Generalized Tolerance Synthesis for Recti-Linear Systems

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Abstract

Minimum cost tolerance allocation is considered for recti-linear systems, in which the dimensions are recti-linear, such as VLSI circuits and peg-hole assemblies. Tolerance synthesis is achieved by non-linear optimization in which the objective is cost minimization and the constraints are design functions, both of which are non-linear. Globally optimal solution for tolerances is obtained by a key development: a proof that those non-linear functions are convex. This enables the utilization of known algorithms for convex programming.

The non-linear constraints are carefully derived through the point of view of tolerance stack-up analysis. The derivation yields a technique that unifies the deterministic approach and the probabilistic approaches by considering asymmetric probability density functions. A model for representing the geometric uncertainty of a dimension is also proposed.

1. INTRODUCTION

Tolerance is the total amount by which a specific dimension in an engineering drawing is permitted to vary [1]. The specification of a tolerance involves opposing considerations in cost and in performance (functionality and interchangeability): the more tolerance, the less cost and performance and vice versa. The resolution of this conflict in tolerancing for recti-linear systems is considered in this paper.

As a global criterion for resolving the conflict, cost is minimized. Minimum cost tolerance allocation, however, is constrained by *design functions* for performance. A design function specifies the scope of permissible variations in the aggregation of component tolerances, or what is commonly referred to as "stack-up". Consider Figure 1. The clearance Y between the two components is represented by the equation $Y=X_1-X_2$, where the variables X_1 and X_2 represent the dimensions of the hole and the shaft, respectively. Suppose the variation of the clearance is to be less than or equal to 0.05 for successful assembly. That is, $T_Y \leq 0.05$, where T_Y is the tolerance for the aggregate Y. Under this constraint, minimum cost tolerance allocation can be formulated as follows:

$$\text{Min} \quad C(T_1, T_2)$$

subject to

$$T_Y = f(T_1, T_2) \leq 0.05$$

where $C(\cdot)$ is the cost function, T_1 and T_2 are, respectively, tolerances for X_1 and X_2, and T_Y is a function of T_1 and T_2. For situations where many design functions are involved, minimum cost tolerance allocation generalizes to:

$$\text{Min} \quad C(T_1, T_2, \cdots, T_n)$$

subject to (1)

$$T_{Y_i} = f_i(T_1, T_2, \cdots, T_n) \leq T_{Y_i}^U \qquad \text{for } i=1,\ldots,m$$

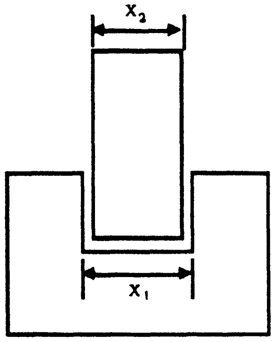
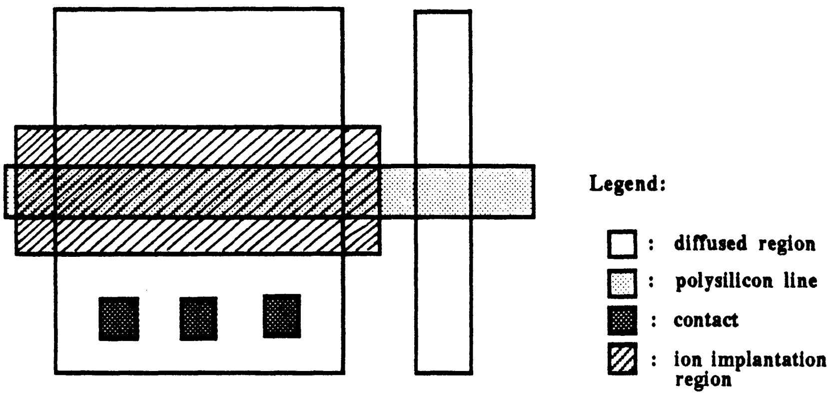
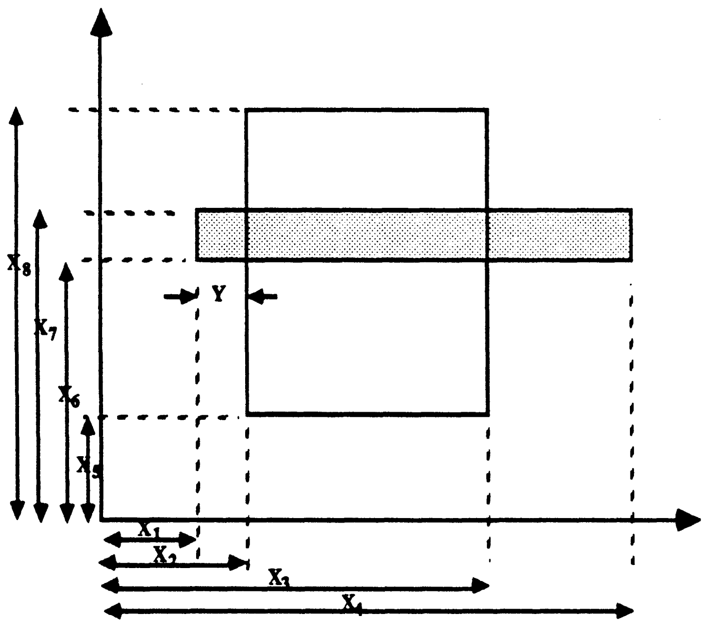


