NONLINEAR FILTERING OF RANDOM FIELDS IN THE PRESENCE OF LONG-MEMORY NOISE AND RELATED PROBLEMS IN STOCHASTIC ANALYSIS

by

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To Mom, Dad and Ry and to Gary Chard whose influence made this work possible.

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TABLE OF CONTENTS

CHAPTER I

Introduction

This work develops new methods in spatial nonlinear stochastic filtering theory and stochastic analysis. In particular, we examine a class of long range dependent stochastic processes and random fields known as fractional Brownian motion and fractional Brownian sheet. The literature on these processes has developed rapidly in recent years. This is at least partially due to the fact that the long range dependence phenomenon has been observed in empirical studies spanning a broad range of disciplines.

Fractional Brownian fields make up a parametric family of processes that evolve over an m-dimensional parameter space and are parameterized by a Hurst multi-index, H, where H is an m-dimensional vector. Depending on the value of the Hurst multi-index, our processes exhibit different levels of dependence. When $H = \frac{1}{2}$ and $m = 1$, the process is a version of the classical Brownian motion which has independent increments. As such, families of fractional Gaussian processes make up a rich class of random fields to which classical Wiener processes belong.

As is well known, Wiener processes assign independent random measures to disjoint increments and exhibit martingale structure. While this makes Wiener processes very attractive as tools to model many real world processes, there are several instances where it is not realistic to assume martingale structure or independence of the process over disjoint increments. Under such circumstances, fractional Gaussian processes and their more complex covariance structure are better suited to model real world phenomena.

When Hurst parameter $H > \frac{1}{2}$, fractional Brownian processes exhibit long range dependence. Such dependence arises often in applications ranging from network communications, where internet traffic has been shown to exhibit long memory, to macroeconomics, where the long range dependence paradigm has been used to analyze the effects of economic shocks on income cycles. Fractional Gaussian processes are often used to model the long range dependence seen in these real world examples. It is important, therefore, that we are able to understand and work with the processes. To this end, a large amount of stochastic analysis of fractional Brownian motion has been done in recent years.

One area of stochastic analysis which is particularly useful in a variety of applications is that of filtering theory. It is often the case that one is able to observe only a noisy version of some process of interest. In this setting one wishes to estimate the underlying process of interest based upon the noisy observation process. It is important, from the theoretical point of view as well as a more applied point of view, that we are able to derive filtering techniques not only for processes distorted by martingale noise, but also for processes exhibiting long memory. Modeling processes as martingales allows one to take advantage of many of the nice mathematical properties associated with such processes. For example, stochastic calculus developed by Itô [12] which is employed in deriving optimal filtering equations, is based on L^2martingale structure. However, in real applications the martingale assumptions are not always justified, hence the recent literature on filtering in the case of fractional Brownian motion noise [14], [5], [6].

Unlike classical nonlinear filtering theory, which is based on an observation model where the observation noise is a single parameter martingale and the signal is a one-parameter Markov process, this dissertation develops results in spatial nonlinear stochastic filtering, where the signal and observation processes are random fields and the spatial observation is distorted by noise in the form of a fractional Brownian sheet.

The classical nonlinear filtering problem based on the model with single parameter martingale noise has been studied by Kushner [17], Zakai [30], Fujisaki, Kallianpur, and Kunita [11]. These papers describe the evolution of the optimal filter over time in terms of stochastic partial differential equations. Many of the results in classical nonlinear filtering have been extended to the setting where the noise is given by fractional Brownian motion [14], [5], [6] rather than standard Brownian motion. In such a model the observation noise is not a martingale.

Here we develop nonlinear filtering theory in the case when the observation noise lacks not only martingale structure, but both our noise and signal processes are random fields with multi-dimensional spatial parameters. Whereas in the one dimensional parameter space setting there is perfect ordering on $\mathbb{R}$ and a straight forward way of defining martingale structure, in higher dimensions defining martingale structure requires a bit more care. In fact, there are a variety of notions of martingales in the plane. When investigating filtering problems in the plane, one must take special care to ensure each notion of martingale can be applied in certain circumstances.

In order to study the continuous nonliner filtering with multidimensional spatial parameters, we apply the theory of stochastic integration in the plane developed by

Renzo Cairoli and John Walsh in 1975 [4]. In this very important paper, integration with respect to two-parameter martingales was developed based on a partial ordering, which will be used to define martingales in the plane.

Nonlinear filtering theory has applications in a wide range of areas. In finance for example, asset prices represent a stochastic process driven by some underlying value of the asset. The actual price of the asset in the market is based on the true underlying unkown value, with observation process based on investors' perception of the assets' value. Filtering of random fields is common in digital image analysis [26]. Typical problems in image analysis involve image restoration, where one tries to recover the true image from blurry or noisy observations of the given image.

In Chapter II, we develop stochastic evolution equations describing the dynamics of the optimal filter in a general nonlinear filtering problem. Much of Chapter II is also devoted to developing the theory that will be used through out the dissertation. By solving the stochastic evolution equations, one can calculate the best estimate of the signal process in the mean square error sense.

In Chapter III, we again investigate the nonlinear filtering problem described in Chapter II. The main goal of Chapter III is to use theoretical methods in stochastic analysis that make applying the filtering theory of Chapter II more feasible. Since it is often the case that the evolution equations of Chapter II are extremely complex and difficult to solve explicitly, we develop alternative representations of the optimal filter. We arrive at representations of the optimal filter through infinite sums of multiple stochastic integrals. These infinite sums can be truncated and discretized in order to numerically implement the optimal filters.

Finally, in Chapter IV, we investigate new theoretical properties involving the multiple integrals used in the integral expansions of the optimal filter described in

Chapter III. Specifically, we look at the relationship between multiple stochastic integrals with respect to standard Brownian sheet and multiple stochastic integrals with respect to fractional Brownian sheet. We give an operator that allows one to transform multiple Wiener integrals with respect to Brownian motion into multiple fractional Wiener integrals with respect to fractional Brownian motion. Both types of multiple integrals are used in the development of Chapter III and are therefore of important interest from both theoretical and applied points of view.

CHAPTER II

Stochastic evolution equations for nonlilnear filtering of random fields

2.1 Introduction

Filtering theory has long been studied by mathematicians, statisticians, scientists and engineers as a way to estimate partially observed processes. The wide-ranging applications of the theory make it a particularly useful area of study for many in the physical sciences and engineering. The mathematical challenges arising in filtering problems often lead to very interesting problems concerning the theoretical underpinnings and structure of the specific model at hand. With so many available applications, there are often new models arising which require slight or sometimes major revisions to existing theory.

In several engineering and physical sciences settings, dynamical systems evolve in space or often in space and time as opposed to random processes that evolve only in time. In such cases we use random fields to model various dynamical systems. Random fields are random processes that evolve over a parameter space $\mathbb{T}$ where $\mathbb{T}$ may be a subset of $\mathbb{R}_+^d$ with $d \geq 2$, as opposed to single parameter random processes that evolve over $\mathbb{T}$ where $\mathbb{T} = [0, \infty)$ or $\mathbb{T} = [0, T]$.

As in the case of single-parameter processes, there is often a need for estimation techniques in dynamical systems that evolve randomly in space and are either corrupted by some random noise or are not directly obervable. One example arises in image processing where denoising of images or video streams obviously requires a multidimensional parameter space.

Mathematically, the goal of filtering is to characterize the conditional distribution of an unobserved "signal" denoted by $(X_t, t \in \mathbb{T})$, given the observation σ-field $\mathcal{F}_t^Y = \sigma\{Y_s,\ s \prec t\}$. Equivalently we can study the conditional distribution of suitable functions of the signal process, given the observation σ-field. In this chapter, we study the dynamics of $\mathbb{E}[F(X_t)|\mathcal{F}_t^Y]$ for a large but sufficiently suitable class of functions. We will examine the multiparameter, nonlinear filtering using the natural observation model

$$Y_{(t_1,\ldots,t_d)} = \int_0^{t_1} \cdots \int_0^{t_d} g\left(X_{(s_1,\ldots,s_d)}\right) ds_1 \ldots ds_d + \mathcal{W}_{(t_1,\ldots,t_d)}, \qquad (t_1,\ldots,t_d) \in \mathbb{T}, \tag{2.1}$$

where g belongs to a suitable class of functions and $\mathcal{W}_{(t_1,\ldots,t_d)}$ is a multiparameter random "noise."

This chapter is devoted to studying the evolution of $\mathbb{E}[F(X_t)|\mathcal{F}_t^Y]$ in the parameter space $\mathbb{T} = [0, T_1] \times [0, T_2]$ where d, the dimension of the parameter space is equal to two. The results are easily extended to general $d > 2$. Perhaps the main difference between the multiparameter evolution of $\mathbb{E}[F(X_t)|\mathcal{F}_t^Y]$ and the single-parameter evolution is the lack of perfect ordering in the multiparmeter setting. That is, for any $t, s \in \mathbb{R}$ with $t \neq s$, it is always the case that either $t < s$ or $s < t$. On the other hand, one needs to take more care in defining an ordering in $\mathbb{R}^d$.

The model (2.1) has been studied in the case where $\mathcal{W}$ is a Wiener (Brownian) sheet in [15] and [16], where several stochastic differential equations governing the evolution of the unnormalized optimal filter were obtained. Here we study the evolution of the unnormalized optimal filter for (2.1) when $\mathcal{W}$ is a fractional Brownian sheet exhibiting long-range dependence along each axis (persistent) and as such $\mathcal{W}$

does not exhibit the martingale-type properties of the standard Wiener sheet.

In Section 2.2 we begin by giving an overview of multiparameter martingales and stochastic calculus with respect to multiparameter martingales. Multiparameter martingales and their properties play a large role in the analysis throughout the rest of this chapter. Most of the properties and definitions in Section 2.2 can be found in [4]. In Section 2.3 we give an introduction to the field of fractional calculus. Fractional calculus is used extensively in the analysis of fractional Brownian motion and fractional Brownian fields. We will use fractional integrals and derivatives throughout our analysis. This calculus allows one to obtain stochastic integral representations of persistent fractional Brownian sheet and fractional Brownian motion. In Section 2.4 we introduce fractional Brownian motion and fractional Brownian sheet. In Section 2.5 we give some known results in the theory of nonlinear filtering. We examine results in the case of single parameter case. Within this framework, we first look at the classical setting, martingale noise and then we look at more recent results involving persistent fractional Brownian motion noise.

Finally in Sections 2.6, 2.7, 2.8 we present the main results of the chapter. We derive two stochastic evolution equations for the optimal unnormalized filter in the plane. The first involves the evolution of the optimal filter along a "increasing" paths. We define increasing paths in terms of the partial ordering described in Section 2.2. The next evolution equation is for arbitrary evolution in the parameter space $\mathbb{T}$. For notational simplicity we restrict our attention to the setting where $\mathbb{T}$ is a bounded, two-dimensional parameter space. The techniques and results developed can be extended to arbitrary d-dimensional parameter spaces, however, the resulting evolution equations will contain many more terms.

2.2 Multiparameter Martingales and Stochastic Integration in the Plane

In this section we state basic definitions and results related to multiparameter martingales and stochastic integration with respect to such processes. These play key roles in the development of stochastic filtering theory in the plane and highlight existing differences and similarities between the one-parameter and multiparameter martingale theories. We will define partial ordering required in order to define martingales in multi-dimensional parameter space. In this section, we give the definitions of different types of martingales in the plane and describe two parameter analogues of quadratic variation and the Doob-Meyer decomposition.

Recall that for a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$, if random process $X := \{X_t,\ t \in \mathbb{R}_+\}$ is adapted to the filtration $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$, and $\mathbb{E}|X_t| < \infty, \forall t \in \mathbb{R}_+$, then $\{X_t, \mathcal{F}_t, t \in \mathbb{R}_+\}$ is a martingale if, for every $0 \le s < t < \infty$, $E(X_t|\mathcal{F}_s) = X_s$ a.s. (P). However in the case of processes of two parameters defining martingale structure is less trivial.

First let us define the partial ordering $\prec$ on the positive quadrant $\mathbb{R}^2_+$, and adopt the following notation for arbitrary points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2_+$:

$$a \preceq b \quad \text{if } a_1 \le b_1 \text{ and } a_2 \le b_2$$

$$a \prec b \quad \text{if } a_1 < b_1 \text{ and } a_2 < b_2$$

$$a \curlywedge b \quad \text{if } a_1 \le b_1 \text{ and } a_2 \ge b_2$$

$$a \wedge b = (\min(a_1, b_1), \min(a_2, b_2))$$

$$a \vee b = (\max(a_1, b_1), \max(a_2, b_2))$$

$$a \odot b = (a_1, b_2)$$

Next, let us define what is meant by a random measure on Borel sets of $\mathbb{R}_+^2$. Martingales in one parameter space are often thought of as having independent increments for disjoint time intervals. For processes X on parameter space $\mathbb{R}_+^2$, we define increments over the rectangle $A = (z, z'] = ((z_1, z_2), (z_1', z_2')]$, where $z \prec z'$ as

$$X(A) \equiv X_{(z_1',z_2')} - X_{(z_1',z_2)} - X_{(z_1,z_2')} + X_{(z_1,z_2)} = X_{z'} - X_{z'\odot z} - X_{z\odot z'} + X_z \tag{2.2}$$

Given a complete probability space $(\Omega, \mathcal{F}, P)$, let $\left\{\mathcal{F}_z, z \in \mathbb{R}_+^2\right\}$ be a family of sub σ-fields of $\mathcal{F}$ satisfying the following properties:

$(F1)$ if $z \prec z'$ then $\mathcal{F}_z \subset \mathcal{F}_{z'}$

$(F2)$ $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$

$(F3)$ for each z,

$$\mathcal{F}_z = \bigcap_{z \prec \prec z'} \mathcal{F}_{z'}$$

$(F4)$ for each z, $\mathcal{F}_z^1$ and $\mathcal{F}_z^2$ are conditionally independent given $\mathcal{F}_z$,

where $\mathcal{F}_z^1$ and $\mathcal{F}_z^2$ are defined as follows: let $z = (z_1, z_2) \in \mathbb{R}_+^2$, then

$$\mathcal{F}_z^1 = \mathcal{F}_{z_1,z_2}^1 = \mathcal{F}_{z_1,\infty} = \bigvee_t \mathcal{F}_{z_1,t} = \sigma\Big\{ \bigcup_{t\in\mathbb{R}_+} \mathcal{F}_{z_1,t} \Big\}$$

$$\mathcal{F}_z^2 = \mathcal{F}_{z_1,z_2}^2 = \mathcal{F}_{\infty,z_2} = \bigvee_s \mathcal{F}_{s,z_2} = \sigma\Big\{ \bigcup_{s\in\mathbb{R}_+} \mathcal{F}_{s,z_2} \Big\}$$

Note 2.2.1. the condition (F4) is equivalent to:

$(F4')$ for all bounded random variables X and all $z \in \mathbb{R}_+^2$

$$E\big\{X|\mathcal{F}_z\big\} = E\big\{E\{X|\mathcal{F}_z^1\}|\mathcal{F}_z^2\big\}$$

It is important to note that for any process $\left\{X(A) : A \text{ is a rectangle in } \mathbb{R}_+^2\right\}$, if $X(A_1), ...X(A_n)$ are independent for all disjoint rectangles $A_1, ...A_n$, then $\mathcal{F}_z := \sigma\big\{X(A) : \forall u \in A, u \prec z\big\}$ satisfies condition (F4) [4].

Note 2.2.2. The condition $(F3)$ is analogous to the definition of a right continuous filtration in one parameter. Similarly,we will call a filtration satisfying $(F3)$ right continuous.

Definition II.1. Let $\mathcal{F}_z$ be a filtration satisfying (F1)-(F4). The process $X = \{X_z, z \in \mathbb{R}_+^2\}$ is a **two − parameter Martingale** if:

1. X_z is adapted to the filtration $\mathcal{F}_z, \ \forall z \in \mathbb{R}_+^2$
2. for each $z \in \mathbb{R}_+^2, X_z$ is integrable
3. for each $z \prec z', E\{X_{z'}|\mathcal{F}_z\} = X_z \quad a.s.$

Proposition II.2. [4] If M_z is a two-parameter martingale with respect to filtration $\mathcal{F}_z$, then for each fixed z_2, $\{M_z, \mathcal{F}_z\} = \{M_{z_1,z_2}, \mathcal{F}_{z_1,z_2}\}$ is a one-parameter martingale.

Definition II.3. Given a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_z), P)$, we say $W = (W_z, \mathcal{F}_z, z \in \mathbb{R}_+^2)$ is a continuous version of a **Brownian Sheet** if it satisfies the following conditions:

1. W is a random measure on $\mathbb{R}_+^2$, assigning to each Borel set A, a Gaussian random variable of mean zero and variance $\lambda(A)$, where λ=Lebesgue measure.
2. for disjoint Borel sets in $\mathbb{R}_+^2$, W assigns independent random variables.
3. $W_z = W(R_z)$ where R_z is the rectangle whose upper right hand corner is z and whose lower left hand corner is the origin and W_z is adapted to $\mathcal{F}_z$.
4. W has continuous trajectories.
5. If $z = (z_1, z_2)$ then for fixed z_1, $(W_{z_1,t})$ is a standard Brownian motion and similarly, for fixed z_2, (W_{s,z_2}) is a Brownian motion.

Definition II.4. Let $X = \{X_z : z \in \mathbb{R}_+^2\}$ be a process such that X_z is integrable for all $z \in \mathbb{R}_+^2$ and let the filtration $\mathcal{F}_z$ satisty $(F1) - (F4)$. Then

1. X is a **Weak Martingale** if

   (i) X_z is adapted to filtration $\mathcal{F}_z$ $\forall z$,

   (ii) $E\{X((z, z'])|\mathcal{F}_z\} = 0 \ \forall \ z \prec\prec z'$.

2. X is an **i-Martingale** (i= 1,2) if,

   (i) X_z is $\mathcal{F}_z^i - adapted$

   (ii) $E\{X((z, z'])|\mathcal{F}_z^i\} = 0 \ \forall \ z \prec\prec z'$

3. X is a **Strong Martingale** if

   (i)X is adapted to the filtration $\mathcal{F}_z$,

   (ii) X vanishes on the axes (i.e. $X_{(0,z_2)} = 0$ and $X_{(z_1,0)} = 0$), $a.s.$

   (iii) $E\{X((z, z'])|\mathcal{F}_z^1 \bigvee \mathcal{F}_z^2\} = 0 \ \forall \ z \prec\prec z'$

Note 2.2.3. Any martingale is a weak martingale and any strong martingale is a martingale. That is:

$\{X : X\text{is a strong martingale}\} \subset \{X : X\text{is a martingale}\} \subset \{X : X \text{ is a weak martingale}\}$

Definition II.5. A process $\{X_z\}$ is **right continuous** if

$$\lim_{\substack{z' \to z \\ z \prec z'}} X_{z'}(\omega) = X_z(\omega) \quad a.s.$$

Definition II.6. A process $X = \{X_z, z \in \mathbb{R}_+^2\}$ is called an *increasing process* if

1. X is right continuous and adapted to $\mathcal{F}_z$

2. $X_z = 0$ on the axes

3. $X(A) \geq 0$ for each rectangle $A \subset \mathbb{R}_+^2$

Definition II.7. For $p \geq 1$

1. $\mathcal{M}^p(\mathbb{T}) :=$ the space of right continuous martingales $M = \{M_z, z \in \mathbb{T}\}$ such that $M = 0$ on the axes and

$$\|M\|_p := \sup_{z \in \mathbb{T}} \left(E\{|M_z|^p\}\right)^{\frac{1}{p}} < \infty.$$

2. $\mathcal{M}^p_S(\mathbb{T}) :=$ the class of all strong martingales in $\mathcal{M}^p(\mathbb{T})$

The following theorem gives the two-parameter analogue to the Doob-Meyer decomposition. In the two-parameter case, uniqueness of the quadratic variation process is not guaranteed, unlike in the one-parameter setting.

Theorem II.8. *Let $M \in \mathcal{M}^2(\mathbb{T})$. There exists an increasing process $A = \{A_z, z \in \mathbb{T}\}$ such that $\{M_z^2 - A_z, z \prec \mathbb{T}\}$ is a weak martingale*

Note 2.2.4. In the one parameter martingale case, the Doob-Meyer decomposition gives a unique increasing process. On the other hand, in the two parameter case, the increasing process A is not necessarily unique. However, for the purposes of stochastic integration this does not pose any serious problem .

Definition II.9. We say that a process $\langle M \rangle$ is a version of the **quadratic variation** of martingale M if $(M_z^2 - \langle M \rangle_z,\ z \in \mathbb{T})$ is a martingale.

Note 2.2.5. By proposition 1.8 and theorem 1.9 of [4], the quadratic variation of a strong martingale $(M_z, \mathcal{F}_z, z \in \mathbb{T})$ is unique if $\mathcal{F}_z$ is a filtration generated by a standard Brownian sheet. However, for general martingales and strong martingales in the plane, the quadratic variation process is not necessarily unique.

Definition II.10. Let $M \in \mathcal{M}^2_S(\mathbb{T})$. There exists unique $\mathcal{F}^1_z$-predictable, increasing process denoted $[M]^{(1)}$ such that $M_z^2 - [M]^{(1)}_z$ is a 1-martingale and there exists a unique $\mathcal{F}_z^2$-predictable increasing process denoted by $[M]^{(2)}$ such that $M_z^2 - [M]_z^{(2)}$ is an 2-martingale.

Integration with respect to two parameter martingales was developed by R. Cairoli and John Walsh in 1975 [4]. Their work is fundamental to the theory of stochastic nonlinear filtering in the plane in much the same way that the work of Itô is pivotal in the classical nonlinear filtering theory. In this section we give a construction of the stochastic integral in the plane and properties that are useful in subsequent sections of the dissertation.

Definition II.11. Consider the space $\mathbb{R}_+^2 \times \Omega$. Let $\left\{\mathcal{G}_z, z \in \mathbb{R}_+^2\right\}$ be an increasing, right continuous family of sub-sigma fields of $\mathcal{F}$. The **sigma field of $\mathcal{G}_\mathbf{z}$ − predictable sets** is defined as

$$\mathcal{D}_{\mathcal{G}_z'} := \sigma - \text{field generated by sets of the form} \quad (z, z'] \times \Lambda$$

where $z \prec z'$ and $\Lambda \in \mathcal{G}_z$

Similarly, a process $X = \left\{X_z, z \in \mathbb{R}_+^2\right\}$ is $\mathcal{G}_\mathbf{z}$ − **predictable** if $(z, \omega) \mapsto X_z(\omega)$ is $\mathcal{D}_\mathcal{G}$-measurable.

We first define the stochastic integral for simple functions in the plane.

Definition II.12. We call ϕ a simple function belonging to class $\mathcal{S}_1^2$ if there exist a finite number of disjoint rectangles $A_i \subset \mathbb{T}$ and bounded random variables α_i such that α_i is $\mathcal{F}_{z_i} - \textit{measurable}$ and ϕ can be written as the finite sum

$$\phi_z = \sum_i \alpha_i 1_{A_i}(z).$$

The superscript 2 and subscript 1 in $\mathcal{S}_1^2$ indicate that z is a 2×1 matrix, i.e. a 2-dimensional vector.

Definition II.13. Let $M \in \mathcal{M}^2(\mathbb{T})$ and let ϕ be a simple function. Then we define the stochastic integral of ϕ with respect to martingale M over rectangle R_z by

$$\int_{R_z} \phi dM = \sum_i \alpha_i M(A_i \cap R_z)$$

Clearly, if ϕ is a simple function and $M \in \mathcal{M}^p(\mathbb{T})$, $M' \in \mathcal{M}^p_S(\mathbb{T})$, then $\int_{R_z} \phi_\zeta dM_\zeta \in \mathcal{M}^p(\mathbb{T})$ and $\int_{R_z} \phi_\zeta dM'_\zeta \in \mathcal{M}^p_S(\mathbb{T})$

In order to extend the definition of the stochastic plane integrals from simple fuctions to a larger class of funtions, let us define such classes of functions $\mathcal{H}_i$, $i = 0, 1, 2$.

Definition II.14. Let $\{\phi_z, z \in \mathbb{T}\}$ be a process such that the following conditions hold:

(a) ϕ is a bimeasurable function of (ω, z),

(b) $\int_{\mathbb{T}} \mathbb{E}\phi_z^2 dz < \infty$,

and for each $z \in \mathbb{T}$, either (c_0) ϕ_z is $\mathcal{F}_z$-measurable,

or (c_1) ϕ_z is $\mathcal{F}_z^1$-measurable,

or (c_2) ϕ_z is $\mathcal{F}_z^2$-measurable.

For $i = 0, 1, 2$, let $\mathcal{H}_i$ denote the space of ϕ satisfying (a),(b) and (c_i).

Under the the norm $\|\phi\|_{\mathcal{H}_0} = \big(E\{\int_{\mathbb{T}} \phi_\zeta^2 d\zeta\}\big)^{\frac{1}{2}}$, the simple functions are dense in $\mathcal{H}_0$ and considered as a function space, $\mathcal{H}_0$ is a Hilbert space. Let $\{W_z, \mathcal{F}_z, z \in \mathbb{T}\}$ be a Wiener sheet. Let us introduce the following classes of integrands. The linear mapping $(\phi_n \circ W)_z := \int_{R_z} \phi_n dW$ maps the simple functions onto $\mathcal{M}^2(\mathbb{T})$ and preserves the norm. Therefore, the mapping can be extended by continuity to a norm-preserving linear map from $\mathcal{H}_0$ into $\mathcal{M}^2(\mathbb{T})$ by appropriately taking limits of sequences of integrals of simple functions.

Proposition II.15. *[28] Let $\phi \in \mathcal{H}_0$ and let $W = \{W_z : z \prec z_0\}$ be a Brownian sheet as in Def. II.3. Then the integral $\int_{R_z} \phi_\zeta dW_\zeta$ is well defined and is a strong martingale. Furthermore, if we let*

$$M_z = \left(\int_{R_z} \phi_\zeta dW_\zeta \right)^2 - \int_{R_z} \phi_\zeta^2 d\zeta$$

then M is a martingale.

Theorem II.16. *[9] Let $\{X_z, \mathcal{G}_z; z \prec z_0\}$ be a strong martingale satisfying either of the following conditions; (i) $E\{X_z^2\} < \infty$ and $\mathcal{G}_z$ is a sigma-field generated by a Brownian sheet; (ii) $E\{X_z^4\} < \infty$ and X is continuous. Then*

$$\exp\left\{X_z^2 - \frac{1}{2}\langle X\rangle_z\right\} \text{ is a martingale iff}$$
$$E\left(\exp\left\{X_{z_0}^2 - \frac{1}{2}\langle X\rangle_{z_0}\right\} \right) = 1$$

One can also show that for $\phi \in \mathcal{H}_i, i = 0, 1, 2$, and W, a standard Brownian sheet, the stochastic integral, denoted

$$(\phi \circ W)_z = \int_{R_z} \phi_\zeta dW_\zeta = \int_{\mathbb{T}} I(\zeta \prec z)\phi_\zeta dW_\zeta, \; z \in \mathbb{T}, \tag{2.3}$$

is a strong martingale for $\phi \in \mathcal{H}_0$, a 1-martingale for $\phi \in \mathcal{H}_1$ and a 2-martingale for $\phi \in \mathcal{H}_2$. Moreover, define a process

$$\xi_z = (\phi \circ W)_z(\psi \circ W)_z - \int_{R_z} \phi_\zeta \psi_\zeta d\zeta, \; z \in \mathbb{T}.$$

Then $\xi = (\xi_z, z \in \mathbb{T})$ is a martingale with respect to $(\mathcal{F}_z)_{z\in\mathbb{T}}$ if $\phi, \psi \in \mathcal{H}_0$, a 1-martingale if $\phi, \psi \in \mathcal{H}_1$ and a 2-martingale if $\phi, \psi \in \mathcal{H}_2$. In all cases continuous versions of the above defined processes can be chosen.

For integration in the plane, we will require the development of another type of stochastic integral, $\iint \psi dMdM$. In order to do so, we first define a new class of simple functions in the plane.

Definition II.17. Let $\mathbb{T} = [0, T_1] \times [0, T_2]$. For fixed n we divide $\mathbb{T}$ into rectangles $\Delta_{i,j} := (z_{i,j}, z_{i+1,j+1}]$, where $z_{i,j} = (2^{-n}i, 2^{-n}j)$. If i, j, k, l satisfy $i < k \leq 2T_1^n - 1,\ l < j \leq 2T_2^n - 1$, we define the class of functions $\mathbb{S}_1$ of the form

$$\chi_{ijkl}(\zeta, \zeta') = \alpha 1_{\Delta_{i,j}}(\zeta) 1_{\Delta_{k,l}}(\zeta') \tag{2.4}$$

where α is bounded and $\mathcal{F}_{z_{k,j}}$-measurable. Similarly, define the class of simple functions $\mathbb{S}$ as finite sums of functions in $\mathbb{S}_1$

$$\chi(\zeta, \zeta') = \sum_{ijkl} \alpha_{ijkl} 1_{\Delta_{i,j}}(\zeta) 1_{\Delta_{k,l}}(\zeta') \tag{2.5}$$

where α_{ijkl} is bounded and $\mathcal{F}_{z_{k,j}}$-measurable

Definition II.18. Let $M = \left(M_z, \mathcal{F}_z, z \in \mathbb{R}_+^2\right)$ be a martingale in the plane. For simple function χ as defined in (2.4), we define the integral $\iint \chi dMdM$ by

$$\int_{\mathbf{R}_z \times \mathbf{R}_z} \chi_{ijkl}(\zeta, \zeta') dM_\zeta dM_{\zeta'} := \alpha M(\Delta_{i,j} \cap \mathbf{R}_z) M(\Delta_{k,l} \cap \mathbf{R}_z), \quad z \in \mathbb{T}. \tag{2.6}$$

The integral $\iint \chi dMdM$ is defined as the corresponding linear combination of integrals of the form (2.6).

Note 2.2.6. $\int_{\mathbf{R}_z \times \mathbf{R}_z} \chi_{ijkl}(\zeta, \zeta') dM_\zeta dM_{\zeta'}$ as defined above, is a martingale.

Definition II.19. Let $\hat{\mathcal{H}}$ denote the space of functions $\psi(\omega, z, z') \equiv \psi_{z,z'}(\omega)$ on $\Omega \times \mathbb{T} \times \mathbb{T}$ which satisfy the following conditions:

(â): ψ is a measurable process and for all $z, z' \in \mathbb{T}$, $\psi_{z,z'}$ is $\mathcal{F}_{z \vee z'}$-measurable, and (b̂): $\iint_{\mathbb{T}^2} I(z \curlywedge z')\mathbb{E}\{\psi^2_{z,z'}\}dzdz' < \infty.$

We equip $\hat{\mathcal{H}}$ with the scalar product

$$\langle \psi, \varphi \rangle := \iint_{\mathbb{T}^2} I(z \curlywedge z')\mathbb{E}\{\psi_{z,z'}\varphi_{z,z'}\}dzdz$$

Note 2.2.7. $\hat{\mathcal{H}}$ is a Hilbert space.

The simple functions $\mathbb{S}$ are dense in $\hat{\mathcal{H}}$. And since the mapping $\iint \psi dWdW$ from $\hat{\mathcal{H}}$ to $\mathcal{M}^2(\mathbb{T})$ is norm-preserving for simple functions, the integral can be extended by continuity of the linear map, to all functions in $\hat{\mathcal{H}}$.

We will also require the notion of "mixed area integrals" as defined in [27].

