. Suppose each secondary work takes a random amount of time. Then, the most reasonable inspection policy in this situation would be to inspect the inventory level at the end of each such secondary work, to determine whether the facility can be used for the

secondary work once more, or whether it should be switched back to the primary mode of operation.

Since the replenishment in our model is made on an item-by-item basis, it is easy to note that our model has an analogy with queueing systems. In fact, the work on control of queueing systems, for instance, the papers by Heyman[4], Yadin and Naor[12], Bell[2], Sobel[8] and Lee and Srinivasan[5], have a close relationship with production/inventory systems. Using this inventory-queueing analogy, some studies have been done on (s,S) policies for the production/inventory systems in which the inventory is replenished item-by-item.

Gavish and Graves[3] considered a continuous review (s,S) production/inventory system with unit Poisson demand arrivals and deterministic processing times. They developed a fairly efficient procedure to find the optimal (s,S) values by using an underlying $M/D/1$ queueing system. Tijms[9] studied a similar continuous review (s,S) system with arbitrary processing times and under a more general cost structure. To find the optimal control values, he used a denumerable state Semi-Markov decision process. However, as pointed out by Gavish and Graves[3], the computational time using a Markov decision process approach for solving such problems is much higher than that using an intelligent search technique. For the batch demand case, Altiok[1] considered a problem with a compound Poisson demand process and phase-type unit processing times. This model uses an underlying birth and death process to obtain the steady state probability of each inventory level for a given policy and found optimal control values by using a search technique. However, in this paper no special properties for the cost function were proved, that would assist in the search for the optimum.

Lee and Srinivasan[6] developed an algorithm to find the optimal control values for the continuous review (s,S) system with unit Poisson demands and arbitrary processing times. In this paper, some properties on the convexity and unimodality of the components of the cost function were proved. These properties result in the development of a very simple and effective search algorithm. In particular, the solution found by this algorithm is always guaranteed to be a global optimum.

The (s,S) policy in this paper extends the work of Lee and Srinivasan[6] to the situation where both bulk arrivals and arbitrary inspection intervals are considered. The general approach used in this paper is basically similar to that used in the earlier work. However, the analysis for this problem is much more complicated since the inventory level at either an inspection point or the beginning of a production period does not have a deterministic value. To overcome this difficulty, we will use some results derived by Lee and Srinivasan[5].

In the following sections, an expression for the expected cost per unit time for given control values will be derived and after that an efficient procedure to find the optimal control values will be presented. In addition to the policy which considers the inspection of the inventory at random points, the situation where the inventory level is continuously monitored during the non-production period will be also discussed.

2. Analysis of the Model

2.1. Notation

For analytical convenience, we set $r=S-s$, and throughout the paper we will use (r,S) instead of (s,S) to describe a policy. Thus, the (r,S) policy represents the policy with $S-r$ as a lower control value and S as an upper control value. Also, for notational convenience, throughout this paper, for any discrete random variable (r.v.) D used in the analysis, we adopt the following notation:

$$d_i = P(D=i),$$

$$D(z) = \sum_{i=0}^{\infty} d_i z^i, \quad \text{the probability generating function (p.g.f.) of } D,$$

$$d^{(1)} = E(D),$$

$$d^{(2)} = E(D(D-1)).$$

Similarly, for any continuous r.v. C used in the analysis, $C(\cdot)$ and $C^*(\cdot)$ will denote the cumulative density function (c.d.f.) and Laplace Stieltjes Transform (LST) of C respectively.

We use the following notation.

X = batch size of a demand, r.v.,

V = inspection interval during the non-production period, r.v.,

A = number of demand requests which arrive during V , r.v.,

B = number of items demanded during V , r.v.,

U = processing time to produce a unit item, r.v.,

G = number of demand requests which arrive during U , r.v.,

H = number of items demanded during U , r.v.,

$X^{*(i)}$ = i -fold convolution of X ,

$x_j^{*(i)}$ = $P\{i\text{-fold convolution of } X \text{ is } j\}$,

Y_r = total number of items demanded during a non-production period when $r=S-s$ is used as a control value, r.v.,

$I(r,S)$ = inventory level when production period begins, when r and S are used as control values, r.v.,

K = set-up cost,

c_h = holding cost/item/unit time,

c_b = back order cost/item/unit time,

$C_N(r,S)$ = expected cost during a non-production period when r and S are used as control values,

$C_P(r,S)$ = expected cost during a production period,

$C(r,S) = C_N(r,S) + C_P(r,S)$,

$L(r)$ = expected length of a cycle when r is used as a control value,

$TC(r,S)$ = expected cost per unit time.

2.2. General Approach

Our first objective is to obtain an expression for $TC(r,S)$, the expected cost per unit time for given control values r and S . To obtain this, we use a regenerative process theorem, since the epoch marking the start of a cycle forms a regeneration point. It follows that what we have is a renewal reward process, and thus from the renewal reward theorem (see, for example, Ross[7]), the expected cost per unit time, when r and S are used as control values, is obtained by

$$TC(r,S) = \frac{C(r,S) + K}{L(r)}. \quad (2.1)$$

Note that we use $L(r)$ in equation (2.1) instead of $L(r,S)$ to denote the expected cycle time when r and S are used as control values. This is because the expected cycle time is a function of r alone as will be explained later. Since the term $C(r,S)$ is expressed as a sum of $C_N(r,S)$ and $C_P(r,S)$, to obtain $TC(r,S)$ by using equation (2.1), we need to find the terms $C_N(r,S)$ and $C_P(r,S)$ as well as the term $L(r)$. We now show how the terms $C_N(r,S)$, $C_P(r,S)$ and $L(r)$ are obtained.

2.3. Computing the terms a_i , b_i , g_i , and h_i :

To obtain $C_N(r,S)$ and $C_P(r,S)$, we need to know a_i , g_i , b_i and h_i as well as the first two moments of B and H . Hence, we will first describe how these terms are calculated.

By definition,

$$a_i = \int_0^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} dV(t), \quad (2.2.a)$$

$$g_i = \int_0^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} dU(t). \quad (2.2.b)$$

However, for the computation of a_i and g_i , we use the following equivalent expressions:

$$a_i = \frac{(-\lambda)^i}{i!} V^{*(i)}(\lambda), \quad (2.3.a)$$

$$g_i = \frac{(-\lambda)^i}{i!} U^{*(i)}(\lambda), \quad (2.3.b)$$

where $V^{*(i)}(\theta)$ and $U^{*(i)}(\theta)$ are the i^{th} derivatives of the LSTs of V and U with respect to θ . Once the terms a_j and g_j have been calculated, the probability that j items are demanded during an inspection interval (or a processing time) is obtained by

$$b_j = \sum_{i=0}^j a_i x_j^{*(i)}, \quad j=0,1,2,\dots, \quad (2.4.a)$$

$$h_j = \sum_{i=0}^j g_i x_j^{*(i)}, \quad j=0,1,2,\dots. \quad (2.4.b)$$

The moments of B and H are obtained by using the following result, stated as lemma 2.1. Since lemma 2.1 can be easily proved, we omit the proof here.