Figure 1. Clearance between Mating Parts



(a) An NMOS VLSI



(b) A Design Rule

Figure 2. An Example of a Recti-Linear System

where $C(\cdot)$: cost function in terms of component tolerances,

T_j : tolerance of the j-th component dimension X_j,

n : total number of component tolerances to be decided,

T_{Y_i} : the i-th assembly tolerance,

$f_i(\cdot)$: design function relating T_{Y_i} to $T_1, \cdots, T_n$,

$T_{Y_i}^U$: upper limit of T_{Y_i} for desired performance in an assembly,

X_j : the j-th component dimension,

Y_i : the i-th assembly dimension, and

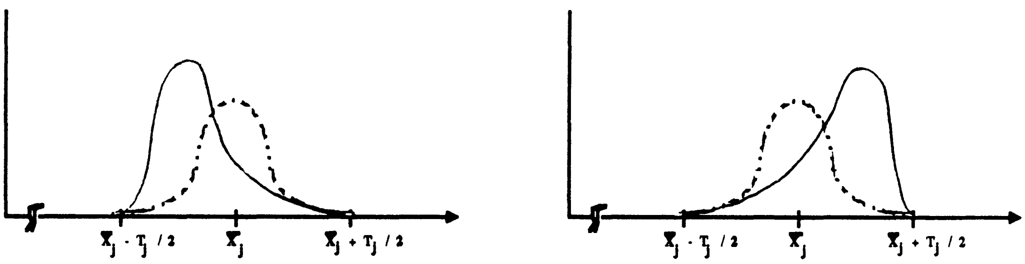
m : number of assembly tolerances (stack-up tolerances) to be considered.

A recti-linear system is one in which an assembly dimension Y_i is defined by a linear function of component dimensions $X_1,\ldots,X_n$:

$$Y_i = \sum_{j=1}^{n} a_{ij} X_j \qquad \text{for } i=1,\ldots,m \tag{2}$$

where a_{ij} is an integer. Recti-linear systems are found not only with mechanical applications, such as peg-hole assemblies illustrated by Figure 1, but also in the design of VLSI (Very Large Scale Integrated) circuits. A mask used in the fabrication of VLSI circuits is composed of a collection of rectangles. Each rectangle is either a portion of a diffused region, a polysilicon line, an ion implantation region, or a "via" contact, as illustrated in Figure 2-(a). These rectangles are laid out according to certain design rules, which specify allowances for certain widths, clearances, extensions, or overlaps. These inter-component relations such as widths, clearances, etc., can be thought of as assembly dimensions Y_i. Consider Figure 2-(b) in which a polysilicon gate crosses a diffused region, thereby forming a transistor. In order to make certain that the diffused region does not short-circuit the drain-to-source path of the transistor, it is necessary for the polysilicon gate to extend a distance Y of at least 2λ beyond the nominal boundary of the diffused area, where λ denotes a unit length in process resolution [2,10,13]. This distance Y can be represented by the difference of the two values X_1 and X_2, i.e., $Y = X_2 - X_1$. In general, by associating all the geometric dimensions X_j in Figure 2-(b), design rules can be described as linear functions of the component dimensions X_j.

Because of the "randomness" of the fabrication process, resulting dimensions follow a distribution. In this paper, asymmetric distribution for dimension X_j is considered. The consideration of asymmetric distribution is motivated by a practice called "maximum material condition" (MMC) [1] (or, LMC for "least material condition" as the case may be). The practice, in effect, encourages the least amount of manufacturing time and produces skewed distributions. MMC tells the machining process to remove as little material as possible to meet the specification, hence leaving the "maximum material". (Alternatively, LMC tells the vapor deposition process to deposit as little material as possible so as to create the "least material condition.") Either way the resulting dimension is far from being distributed normally as illustrated in Figure 3.



(a) Least Material Condition for Shaft (Maximum Material Condition for Hole)

(b) Maximum Material Condition for Shaft (Least Material Condition for Hole)

Figure 3. Asymmetric Distribution due to MMC or LMC

In solving formulation (1), the tolerances T_j in the objective function are computed, hence synthesized. But they must obey the constraint which specifies the functional relation $f_i(T_1, T_2, \cdots, T_n)$ between assembly tolerance T_{Y_i} and component tolerances $T_1, T_2, \cdots, T_n$. The study of the aggregate behavior of given individual tolerance $T_1, T_2, \cdots, T_n$ is referred to as *tolerance analysis* or, more commonly, as *stack-up analysis*. Hence, synthesis is performed through analysis. Tolerance analysis is typically conducted in one of two ways: deterministic or probabilistic. Neither approach is considered as appropriate for dealing with asymmetric tolerances. The deterministic approach (worst-case analysis) evaluates an assembly tolerance to a value larger than necessary. In the work by Balling, *et al.* [3] and Michael and Siddall [14], the i-th assembly tolerance T_{Y_i} is represented by

$$T_{Y_i} = \sum_{j=1}^{n} |a_{ij}| \, T_j \qquad \text{for } i=1,\ldots,m. \tag{3}$$

While (3) is very simple to compute, the resulting assembly tolerance is often pessimistic. For example, five components each with 0.001 inch tolerance will stack up to 0.005 inches assembly tolerance according to (3). If the sum exceeds the specified upper bound for assembly tolerance (of, say, 0.0017 inches), what typically happens next is that the component tolerances T_j are reduced. To continue the example, suppose the component tolerances are reduced uniformly from 0.001 inch to 0.0003 inches by the designer. As common sense dictates - higher cost for lower tolerance - worst case analysis incurs a component cost higher than otherwise possible. With the probabilistic approach, partial satisfaction of the design function, e.g., 99% yield, is possible.