Definition II.20. For function χ_{ijkl} in $\mathbb{S}$ given by (2.5), define the integrals $\iint \chi dMd\zeta$ and $\iint \chi d\zeta dM$ by

$$\int_{\mathbf{R}_z \times \mathbf{R}_z} \chi_{ijkl}(\zeta, \zeta')dM_\zeta d\zeta' := \sum_{ijkl} \alpha_{ijkl} M(\Delta_{i,j} \cap \mathbf{R}_z)\lambda(\Delta_{k,l} \cap \mathbf{R}_z) \quad z \in \mathbb{T}, \tag{2.7}$$

and

$$\int_{\mathbf{R}_z \times \mathbf{R}_z} \chi_{ijkl}(\zeta, \zeta')d\zeta dM_{\zeta'} := \sum_{ijkl} \alpha_{ijkl} \lambda(\Delta_{i,j} \cap \mathbf{R}_z)M(\Delta_{k,l} \cap \mathbf{R}_z) \quad z \in \mathbb{T}, \tag{2.8}$$

where λ is Lebesgue measure on $\mathbb{R}^2$.

Note 2.2.8. $\int_{\mathbf{R}_z \times \mathbf{R}_z} \chi_{ijkl}(\zeta, \zeta')dM_\zeta d\zeta'$ and $\int_{\mathbf{R}_z \times \mathbf{R}_z} \chi_{ijkl}(\zeta, \zeta')d\zeta dM_{\zeta'}$ as defined above, are a 1-martingale and 2-martingale respectively.

Again, since $\mathbb{S}$ is dense in $\hat{\mathcal{H}}$, the mixed area integrals $\iint \psi dzdW$ and $\iint \psi dWdz$, (where W is a standard Brownian sheet) can be extended to all functions in $\hat{\mathcal{H}}$. Then for arbitrary $\psi \in \hat{\mathcal{H}}$, the stochastic integrals

$$X_z := \iint_{R_z \times R_z} \psi_{\zeta,\zeta'} dW_\zeta dW_{\zeta'}, \; Y^1_z := \iint_{R_z \times R_z} \psi_{\zeta,\zeta'} d\zeta dW_{\zeta'}, \; Y^2_z := \iint_{R_z \times R_z} \psi_{\zeta,\zeta'} dW_\zeta d\zeta'$$

are well-defined (as in [29]) for all $z \in \mathbb{T}$ and X, Y^1, Y^2 are respectively a martingale, an (adapted) 1-martingale and an (adapted) 2-martingale, and in all the cases the sample-continuous versions can be chosen.

2.3 Fractional Calculus

Fractional Brownian motion and fractional Brownian fields can be represented as integrals of deterministic kernels with respect to standard Brownian motion and Brownian sheet respectively (see [20], [23], [14], [25]). We will introduce operators known as fractional integrals and derivatives that can be used to derive desired kernels for integral transforms of fractional Gaussian processes. These operators and the kernels they produce will play a large role in the remainder of the dissertation.

To motivate the use of fractional integrals and operators, let us introduce an n-fold iterated integral and show that it can be expressed as a single integral described by a parameter n.

Let a, b be real numbers with $a < b$. For a function f on $[a, b]$, the multiple n-fold integral

$$\int_a^t \left\{ \int_a^t ... \left\{ \int_a^t \left\{ \int_a^t f(x)dx \right\} dt \right\} ....dt \right\} dt = \frac{1}{(n-1)!} \int_a^t f(y)(t-y)^{n-1} dy$$

$$= \frac{1}{\Gamma(n)} \int_a^t \frac{\varphi(t)}{(x-t)^{1-n}} dt, \tag{2.9}$$

where $t_n \in [a, b]$ and $n \geq 1$.

Clearly, the right hand side of (2.9) makes sense even when n is not an integer, which motivates the definition of the fractional integral of order $\alpha \in \mathbb{R}_+$.

Definition II.21. Let $\varphi(x) \in L_1(a, b)$ and let $\alpha > 0$. The integrals

$$(2.10)\qquad (I^{\alpha}_{a+}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt, \quad x < b,$$

$$(2.11)\qquad (I^{\alpha}_{b-}\varphi)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)}{(t-x)^{1-\alpha}}dt, \quad x > a,$$

are called left-handed and right-handed Riemann-Liouville fractional integrals of order α.

Definition II.22. For functions $f(x)$ on interval $[a, b]$, the expressions

$$(2.12)\qquad (\mathcal{D}^{\alpha}_{a+}f)(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha}}dt,$$

$$(2.13)\qquad (\mathcal{D}^{\alpha}_{b-}f)(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(t-x)^{\alpha}}dt$$

where $\alpha > 0$ are called the left-handed and the right-handed fractional Riemann-Liouville derivatives of order α, provided the the right hand side exists.

For $\alpha \in (-1, 0)$ we define

$$(I^{\alpha}_{a+}\varphi)(x) := (\mathcal{D}^{-\alpha}_{a+}\varphi)(x) \quad \textit{and}$$

$$(I^{\alpha}_{b-}\varphi)(x) := (\mathcal{D}^{-\alpha}_{b-}\varphi)(x).$$

Note that when $\alpha = 0$ we have the identity operators $I^0_{a+}\varphi = \varphi$ and $I^0_{b-}\varphi = \varphi$.

One should be aware that it is not always clear whether such integrals and derivatives exist. In many cases, a sufficient condition for fractional operators to be well defined and for manipulation of these operators to be valid is that our function being operated on is in the class denoted $I^{\alpha}_{a+}(L_p)$, defined in the following way:

Definition II.23. For $\alpha > 0$, $I^{\alpha}_{a+}(L_p)$ and $I^{\alpha}_{b-}(L_q)$ are defined as the spaces of functions $f(x)$ and $g(x)$ which can be represented as the left-handed and right-handed fractional integrals, respectively, of order α of summable functions φ, ψ,

$$f = I^{\alpha}_{a+}\varphi,\ \varphi \in L_p(a,b),\ 1 \le p < \infty \textit{ and} \tag{2.14}$$

$$g = I^{\alpha}_{b-}\psi,\ \psi \in L_q(a,b),\ 1 \le q < \infty. \tag{2.15}$$

Definition II.24. Let $\mathcal{X}$ be a finite interval in $\mathbb{R}$. The function $f : \mathcal{X} \to \mathbb{R}$ is said to satisfy the **Hölder condition of order** λ on $\mathcal{X}$ if there exists a constant $C > 0$, such that

$$|f(x_1) - f(x_2)| \le C|x_1 - x_2|^{\lambda} \tag{2.16}$$

for all $x_1, x_2 \in \mathcal{X}$. We denote by $H^{\lambda}(\mathcal{X})$, the space of functions satisfying (2.16). More generally, for a d-dimensional rectangle $\mathcal{X}_d$, a function $f : \mathcal{X}_d \to \mathbb{R}$, is said to satisfy the Hölder condition of order $(\lambda_1, ..., \lambda_d)$ (where $\lambda_i \in (0,1)\ for\ i \in \{1, .., d\}$), if there exists constant $C > 0$ such that

$$|f(x) - f(y)| \le C(|x_1 - y_1|^{\lambda_1} + \cdots + |x_d - y_d|^{\lambda_d}), \tag{2.17}$$

for all $x, y \in \mathcal{X}_d$. We denote by $H^{\lambda_1,...,\lambda_d}(\mathcal{X})$, the space of functions satisfying (2.17).

The following theorem will be useful for us later in this chapter:

Theorem II.25. *[24] Let* $f(x) = (x-a)^{-\mu} g(x)$, *where* $g(x) \in H^{\lambda}([a,b])$, $-\infty < a < b < \infty$, $\lambda > \alpha$, $-\alpha < \mu < 1$. *Then,* $f(x) \in I^{\alpha}_{a+}(L_p)$ *if* $\mu + \alpha < \frac{1}{p}$ *for* $1 \le p < \infty$.

Theorem II.26. ***Semigroup Property*** *[24] For real-valued* α, β *the relation*

$$I^{\alpha}_{a+} I^{\beta}_{a+}\varphi = I^{\alpha+\beta}_{a+}\varphi \tag{2.18}$$

holds in each of the following cases:

i) $\beta \geq 0,\ \alpha+\beta \geq 0,\ \varphi \in L_1(a,b)$*;*

ii) $\beta \leq 0,\ \alpha \geq 0,\ \varphi \in I_{a+}^{-\beta}(L_1)$*:*

iii) $\alpha \leq 0,\ \alpha+\beta \leq 0,\ \varphi \in I_{a+}^{-\alpha-\beta}(L_1)$

Theorem II.27. ***Fractional Integration by Parts*** *Let* $\varphi \in L_p(a,b)$*,* $\psi \in L_q(a,b)$*, then*

$$\int_a^b \varphi(x)(I_{a+}^{\alpha}\psi)(x)dx = \int_a^b \psi(x)(I_{b-}^{\alpha}\varphi)(x)dx \tag{2.19}$$

for any $p, q \geq 1$ *such that* $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$*.*

Corollary II.28. *The formula*

$$\int_a^b f(x)(\mathcal{D}_{a+}^{\alpha}g)(x)dx = \int_a^b g(x)(\mathcal{D}_{b-}^{\alpha}f)(x)dx, \quad 0<\alpha<1, \tag{2.20}$$

is valid under the assumption that $f \in I_{b-}^{\alpha}(L_p),\ g \in I_{a+}^{\alpha}(L_q),\ \frac{1}{p}+\frac{1}{q} \leq 1+\alpha$*.*

Corollary II.29. *[24] A sufficient condition for* $f(x)$ *and* $g(x)$ *to satisfy* (2.20) *is that the following conditions hold:*

1. $f, g \in C([a,b])$*, and*

2. $(\mathcal{D}_{a+}^{\alpha}g)(x), (\mathcal{D}_{b-}^{\alpha}f)(x)$ *exist at every point* $x \in [a,b]$ *and are continuous.*

2.4 Fractional Brownian motion and fractional Brownian sheet

In this chapter we define fractional Brownian motion and describe useful relationships between fractional Brownian motion and standard Brownian motion.

Definition II.30. Let $(\Omega, \mathcal{F}, P)$ be a complele probability space and $\{\mathcal{F}_t, t \in [0,T]\}$ be a flitration of sub σ-fields of $\mathcal{F}$. **Fractional Brownian Motion** with Hurst parameter $H \in (0,1)$ is a continuous Gaussian process $(W_t^H,\ t \in \mathbb{R}_+)$ such that

(i) $W_0^H = 0 \quad$ a.s.

(ii) $E(W_t^H) \equiv 0$

(iii) $R_H(s,t) := E(W_s^H W_t^H) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}), \forall s,t \in \mathbb{R}_+$

Note that when $H = \frac{1}{2}$, W_t^H is a standard Brownian motion which we denote $W = \{W_t, t \in \mathbb{R}_+\}$. For the standard Wiener process, the covariance function $R(s,t)$ is given by

$$R(s,t) = E(W_s W_t) = \min\{s,t\}, \forall s,t \in \mathbb{R}_+$$

In the case $H > \frac{1}{2}$, the increments $(W_t^H - W_s^H)$ and $(W_{t+h}^H - W_t^H)$ are positively correlated, for all $t, h, s > 0$ and $t > s$. We should also note that such a process with $H > \frac{1}{2}$ will exhibit long range dependence, i.e.

$$\sum_{i=1}^{\infty} E[W_1^H (W_{i+1}^H - W_i^H)] = \infty.$$

When $H \neq \frac{1}{2}$, $\{W_t^H, t \in \mathbb{R}_+\}$ is not a semimartingale. As a result, standard techniques of stochastic calculus and stochastic integration cannot be appied directly to fractional Brownian motion. In order to circumvent problems with the lack of martingale structure, one uses a kernel, $K_H(t,s)$ to represent fractional Brownian motion as a stochastic integral with respect to a standard Wiener process, The kernel is given by

$$K_H(t,s) = C_H\left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\right)$$

where

$$C_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}. \tag{2.21}$$

Namely, it has been shown [20] that the fractional Brownian motion W^H with Hurst parameter $H \in (0,1)$ can be represented as the stochastic integral of $K_H(t,s)$ with respect to the standard Wiener process $W = W^{\frac{1}{2}}$

$$W_t^H = \int_0^t K_H(t,s)dW_s.$$

It will be useful for us to note that $K_H(t,s)$ can also be represented in the following way [25], [23]:

$$K_H(s,t) = C_H^* s^{\frac{1}{2}-H}\Big(I_{t-}^{H-\frac{1}{2}} u^{H-\frac{1}{2}} 1_{[0,t]}(u)\Big)(s) \tag{2.22}$$

where

$$C_H^* = \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\right)^{\frac{1}{2}}. \tag{2.23}$$

The Wiener process W can also be constructed from the fractional Brownian motion via the integral of another deterministic kernel with respect to the fractional Brownian motion

$$W_t = \int_0^t K_H^{-1}(t,s)dW_s^H,$$

where the kernel K_H^{-1} is given by

$$K_H^{-1}(t,s) = C'_H\left(\left(\frac{t}{s}\right)^{H-\frac{1}{2}}(t-s)^{\frac{1}{2}-H} - (H-\frac{1}{2})s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{\frac{1}{2}-H}du\right) \tag{2.24}$$

where

$$C'_H = \frac{1}{\Gamma(\frac{3}{2}-H)}\left(\frac{\Gamma(2-2H)}{2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2})}\right)^{\frac{1}{2}}. \tag{2.25}$$

K_H^{-1} can also be represented in terms of fractional integrals which will often give more insight into the kernel's properties than does (2.22);

$$K_H^{-1}(s,t) = \frac{1}{C_H^*} s^{\frac{1}{2}-H}\Big(I_{t-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} 1_{[0,t]}(u)\Big)(s). \tag{2.26}$$

We now define fractional Brownian sheet (fBs), which is the multi-parameter analogue of a fractional Brownian motion. Chapter 6 will be devoted to the filtering model where observation noise is given by fractional Brownian sheet (fBs).

Definition II.31. The **fractional Brownian sheet** with Hurst parameters $\alpha, \beta \in (0,1)$ is a continuous Gaussian random field $\left(B_z^{\alpha,\beta}, z \in \mathbb{R}_+^2\right)$ with parameters $\alpha, \beta \in (0,1)$ such that

i) $B_z^{\alpha,\beta} = 0$ a.s. for $z = (0,0)$,

ii)$E\left(B_z^{\alpha,\beta}\right) = 0 \quad \forall z \in \mathbb{R}_+^2$,

iii)$E\left(B_z^{\alpha,\beta} B_{z'}^{\alpha,\beta}\right) = R_\alpha(z_1, z_1') R_\beta(z_2, z_2') = \frac{1}{2}\left(z_1^{2\alpha} + (z_1')^{2\alpha} - |z_1 - z_1'|^{2\alpha}\right) \frac{1}{2}\left(z_2^{2\beta} + (z_2')^{2\beta} - |z_2 - z_2'|^{2\beta}\right)$.

We note that fBs can be obtained from a Brownian sheet via the integral transform

$$B_z^{\alpha,\beta} = \int_{R_z} K_\alpha(z_1, \zeta_1) K_\beta(z_2, \zeta_2) dW_\zeta \tag{2.27}$$

where $z, \zeta \in \mathbb{T}$ and where W_ζ is the Brownian sheet. On the other hand, if we let $K_{\alpha,\beta}^{-1}(z;\zeta) = K_\alpha^{-1}(z_1, \zeta_1) K_\beta^{-1}(z_2; \zeta_2)$, where $K_\alpha^{-1}(s,t)$ is the kernel defined in (2.24). Then we get the relationship

$$W_z = \int_{R_z} K_{\alpha,\beta}^{-1}(z;\zeta) dB_\zeta^{\alpha,\beta}.$$

2.5 Nonlinear filtering theory

The classical stochastic nonlinear filtering problem is to obtain an optimal estimate of an unobserved signal process $X_t = \left\{X(t,\omega), t \in [0,T], \omega \in \Omega\right\}$ from an observation process which is a function of the signal observed in the presence of noise.

The process we observe is generally given by a function h of the signal function, distorted by a noise process.

The classical non-linear problem is modelled by the observation process

$$Y_t := \int_0^t h(X_s)ds + \widetilde{W}_t \tag{2.28}$$

where $\widetilde{W_t}$ is a Brownian motion independent of the signal process X_t and h is a measurable function satisfying

$$\int_0^T E|h(X_t)|^2 dt < \infty. \tag{2.29}$$

The objective is to estimate the signal X_t in a manner that will minimize mean squared error loss. As conditional expectation minimizes mean squared error loss, the estimate known as the "optimal filter" of our observation is given by the conditional expectation of X_t, given the sigma field generated by observation Y up to time t, i.e. given $\mathcal{F}_t^{\mathcal{Y}} = \sigma\{Y_s : 0 \leq s \leq t\}$.

Define the random measure

$$\pi_t(dy, \omega) := P(X_t \in dy|\mathcal{F}_t^{\mathcal{Y}})(\omega). \tag{2.30}$$

For any measure μ, define $\mu(f) = \int_{\mathcal{R}} f(x)\mu(dx)$. Then for a sufficiently wide class of funtions, f, we study the evolution of

$$\pi_t(f) = E\big[f(X_t)|\mathcal{F}_t^{\mathcal{Y}}\big] \tag{2.31}$$

Section 2.5 gives stochastic differential equations describing the evolution of the optimal filter in the classical nonlinear filtering problem described above. In Section 2.5 we give equations for the optimal filter in the nonlinear filtering problem where the noise term is given by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ so that the noise term exhibits long range dependence.

Kushner, Fujisaki-Kallianpur-Kunita equation and Zakai equation

In this section the signal process is assumed to be a Markov process. When $\big(X_t\big)_{t\in[0,T]}$ is a Markov process, the normalized filter $\pi_t(f)$ can be written as the solution to a stochastic differential equation involving the infinitesimal generator of X_t.

Definition II.32. Let $(X_t)_{t\in[0,T]}$ be a Markov process. The **Infinitesimal Generator** $\mathcal{L}(s)$ of $(X_t)_{t\in[0,T]}$ is defined by:

$$\Big(\mathcal{L}(s)f\Big)(x) = \lim_{t\downarrow 0} \frac{E[f(X_{s+t})|X_s = x] - f(x)}{t}, \quad x \in \mathcal{R} \quad s \in [0,T) \tag{2.32}$$

Let $\mathcal{D}(\mathcal{L})$ denote the set of all functions f for which the limit exists for all $x \in \mathcal{R}$, $0 < s < T$. Similarly, let $\mathcal{D}(\mathcal{L})(x)$ denote the set of functions $f : \mathcal{R} \to \mathcal{R}$ such that, for given $x \in \mathcal{R}$, the limit exits.

It is often assumed that the signal process is a time homogeneous diffusion satisfying:

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.33}$$

where $\sigma(t, x)$ and $b(t, x)$ are called the diffusion and the drift coefficients respectively, and W_t is a Brownian motion independent of the noise process $\widetilde{W}_t$.

When X_t is given by (2.33), the infinitesimal generator is given by:

$$\mathcal{L}(s)f(x) = b(s, x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(s, x)\frac{\partial^2 f}{\partial x^2} \quad \forall f \in \mathcal{C}_b^2(\mathcal{R}). \tag{2.34}$$

Consider filtering model (2.28), where X is a Markov process with the infinitesimal generator $\mathcal{L}$. The following results for the optimal filter hold.

Theorem II.33. *[11], [17] For $f \in \mathcal{D}(\mathcal{L})$ such that $\int_0^T E[f(X_t)h(X_t)]^2 dt < \infty$, the random measure π satisfies:*

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(\mathcal{L}(s)f)ds + \int_0^t [\pi_s(fh) - \pi_s(f)\pi_s(h)]d\nu_s \tag{2.35}$$

where ν_t is the innovation process given by:

$$\nu_t = Y_t - \int_0^t E[h(X_s)|\mathcal{F}_s^{\mathcal{Y}}]ds. \tag{2.36}$$

It is well known that $\exists$ stochastic process $(\rho_t)_{t\in[0,T]}$, taking values in the space of finite positive measures on $\mathcal{R}$, such that $\forall f \in \mathcal{C}_b^2(\mathcal{R})$

$$\pi_t(f) = \frac{\rho_t(f)}{\rho_t(1)}. \tag{2.37}$$

In light of (2.37), $\pi_t(f)$ is called the normalized optimal filter for $f(X_t)$ while $\rho_t(f)$ is the unnormalized optimal filter.

We can define $\rho_t(f)$ as

$$\rho_t(f) := \pi_t(f) \exp\left\{ \int_0^t \pi_s(h)dY_s - \frac{1}{2}\int_0^t |\pi_s(h)|^2 ds \right\} \tag{2.38}$$

Theorem II.34. ***(Zakai Equation for the Unnormalized Optimal Filter)*** *The process $\rho_t(f)$ defined by (2.37) satisfies the following stochastic differential equation:*

$$\rho_t(f) = \pi_0(f) + \int_0^t \rho_s(\mathcal{L}_s f)ds + \int_0^t \rho_s(hf)dY_s. \tag{2.39}$$

The Zakai equation for the unnormalized optimal filter (2.39) is a stochastic differential equation which is linear in $\rho_t(\cdot)$. On the other hand the measure-valued stochastic differential equation (2.35) is not linear in the $\pi_s(\cdot)$. It is also worth noting that the innovation process (2.36) is a semi-martingale, which allows one to define the stochastic Itô integral $\int_0^t [\pi_s(fh) - \pi_s(f)\pi_s(h)]d\nu_s$ used in Theorem II.33.

Filtering with fractional Brownian motion

The classical filtering problem can be extended to a wider class of observation processes. One can change the noise term to exhibit long range dependence so that the noise is no longer a martingale with independent increments. It is also possible to extend the model to examine spatial filtering where both the signal and noise are multi-parameter processes, resulting in a multiparameter observation process. In the case of multiparameter filtering, it is again interesting to look first at the case where the noise term has properties analogous to the independent increments of Brownian motion. Next, in the multi-parameter setting the filtering model can be further extended to include a noise term exhibiting long memory. The goal of this chapter is to give filtering equations similar to (2.39) and (2.37) for the optimal filter in the multiparameter setting with long-memory observation noise.

In this section we present filtering equations for the case when observation noise is given by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Here Y is a one parameter observation process with long range dependence in the noise. In this setting the model is given by:

$$Y_t = \int_0^t h(X_s)ds + B_t^H, \quad t \in [0,T] \tag{2.40}$$

where $h \in C([0,T])$, B_t^H is a fractional Brownian motion, independent of the signal, with $H > \frac{1}{2}$ and the signal is given by (2.33).

Proposition II.35. *[14] Let B^H be fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. For continuous function $C : [0,T] \to \mathbb{R}$, define the function $k_C^t = (k_C^t(s), 0 < s < t)$ as follows:*

$$k_C^t(s) \equiv -\rho_H^{-1} s^{\frac{1}{2}-H} \frac{d}{ds}\left(\int_s^t \frac{d}{dw}\left\{ \int_0^w z^{\frac{1}{2}-H}(w-z)^{\frac{1}{2}-H} C(z)dz \right\} w^{2H-1}(w-s)^{\frac{1}{2}-H} dw \right)$$

where $\rho_H = \Gamma^2(\frac{3}{2}-H)\Gamma(2H+1)sin(\pi H)$. *Then the function* k_C^t *satisfies the equation*

$$H(2H-1)\int_0^t k_C^t(s)|s-r|^{2H-2}ds = C(r); \quad 0 < r < t.$$

For $0 \le t \le T$, *define*

$$N_t^C \equiv \int_0^t k_C^t(s)dB_s^H \tag{2.41}$$

and

$$\langle N^C\rangle_t \equiv \int_0^t C(s)k_C^t(s)ds \tag{2.42}$$

Then the process $N^C = (N_t^C, 0 \le t \le T)$ *is a Gaussian martingale with variance function given by* $\langle N^C\rangle = (\langle N^C\rangle_t, t \in [0,T])$.

For the particular case when $C \equiv 1$, the function k_C^t is given by

$$k_*^t(s) = \kappa_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \quad 0 < s < t \tag{2.43}$$

where

$$\kappa_H = 2H\Gamma(\frac{3}{2}-H)\Gamma(H+\frac{1}{2}).$$

The corresponding Gaussian martingale, called the fundamental martingale associated with W^H, takes the form

$$N_t^* = \kappa_H^{-1}\int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}dW_s^H = \int_0^t k_*^t(s)dB_s^H \quad 0 \le t \le T. \tag{2.44}$$

The variance function corresponding to the fundamental martingale is given by

$$(2.45)\qquad \langle N^*\rangle_t = \kappa_H^{-1}\int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}ds = \int_0^t k_*^t ds = \lambda_H^{-1}t^{2-2H}, \quad \forall t\in[0,T]$$

where

$$\lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}.$$

Note that

$$(2.46)\qquad Cov(N_t^C, N_t^*) = \langle N^C, N^*\rangle_t = \int_0^t k_*^t(s)C(s)ds,$$

where $\langle N^C, N^*\rangle$ is a cross-quadratic variation process between N^C and N^*. The representations (2.45) and (2.46) make it clear that the measures induced by $\langle N^C, N^*\rangle_t$ and $\langle N^*\rangle$ are absolutely continuous with respect to Lebesgue measure.

Proposition II.36. *[14] Let N^* and $\langle N^*\rangle$ be as defined in* (2.44) *and* (2.45)*. Then for continuous function C on $[0,T]$, the martingale N^C defined by* (2.41) *is continuous. Moreover, there exists a measurable function $q^C = \big(q_t^C, t\in[0,T]\big)$ such that*

$$(2.47)\qquad \int_0^T (q_t^C)^2 d\langle N^*\rangle_t < \infty,$$

and for all $t\in[0,T]$ the following representations hold

$$(2.48)\qquad N_t^C = \int_0^t q_s^C dN_s^* \quad \textit{and} \quad \langle N^C\rangle_t = \int_0^t (q_s^C)^2 d\langle N^*\rangle_s.$$

Furthermore, the above representations hold for q^C given by the Radon-Nikodym derivative

$$(2.49)\qquad q_t^C = \frac{d\langle N^C, N^*\rangle_t}{d\langle N^*\rangle_t}.$$

Note that since the measure induced by $\langle N^C, N^*\rangle_t$ is absolutely continuous with respect to Lebesgue measure and (from (2.45)) Lebesgue measure is absolutely continuous with respect to the measure induced by $\langle N^*\rangle_t$, then

$$q_t^C = \frac{d\langle N^C, N^*\rangle_t}{d\langle N^*\rangle_t} = \left(\frac{d\langle N^C, N^*\rangle_t}{dt}\right).\left(\frac{dt}{d\langle N^*\rangle_t}\right) \tag{2.50}$$

Similarly, since the measure, induced by $\langle N^*\rangle_t$ and Lebesgue measures are equivalent measures (each is absolutely continuous with respect to the other), then

$$\left(\frac{dt}{d\langle N^*\rangle_t}\right) = \left(\frac{d\langle N^*\rangle_t}{dt}\right)^{-1} \tag{2.51}$$

As a result we can express q^C in the following way

$$q_t^C = \left(\frac{d}{dt}\langle N^C, N^*\rangle_t\right).\left(\frac{d}{dt}\langle N^*\rangle_t\right)^{-1} \tag{2.52}$$

From (2.43), (2.46), (2.45) and (2.52) we can represent q^C by

$$q_t^C = \frac{\Gamma(2-2H)}{\Gamma(\frac{3}{2}-H)} t^{2H-1}\left(\mathcal{D}_{0+}^{H-\frac{1}{2}} u^{\frac{1}{2}-H} C(u)\right)(t). \tag{2.53}$$

For the purposes of the filtering problem described above, one needs to consider the case when $h(X_t(\omega))$ plays the role of $C(t)$. We will write the corresponding N_t^C as:

$$N_t(\omega) = \int_0^t k_{h(X.(\omega))}^t(s) dB_s^H(\omega), \tag{2.54}$$

$$\langle N, N^*\rangle_t(\omega) := \int_0^t k_*^t(s) h(X_s(\omega)) ds \tag{2.55}$$

and

$$q_t(X)(\omega) := \frac{d\langle N, N^*\rangle(\omega)}{d\langle N^*\rangle_t} \tag{2.56}$$

for all $t \in [0, T]$.

Bayes' formula for fractional Brownian motion Since X and B^H are independent, we consider the signal process X_t as being defined on the probability space $(\Omega_X, \mathcal{F}_X, P_X)$ and noise, B_t^H as being defined on the probability space $(\Omega_B, \mathcal{F}_B, P_B)$. Consider $(\Omega, \mathcal{F}, P) = (\Omega_X \times \Omega_B, \mathcal{F}_X \times \mathcal{F}_B, P_X \times P_B)$. Then

$$\pi_t(f)(\omega) = E[f(X_t)|\mathcal{F}_t^{\mathcal{Y}}](\omega) = \frac{\int_{\Omega_X} f(X_t(u)) \exp\left\{\alpha_u(t)(\omega)\right\} P_X(du)}{\int_{\Omega_X} \exp\left\{\alpha_u(t)(\omega) \exp\right\} P_X(du)} \quad a.s. \tag{2.57}$$

where $\alpha_u(t)(\omega) := \int_0^t k_{h(X.(u))}^t(s) dY_s(\omega) - \frac{1}{2}\int_0^t k_{h(X.(u))}^t(s) h(X_s(u)) ds.$

Similarly,

$$\pi_t(f)(\omega) = E[f(X_t)|\mathcal{F}_t^{\mathcal{Y}}](\omega) = \frac{E_{\widetilde{P}}\left[f(X_t)\frac{dP}{d\widetilde{P}}|\mathcal{F}_t^{\mathcal{Y}}\right](\omega)}{E_{\widetilde{P}}\left[\frac{dP}{d\widetilde{P}}|\mathcal{F}_t^{\mathcal{Y}}\right](\omega)}, \tag{2.58}$$

where

$$\frac{dP}{d\widetilde{P}}(\omega) := \exp\left\{N_t(\omega) + \frac{1}{2}\int_0^T k_{h(X(\omega))}^T(s) h(X_s(\omega)) ds\right\} \tag{2.59}$$

and $E_{\widetilde{P}}\left[\,\cdot\,|\mathcal{F}_t^Y\right]$ is the conditional expectation under the new measure $\widetilde{P}$. Note that under $\widetilde{P}$, Y is a standard fractional Brownian motion independent of X. Furthermore, the distribution of X is the same under P and $\widetilde{P}$.

Theorem II.37. *[14] Suppose X is a Markov process and the observation model satisfies* (2.40) *where $f \in \mathcal{C}_b(\mathbb{R})$, where W^H is a fractional Brownian motion, independent of the signal, with $H > \frac{1}{2}$, then*

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(\mathcal{L}(s)f) ds + \int_0^t [\pi_s(fq) - \pi_s(f)\pi_s(q)] d\nu_s \tag{2.60}$$

where $\nu_t = \int_0^t [q_s(X) - \pi_s(q)] d\langle N^*\rangle_s$ and $\pi_s(fq)(\omega) := E[f(X_s)q_s(X)|\mathcal{F}_t^Y](\omega)$

Note that (2.60) is not a measure-valued stochastic partial differential equation, since $q_s(X)$ is not a function of X_s alone, but a function of $\{X_u, 0 \le u \le s\}$ i.e. the entire past of the process X up to time s.

We can also give a fractional analogue of the Zakai equation in the case where the observation noise is a fractional Brownian motion (rather than an ordinary Brownian motion).

Theorem II.38. ***Fractional Zakai Equation****[14]*

$$\rho_t(f) = \pi_0(f) + \int_0^t \rho_s(\mathcal{L}_s f)ds + \int_0^t \rho_s(fq)dZ_s^* \tag{2.61}$$

where $Z_t^* := \int_0^t k_*^t(s)dY_s, \quad t \in [0,T]$ *is a semi-martingale with the decomposition:*

$$Z_t^* = N_t^* + \langle N, N^* \rangle_t \quad t \in [0,T] \tag{2.62}$$

and $\rho_t(fq) = E_{\widetilde{P}}\Big[f(X_t)q_t(X)\frac{dP}{d\widetilde{P}}|\mathcal{F}_t^Y\Big](\omega)$

2.6 Fractional-spatial Bayes' formula with persistent fBs noise.

In this section we derive a Bayes type formula for the multiparameter nonlinear filtering problem with fractional Brownian sheet observation noise with Hurst parameters $\alpha, \beta \in (\frac{1}{2}, 1)$.