Lemma 2.1. The probability generating function of B and H are given by

$$B(z) = V^*(\lambda - \lambda X(z)). \quad (2.5.a)$$

$$H(z) = U^*(\lambda - \lambda X(z)). \quad (2.5.b)$$

■

From equation (2.5), we can obtain the first and second factorial moments of B and H as

$$b^{(1)} = E(B) = \lambda E(V) x^{(1)}, \quad (2.6.a)$$

$$b^{(2)} = E(B(B-1)) = (\lambda x^{(1)})^2 E(V^2) + \lambda x^{(2)} E(V), \quad (2.6.b)$$

$$h^{(1)} = E(H) = \lambda E(U) x^{(1)}, \quad (2.6.c)$$

$$h^{(2)} = E(H(H-1)) = (\lambda x^{(1)})^2 E(U^2) + \lambda x^{(2)} E(U). \quad (2.6.d)$$

2.4. Computing the term $C_N(r,S)$:

In this section, we will compute the term $C_N(r,S)$, the expected total cost during a non-production period when r and S are used as control values. Let EV_k be the expected cost incurred during an inspection interval V initiated with k inventories. Then, the expected cost during a non-production period when r and S are used as control values is obtained by the following recursive equation:

$$C_N(r,S) = EV_S + \sum_{j=0}^{r-1} b_j C_N(r-j, S-j). \quad (2.7)$$

The explanation of equation (2.7) is as follows.

For any value of r and S , the (r,S) policy must have at least one inspection. Therefore, $C_N(r,S)$ is expressed by the expected cost during the interval upto the first inspection point plus the expected cost from this inspection point onwards until the end of non-production period. The first term EV_S in (2.7) represents the expected cost during the interval upto the first inspection point, and the second term represents the expected cost from this point onward until the end of non-production period. The explanation for the second term is based on the following reasoning:

- 1) If the number of items demanded during the interval upto the first inspection point is more than or equal to r , the non-production period ends here, so that there is no further cost incurred during the non-production period.
- 2) Suppose j items are demanded during the interval upto the first inspection point. Then, the inventory level at the point of the first inspection becomes $S-j$. If j is less than r , the inventory level at the first inspection point is greater than the lower control value s and the non-production period does not end at this point. Since the inventory level at this point is $S-j$, an expected cost of $C_N(r-j, S-j)$ is incurred from this point onward until the end of the non-production period. Consequently, since the probability that j items are demanded during the interval upto the first inspection point is b_j , the second term of (2.7) is obtained.

If we solve equation (2.7) for $C_N(r,S)$, we obtain

$$C_N(r,S) = \frac{1}{1-b_0} \left(EV_S + \sum_{j=1}^{r-1} b_j C_N(r-j, S-j) \right) \quad (2.8)$$

where an empty sum is defined to be 0.

In equation (2.8), we see that the term $C_N(r,S)$ can be computed if the value of EV_S is known. The term EV_S can be obtained by using the following lemma, stated as lemma 2.2. The proof of lemma 2.2 is given in appendix A.

Lemma 2.2

EV_k is expressed, recursively, as

$$EV_k = EV_{k-1} + \bar{d}_k(c_b+c_h) - E(V)c_b, \quad (2.9.a)$$

where

$$\bar{d}_k = \frac{1}{\lambda} \left\{ \sum_{j=1}^k \Pr(X_1 + \dots + X_{j-1} \leq k-1, X_1 + \dots + X_j \geq k) \left(1 - \sum_{i=0}^j a_i\right) + \sum_{j=0}^{k-1} \Pr(X_1 + \dots + X_j \leq k-1) (j+1) a_{j+1} \right\}, \quad (2.9.b)$$

and

$\Pr(X_0 \leq k-1)$ is defined to be 1.

For $k \leq 0$, EV_k can be recursively expressed as

$$EV_k = EV_{k+1} + E(V)c_b. \quad (2.10)$$

For an initial value for equations (2.9) and (2.10), we use

$$EV_0 = \frac{1}{2} x^{(1)} \lambda E(V^2) c_b. \quad (2.11)$$

■

Note that since (2.9.a) and (2.10) are recursive equations, once EV_{k-1} for $k \geq 1$ or EV_{k+1} for $k < 0$ is obtained, EV_k can be computed very easily, that is, only \bar{d}_k is needed to be calculated to obtain EV_k . To obtain $\Pr(X_1 + \dots + X_{j-1} \leq k-1, X_1 + \dots + X_j \geq k)$ in lemma 2.2, we use the following equation:

$$\Pr(X_1 + \dots + X_{j-1} \leq k-1, X_1 + \dots + X_j \geq k) = \sum_{i=1}^{k-j+1} x_{k-i}^{*(j-1)} \Pr(X \geq i). \quad (2.12)$$

2.5. Computing the term $C_p(r,S)$

During the production period, the production completion epochs are the times at which the inventory is replenished. To compute $C_p(r,S)$, we restrict our attention only to these epochs. Let $f_{i,j}$ denote the expected cost from the epoch at which the inventory level reaches i , to the epoch at which the inventory level is raised to j ($j > i$) for the first time. If i is equal to j , $f_{i,i}$ is defined to be 0. Then the expected total cost incurred during the production period, $C_p(r,S)$, is just $f_{I(r,S),S}$.

To obtain $f_{I(r,S),S}$ we first need an expression for the probability that the inventory level at the beginning of the production period is j when r and S are used as control values.

By conditioning on the number of items demanded during the first inspection interval, we can obtain a recursive equation that yields $P\{I(r,S)=j\}$ as

$$P\{I(r,S)=j\} = \sum_{k=0}^{r-1} b_k P\{I(r-k,S-k)=j\} + b_{S-j}, \quad \text{for } j \leq S-r. \quad (2.13)$$

Let us define a new random variable \tilde{B} such that

$$P\{\tilde{B}=j\} = P\{B=j \mid B \geq 1\}, \quad j=1,2,\dots$$

Note that the first and second factorial moments of \tilde{B} are

$$\tilde{b}^{(1)} = \frac{b^{(1)}}{1-b_0}, \quad (2.14.a)$$

$$\tilde{b}^{(2)} = \frac{b^{(2)}}{1-b_0}. \quad (2.14.b)$$

By solving equation (2.13) in terms of $P\{I(r,S)=j\}$, we obtain

$$P\{I(r,S)=j\} = \sum_{k=1}^{r-1} \tilde{b}_k P\{I(r-k,S-k)=j\} + \tilde{b}_{S-j}. \quad (2.15)$$

Using equation (2.15), after some algebra, we can now express $C_p(r,S)$ recursively as stated in the following lemma. The proof of lemma 2.3 is given in appendix B.

Lemma 2.3

$$C_p(r,S) = \sum_{j=1}^{r-1} \tilde{b}_j C_p(r-j,S-j) + C_p(1,S). \quad (2.16)$$

■

From lemma 2.3 and equation (2.8), we can also express $C(r,S)$ in a similar manner as $C_p(r,S)$. Lemma 2.4 states this result. The proof of lemma 2.4 is omitted.

Lemma 2.4

$$C(r,S) = \sum_{j=1}^{r-1} \tilde{b}_j C(r-j,S-j) + C(1,S). \quad (2.17)$$

■

From lemma 2.4, we see that, for any value of r and S , $C(r,S)$ can be computed recursively starting from $r=1$. Hence, $C(r,S)$ can be obtained if only the values of $C(1,k)$ are known for any k . Since $C_N(1,k)$ is just $\frac{EV_k}{1-b_0}$, it can be easily calculated. Thus, if we can compute $C_p(1,k)$, we can compute $C(r,S)$ for any value of r and S . Therefore, we will describe below how the term $C_p(1,S)$ can be obtained for any given S .