The probabilistic approach is well-defined under the assumption of normality for the random variable X_j by the following property: when the constituting dimensions follow the normal distributions, the linear sum for an assembly dimension also follows a normal distribution . However, distributions are assumed to be asymmetric in this paper. While an asymmetric p.d.f. (probability density function) may be modeled by the beta distribution,[4,5] there is no general rule for representing the distribution of the linear sum for an assembly dimension Y. The challenge of the optimization problem formulated as (1) may be now summarized. Because of the problem domain, the assembly dimension Y_i in a rect-linear system is a linear function of the component dimensions as described by (2). But because of process considerations, asymmetric distributions for component dimensions are assumed as shown in Figure 3. However, there is no known technique for computing the linear sum of asymmetric distributions.

To overcome the difficulty, a new model for a dimension is proposed. The basic idea is to decompose the manufacturing uncertainty into two parts: uncertainty due to mean shift and uncertainty due to other noises. Then, tolerance is represented by the sum of these two parts. The concept of decomposition has been used in market forecasting [16] to explain the variation of market demand influenced by predictable factors such as seasonal effects and less predictable factors due to the introduction of competitions or the discovery of new materials and processes. The first application of decomposition to tolerancing was due to Greenwood and Chase [8]. This paper, in addition to providing statistical justification, proves $f_i(T_1, T_2, \cdots, T_n)$ to be a convex function in terms of the component tolerances. Because of this proof of convexity, the tolerance synthesis problem of formulation (1) becomes a convex programming (CP) problem. This observation leads to a pleasant result: that the globally minimum cost tolerances for a recti-linear system can always be obtained. There exist several algorithms to guarantee an optimum for a CP problem, such as the homotopy procedure of Eaves and Saigal for finding a fixed point [20] and the Lagrangean method [23]. The cost function adopts existing models [21,22].

The paper is organized as follows. Section 2 presents a new approach and its justification. Section 3 investigates the convexity of $f_i(T_1, T_2, \cdots, T_n)$ for recti-linear systems. The result of convexity is then applied to the solution procedure of problem (1) and the whole procedure is illustrated with an example.

2. HYBRID TOLERANCE ANALYSIS

In the probabilistic approach, a random variable and its confidence interval are associated with a dimension and its tolerance. The confidence interval (tolerance T_j) for the j-th dimension X_j is determined by two attributes; the confidence coefficient k_j and the standard deviation σ_j such that $T_j = k_j \sigma_j$ for j=1,...,n. Then,

$$\sigma_j = \frac{T_j}{k_j} \qquad \text{for } j=1,\ldots,n \tag{4}$$

Since the variance of the linear sum Y_i of (2) is the sum of the squares of the variances of the constituting variables, i.e., $(\sigma_{Y_i})^2 = \sum_{j=1}^{n} (a_{ij})^2(\sigma_j)^2$, the i-th assembly tolerance can be represented by

$$T_{Y_i} = k_{Y_i} \sqrt{\sum_{j=1}^{n} (a_{ij})^2 (\frac{T_j}{k_j})^2} \qquad \text{for } i=1,\ldots,m \tag{5}$$

where k_{Y_i} is a confidence coefficient for the i-th assembly dimension Y_i. Since tolerance is the amount by which a dimension is permitted to vary, as in 1.000 ± 0.002, it is an unsigned magnitude. The square root in equation (5) is always positive.

To illustrate the approach, the example from Figure 1 is again considered. Suppose the tolerances for X_1 and X_2 are 0.001 with confidence coefficients k_1=k_2=6 and they follow (for the time being) the normal distribution. The worst-case tolerance for Y by the deterministic approach is 0.002. Assume that in the probabilistic approach a 1% defect is allowed in the linear sum. Then 5.15 σ_Y = 0.002, since $Pr(Z \leq \frac{5.15}{2}) = 1 - \frac{0.01}{2}$, where Z denotes a random variable following the standard normal distribution. This leads to $\sigma_Y = \frac{0.002}{5.15}$. If equal distribution of this linear sum tolerance into components is assumed, then $\sigma_1 = \sigma_2 = 0.000275$ since $(\sigma_Y)^2 = (\sigma_1)^2 + (\sigma_2)^2$. As a result, the tolerances for X_1 and X_2 can be increased by 65% from 0.001 to 0.001647 (= 6 x 0.000275) by allowing a 1% defect in the linear sum (i.e., stack-up).

However, the above procedure can not be applied since the distribution of the dimension X_j, hence that of the linear sum Y_i, are not symmetric. To resolve the difficulty from asymmetry, a hybrid approach, which combines the deterministic and probabilistic approaches, is proposed.

The concept of a hybrid approach is based on the decomposition of a tolerance into a known quantity (deterministic) and other uncertain quantities (probabilistic). The decomposition of a tolerance is explained by the following additive model for a dimension:

$$X_j = C_j + M_j + E_j \qquad \text{for } j = 1,\ldots,n. \tag{6}$$

C_j is a constant representing the center of the j-th tolerance interval. M_j and E_j are, respectively, independent random variables for representing the mean shift from C_j and the remaining error term. To capture the general asymmetric case, the distribution form of M_j is assumed to be unknown. However, the error term E_j is assumed to follow the normal distribution around the shifted mean $C_j + M_j$ (i.e., $E(E_j)=0$) based on the central limit theorem since it is due to many independent sources. The relationship among C_j, M_j, and E_j is illustrated in Figure 4. If $M_j=M_j^*$, the p.d.f. of E_j is symmetric around $C_j+M_j^*$, but the p.d.f. for X_j could be asymmetric because of the unknown distribution of M_j.