We consider the following observation model:

$$Y_z = \int_{R_z} g(X_\zeta)d\zeta + B_z^{\alpha,\beta}, \quad z \in \mathbb{T} \equiv [0,T_1] \times [0,T_2] \subset \mathbb{R}_+^2 \tag{2.63}$$

where $R_z \equiv (0, z_1] \times (0, z_2]$, the signal process X_z and observation process Y_z are measurable, $\mathcal{F}_z$-adapted random fields defined on the complete, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_z), P)$, where the filtration $(\mathcal{F}_z)$ satisfies (F1)-(F4), given in section 2.2. The observation noise, $B^{\alpha,\beta}$ is a fractional Brownian sheet on $(\Omega, \mathcal{F}, (\mathcal{F}_z), P)$ with Hurst parameters $\alpha, \beta \in (\frac{1}{2}, 1)$ and $B^{\alpha,\beta}$ is assumed to be independent of the signal process. We also assume that the function g satisfies the following technical assumptions:

(A_1) The function $g : \mathbb{R} \to \mathbb{R}$ is Hölder continuous of order λ on any finite interval in $\mathbb{R}$, where $2\max\{\alpha, \beta\} - 1 < \lambda$;

(A_2) The function $g : \mathbb{R} \to \mathbb{R}$ satisfies the integrability condition:

$$\int_{R_{z_0}} \left(\zeta_1^{\alpha-\frac{1}{2}}\zeta_2^{\beta-\frac{1}{2}}\right)^2 \mathbb{E}\left[\left(\mathcal{D}_{0+}^{\beta-\frac{1}{2}} \otimes \mathcal{D}_{0+}^{\alpha-\frac{1}{2}} g^*\right)(\zeta_1, \zeta_2)\right]^2 d\zeta_1 d\zeta_2 < \infty, \tag{2.64}$$

where

$$g^*(z_1, z_2) = z_1^{\frac{1}{2}-\alpha} z_2^{\frac{1}{2}-\beta} g(X_{z_1,z_2}), \tag{2.65}$$

and "$\otimes$" denotes the tensor product of operators. In particular, given a function $f : \mathbb{R}^2 \to \mathbb{R}$, a pair of operators $\mathcal{L}_1,\ \mathcal{L}_2$ defined on functions $g : \mathbb{R} \to \mathbb{R}$,

$$\big(\mathcal{L}_1 \otimes \mathcal{L}_2 f\big)(x_1, x_2) := \mathcal{L}_1(\mathcal{L}_2 f(\cdot, x_2))(x_1).$$

Lemma II.39. *For fixed $\alpha, \beta \in (\frac{1}{2}, 1)$, let $h : \mathbb{T} \to \mathbb{R}$ be a Hölder continuous function of order (λ_1, λ_2), where $\lambda_1 > \alpha - \frac{1}{2}$ and $\lambda_2 > \beta - \frac{1}{2}$. Then there exists a function $\delta_h : \mathbb{T} \to \mathbb{R}$ such that $\delta_h \in L^2(\mathbb{T})$ while satisfying both*

$$\int_{R_z} h(\zeta) d\zeta = \int_{R_z} K_{\alpha,\beta}(z;\zeta)\delta_h(\zeta) d\zeta \quad \forall z \in \mathbb{T} \tag{2.66}$$

and

$$\int_{R_z} \delta_h(\zeta) d\zeta = \int_{R_z} K^{-1}_{\alpha,\beta}(z;\zeta) h(\zeta) d\zeta \quad \forall z \in \mathbb{T}. \tag{2.67}$$

Furthermore, the function

$$\delta_h(z_1, z_2) = (C^*_\alpha C^*_\beta)^{-1} z_1^{\alpha-\frac{1}{2}} z_2^{\beta-\frac{1}{2}} \left(\mathcal{D}_{0+}^{\alpha-\frac{1}{2}} \otimes \mathcal{D}_{0+}^{\beta-\frac{1}{2}} h^*\right)(z_1, z_2), \tag{2.68}$$

satisfies (2.66) *where $h^*(z_1, z_2) = z_1^{\frac{1}{2}-\alpha} z_2^{\frac{1}{2}-\beta} h(z_1, z_2)$.*

Proof By using the polarization technique [19], it is sufficient to consider functions h of the form $h(z_1, z_2) = h_1(z_1)h_2(z_2)$, where $h_1 \in H^{\lambda_1}([0, T_1]), h_2 \in H^{\lambda_2}([0, T_2])$. For notational simplicity set $k = \beta - \frac{1}{2}$ and $d = \alpha - \frac{1}{2}$ so that $k, d \in (0, \frac{1}{2})$. Let δ_h be as given in (2.68). Then using representations (2.22),

$$\int_{R_z} K_{\alpha,\beta}(z;\zeta)\delta_H(\zeta)d\zeta$$

$$= \int_{u=0}^{z_1}\int_{v=0}^{z_2} u^{-d}\left(I^d_{z_1-}t^d 1_{[0,z_1]}(t)\right)(u)v^{-k}\left(I^k_{z_2-}w^k 1_{[0,z_2]}(w)\right)(v)u^d v^k$$

$$\times\left(D^d_{0+}\otimes D^k_{0+}h^*(z,w)\right)(u,v)dvdu$$

$$= \int_{u=0}^{z_1}\left(I^d_{z_1-}t^d 1_{[0,z_1]}(t)\right)(u)\left(D^d_{0+}t^{-d}h_1(t)\right)(u)du$$

$$\times\int_{v=0}^{z_2}\left(I^k_{z_2-}w^k 1_{[0,z_2]}(w)\right)(v)\left(D^k_{0+}w^{-k}h_2(w)\right)(v)dv$$

$$= \int_{u=0}^{z_1} u^{-d}h_1(u)\left(D^d_{z_1-}\left(I^d_{z_1-}(t')^d 1_{[0,z_1]}(t')\right)(t)\right)(u)du$$

$$\times\int_{v=0}^{z_2} v^{-k}h_2(v)\left(D^k_{z_2-}\left(I^k_{z_2-}(w')^k 1_{[0,z_2]}(w')\right)(w)\right)(v)dv \tag{2.69}$$

$$= \int_{u=0}^{z_1}\int_{v=0}^{z_2} h_1(u)h_2(v)dvdu = \int_{R_z} h(\zeta)d\zeta. \tag{2.70}$$

Equality (2.69) follows from an application of fractional differentiation by parts (see corollary of (2.19) in [24]) since $u^{-d}h_1(u) \in I^d_{0+}(L_1[0,t])$ and $v^{-k}h_2(v) \in I^k_{0+}(L_1[0,s])$ by theorem II.25. Similarly, one can show (2.67) by using representation (2.26). Next, notice that since $h_1 \in H^{\lambda_1}([0,T_1])$, where $\lambda_1 > \alpha - \frac{1}{2}$, by theorem II.25, $h_1 \in I^{\alpha-\frac{1}{2}}_{0+}(L_2([0,T_1]))$, and hence $\int_0^{\cdot} h_1(u)du \in I^{\alpha+\frac{1}{2}}_{0+}(L_2([0,T_1]))$. If we let

$$\delta_{h_1}(z_1) = (C^*_\alpha)^{-1}z_1^d\left(\mathcal{D}^d_{0+}t^{-d}h_1(t)\right)(z_1)\ \forall z_1 \in [0,T_1],$$

then it easily follows from (2.70), that

$$\int_0^{z_1} h_1^i(t)dt = \int_0^{z_1} K_\alpha(z_1,t)\delta_{h_1^i}(t)dt,\ \forall z_1 \in [0,T_1].$$

Since the integral operator $\mathfrak{K}_\alpha$ associated with the kernel K_α,

$$\big(\mathfrak{K}_\alpha f\big)(z_1) := \int_0^{z_1} K_\alpha(z_1,t)f(t)dt,\ \ f \in L^2([0,T_1]),$$

is an isomorphism from $L^2([0,T_1])$ onto $I_{0+}^{\alpha+\frac{1}{2}}(L^2([0,T_1]))$ [3], it follows that $f \in L^2([0,T_1])$ if and only if $\mathfrak{K}_\alpha f \in I_{0+}^{\alpha+\frac{1}{2}}(L^2([0,T_1]))$. Therefore $\delta_{h_1^i} \in L^2([0,T_1])$. Similarly, if $h_2 \in H^{\lambda_1}([0,T_2])$, where $\lambda_2 > \beta - \frac{1}{2}$, then $\delta_{h_2} \in L^2([0,T_2])$, where

$$\delta_{h_2}(z_2) = (C^*_\beta)^{-1} z_2^k \left(\mathcal{D}^k_{0+} t^{-k} h_1(t)\right)(z_2)\ \forall z_2 \in [0,T_2].$$

Therefore, $\delta_{(h_1 \otimes h_2)} = \delta_{h_1} \otimes \delta_{h_2} \in L^2([0,T_1]\times[0,T_2])$, and the result follows.

$\square$

Corollary II.40. *Fix* $\lambda_0 > \frac{\max(\alpha,\beta)-\frac{1}{2}}{\lambda}$. *Suppose the signal* $X = (X_z, z \in \mathbb{T})$ *has almost surely Hölder-continuous sample paths of order* (λ_0,λ_0) *and* g *satisfies condition* (A_1). *Then for almost all* $\omega \in \Omega$ *one can define function* $(\delta_z(X), z \in \mathbb{T})$ *by*

$$(2.71)\quad \delta_z(X)(\omega) := \frac{1}{C^*_\alpha C^*_\beta} z_1^{\alpha-\frac{1}{2}} z_2^{\beta-\frac{1}{2}} \big(\mathcal{D}_{0+}^{\alpha-\frac{1}{2}} \otimes \mathcal{D}_{0+}^{\beta-\frac{1}{2}} g^*_\cdot(X)(\omega)\big)(z),\quad z=(z_1,z_2)\in\mathbb{T},$$

with $g^*_\cdot(X)$ *defined by* (2.65). *Then, assuming also that* (A_2) *holds,* $\delta(X) = (\delta_z(X), z \in \mathbb{T})$ *has the following properties:*

(i) $\delta_\cdot(X)(\omega) \in L^2(\mathbb{T})$ *for almost all* $\omega \in \Omega$ *and* $\mathbb{E}\int_{\mathbb{T}}(\delta_z(X))^2 dz < \infty$;

(ii) For every rectangle $R_z = [\underline{0}, z] \subset \mathbb{T}$,

$$(2.72)\qquad \int_{R_z} K^{-1}_{\alpha,\beta}(z;\zeta) g(X_\zeta) d\zeta = \int_{R_z} \delta_\zeta(X) d\zeta \ \text{ a.s.}$$

and

$$(2.73)\qquad \int_{R_z} K_{\alpha,\beta}(z;\zeta)\delta_\zeta(X) d\zeta = \int_{R_z} g(X_\zeta) d\zeta \ \text{ a.s.}$$

Proof The result follows directly from Lemma II.39 noticing that for $\lambda_0 > \frac{\max(\alpha,\beta)-\frac{1}{2}}{\lambda}$ and g satisfying (A$_1$), $g(X_z) \in H^{\lambda_1,\lambda_2}(\mathbb{T})$. Notice that we've used the fact that for all $x, y \in \mathbb{R}_+$, and all $a, b \in (0,1)$, $(x^a + y^a)^b \leq x^{ab} + y^{ab}$.

□

For the remainder of this chapter, assume that the assumptions of Corollary II.40 are satisfied.

Define the following processes

$$W_z^Y := \int_{R_z} K_{\alpha,\beta}^{-1}(z;\zeta)dY_\zeta \ \text{ and } \ W_z^B := \int_{R_z} K_{\alpha,\beta}^{-1}(z;\zeta)dB_\zeta^{\alpha,\beta}, \quad z \in \mathbb{T}. \tag{2.74}$$

It easily follows that $W_z^Y = \int_{R_z} \delta_\zeta(X)d\zeta + W_z^B$.

Next let us define the process

$$V_z = \exp\left\{ - \int_{R_z} \delta_\zeta(X)dW_\zeta^B - \frac{1}{2}\int_{R_z} (\delta_\zeta(X))^2 d\zeta \right\} \quad z \in \mathbb{T}. \tag{2.75}$$

We can write V_z in terms of our observation process in the following manner:

$$V_z = \exp\left\{ - \int_{R_z} \delta_\zeta(X)dW_\zeta^Y + \frac{1}{2}\int_{R_z} (\delta_\zeta(X))^2 d\zeta \right\} \quad z \in \mathbb{T} \tag{2.76}$$

and hence

$$V_z = \exp\left\{ - \int_{R_z} \delta_\zeta(X)d\Big(\int_{R_\zeta} K_{\alpha,\beta}^{-1}(\zeta,\zeta')dY_{\zeta'}\Big) + \frac{1}{2}\int_{R_z} (\delta_\zeta(X))^2 d\zeta \right\} \quad z \in \mathbb{T}$$

Lemma II.41. *Let $V = (V_z, z \in \mathbb{T})$ be defined by* (2.76)*. Then* $\mathbb{E}\big(V_{(T_1,T_2)}\big) = 1$.

Proof Since $B^{\alpha,\beta}$ and X are independent it follows that W^B and X are independent. We can therefore define a standard Wiener sheet W^B on a complete probability space $(\Omega_2, \mathcal{F}_2, P_2)$, define X on a complete probability space $(\Omega_1, \mathcal{F}_1, P_1)$ and then consider the processes on a product probability space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$, with $W^B(\omega) = W^B(\omega_2)$ and $X(\omega) = X(\omega_1)$ for all $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$. Let $M(\omega_1, \omega_2) \equiv \int_{\mathbb{T}} \delta_\zeta(X(\omega_1)) dW^B_\zeta(\omega_2)$ on $(\Omega, \mathcal{F}, P)$. For fixed $\omega_1 \in \Omega_X$, $M(\omega_1, \cdot)$ is a mean zero, Gaussian random variable with variance given by $\int_{\mathbb{T}} (\delta_\zeta(X))^2(\omega_1) d\zeta$. So for almost all fixed $\omega_1 \in \Omega_1$, we have

$$\mathbb{E}_{P_2}\bigg(V_{T_1,T_2}(\omega_1, \cdot)\bigg) = \mathbb{E}_{P_2}\bigg[\exp\Big\{ - M(\omega_1, \cdot) - \frac{1}{2}\int_{\mathbb{T}} (\delta_\zeta(X))^2(\omega_1) d\zeta\Big\}\bigg] = 1,$$

and hence

$$\mathbb{E}(V_{(T_1,T_2)}) = \int_{\Omega_1 \times \Omega_2} V_{(T_1,T_2)}(\omega_1, \omega_2)(P_1 \times P_2)(d\omega_1, d\omega_2) = \int_{\Omega_1} 1\, P_1(d\omega_1) = 1. \ \square$$

Theorem II.42. *Consider observation model* (2.63)*, where the signal* $X = (X_z, z \in \mathbb{T})$ *has almost surely Hölder-continuous sample paths of order* (λ_0, λ_0)*, where* $\frac{\max(\alpha,\beta) - \frac{1}{2}}{\lambda} < \lambda_0$*, and* g *satisfies conditions* (A_1)*-*(A_2)*. Under probability measure* $\widetilde{P}$*, defined in terms of the Radon-Nikodym derivative as*

$$\frac{d\widetilde{P}}{dP} = V_{(T_1,T_2)}$$

defined on $(\Omega, \mathcal{F}, P)$*,* X *and* Y *are independent and* X *has the same distribution under* P *as under* $\widetilde{P}$*.* P *and* $\widetilde{P}$ *are also equivalent probability measures. Furthermore, under* $\widetilde{P}$*,* Y *is fractional Brownian sheet with Hurst indices* α, β*.*

Proof The proof follows directly from the multiparameter Girsanov-type theorem for the standard Wiener sheet (see e.g. Theorem 1 in [7], p. 89), Proposition II.16 and Lemma II.41.

$\square$

We will use the following lemma to prove a Bayes-type formula for the spatial filtering model (2.63).

Lemma II.43. *Let* $(\Omega, \mathcal{F})$ *be a probability space and let* P *and* $\widetilde{P}$ *be equivalent probability measures on* $\mathcal{F}$. *Denote by* $E(\cdot|\mathcal{G})$ *and* $\tilde{E}(\cdot|\mathcal{G})$ *the conditional expectations with respect to* $\mathcal{G} \subset \mathcal{F}$ *under* P *and* $\widetilde{P}$. *Then for any* X *such that* $E|X| < \infty$,

$$E(X|\mathcal{G})(\omega) = \frac{\tilde{E}\big(X\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}{\tilde{E}\big(\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)} \quad a.s.$$

Proof: Since

$$P\Bigg(\tilde{E}\bigg(\frac{dP}{d\widetilde{P}}|\mathcal{G}\bigg)(\omega) = 0\Bigg) = \tilde{E}\Bigg[1_{\Big\{\tilde{E}\big(\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)=0\Big\}}(\omega)\frac{dP}{d\widetilde{P}}(\omega)\Bigg] =$$

$$\tilde{E}\Bigg[1_{\Big\{\tilde{E}\big(\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)=0\Big\}}(\omega)\tilde{E}\bigg(\frac{dP}{d\widetilde{P}}|\mathcal{G}\bigg)(\omega)\Bigg] = 0.$$

Note that $\frac{\tilde{E}\big(X\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}{\tilde{E}\big(\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}$ is $\mathcal{G} - measurable$ and for any $A \in \mathcal{G}$

$$E\Bigg[\Bigg(X(\omega) - \frac{\tilde{E}\big(X\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}{\tilde{E}\big(\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}\Bigg)1_A(\omega)\Bigg] = \tilde{E}\Bigg[\Bigg(X(\omega) - \frac{\tilde{E}\big(X(\omega)\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}{\tilde{E}\big(\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}\Bigg)1_A(\omega)\frac{dP}{d\widetilde{P}}(\omega)\Bigg]$$

$$= \tilde{E}\bigg(X(\omega)\frac{dP}{d\widetilde{P}}(\omega)1_A(\omega)\bigg) - \tilde{E}\Bigg[\frac{\tilde{E}\big(X\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}{\tilde{E}\big(\frac{dP}{d\widetilde{P}}|\mathcal{G}\big)(\omega)}1_A(\omega)\tilde{E}\bigg(\frac{dP}{d\widetilde{P}}|\mathcal{G}\bigg)(\omega)\Bigg] = 0$$

And since this holds for all $A \in \mathcal{G}$, the result follows.

$\square$

Theorem II.44. *Consider observation model* (2.63), *where the signal* $X = (X_z, z \in \mathbb{T})$ *has almost surely Hölder-continuous sample paths of order* (λ_0, λ_0), *where* $\frac{\max(\alpha,\beta)-\frac{1}{2}}{\lambda} < \lambda_0$, *and* g *satisfies conditions* (A_1)*-*(A_2). *The following "spatial-fractional" version of the Bayes' formula holds: For any* $F \in C_b(\mathbb{R})$,

$$\mathbb{E}\big(F(X_z)|\mathcal{F}_z^Y\big) = \frac{\tilde{\mathbb{E}}\big[F(X_z)V^{-1}_{(T_1,T_2)}|\mathcal{F}_z^Y\big]}{\tilde{\mathbb{E}}\big[V^{-1}_{(T_1,T_2)}|\mathcal{F}_z^Y\big]} = \frac{\tilde{\mathbb{E}}\big[F(X_z)V_z^{-1}|\mathcal{F}_z^Y\big]}{\tilde{\mathbb{E}}\big[V_z^{-1}|\mathcal{F}_z^Y\big]} \quad a.s. \tag{2.77}$$

where $\tilde{\mathbb{E}}$ denotes the expectation under probability $\tilde{P}$, $\mathcal{F}_z^Y$ denotes the filtration generated by the observation process in the rectangle $R_z = [0, z_1] \times [0, z_2] \subset \mathbb{T}$, i.e.

$$\mathcal{F}_z^Y := \sigma\{Y_\zeta : \underline{0} \prec \zeta \prec z\}, \quad z \in \mathbb{T},$$

and $V = (V_z, z \in \mathbb{T})$ is defined by (2.75) *(in terms of* (4.4)*).*

Proof Since $P \sim \widetilde{P}$, we have

$$\frac{dP}{d\widetilde{P}} = V_{\mathbb{T}}^{-1} = \exp\left\{ \int_{\mathbb{T}} \delta_\zeta dW_\zeta^Y - \frac{1}{2}\int_{\mathbb{T}} [\delta_\zeta(X)]^2 d\zeta \right\}$$

From lemma II.43, it follows that for any $F \in C_b(\mathbb{R})$ we have

$$E[F(X_z)|\mathcal{F}_z^Y = \frac{\tilde{E}[F(X_z)V_{\mathbb{T}}^{-1}|\mathcal{F}_z^Y]}{\tilde{E}[V_{z_0}^{-1}|\mathcal{F}_z^Y} \quad a.s.$$

giving us the first equality in (2.77). By Theorem II.42, Y and X are independent under $\widetilde{P}$. Furthermore, Y is a fractional Brownian sheet with Hurst parameters α, β under $\widetilde{P}$. As in the proof of Lemma II.41 we can define a standard Wiener sheet W^Y on a complete probability space $(\Omega_2, \mathcal{F}_2, P_2)$, and define X on a complete probability space $(\Omega_1, \mathcal{F}_1, P_1)$ and then consider the processes on a product probability space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \times P_2)$, with $W^Y(\omega) = W^Y(\omega_2)$ and $X(\omega) = X(\omega_1)$ for all $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$. For each fixed ω_1,

$$V_{\mathbb{T}}^{-1}(\omega_1, \cdot) = \exp\left\{ \int_{\mathbb{T}} \delta_\zeta(X(\omega_1)) dW_\zeta^Y(\cdot) - \frac{1}{2}\int_{\mathbb{T}} [\delta_\zeta(X(\omega_1))]^2 d\zeta \right\}.$$

It follows from Proposition II.15 and Corollary II.40, that for fixed

$\omega_1 \in \Omega_1$, $\int_{R_z} \delta_\zeta(X(\omega_1)) dW_\zeta^Y(\cdot)$ is a mean zero, Gaussian strong martingale with respect to $\mathcal{F}_z^{W^Y} := \sigma\{W_{z'}^Y : 0 \prec z' \prec z \in \mathbb{T}\}$ with quadratic variation process $\int_{R_z} [\delta_\zeta(X(\omega_1))]^2 d\zeta$. Thus, applying Theorem II.16, for fixed ω_1, $\left(V_z^{-1}(\omega_1, \cdot), \mathcal{F}_z^{W^Y}\right)$ is a martingale with respect to $\widetilde{P}$. For any $z \in \mathbb{T}$, the sigma field generated by W_z^Y is equal to the sigma field generated by Y_z, so it follows that $\left(V_z^{-1}(\omega_1, \cdot), \mathcal{F}_z^{\mathcal{Y}}\right)$ is a strong martingale under $\widetilde{P}$ for fixed ω_1. The second equality in (2.77) now can easily be obtained using arguments similar to the classical reference measure approach (see e.g. [13], pg. 282).

□

2.7 Fractional Duncan-Mortensen-Zakai-type evolution equation along increasing path in the plane

We are now ready to present evolution equations for the optimal filter in the multi parameter non-linear filtering problem. Similarly to the single parameter case, we will derive a stochastic evolution equation describing the dynamics of the unnormalized optimal filter for an observation process in the plane corrupted by noise term $B_z^{\alpha,\beta}$, a fractional Brownian sheet.

Let Δ be an increasing path connecting the origin to $\mathbb{T}$. By increasing path we mean that Δ is non decreasing in both the z_1 direction and the z_2 direction. For each $z \in \mathbb{T}$, let z_Δ be the "smallest" point on Δ larger than z with respect to the partial ordering $\prec$. The path Δ divides the R_z into two regions; the region below Δ which we will denote D_1^Δ and the region above, denoted D_2^Δ. More formally, we can express D_1^Δ and D_2^Δ as $D_1^\Delta = \left\{\zeta \in \mathbb{T} : \zeta \odot \zeta_\Delta = \zeta_\Delta\right\}$ and $D_2^\Delta = \left\{\zeta \in \mathbb{T} : \zeta_\Delta \odot \zeta = \zeta_\Delta\right\}$.

Definition II.45. Let $(\mathcal{F}_z, z \in \mathbb{T})$ be a filtration satisfying conditions (F1)-(F4). Suppose Δ is a monotone nondecreasing continuous 1-dim curve connecting the origin to point $T = (T_1, T_2) \in \mathbb{R}_+^2$. Then

i) A process $\phi = (\phi_z, z \in \mathbb{T})$ is called Δ-adapted if ϕ_z is $\mathcal{F}_{z_\Delta}$-measurable for all $z \in \mathbb{T}$.

ii) A process $X = (X_z, z \in \mathbb{T})$ is called a Δ-martingale if X is Δ-adapted and

$$\mathbb{E}\left[X\big(z, z'\big] \,\big|\, \mathcal{F}_{z_\Delta}\right] = 0 \;\; for \;\; all \;\; \underline{0} \prec z \prec z' \prec T.$$

Definition II.46. Let $\mathcal{H}_\Delta$ be the space of processes $\phi = (\phi_z, z \in \mathbb{T})$ satisfying the following conditions:

(a) ϕ is a bimeasurable function of (ω, z);

(b) $\int_{\mathbb{T}} \mathbb{E}\phi_z^2 dz < \infty$;

(c_Δ) ϕ is Δ-adapted.

For $\phi \in \mathcal{H}_\Delta$, define processes $\phi_i^\Delta = (\phi_{iz}^\Delta, z \in \mathbb{T}) \in \mathcal{H}_i$, $i = 1, 2$ by:

$$\phi_{1z}^\Delta = \begin{cases} \phi_z, & \text{if } z \in D_1^\Delta, \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad \phi_{2z}^\Delta = \begin{cases} \phi_z, & \text{if } z \in D_2^\Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\phi_z = \phi_{1z}^\Delta + \phi_{2z}^\Delta$ for almost all $z \in \mathbb{T}$ and one can construct stochastic integral $\int_{\mathbb{T}} \phi_z dW_z = (\phi \circ W)_T^\Delta$ for $\phi \in \mathcal{H}_\Delta$ and show the following properties for the resulting integral (see [28] for details):

Proposition II.47. *Let Δ be a monotone nondecreasing 1-dim continuous curve connecting the origin to the final point T. Let $\phi \in \mathcal{H}_\Delta$ and define the stochastic integral of ϕ with respect to a standard Wiener sheet $(W_z, \mathcal{F}_z, z \in \mathbb{T})$ by*

$$(\phi \circ W)_z^\Delta = (\phi_1^\Delta \circ W)_z + (\phi_2^\Delta \circ W)_z, \quad z \in \mathbb{T}. \tag{2.78}$$

Then the integral has the following properties:

i) $(\phi \circ W)^\Delta$ is a Δ-martingale;

ii) $(\phi \circ W)^\Delta$ is a one-parameter martingale on the path Δ;

iii) If Δ and Δ' are two monotone nondecreasing paths connecting the origin to T and both passing through a point $z_0 \in \mathbb{T}$, and ϕ is both Δ and Δ'-adapted, then $(\phi \circ W)_{z_0}^\Delta = (\phi \circ W)_{z_0}^{\Delta'}$.

Proposition II.48. *[28] Suppose X_z is a semi-martingale in the plane taking the form*

$$X_z = X_0 + \int_{R_z} \phi_\zeta dW_\zeta + \int_{R_z} \theta_\zeta d\zeta + \int_{R_z \times R_z} \psi_{\zeta,\zeta'} dW_\zeta dW_{\zeta'}$$

$$+\int_{R_z\times R_z} f_{\zeta,\zeta'}d\zeta dW_{\zeta'}+\int_{R_z\times R_z} g_{\zeta,\zeta'}dW_{\zeta}d\zeta',\quad z\in\mathbb{T}, \tag{2.79}$$

where $R_z=[0,z_1]\times[0,z_2]$ *and* $\phi\in\mathcal{H}_0$ *and* $\psi,f,g\in\hat{\mathcal{H}}$, *where spaces* $\mathcal{H}_0,\hat{\mathcal{H}}$ *are defined as in Definition II.14 and Definition II.19. Then for any increasing path* Δ, *connecting the origin to* z, *there exist* $\eta_\zeta=\eta(\Delta,\zeta)$ *and* $\nu_\zeta=\nu(\Delta,\zeta)$ *such that* $\eta\in\mathcal{H}_\Delta$ *and*

$$X_z=X_0+\int_{R_z}\eta(\Delta,\zeta)dW_\zeta+\int_{R_z}\nu(\Delta,\zeta)d\zeta\quad x\in\Delta. \tag{2.80}$$

As such, X *is a sample-continuous semimartingale along the increasing path* Δ.

We now consider the spatial nonlinear filtering model (2.63) with signal process given by the general multiparameter semimartingale (2.79).

Theorem II.49. *Let* Δ *be an arbitrary monotone nondecreasing continuous 1-dim curve connecting the origin to the final point* $T\in\mathbb{R}^2_+$. *Let us assume that the observation model* (2.63) *holds, along with conditions* $(\mathrm{A}_1),(\mathrm{A}_2)$, *and suppose that the signal* X *is a two-parameter semimartingale in the plane, which is written in the form* (2.80), *where* $\eta\in\mathcal{H}_\Delta$ *and* ν *is* Δ*-adapted, and whose trajectories are Hölder-continuous of order* (λ_0,λ_0), *where* $\lambda_0>\frac{\max\{\alpha,\beta\}-\frac{1}{2}}{\lambda}$. *For* $F\in C_b^2(\mathbb{R})$, *consider the unnormalized optimal filter*

$$\sigma_z(F):=\tilde{\mathbb{E}}\big[F(X_z)V_z^{-1}|\mathcal{F}_z^Y\big],\quad z\in\mathbb{T}, \tag{2.81}$$

introduced in Theorem II.44. Then the following stochastic evolution equation, governing the dynamics of the unnormalized optimal filter along the monotone increasing path Δ, *is satisfied:*

(2.82)

$$\sigma_z(F)=\sigma_{\underline{0}}(F)+\int_{R_z}\sigma_{\zeta_\Delta}\big(\nu F'+\frac{1}{2}\eta^2F''\big)d\zeta+\int_{R_z}\sigma_{\zeta_\Delta}(F\delta)d\Big(\int_{R_\zeta}K^{-1}_{\alpha,\beta}(\zeta;\zeta')dY_{\zeta'}\Big),$$

$\forall z \in \Delta$, *where*

$$\sigma_{\zeta_\Delta}(F\delta) := \tilde{\mathbb{E}}\big[F\left(X_{\zeta_\Delta}\right)\delta_{\zeta_\Delta}(X)V_{\zeta_\Delta}^{-1}\,\big|\,\mathcal{F}_{\zeta_\Delta}^{Y}\big], \tag{2.83}$$

and $(\delta_z(X), z \in \mathbb{T})$ *is defined in* (4.4).