Computing the term $C_p(1,S)$

To compute the term $C_p(1,S)$, we consider two possible cases:

(a) Case $S \geq 1$:

In this case, the inventory level at the beginning of a production period can be either positive, zero, or negative. That is, if $\tilde{B} < S$, the inventory level $I(1,S)$ is positive and if $\tilde{B} \geq S$, $I(1,S)$ is zero or negative. In particular, we can find the probability that the inventory level at the beginning of a production period will take on value k as

$$P\{I(1,S) = k\} = P\{\tilde{B} = S - k\} = \tilde{b}_{S-k}, \quad k = -\infty, \dots, S-1. \quad (2.18)$$

Consequently, the probability that $I(1,S)$ will be non-positive is

$$P\{I(1,S) \leq 0\} = 1 - \sum_{j=1}^{S-1} \tilde{b}_j. \quad (2.19)$$

Thus, the term $C_p(1,S)$ is expressed as

$$\begin{aligned} C_p(1,S) &= f_{I(1,S),S} \\ &= \sum_{j=-\infty}^{S-1} P\{I(1,S) = j\} f_{j,S} = \sum_{j=-\infty}^{S-1} P\{I(1,S) = j\} (f_{j,S-1} + f_{S-1,S}) \\ &= \sum_{j=-\infty}^0 P\{I(1,S) = j\} f_{j,S-1} + \sum_{j=1}^{S-1} P\{I(1,S) = j\} f_{j,S-1} + f_{S-1,S} \\ &= \sum_{j=-\infty}^0 P\{I(1,S) = j\} f_{j,0} + P\{I(1,S) \leq 0\} f_{0,S-1} + \sum_{j=1}^{S-1} P\{I(1,S) = j\} f_{j,S-1} + f_{S-1,S} \\ &= \sum_{j=-\infty}^0 P\{I(1,S) = j\} f_{j,0} + (1 - \sum_{j=1}^{S-1} \tilde{b}_j) f_{0,S-1} + \sum_{j=1}^{S-1} \tilde{b}_{S-j} f_{j,S-1} + f_{S-1,S}. \end{aligned} \quad (2.20)$$

Suppose that we can compute the value of $f_{i,j}$ for any $i \leq j$ and $i \geq 0$. Then we see from equation (2.20) that, except for the first term, namely, $\sum_{j=-\infty}^0 P\{I(1,S) = j\} f_{j,0}$, all other terms can be calculated easily by simple operations, i.e., as sums of finite terms. The first term which consists of infinite terms, can be calculated without adding these infinite terms by exploiting the analogy between our production/inventory system and a queueing system. In particular, we will use a result on the control of $M^X/G/1$ queueing systems with server vacations that has been obtained by Lee and Srinivasan [5]. This result is stated below as lemma 2.5. A proof of lemma 2.5 can be found in [5].

Lemma 2.5 Consider an $M^X/G/1$ queue with vacation model with exhaustive service discipline. Let U be a service time for an individual customer and $i^{(1)}, i^{(2)}$ be the first and second factorial moments of the number of customers when service begins. Then the mean waiting time of an arbitrary unit during a busy period is represented by

$$W_b(i^{(1)}, i^{(2)}) = \frac{i^{(2)}}{2i^{(1)}} E(U) + \frac{\lambda\{x^{(1)}E(U^2) + x^{(2)}(E(U))^2\}}{2(1-\rho)}, \quad (2.21)$$

where $\rho = \lambda E(X)E(U)$. ■

From lemma 2.5 and the well known fact that the mean number of customers served in a busy period initiated with one customer in an $M^X/G/1$ queueing system is $\frac{1}{1-\rho}$, we get the following result stated as lemma 2.6.

Lemma 2.6: Suppose that the inventory level at the epoch when either a production period begins or the processing of an item ends, is known to be non-positive, and that the first and second factorial moments of the shortage size at this epoch is known to be $i^{-(1)}$ and $i^{-(2)}$ respectively. Then the expected total backorder time from this epoch to the epoch when the inventory level is raised to zero for the first time is expressed as

$$\alpha(i^{-(1)}, i^{-(2)}) = (W_b(i^{-(1)}, i^{-(2)}) + E(U)) \frac{i^{-(1)}}{1-\rho}$$

$$= \frac{i^{-(2)}E(U) + 2i^{-(1)}\gamma}{2(1-\rho)}, \quad (2.22)$$

where

$$\gamma = \frac{\lambda\{x^{(1)}E(U^2) + x^{(2)}(E(U))^2\}}{2(1-\rho)} + E(U).$$

■

Now, using this result, we can compute the term, $\sum_{j=-\infty}^0 P(I(1,S) = j)f_{j,0}$, as described below.

Let $I^-(1,S)$ be the shortage size at the beginning of production period given that the inventory level at the beginning of production period is non-positive and let $i^{-(1)}(1,S)$ and $i^{-(2)}(1,S)$ be the first and second factorial moments of $I^-(1,S)$. Then, if the condition is given that shortage occurs at the beginning of the production period, the expected cost that is incurred from the beginning of the production period to the time when the inventory level reaches 0 for the first time, is given by $\alpha(i^{-(1)}(1,S), i^{-(2)}(1,S))c_b$. Since the probability that shortage occurs at the beginning of the production period is $P(I(1,S) \leq 0)$, the term $\sum_{j=-\infty}^0 P(I(1,S) = j)f_{j,0}$ is computed by

$$\sum_{j=-\infty}^0 P(I(1,S) = j)f_{j,0} = P(I(1,S) \leq 0) \alpha(i^{-(1)}(1,S), i^{-(2)}(1,S))c_b, \quad (2.23)$$

where $i^{-(1)}(1,S)$ and $i^{-(2)}(1,S)$ are given by

$$i^{-(1)}(1,S) = E(\tilde{B} \mid \tilde{B} \geq S) - S, \quad (2.24.a)$$

$$i^{-(2)}(1,S) = E(\tilde{B}^2 \mid \tilde{B} \geq S) - E(\tilde{B} \mid \tilde{B} \geq S) - 2S E(\tilde{B} \mid \tilde{B} \geq S) + S(S+1). \quad (2.24.b)$$

In equation (2.24), the terms $E(\tilde{B} \mid \tilde{B} \geq S)$ and $E(\tilde{B}^2 \mid \tilde{B} \geq S)$ are obtained from the following equations:

$$E(\tilde{B}) = P(\tilde{B} \geq S) E(\tilde{B} \mid \tilde{B} \geq S) + \sum_{j=1}^{S-1} j \tilde{b}_j, \quad (2.25.b)$$

$$E(\tilde{B}^2) = P(\tilde{B} \geq S) E(\tilde{B}^2 \mid \tilde{B} \geq S) + \sum_{j=1}^{S-1} j^2 \tilde{b}_j. \quad (2.25.b)$$

(b) Case $S \leq 0$:

In this case, to obtain $C_P(1,S)$, we use an analogy between our production/inventory system and an $M^X/G/1$ queueing system. Observe that the length of the elapsed time from the epoch where the inventory level reaches k to the epoch where the inventory level is raised to $k+1$ for the first time is equivalent to the busy period in an $M^X/G/1$ queueing system that is initiated with one customer. Note that, if $S \leq 0$, the inventory level at the beginning of a production period is always zero or negative. Using this observation and the well known fact that the expected length of a busy period in an $M^X/G/1$ queueing system with mean service time $E(U)$, is $\frac{E(U)}{1-\rho}$, where $\rho = \lambda E(U)x^{(1)}$, we can obtain a recursive expression for $C_P(1,k)$ as

$$C_P(1,k) = C_P(1,k+1) + \frac{E(U)}{1-\rho}c_b, \quad \text{for } k < 0. \quad (2.26)$$

The initial value for equation (2.26), namely, $C_P(1,0)$, can be obtained using lemma 2.6 as follows:

$$\begin{aligned} C_P(1,0) &= \alpha(i^{-(1)}(1,0), i^{-(2)}(1,0))c_b \\ &= \alpha(\tilde{b}^{(1)}, \tilde{b}^{(2)})c_b. \end{aligned} \quad (2.27)$$

Computing $f_{k,k+1}$

Note from equation (2.20) that we can now compute the term $C_P(1,S)$ if we can calculate $f_{i,j}$ for $i \leq j$ and $i \geq 0$. Since the term $f_{i,j}$ can be expressed as $f_{i,j} = \sum_{k=i}^{j-1} f_{k,k+1}$, we can compute $f_{i,j}$ if only we can compute $f_{k,k+1}$ for any k . Thus, we will describe the procedure to obtain $f_{k,k+1}$, for the case of $k \geq 0$ since we do not need to calculate $f_{k,k+1}$ for $k < 0$.