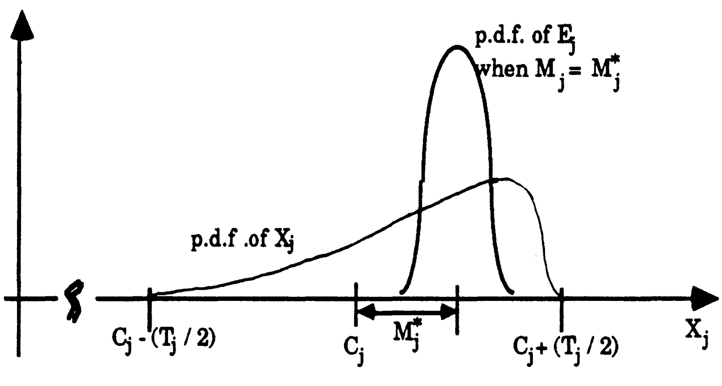


Figure 4. Additive Model for Dimension X_j

3.1 Representation of the First and Second Moments of a Dimension

To compensate for the lack of information about M_j, its first and second moments and those of E_j are determined in terms of tolerance T_j. Suppose the mean of M_j is rewritten as a convex combination of the two limits of the tolerance interval:

$$E(M_j) = \frac{T_j}{2} p_j - \frac{T_j}{2}(1-p_j) = T_j\,(p_j - 0.5)$$

where p_j is a positive scalar such that $0 \le p_j \le 1$. The scalar p_j reflects the skewness of the tolerance interval:

if $0 \le p_j < 0.5$, the mean value of X_j is to the left of C_j,

if $p_j = 0.5$, the mean value of X_j coincides with C_j, and

if $0.5 < p_j \le 1$, the mean value of X_j is to the right of C_j.

Hence, the measure of skewness of $E(M_j)$ can be obtained from the two variables T_j and p_j.

The variability of M_j and E_j also can be represented in terms of T_j and p_j by taking the following two steps: (i) represent the standard deviations of M_j and E_j in terms of T_j and a new scalar w_j, and (ii) relate w_j to p_j.

For the first step, denote the standard deviations of M_j and E_j by σ_{M_j} and σ_{E_j}, respectively, and define the proportion of σ_{M_j} to σ_j as the weighting factor w_j, i.e.,

$$w_j = \frac{\sigma_{M_j}}{\sigma_j} \qquad \text{for } j=1,\ldots,n \qquad (7)$$

Since σ_{M_j} and σ_{E_j} are usually very small†, the standard deviation σ_j can be approximated by the sum of σ_{M_j} and σ_{E_j}††. That is,

$$\sigma_j \sim \sigma_{M_j} + \sigma_{E_j} \qquad \text{for } j=1,\ldots,n \tag{8}$$

By combining (8) with equations (4) and (7), σ_{M_j} and σ_{E_j} can be represented in terms of T_j:

$$\begin{aligned} \sigma_{M_j} &= w_j \frac{T_j}{k_j} \\ \sigma_{E_j} &= (1-w_j) \frac{T_j}{k_j} \end{aligned} \tag{9}$$

Now, the second step is to relate w_j to p_j. The relation is based on the observation that the variability by random noise is proportional to the minimum distance to the tolerance limit. The essence of this observation is illustrated in Figure 5: the further the shifted mean is from the center, the smaller the variability becomes (in order to keep the manufacturing process in control). This observation can be expressed as follows:

$$\begin{aligned} \sigma_{E_j} &= 2\, p_j\, \sigma_j && \text{if } 0.0 \le p_j \le 0.5 \\ &= (2 - 2p_j)\, \sigma_j && \text{if } 0.5 < p_j \le 1.0 \end{aligned} \tag{10}$$

Since σ_{E_j} is approximated by $(1-w_j)\sigma_j$ in (9), equation (10) leads to the following relation:

$$w_j = |1 - 2p_j| \tag{11}$$

These two steps combined are summarized in the following lemma.

† For instance, the tolerance for a hole or shaft of diameter 1.00 is recommended by ANSI to be 0.0025 (or 0.0006) in case of loose fit (or tight fit).

†† Since $\sigma_j^2 = \sigma_{M_j}^2 + \sigma_{E_j}^2$, $\sigma_j = \sqrt{(\sigma_{M_j} + \sigma_{E_j})^2 - 2\sigma_{M_j}\sigma_{E_j}}$. When $\sigma_{M_j} \ll 1.0$ and $\sigma_{E_j} \ll 1.0$, $\sigma_{M_j}\sigma_{E_j}$ becomes almost zero and hence $\sigma_j \sim \sigma_{M_j} + \sigma_{E_j}$.