Proof First we reparameterize Δ by $\{z(t); 0 \le t \le 1\}$ so that the process $\{X_z, z \in \Delta\}$ can be rewritten as $\{X_{z(t)}, 0 \le t \le 1\}$. By Proposition II.47 and Proposition II.48, X is a continuous one-parameter semimartingale on Δ. Therefore, for any $F \in C_b^2(\mathbb{R})$,

$$F(X_{z(t)}) = F(X_{z(0)}) + \int_0^t F'(X_{z(s)})dX_{z(s)} + \frac{1}{2}\int_0^t F''(X_{z(s)})d\langle X, X\rangle_{z(s)}, \quad t \in [0,1],$$

where $\langle X, X\rangle_{z(t)} = \int_{R_{z(t)}} \eta_\zeta^2 d\zeta$. Note that one can re-express F along Δ free of the earlier parametrization as follows:

$$F(X_z) = F(X_{\underline{0}}) + \int_{R_z} F'(X_{\zeta_\Delta})dX_\zeta + \frac{1}{2}\int_{R_z} F''(X_{\zeta_\Delta})\eta_\zeta^2 d\zeta, \quad z \in \Delta.$$

Similarly, since

$$V_{z(t)}^{-1} = \exp\left\{\int_0^t \delta_{z(s)}(X)dW_{z(s)}^B + \frac{1}{2}\int_0^t [\delta_{z(s)}(X)]^2 dz(s)\right\}, \quad t \in [0,1],$$

the (single parameter) Itô formula gives

$$V_{z(t)}^{-1} = 1 + \int_0^t V_{z(s)}^{-1}\delta_{z(s)}(X)dW_{z(s)}^B + \int_0^t V_{z(s)}^{-1}[\delta_{z(s)}(X)]^2 dz(s), \quad t \in [0,1],$$

where the latter equation can also be rewritten free of parametrization as

$$V_z^{-1} = 1 + \int_{R_z} V_{\zeta_\Delta}^{-1}\delta_{\zeta_\Delta}(X)dW_\zeta^Y, \quad z \in \Delta.$$

We can apply the one-parameter Itô product rule to $V_{z(s)}^{-1}F(X_{z(s)})$ giving

$$V_{z(t)}^{-1}F(X_{z(t)}) = F(X_{z(0)}) + \int_0^t V_{z(s)}^{-1}\left(\nu_{z(s)}F'(X_{z(s)}) + \frac{1}{2}\eta_{z(s)}^2 F''(X_{z(s)})\right)dz(s)$$

$$+\int_0^t V_{z(s)}^{-1} F'(X_{z(s)})\eta_{z(s)} dW_{z(s)} + \int_0^t F(X_{z(s)}) V_{z(s)}^{-1} \delta_{z(s)}(X) dW_{z(s)}^Y, \quad t \in [0,1].$$

Taking conditional expectations of both sides of the above one-parameter equation with respect to $\mathcal{F}_{z(t)}^Y = \mathcal{F}_{z(t)}^{W^Y}$ under $\tilde{P}$, we obtain the following equation along the path Δ:

$$\sigma_{z(t)}(F) = \sigma_{z(0)}(F) + \int_0^t \sigma_{z(s)}(\nu F' + \frac{1}{2}\eta^2 F'') dz(s) + \int_0^t \sigma_{z(s)}(F\delta) dW_{z(s)}^Y, \quad t \in [0,1],$$

where

$$\sigma_{z(s)}(F\delta) := \tilde{E}[V_{z(s)}^{-1} F(X_{z(s)}) \delta_{z(s)}(X) | \mathcal{F}_{z(s)}^Y].$$

The latter evolution along the 1-dimensional path Δ can be expressed free of parametrization as follows:

$$\sigma_z(F) = \sigma_{\underline{0}}(F) + \int_{R_z} \sigma_{\zeta_\Delta}(\nu F' + \frac{1}{2}\eta^2 F'') d\zeta + \int_{R_z} \sigma_{\zeta_\Delta}(F\delta) dW_\zeta^Y, \quad z \in \Delta,$$

or, equivalently, in terms of the observation process

(2.84)

$$\sigma_z(F) = \sigma_{\underline{0}}(F) + \int_{R_z} \sigma_{\zeta_\Delta}(\nu F' + \frac{1}{2}\eta^2 F'') d\zeta + \int_{R_z} \sigma_{\zeta_\Delta}(F\delta) d\Big(\int_{R_\zeta} K_{\alpha,\beta}^{-1}(\zeta;\zeta') dY_{\zeta'}\Big), \quad z \in \Delta,$$

where the equations hold almost surely under $\widetilde{P}$ and P.

$\square$

Remark The above stochastic evolution equation cannot be interpreted as a measure-valued stochastic differential equation because of the special meaning assigned to $\sigma_{\zeta_\delta}(F\delta)$ in (2.83). This form of the optimal filter takes into account the fact that $\delta_z(X)$ is not a function of X_z alone but rather a function of the entire past $(X_\zeta, 0 \prec \zeta \prec z)$.

2.8 Fractional Duncan-Mortensen-Zakai-type evolution equation in the plane

The stochastic evolution equation (2.82) of Section 2.7, governing the dynamics of the unnormalized optimal filter, holds only for "increasing" evolution in the parameter space. Our objective in this section is to develop a "fractional-spatial" analogue of the Duncan-Mortensen-Zakai equation for the unnormalized optimal filter which holds along arbitrary paths in the plane.

Let $\mathfrak{a} : \mathbb{R} \to \mathbb{R}$ and $\mathfrak{b} : \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying the following Lipshitz and growth conditions: there exists a finite constant $C > 0$ such that for all $x, y \in \mathbb{R}$,

$$|\mathfrak{a}(x) - \mathfrak{a}(y)| + |\mathfrak{b}(x) - \mathfrak{b}(y)| \le C|x - y| \tag{2.85}$$

and

$$|\mathfrak{a}(x)| + |\mathfrak{b}(x)| \le C(1 + |x|). \tag{2.86}$$

Then there exists a unique strong solution to the following multiparameter SDE (see e.g. [7]):

$$X_z = X_{\underline{0}} + \int_{R_z} \mathfrak{a}(X_\zeta) d\zeta + \int_{R_z} \mathfrak{b}(X_\zeta) dW_\zeta, \quad z \in \mathbb{T},$$

where W denotes a standard Wiener sheet. Moreover, the solution has Hölder-continuous sample path of order (λ_1, λ_2) for all $\lambda_1, \lambda_2 \in \left(0, \frac{1}{2}\right)$.

Theorem II.50. *Consider the observation model* (2.63)*, and that* $\mathfrak{a}(\cdot)$ *and* $\mathfrak{b}(\cdot)$ *satisfy the Lipshitz conditions* (2.85) *and* (2.86)*. Furthermore, assume conditions* $(\mathrm{A}_1), (\mathrm{A}_2)$ *are valid. Suppose that the signal* X *is the unique strong solution of the* *following SDE:*

$$X_z = X_{\underline{0}} + \int_{R_z} \mathfrak{a}(X_\zeta)d\zeta + \int_{R_z} \mathfrak{b}(X_\zeta)dW_\zeta, \quad z \in \mathbb{T}, \tag{2.87}$$

where W is a standard Wiener sheet independent of the observation Y. Let $\sigma_z(F) := \tilde{\mathbb{E}}\big[F(X_z)V_z^{-1} \,|\, \mathcal{F}_z^Y\big]$, i.e. $\sigma_z(F)$ is the unnormalized conditional expectation corresponding to the optimal filter. Then for all $F \in C_b^4(\mathbb{R})$, evolution of the unnormalized optimal filter satisfies the following:

$$\begin{aligned}
\sigma_z(F) = \sigma_{\underline{0}}(F) &+ \int_{R_z} \sigma_\zeta\big(\mathfrak{a}F' + \frac{1}{2}\mathfrak{b}^2F''\big)d\zeta + \int_{R_z} \sigma_\zeta\big(F\delta\big)dW_\zeta^Y \\
&+ \iint_{R_z\times R_z} \sigma_{\zeta,\zeta'}\big(F;\delta\otimes\delta\big)dW_\zeta^Y dW_{\zeta'}^Y \\
&+ \iint_{R_z\times R_z} \big[\sigma_{\zeta,\zeta'}(F';\mathfrak{a}\otimes\delta) + \frac{1}{2}\sigma_{\zeta,\zeta'}(F'';\mathfrak{b}^2\otimes\delta)\big]d\zeta dW_{\zeta'}^Y \\
&+ \iint_{R_z\times R_z} \big[\sigma_{\zeta,\zeta'}(F';\delta\otimes\mathfrak{a}) + \frac{1}{2}\sigma_{\zeta,\zeta'}(F'';\delta\otimes\mathfrak{b}^2)\big]dW_\zeta^Y d\zeta'
\end{aligned}$$

$$+\iint_{R_z\times R_z} I(\zeta\curlywedge\zeta')\Big[\sigma_{\zeta,\zeta'}\big(F'';\mathfrak{a}\otimes\mathfrak{a}\big) + \frac{1}{2}\sigma_{\zeta,\zeta'}\big(F''';\mathfrak{b}^2\otimes\mathfrak{a}+\mathfrak{a}\otimes\mathfrak{b}^2\big) + \frac{1}{4}\sigma_{\zeta,\zeta'}\big(F^{(iv)};\mathfrak{b}^2\otimes\mathfrak{b}^2\big)\Big]d\zeta d\zeta', \tag{2.88}$$

where $\otimes$ denotes the tensor product of functions, $\sigma_z(F\delta) := \tilde{\mathbb{E}}\big[F(X_z)\delta_z(X)V_z^{-1} \,|\, \mathcal{F}_z^Y\big]$, $\sigma_{z,z'}\big(F;\delta\otimes\delta\big) := \tilde{\mathbb{E}}\big[F(X_{z\vee z'})\delta_z(X)\delta_{z'}(X)V_{z\vee z'}^{-1} \,|\, \mathcal{F}_{z\vee z'}^Y\big]$, and for arbitrary functions $f_1 : \mathbb{R}\to\mathbb{R}$, $f_2 : \mathbb{R}^2\to\mathbb{R}$, we put $\sigma_{z,z'}(f_1;f_2) := \tilde{\mathbb{E}}\big[f_1(X_{z\vee z'})f_2(X_z,X_{z'})V_{z\vee z'}^{-1} \,|\, \mathcal{F}_{z\vee z'}^Y\big]$ for all $z,z' \in \mathbb{T}$. (In (2.88), $W_z^Y = \int_{R_z} K_{\alpha,\beta}^{-1}(z;\zeta)dY_\zeta$ and δ is given by (4.4), as before.)

Remark In Theorem II.50, we could write $\sigma_z(F) = \sigma_{z,z}(F;1)$, where 1 denotes function on $\mathbb{R}^2$ which is identically equal to one.

Proof : First note that, under $\tilde{P}$, Y is a fractional Brownian sheet with Hurst indices (α,β), while the corresponding field W^Y, given by $W_z^Y = \int_{R_z} K_{\alpha,\beta}^{-1}(z;\zeta)dY_\zeta$, is a standard Wiener sheet and the two random fields generate the same natural filtration, thus, the observation sigma-field $(\mathcal{F}_z^Y)_{0\prec z\prec T}$ has properties $(F1)$–$(F4)$. Similarly, $(\mathcal{F}_z^X)_{z\in\mathbb{T}}$ and $(\mathcal{F}_z^{X,Y})_{z\in\mathbb{T}}$ have properties $(F1)$–$(F4)$ under reference probability measure $\tilde{P}$. Note also that the paths of $X = (X_z, z \in \mathbb{T})$ are almost surely Hölder-continuous of arbitrary order (λ_1, λ_2), where $\lambda_1, \lambda_2 < \frac{1}{2}$, thus δ is well-defined and the conclusions of Corollary II.40 hold. Next, by a version of the Itô's formula for multiparameter semimartingales (see [29]), we obtain that for arbitrary $F \in C_b^4(\mathbb{R})$,

$$
\begin{aligned}
F(X_z) = F(X_{\underline{0}}) &+ \int_{R_z} F'(X_\zeta)\big[\mathfrak{a}(X_\zeta)d\zeta + \mathfrak{b}(X_\zeta)dW_\zeta\big] + \frac{1}{2}\int_{R_z} F''(X_\zeta)\mathfrak{b}^2(X_\zeta)d\zeta \\
&+ \iint_{R_z\times R_z} F''(X_{\zeta\vee\zeta'})\mathfrak{b}(X_\zeta)\mathfrak{b}(X_{\zeta'})dW_\zeta dW_{\zeta'} \\
&+ \iint_{R_z\times R_z} \Big[F''(X_{\zeta\vee\zeta'})\mathfrak{b}(X_\zeta)\mathfrak{a}(X_{\zeta'}) + \frac{1}{2}F'''(X_{\zeta\vee\zeta'})\mathfrak{b}(X_\zeta)\mathfrak{b}^2(X_{\zeta'})\Big]d\zeta dW_{\zeta'} \\
&+ \iint_{R_z\times R_z} \Big[F''(X_{\zeta\vee\zeta'})\mathfrak{b}(X_\zeta)\mathfrak{a}(X_{\zeta'}) + \frac{1}{2}F'''(X_{\zeta\vee\zeta'})\mathfrak{b}(X_\zeta)\mathfrak{b}^2(X_{\zeta'})\Big]dW_\zeta d\zeta' \\
&+ \iint_{R_z\times R_z} I(\zeta\curlywedge\zeta')\Big[F''(X_{\zeta\vee\zeta'})\mathfrak{a}(X_\zeta)\mathfrak{a}(X_{\zeta'}) + \frac{1}{2}F'''(X_{\zeta\vee\zeta'})\big(\mathfrak{a}(X_\zeta)\mathfrak{b}^2(X_{\zeta'}) + \mathfrak{a}(X_{\zeta'})\mathfrak{b}^2(X_\zeta)\big) \\
&\qquad + \frac{1}{4}F^{(iv)}(X_{\zeta\vee\zeta'})\mathfrak{b}^2(X_\zeta)\mathfrak{b}^2(X_{\zeta'})\Big]d\zeta d\zeta'.
\end{aligned}
$$

Similarly, under $\tilde{P}$, one shows that

$$
V_z^{-1} = 1 + \int_{R_z} V_\zeta^{-1}\delta_\zeta(X)dW_\zeta^Y + \iint_{R_z\times R_z} V_{\zeta\vee\zeta'}^{-1}\,\delta_\zeta(X)\delta_{\zeta'}(X)dW_\zeta^Y dW_{\zeta'}^Y \quad \text{a.s.}
$$

Then the multiparameter version of the stochastic integration-by-parts formula (together with independence of W and W^Y under $\tilde{P}$) yields a corresponding equation for the product $F(X_z)V_z^{-1}$. Upon taking conditional expectation of both sides of the latter equation for $F(X_z)V_z^{-1}$ with respect to $\mathcal{F}_z^Y$ (note that $\mathcal{F}_z^Y = \mathcal{F}_z^{W^Y}$) and using

Lemma II.51, which is proved below, one arrives at the following equation:

$$
\begin{aligned}
&\tilde{\mathbb{E}}\big(F(X_z)V_z^{-1}|\mathcal{F}_z^Y\big) \\
&= \tilde{\mathbb{E}}\big(F(X_{\underline{0}})|\mathcal{F}_{\underline{0}}^Y\big) + \int_{R_z} \tilde{\mathbb{E}}\Big(\big[\mathfrak{a}(X_\zeta)F'(X_\zeta) + \frac{1}{2}\mathfrak{b}^2(X_\zeta)F''(X_\zeta)\big]V_\zeta^{-1}\big|\mathcal{F}_z^Y\Big)d\zeta \\
&+ \int_{R_z} \tilde{\mathbb{E}}\big(F(X_\zeta)\delta_\zeta(X)V_\zeta^{-1}\,|\,\mathcal{F}_\zeta^Y\big)dW_\zeta^Y \\
&+ \iint_{R_z\times R_z} \tilde{\mathbb{E}}\left(F(X_{\zeta\vee\zeta'})\delta_\zeta(X)\delta_{\zeta'}(X)V_{\zeta\vee\zeta'}^{-1}\,\middle|\,\mathcal{F}_{\zeta\vee\zeta'}^Y\right)dW_\zeta^Y dW_{\zeta'}^Y \\
&+ \iint_{R_z\times R_z} \tilde{\mathbb{E}}\left(\big[\mathfrak{a}(X_\zeta)F'(X_{\zeta\vee\zeta'}) + \frac{1}{2}\mathfrak{b}^2(X_\zeta)F''(X_{\zeta\vee\zeta'})\big]\delta_{\zeta'}V_{\zeta\vee\zeta'}^{-1}\middle|\mathcal{F}_{\zeta\vee\zeta'}^Y\right)d\zeta dW_{\zeta'}^Y \\
&+ \iint_{R_z\times R_z} \tilde{\mathbb{E}}\left(\big[\mathfrak{a}(X_{\zeta'})F'(X_{\zeta\vee\zeta'}) + \frac{1}{2}\mathfrak{b}^2(X_{\zeta'})F''(X_{\zeta\vee\zeta'})\big]\delta_\zeta V_{\zeta\vee\zeta'}^{-1}\middle|\mathcal{F}_{\zeta\vee\zeta'}^Y\right)dW_\zeta^Y d\zeta' \\
&+ \iint_{R_z\times R_z} I(\zeta \curlywedge \zeta')\tilde{\mathbb{E}}\Bigg(\Big[F''(X_{\zeta\vee\zeta'})\mathfrak{a}(X_\zeta)\mathfrak{a}(X_{\zeta'}) + \frac{1}{4}F^{(iv)}(X_{\zeta\vee\zeta'})\mathfrak{b}^2(X_\zeta)\mathfrak{b}^2(X_{\zeta'}) \\
&\qquad + \frac{1}{2}F'''(X_{\zeta\vee\zeta'})\big\{\mathfrak{b}^2(X_{\zeta'})\mathfrak{a}(X_\zeta) + \mathfrak{a}(X_{\zeta'})\mathfrak{b}^2(X_\zeta)\big\}\Big]V_{\zeta\vee\zeta'}^{-1}\,\Bigg|\,\mathcal{F}_{\zeta\vee\zeta'}^Y\Bigg)d\zeta d\zeta' \text{ a.s.},
\end{aligned}
$$

thus, the required conclusion follows. □

Lemma II.51. *Let W and W^Y be independent standard Wiener sheets on a probability space $(\Omega, \mathcal{F}_T, \tilde{P})$ and $\mathcal{F}_z^{W,W^Y} := \sigma(W_\zeta, W_{\zeta'}^Y : \underline{0} \prec \zeta \prec z, 0 \prec \zeta' \prec z)$, $z \in \mathbb{T}$. Also let $(\mathcal{F}_z^W)$ and $(\mathcal{F}_z^{W^Y})$ denote the natural filtrations generated by W and W^Y, respectively. Consider a process M (which is $\big(\mathcal{F}_z^{W,W^Y}\big)$-measurable), given by*

$$M_z := \int_{R_z} \phi_\zeta dW_\zeta^Y + \iint_{R_z\times R_z} \psi_{\zeta,\zeta'}dW_\zeta^Y dW_{\zeta'}^Y,$$

where $\phi \in \mathcal{H}_0$ and $\psi \in \hat{\mathcal{H}}$, with $\mathcal{H}_0$ and $\hat{\mathcal{H}}$ being defined with respect to filtration $(\mathcal{F}_z^{W,W^Y})$. Then

i) For any process $\psi \in \hat{\mathcal{H}}$,

$$\tilde{\mathbb{E}}\left(\int_{R_z\times R_z} \psi_{\zeta,\zeta'}dW_\zeta dW_{\zeta'}\,\middle|\,\mathcal{F}_z^{W^Y}\right) = 0 \text{ a.s. } \tilde{P},$$

$$\tilde{\mathbb{E}}\left(\int_{R_z\times R_z} \psi_{\zeta,\zeta'}dW_\zeta d\zeta'\,\middle|\,\mathcal{F}_z^{W^Y}\right) = 0 \text{ a.s. } \tilde{P},$$

$$\tilde{\mathbb{E}}\left(\int_{R_z\times R_z}\psi_{\zeta,\zeta'}d\zeta dW_{\zeta'}\,\middle|\,\mathcal{F}_z^{W^Y}\right)=0\ \text{ a.s. }\ \tilde{P},$$

$$\tilde{\mathbb{E}}\left(\int_{R_z\times R_z}\psi_{\zeta,\zeta'}dW_\zeta dW^Y_{\zeta'}\,\middle|\,\mathcal{F}_z^{W^Y}\right)=0\ \text{ a.s. }\ \tilde{P},$$

$$\tilde{\mathbb{E}}\left(\int_{R_z\times R_z}\psi_{\zeta,\zeta'}dW^Y_\zeta dW_{\zeta'}\,\middle|\,\mathcal{F}_z^{W^Y}\right)=0\ \text{ a.s. }\ \tilde{P}.$$

ii) The following equation holds almost surely with respect to $\tilde{P}$:

$$\tilde{\mathbb{E}}\left(M_z\,|\,\mathcal{F}_z^{W^Y}\right)=\int_{R_z}\tilde{\mathbb{E}}\left(\phi_\zeta\,|\,\mathcal{F}_\zeta^{W^Y}\right)dW^Y_\zeta+\int_{R_z\times R_z}\tilde{\mathbb{E}}\left(\psi_{\zeta,\zeta'}\,|\,\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\right)dW^Y_\zeta dW^Y_{\zeta'}.$$

Also,

$$\tilde{\mathbb{E}}\left(\int_{R_z\times R_z}\psi_{\zeta,\zeta'}d\zeta dW^Y_{\zeta'}\,\middle|\,\mathcal{F}_z^{W^Y}\right)=\int_{R_z\times R_z}\tilde{\mathbb{E}}\big(\psi_{\zeta,\zeta'}|\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big)d\zeta dW^Y_{\zeta'}\ \text{ a.s. }\ \tilde{P},$$

$$\tilde{\mathbb{E}}\left(\int_{R_z\times R_z}\psi_{\zeta,\zeta'}dW^Y_\zeta d\zeta'\,\middle|\,\mathcal{F}_z^{W^Y}\right)=\int_{R_z\times R_z}\tilde{\mathbb{E}}\big(\psi_{\zeta,\zeta'}|\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big)dW^Y_\zeta d\zeta'\ \text{ a.s. }\ \tilde{P}.$$

Proof: Let us start by showing that the first equality in (i) holds, i.e. that

$$\tilde{\mathbb{E}}\bigg(\int_{R_z\times R_z}\psi_{\zeta,\zeta'}dW_\zeta dW_{\zeta'}|\mathcal{F}_z^{W^Y}\bigg)=0$$

almost surely under $\tilde{P}$. By independence of W and W^Y, we may assume that W is a standard Wiener sheet on a filtered complete probability space $(\Omega^X,\breve{\mathcal{F}}^W_T,(\breve{\mathcal{F}}^W_z)_{z\in\mathbb{T}},\tilde{P}^X)$, whereas W^Y is a standard Wiener sheet on another filtered complete probability space $(\Omega^Y,\breve{\mathcal{F}}^{W^Y}_T,(\breve{\mathcal{F}}^{W^Y}_z)_{z\in\mathbb{T}},\tilde{P}^Y)$, where $(\breve{\mathcal{F}}^W_z)$ and $(\breve{\mathcal{F}}^{W^Y}_z)$ are, respectively, natural filtrations generated by processes W and W^Y (in Ω^X and Ω^Y, respectively), and $(\Omega,\mathcal{F}_T,\tilde{P})=(\Omega^X\times\Omega^Y,\breve{\mathcal{F}}^W_T\times\breve{\mathcal{F}}^{W^Y}_T,P^X\times P^Y)$, i.e. the product probability space. Then W and W^Y are defined on $(\Omega,\mathcal{F}_T,\tilde{P})$ by $W_z(\omega)=W_z(\omega_1)$ and $W^Y_z(\omega)=W^Y_z(\omega_2)$ for all $\omega=(\omega_1,\omega_2)\in\Omega$. Then, clearly, $\mathcal{F}^W_z=\breve{\mathcal{F}}^W_z\times\{\emptyset,\Omega^Y\}$ and $\mathcal{F}^{W^Y}_z=\{\emptyset,\Omega^X\}\times\breve{\mathcal{F}}^{W^Y}_z$.

Next let us fix some $n\in\mathbb{N}$ and consider a partition of rectangle $R_T=(\underline{0},T]$ (where $T=(T_1,T_2)$) into rectangles $\Delta_{i,j}:=\big(z_{(i,j)},z_{(i+1,j+1)}\big]$, where $z_{(i,j)}=(2^{-n}iT_1,2^{-n}jT_2)$.

Let $\mathcal{S}$ be the class of processes ψ of the form:

(2.89)
$$\psi(\zeta,\zeta') = \sum_{i,j,k,\ell=0}^{2^n-1} \alpha_{ijk\ell} 1_{\Delta_{i,j}}(\zeta) 1_{\Delta_{k,\ell}}(\zeta'),$$

where $\alpha_{ijk\ell}$ is $\mathcal{F}^{W,W^Y}_{z_{(i,j)}\vee z_{(k,\ell)}}$-measurable. By definition of the double integral,

$$\int_{R_z\times R_z} \psi(\zeta,\zeta') dW_\zeta dW_{\zeta'} := \sum_{i,j,k,\ell=0}^{2^n-1} \alpha_{ijk\ell}\, 1_{\{i<k\}} 1_{\{\ell<j\}} W(R_z\cap\Delta_{i,j}) W(R_z\cap\Delta_{k,\ell}).$$

Then,

$$\begin{aligned}
&\tilde{\mathbb{E}}\Big(\int_{R_z\times R_z} \psi(\zeta,\zeta') dW_\zeta dW_{\zeta'} | \mathcal{F}^{W^Y}_z\Big) \\
&= \sum_{i,j,k,\ell=0}^{2^n-1} \tilde{\mathbb{E}}\Big[\alpha_{ijk\ell} 1_{\{i<k\}\cap\{\ell<j\}} W(R_z\cap\Delta_{i,j}) W(R_z\cap\Delta_{k,\ell}) \,\big|\, \{\emptyset,\Omega^X\}\times\breve{\mathcal{F}}^{W^Y}_z\Big].
\end{aligned}$$

Note that $\forall Q\in\breve{\mathcal{F}}^{W^Y}_z$, we have

$$\begin{aligned}
&\int_{\Omega^X\times Q} \alpha_{ijk\ell}(\omega_1,\omega_2) 1_{\{i<k\}\cap\{\ell<j\}} W(R_z\cap\Delta_{i,j})(\omega_1) W(R_z\cap\Delta_{k,\ell})(\omega_1) d\tilde{P}(\omega_1,\omega_2) \\
&= \int_Q \Big[\int_{\Omega^X} \alpha_{ijk\ell}(\omega_1,\omega_2) 1_{\{i<k\}\cap\{\ell<j\}} W(R_z\cap\Delta_{i,j})(\omega_1) W(R_z\cap\Delta_{k,\ell})(\omega_1) d\tilde{P}^X(\omega_1)\Big] d\tilde{P}^Y(\omega_2) = 0,
\end{aligned}$$

since for fixed $\omega_2\in\Omega_2$ and for all $i<k$ and $\ell<j$, random variables $\alpha_{ijk\ell}(\cdot,\omega_2)$, $W(R_z\cap\Delta_{k,\ell})$ and $W(R_z\cap\Delta_{i,j})$ are mutually independent. Thus,

$$\tilde{\mathbb{E}}\Big[\alpha_{ijk\ell} 1_{\{i<k\}\cap\{\ell<j\}} W(R_z\cap\Delta_{i,j}) W(R_z\cap\Delta_{k,\ell}) \,\big|\, \{\emptyset,\Omega^X\}\times\breve{\mathcal{F}}^{W^Y}_z\Big] = 0 \ \text{ a.s.}(\tilde{P}),$$

implying that for any simple process $\psi\in\mathcal{S}$,

$$\tilde{\mathbb{E}}\Big(\int_{R_z\times R_z} \psi(\zeta,\zeta') dW_\zeta dW_{\zeta'} | \mathcal{F}^{W^Y}_z\Big) = 0 \ \text{ a.s. } (\tilde{P}).$$

Since $\mathcal{S}$ is dense in $\hat{\mathcal{H}}$, the required equality follows for arbitrary $\psi\in\hat{\mathcal{H}}$ by taking appropriate limits. Similar arguments show that the remaining equalities in (i) are also valid.

To prove (ii), let us show that $\forall\psi\in\hat{\mathcal{H}}$,

(2.90)
$$\tilde{\mathbb{E}}\Big(\int_{R_z\times R_z} \psi(\zeta,\zeta') dW^Y_\zeta dW^Y_{\zeta'} | \mathcal{F}^{W^Y}_z\Big) = \int_{R_z\times R_z} \tilde{\mathbb{E}}\big[\psi(\zeta,\zeta') | \mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big] dW^Y_\zeta dW^Y_{\zeta'} \ \text{ a.s.}$$

First, consider $\psi \in \mathcal{S}$ of the form (2.89). Then

$$\begin{aligned} &\tilde{\mathbb{E}}\Big(\int_{R_z\times R_z}\psi(\zeta,\zeta')dW^Y_\zeta dW^Y_{\zeta'}|\mathcal{F}^{W^Y}_z\Big)\\ (2.91)\qquad &= \sum_{i,j,k,\ell=0}^{2^n-1}\tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_z\big]1_{\{i<k\}\cap\{\ell<j\}}W^Y(R_z\cap\Delta_{i,j})W^Y(R_z\cap\Delta_{k\ell}), \end{aligned}$$

where note that

$$1_{\{i<k\}\cap\{\ell<j\}}W^Y(R_z\cap\Delta_{i,j})W^Y(R_z\cap\Delta_{k\ell}) = 0 \;\text{ unless }\; z_{(k,j)} = \big(z_{(i,j)}\vee z_{(k,\ell)}\big) \prec\prec z.$$

Since $\alpha_{ijk\ell}$ is $\mathcal{F}^{W,W^Y}_{z_{(k,j)}}$-measurable and $z_{(k,j)} \prec\prec z$, then the conditional expectation in the right-hand side of (2.91) satisfies equation

$$\tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_z\big] = \tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_{z_{(k,j)}}\big] \;\text{ a.s.},$$

by independence of W and W^Y and since Wiener sheets generate independently scattered measures. Thus,

$$\begin{aligned} &\tilde{\mathbb{E}}\Big(\int_{R_z\times R_z}\psi(\zeta,\zeta')dW^Y_\zeta dW^Y_{\zeta'}|\mathcal{F}^{W^Y}_z\Big)\\ (2.92)\quad &= \sum_{i,j,k,\ell=0}^{2^n-1}\tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_{z_{(k,j)}}\big]1_{\{i<k\}\cap\{\ell<j\}}W^Y(R_z\cap\Delta_{i,j})W^Y(R_z\cap\Delta_{k\ell}) \;\text{ a.s.} \end{aligned}$$

On the other hand,

$$\begin{aligned} &\int_{R_z\times R_z}\tilde{\mathbb{E}}\big[\psi(\zeta,\zeta')|\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big]dW^Y_\zeta dW^Y_{\zeta'}\\ &= \int_{R_z\times R_z}\sum_{i,j,k,\ell=0}^{2^n-1}\tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big]1_{\Delta_{i,j}}(\zeta)1_{\Delta_{k,\ell}}(\zeta')dW^Y_\zeta dW^Y_{\zeta'}\\ (2.93)\quad &= \int_{R_z\times R_z}\sum_{i,j,k,\ell=0}^{2^n-1}1_{\{i<k\}\cap\{\ell<j\}}\tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big]1_{\Delta_{i,j}}(\zeta)1_{\Delta_{k,\ell}}(\zeta')dW^Y_\zeta dW^Y_{\zeta'}, \end{aligned}$$

where the last equality holds by definition of the double integral. Note that

$$1_{\{i<k\}\cap\{\ell<j\}}1_{\Delta_{i,j}}(\zeta)1_{\Delta_{k,\ell}}(\zeta') \neq 0 \;\text{ implies that }\; \zeta\vee\zeta' \in \Delta_{z_{(k,j)}},$$

which, in turn, implies that $\tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big]$ (in the right-hand side of (2.93)) equals almost surely to $\tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_{z_{(k,j)}}\big]$, since $\alpha_{ijk\ell}$ is $\mathcal{F}^{W,W^Y}_{z_{(k,j)}}$-measurable. Thus, from (2.93) by definition of the double integral,

$$\begin{aligned} & \int_{R_z\times R_z} \tilde{\mathbb{E}}\big[\psi(\zeta,\zeta')|\mathcal{F}^{W^Y}_{\zeta\vee\zeta'}\big]dW^Y_\zeta dW^Y_{\zeta'} \\ (2.94) \quad = \quad & \sum_{i,j,k,\ell=0}^{2^n-1} \tilde{\mathbb{E}}\big[\alpha_{ijk\ell}|\mathcal{F}^{W^Y}_{z_{(k,j)}}\big]1_{\{i<k\}\cap\{\ell<j\}}W^Y(R_z\cap\Delta_{i,j})W^Y(R_z\cap\Delta_{k\ell}) \text{ a.s.} \end{aligned}$$

From (2.92) and (2.94), it follows that (2.90) holds for all $\psi \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $\hat{\mathcal{H}}$, it follows that (2.90) holds for all $\psi \in \hat{\mathcal{H}}$ by taking appropriate limits. Similarly one establishes that $\forall\phi \in \mathcal{H}_0$,

$$\tilde{\mathbb{E}}\Big(\int_{R_z}\phi_\zeta dW^Y_\zeta\Big|\mathcal{F}^{W^Y}_z\Big) = \int_{R_z}\tilde{\mathbb{E}}(\phi_\zeta|\mathcal{F}^{W^Y}_\zeta)dW^Y_\zeta \text{ a.s.,}$$

thus, the first statement in (ii) is proved. The remaining two statements in (ii) can be established by analogous arguments. $\square$

CHAPTER III

Representations of the optimal filter in the context of nonlinear filtering of random fields with fractional noise

3.1 Introduction

In Chapter II, we considered the nonlinear filtering model (2.63) and derived the Duncan-Mortensen-Zakai-type evolution equation describing the evolution of the best mean square estimate of (a function of) the signal. While this evolution equation is useful in describing the dynamics of the optimal filter, one may or may not be able to work with the evolution equation which is analytically very complex. In this chapter we develop methods for representing the optimal filter in terms of ratios of infinite sums of multiple stochastic integrals.