To this end, we first define EU_k to be the expected cost incurred during a unit processing time initiated with k inventories. Then, $f_{k,k+1}$ is obtained by the following equation:

$$f_{k,k+1} = EU_k + \sum_{j=0}^{\infty} h_j f_{k+1-j,k+1}. \quad (2.28)$$

The first term in (2.28) is the expected cost incurred during the processing time and the second term in (2.28) is the expected cost incurred from the end of that processing time to the time at which the inventory level is raised to $k+1$ for the first time. The term EU_k , which is needed in (2.28) can be obtained in the same way as EV_k , that is, we can compute EU_k by simply replacing V and a_i with U and g_i respectively in lemma 2.2.

To obtain $f_{k,k+1}$ efficiently, we rewrite (2.28) as

$$f_{k,k+1} = EU_k + \sum_{j=1}^k h_j (f_{k+1-j,k} + f_{k,k+1}) + \sum_{j=k+1}^{\infty} h_j (f_{k+1-j,0} + f_{0,k} + f_{k,k+1}), \quad k \geq 0. \quad (2.29)$$

If we solve equation (2.29) for $f_{k,k+1}$, we obtain

$$f_{k,k+1} = \frac{1}{h_0} \left(EU_k + \sum_{j=1}^k h_j f_{k+1-j,k} + \sum_{j=k+1}^{\infty} h_j f_{k+1-j,0} + f_{0,k} P\{H \geq k+1\} \right). \quad (2.30)$$

In equation (2.30), only the term $\sum_{j=k+1}^{\infty} h_j f_{k+1-j,0}$ is expressed as the sum of infinite elements.

This term, however, can be obtained without adding infinite terms in a manner similar to the way we obtained the first term in equation (2.20), as will be described below.

Let $I^-(k)$ be the shortage size at the end of a processing time initiated with k inventories, given that inventory level at the end of the processing time is non-positive, and let $i^{-(1)}(k)$ and $i^{-(2)}(k)$ be the first and second factorial moments of $I^-(k)$. Then, similar to equation (2.23), the

term $\sum_{j=k+1}^{\infty} h_j f_{k+1-j,0}$ is expressed as follows:

$$\sum_{j=k+1}^{\infty} h_j f_{k+1-j,0} = P\{H \geq k+1\} \alpha(i^{-(1)}(k), i^{-(2)}(k)) c_b, \quad (2.31)$$

where $i^{-(1)}(k)$ and $i^{-(2)}(k)$ are obtained from

$$i^{-(1)}(k) = E(H | H \geq k+1) - (k+1), \quad (2.32.a)$$

$$i^{-(2)}(k) = E(H^2 | H \geq k+1) - E(H | H \geq k+1) - 2(k+1) E(H | H \geq k+1) + (k+1)(k+2). \quad (2.32.b)$$

The terms $E(H | H \geq k+1)$ and $E(H^2 | H \geq k+1)$ which are needed in equation (2.32) can be obtained from

$$E(H) = E(H | H \geq k+1)P(H \geq k+1) + \sum_{j=0}^k j h_j, \quad (2.33.a)$$

$$E(H^2) = E(H^2 | H \geq k+1)P(H \geq k+1) + \sum_{j=0}^k j^2 h_j. \quad (2.33.b)$$

As we see in equation (2.30), to obtain $f_{k,k+1}$, the terms $f_{0,k}$ and $f_{i,k}$ for $0 \leq i < k$ should be available beforehand. This implies that to obtain the term $f_{k,k+1}$, we must calculate $f_{i,i+1}$ in the order of $i=0,1,2,\dots,k$, that is, $f_{i,i+1}$ is calculated recursively starting from $i=0$. This implies that $C_p(1,S)$ is calculated recursively starting from $S=0$. Since $C_N(1,S)$, which is nothing but EV_S , is also calculated recursively starting from $S=0$, we observe that $C(1,S)$ is in turn calculated recursively starting from $S=0$.

2.6. Computing $L(r)$ and $TC(r,S)$

The expected cycle time in the (r,S) system can be obtained by again exploiting the analogy between production/inventory systems and queueing systems. Let Y_r be the total number of items demanded during a non-production period when $r=S-s$ is used as a control value. Then, as described in section 2.5, the length of production period in our (r,S) system is just the convolution of Y_r busy periods in an $M^X/G/1$ queueing system in which each busy period is initiated with one customer. Therefore, the expected length of a production period, $L_p(r)$, is obtained as

$$L_p(r) = \frac{E(Y_r)E(U)}{(1-\rho)}, \quad (2.34)$$

where $E(Y_r)$ can be computed from

$$E(Y_r) = \sum_{j=1}^{r-1} \tilde{b}_j E(Y_{r-j}) + \tilde{b}^{(1)}, \quad r=1,2,\dots \quad (2.35)$$

Similarly, the expected length of a non-production period, $L_N(r)$, can be computed from

$$L_N(r) = \frac{E(Y_r)}{\lambda_x(1)}. \quad (2.36)$$

For more details on how equations (2.35) and (2.36) can be obtained, refer to Lee and Srinivasan[5].

From (2.34) and (2.36), we can now obtain the expected length of a cycle as

$$L(r) = L_P(r) + L_N(r) = \frac{E(Y_r)}{(1-\rho)\lambda_x(1)}. \quad (2.37)$$

Note that, similar to the terms $C(r,S)$, $L(r)$ can be also computed recursively as

$$L(r) = \sum_{j=1}^{r-1} \tilde{b}_j L(r-j) + \frac{\tilde{b}(1)}{(1-\rho)\lambda_x(1)}. \quad (2.38)$$

Since the term $C(r,S)$ is obtained as $C(r,S) = C_N(r,S) + C_P(r,S)$, by substituting (2.36) and (2.37) into (2.1), we finally obtain $TC(r,S)$, the expected cost per unit time.

3. Finding the Optimal Control Values

In order to find the optimal control values (r^*, S^*) , a two-dimensional search over the integer parameter space must be made. Usually, the search for the optimal point in two dimensional space is not easy. However, if we exploit some properties of the cost functions, we can find a reasonably efficient search procedure. Let us denote the optimal S value for a given r by $S^*(r)$ and the optimal r value for a given S by $r^*(S)$. Then, we can prove the following properties which enable us to devise an efficient search procedure. The proofs of these properties are given in appendix C.

Property 1 For a given value of r , $C(r,S)$ is convex with respect to S .

Property 2 If $C(r,k) < C(r,k-1)$, then, $C(r+1,k) < C(r+1,k-1)$.

Property 3 If $S^*(1)=n$, then, $S^*(r) \leq n+r-1$ for $r \geq 2$.

Property 4 If $S^*(1)=n$ and $S^*(r)=k$, then, $k \leq S^*(r+1) \leq n+r$.

Property 5 $S^*(r) \geq 0$ for any $r \geq 1$.