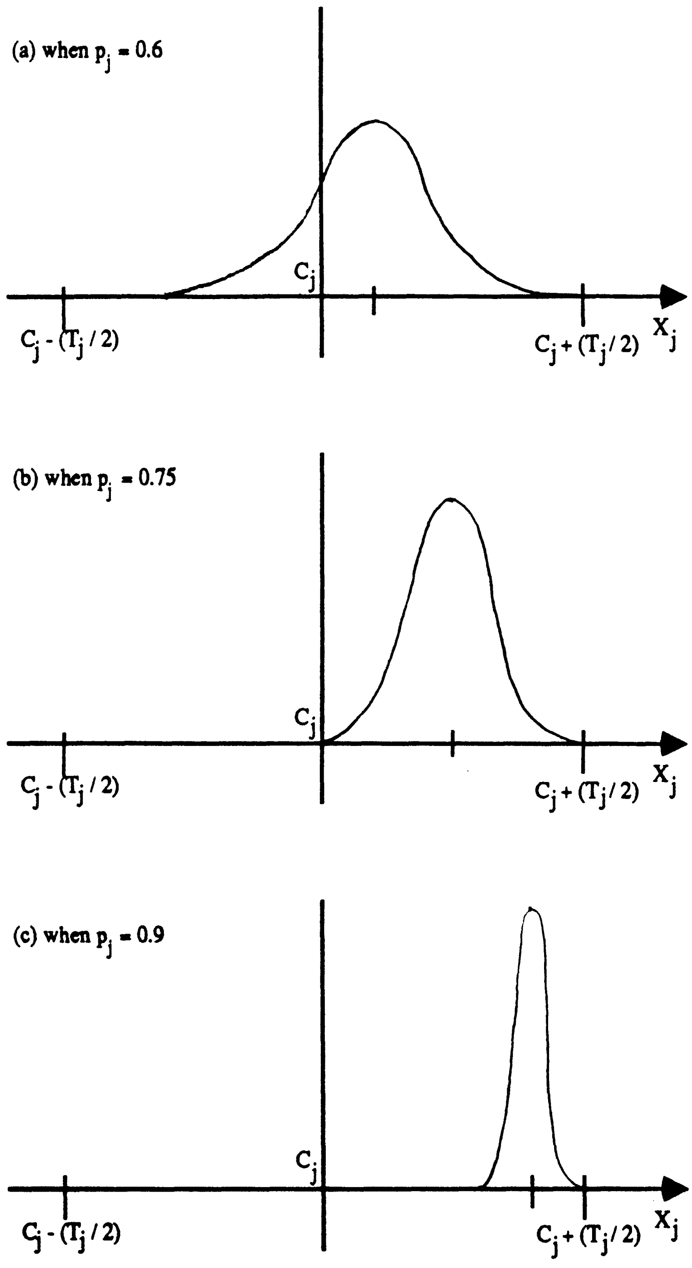


Figure 5. Variability due to Random Noise after Mean Shift

Lemma 1: In the additive model of (6), the first two moments of a dimension can be represented in terms of four parameters C_j, T_j, p_j, and k_j as follows:

$$E(X_j) = C_j + E(M_j) = C_j + T_j\,(p_j - 0.5)$$

$$\sigma_{M_j} = |1 - 2p_j|\,\frac{T_j}{k_j}$$

$$\sigma_{E_j} = (1 - |1 - 2p_j|)\,\frac{T_j}{k_j}$$

3.2 Representation of Assembly Tolerance

Based on Lemma 1, the tolerance of an assembly dimension T_{Y_i} can be expressed in terms of σ_{M_j} and σ_{E_j}. By combining equations (2) and (6), an assembly dimension has three parts: sum of constant centers CS_i, sum of mean shifts MS_i, and sum of error terms ES_i. That is,

$$\begin{aligned} Y_i &= \sum_{j=1}^{n} a_{ij}\,C_j + \sum_{j=1}^{n} a_{ij}\,M_j + \sum_{j=1}^{n} a_{ij}\,E_j \\ &= CS_i + MS_i + ES_i \end{aligned} \qquad \text{for } i=1,\ldots,m$$

Since the first term CS_i is a constant, it does not influence the assembly tolerance T_{Y_i}. Hence, the assembly tolerance T_{Y_i} can be determined by tolerances due to MS_i and ES_i. Denote those two tolerances by T_{MS_i} and T_{ES_i}, respectively. Then, the assembly tolerance T_{Y_i} can be represented by

$$T_{Y_i} = T_{MS_i} + T_{ES_i} \qquad \text{for } i=1,\ldots,m \tag{12}$$

Recall that the distribution of MS_i is unknown hence the distribution of Y_i is also unknown. The derivation for T_{Y_i} is as follows.

To compute T_{Y_i}, first consider T_{ES_i}. Note that T_{ES_i} is the tolerance for the linear sum of error terms, i.e., $\sum_{j=1}^{n} a_{ij}E_j$. Since each error term E_j follows the normal distribution, the tolerance of the linear sum can be described by using (5). That is, †††

$$T_{ES_i} = k_{ES_i} \sqrt{\sum_{j=1}^{n} (a_{ij})^2 (1 - |1 - 2p_j|)^2 (\frac{T_j}{k_j})^2} \qquad \text{for } i=1,\ldots m \qquad (13)$$

where k_{ES_i} is a confidence coefficient for T_{ES_i}. Note that (13) is probabilistic.

Next, consider T_{MS_i}. It is the tolerance for the linear sum of the mean shift M_j's. i.e., $\sum_{j=1}^{n} a_{ij}M_j$. Since the distribution of each component M_j is unknown, the tolerance of the linear sum is taken to be the worst case. That is,

$$T_{MS_i} = \sum_{j=1}^{n} |a_{ij}|\, |1 - 2p_j|\, \frac{T_j}{k_j} k_{M_j} \qquad \text{for } i=1,\ldots,m \qquad (14)$$

where k_{M_j} is a confidence coefficient for the j-th tolerance due to M_j. Note that (14) is deterministic. By substituting (13) and (14) into (12), an assembly tolerance is obtained.