Here we will extend the model (2.63) to the analogous, more general model in m-dimensions where $m \geq 1$. For this model, we develop expansions based on the integral transformations and change of measure associated with V_z as defined by (2.75), that was used in the Duncan-Mortensen-Zakai-type evolution equation. We also derive expansions based on the multiparameter version of the transformation N^C defined in (2.41).

The expansions described in Theorems III.31, III.32 and III.33 can be truncated and discretized in order to provide a method for numerically implementing the optimal filter in practice. In Theorem III.34, we show that the ratio of a truncated and discretized version of the multiple integral expansion converges to the multiple optimal filter. This truncation and discretization technique describes a method which can be used to compute accurate estimates of the optimal filters in a very general class of spatial nonlinear filtering problems with long range dependent noise.

In section 3.2 we introduce several notions of multiple stochastic integrals. Be begin with the multiple stochastic integral with respect to Brownian sheet in m dimensional parameter space. We then develop the concept of the Wiener integral with respect to persistent fractional Brownian motion and then move on to multiple stochastic integrals (of both Wiener and Stratonovich type) with respect to fractional Brownian motion and fractional Brownian sheet. In section 3.3 we describe the model being used and extend some of the methods developed in Chapter II to the m dimensional parameter space. In section 3.4 we extend the notion of integral transform N^C and the results of section 2.5 from the single parameter case to the m-dimensional parameter space. Finally, in section 3.5 we give expansions of the optimal filter and truncated, discretized expansions of these representations.

3.2 Multiple stochastic integrals

In this section we outline the theory of multiple stochastic integrals with respect to Brownian sheet and with respect to persistent *fractional* Brownian sheet. We will use these processes to express the optimal filter discussed earlier in Chapter II as infinite sums. Specifically, we will develop several methods for expressing the conditional expectation in (II.44) in the framework of the spatial filtering model (2.63). Here we will be able to work in the general m-dimensional bounded space $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_m]$. Since we will not be examining evolution equations, the notation is not greatly affected by the additional dimensions. We begin by introducing a few important tools in Gaussian analysis.

Definition III.1. The nth Hermite polynomial, $\mathcal{H}_n(x)$, is defined by

$$\mathcal{H}_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}), \quad n \geq 1,$$

and $\mathcal{H}_0(x) = 1$.

These polynomials are coefficients of powers of t in the expansion of $\exp(tx - \frac{t^2}{2})$. That is

$$\begin{aligned} \exp(tx - \frac{t^2}{2}) &= \exp\left(\frac{x^2}{2} - \frac{1}{2}(x-t)^2\right) \\ &= e^{\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{d^n}{dt^n} e^{-\frac{(x-t)^2}{2}}\right)\Bigg|_{t=0} \end{aligned}$$

$$= \sum_{n=0}^{\infty} t^n \mathcal{H}_n(x). \tag{3.1}$$

It follows that we can characterize the Hermite polynomials as follows:

$$\mathcal{H}_n'(x) = n\mathcal{H}_{n-1}(x), \quad n \geq 1 \tag{3.2}$$

$$\mathcal{H}_{n+1}(x) = x\mathcal{H}_n(x) - n\mathcal{H}_{n-1}(x), \quad n \geq 1. \tag{3.3}$$

The first Hermite polynomials are:

$\mathcal{H}_0(x) = 1$

$\mathcal{H}_1(x) = x$

$\mathcal{H}_2(x) = x^2 - 1$

$\mathcal{H}_3(x) = x^3 - 3x$

$\mathcal{H}_4(x) = x^4 - 6x^2 + 3$

$\mathcal{H}_5(x) = x^5 - 10x^3 + 15x$

Proposition III.2. *Let random variables Z_1, Z_2 have joint Gaussian distribution such that $E(Z_1) = E(Z_2) = 0$ and $Var(Z_1) = Var(Z_2) = 1$. If $n, m \geq 0$ then,*

$$E\left(\mathcal{H}_n(Z_1)\mathcal{H}_m(Z_2)\right) = \begin{cases} n!\left(E\left(Z_1Z_2\right)\right)^n & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Definition III.3. For any n random variables, $X_1, ..., X_n$, the Wick product $X_1 \diamond \cdots \diamond X_n$ is defined recursively for $k_1 + ... + k_n = N$, as satisfying

1. $X_1^0 \diamond \cdots \diamond X_n^0 = 1$
2. $E\left(X_1^{k_1} \diamond \cdots \diamond X_n^{k_n}\right) = 0$
3. $\frac{\partial}{\partial X_i}\left(X_1^{k_1} \diamond \cdots \diamond X_n^{k_n}\right) = k_i\left(X_1^{k_1} \diamond \cdots \diamond X_i^{k_i - 1} \diamond \cdots \diamond X_n^{k_n}\right)$

We also define the nth Wick power of a random variable X, by $X^{\diamond n} := \underbrace{X \diamond \cdots \diamond X}_{n}$. The following identities follow from the definition of the Wick product and Hermite polynomial.

For $X, Y, Z \in L^2(\Omega, \mathcal{F}, P)$, and $a \in \mathbb{R}$

$$(aX) \diamond Y = X \diamond (aY) = a(X \diamond Y)$$

$$X \diamond (Y + Z) = X \diamond Y + X \diamond Z = (Y + Z) \diamond X$$

Proposition III.4. *For any mean zero Gaussian random variable X with variance given by σ^2,*

$$X^{\diamond n} = \sigma^n \mathcal{H}_n\left(X/\sigma\right) \tag{3.4}$$

where $\mathcal{H}_n(x)$ is the Hermite polynomial of order n.

In order to examine stochastic integrals in the m-dimensional hyperplane, we first need to generalize the notions of increments in the plane and hence random measures in the plane. Using these generalizations we will give the m-dimensional analogue of the Brownian sheet given in Definition II.3.

Definition III.5. Let $\mathbb{T} = [\underline{0}, \underline{T}] = [0, T_1] \times \cdots \times [0, T_m]$ and take $t, s \in \mathbb{T}$ such that $s \preceq t$. Then for process $X = \big(X_z, z \in \mathbb{T}\big)$ define the increment of X over the rectangle $R = (s, t] = (s_1, t_1] \times \cdots \times (s_m, t_m]$ by

$$X(R) \equiv \sum_{\alpha \in \{0,1\}^m} (-1)^{m - \sum_{i=1}^m \alpha_i} X\Big(s + \sum_{i=1}^m \alpha_i (t_i - s_i)\Big), \tag{3.5}$$

where $X\Big(s + \sum_{i=1}^m \alpha_i (t_i - s_i)\Big)$ denotes the value X assumes at the point $s + \sum_{i=1}^m \alpha_i (t_i - s_i)$ in $\mathbb{T}$. $X(R)$ is also referred to as the variation of X over (hyper) rectangle R.

Definition III.6. On a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_z), P)$, let $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_m]$. $W = \big(W_z, \mathcal{F}_z, z \in \mathbb{T}\big)$ is called a continuous version of a Brownian (Wiener) sheet if it satisfies the following conditions:

1. W is a random measure on $\mathbb{T}$, assigning to each (hyper) rectangle R, a Gaussian random variable of mean zero;
2. for disjoint Borel sets in $\mathbb{T}$, W assigns independent random variables;
3. $W_z = W(R_z)$ where R_z is the rectangle whose upper right hand corner is z and whose lower left hand corner is the origin and W_z is adapted to $\mathcal{F}_z$;
4. W has continuous trajectories;
5. for all $t, s \in \mathbb{T}$, $\mathbb{E}(W_t W_s) = (s_1 \wedge t_1) \cdots (s_m \wedge t_m)$.

Now suppose our underlying probability space, $(\Omega, \mathcal{F}, P)$ is complete. We denote by $\mathscr{H}_m = \overline{\operatorname{span}}\,\{W(R_z) \ : \ z \in \mathbb{T}\}$ the closed Gaussian subspace of $L^2(\Omega, \mathcal{F}, P)$

whose elements are mean zero Guassian random variables. We have an isometry between $\mathcal{H}_m$ and the Hilbert space $L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \lambda)$, where λ is Lebesgue measure on $\mathbb{R}^d$. We consider $W(R)$ an $L^2(\Omega, \mathcal{F}, P)$-valued random signed measure on the parameter space $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, assigning independent random variables to disjoint Borel sets in the plane such that $W(R)$ is a normal mean zero random variable with variance given by $\lambda(R)$. We say W is an $L^2(\Omega)$-valued Gaussian measure on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$.

Let us define Hilbert space

$$L^2(\mathbb{T}^n) := \{f : \mathbb{T}^n \to \mathbb{R}, s.t. \ \|f\|_{L^2(\mathbb{T}^n)} < \infty\} \tag{3.6}$$

where the norm $\|\cdot\|_{L^2(\mathbb{T}^n)}$ is induced by the scalar product,

$$\langle f, g\rangle_{L^2(\mathbb{T}^n)} := \int_{\mathbb{T}^n} f(z_1, ..., z_n)g(z_1, ..., z_n)\lambda(z_1)...\lambda(z_n) \tag{3.7}$$

We first define the multiple stochastic integral for the class of special elementary functions.

Definition III.7. For fixed $n \geq 1$ let $\mathcal{S}_n^m$ denote the class of symmetric deterministic functions of the form

$$f(z_1, z_2, ..., z_n) = \sum_{i_1,...,i_n=1}^{k} a_{i_1...i_n} 1_{A_{i_1}\times...\times A_{i_n}}(z_1, ..., z_n)$$

where $A_1, ..., A_k$ are pairwise disjoint Borel sets in $\mathbb{T} \subset \mathbb{R}_+^m$ and $a_{i_1...i_n} = 0$ if any two of the indices $i_1, ...i_m$ are equal. So if $z_i = z_j$ for $i \neq j$ then f vanishes. For this reason we say that f vanishes along the "diagonals."

Definition III.8. For $f \in \mathcal{S}_n^m$, the multiple stochastic integral $I_m(f)$ of f is defined by

$$I_n(f) = \sum_{i_1,...,i_n=1}^{k} a_{i_1...i_n} W(A_{i_1})...W(A_{i_n}). \tag{3.8}$$

For non-symmetric functions $g \in L^2(\mathbb{T}^n)$, define the symmeterization of g as

$$\widetilde{g}(z_1, ..., z_n) = \frac{1}{n!} \sum_{\pi} g(z_{\pi(1)}, ..., z_{\pi(n)}), \tag{3.9}$$

where π is the set of all permutations of the set $\{1, 2, ..., n\}$.

Proposition III.9. *[21] The class $\mathcal{S}_n^m$ is dense in $L^2(\mathbb{T}^n)$.*

Proposition III.9 allows us to extend the integrals $I_n(f)$ to the class of all such functions f in $L^2(\mathbb{T}^n)$. For the remainder of the thesis, $I_n(f)$ will denote the multiple stochastic integral with respect to Brownian sheet, also known as the Wiener integral with respect to Brownian sheet. We will not specify the dimension of the parameter space of the Brownian sheet corresponding to the integral $I_n(f)$ but this should be clear from the context in which the integral is described.

Proposition III.10. *The following properties hold for $I_n(\cdot) : L^2(\mathbb{T}^n) \to \mathbb{R}$:*

1. *I_n is linear and $E[I_n(f)] = 0 \ \forall n \geq 1, \ f \in L^2(\mathbb{T}^n)$.*
2. *for any $f \in L^2(\mathbb{T}^n)$ the following relationship holds:*

$$I_n(f^{\otimes n}) = \left(\|f\|_{L^2(\mathbb{T}^n)}\right)^n \mathcal{H}_n\left(\frac{I_1^{R_{\mathbb{T}}}(f)}{\|f\|_{L^2(\mathbb{T})}}\right) \tag{3.10}$$

where the tensor product of a function, $f^{\otimes m}(z_1, ..., z_n) = f(z_1)f(z_2)...f(z_n)$.

3. *for $f \in L^2(\mathbb{T}^n)$ and $g \in L^2(\mathbb{T}^k)$, then*

$$E[I_n(g)I_k(f)] = \begin{cases} n!\langle \widetilde{f}, \widetilde{g} \rangle_{L^2(\mathbb{T}^n)} & \textit{if } n = k \\ 0 & \textit{if } n \neq k \end{cases} \tag{3.11}$$

Next, let us turn to stochastic integration with respect to *fractional* Brownian motion and *fractional* Brownian sheet. Recall that for $t, s \in \mathbb{R}_+, H \in (\frac{1}{2}, 1)$, we call the mean zero Gaussian process $B^H(t)$, with with covariance

$$R_H(t,s) := E\Big[B^H(t)B^H(s)\Big] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \tag{3.12}$$

a fractional Brownian motion.

We define the m-parameter fractional Brownian sheet in an analogous manner to the 2-parameter fractional Brownian sheet of Definition II.31.

Definition III.11. On a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_z), P)$, let $H = (H_1, ..., H_m) \in (0,1)^m$ and let $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_m]$. $B^H = \left(B^H_z, \mathcal{F}_z, z \in \mathbb{T}\right)$ is called a continuous version of a fractional Brownian sheet (field), if it satisfies the following conditions:

1. B^H is a random measure on $\mathbb{T}$, assigning to each (hyper) rectangle R, a Gaussian random variable of mean zero;
2. $B^H_z = B^H(R_z)$ where R_z is the rectangle whose upper right hand corner is z and whose "lower left hand corner" is the origin and B^H_z is adapted to $\mathcal{F}_z$;
3. B^H has continuous trajectories;
4. for all $z, y \in \mathbb{T}$ with $z = (z_1, ..., z_m),\ y = (y_1, ..., y_m)$, $\mathbb{E}(B^H_z B^H_y) = \prod_{i=1}^m R_{H_i}(z_i, y_i)$.

For $t, s \in \mathbb{R}_+$, define

$$\phi_H(t,s) := \frac{\partial^2}{\partial t \partial s} R_H(t,s) = H(2H-1)\,|t-s|^{2H-2}. \tag{3.13}$$

The stochastic integral with respect to fractional Brownian motion is constructed by first defining the integral for $\mathcal{S}_1^1$, a class of simple functions on $[0,T]$ of the form

$$f(s) = \sum_{i=1}^{n} a_i 1_{(c_i,d_i]}(s),$$

where $a_i \in \mathbb{R}$ are constants and $(c_i, d_i]$ $i = 1, ..., n$ are disjoint intervals in $[0,T]$.

Definition III.12. For $k \in \mathbb{N}_+$, define the space $\mathscr{H}_k^H$ by

$$\mathscr{H}_k^H := \overline{\text{span}}\left\{B^H(R_z) \ : \ z \in [0,T]^k\right\}$$

where R_z is the (hyper) rectangle with "lower left" endpoint at the origin and "upper right" endpoint at z. Equip this space with the scalar product

$$\langle X, Y\rangle_{\mathscr{H}_k^H} = \mathbb{E}(XY),$$

for any $X, Y \in \mathscr{H}_k^H$.

For $f \in \mathcal{S}_1^1$, define the Itô (Wiener) integral with respect to B^H as the map from $\mathcal{S}_1^1$ to $\mathscr{H}_1^H$ given by

$$\mathcal{I}_1^{B^H}(f) := \int_{[0,T]} f(s) dB_s^H := \sum_{i=1}^{n} a_i (B_{d_i}^H - B_{c_i}^H).$$

Definition III.13. Define the space $L_{H,1}^2([0,T])$ by

$$L_{H,1}^2([0,T]) = \left\{f : [0,T] \to \mathbb{R}, \ s.t. \ \left\|f\right\|_{L_{H,1}^2([0,T])} < \infty\right\}.$$

where the $L_{H,1}^2([0,T])$-norm is induced by the scalar product

$$\langle f, g\rangle_{L_{H,1}^2([0,T]^2)} := \int\int_{[0,T]} f(s)g(t)\phi_H(s,t) ds dt, \tag{3.14}$$

and $\phi_H(\cdot,\cdot)$ is defined by (3.13).

For all $f, g \in \mathcal{S}_1^1$, we have the isometry $\langle f, g\rangle_{L_{H,1}^2([0,T])} = \langle \mathcal{I}_1^{B^H}(f), \mathcal{I}_1^{B^H}(g)\rangle_{\mathscr{H}_1^H}$. Define Λ_1^H to be the completion of $\mathcal{S}_1^1$ under the norm induced by $\langle\cdot,\cdot\rangle_{L_{H,1}^2([0,T])}$, so that $\{\Lambda_1^H, \langle\cdot,\cdot\rangle_{L^2_{H,1}([0,T])}\}$ is a Hilbert space and $\mathcal{I}_1^{B^H}$ can be extended to an isometric isomorphism mapping Λ_1^H to $\mathscr{H}_1^H$. We will call this extension $\mathcal{I}_1^{B^H}$ as well. For notational simplicity we will often drop the superscript and simply write $\mathcal{I}_1$ whenever it is clear which process we are integrating with respect to.

Proposition III.14. *[23] For $H > \frac{1}{2}$, the class $L^2_{H,1}([0,T])$ is an incomplete linear subspace of Λ_1^H, and $\mathcal{S}_1^1 \subset L^2_H$. However, the closure $\overline{(L^2_{H,1}([0,T]),\ \langle\cdot,\cdot\rangle_{L^2_{H,1}([0,T])})}$ is equal to Λ_1^H and is a Hilbert space.*

Note 3.2.1. The elements of Λ_1^H may not be functions but rather generalized functions. For this reason, it is convenient to work with functions in $L^2_{H,1}([0,T])$.

Definition III.15. Let $\{e_i\}_{i=1}^\infty$ be a complete orthonormal system in $L^2_{H,1}([0,T])$. Define $\mathscr{S}_n^1$, the class of deterministic symmetric functions of the form:

$$f(t_1,...,t_n) = \sum_{1\le k_i\le k} a_{k_1,...,k_n} e_{k_1}(t_1)...e_{k_n}(t_n). \tag{3.15}$$

Definition III.16. For $f \in \mathscr{S}_n^1$, define the m-fold multiple Itô integral with respect to B^H as map $\mathcal{I}_n^{B^H}$ by

$$\mathcal{I}_n^{B^H}(f) := \sum_{1\le k_i\le k} a_{k_1,...,k_2} \mathcal{I}_1^{B^H}(e_{k_1}) \diamond ... \diamond \mathcal{I}_1^{B^H}(e_{k_n}), \tag{3.16}$$

where $\mathcal{I}_1^{B^H}(e_{k_j})$ is the stochastic integral of e_{k_j} with respect to fractional Brownian motion B^H. $\mathcal{I}_n^{B^H}$ is the nth order Wiener integral with respect to fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$. For non-symmetric f, $\mathcal{I}_n^{B^H}(f)$ is defined by $\mathcal{I}_n^{B^H}(\widetilde{f})$, where $\widetilde{f}$ is the symmetrization of f, defined by (3.9).

Definition III.17. Define the space of functions $L^2_{H,n}([0,T]^n)$ by

$$L^2_{H,n}([0,T]^n) := \left\{ f : [0,T]^n \to \mathbb{R} \text{ s.t. } \|f\|_{L^2_{H,n}([0,T]^n)} < \infty \right\} \tag{3.17}$$

where the norm $\|\cdot\|_{L^2_{H,n}([0,T]^n)}$ is induced by the scalar product

(3.18)

$$\langle f,g\rangle_{L^2_{H,n}([0,T]^n)} = \int_{[0,T]^{2n}} f(t_1,..,t_n)g(s_1,..,s_n)\phi_H(s_1,t_1)\cdots\phi_H(s_n,t_n)ds_1dt_1\cdot\cdot ds_ndt_n.$$

Note 3.2.2. Let $(L^2_{H,1}([0,T]))^{\otimes n}$ denote the *nth* symmetric tensor power of $L^2_{H,1}([0,T])$. Then $(L^2_{H,1}([0,T]))^{\otimes n} \subset L^2_{H,n}([0,T]^n)$.

Any $f \in (L^2_{H,1}([0,T]))^{\otimes n}$, can be written as the limit of functions $f_j \in \mathscr{S}^1_n$ where the limit is taken in the $\langle\cdot,\cdot\rangle_{L^2_{H,n}}$-sense. Hence for any $f \in (L^2_{H,1}([0,T]))^{\otimes n}$ the multiple stochastic integral $\mathcal{I}^{B^H}_n(f)$ can be defined uniquely as the limit (in the Gaussian space) of sequences of integrals of $\mathscr{S}^1_n$ functions, $\mathcal{I}^{B^H}_n(f_j)$, where $f_j \to f$. The integral can be extended to non-symmetric functions by defining $\mathcal{I}^{B^H}_n(f)$ by $\mathcal{I}^{B^H}_n(\widetilde{f})$ for non-symmetric functions f. We will henceforth denote the extension by $\mathcal{I}^{B^H}_n$.

Proposition III.18. *[22] For $H > \frac{1}{2}$, $L^2([0,T]^n)$ is a dense subclass of $L^2_{H,n}([0,T]^n)$.*

Finally, for the Itô integral with respect to fractional Brownian *sheet* and for the multiple Wiener integral with respect to fractional Brownian *sheet*, the development is very similar to the development of the corresponding integrals with respect to fractional Brownian motion. Define the space $\mathcal{S}^m_1$ of simple functions on $\mathbb{T}$, where $\mathbb{T}$ is in the m-dimensional hyperplane.

Definition III.19. Let f be a simple function in $\mathcal{S}^m_1$. Then we define the Itô integral of f with respect to fractional Brownian sheet B^H over rectangles $R_z \in \mathbb{T}$ by

$$\mathcal{I}^{B^H}_1(f) = \int_{R_z} f dB^H_y = \sum_i \alpha_i B^H(A_i \cap R_z)$$

Definition III.20. Let $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_m]$. Define the space of functions $L^2_{H,1}(\mathbb{T})$ by

$$L^2_{H,1}(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{R} \text{ s.t. } \|f\|_{L^2_{H,1}(\mathbb{T})} < \infty \right\} \tag{3.19}$$

where the norm $\|\cdot\|_{L^2_{H,1}(\mathbb{T})}$ is induced by the scalar product

$$\langle f, g\rangle_{L^2_{H,1}(\mathbb{T})} := \int_{\mathbb{T}^2} f(t_1, ..., t_m) g(s_1, ..., s_m) \prod_{i=1}^{m} \phi_H(t_i, s_i) dt_1 ds_1 \cdots dt_m ds_m, \tag{3.20}$$

$t_i, s_i \in [0, T_i]$ $\forall i$ and where $\phi_H(\cdot, \cdot)$ is given by (3.13).

For all $f, g \in \mathcal{S}_1^m$, we have the isometry $\langle f, g\rangle_{L^2_{H,1}(\mathbb{T})} = \mathbb{E}\big(\mathcal{I}_1^{B^H}(f)\mathcal{I}_1^{B^H}(g)\big)$. Define Λ_m^H to be the completion of $\mathcal{S}_1^m$ under the norm induced by $\langle \cdot, \cdot\rangle_{L^2_{H,1}(\mathbb{T})}$, so that $\{\Lambda_m^H, \langle \cdot, \cdot\rangle_{L^2_{H,1}(\mathbb{T})}\}$ is a Hilbert space and $\mathcal{I}_1^{B^H}$ can be extended to an isometric isomorphism mapping Λ_m^H to the $\mathscr{H}_1^H$. We will call this extension $\mathcal{I}_1^{B^H}(f)$ as well. For notational simplicity we will often drop the superscript and simply write $\mathcal{I}_1(f)$ whenever integrating process is clear.

Note 3.2.3. As was true in the case of Λ_1^H, the elements of Λ_m^H may not be functions but rather generalized functions. For this reason, it is convenient to work with functions in $L^2_{H,1}(\mathbb{T})$ which is a linear subspace of Λ_m^H containing $\mathcal{S}_1^m$ and satisfies $\overline{(L^2_{H,1}(\mathbb{T}),\ \langle \cdot, \cdot\rangle_{L^2_{H,1}(\mathbb{T})})} = \Lambda_m^H$.

Definition III.21. Let $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_m]$. Define the space of functions $L^2_{H,m}(\mathbb{T}^n)$ by

$$L^2_{H,m}(\mathbb{T}^n) := \left\{ f : \mathbb{T}^n \to \mathbb{R} \text{ s.t. } \|f\|_{L^2_{H,m}(\mathbb{T}^n)} < \infty \right\} \tag{3.21}$$

where the norm $\|\cdot\|_{L^2_{H,m}(\mathbb{T}^n)}$ is induced by the scalar product

(3.22)

$$\langle f, g\rangle_{L^2_{H,m}(\mathbb{T}^n)} := \int_{\mathbb{T}^{2n}} f(z^1, ..., z^n) g(y^1, ..., y^n) \Big(\prod_{i=1}^{n} \phi_H^{\otimes m}(z^i, y^i) \Big) dz^1 dy^1 \cdots dz^n dy^n,$$

$z^i = (z^i_1, ..., z^i_m), y^i = (y^i_1, ..., y^i_m)$ $\forall i$ and

$$\phi_H^{\otimes m}(z^r, y^r) = \prod_{j=1}^{m} \phi_H(z^r_j, y^r_j) \tag{3.23}$$

where $\phi_H(\cdot,\cdot)$ is given by (3.13).

Definition III.22. Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal system in $L^2_{H,1}(\mathbb{T})$. Define $\mathscr{S}^m_n$, the class of deterministic symmetric functions of the form:

$$f(z^1, ..., z^n) = \sum_{1 \le k_i \le k} a_{k_1,...,k_n} e_{k_1}(z^1)...e_{k_n}(z^n). \tag{3.24}$$

The superscript n and subscript m, indicate that f is a function of an $m \times n$ matrix where the ith column is z^i.

Denote the nth symmetric tensor power of $L^2_{H,1}(\mathbb{T})$, by $L^2_{H,1}(\mathbb{T})^{\otimes n}$

Definition III.23. For $h \in \mathscr{S}^m_n$, define the map $\mathcal{I}^{B^H}_n(h)$ by

$$\mathcal{I}^{B^H}_n(h) := \sum_{1 \le k_i \le k} a_{k_1,...,k_2} \mathcal{I}^{B^H}_1(e_{k_1}) \diamond ... \diamond \mathcal{I}^{B^H}_1(e_{k_n}), \tag{3.25}$$

where $\mathcal{I}^{B^H}_1(\cdot)$ is the stochastic integral with respect to fractional Brownian sheet B^H. $\mathcal{I}^{B^H}_n(h)$ is the nth order Wiener integral with respect to fractional Brownian sheet with Hurst parameter $H \in (\frac{1}{2}, 1)^n$.

Any $g \in (L^2_{H,1}(\mathbb{T}))^{\otimes n}$, can be written as the limit of functions $g_j \in \mathscr{S}^m_n$ where the limit is taken in the $\langle \cdot,\cdot \rangle_{L^2_{H,n}(\mathbb{T}^n)}$-sense. Hence for any $g \in (L^2_{H,n}(\mathbb{T}))^{\otimes n}$ the multiple stochastic integral $\mathcal{I}^{B^H}_n(g)$ with respect to fractional Brownian sheet B^H can be defined uniquely as the limit (in the Gaussian space) of sequences of integrals, $\mathcal{I}^{B^H}_n(g_j)$, where $g_j \to g$ (in $\langle \cdot,\cdot \rangle_{L^2_{H,n}(\mathbb{T}^n)}$-sense). The integral can be extended to non-symmetric function by defining $\mathcal{I}^{B^H}_n(g)$ by $\mathcal{I}^{B^H}_n(\widetilde{g})$. We will denote the extension by $\mathcal{I}^{B^H}_n$.

The final stochastic integral we will need is the multiple fractional integral of the *Stratonovich* type.

Definition III.24. Let $g \in \mathscr{S}_n^m$. The nth order Stratonovich integral of g with respect to fractional Brownian sheet B^H is defined by

$$\mathfrak{I}_n^{B^H}(g) := \sum_{1 \le k_i \le k} a_{k_1,\ldots,k_n} \mathcal{I}_1^{B^H}(e_{k_1}) \cdots \mathcal{I}_1^{B^H}(e_{k_n}), \tag{3.26}$$

where $\mathcal{I}_1^{B^H}(\cdot)$ is the Itô integral with respect to fractional Brownian sheet B^H.

Again, since any $f \in (L^2_{H,1}(\mathbb{T}))^{\otimes n}$ can be written as the limit of functions $f_j \in \mathscr{S}_n^m$ where the limit is taken in the $\langle \cdot, \cdot \rangle_{L^2_{H,n}(\mathbb{T}^n)}$-sense, we can again extend the integral mapping (3.26) to all functions $\mathfrak{I}_n^{B^H} : (L^2_{H^m})^{\otimes n} \to L^2(\Omega)$. For non-symmetric $f \in L^2_{H,n}(\mathbb{T}^n)$ we define $\mathfrak{I}_n^{B^H}(f) := \mathfrak{I}_n^{B^H}(\widetilde{f})$.

We will make use of the multiparameter version of the *fractional Hu-Meyer formula* [8] which describes the relationship between the multiple stochastic integrals of Stratonovich type and multiple stochastic integrals of the (Wiener) Itô-type.

Proposition III.25. *For symmetric function $f \in L^2_{H^m}(\mathbb{T}^n)$, we define the trace $Tr^k_\phi(f)$ $\forall u^1, \ldots, u^{n-2k} \in R_z$,*

$$\begin{aligned} &Tr^k_\phi(f)(u^1, \ldots, u^{n-2k}) \\ := &\int_{\mathbb{T}2k} f(s^1, t^1, \ldots, s^k, t^k, u^1, \ldots, u^{n-2k}) \left[\prod_{\ell=1}^k \phi_H^{\otimes m}(s^\ell, t^\ell) \right] ds^1 dt^1 \ldots ds^k dt^k. \end{aligned} \tag{3.27}$$

For $H \in (\frac{1}{2}, 1)$ the following hold

$$\mathfrak{I}_n^{B^H}(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{k,n} \mathcal{I}_{n-2k}^{B^H}(Tr^k_\phi f) \quad a.s.,$$

$$\mathcal{I}_n^{B^H}(g) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k c_{k,n} \mathfrak{I}_{n-2k}^{B^H}(Tr^k_\phi g) \quad a.s.$$

where $c_{k,n} = \frac{n!}{k!(n-2k)!2^k}$. By convention, $Tr^0_\phi(f) = 0$ and $\mathcal{I}_n^{B^H} = \mathfrak{I}_0^{B^H} = Id$, the identity operator.

Throughout the remainder of the dissertation, we will make extensive use of the integrals defined in this section. In this chapter we will use the integrals in multidimensional parameter space to derive representations of the optimal filter described in Chapter II, where we extend the filtering to general m-parameter space for all finite $m \geq 1$. In Chapter IV we will study properties related to the multiple fractional integrals in the 1-dimensional parameter space.

3.3 Filtering model in m-dimensional parameter space

We will consider the same filtering model as discussed in Chapter II. However, in this chapter, since we are not concerned with evolution equations describing the dynamics of the filter, we can extend the model to the general m-dimensional parameter space without significantly complicating notation. For the sake of completeness, we will state the new model and the underlying assumption in the m-dimension space even though the extensions should be clear.