Now we are in a position to describe the search procedure. To find the optimal control values (r^*, S^*) , our algorithm makes use of properties 1-5 as well as the recursive nature of the cost function $C(r, S)$. Recall that $C(r, S)$ is calculated recursively starting from $r=1, S=0$ and that it can be calculated very quickly once we know $C(i, j)$ for $i \leq r-1$ and $j \leq S$. Therefore our algorithm starts with $r=1$ and finds $C(1, S^*(1))$ first. To compute $C(1, S^*(1))$, we simply compute $C(1, k)$ from $k=0$ up to the point $k = S^*(1) + 1$, at which $C(1, k)$ is first increased. Then, due to the convexity of $C(1, k)$, the minimum value found is at $C(1, S^*(1))$. After we find $C(1, S^*(1))$, we compute $TC(1, S^*(1))$ using equation (2.1).

Note that the search starts from $k=0$ since by property 5, $C(1, k)$ for $k < 0$ cannot be a minimum. Once $C(1, S^*(1))$ is found, we can find $C(r, S^*(r))$ (and therefore, $TC(r, S^*(r))$) sequentially in the order of $r=2, 3, 4, \dots$ using property 4. Note that from property 4, to find $C(r+1, S^*(r+1))$, we do not have to consider $C(r+1, k)$ for $k < S^*(r)$: we just compute $C(r+1, k)$ starting from $k=S^*(r)$ until we encounter a point at which $C(r+1, k)$ is first increased. Although the upper bound for $S^*(r+1)$ in property 4 is $S^*(1)+r$, our computational experiments show that $S^*(r+1)$ is always either $S^*(r)$ or $S^*(r)+1$. Thus, very few evaluations of $C(r, k)$ are, in fact, needed in order to obtain $C(r, S^*(r))$ for each r .

The function $TC(r, S^*(r))$ is very complex and we could not prove any convexity/unimodality properties for the cost function $TC(r, S^*(r))$. However, extensive numerical tests suggest that $TC(r, S^*(r))$ is unimodal with respect to r . (For the continuous review (s, S) policy with unit Poisson demands, it has been proved [6] that $TC(r, S^*(r))$ is unimodal with respect to r .) Based on this observation, we can devise an extremely simple search procedure: start with $r=1$ and find $C(r, S^*(r))$ and $TC(r, S^*(r))$; increase r by 1 and compute $C(r, S^*(r))$ and $TC(r, S^*(r))$ for new r until $TC(r, S^*(r))$ is first increased. Our algorithm is fairly efficient since very few evaluations of $C(r, k)$ yield $C(r, S^*(r))$, and the term $C(r, k)$ can be computed very quickly using the values of $C(i, j)$ for $i < k$, which were previously

obtained. In fact, to obtain $C(r,k)$, the new terms we should compute are only EV_k and $f_{k-1,k}$, which are also easily calculated using EV_j and $f_{j-1,j}$ for $j < k$, that were obtained earlier.

Algorithm to find the optimal control values (r^*, S^*)

0. Set $r=1, k=0$.
 Compute $C(1,k)$
1. Set $k=k+1$
 Compute $C(1,k)$
 If $C(1,k) > C(1,k-1)$, then
 $S^*(1)=k-1$. Compute $TC(1, S^*(1))$
 go to step 2
 else
 go to step 1
 endif
2. Set $r=r+1, k=S^*(r-1)$.
 Compute $C(r,k)$
3. Set $k=k+1$
 If $C(r,k) > C(r,k-1)$, then
 $S^*(r)=k-1$. Compute $TC(r, S^*(r))$
 If $TC(r, S^*(r)) > TC(r-1, S^*(r-1))$ then
 optimal control values have been found; $(r-1, S^*(r-1))$.
 stop
 else
 go to step 2
 endif
 else
 go to step 3
 endif

4. Numerical Examples

In order to verify the efficiency of our algorithm and to check the unimodality of $TC(r, S^*(r))$, we made extensive numerical tests. Among these, we present two examples below. In example one, the inspection interval is assumed to follow a uniform distribution and the processing time is assumed to follow an Erlang distribution. In example 2, the inspection interval follows an exponential distribution and the processing time is assumed to follow a special distribution, which is encountered frequently in manufacturing situations. The

processing time in example 2 has a deterministic value t_0 if the production facility does not fail during the processing time. However, if the production facility fails during the processing time, which is assumed to occur with probability p , the processing time is the sum of t_0 plus a random repair time R which follows an exponential distribution with rate μ .

(example 1)

$$\lambda = 0.1, K=1000, c_h = 1, c_b = 20,$$

$$x_1 = 0.5, x_2 = 0.3, x_3 = 0.2,$$

V follows a uniform distribution on $[2,3]$ and

U follows a three stage Erlang distribution with mean 0.5.

(example 2)

$$\lambda = 0.1, K=1000, c_h = 1, c_b = 10$$

$$x_1 = 0.4, x_2 = 0.4, x_3 = 0.2,$$

V follows an exponential distribution with mean $3\frac{1}{3}$,

$$U = t_0 \quad \text{with probability } 1-p,$$

$$= t_0 + R \quad \text{with probability } p,$$

where $p=0.03$, $t_0=1.2$ and $\mu=0.1$.

The results of the policy comparisons for these two test examples are presented in tables 1 through 2. In each table, the values of r , $s^*(r)$, $S^*(r)$ and $TC(r, S^*(r))$ are shown for each value of r . As shown in the tables, in all these two examples, $TC(r, S^*(r))$ shows a unimodal property as was expected. Finally, although only three distributions for the inspection intervals and the processing times are demonstrated in the examples, many other distributions can be implemented easily if their Laplace-Stieltjes transform functions are well differentiable.

Table 1.

Result of Example 1

$r=S-s$	$s^*(r)$	$S^*(r)$	$TC(r, S^*(r))$
13	-1	12	18.2235
14	-1	13	17.8957
15	-1	14	17.6731
16	-1	15	17.5367
17	-1	16	17.4721
18*	-1	17	17.4677
19	-1	18	17.5144
20	-1	19	17.6048
21	-1	20	17.7329

(* means the optimal policy)

Table 2.

Result of Example 2

$r=S-s$	$s^*(r)$	$S^*(r)$	$TC(r, S^*(r))$
12	0	12	17.5078
13	-1	12	17.1587
14	-1	13	16.8800
15	-1	14	16.6971
16	-1	15	16.5934
17*	-1	16	16.5558
18	-1	17	16.5742
19	-1	18	16.6403
20	-1	19	16.7473

(* means the optimal policy)

5. Continuous Review Case

In this section, we consider the case in which the inventory level is continuously monitored during the non-production period, so that the production period begins as soon as the inventory level drops to or below the lower control value s . In this case, the general approach to obtaining $C(r,S)$ is basically the same as that outlined in the case of irregular inspection intervals and the search procedure to find the optimal control values is identical to that described in section 3 since all five properties in section 3 continue to hold. Therefore, in this section, we will only discuss those expressions which are different from the corresponding expressions in section 2.

The basic difference between the continuous review model and the irregular inspection interval model is that, in the continuous review model, x_j replaces \tilde{b}_j in every equation involving \tilde{b}_j . Thus, to obtain $C_N(r,S)$ for example, the following equation is used instead of equation (2.8):

$$C_N(r,S) = EV_S + \sum_{j=1}^{r-1} x_j C_N(r-j, S-j), \quad (5.1)$$

$$\begin{aligned} \text{where } EV_S &= S C_h \frac{1}{\lambda} \quad \text{if } S \geq 0, \\ &= -S C_b \frac{1}{\lambda} \quad \text{if } S < 0. \end{aligned}$$

Note that the term EV_S is the expected cost incurred from the instant the non-production period begins to the instant the first demand arrives, and the second term represents the expected cost incurred from the first demand arrival to the end of the non-production period.