<u>Lemma 2</u>: In the additive model of (6), the tolerance of the linear sum of dimensions can be computed by

$$T_{Y_i} = \sum_{j=1}^{n} |a_{ij}|\, |1 - 2p_j|\, \frac{T_j}{k_j} k_{M_j} + k_{ES_i} \sqrt{\sum_{j=1}^{n} (a_{ij})^2 (1 - |1 - 2p_j|)^2 (\frac{T_j}{k_j})^2} \qquad \text{for } i=1,\ldots,m \qquad (15)$$

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††† $ES_i = \sum_{j=1}^{n} a_{ij}E_j$. Hence, $(\sigma_{ES_i})^2 = \sum_{j=1}^{n} (a_{ij})^2 (\sigma_{E_j})^2$. Since $\sigma_{E_j} = (1 - |1 - 2p_j|) \frac{T_j}{k_j}$ by Lemma 1, $(\sigma_{ES_i})^2 = \sum_{j=1}^{n} (a_{ij})^2 (1 - |1 - 2p_j|)^2 (\frac{T_j}{k_j})^2$. This leads to (13) since T_{ES_i} is defined as $k_{ES_i} \sigma_{ES_i}$.

In (15), the natural choice of k_{M_j} is to make it equal to k_j since both distributions of M_j and X_j are unknown. In addition, k_{ES_i} can be set to 6 such that a 99.73% confidence level is covered under the normal distribution. Then, equation (15) is simplified to

$$T_{Y_i} = \sum_{j=1}^{n} |a_{ij}|\,|1 - 2p_j|\, T_j + 6\sqrt{\sum_{j=1}^{n} (a_{ij})^2 (1 - |1 - 2p_j|)^2 (\frac{T_j}{k_j})^2} \qquad \text{for } i=1,\ldots,m \qquad (16)$$

This hybrid approach unifies the existing approaches [4,5,6,9,11,12,14,15] to tolerance analysis. That is, both the deterministic and the probabilistic approaches can be explained by equation (16): if $p_j = 0$ or 1 (i.e., w_j=1) for j=1,...,n, the equation is the same as that for the worst-case analysis; if $p_j = 0.5$ (i.e., if w_j=0) for j=1,...,n, the equation is the same as that for the probabilistic analysis.

3. TOLERANCE SYNTHESIS

This section presents the solution to tolerance synthesis. Based on the derivations in the previous section, formulation (1) for tolerance synthesis can be written as a non-linear programming (NLP) problem:

$$\text{Min} \qquad C(T_1, T_2, \cdots, T_n)$$

subject to (17)

$$\sum_{j=1}^{n} w_j\,|a_{ij}|\,T_j + 6\sqrt{\sum_{j=1}^{n} (1-w_j)^2\,(a_{ij})^2\,(\frac{T_j}{k_j})^2} \le T_{Y_i}^U \qquad \text{for } i=1,\ldots,m$$

In this formulation, the decision variables are T_j's for $1 \le j \le n$. The constraint is derived through tolerance analysis. The scalars a_{ij} and k_j are given, and w_j is computed from the given p_j using equation (11).

Now, consider the solution scheme for this NLP problem. To determine the existence of an algorithm that guarantees global convergence to the optimum, the objective function $C(T_1, T_2, \cdots, T_n)$ and the constraint functions must be examined. If these functions turn out to be convex, the corresponding NLP then becomes a convex programming (CP) problem. Then, existing algorithms for CP can be used to solve problem (17). Convexity of the cost function $C(T_1, T_2, \cdots, T_n)$ is generally understood in tolerancing [4,17,21,22]; the more tolerance the less cost. Hence, this section is devoted to showing the convexity of the constraint.

The first part of the constraint, i.e., $\sum_{j=1}^{n} w_j\,|a_{ij}|\,T_j$, is a linear function in terms of T_j's. So it is a convex function. Since the linear sum of two convex functions is also convex, the remaining work is to show the convexity of the second part, i.e., $\sqrt{\sum_{j=1}^{n} (1-w_j)^2\,(a_{ij})^2\,(\frac{T_j}{k_j})^2}$. For its proof, the following theorem is used.

Theorem (Cauchy-Schwartz Inequality): Let x and y be two column vectors in R^n. Then,

$$| x^t y | \leq \| x \| \; \| y \|$$

where $\|\cdot\|$ denotes the Euclidean norm.

(Proof) The proof can be found in page 130 of [19].

Lemma 3:

$$\sqrt{\sum_{j=1}^{n} (1-w_j)^2 (a_{ij})^2 (\frac{T_j}{k_j})^2}$$ is a convex function in terms of T_j's.

(Proof) For notational convenience, denote the function $\sqrt{\sum_{j=1}^{n} (1-w_j)^2 (a_{ij})^2 (\frac{T_j}{k_j})^2}$ by G(**T**) and $(1-w_j)^2 (\frac{a_{ij}}{k_j})^2$ by h_{ij}. Notice that $h_{ij} \geq 0$. Then,

$$G(T) = \sqrt{\sum_{j=1}^{n} h_{ij} \, T_j^2} \quad .$$

The function G(T) is convex if

$$\alpha \, G(\mathbf{T}_1) + (1-\alpha) \, G(\mathbf{T}_2) \geq G(\alpha \mathbf{T}_1 + (1-\alpha)\mathbf{T}_2) \qquad (18)$$

for two arbitrarily chosen tolerance vectors $\mathbf{T}_1=(T_{11}, T_{21}, \ldots , T_{n1})^t$, $\mathbf{T}_2=(T_{12}, T_{22}, \ldots , T_{n2})^t$ and a scalar α such that $0 \leq \alpha \leq 1$. To see if inequality (18) holds, the difference of the squares of both sides, i.e., $(\alpha G(\mathbf{T}_1) + (1-\alpha)G(\mathbf{T}_2))^2 - G(\alpha \mathbf{T}_1 + (1-\alpha)\mathbf{T}_2)^2$, is checked. (Squaring both sides of inequality (18) is permissible since they are nonnegative. Refer to equation (5).) Invoking the Cauchy-Schwartz inequality proves the convexity of G(T) as follows:

$$(\alpha G(T_1) + (1-\alpha)G(T_2))^2 - G(\alpha T_1 + (1-\alpha)T_2)^2$$

$$= 2\alpha(1-\alpha)\left\{\sqrt{\sum_{j=1}^{n}(\sqrt{h_{ij}}\,T_{j1})^2}\sqrt{\sum_{j=1}^{n}(\sqrt{h_{ij}}\,T_{j2})^2} - \sum_{j=1}^{n}(\sqrt{h_{ij}}\,T_{j1})(\sqrt{h_{ij}}\,T_{j2})\right\}$$

$$= 2\alpha(1-\alpha)\left\{ \| T_1^* \| \| T_2^* \| - | T_1^{*t} T_2^* | \right\} \geq 0$$

where $T_l^* = (\sqrt{h_{ij}}\,T_{1l}, \sqrt{h_{ij}}\,T_{2l}, \ldots, \sqrt{h_{ij}}\,T_{nl})$ for l=1,2. Q.E.D.

Since both the objective function and the constraints are convex, the tolerance synthesis problem of (17) is a CP problem. This leads to a desirable result: that the minimum cost tolerances can be always found by using the existing algorithms for a CP problem [23].

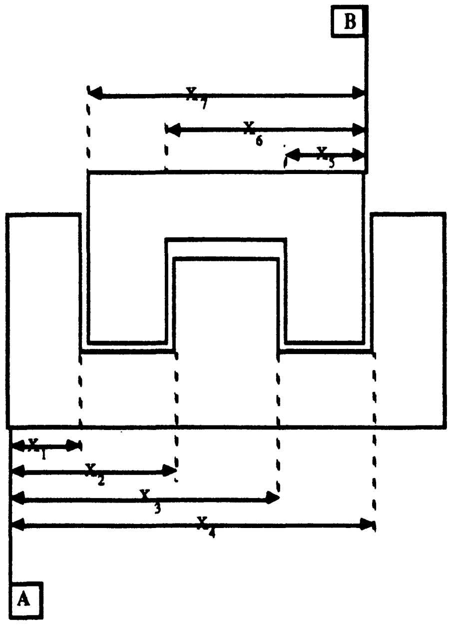
Example

Consider the assembly in Figure 6 in which three assembly dimensions $Y_1 \sim Y_3$ are described as linear functions of component dimensions $X_1 \sim X_7$. (In this example, the component dimensions are specified with respect to datums A and B, which are the surfaces labelled in the drawing.) The three assembly dimensions are for clearances between the two parts. To synthesize the tolerances of component dimensions, three kinds of input are needed: (i) bounds on the assembly dimensions T_{Y_i}, (ii) probabilistic data on component dimension X_j, and (iii) cost function $C(T_1, T_2, \ldots, T_n)$.

The maximum allowable tolerances of the assembly dimensions specify the performance of the system. In this example, the three assembly tolerances have the following upper bounds:

$$T_{Y_1} \leq 0.005, \qquad T_{Y_2} \leq 0.003, \qquad \text{and} \qquad T_{Y_3} \leq 0.005.$$

As each dimension is represented by the additive model of (6), the center C_j of tolerance limit and scalar p_j for mean shift are read in for each dimension X_j. Table 1 gives the data. In it, the scalar w_j is computed from p_j using (11). Although C_j does not influence the determination of T_j, it is used to compute the resultant tolerance limits $[C_j - \frac{T_j}{2}, C_j + \frac{T_j}{2}]$. The coefficient p_j is used not only for the determination of T_j, but also for the computation of the mean value of X_j since $E(X_j)=C_j+T_j(p_j-0.5)$ according to Lemma 1.



$$Y_1 = X_2 - X_1 - X_7 + X_6$$
$$Y_2 = X_6 - X_5 - X_3 + X_2$$
$$Y_3 = X_4 - X_3 - X_5$$

Figure 6. Assembly of Two Mating Parts

Dimension Index	1	2	3	4	5	6	7
C_j	1.0	2.0	3.0	4.0	0.998	2.0	2.998
p_j	0.6	0.1	0.9	0.4	0.6	0.4	0.5
w_j	0.2	0.8	0.8	0.2	0.2	0.2	0.0

Table 1. Data for the Component Dimensions

Cost functions typically fall into two groups: inverse-square model [21] and exponential model [22]. Both models describe a convex relationship between cost and tolerance:

$$C(T_j) = \frac{\alpha_j}{T_j^2} + f_j \qquad \text{(inverse-square model)}$$

$$C(T_j) = \alpha_j \exp\left(-\frac{T_j}{\beta_j}\right) + f_j \qquad \text{(exponential model)}$$

where $C(T_j)$ denotes the manufacturing cost for tolerance T_j, while α_j and β_j are the coefficients for variable cost, and f_j indicates the fixed cost. For this example, the inverse-square model is adopted and the corresponding coefficients are given in Table 2:

Dimension Index	1	2	3	4	5	6	7
α_j	1.0	1.0	1.5	1.5	1.4	1.4	0.6
f_j	0.1	0.1	0.1	0.1	0.1	0.1	0.1

Table 2. Coefficients of Tolerance-Cost Model

Then, the total manufacturing cost is:

$$C(T_1,T_2,\ldots,T_n) = \sum_{j=1}^{n} C(T_j) = \sum_{j=1}^{n} \left(\frac{\alpha_j}{T_j^2} \times 10^{-6}\right) + \sum_{j=1}^{n} f_j$$