Throughout the analysis, we consider a fixed, complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_z), P)$ where filtration $(\mathcal{F}_z)$ satisfies the following conditions:

F(i) If $z \preceq z'$ then $\mathcal{F}_z \subset \mathcal{F}_{z'}$;

F(ii) $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$;

F(iii) $\mathcal{F}_z = \bigcap_{z \prec z'} \mathcal{F}_{z'}$;

F(iv) The sigma-fields $\mathcal{F}_z^1, ..., \mathcal{F}_z^m$ are mutually independent conditional on $\mathcal{F}_z$, where

$$\mathcal{F}_z^i := \bigvee_{\substack{t_j \geq 0 \\ j \neq i}} \mathcal{F}_{(t_1, \ldots z_i, \ldots t_m)}.$$

On the probability space, consider m-parameter random fields $X = (X_z, z \in \mathbb{T})$, $Y = (Y_z, z \in \mathbb{T})$ and $B^H = (B_z^H, z \in \mathbb{T})$ where $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_m]$, $m \geq 2$.

Assume each of these processes is measurable, $(\mathcal{F}_z)$-adapted. Instead of directly observing X, suppose that a noisy field Y is observed and that the observation model is given by:

$$Y_z = \int_{R_z} h(X_\zeta)d\zeta + B_z^H, \quad z \in \mathbb{T}, \tag{3.28}$$

where R_z denotes a rectangle $[0, z_1) \times \cdots \times [0, z_m)$ for $z = (z_1, \ldots, z_m) \in \mathbb{T}$, h is a measurable function, process $B^H = (B_z^H, z \in \mathbb{T})$ is an m-parameter fractional Brownian sheet with Hurst index $H = (H_1, \ldots, H_m) \in (\frac{1}{2}, 1)^m$ (thus, the fBs is persistent, i.e. has long-range dependence in all spatial directions). Finally, we make the usual assumption that "signal" X is independent of the observation noise B^H.

As in Chapter II, the goal is to characterize the best mean square estimate of the signal given the σ-field generated by the observation process. Equivalently, the goal is to identify $\mathbb{E}[f(X_z)|\mathcal{F}_z^Y]$ where $\mathcal{F}_z^Y = \sigma(\{Y_\zeta, \underline{0} \prec \zeta \prec z\})$. However, in this chapter we will be concerned with the estimation and representation of $\mathbb{E}[f(X_z)|\mathcal{F}_z^Y]$ rather than studying it's dynamics.

Introduce the following conditions on the function $h : \mathbb{R} \to \mathbb{R}$, analogous to the conditions imposed on h in Chapter II:

(A_1^m) h is Hölder-continuous of order λ on any finite subinterval in $\mathbb{R}$, where $\lambda > 2\max(H_1, \ldots, H_m) - 1$; and

(A_2^m) For $h_z^*(X)(\omega) := (\prod_{j=1}^m z_j^{\frac{1}{2}-H_j})h(X_z(\omega))$ (where $z = (z_1, \ldots, z_m) \in \mathbb{T}$, $\omega \in \Omega$),

$$\int_{\mathbb{T}} (\prod_{j=1}^m \zeta_j^{2H_j-1})\mathbb{E}\Big[\big(\mathcal{D}_{0+,\ldots,0+}^{H_1-\frac{1}{2},\ldots,H_m-\frac{1}{2}} h_\cdot^*(X)\big)(\zeta)\Big]^2 d\zeta < \infty, \tag{3.29}$$

where $\mathcal{D}^{\alpha_1,\ldots,\alpha_m}_{a_1+,\ldots,a_m+}$ denotes the left-handed mixed fractional Riemann-Liouville derivative of order $\alpha = (\alpha_1, \ldots, \alpha_m) \in (0,1)^m$ defined by

$$\begin{aligned}&(\mathcal{D}^{\alpha_1,\ldots,\alpha_m}_{a_1+,\ldots,a_m+} f)(x)\\ =\ &\frac{1}{\prod_{j=1}^m \Gamma(1-\alpha_j)} \frac{\partial^m}{\partial x_1 \ldots \partial x_m} \int_{a_1}^{x_1} \cdots \int_{a_m}^{x_m} \frac{f(\zeta_1, \ldots, \zeta_m)}{\prod_{j=1}^m (x_j - \zeta_j)^{\alpha_j}} d\zeta_1 \ldots d\zeta_m,\end{aligned}$$

for all $x = (x_1, \ldots, x_m)$ (where $a = (a_1, ..., a_m) \prec x$).

Also, introduce the following condition on the signal:

(A_3^m) X has Hölder-continuous trajectories of order $(\lambda_0, \ldots, \lambda_0)$, where

$$\lambda_0 > \frac{\max\{H_1, \ldots, H_m\} - \frac{1}{2}}{\lambda},$$

where λ is as given in condition (A_1) above.

For almost all $\omega \in \Omega$ define function $(\delta_z(X), z = (z_1, \ldots, z_m) \in \mathbb{T})$ by

$$\delta_z(X)(\omega) := \frac{1}{\prod_{j=1}^m c^*_{H_j}} \big(\prod_{j=1}^m z_j^{H_j - \frac{1}{2}}\big) \big(\mathcal{D}^{H_1-\frac{1}{2},\ldots,H_m-\frac{1}{2}}_{0+,\ldots,0+} h^*_\cdot(X)(\omega)\big)(z), \quad z \in \mathbb{T}, \tag{3.30}$$

where $c^*_{H_j} = \Gamma(H_j + \frac{1}{2}) \sqrt{\frac{2H_j \Gamma(\frac{3}{2} - H_j)}{\Gamma(H_j + \frac{1}{2}) \Gamma(2 - 2H_j)}}$. By the same argument as in [2], one can show the following property:

$$\int_{R_z} \big(\prod_{j=1}^m K^{-1}_{H_j}(z_j; \zeta_j)\big) h(X_\zeta) d\zeta = \int_{R_z} \delta_\zeta(X) d\zeta \ \ a.s.,$$

where

$$K^{-1}_{H_j}(t;s) = c'_{H_j} \left(\left(\frac{t}{s}\right)^{H_j - \frac{1}{2}} (t-s)^{\frac{1}{2} - H_j} - (H_j - \frac{1}{2}) s^{\frac{1}{2} - H_j} \int_s^t u^{H_j - \frac{3}{2}} (u-s)^{\frac{1}{2} - H_j} du \right), \tag{3.31}$$

where $c'_{H_j} = \frac{1}{\Gamma(\frac{3}{2} - H_j)} \sqrt{\frac{\Gamma(2 - 2H_j)}{2H_j \Gamma(\frac{3}{2} - H_j) \Gamma(H_j + \frac{1}{2})}}$.

Next define the following random fields:

$$W_z^Y := \int_{R_z} \prod_{j=1}^{m} K_{H_j}^{-1}(z_j;\zeta_j)dY_\zeta, \quad z \in \mathbb{T}, \tag{3.32}$$

$$V_z = \exp\left\{ - \int_{R_z} \delta_\zeta(X)dW_\zeta^Y + \frac{1}{2}\int_{R_z} (\delta_\zeta(X))^2 d\zeta \right\}, \quad z \in \mathbb{T}. \tag{3.33}$$

Then, as in Chapter II:

Lemma III.26. *Let $\tilde{P}$ be a new probability measure on $(\Omega, \mathcal{F})$ given by:*

$$\frac{d\tilde{P}}{dP} = V_{(T_1,\dots,T_m)} \;\; a.s.(P) \tag{3.34}$$

Then $\tilde{P}$ is equivalent to P and, under $\tilde{P}$, (3.28) *holds a.s., Y is a standard fBs with Hurst multi-index $H = (H_1, \dots, H_m)$, X has the same law as under P, and processes X and Y are independent under $\tilde{P}$. Moreover, the following "spatial-fractional" version of the Bayes' formula holds: For any $F \in C_b(\mathbb{R})$,*

$$\mathbb{E}\big(F(X_z)|\mathcal{F}_z^Y\big) = \frac{\tilde{\mathbb{E}}\big[F(X_z)V_z^{-1}|\mathcal{F}_z^Y\big]}{\tilde{\mathbb{E}}\big[V_z^{-1}|\mathcal{F}_z^Y\big]} \;\; a.s. \tag{3.35}$$

where $\tilde{\mathbb{E}}$ denotes the mathematical expectation under $\tilde{P}$.

3.4 Multiparameter strong martingale transforms associated to a persistent fractional Brownian sheet

Denote the space of continuous, real-valued functions mapping $[0, T_1]\times\cdots\times[0, T_m]$ to $\mathbb{R}$ by $C([0, T_1] \times \cdots \times [0, T_m], \mathbb{R})$.

Lemma III.27. *For arbitrary $\mathcal{C} \in C([0, T_1]\times\cdots\times[0, T_m], \mathbb{R})$ and $\forall z = (z_1, \dots, z_m) \in [0, T_1]\times\cdots\times[0, T_m]$, $\forall \zeta = (\zeta_1, \dots, \zeta_m) \in (0, z_1)\times\cdots\times(0, z_m)$, $\forall H = (H_1, \dots, H_m) \in (0.5, 1)^m$, define kernel $k_{\mathcal{C};H}^z(\cdot)$ by*

$$\begin{aligned} k_{\mathcal{C};H}^z(\zeta) \;\; &= \;\; \alpha_H(\zeta)\frac{\partial^m}{\partial\zeta_1\dots\partial\zeta_m}\Big[\int_{\zeta_1}^{z_1}\dots\int_{\zeta_m}^{z_m} \mathcal{A}_{\mathcal{C};H}(v_1, \dots, v_m) \\ &\qquad \times \prod_{j=1}^{m} v_j^{2H_j-1}(v_j - \zeta_j)^{\frac{1}{2}-H_j} dv_1 \dots dv_m\Big], \end{aligned} \tag{3.36}$$

where

$$\mathcal{A}_{\mathcal{C};H}(v) \;:=\; \frac{\partial^m}{\partial v_1 \dots \partial v_m} \int_0^{v_1} \dots \int_0^{v_m} \Big(\prod_{j=1}^m u_j^{\frac{1}{2}-H_j}(v_j - u_j)^{\frac{1}{2}-H_j}\Big) \times \mathcal{C}(u_1, \dots, u_m) du_1 \dots du_m,$$

$$\alpha_H(\zeta) := \frac{(-1)^m}{\rho_H} \prod_{j=1}^m \zeta_j^{\frac{1}{2}-H_j}, \quad \rho_H := \prod_{j=1}^m \Big[\Gamma\big(\frac{3}{2} - H_j\big)\Big]^2 \Gamma(2H_j + 1) \sin(\pi H_j).$$

Then the following equation holds: $\forall 0 < r_j < z_j$,

$$(3.37) \qquad \int_0^{z_1} \dots \int_0^{z_m} k^z_{\mathcal{C};H}(v_1, \dots, v_m) \prod_{j=1}^m \phi_{H_j}(v_j, r_j) dv_1 \dots dv_m = \mathcal{C}(r_1, \dots, r_m),$$

for all $z = (z_1, \dots, z_m) \in [0, T_1] \times \dots \times [0, T_m]$, *where*

$$\phi_{H_j}(v_j, r_j) := H_j(2H_j - 1)|v_j - r_j|^{2H_j - 2}, \;\; j = 1, \dots, m.$$

Moreover, for all $z, z' \in \mathbb{T}$,

$$(3.38) \qquad \int_{R_z \times R_{z'}} k^z_{\mathcal{C};H}(v) k^{z'}_{\mathcal{C};H}(v') \phi_H(v, v') dv dv' = \int_{R_{z \wedge z'}} \mathcal{C}(\zeta) k^{z \wedge z'}_{\mathcal{C};H}(\zeta) d\zeta,$$

where $z \wedge z' := (\min(z_1, z'_1), \dots, \min(z_m, z'_m))$, $R_z := [0, z_1) \times \dots \times [0, z_m)$ *for all* $z = (z_1, \dots, z_m) \in \mathbb{T}$, *and*

$$\phi_H(z, z') := \prod_{j=1}^m \phi_{H_j}(z_j, z'_j).$$

Proof: First we show that (3.37) holds for arbitrary $\mathcal{C} \in C([0, T_1] \times \dots \times [0, T_m], \mathbb{R})$ of the form $\mathcal{C}(r_1, \dots, r_m) = \mathcal{C}_1(r_1, \dots, r_{m-1})\mathcal{C}_2(r_m)$. When $m = 1$, (3.37) is valid by Proposition II.35. Assume that (3.37) is true for an arbitrary continuous function of $m - 1$ variables. Then

$$\int_0^{z_1} \dots \int_0^{z_m} k^z_{\mathcal{C};H}(v_1, \dots, v_m) \prod_{j=1}^m \phi_{H_j}(v_j, r_j) dv_1 \dots dv_m$$

$$= \Big[\int_0^{z_1} \dots \int_0^{z_{m-1}} k^{(z_1, \dots, z_{m-1})}_{\mathcal{C}_1;(H_1, \dots, H_{m-1})}(v_1, \dots, v_{m-1}) \prod_{j=1}^{m-1} \phi_{H_j}(v_j, r_j) dv_1 \dots dv_{m-1}\Big]$$

$$\times \Big[\int_0^{z_m} k^{z_m}_{\mathcal{C}_2;H_m}(v_m)\phi_{H_m}(v_m, r_m)dv_m \Big]$$
$$= \mathcal{C}_1(r_1, \dots, r_{m-1})\mathcal{C}_2(r_m) = \mathcal{C}(r_1, \dots, r_m)$$

for all $0 < r_j < z_j$, $j = 1, \dots, m$. Thus, the desired result holds by induction. In the case of a more general $\mathcal{C}$, the required equality follows by the polarization technique. To prove (3.38) we first use (3.37) from which it follows that for $s < t$,

$$\int_0^s \int_0^t K^s_{\mathcal{C}_2;H} \Big[K^t_{\mathcal{C}_2;H}(v)\phi(v, r)drdv \Big] dr = \int_0^s K^s_{\mathcal{C}_2;H}\mathcal{C}_2(r)dr,$$

and hence, for $m = 1$, (3.38) holds.

Next we show that (3.38) holds for arbitrary $\mathcal{C} \in C([0, T_1] \times \cdots \times [0, T_m], \mathbb{R})$ of the form $\mathcal{C}(r_1, \dots, r_m) = \mathcal{C}_1(r_1, \dots, r_{m-1})\mathcal{C}_2(r_m)$. Assume (3.38) holds for arbitrary continuous function of $m-1$ variables. Then

$$\int_{R_z \times R_{z'}} k^z_{\mathcal{C};H}(v)k^{z'}_{\mathcal{C};H}(v')\phi_H(v, v')dvdv'$$
$$= \Big[\int_0^{z_1} \cdots \int_0^{z_{m-1}} \int_0^{z'_1} \cdots \int_0^{z'_{m-1}} k^{(z_1, \cdot\cdot, z_{m-1})}_{\mathcal{C}_1;(H_1, \dots, H_{m-1})}(v_1, .., v_{m-1})k^{(z'_1, \cdot\cdot, z'_{m-1})}_{\mathcal{C}_1;(H_1, \dots, H_{m-1})}(v'_1, .., v'_{m-1})$$
$$\times \prod_{j=1}^{m-1} \phi_{H_j}(v_j, v'_j)dv'_1 \dots dv'_{m-1}dv_1 \dots dv_{m-1} \Big]$$
$$\times \Big[\int_0^{z_m} \int_0^{z'_m} k^{z_m}_{\mathcal{C}_2;H_m}(v_m)k^{z'_m}_{\mathcal{C}_2;H_m}(v'_m)\phi_{H_m}(v_m, v'_m)dv_m \Big]$$
$$= \Big[\int_0^{z_1 \wedge z'_1} \cdot\cdot \int_0^{z_{m-1} \wedge z'_{m-1}} k^{(z_1, .., z_{m-1}) \wedge (z'_1, .., z'_{m-1})}_{\mathcal{C}_1;(H_1, .., H_{m-1})}(v_1, .., v_{m-1})\mathcal{C}_1(v_1, .., v_{m-1})dv_1 \cdot\cdot dv_{m-1} \Big]$$
$$\times \Big[\int^{z_m \wedge z'_m} \mathcal{C}_2(v_m)k^{z_m \wedge z'_m}_{\mathcal{C}_2;H}(v_m)dv_m \Big]$$
$$= \int_0^{z_1 \wedge z'_1} \cdots \int_0^{z_m \wedge z'_m} k^{z \wedge z'}_{\mathcal{C};H}(v_1, \dots, v_m)\mathcal{C}_1(v_1, \dots, v_{m-1})\mathcal{C}_2(v_m)dv_1 \dots dv_m$$
$$= \int_{R_{z \wedge z'}} \mathcal{C}(\zeta)k^{z \wedge z'}_{\mathcal{C};H}(\zeta)d\zeta.$$

The result follows by induction and the polarization technique.

□

Next we present the multiparameter version of Proposition II.36, which will be used in the multiple integral expansions of the optimal filter.

Theorem III.28. *Let $\mathbb{T} = [0, T_1] \times \cdots \times [0, T_m]$ and let $(B_z^H, z \in \mathbb{T})$ be a normalized m-parameter fractional Brownian sheet of Hurst index $H = (H_1, \ldots, H_m) \in (0.5, 1)^m$, defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_z)_{z\in\mathbb{T}}, P)$. Let*

$$N_z^{\mathcal{C}} := \int_{[0,z_1)\times\cdots\times[0,z_m)} k_{\mathcal{C};H}^z(\zeta) dB_\zeta^H \tag{3.39}$$

for all $z = (z_1, \ldots, z_m) \in \mathbb{T}$, where $k_{\mathcal{C};H}^z(\cdot)$ is given by (3.36). Then $N^{\mathcal{C}} = (N_z^{\mathcal{C}})_{z\in\mathbb{T}}$ is a Gaussian strong m-parameter martingale with variance function given by

$$\langle N^{\mathcal{C}} \rangle_z = \int_{[0,z_1)\times\cdots\times[0,z_m)} k_{\mathcal{C};H}^z(\zeta)\mathcal{C}(\zeta) d\zeta,$$

and $(\mathcal{F}_z^{N^{\mathcal{C}}}) = (\mathcal{F}_z^{B^H})$, where $(\mathcal{F}_z^{N^{\mathcal{C}}})$ and $(\mathcal{F}_z^{B^H})$ are the natural filtrations generated by the processes $N^{\mathcal{C}}$ and B^H, respectively. Moreover, let $\mathrm{id}(z) \equiv 1$, i.e. $\mathrm{id}(\cdot)$ denotes a function on $\mathbb{T}$ which is identically equal to 1. Then

$$\langle N^{\mathcal{C}}, N^{\mathrm{id}} \rangle_z \equiv \mathrm{Cov}(N_z^{\mathcal{C}}, N_z^{\mathrm{id}}) = \int_{[0,z_1)\times\cdots\times[0,z_m)} k_{\mathrm{id};H}^z(\zeta)\mathcal{C}(\zeta) d\zeta, \tag{3.40}$$

where $k_{\mathrm{id};H}^z(\cdot)$ reduces to

$$k_{\mathrm{id};H}^z(\zeta) = \prod_{j=1}^m \frac{\zeta_j^{\frac{1}{2}-H_j}(z_j - \zeta_j)^{\frac{1}{2}-H_j}}{2H_j\Gamma(\frac{3}{2} - H_j)\Gamma(H_j + \frac{1}{2})}. \tag{3.41}$$

Moreover,

$$N_z^{\mathcal{C}} = \int_{[0,z_1)\times\cdots\times[0,z_m)} q_\zeta^{\mathcal{C}} dN_\zeta^{\mathrm{id}} \quad \text{and} \quad \langle N^{\mathcal{C}} \rangle_z = \int_{[0,z_1)\times\cdots\times[0,z_m)} \left(q_\zeta^{\mathcal{C}}\right)^2 d\langle N^{\mathrm{id}} \rangle_\zeta, \tag{3.42}$$

where

$$q_z^{\mathcal{C}} := \frac{d\langle N^{\mathcal{C}}, N^{\mathrm{id}} \rangle_z}{d\langle N^{\mathrm{id}} \rangle_z}. \tag{3.43}$$

Proof: Since $N_z^{\mathcal{C}}$ and N_z^{id} are, by definition, integrals with respect to fractional Brownian sheet, B^H, $N^{\mathcal{C}}$ is a mean-zero Gaussian process with

$$\begin{aligned}
\mathbb{E}\big(N_z^{\mathcal{C}} N_{z'}^{\mathcal{C}}\big) &= \Big\langle k_{\mathcal{C};H}^{z} 1_{R_z}, k_{\mathcal{C};H}^{z'} 1_{R_{z'}} \Big\rangle_{L^2_{H,1}(\mathbb{T})} \\
&= \int_{R_z \times R_{z'}} k_{\mathcal{C};H}^{z}(\zeta) k_{\mathcal{C};H}^{z'}(\zeta') \phi_H(\zeta,\zeta') d\zeta d\zeta' \\
&= \int_{R_{z\wedge z'}} \mathcal{C}(v) k_{\mathcal{C};H}^{z\wedge z'}(v) dv, \quad \forall z, z' \in \mathbb{T}, \qquad (3.44)
\end{aligned}$$

by Lemma III.27.

$$\begin{aligned}
\mathrm{Cov}(N_z^{\mathcal{C}}, N_z^{\mathrm{id}}) &= \big\langle k_{\mathcal{C};H}^{z} 1_{R_z}, k_{\mathrm{id};H}^{z} 1_{R_z} \big\rangle_{L^2_{H,1}(\mathbb{T})} \\
&= \int_{R_z \times R_z} k_{\mathcal{C};H}^{z}(\zeta) k_{\mathrm{id};H}^{z}(\zeta') \phi_H(\zeta,\zeta') d\zeta d\zeta' \\
&= \int_{R_z} \left\{ \int_{R_z} k_{\mathcal{C};H}^{z}(\zeta) \phi_H(\zeta,\zeta') d\zeta \right\} k_{\mathrm{id};H}^{z}(\zeta') d\zeta' \\
&= \int_{R_z} \mathcal{C}(\zeta') k_{\mathrm{id};H}^{z}(\zeta') d\zeta'
\end{aligned}$$

by Lemma III.27, where $k_{\mathrm{id};H}^{z}$ satisfies (3.41), since

$$\begin{aligned}
\mathcal{A}_{\mathrm{id};H}(v) &= \frac{\partial^m}{\partial v_1 \dots \partial v_m} \int_0^{v_1} \cdots \int_0^{v_m} \prod_{j=1}^{m} u_j^{\frac{1}{2}-H_j} (v_j - u_j)^{\frac{1}{2}-H_j} du_1 \dots du_m \\
&= \prod_{j=1}^{m} \frac{\partial}{\partial v_j} \int_0^{v_j} u_j^{\frac{1}{2}-H_j} (v_j - u_j)^{\frac{1}{2}-H_j} du_j \\
&= \prod_{j=1}^{m} \frac{\partial}{\partial v_j} v_j^{2-2H_j} B\big(\frac{3}{2} - H_j, \frac{3}{2} - H_j\big) \\
&= \prod_{j=1}^{m} \frac{(2-2H_j)\big[\Gamma(\frac{3}{2} - H_j)\big]^2}{\Gamma(3-2H_j)} v_j^{1-2H_j} \equiv \hat{c}_H \prod_{j=1}^{m} v_j^{1-2H_j},
\end{aligned}$$

with

$$\hat{c}_H = \prod_{j=1}^{m} \frac{(2-2H_j)\big[\Gamma(\frac{3}{2} - H_j)\big]^2}{\Gamma(3-2H_j)},$$

which implies that

$$\begin{aligned}
k^z_{\mathrm{id};H}(\zeta) &= \alpha_H(\zeta)\frac{\partial^m}{\partial\zeta_1\ldots\partial\zeta_m}\Big[\int_{\zeta_1}^{z_1}\cdots\int_{\zeta_m}^{z_m}\hat{c}_H\big(\prod_{j=1}^m v_j^{1-2H_j}\big)\\
&\quad\times\bigg(\prod_{j=1}^m v_j^{2H_j-1}(v_j-\zeta_j)^{\frac{1}{2}-H_j}\bigg)dv_1\ldots dv_m\Big]\\
&= \hat{c}_H\alpha_H(\zeta)\prod_{j=1}^m\frac{\partial}{\partial\zeta_j}\int_{\zeta_j}^{z_j}(v_j-\zeta_j)^{\frac{1}{2}-H_j}dv_j\\
&= \hat{c}_H\alpha_H(\zeta)\prod_{j=1}^m\frac{\partial}{\partial\zeta_j}\left[\frac{(z_j-\zeta_j)^{\frac{3}{2}-H_j}}{\frac{3}{2}-H_j}\right]\\
&= \hat{c}_H\frac{(-1)^m}{\rho_H}\big(\prod_{k=1}^m\zeta_k^{\frac{1}{2}-H_k}\big)(-1)^m\prod_{j=1}^m(z_j-\zeta_j)^{\frac{1}{2}-H_j}\\
&= \frac{\hat{c}_H}{\rho_H}\prod_{j=1}^m\zeta_j^{\frac{1}{2}-H_j}(z_j-\zeta_j)^{\frac{1}{2}-H_j},
\end{aligned}$$

where

$$\frac{\hat{c}_H}{\rho_H}=\prod_{j=1}^m\frac{2-2H_j}{\Gamma(3-2H_j)\Gamma(2H_j+1)\sin(\pi H_j)}=\prod_{j=1}^m\frac{1}{2H_j\Gamma(H_j+\frac{1}{2})\Gamma(\frac{3}{2}-H_j)}$$

the last equality follows from the relations $\Gamma(z+1)=z\Gamma(z)$, $\Gamma(z)\Gamma(1-z)=\pi/\sin(z\pi)$ and the Legendre formula:

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

Representation (3.41) is thus proved.

Next we show that $N^{\mathcal{C}}$ is an m-parameter strong martingale. For $z\prec z'$, let $(z,z']:=(z_1,z_1']\times\cdots\times(z_m,z_m']\}$, and let $N^{\mathcal{C}}((z,z'])$ be the increment of $N^{\mathcal{C}}$ over $(z,z']$ as defined by (3.5). Then for all $\zeta=(\zeta_1,...,\zeta_m)\in[0,T_1]\times\cdots\times[0,T_m]$ such that $0\le\zeta_i\le z_i$ for at least one i, it follows by (3.44) that

$$\mathbb{E}\left[N^{\mathcal{C}}((z,z'])N^{\mathcal{C}}_\zeta\right]=0. \tag{3.45}$$

Since $N^{\mathcal{C}}((z,z'])$ and $N^{\mathcal{C}}_{\zeta}$ are mean-zero Gaussian random variables, (3.45) implies that $N^{\mathcal{C}}((z,z'])$ and $N^{\mathcal{C}}_{\zeta}$ are independent for all ζ such that $\zeta_i \leq z_i$ for at least one $i \in \{1,...,m\}$. Therefore, $N^{\mathcal{C}}((z,z'])$ is independent of $\mathcal{F}_z^{N^{\mathcal{C}},1} \vee \cdots \vee \mathcal{F}_z^{N^{\mathcal{C}},m}$ for all $z \prec z'$, where $\mathcal{F}_z^{N^{\mathcal{C}},i} := \sigma(N^{\mathcal{C}}_{\zeta} : \zeta \in [0,T_1] \times \cdots \times [0,z_i] \times \cdots \times [0,T_m])$. Thus,

$$\mathbb{E}\left[N^{\mathcal{C}}((z,z'])\middle|\mathcal{F}_z^{N^{\mathcal{C}},1} \bigvee \cdots \bigvee \mathcal{F}_z^{N^{\mathcal{C}},m}\right] = \mathbb{E}\left[N^{\mathcal{C}}(z,z'])\right] = 0 \text{ a.s. } \forall z \prec z'. \tag{3.46}$$

Moreover, note that $\mathcal{F}_z^{N^{\mathcal{C}}} = \mathcal{F}_z^{B^H} = \mathcal{F}_z^{W^b}$, where W is the standard Wiener sheet associated with B^H, via the integral transform $W_z^B = \int_{R_z} \prod_{i=1}^m K_{H_1}^{-1}(z_i;\zeta_i) dB_{\zeta}^H$, $z \in \mathbb{T}$. Therefore, $(\mathcal{F}_z^{N^{\mathcal{C}}})_{z\in\mathbb{T}}$ satisfies conditions $F(i)$, $F(ii)$, $F(iii)$ and $F(iv)$ of the multiparameter martingale theory. By (3.46) we conclude that $N^{\mathcal{C}}$ is a two-parameter strong martingale.

Since $N^{\mathcal{C}}$ and N^{id} are Gaussian strong martingales with $(\mathcal{F}_z^{N^{\mathcal{C}}}) = (\mathcal{F}_z^{W^B}) = (\mathcal{F}_z^{N^{\mathrm{id}}})$, then, by the multiparameter version of martingale representation theorem, there exists $q^{\mathcal{C}} = (q_z^{\mathcal{C}}, z \in \mathbb{T})$ such that $\int_{\mathbb{T}} \left(q_z^{\mathcal{C}}\right)^2 d\langle N^{\mathrm{id}}\rangle_z < \infty$ and

$$N_z^{\mathcal{C}} = \int_{R_z} q_{\zeta}^{\mathcal{C}} dN_{\zeta}^{\mathrm{id}}, \quad \forall z \in \mathbb{T}. \tag{3.47}$$

It follows that

$$\langle N^{\mathcal{C}}, N^{\mathrm{id}}\rangle_z = \int_{R_z} q_{\zeta}^{\mathcal{C}} d\langle N^{\mathrm{id}}\rangle_{\zeta}, \quad \text{which implies } \; q_z^{\mathcal{C}} = \frac{d\langle N^{\mathcal{C}}, N^{\mathrm{id}}\rangle_z}{d\langle N^{\mathrm{id}}\rangle_z} \; \text{ a.e.}$$

and also $\langle N^{\mathcal{C}}\rangle_z = \int_{R_z} (q_{\zeta}^{\mathcal{C}})^2 d\langle N^{\mathrm{id}}\rangle_{\zeta}$.