To obtain $C_P(r,S)$ and $C(r,S)$, we use the following equations which correspond to equations (2.16) and (2.17):

$$C_P(r,S) = \sum_{j=1}^{r-1} x_j C_P(r-j, S-j) + C_P(1,S), \quad (5.2)$$

$$C(r,S) = \sum_{j=1}^{r-1} x_j C(r-j, S-j) + C(1,S). \quad (5.3)$$

To obtain $C_p(1,k)$ which is needed to obtain $C_p(r,S)$, all the equations described in section 2 are used with only slight modification. Besides replacing \tilde{b}_j with x_j here, the following expression is used for $P\{I(1,S)=j\}$:

$$P\{I(1,S)=j\}=P\{X=S-j\}=x_{S-j}. \quad (5.4)$$

The expected cycle time can be obtained in the same way: instead of using equation (2.35), $E(Y_r)$ must be calculated from $E(Y_r) = \sum_{j=1}^{r-1} x_j E(Y_{r-j}) + x^{(1)}$; \tilde{b}_j and $\tilde{b}^{(1)}$ should be substituted by x_j and $x^{(1)}$ in equation (2.38). All the other equations in section 2.6 except these two equations continue to hold.

6. Conclusion

We have introduced (s,S) control policies for production/inventory systems with compound Poisson arrival process and with irregular inspection intervals. We obtained an expression for the expected cost per unit time for given control values and developed an extremely simple, yet efficient search procedure to find the optimal control values. In devising search procedures, convexity property as well as the recursive nature of the cost functions were effectively exploited. We also showed that the same procedure can be directly applied to the situation in which inventory level is continuously monitored during a non-production period.

Appendix A.

Proof of lemma 2.2

Let V_i be the length of an inspection interval given that the number of demand requests which arrive during that inspection interval is i and let \bar{v}_i be $E[V_i]$. If we apply Bayes' formula, \bar{v}_i is expressed as

$$\bar{v}_i = E[V | A=i] = \frac{1}{a_i} \int_0^{\infty} \frac{(\lambda t)^i}{i!} e^{-\lambda t} t dV(t). \quad (A.1)$$

Using equation (A.1), we can obtain the following relationship after some algebraic manipulations:

$$\frac{a_j \bar{v}_j}{j+1} = \frac{a_{j+1}}{\lambda}. \quad (A.2)$$

We can now obtain the term EV_k by using equations (A.1) and (A.2). Note that the term EV_k consists of two components: the expected holding cost and the expected backorder cost during an inspection interval. Let us denote by \bar{D}_k the expected total holding time during an inspection interval which is initiated with k inventories and denote by \bar{E}_k the expected total backorder time during the inspection interval initiated with k inventories. Then EV_k is expressed as

$$EV_k = c_h \bar{D}_k + c_b \bar{E}_k. \quad (A.3)$$

We will first show how the expected holding time \bar{D}_k is obtained. To this end, we will call the inventory which is used for the i^{th} demand during the inspection interval as the i^{th} inventory.

Let \bar{d}_i denote the expected holding time of the i^{th} inventory during an inspection interval. Then

\bar{d}_i is given by

$$\begin{aligned} \bar{d}_i &= \sum_{j=0}^{i-1} a_j \Pr(X_1 + \dots + X_j \leq i-1) \bar{v}_j \\ &+ \sum_{j=1}^i \left[\Pr(X_1 + \dots + X_{j-1} \leq i-1, X_1 + \dots + X_j \geq i) j \sum_{k=j}^{\infty} \frac{a_k \bar{v}_k}{k+1} \right]. \quad i=1,2,\dots \end{aligned} \quad (A.4)$$

If we use equation (A.2), we can rewrite \bar{d}_i as

$$\bar{d}_i = \frac{1}{\lambda} \left[\sum_{j=0}^{i-1} \Pr(X_1 + \dots + X_j \leq i-1) (j+1) a_{j+1} \right]$$

$$+ \sum_{j=1}^i \{ \Pr(X_1 + \dots + X_{j-1} \leq i-1, X_1 + \dots + X_j \geq i) \sum_{k=j}^{\infty} a_{k+1} \}] . \quad i=1,2,\dots \quad (\text{A.5})$$

Thus the expected total holding time during the inspection interval initiated with k inventories is just

$$\bar{D}_k = \sum_{i=1}^k \bar{d}_i. \quad (\text{A.6})$$

Let the expected time from the instant the i^{th} demand arrives onwards until the instant the inspection interval ends be \bar{e}_i . Then, \bar{E}_k is expressed as

$$\bar{E}_k = \sum_{i=k+1}^{\infty} \bar{e}_i. \quad (\text{A.7})$$

Since the sum of \bar{d}_i and \bar{e}_i is the expected length of an inspection interval, \bar{e}_i can be computed from the following relationship:

$$\bar{e}_i = E(V) - \bar{d}_i, \quad i=1,2,\dots, \quad (\text{A.8})$$

with $\bar{d}_0 = \bar{e}_0 = 0$.

Also from the renewal theorem, the expected total time from the instant each demand arrives onwards to the instant the inspection interval ends is given by

$$\sum_{i=0}^{\infty} \bar{e}_i = \frac{\lambda_x(1)E(V^2)}{2}. \quad (\text{A.9})$$

From equations (A.7), (A.8) and (A.9), \bar{E}_k is expressed as

$$\bar{E}_k = \sum_{i=0}^{\infty} \bar{e}_i - \sum_{i=0}^k \bar{e}_i = \frac{\lambda_x(1)E(V^2)}{2} - \{kE(V) - \bar{D}_k\}. \quad (\text{A.10})$$

Thus, from equation (A.3), EV_k is given by

$$EV_k = \bar{D}_k(c_h + c_b) + c_b \left(\frac{\lambda_x(1)E(V^2)}{2} - kE(V) \right). \quad (\text{A.11})$$

Finally, from equation (A.11), EV_k is expressed as

$$EV_k = EV_{k-1} + \bar{d}_k(c_h + c_b) - E(V)c_b. \quad (\text{A.12})$$

For $k \leq 0$, since $\bar{d}_k = 0$, from equation (A.12), EV_k is expressed as

$$EV_k = EV_{k+1} + E(V)c_b.$$

EV_0 , used as an initial value, can be obtained directly from equation (A.9). ■

Appendix B.

Proof of Lemma 2.4

Lemma 2.4

$$C_P(r,S) = \sum_{j=1}^{r-1} \tilde{b}_j C_P(r-j,S-j) + C_P(1,S)$$

Proof $C_P(r,S) = f_{I(r,S),S}$

$$\begin{aligned}
 &= \sum_{j=-\infty}^{S-r} P\{I(r,S)=j\} f_{j,S} \\
 &= \sum_{j=-\infty}^{S-r} \left[\sum_{k=1}^{r-1} \tilde{b}_k P\{I(r-k,S-k)=j\} + \tilde{b}_{S-j} \right] f_{j,S} \\
 &= \sum_{k=1}^{r-1} \tilde{b}_k \sum_{j=-\infty}^{S-r} [P\{I(r-k,S-k)=j\} (f_{j,S-k} + f_{S-k,S})] + \sum_{j=-\infty}^{S-r} \tilde{b}_{S-j} f_{j,S} \\
 &= \sum_{k=1}^{r-1} \tilde{b}_k \left[C_P(r-k,S-k) + \sum_{j=-\infty}^{S-r} P\{I(r-k,S-k)=j\} f_{S-k,S} \right] + \sum_{j=-\infty}^{S-r} \tilde{b}_{S-j} f_{j,S}
 \end{aligned}$$

Since $\sum_{j=-\infty}^{S-r} P\{I(r-k,S-k)=j\} = 1$, the equation becomes

$$\begin{aligned}
 &= \sum_{k=1}^{r-1} \tilde{b}_k [C_P(r-k,S-k) + f_{S-k,S}] + \sum_{j=-\infty}^{S-r} \tilde{b}_{S-j} f_{j,S} \\
 &= \sum_{k=1}^{r-1} \tilde{b}_k C_P(r-k,S-k) + \sum_{j=1}^{\infty} \tilde{b}_j f_{S-j,S} \\
 &= \sum_{k=1}^{r-1} \tilde{b}_k C_P(r-k,S-k) + C_P(1,S).
 \end{aligned}$$

■

Appendix C.