Tolerance synthesis by cost minimization (17) instantiates to:

$$\text{Min} \quad \sum_{j=1}^{7} \left(\frac{\alpha_j}{T_j^2} \times 10^{-6} \right) + \sum_{j=1}^{7} f_j$$

subject to

$$w_1 T_1 + w_2 T_2 + w_6 T_6 + w_7 T_7 + \sqrt{(1-w_1)^2 T_1^2 + (1-w_2)^2 T_2^2 + (1-w_6)^2 T_6^2 + (1-w_7)^2 T_7^2} \leq 0.005$$

$$w_2 T_2 + w_3 T_3 + w_5 T_5 + w_6 T_6 + \sqrt{(1-w_2)^2 T_2^2 + (1-w_3)^2 T_3^2 + (1-w_5)^2 T_5^2 + (1-w_6)^2 T_6^2} \leq 0.003$$

$$w_3 T_3 + w_4 T_4 + w_5 T_5 + \sqrt{(1-w_3)^2 T_3^2 + (1-w_4)^2 T_4^2 + (1-w_5)^2 T_5^2} \leq 0.005$$

To solve this CP problem, the software package MP5-FIXPT, which implements the homotopy procedure of Eaves and Saigal for finding a fixed point [20], is used. It runs on the IBM AT personal computer. The result of executing the algorithm is given in Table 3. And, this result is compared in Table 4 with results from two other methods, worst-case [3,14] and probabilistic [4,11,15], which were also implemented by adjusting w_j. The hybrid approach generates tolerances looser than the worst-case analysis but tighter than the probabilistic approach, for a total cost 6.849. For instance, the tolerance T_7 is 25.095 x 10^{-4} units when the hybrid approach is used. If the worst-case analysis (or the probabilistic approach) is used for tolerance analysis, then T_7 decreases (or increases) by 35.1% (by 19.9%). This decrease (or increase) in turn incurs an increase (or decrease) in manufacturing cost. The difference in cost among the approaches is due to the difference in the available information about the distribution of the dimensions: if the distribution is fully known, then the probabilistic approach should be used to reduce the manufacturing cost; if the distribution is totally unknown as in the worst-case analysis, then a higher manufacturing cost is incurred to insure against uncertainty. The hybrid approach deals with partial information.

dimension index j	$T_j^{(*)}$	Mean $E(X_j)^{(*)}$	Tolerance Limits $^{(*)}$	
			Lower	Upper
1	29.4381	10002.9400	9985.2810	10014.7200
2	8.4896	19996.6000	19995.7600	20004.2400
3	9.7463	30003.9000	29995.1300	30004.8700
4	39.1872	39996.0800	39980.4100	40019.5900
5	9.8801	9980.9880	9975.0600	9984.9400
6	9.8617	19999.0100	19995.0700	20004.9300
7	25.0950	29980.0000	29967.4500	29992.5500

Note: (*) unit: 10^{-4}

Table 3. Result of the Example

T_j	Worst-Case $^{(*)}$	Probabilistic $^{(**)}$	Hybrid$^{(***)}$
T_1	19.299 (65.6%)	34.183 (116.1%)	29.438 (100.0%)
T_2	6.807 (80.2%)	13.975 (164.6%)	8.490 (100.0%)
T_3	7.878 (80.8%)	15.521 (159.3%)	9.746 (100.0%)
T_4	34.423 (87.8%)	45.015 (114.9%)	39.187 (100.0%)
T_5	7.699 (77.9%)	15.255 (154.4%)	9.880 (100.0%)
T_6	7.615 (77.2%)	15.202 (154.1%)	9.862 (100.0%)
T_7	16.278 (64.9%)	30.085 (119.9%)	25.095 (100.0%)
Total Cost	10.672 (155.8%)	3.268 (47.7%)	6.849 (100.0%)

Note: - The values in parenthesis are normalized ones based on the values of the hybrid approach
- (*) when all w_j=1.0, unit : 10^{-4}
- (**) when all w_j=0.0, unit : 10^{-4}
- (***) unit : 10^{-4}

Table 4. Comparison with Other Methods

4. SUMMARY AND DISCUSSION

The contribution of this paper is two-fold. In theory, the hybrid technique in section 2 unifies the deterministic and the probabilistic approaches to tolerance analysis. This is made possible by the additive model (6) that handles both the symmetric and the asymmetric distributions, the properties of which have been rigorously derived. In practice, globally optimum tolerances for recti-linear systems that are prevalent in simple assembly tasks and in VLSI design are synthesized by cost minimization. The globally optimum solution is enabled by the formulation of an NLP (17) which is proven to be convex in section 3.

In the early days of computer aided design, development of finite element analysis and computer graphics proceeded in parallel. It was not until both areas were sufficiently mature did automatic meshing emerge. Here, a technique for synthesizing tolerances in recti-linear electro-mechanical systems is reported. It requires "non-geometric" data such as tolerance-cost models and probabilistic parameters. If tolerance synthesis turns out to be useful, this and other recent efforts [11,12] merely underscore the need to incorporate manufacturing related data in product models. Perhaps more fundamental is the issue of representing geometric uncertainties. While developments in computer hardware and application software have advanced, the representation of a non-ideal world, in which noise abounds and in particular dimensions are not nominal, has not been fully embraced by CAD/CAM researchers. Work in geometric modeling such as [7,18] and probabilistic models in section 3.1 and 3.2 are but the first steps from different directions. It is hoped that this work will generate a critical mass of researchers not only in the modeling but also in the computation of geometric uncertainties. In turn, such an effort will enable designers to fold some of the "downstream" considerations such as manufacturability and cost into the design stage.

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