□

3.5 Representations of the optimal filter

For the remainder of this chapter, we assume the observation model described by (3.28) holds. Theorems III.29–III.33, presented below, generalize representations found in [5] to the case of multiparameter random fields. By the same arguments as Theorem II.42, one can show that under the probability measure P_0, where

$$\frac{dP}{dP_0} = \exp\left\{ \int_{\mathbb{T}} k^T_{h(X_\cdot);H}(\zeta) dY_\zeta - \frac{1}{2} \int_{\mathbb{T}} k^T_{h(X_\cdot);H}(\zeta) h(X_\zeta) d\zeta \right\}, \tag{3.48}$$

Y is a normalized fractional Wiener sheet with Hurst index $H = (H_1, \dots, H_m) \in (\frac{1}{2}, 1)^m$, Y and X are independent, and the law of X is the same under P and P_0. Let μ_X denote the probability law induced by X on $\mathcal{B}(C(\mathbb{T}))$, where $\mathcal{B}(C(\mathbb{T}))$ denotes the σ-field generated by all finite-dimensional Borel cylinder subsets of the space of continuous functions on $\mathbb{T}$. Also let μ_H denote the probability law induced by a standart fractional Brownian sheet with Hurst index H on $\mathcal{B}(C(\mathbb{T}))$. It will be convenient for us to consider the canonical probability space $(\Omega, \mathcal{F}, P_0) = (C(\mathbb{T}) \times C(\mathbb{T}), \overline{\mathcal{B}(C(\mathbb{T})) \times \mathcal{B}(C(\mathbb{T}))}, \mu_X \times \mu_H)$, where $\overline{\mathcal{B}(C(\mathbb{T})) \times \mathcal{B}(C(\mathbb{T}))}$ denotes the completion of the Borel product sigma field. In this case $X_z(\omega) = X_z(\omega_1, \omega_2) := \omega_1(z)$ and $Y_z(\omega) = Y_z(\omega_1, \omega_2) := \omega_2(z)$ for all $\omega = (\omega_1, \omega_2) \in C(\mathbb{T}) \times C(\mathbb{T})$. Similarly, we consider the canonical space $(\Omega, \mathcal{F}, \widetilde{P}) = (C(\mathbb{T}) \times C(\mathbb{T}), \overline{\mathcal{B}(C(\mathbb{T})) \times \mathcal{B}(C(\mathbb{T}))}, \mu_X \times \mu_W)$, where μ_W denotes the measure induced by the Wiener process W^Y. On this space, $X_z(\omega) = X_z(\omega_1, \omega_2) := \omega_1(z)$ and $W^Y_z(\omega) = W^Y_z(\omega_1, \omega_2) := \omega_2(z)$ for all $\omega = (\omega_1, \omega_2) \in C(\mathbb{T}) \times C(\mathbb{T})$. The following versions of the Bayes' formula are valid and follow directly from Lemma III.26

Theorem III.29. *Assume the filtering model* (3.28) *and conditions* (A_1)–(A_3) *of Section 1. Then* $\forall f \in C_b(\mathbb{R})$,

$$\mathbb{E}[f(X_z)|\mathcal{F}^Y_z](\omega) = \frac{\mathbb{E}_{\mu_X}[f(X_z)V_z^{-1}(\cdot, \omega_2)]}{\mathbb{E}_{\mu_X}[V_z^{-1}(\cdot, \omega_2)]} \tag{3.49}$$

and similarly

$$\mathbb{E}\big[f(X_z)|\mathcal{F}^Y_z\big](\omega) = \frac{\mathbb{E}_{\mu_X}[f(X_z)L_z(\cdot, \omega_2)]}{\mathbb{E}_{\mu_X}[L_z(\cdot, \omega_2)]}, \quad \forall z \in \mathbb{T} = [0, T_1] \times \cdots \times [0, T_m], \tag{3.50}$$

for almost all $\omega = (\omega_1, \omega_2) \in \Omega$, *where*

$$L_z(\omega_1, \omega_2) = e^{\int_{R_z} k^z_{h(X_\cdot(\omega_1));H}(\zeta) dY_\zeta(\omega_2) - \frac{1}{2}\int_{R_z} k^z_{h(X_\cdot(\omega_1));H}(\zeta) h(X_\zeta(\omega_1)) d\zeta}, \ z \in \mathbb{T},$$

where $k^z_{h(X_\cdot(\omega_1));H}(\cdot)$ *is the kernel* $k^z_{\mathcal{C};H}(\cdot)$ *(given by equation (3.36)) with* $\mathcal{C}(z) = h(X_z(\omega_1))$, *and* R_z *denotes the rectangle* $[0, z_1) \times \cdots \times [0, z_m)$.

Note: For fixed ω_1, using the notation of Section 3.4, $L_z(\omega_1, \cdot)$ can be written as

$$L_z(\omega_1, \cdot) = \exp\left\{ N_z^{h(X_\cdot(\omega_1))} - \frac{1}{2}\left\langle N^{h(X_\cdot(\omega_1))}\right\rangle_z \right\},$$

i.e. in terms of $N^{\mathcal{C}}$ with curve $\mathcal{C} = h(X_\cdot(\omega_1))$.

The proofs of theorems III.32 III.33 and III.31 use the following results.

Lemma III.30. *(i) Let* $(V_z, z \in \mathbb{T})$ *be defined by* (3.33) *and let* f *be a Borel-measurable function satisfying*

$$\mathbb{E}\left(f^2(X_z) e^{\int_{R_z} (\delta_\zeta(X))^2 d\zeta}\right) < \infty, \ \ \forall z \in \mathbb{T}. \tag{3.51}$$

Then

$$f(X_z)V_z^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} f(X_z) \left(\int_{R_z} \delta_\zeta(X) dW^Y_\zeta\right)^n \exp\left\{ -\frac{1}{2}\int_{R_z} \delta^2_\zeta(X) d\zeta \right\}$$

$$= \sum_{n=0}^{\infty} \frac{f(X_z)\left(\int_{R_z} (\delta_\zeta(X))^2 d\zeta\right)^{n/2}}{n!} H_n\left(\frac{\int_{R_z} \delta_\zeta(X) dW^Y_\zeta}{\sqrt{\int_{R_z} (\delta_\zeta(X))^2 d\zeta}} \right),$$

where the series converge in $L^2(\tilde{P})$.

(ii) Let $(L_z, z \in \mathbb{T})$ *be defined as in Theorem III.29 and let* f *be a Borel measurable function satisfying:*

$$\mathbb{E}\left(f^2(X_z) e^{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta) h(X_\zeta) d\zeta}\right) < \infty, \ \ \forall z \in \mathbb{T}. \tag{3.52}$$

Then

$$f(X_z)L_z = \sum_{n=0}^{\infty} \frac{1}{n!} f(X_z) e^{-\frac{1}{2}\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta} \left[\int_{R_z} k^z_{h(X_\cdot);H}(\zeta) dY_\zeta \right]^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f(X_z) \left(\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta \right)^{n/2}$$

$$\times \mathcal{H}_n \left(\frac{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)dY_\zeta}{\sqrt{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}} \right), \tag{3.53}$$

where $\mathcal{H}_n$ denotes the nth Hermite polynomial and both series converge in $L^2(P_0)$.

Proof (i) A Taylor expansion gives

$$V_z^{-1} = \exp\left\{ \frac{1}{2} \int_{R_z} \delta^2_\zeta(X)d\zeta \right\} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{R_z} \delta_\zeta(X)dW^Y_\zeta \right)^n .$$

Notice that under $\widetilde{P}$, W^Y is a Brownian sheet and as δ is a function of the random variable X but not Y, for fixed, $\int_{R_z} \delta_\zeta(X)(\tilde{\omega}_1, \cdot)dW^Y_\zeta$ is a mean zero Gaussian martingale and hence its variance is given by the quadratic variation which is $\int_{R_z} \delta^2_\zeta(X(\tilde{\omega}_1, \cdot))d\zeta$. Equivalently, the conditional distribution of $\int_{R_z} \delta_\zeta(X)dW^Y_\zeta$ given $\mathcal{F}^X$ under $\widetilde{P}$ is a mean zero Gaussian martingale with variance given by $\int_{R_z} \delta^2_\zeta(X)d\zeta$. The moments of $\int_{R_z} \delta_\zeta(X)dW^Y_\zeta$ conditioned on $\mathcal{F}^X$ are therefore given by:

$$\widetilde{\mathbb{E}} \left[\left(\int_{R_z} \delta_\zeta(X)dW^Y_\zeta \right)^{2j} \middle| \mathcal{F}^X_z \right] = \frac{(2j)!}{j!2^j} \left(\int_{R_z} \delta^2_\zeta(X)d\zeta \right)^j \quad \forall j \geq 1$$

$$\widetilde{\mathbb{E}} \left[\left(\int_{R_z} \delta_\zeta(X)dW^Y_\zeta \right)^{2j+1} \middle| \mathcal{F}^X_z \right] = 0 \quad \forall j \geq 1$$

So for any $a < b$, we have

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left(\left[\sum_{n=a}^{b}\frac{1}{n!}\left(\int_{R_z}\delta_\zeta(X)dW_\zeta^Y\right)^n\right]^2\middle|\mathcal{F}_z^X\right)\\
&=\sum_{\substack{a\le l,n\le b\\ l+n\ even}}\frac{1}{n!l!}\widetilde{\mathbb{E}}\left[\left(\int_{R_z}\delta_\zeta(X)dW_\zeta^Y\right)^{l+n}\middle|\mathcal{F}_z^X\right]\\
&=\sum_{i=a}^{b}\sum_{n=(2i-b)\vee a}^{(2i-a)\wedge b}\frac{1}{(2i-n)!n!}\widetilde{\mathbb{E}}\left[\left(\int_{R_z}\delta_\zeta(X)dW_\zeta^Y\right)^{2i}\middle|\mathcal{F}_z^X\right]\\
&=\sum_{i=a}^{b}\sum_{n=(2i-b)\vee a}^{(2i-a)\wedge b}\binom{2i}{n}\frac{\left(\int_{R_z}\delta_\zeta^2(X)d\zeta\right)^i}{i!2^i}\\
&\le\sum_{i=a}^{b}\frac{\left(\int_{R_z}\delta_\zeta^2(X)d\zeta\right)^i}{i!2^i}\left(\sum_{n=0}^{2i}\binom{2i}{n}\right)\\
&=\sum_{i=a}^{b}\frac{\left(\int_{R_z}\delta_\zeta^2(X)d\zeta\right)^i}{i!2^i}2^{2i}\\
&=\sum_{i=a}^{b}\frac{\left(2\int_{R_z}\delta_\zeta^2(X)d\zeta\right)^i}{i!}.
\end{aligned}$$

Using smoothing and the above inequality, we get, for any Borel measurable function F,

$$\begin{aligned}
&\widetilde{\mathbb{E}}\left(f(X_z)\exp\left\{-\frac{1}{2}\int_{R_z}\delta_\zeta^2(X)d\zeta\right\}\sum_{n=a}^{b}\frac{1}{n!}\left(\int_{R_z}\delta_\zeta(X)dW_\zeta^Y\right)^n\right)^2\\
&=\widetilde{\mathbb{E}}\left\{f^2(X_z)\exp\left\{\int_{R_z}\delta_\zeta^2(X)d\zeta\right\}\widetilde{\mathbb{E}}\left(\left[\sum_{n=a}^{b}\frac{1}{n!}\left(\int_{R_z}\delta_\zeta(X)dW_\zeta^Y\right)^n\right]^2\middle|\mathcal{F}_z^X\right)\right\}\\
&\le\widetilde{\mathbb{E}}\left\{f^2(X_z)\exp\left\{\int_{R_z}\delta_\zeta^2(X)d\zeta\right\}\sum_{i=a}^{b}\frac{\left(2\int_{R_z}\delta_\zeta^2(X)d\zeta\right)^i}{i!}\right\}\\
&=\widetilde{\mathbb{E}}\left\{f^2(X_z)\exp\left\{\int_{R_z}\delta_\zeta^2(X)d\zeta\right\}\sum_{i=a}^{b}\frac{2^i}{i!}\left(\int_{R_z}\delta_\zeta^2(X)d\zeta\right)^i\right\}\to 0 \text{ as } a\to\infty,
\end{aligned}$$

where the convergence relies on assumption (3.51) and follows by the dominated convergence theorem. Hence the sum, when taken from zero to ∞ converges in $L^2(\widetilde{P})$.

Applying relation (3.1) with

$$x = \frac{\int_{R_z} \delta_\zeta(X) dW_\zeta^Y}{\sqrt{\int_{R_z} \delta_\zeta^2(X) d\zeta}} \quad \text{and} \quad t = \sqrt{\int_{R_z} \delta_\zeta^2(X) d\zeta}$$

gives

$$V_z^{-1} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{n}{2}} \mathcal{H}_n \left(\frac{\int_{R_z} \delta_\zeta(X) dW_\zeta^Y}{\left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{1}{2}}} \right).$$

Taking $a < b$ again, and applying Proposition III.2 gives

$$\widetilde{\mathbb{E}} \left(\left[\sum_{n=a}^{b} \frac{1}{n!} \left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{n}{2}} \mathcal{H}_n \left(\frac{\int_{R_z} \delta_\zeta(X) dW_\zeta^Y}{\left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{1}{2}}} \right) \right]^2 \middle| \mathcal{F}_z^X \right)$$

$$= \sum_{n=a}^{b} \sum_{l=a}^{b} \frac{1}{n! l!} \left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{n+l}{2}} \widetilde{\mathbb{E}} \left\{ \mathcal{H}_n \left(\frac{\int_{R_z} \delta_\zeta(X) dW_\zeta^Y}{\sqrt{\int_{R_z} \delta_\zeta^2(X) d\zeta}} \right) \mathcal{H}_l \left(\frac{\int_{R_z} \delta_\zeta(X) dW_\zeta^Y}{\sqrt{\int_{R_z} \delta_\zeta^2(X) d\zeta}} \right) \middle| \mathcal{F}_z^X \right\}$$

$$= \sum_{n=a}^{b} \frac{\left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^n}{n!}.$$

Therefore, by smoothing we see

$$\widetilde{\mathbb{E}} \left[\sum_{n=a}^{b} \left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{n}{2}} \mathcal{H}_n \left(\frac{\int_{R_z} \delta_\zeta(X) dW_\zeta^Y}{\left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{1}{2}}} \right) \right]^2$$

$$= \widetilde{\mathbb{E}} \left\{ f^2(X_z) \widetilde{\mathbb{E}} \left(\left[\sum_{n=a}^{b} \frac{1}{n!} \left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{n}{2}} \mathcal{H}_n \left(\frac{\int_{R_z} \delta_\zeta(X) dW_\zeta^Y}{\left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^{\frac{1}{2}}} \right) \right]^2 \middle| \mathcal{F}_z^X \right) \right\}$$

$$= \widetilde{\mathbb{E}} \left\{ f^2(X_z) \sum_{n=a}^{b} \frac{\left(\int_{R_z} \delta_\zeta^2(X) d\zeta \right)^n}{n!} \right\} \to 0 \text{ as } a \to \infty,$$

by assumption (3.51) and the dominated convergence theorem and giving convergence of the series in $L^2(\widetilde{P})$.

The proof of (ii) follows similarly by taking $\int_{R_z} k^z_{h(X_\cdot);H}(\zeta) dY_\zeta$ in place of $\int_{R_z} \delta_\zeta(X) dW_\zeta^Y$ and taking $\int_{R_z} k^z_{h(X_\cdot(\omega_1));H}(\zeta) h(X_\zeta(\omega_1)) d\zeta$ in place of $\int_{R_z} \delta_\zeta^2(X) d\zeta$.

□

We now give three different expansions of the optimal filter $\mathbb{E}[f(X_z)|\mathcal{F}_z^Y]$, each of which is expressed in terms of the ratio of infinite series of different multiple stochastic integrals.

Theorem III.31. *Assume the filtering model* (3.28) *and the corresponding conditions. Moreover, assume conditions* (A_1^m), (A_2^m) *and* (A_3^m). *If* $f \in C_b(\mathbb{R})$ *and the following conditions hold:*

$$\mathbb{E}\left[f^2(X_z)e^{\int_{R_z}(\delta_\zeta(X))^2 d\zeta}\right] < \infty \tag{3.54}$$

and

$$\mathbb{E}\left[e^{\frac{1}{2}\int_{R_z}(\delta_\zeta(X))^2 d\zeta}\right] < \infty, \quad \forall z \in \mathbb{T}, \tag{3.55}$$

then the following representation for the optimal filter holds:

$$\mathbb{E}\big[f(X_z)|\mathcal{F}_z^Y\big] \overset{a.s.}{=} \frac{\sum_{p=0}^{\infty}\frac{1}{p!}I_p^{W^Y,z}\left(\mathbb{E}\left[f(X_z)(\delta_{\cdot}(X))^{\otimes p}\right]\right)}{\sum_{p=0}^{\infty}\frac{1}{p!}I_p^{W^Y,z}\left(\mathbb{E}\left[(\delta_{\cdot}(X))^{\otimes p}\right]\right)}, \tag{3.56}$$

where the series converge in $L^2(\tilde{P})$ *(and in* $L^1(P)$*) and* $I_p^{W^Y,z}$ *denotes the pth order multiple stochastic fractional integral of the Itô type with respect to m-parameter field* W^Y, *given by* ((3.32)), *with the integral taken over rectangle* $R_z = [0, z_1) \times \cdots \times [0, z_m)$, *and where* δ *is given by* (4.4).

Proof First we prove the following convergence for arbitrary $p \geq 1$, where $\{e_i\}_{i=1}^{\infty}$ is a complete orthonormal system in $L^2_{H,1}(R_z)$

$$\lim_{M\to\infty}\sum_{i_1,\ldots,i_p=1}^{M}\left\langle(\delta_z)^{\otimes n}, e_{i_1}\otimes\ldots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)} I_p^{W^Y,z}(e_{i_1}\otimes\ldots\otimes e_{i_p})$$

$$= \left(\int_{R_z}(\delta_\zeta(X))^2 d\zeta\right)^{p/2}\mathcal{H}_p\left(\frac{\int_{R_z}\delta_\zeta(X)dW_\zeta^Y}{\sqrt{\int_{R_z}(\delta_\zeta(X))^2 d\zeta}}\right), \tag{3.57}$$

where the limit is in $L^2(\widetilde{P})$. To prove (3.57), note that for fixed $\widetilde{\omega}_1 \in C[R_{z_0}]$, $\delta.(\widetilde{\omega}_1) \in L^2(\mathbb{T}^p)$ so the integral $I_p^{W^Y,z}([\delta.(\widetilde{\omega}_1)]^{\otimes n})$ is well defined. By linearity of the multiple stochastic integral and (3.10),

$$\mathbb{E}_{\mu_W}\Bigg[\left(\int_{R_z}\delta_\zeta^2(X(\tilde{\omega}_1,\cdot))d\zeta\right)^{p/2}\mathcal{H}_p\left(\frac{\int_{R_z}\delta_\zeta(X)(\tilde{\omega}_1,\cdot)dW_\zeta^Y}{\left(\int_{R_z}\delta_\zeta^2(X(\tilde{\omega}_1,\cdot))d\zeta\right)^{\frac{1}{2}}}\right)$$

$$-\sum_{i_1,\ldots,i_p=1}^{M}\left\langle(\delta_z)^{\otimes p},e_{i_1}\otimes\ldots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)}I_p^{W^Y,z}(e_{i_1}\otimes\ldots\otimes e_{i_p})\Bigg]^2$$

$$=\mathbb{E}_{\mu_W}\Bigg[I_p^{W^Y,z}\Bigg((\delta.(X(\widetilde{\omega}_1,\cdot)))^{\otimes p}-$$

$$\sum_{i_1,..,i_p=1}^{M}\left\langle(\delta_z(X))^{\otimes p},e_{i_1}\otimes\ldots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)}e_{i_1}\otimes\ldots\otimes e_{i_p}\Bigg)\Bigg]^2$$

$$\leq p!\left\|(\delta.(X(\widetilde{\omega}_1,\cdot)))^{\otimes p}-\sum_{i_1,\ldots,i_p=1}^{M}\left\langle(\delta_z(X))^{\otimes p},e_{i_1}\otimes\ldots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)}e_{i_1}\otimes\ldots\otimes e_{i_p}\right\|_{L^2(R_z)}^2\to 0$$

as $M\to\infty$. The last inequality follows from (3.11) and the fact that $\|\tilde{f}\|_{L^2(R_z)}^2\leq p!\|f\|_{L^2(R_z)}^2$. To show convergence in $L^2(\widetilde{P})$, note that by (3.11),

$$\mathbb{E}_{\mu_W}\left[\sum_{i_1,\ldots,i_p=1}^{M}\left\langle(\delta_z(X))^{\otimes p},e_{i_1}\otimes\ldots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)}I_p^{W^Y,z}(e_{i_1}\otimes\ldots\otimes e_{i_p})\right]^2$$

$$=\sum_{i_1,\ldots,i_p=1}^{M}\left\langle(\delta_z(X))^{\otimes p},e_{i_1}\otimes\ldots\otimes e_{i_n}\right\rangle_{L^2(R_z^p)}p!\|e_{i_1}\otimes\ldots\otimes e_{i_n})\|_{L^2(R_z)}^2$$

$$\leq p!\|\delta.(X(\widetilde{\omega}_1,\cdot))^{\otimes p}\|_{L^2(R_z)}^2=p!\|(\delta.(X(\widetilde{\omega}_1,\cdot)))\|_{L^2(R_z)}^{2p}.$$

And $E_{\mu_X}\|\delta.\|_{L^2(R_z)}^{2p}<\infty$ follows by assumption (3.55) since $\|\delta.(\omega_1)\|_{L^2(R_z)}^2=\int_{R_z}\delta_\zeta^2(X)d\zeta$. Now we apply dominated convergence theorem, proving the convergence (3.57) in $L^2(\widetilde{P})$ since $\widetilde{P}=\mu_X\times\mu_W$.

Now by Lemma III.30 and (3.57) we have

$$
\begin{aligned}
&\mathbb{E}_{\mu_X}[f(X_z)V_z^{-1}(\cdot,\omega_2)] \\
&= \mathbb{E}_{\mu_X}\left[f(X_z)\sum_{p=0}^{\infty}\frac{1}{p!}\left(\int_{R_z}\delta_\zeta^2(X(\cdot,\omega_2))d\zeta\right)^{p/2}\mathcal{H}_p\left(\frac{\int_{R_z}\delta_\zeta(X)(\cdot,\omega_2)dW_\zeta^Y}{\left(\int_{R_z}\delta_\zeta^2(X(\cdot,\omega_2))d\zeta\right)^{\frac{1}{2}}}\right)\right] \\
&= \sum_{p=0}^{\infty}\frac{1}{p!}\mathbb{E}_{\mu_X}\left[f(X_z)\left(\int_{R_z}\delta_\zeta^2(X(\cdot,\omega_2))d\zeta\right)^{p/2}\mathcal{H}_p\left(\frac{\int_{R_z}\delta_\zeta(X)(\cdot,\omega_2)dW_\zeta^Y}{\left(\int_{R_z}\delta_\zeta^2(X(\cdot,\omega_2))d\zeta\right)^{\frac{1}{2}}}\right)\right] \\
&= \mathbb{E}_{\mu_X}[f(X_z)] + \sum_{p=1}^{\infty}\frac{1}{p!}\mathbb{E}_{\mu_X}\left[f(X_z)\sum_{i_1,\dots,i_p=1}^{\infty}\left\langle(\delta_z(X))^{\otimes p}, e_{i_1}\otimes\dots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)} I_p^{W^Y,z}(e_{i_1}\otimes\dots\otimes e_{i_p})\right] \\
&= \mathbb{E}_{\mu_X}[f(X_z)] + \sum_{p=1}^{\infty}\frac{1}{p!}\sum_{i_1,\dots,i_p=1}^{\infty}\mathbb{E}_{\mu_X}\left[f(X_z)\left\langle(\delta_z(X))^{\otimes p}, e_{i_1}\otimes\dots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)}\right] I_p^{W^Y,z}(e_{i_1}\otimes\dots\otimes e_{i_p}) \\
&= \mathbb{E}_{\mu_X}[f(X_z)] + \sum_{p=1}^{\infty}\frac{1}{p!}\sum_{i_1,\dots,i_p=1}^{\infty}\left\langle\mathbb{E}_{\mu_X}\left[f(X_z)(\delta_z(X))^{\otimes p}\right], e_{i_1}\otimes\dots\otimes e_{i_p}\right\rangle_{L^2(R_z^p)} I_p^{W^Y,z}(e_{i_1}\otimes\dots\otimes e_{i_p}) \\
&= \mathbb{E}_{\mu_X}[f(X_z)] + \sum_{p=1}^{\infty}\frac{1}{p!}I_n^{W^Y,z}\left(\mathbb{E}_{\mu_X}\left[f(X_z)\,(\delta_\cdot(X))^{\otimes p}\right]\right) = \sum_{p=0}^{\infty}\frac{1}{p!}I_p^{W^Y,z}\left(\mathbb{E}_{\mu_X}\left[f(X_z)\,(\delta_\cdot(X))^{\otimes p}\right]\right) \\
&= \sum_{p=0}^{\infty}\frac{1}{p!}I_p^{W^Y,z}\left(\mathbb{E}\left[f(X_z)\,(\delta_\cdot(X))^{\otimes p}\right]\right)
\end{aligned}
$$

where the series converges in $L^2(\widetilde{P})$. Notice that since $V_{T_1,\dots,T_m}^{-1} = \frac{dP}{d\widetilde{P}}$, it follows that $\widetilde{\mathbb{E}}(V_{T_1,\dots,T_m}^{-1})^2 = \mathbb{E}(V_{T_1,\dots,T_m}^{-1}) < \infty$, which implies $V_{T_1,\dots,T_m}^{-1} \in L^2(\widetilde{P})$. Since the sequence of partial sums

$$
S_N := \sum_{p=0}^{N}\frac{1}{p!}I_p^{W^Y,z}\left(\mathbb{E}\left[f(X_z)\,(\delta_\cdot(X))^{\otimes p}\right]\right)
$$

converges in $L^2(\widetilde{P})$, clearly $|S_N|$ converges in $L^2(\widetilde{P})$ as $N \to \infty$. Again we use the fact that $V_{T_1,\dots,T_m}^{-1} = \frac{dP}{d\widetilde{P}}$, giving

$$
\langle V_{T_1,\dots,T_m}^{-1}, |S_N|\rangle_{L^2(\widetilde{P})} = \|S_N\|_{L^1(P)} \tag{3.58}
$$

and by the Cauchy-Schwarz inequality $\langle V_{T_1,\dots,T_m}^{-1}, |S_N|\rangle_{L^2(\widetilde{P})}$ converges and so by (3.58), S_N converges in $L^1(P)$. Taking $f = 1$ gives the denominator of (III.31). Plugging into (3.49) gives (3.56). □

Theorem III.32. *Assume the filtering model* (3.28) *and the conditions therein.*

Moreover, assume that $f \in C_b(\mathbb{R})$ *and the following conditions hold:*

$$\mathbb{E}\left[f^2(X_z)e^{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}\right] < \infty \tag{3.59}$$

and

$$\mathbb{E}\left[e^{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}\right] < \infty. \tag{3.60}$$

Then the following representation holds:

(3.61)
$$\mathbb{E}\big[f(X_z)|\mathcal{F}^Y_z\big] \overset{a.s.}{=} \frac{\sum_{p=0}^{\infty} \frac{1}{p!}\mathfrak{I}^{Y,z}_p\left(\mathbb{E}\left[f(X_z)e^{-\frac{1}{2}\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}(k^z_{h(X_\cdot);H}(\cdot))^{\otimes p}\right]\right)}{\sum_{p=0}^{\infty} \frac{1}{p!}\mathfrak{I}^{Y,z}_p\left(\mathbb{E}\left[e^{-\frac{1}{2}\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}(k^z_{h(X_\cdot);H}(\cdot))^{\otimes p}\right]\right)},$$

where the series converge in $L^2(P_0)$ *(and in* $L^1(P)$*) and* $\mathfrak{I}^{Y,z}_p$ *denotes the pth order multiple stochastic fractional integral of the Stratonovich type with respect to the observation m-parameter field* Y *taken over the rectangle* $R_z = [0, z_1) \times \cdots \times [0, z_m)$.

Proof: Let

$$\widetilde{N}_z := \int_{R_z} k^z_{h(X_\cdot);H}(\zeta)dY_\zeta, \quad \text{and} \quad \langle\widetilde{N}\rangle_z := \int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta.$$

Lemma III.30(i), gives

$$\mathbb{E}_{\mu_X}\left[f(X_z)L_z(\cdot,\omega_2)\right] = \sum_{p=0}^{\infty}\frac{1}{p!}\mathbb{E}_{\mu_X}\left[f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\widetilde{N}^p_z(\cdot,\omega_2)\right], \tag{3.62}$$

where the series converges in $L^2(P_0)$. By the relation $x^p = \sum_{k=0}^{\lfloor p/2\rfloor} c_{k,p}\mathcal{H}_{p-2k}(x)$ for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$, where $c_{k,p} = p!/(k!(p-2k)!2^k)$, we have

$$\widetilde{N}^p_z = \sum_{k=0}^{\lfloor p/2\rfloor} c_{k,p}\langle\widetilde{N}\rangle^{p/2}_z\mathcal{H}_{p-2k}\left(\frac{\widetilde{N}_z}{\langle\widetilde{N}\rangle^{1/2}_z}\right) \quad \text{a.s.} \tag{3.63}$$

Choose a complete orthonormal system $\{e_i\}_{i=1}^{\infty}$ in $L^2_{H,1}(R_z)$. Then it is easy to show by arguments similar to those used to show (3.57), that for all $p \geq 1$ and all $z \in \mathbb{T}$,

$$\lim_{M\to\infty}\mathbb{E}_{P_0}\left[\langle\widetilde{N}\rangle^{p/2}_z\mathcal{H}_p\left(\frac{\widetilde{N}_z}{\langle\widetilde{N}\rangle^{1/2}_z}\right)\right.$$

$$- \sum_{i_1,\dots,i_p=1}^{M} \left\langle (k^z_{h(X_\cdot);H})^{\otimes p}, e_{i_1} \otimes \cdots \otimes e_{i_p} \right\rangle_{L^2_{H,p}(R^p_z)} I^{Y,z}_p(e_{i_1} \otimes \cdots \otimes e_{i_p}) \Big]^2 = 0.$$

Applying the above equality and equation (3.63) gives

$$\begin{aligned}
&\mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\widetilde{N}^p_z\right] \\
&= \sum_{k=0}^{\lfloor p/2\rfloor} c_{k,p}\,\mathbb{E}_{\mu_X}\left[f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\langle\widetilde{N}\rangle_z^{p/2}\mathcal{H}_{p-2k}\left(\frac{\widetilde{N}_z}{\langle\widetilde{N}\rangle_z^{1/2}}\right)\right]
\end{aligned}$$

$$\begin{aligned}
= \quad & \sum_{\substack{k\in\{0,\cdots,\lfloor p/2\rfloor\}:\\ p-2k\geq 1}} c_{k,p} \sum_{i_1,\cdots,i_{p-2k}=1}^{\infty} \Big\{ \mathbb{E}_{\mu_X}\Big[f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\langle\widetilde{N}\rangle_z^k \\
\times \quad & \left\langle (k^z_{h(X_\cdot);H})^{\otimes p-2k}, e_{i_1}\otimes\cdots\otimes e_{i_{p-2k}}\right\rangle_{L^2_{H,p-2k}(R_z^{p-2k})}\Big]\, I^{Y,z}_{p-2k}\left(e_{i_1}\otimes\cdots\otimes e_{i_{p-2k}}\right)\Big\} \\
+ \quad & c_{\frac{p}{2},p}\,1_{\{p\text{ is even}\}}\mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\langle\widetilde{N}\rangle_z^{p/2}\right] \\
= \quad & \sum_{\substack{k\in\{0,\cdots,\lfloor p/2\rfloor\}:\\ p-2k\geq 1}} c_{k,p} \sum_{i_1,\cdots,i_{p-2k}=1}^{\infty} \Big\{ \Big\langle \mathbb{E}_{\mu_X}\left[f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\langle\widetilde{N}\rangle_z^k\,(k^z_{h(X_\cdot);H})^{\otimes p-2k}\right], \\
& e_{i_1}\otimes\cdots\otimes e_{i_{p-2k}} \Big\rangle_{L^2_{H,p-2k}(R_z^{p-2k})} I^{Y,z}_{p-2k}\left(e_{i_1}\otimes\cdots\otimes e_{i_{p-2k}}\right)\Big\} \\
+ \quad & c_{\frac{p}{2},p}1_{\{p\text{ is even}\}}\mathbb{E}_{\mu_X}\left[f_z\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\langle\widetilde{N}\rangle_z^{p/2}\right] \\
= \quad & \sum_{k=0}^{\lfloor p/2\rfloor} c_{k,p}\, I^{Y,z}_{p-2k}\left(\mathbb{E}_{\mu_X}\left[f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\langle\widetilde{N}\rangle_z^k\,(k_{h(X_\cdot);H})^{\otimes p-2k}\right]\right),
\end{aligned}$$

with the convention: $x^{\otimes 0} \equiv 1$, and $I^{Y,z}_0(\varphi) = \varphi$. Note that for $p > 0$ and $k = 0, 1, \dots, \lfloor p/2\rfloor$, and every fixed $z \in \mathbb{T}$,

$$\begin{aligned}
&Tr^{k,z}_{\phi}\left(\mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\left(k_{h(X_\cdot);H}\right)^{\otimes p}\right]\right)(u^1,\dots,u^{p-2k}) \\
&= \mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\langle\widetilde{N}\rangle_z^k\left(k^z_{h(X_\cdot);H}\right)^{\otimes p-2k}(u^1,\dots,u^{p-2k})\right], \qquad (3.64)
\end{aligned}$$

where, for a symmetric function $g \in L^2_{H,p}(R^p_z)$, the trace $Tr^{k,z}_{\phi}(g)$ is defined by (3.27). Therefore, it follows that

$$\begin{aligned}
& \mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\langle \widetilde{N}\rangle_z}\widetilde{N}^p_z\right] \\
&= \sum_{k=0}^{\lfloor p/2 \rfloor} c_{k,p} I^{Y,z}_{p-2k}\left(Tr^{k,z}_{\phi}\left(\mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\langle \widetilde{N}\rangle_z}(k^z_{h(X_\cdot);H})^{\otimes p}\right]\right)\right) \\
&= \mathcal{I}^{Y,z}_p\left(\mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\langle \widetilde{N}\rangle_z}(k^z_{h(X_\cdot);H})^{\otimes p}\right]\right), \qquad (3.65)
\end{aligned}$$

where (3.65) follows by direct application of the multiparameter version of the “fractional” Hu-Meyer formula (which is a simple extension of the one-parameter “fractional” Hu-Meyer formula found in [8]). From (3.62), (3.65) and $\langle \widetilde{N}\rangle_z = \|k^z_{h(X_\cdot);H}\|^2_{L^2_{H,1}(R_z)}$, we obtain that

$$\begin{aligned}
\mathbb{E}_{\mu_X}\left[f(X_z)L_z(\cdot,\omega_2)\right] &= \sum_{p=0}^{\infty}\frac{1}{p!}\mathcal{I}^{Y,z}_p\Bigg(\mathbb{E}_{\mu_X}\Bigg[f(X_z)e^{-\frac{1}{2}\int_{R_z}k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta} \\
&\quad \times \ (k^z_{h(X_\cdot);H})^{\otimes p}\Bigg]\Bigg)(\omega_2), \qquad (3.66)
\end{aligned}$$

where the series converges in $L^2(P_0)$. Substituting $f \equiv 1$ in the above equation leads to corresponding expansion for $\mathbb{E}_{\mu_X}\left[L_z(\cdot,\omega_2)\right]$.