Proofs of properties 1-5

Proof of Property 1 Since $C(r,S) = C_N(r,S) + C_P(r,S)$, note that for a given r , $C(r,S)$ is convex in S if $C_N(r,S)$ and $C_P(r,S)$ are both convex in S . To prove the convexity of $C_N(r,S)$ and $C_P(r,S)$, we need the following properties stated as propositions a, b, and c.

Proposition a : EV_k is convex with respect to k .

Proof To prove the convexity of EV_k , we need to show that:

$$\Delta EV(k) = (EV_{k+1} - EV_k) - (EV_k - EV_{k-1}) \geq 0, \quad \text{for all } k. \quad (C.1)$$

From equation (2.9), we obtain

$$\Delta EV(k) = (\bar{d}_{k+1} - \bar{d}_k)(c_h + c_b), \quad \text{for all } k. \quad (C.2)$$

Since \bar{d}_k is an expected holding time of the k^{th} inventory, $\bar{d}_k > \bar{d}_{k-1}$ for $k > 0$ and $\bar{d}_k = \bar{d}_{k-1} = 0$ for $k \leq 0$ (or it can be proved from equation (2.9.a)). Also, since both c_h and c_b are positive, $\Delta EV(k) \geq 0$ for all k . Thus, E_k is convex with respect to k . ■

Proposition b: EU_k is convex with respect to k .

Proof The proof is obtained using an approach identical to that used in proposition a. ■

Proposition c : $f_{k,k+1}$ is convex with respect to k .

Proof Let $\Delta f(k) = (f_{k+1,k+2} - f_{k,k+1}) - (f_{k,k+1} - f_{k-1,k})$. To prove the convexity of $f_{k,k+1}$ we need to prove $\Delta f(k) \geq 0$ for all k . We will prove this by induction on k .

For $k \leq -2$, $\Delta f(k) = 0$. Suppose $\Delta f(k) \geq 0$ holds for $k \leq n-1$, for some $n > 0$. We now prove that $\Delta f(k) \geq 0$ for $k=n$ using the induction hypothesis. From equation (2.28), we have

$$f_{n,n+1} = \frac{1}{h_0} (EU_n + \sum_{j=1}^{\infty} h_j f_{n-j+1,n}). \quad (C.3)$$

By substituting (C.3) into $\Delta f(n)$, we obtain

$$\Delta f(n) = \frac{1}{h_0} \{ (EU_{n+1} - EU_n) - (EU_n - EU_{n-1}) \} + \sum_{j=1}^{\infty} h_j \{ (f_{n-j+2,n+1} - f_{n-j+1,n}) - (f_{n-j+1,n} - f_{n-j,n-1}) \}$$

$$= \frac{1}{b_0} ((EU_{n+1} - EU_n) - (EU_n - EU_{n-1})) + \sum_{j=1}^{\infty} h_j (f_{n,n+1} - f_{n-1,n}) - (f_{n-j+1,n-j+2} - f_{n-j,n-j+1}). \quad (C.4)$$

In equation (C.4), since EU_n is convex, $(EU_{n+1} - EU_n) - (EU_n - EU_{n-1}) \geq 0$. Also, from the induction hypothesis, $(f_{n,n+1} - f_{n-1,n}) - (f_{n-j+1,n-j+2} - f_{n-j,n-j+1}) \geq 0$ for all $j \geq 1$. Since $h_j \geq 0$ for $j \geq 0$, it follows that $\Delta f(n) \geq 0$.

So, by the principle of mathematical induction, $\Delta f(k) \geq 0$ for all k , hence, $f_{k,k+1}$ is convex with respect to k . ■

Proposition d : For a given r , $C_N(r,S)$ is convex with respect to S .

Proof From equation (2.8), the term $C_N(r,S)$ is expressed as

$$C_N(r,S) = \frac{1}{1-b_0} (EV_S + \sum_{j=1}^{r-1} e_j C_N(r-j,S-j)). \quad (C.5)$$

To prove the convexity of $C_N(r,S)$ for given r , we need to show that

$$\Delta C_N(r,S) = \{ C_N(r,S+1) - C_N(r,S) \} - \{ C_N(r,S) - C_N(r,S-1) \} \geq 0 \text{ for any given } r. \quad (C.6)$$

This will be proved by induction on r .

i) For $r=1$, from equation (2.8), we have

$$\begin{aligned} \Delta C_N(1,S) &= \{ C_N(1,S+1) - C_N(1,S) \} - \{ C_N(1,S) - C_N(1,S-1) \} \\ &= \frac{1}{1-b_0} \{ (EV_{S+1} - EV_S) - (EV_S - EV_{S-1}) \} \end{aligned} \quad (C.7)$$

Since EV_k is convex with respect to k , $\Delta C_N(1,S) \geq 0$.

ii) Suppose $\Delta C_N(r,S) \geq 0$ holds for $r \leq n-1$. We now prove $\Delta C_N(r,S) \geq 0$ for $r=n$.

$$\begin{aligned} \Delta C_N(n,S) &= \{ C_N(n,S+1) - C_N(n,S) \} - \{ C_N(n,S) - C_N(n,S-1) \} \\ &= \frac{1}{1-b_0} \{ (EV_{S+1} - EV_S) - (EV_S - EV_{S-1}) \} \\ &\quad + \sum_{j=1}^{n-1} \tilde{b}_j [\{ C_N(n-j,S+1-j) - C_N(n-j,S-j) \} - \{ C_N(n-j,S-j) - C_N(n-j,S-1-j) \}] \end{aligned} \quad (C.8)$$

Since EV_k is convex, $(EV_{S+1} - EV_S) - (EV_S - EV_{S-1}) \geq 0$ and from the induction hypothesis,

$\{ C_N(n-j, S+1-j) - C_N(n-j, S-j) \} - \{ C_N(n-j, S-j) - C_N(n-j, S-1-j) \} \geq 0$ for $j=1, \dots, n-1$. So $\Delta C_N(n, S) \geq 0$. So by the principle of mathematical induction, $\Delta C_N(r, S) \geq 0$ for all r , hence, $C_N(r, S)$ is convex with respect to S for given r . ■

Proposition e : For a given r , $C_p(r, S)$ is convex with respect to S .

Proof From equation (2.24), the term $C_p(r, S)$ is expressed as

$$C_p(r, S) = f_{I(r, S), S} = \sum_{j=-\infty}^{S-r} P\{I(r, S) = j\} f_{j, S} \quad (C.9)$$

To prove the convexity of $C_p(r, S)$ with respect to S , we need to show that

$$\Delta C_p(r, S) = \{ C_p(r, S+1) - C_p(r, S) \} - \{ C_p(r, S) - C_p(r, S-1) \} \geq 0 \text{ for any } r. \quad (C.10)$$

From equation (C.9), this reduces to

$$\begin{aligned} \Delta C_p(r, S) = & \left\{ \sum_{j=-\infty}^{S+1-r} P\{I(r, S+1) = j\} f_{j, S+1} - \sum_{j=-\infty}^{S-r} P\{I(r, S) = j\} f_{j, S} \right\} \\ & - \left\{ \sum_{j=-\infty}^{S-r} P\{I(r, S) = j\} f_{j, S} - \sum_{j=-\infty}^{S-1-r} P\{I(r, S-1) = j\} f_{j, S-1} \right\} \end{aligned}$$

Using the fact that $P\{I(r, S+1) = j\} = P\{I(r, S) = j-1\}$, $C_p(r, S)$ is expressed as

$$\begin{aligned} \Delta C_p(r, S) = & \left\{ \sum_{j=-\infty}^{S+1-r} P\{I(r, S+1) = j\} (f_{j, S+1} - f_{j-1, S}) \right\} - \left\{ \sum_{j=-\infty}^{S-r} P\{I(r, S) = j\} (f_{j, S} - f_{j-1, S-1}) \right\} \\ = & \sum_{j=-\infty}^{S+1-r} P\{I(r, S+1) = j\} \{ (f_{j, S+1} - f_{j-1, S}) - (f_{j-1, S} - f_{j-2, S-1}) \}. \end{aligned} \quad (C.11)$$

In equation (C.11), since $f_{k, k+1}$ is convex, $(f_{j, S+1} - f_{j-1, S}) - (f_{j-1, S} - f_{j-2, S-1}) \geq 0$.

Thus $\Delta C_p(r, S) \geq 0$ holds for any given r , hence $C_p(r, S)$ is convex with respect to S . From propositions d and e, the convexity of $C(r, S)$ is proved. ■

Property 2 If $C(r, k) < C(r, k-1)$, then $C(r+1, k) < C(r+1, k-1)$.

Proof If we apply lemma 2.4 to the given condition $C(r, k) - C(r, k-1) < 0$, we obtain the following inequality:

$$\sum_{j=1}^{r-1} \tilde{b}_j \{ C(r-j, k-j) - C(r-j, k-j-1) \} + C(1, k) - C(1, k-1) < 0 \quad (C.12)$$

If we continue to apply lemma 2.4 to inequality (C.12), it reduces to the following form:

$$\sum_{j=1}^{r-1} \alpha_j \{ C(1, k-j) - C(1, k-j-1) \} + C(1, k) - C(1, k-1) < 0, \quad (C.13)$$

where $\alpha_j > 0$ for $j=1, \dots, r-1$.

Note that since $C(1,j)$ is convex with respect to j , $C(1,k)-C(1,k-1)$ is a non-decreasing function of j . Thus, in order for the LHS of (C.13) to be negative, the following inequality must be satisfied:

$$C(1,k-j) - C(1,k-j-1) < 0 \quad \text{for } j \geq r-1. \quad (\text{C.14})$$

Now under this condition, we must prove that

$$C(r+1,k) - C(r+1,k-1) < 0. \quad (\text{C.15})$$

If we apply lemma 2.4, inequality (C.15) becomes

$$\sum_{j=1}^r \tilde{b}_j \{C(r+1-j,k-j) - C(r+1-j,k-j-1)\} + C(1,k) - C(1,k-1) < 0 \quad (\text{C.16})$$

We now compare each term of inequality (C.12) with that of (C.16). Since $C(1,k-r)-C(1,k-r-1) < 0$, inequality (C.16) is true if the following inequality holds for $j=r-1, r-2, \dots, 1$:

$$C(r+1-j,k-j) - C(r+1-j,k-j-1) < C(r-j,k-j) - C(r-j,k-j-1), \quad j=r-1, r-2, \dots, 1. \quad (\text{C.17})$$

We will prove this by induction on j .

(i) $j=r-1$

In this case inequality (C.17) is

$$C(2,k-r+1) - C(2,k-r) < C(1,k-r+1) - C(1,k-r). \quad (\text{C.18})$$

Since LHS of (C.18) is $\tilde{b}_1 \{C(1,k-r) - C(1,k-r-1)\} + C(1,k-r+1) - C(1,k-r)$ and noting that $C(1,k-r) - C(1,k-r-1) < 0$, inequality (C.18) proves to be true.

(ii) Assume inequality (C.17) holds for $j=r-1, r-2, \dots, n+1$. Under this induction hypothesis, we will prove that (C.17) also holds for $j=n$, that is, we will show that

$$C(r+1-n,k-n) - C(r+1-n,k-n-1) < C(r-n,k-n) - C(r-n,k-n-1). \quad (\text{C.19})$$

Inequality (C.19) can be rewritten as

$$\begin{aligned} & \sum_{j=1}^{r-n} \tilde{b}_j \{C(r+1-n-j,k-n-j) - C(r+1-n-j,k-n-j-1)\} + C(1,k-n) - C(1,k-n-1) \\ & < \sum_{j=1}^{r-n-1} \tilde{b}_j \{C(r-n-j,k-n-j) - C(r-n-j,k-n-j-1)\} + C(1,k-n) - C(1,k-n-1). \end{aligned} \quad (\text{C.20})$$

From the induction hypothesis together with the fact that $C(1,k-r) - C(1,k-r-1) < 0$, inequality (C.20) is true. Therefore, from the principle of mathematical induction, inequality (C.17) is true for $j=r-1, r-2, \dots, 1$, and hence, inequality (C.16) is true. ■

Property 3 If $S^*(1)=n$, then, $S^*(r) \leq n+r-1$ for $r \geq 2$.

Proof Proof is done by induction on r .

(i) $r=2$

Note that proving inequality $S^*(2) \leq n+1$ is equivalent to showing that

$$C(2,n+2) > C(2,n+1), \quad (C.21)$$

which can be rewritten as

$$\tilde{b}_1 C(1,n+1) + C(1,n+2) > \tilde{b}_1 C(1,n) + C(1,n+1). \quad (C.22)$$

This inequality is true since $C(1,k)$ is convex in k and $S^*(1)=n$.

(ii) Assume that if $S^*(1)=n$, then $S^*(j) \leq n+j-1$ holds for $j=1, \dots, r-1$. We now prove that $S^*(r) \leq n+r-1$ under this induction hypothesis. Note that we have only to prove that

$$C(r,n+r) > C(r,n+r-1). \quad (C.23)$$

Inequality (C.23) can be rewritten as

$$\sum_{k=1}^{r-1} \tilde{b}_k \{C(r-k,n+r-k) - C(r-k,n+r-1-k)\} + C(1,n+r) - C(1,n+r-1) > 0. \quad (C.24)$$

From the induction hypothesis, we have $C(r-k,n+r-k) > C(r-k,n+r-1-k)$ for $k=1, \dots, r-1$, and from the given condition that $S^*(1)=n$, we have $C(1,n+r) > C(1,n+r-1)$. Thus, inequality (C.24) is true. Therefore, from the principle of mathematical induction, property 3 has been proved. ■

Property 4 If $S^*(1)=n$ and $S^*(r)=k$, then, $k \leq S^*(r+1) \leq n+r$ holds.

Proof Direct result of properties 2 and 3.

Property 5 $S^*(r) \geq 0$ for any $r \geq 1$.

Proof Consider a policy (r,S) where $S < 0$. Then, policy $(r,0)$ always dominates this policy since only non-positive inventory levels exist for these two policies and the expected shortage level of policy $(r,0)$ is always less than that of policy (r,S) by $-S$. Thus, $S^*(r)$ cannot be negative for any r . ■

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