Let us denote by $\mathbb{E}_{P_0}$, the expectation under probability measure P_0. Assumption (3.60) implies $\mathbb{E}(\frac{dP}{dP_0}) = 1$. Consequently, since $\mathbb{E}_{P_0}(\frac{dP}{dP_0})^2 = \mathbb{E}(\frac{dP}{dP_0})$, we have $\frac{dP}{dP_0} \in L^2(P_0)$. Since the sequence of partial sums

$$\mathcal{S}_N := \sum_{p=0}^{N}\frac{1}{p!}\mathcal{I}^{Y,z}_p\left(\mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\int_{R_z}k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}(k^z_{h(X_\cdot);H})^{\otimes p}\right]\right)$$

converges in $L^2(P_0)$, clearly $|\mathcal{S}_N|$ converges in $L^2(P_0)$ as $N \to \infty$. By the Cauchy Schwarz inequality and the fact that $\langle \frac{dP}{dP_0}, |\mathcal{S}_N|\rangle_{L^2(P_0)} = \mathbb{E}(|\mathcal{S}_N|)$ we also have convergence in $L^1(P)$. Substituting (3.66) in (3.50), gives the required result.

$\square$

Theorem III.33. *Under assumptions of Theorem III.32,* $\forall f \in C_b(\mathbb{R})$,

$$\mathbb{E}\big[f(X_z)|\mathcal{F}_z^Y\big] \overset{a.s.}{=} \frac{\sum_{p=0}^{\infty} \frac{1}{p!} I_p^{Y,z}\left(\mathbb{E}\left[f(X_z)(k^z_{h(X_\cdot);H}(\cdot))^{\otimes p}\right]\right)}{\sum_{p=0}^{\infty} \frac{1}{p!} I_p^{Y,z}\left(\mathbb{E}\left[(k^z_{h(X_\cdot);H}(\cdot))^{\otimes p}\right]\right)}, \tag{3.67}$$

where the series converge in $L^2(P_0)$ *(and in* $L^1(P)$*) and* $I_p^{Y,z}$ *denotes the pth order multiple stochastic fractional integral of the Itô type (defined as in [8]) with respect to the m-parameter observation field* Y *taken over the rectangle* $R_z = [0, z_1) \times \cdots \times [0, z_m)$.

Proof: Choose a complete orthonormal system $\{e_i\}_{i=1}^{\infty}$ in $L^2_{H,1}(R_z)$. Using arguments similar to those employed in proving (3.57), we can show that for all $p \geq 1$,

$$\lim_{M\to\infty} \mathbb{E}_{P_0}\left[\left(\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta\right)^{p/2} \mathcal{H}_p\left(\frac{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)dY_\zeta}{\sqrt{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}}\right)\right.$$

$$\left. - \sum_{i_1,\ldots,i_p=1}^{M} \langle (k^z_{h(X_\cdot);H})^{\otimes p}, e_{i_1}\otimes\cdots\otimes e_{i_p}\rangle_{L^2_{H,p}(R_z^p)} I_p^{Y,z}\left(e_{i_1}\otimes\cdots\otimes e_{i_p}\right)\right]^2 = 0. \tag{3.68}$$

By Lemma III.30(i) and (3.68),

$$\begin{aligned}
\mathbb{E}_{\mu_X}\left[f(X_z)L_z\right] &= \sum_{p=0}^{\infty} \frac{1}{p!}\mathbb{E}_{\mu_X}\left[f(X_z)\left(\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta\right)^{p/2}\right.\\
&\qquad \left.\times\mathcal{H}_p\left(\frac{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)dY_\zeta}{\sqrt{\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}}\right)\right]\\
&= \sum_{p=1}^{\infty} \frac{1}{p!}\mathbb{E}_{\mu_X}\left[f(X_z)\sum_{i_1,\cdots,i_p=1}^{\infty} \langle (k^z_{h(X_\cdot);H})^{\otimes p}, e_{i_1}\otimes\cdots\otimes e_{i_p}\rangle_{L^2_{H,p}(R_z^p)}\right.\\
&\qquad \left.\times I_p^{Y,z}\left(e_{i_1}\otimes\cdots\otimes e_{i_p}\right)\right] + \mathbb{E}\left(f(X_z)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{i_1,\cdots,i_p=1}^{\infty} \mathbb{E}_{\mu_X}\Big[f(X_z)\langle (k^z_{h(X.);H})^{\otimes p}, e_{i_1} \otimes \cdots \otimes e_{i_p}\rangle_{L^2_{H,p}(R^p_z)}\Big] \\
&\quad \times I^{Y,z}_p \left(e_{i_1} \otimes \cdots \otimes e_{i_p}\right) + \mathbb{E}\left(f(X_z)\right) \\
&= \sum_{p=1}^{\infty} \frac{1}{p!} \sum_{i_1,\cdots,i_p=1}^{\infty} \left\langle \mathbb{E}_{\mu_X}\big[f(X_z)(k^z_{h(X.);H})^{\otimes p}\big], e_{i_1} \otimes \cdots \otimes e_{i_p}\right\rangle_{L^2_{H,p}(R^p_z)} \\
&\quad \times I^{Y,z}_p \left(e_{i_1} \otimes \cdots \otimes e_{i_p}\right) + \mathbb{E}\left(f(X_z)\right) \\
&= \sum_{p=0}^{\infty} \frac{1}{n!} I^{Y,z}_p \left(\mathbb{E}\left[f(X_z)\,(k^z_{h(X.);H})^{\otimes p}\right]\right), \qquad (3.69)
\end{aligned}
$$

where the series converges in $L^2(P_0)$. As we've shown (in the proof of Theorem III.32) convergence in $L^2(P_0)$ implies convergence in $L^1(P)$. Substituting (3.69) in the numerator and denominator (with $f \equiv 1$) of (3.50) gives the required result.

□

The multiple stochastic integrals in (3.56) are mathematically simpler then those described in (3.61) and (3.5) since they are multiple stochastic integrals with respect to a standardized Brownian sheet. However, from the point of view of implementation of the filter, representations (3.61) and (3.5) are more practical. The reason being that the integrals in (3.56), taken with respect to Brownian sheet W^Y require that we compute W^Y via the integral transform

$$
W^Y_z := \int_{R_z} \prod_{j=1}^{m} K^{-1}_{H_j}(z_j;\zeta_j)dY_\zeta, \quad z \in \mathbb{T}.
$$

Notice that the kernels $K^{-1}_{H_j}(z_j;\zeta_j)$ take different forms depending on the value of z_j. That is, the kernel used in the transformation must be continuously updated as the value of the current state z in the parameter space $\mathbb{T}$ changes. On the other hand, integrals in (3.61) and are simply taken with respect to the observation random field Y. The additional layer of complexity makes taking pathwise integrals with respect to W^Y_z computationally expensive. However, (3.56) may be used to obtain a representation of the optimal filter in terms of integrals with respect to Y if there exists a suitable version of the stochastic transfer principle (similar to those found in [22],[5]). That is, if we can identify an operator $\Gamma^z_{p;H}$ such that $I_p^{W^Y,z}(F) = I_p^{Y,z}(\Gamma^z_{p;H}F)$ for all suitably regular and integrable functions F from the domain of $\Gamma^z_{p;H}$, and then rewrite (3.56) in the form:

$$\mathbb{E}\big[f(X_z)|\mathcal{F}^Y_z\big] \overset{a.s.}{=} \frac{\sum_{p=0}^{\infty} \frac{1}{p!} I_p^{Y,z}\left(\Gamma^z_{p;H}\mathbb{E}\Big[f(X_z)(\delta_{\cdot}(X))^{\otimes p}\Big]\right)}{\sum_{p=0}^{\infty} \frac{1}{p!} I_p^{Y,z}\left(\Gamma^z_{p;H}\mathbb{E}\Big[(\delta_{\cdot}(X))^{\otimes p}\Big]\right)}. \tag{3.70}$$

In Chapter IV we will explore this idea of the operator $\Gamma^z_{p;H}$ and conditions under which such an operator exists.

While evolution equations such as those described in Chapter II are useful in describing the dynamics, they are often not easy to use in practice. The evolution equations have complicated analytical structure and are not proper stochastic differential equations. The expansions described in Theorems III.31, III.33 and III.31 can be used as a tool to approximate the optimal filter in the general model described by (3.28). By truncating the series and discretizing the integrals in the series, we can obtain a class of suboptimal filters which converge to the best mean square filter. These suboptimal filters can be implemented numerically. Here we extend results given in [5] to the multiparameter setting described in the model (3.28). The following theorem describes the truncated, discretized suboptimal filter and gives convergence results for this approximation.

Theorem III.34. *Let $\{\pi_M\}$ be a sequence of partitions of $R_z = [0, z_1] \times \cdots \times [0, z_m]$ into m-dimensional rectangles with mesh $|\pi_M| \to 0$ as $M \to \infty$. Assume conditions* *of Theorem III.32, then*

$$\mathbb{E}\left[f(X_z)|\mathcal{F}_z^Y\right] = \frac{L^1(P) - \lim\limits_{M\to\infty} \sum\limits_{p=0}^{M} \frac{1}{p!}\mathcal{S}_p(\ell_p(f,z|\cdot),\pi_M,Y)}{L^1(P) - \lim\limits_{M\to\infty} \sum\limits_{p=0}^{M} \frac{1}{p!}\mathcal{S}_p(\ell_p(\tilde{1},z|\cdot),\pi_M,Y)}, \tag{3.71}$$

where $\mathcal{S}_0(\ell_0(f,z|\cdot),\cdot) \equiv \mathbb{E}_{\mu_X}\left[f(X_z)e^{-\frac{1}{2}\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta}\right]$, $\tilde{1}$ *denotes the function identically equal to 1, and for all functions* $g_p \in L^2_{H,p}(R_z)$, $\forall p \geq 1$,

$$\begin{aligned}\mathcal{S}_p(g_p,\pi_M,Y) \;&:=\; \sum_{i_1,\cdots,i_p=1}^{n(M)} \left[\int_{R_z^p} g_p(u^1,\ldots,u^p)\beta^M_{i_1,\cdots,i_p}(u^1,\ldots,u^p)du^1\ldots du^p\right] \\ &\quad\times Y(\square^M_{i_1})\ldots Y(\square^M_{i_p}),\end{aligned}$$

with $Y(\square_i^M)$ *represents the variation of* Y *over partition rectangle* $\square_i^M \in \pi_M$ *as defined in* (3.5), $n(M)$ *is the number of rectangles in partition* π_M, *and for all* $u^j = (u^j_1,\ldots,u^j_m) \in R_z$, *where* $j = 1,\ldots,p$, *we define*

$$\begin{aligned}\beta^M_{i_1,\cdots,i_p}(u^1,\cdots,u^p) \;&:=\; \sum_{j_1,\cdots,j_p=1}^{n(M)} \Big\{ \left((E_M)^{-1}\right)_{i_1,j_1}\ldots\left((E_M)^{-1}\right)_{i_p,j_p} \\ &\quad\times\left(\int_{\square^M_{j_1}} \phi_H(u^1,\zeta)d\zeta\right)\times\cdots\times\left(\int_{\square^M_{j_p}} \phi_H(u^p,\zeta)d\zeta\right)\Big\},\end{aligned}$$

where $\phi_H(u,\zeta) := \prod_{i=1}^{m} H_i(2H_i-1)|\zeta_i-u_i|^{2H_i-2}$ *for all* $u = (u_1,\ldots,u_m)$, $\zeta = (\zeta_1,\ldots,\zeta_m) \in R_z$,

$$\begin{aligned}\ell_p(f,z|u^1,\cdots,u^p) \;&=\; \mathbb{E}_{\mu_X}\Big[f(X_z)\exp\left\{-\frac{1}{2}\int_{R_z} k^z_{h(X_\cdot);H}(\zeta)h(X_\zeta)d\zeta\right\} \\ &\quad\times\left(k^z_{h(X_\cdot);H}\right)^{\otimes p}(u^1,\cdots,u^p)\Big]\end{aligned} \tag{3.72}$$

and the matrix E_M *is defined by* $E_M := \left(\left\langle 1_{\square_i^M}, 1_{\square_j^M}\right\rangle_{L^2_{H,1}(R_z)}\right)_{1\leq i,j\leq n(M)}$.

Proof: Consider a sequence of partitions $\pi_M = \{\square_i^M\}_{i=1,\ldots,n(M)}$ of the rectangle R_z into $n(M)$ rectangles, and let P_M denote the orthogonal projection operator (in

$L^2_{H,1}(R_z))$ onto the linear span of $\{1_{\square_i^M}; i = 1, 2, \cdots, n(M)\}$. Then P_M converges to the identity operator as $M \to \infty$ in the sense that $\forall g \in L^2_{H,1}(R_z)$

$$\|(P_M - I)g\|_{L^2_{H,1}(R_z)} \longrightarrow 0 \text{ as } M \to \infty.$$

Let us introduce the notation

$$\langle \tilde{N} \rangle_z := \int_{R_z} k^z_{h(X.);H}(\zeta) h(X_\zeta) d\zeta.$$

By arguments used in proving (3.53), we have that as $a, b \to \infty$,

$$\mathbb{E}_{P_0}\left[f(X_z) e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} \sum_{p=a}^{b} \frac{1}{p!} \left(\int_{R_z} P_M k^z_{h(X.);H}(\zeta) dY_\zeta \right)^p \right]^2 \longrightarrow 0,$$

uniformly in M. Therefore $\sum_{p=0}^{\infty} \frac{1}{p!} f(X_z)\, e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} \left(\int_{R_z} P_M k^z_{h(X.);H}(\zeta) dY_\zeta \right)^p$ converges in $L^2(P_0)$ and hence,

$$\sum_{p=0}^{\infty} \frac{1}{p!} \mathbb{E}_{\mu_X}\left[f(X_z)\, e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} \left(\int_{R_z} P_M k^z_{h(X.);H}(\zeta) dY_\zeta \right)^p \right] \to 0 \tag{3.73}$$

in $L^2(P_0)$ where the convergence is uniform in M. We also note that since we can represent the projection by $P_{_M} k^z_{h(X.);H} = \sum_{k=1}^{n(M)} \overset{M}{\alpha_k} 1_{\square_k^M}$, we have

$$\begin{aligned}
&\mathbb{E}_{\mu_X}\left[f(X_z)\, e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} \left(\int_{R_z} P_M k^z_{h(X.);H}(\zeta) dY_\zeta \right)^p \right] \\
&= \mathbb{E}_{\mu_X}\left[f(X_z)\, e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} \left(\sum_{k=1}^{n(M)} \overset{M}{\alpha_k} Y(1_{\square_k^M}) \right)^p \right] \\
&= \mathbb{E}_{\mu_X}\left[f(X_z)\, e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} \sum_{i_1 \cdots i_p = 1}^{n(M)} \overset{M}{\alpha_{i_1}} \cdots \overset{M}{\alpha_{i_p}} Y(1_{\square_{i_1}^M}) \cdots Y(1_{\square_{i_p}^M}) \right] \\
&= \sum_{i_1 \cdots i_p = 1}^{n(M)} \mathbb{E}_{\mu_X} f(X_z)\, e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} \overset{M}{\alpha_{i_1}} \cdots \overset{M}{\alpha_{i_p}} Y(\square_{i_1}^M) \cdots Y(\square_{i_p}^M) \\
&= \mathfrak{I}_p^{Y,z}\left(\mathbb{E}_{\mu_X}\left[f(X_z)\, e^{-\frac{1}{2}\langle \tilde{N} \rangle_z} (P_M k^z_{h(X.);H})^{\otimes p} \right] \right). \qquad (3.74)
\end{aligned}$$

Let us fix arbitrary $M \geq 1$. Note that $\{1_{\square_i^M}; i = 1, 2, \cdots, n(M)\}$ are linearly independent (as elements of $L^2_{H,1}(R_z)$) and $\forall g \in L^2_{H,1}(R_z)$, the projection $P_M g$ can be represented as:

$$P_{M} g = \sum_{k=1}^{n(M)} \alpha_k^{M} 1_{\square_k^M},$$

where

$$\begin{pmatrix} \alpha_1^{M} \\ \cdots \\ \cdots \\ \alpha_{n(M)}^{M} \end{pmatrix} = (E_{M})^{-1} \begin{pmatrix} \left\langle 1_{\square_1^M}, g \right\rangle_{L^2_{H,1}(R_z)} \\ \cdots \\ \cdots \\ \left\langle 1_{\square_{n(M)}^M}, g \right\rangle_{L^2_{H,1}(R_z)} \end{pmatrix},$$

and E_{M} denotes the matrix $\left(\left\langle 1_{\square_i^M}, 1_{\square_j^M} \right\rangle_{L^2_{H,1}(R_z)} \right)_{1 \leq i,j \leq n(M)}$.

Setting $g = k^z_{h(X_.);H}$, define

$$\alpha_i^{M}(X) := \sum_{j=1}^{n(M)} \left((E_{M})^{-1}\right)_{ij} \left\langle 1_{\square_j^M}, k^z_{h(X_.);H} \right\rangle_{L^2_{H,1}(R_z)}, \quad i = 1, \cdots, n(M). \tag{3.75}$$

Then by (3.74)

$$\sum_{p=0}^{\infty} \frac{1}{p!} \mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \left(\int_{R_z} P_M k^z_{h(X_.);H}(\zeta) dY_\zeta \right)^p \right]$$

$$
\begin{aligned}
&= \sum_{p=0}^{\infty} \frac{1}{p!} \mathfrak{I}_p^{Y,z} \left(\mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} (P_M k^z_{h(X_\cdot);H})^{\otimes p} \right] \right) \\
&= \sum_{p=1}^{\infty} \frac{1}{p!} \mathfrak{I}_p^{Y,z} \Bigg(\mathbb{E}_{\mu_X} \Bigg[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \sum_{i_1,\cdots,i_p=1}^{n(M)} \left(\prod_{k=1}^{p} \alpha^{M}_{i_k}(X) \right) \\
&\quad \times 1_{\square^M_{i_1}} \otimes \cdots \otimes 1_{\square^M_{i_p}} \Bigg] \Bigg) + \mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \right] \\
&= \sum_{p=1}^{\infty} \frac{1}{p!} \mathfrak{I}_p^{Y,z} \Bigg(\sum_{i_1,\cdots,i_p=1}^{n(M)} \mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \prod_{k=1}^{p} \alpha^{M}_{i_k}(X) \right] 1_{\square^M_{i_1}} \otimes \cdots \otimes 1_{\square^M_{i_p}} \Bigg) \\
&\quad + \mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \right] \\
&= \sum_{p=1}^{\infty} \frac{1}{p!} \Bigg(\sum_{i_1,\cdots,i_p=1}^{n(M)} \mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \prod_{k=1}^{p} \alpha^{M}_{i_k}(X) \right] Y(\square^M_{i_1}) \dots Y(\square^M_{i_p}) \Bigg) \\
&\quad + \mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \right], \qquad (3.76)
\end{aligned}
$$

where the series converge in $L^2(P_0)$ uniformly in M by (3.73) and (3.74). Note that

$$
\begin{aligned}
\prod_{k=1}^{p} \alpha^{M}_{i_k}(X) &= \sum_{j_1,\cdots,j_p=1}^{n(M)} \left[\prod_{k=1}^{p} \left((E_M)^{-1} \right)_{i_k, j_k} \right] \int_{R^p_z} \Big\{ \left(k^z_{h(X_\cdot);H} \right)^{\otimes p} (u^1, \cdots, u^p) \\
&\quad \times \left(\int_{\square^M_{j_1}} \phi_H(u^1, \zeta) d\zeta \right) \times \cdots \times \left(\int_{\square^M_{j_p}} \phi_H(u^p, \zeta) d\zeta \right) \Big\} du^1 \dots du^p,
\end{aligned}
$$

therefore,

$$
\begin{aligned}
\mathbb{E}_{\mu_X} \left[f(X_z)\, e^{-\frac{1}{2}\langle \widetilde{N} \rangle_z} \prod_{k=1}^{p} \alpha^{M}_{i_k}(X) \right] &= \sum_{j_1,\cdots,j_p=1}^{n(M)} \Big\{ \left((E_M)^{-1} \right)_{i_1, j_1} \cdots \left((E_M)^{-1} \right)_{i_p, j_p} \\
&\quad \times \int_{R^p_z} \ell_p(f, z | u^1, \cdots, u^p) \left(\prod_{k=1}^{p} \int_{\square^M_{j_k}} \phi_H(u^k, \zeta) d\zeta \right) du^1 \dots du^p \Big\} \\
&= \int_{R^p_z} \ell_p(f, z | u^1, \cdots, u^p) \beta^{M}_{i_1,\cdots,i_p}(u^1, \cdots, u^p) du^1 \dots du^p, \qquad (3.77)
\end{aligned}
$$

where $\ell_p(f, z|\cdot)$ is defined as in (3.72) and

$$
\begin{aligned}
\beta^{M}_{i_1,\cdots,i_p}(u^1, \cdots, u^p) \;&:=\; \sum_{j_1,\cdots,j_p=1}^{n(M)} \Big\{ \left((E_M)^{-1} \right)_{i_1, j_1} \dots \left((E_M)^{-1} \right)_{i_p, j_p} \\
&\quad \times \left(\int_{\square^M_{j_1}} \phi_H(u^1, \zeta) d\zeta \right) \times \cdots \times \left(\int_{\square^M_{j_p}} \phi_H(u^p, \zeta) d\zeta \right) \Big\}.
\end{aligned}
$$

Therefore, we obtain that

$$\sum_{p=0}^{\infty}\frac{1}{p!}\mathbb{E}_{\mu_X}\left[f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\left(\int_{R_z}P_M k^z_{h(X.);H}(\zeta)dY_\zeta\right)^p\right]$$

$$=\sum_{p=0}^{\infty}\frac{1}{p!}S_p\left(\ell_p(f,z|\cdot),\pi_M,Y\right), \tag{3.78}$$

where $S_0(\cdot,\cdot,\cdot)\equiv\mathbb{E}_{\mu_X}\left[f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\right]$, and for arbitrary $g_p\in L^2_{H,p}(R^p_z)$, $p\geq 1$,

$$\begin{aligned} S_p\left(g_p,\pi_M,Y\right) \;&:=\; \sum_{i_1,\cdots,i_p=1}^{n(M)}\left[\int_{R^p_z}g_p(u^1,\cdots,u^p)\beta^M_{i_1,\cdots,i_p}(u^1,\cdots,u^p)du^1\ldots du^p\right] \\ &\times\;\; Y(\square^M_{i_1})\ldots Y(\square^M_{i_p}), \end{aligned}$$

and the series $\sum_{p=0}^{k}\frac{1}{p!}S_p\left(\ell_p(f,z|\cdot),\pi_M,Y\right)$, converges in $L^2(P_0)$ uniformly in M as $k\to\infty$, since the convergence in (3.76) is uniform in M. Notice that in showing (3.78), we also have shown that

$$S_p(\ell_p(f,z|\cdot),\pi_M,Y)=\mathfrak{I}^{Y,z}_p\left(\mathbb{E}_{\mu_X}f(X_z)\,e^{-\frac{1}{2}\langle\widetilde{N}\rangle_z}\left(P_M k^z_{h(X.);H}\right)^{\otimes p}\right)$$

$\forall\ p=1,2,.....$

Since for each $p\geq 1$,

$$S_p(\ell_p(f,z|\cdot),\pi_M,Y)\overset{L^2(P_0)}{\longrightarrow}\mathfrak{I}^{Y,z}_p\left(\ell_p(f,z|\cdot)\right)\ \text{ as } M\to\infty, \tag{3.79}$$

it follows by the uniform convergence of the series that

$$\sum_{p=0}^{M}\frac{1}{p!}S_p(\ell_p(f,z|\cdot),\pi_M,Y)\overset{L^2(P_0)}{\longrightarrow}\sum_{p=0}^{\infty}\frac{1}{p!}\mathfrak{I}^{Y,z}_p(\ell_p(f,z|\cdot))\text{ as }M\to\infty,$$

and since we have shown that convergence in $L^2(P_0)$ implies convergence in $L^1(P)$, we have the result that

$$\sum_{p=0}^{M}\frac{1}{p!}S_p(\ell_p(f,z|\cdot),\pi_M,Y)\overset{L^1(P)}{\longrightarrow}\sum_{p=0}^{\infty}\frac{1}{p!}\mathfrak{I}^{Y,z}_p(\ell_p(f,z|\cdot))\text{ as }M\to\infty. \tag{3.80}$$

Substituting (3.80) into (3.61) and setting $f\equiv 1$ in the denominator gives (3.71).

$\square$

CHAPTER IV

Inverse stochastic transfer principle

4.1 Introduction

In their 2002 paper [22], Pérez-Abreu and Tudor give a stochastic transfer principle enabling one to represent the multiple fractional stochastic integrals of deterministic functions f as Itô type multiple stochastic integrals of a deterministic operator $\Gamma_{H,T}^{(n)}(f)$. The operator $\Gamma_{H,T}^{(n)} : L_{H,n}^2(T^n) \to L^2(T^n)$ is defined in terms of multivariate Riemann-Liouville fractional integrals, which will be discussed in section 4.2. These very nice results allows one to take many of the known properties of multiple stochastic integrals with respect to Brownian motion and apply these properties to multiple fractional stochastic integrals. A similar transfer principle for general Volterra Gaussian process has been developed in [1].

However, the inverse relation, giving a scheme for representing multiple Wiener Itô integrals of a deterministic function in terms of a multiple *fractional* Wiener integral is not immediately clear. This stems from the fact that the Riemann-Liouville fractional integrals of functions $f \in L^p$ are invertible only for a certain subclass of L^p. As these fractional integrals play a key role in the definition of $\Gamma_{H,T}^{(n)}$, whether there exists an inverse operator depends on the inversion properties of the Riemann-Liouville fractional integrals.

From the transfer principle in [22], the form of the left inverse of the $\Gamma_{H,T}^{(n)}$ seems very intuitive. However, $\Gamma_{H,T}^{(n)}$ may not be invertible. Obviously for functions f which can be represented as $\Gamma_{H,T}^{(n)}\phi$ for some $\phi \in L_{H,n}^2(T^n)$, $\Gamma_{H,T}^{(n)}$ is invertible. The fact that the operator involves n-dimensional fractional integrals however, makes the invertibility non-trivial. In this chapter we give two main results. The first result gives a class of functions for which the multiparameter Riemann-Liouville fractional integral operator is invertible on a compact domain. In proving this result, we give an explicit method for identifying a function φ such that f is equal to the Riemann-Liouville fractional integral of φ, provided the function f lives in $I^\alpha(L^p)$. The second result gives an explicit class of functions $L_{\Phi,H}^2$ for which $\Gamma_{H,T}^{(n)}$ from [22] is invertible. Inclusion in this class can be verified by checking the existence of the limits specified in section 4.4. For such functions we establish the Inverse Stochastic Transfer Principle (ISTP).

The ISTP is applicable in a number of settings. Firstly, in combination with the direct stochastic transfer principle, it allows one to obtain general representations of multiple fractional stochastic integrals of Hurst index H_1 in terms of multiple fractional stochastic integrals of Hurst index H_2, for arbitrary $H_1, H_2 \in (0,1)$. Secondly, it is a useful tool in certain problems where fractional chaos decomposition is desired. For example, in many continuous time nonlinear filtering applications with fractional noise, the goal is to describe the optimal filter (which is the best mean-square estimate of an underlying signal of interest) in terms of the trajectory of the observation process (call it, Y). When the noise corrupting the observation is fractional white noise, then, under an appropriate reference measure, Y is a fractional Brownian motion. In [5] and [18] it has been shown that the optimal filter can, with probability one, be represented as a ratio of infinite series of multiple stochastic integrals of various types. The latter integrals can be taken with respect to a suitably constructed ordinary Brownian motion (with an appropriately defined integrand). However, it is much more natural and computationally efficient to describe the optimal filter in terms of multiple fractional integrals with respect to observation process Y itself ([2], [5]). The inverse transfer principle allows to easily transform integral expansions of the optimal filter with respect to ordinary Brownian motion to the more natural and numerically convenient expansion involving multiple fractional integrals with respect to the observation process. Such a transformation of the integral expansions makes numerical approximations of the optimal filter possible whereas the implementation of the original filter approximation is computationally inefficient.

This chapter is organized as follows. In section 4.2, we give an introduction to multidimensional fractional calculus, laying out the tools we will need throughout our analysis. In section 4.3 we give criteria that guarantee a function f belongs to $I(L^p)$ on a compact domain. The criteria also establish a way of finding such a function $\varphi \in L^p$ such that f is equal to the fractional integral of φ. In section 4.4, we establish the Inverse Stochastic Transfer Principle. In section 4.5 we give an example of an application of the ISTP applied to integral expansions of optimal filters for a nonlinear filtering problem with fractional noise.

4.2 Multivariate fractional calculus

In this section we discuss concepts of fractional calculus which will be used throughout the chapter, namely fractional integration and differentiation of functions of many variables. As in [24] (section 24, Chapter 5), let us make use of the following extensions of the Riemann-Liouville fractional integral and derivative.

Definition IV.1. Let $[a, b] = [a_1, b_1] \times \cdots \times [a_n, b_n]$, where each $[a_i, b_i]$ are fixed intervals in $\mathbb{R}$. For the function $\varphi(x) : [a,b] \to \mathbb{R}$, $\varphi \in L^1([a,b])$, the right-sided mixed fractional integral of order $\alpha = (\alpha_1, ..., \alpha_n)$, is given by

$$(4.1)\quad (I^{\alpha,n}_{b-}\varphi)(x) = (I^{\alpha_1}_{b_1-} \circ ... \circ I^{\alpha_n}_{b_n-}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b_1} ... \int_{x_n}^{b_n} \frac{\varphi(t)dt}{(t-x)^{1-\alpha}}, \quad 0 \prec \alpha \prec 1$$

where $(t-x)^{1-\alpha} = (t_1 - x_1)^{1-\alpha_1}....(t_n - x_n)^{1-\alpha_n}$, $\Gamma(\alpha) = \Gamma(\alpha_1)\cdots\Gamma(\alpha_n)$ and $dt = dt_1...dt_n$.

Definition IV.2. For φ as in definition IV.1, the right-sided mixed Riemann-Liouville fractional derivative of order $\alpha = (\alpha_1, ..., \alpha_n)$ is given by

$$(4.2)\qquad (\mathcal{D}^{\alpha,n}_{b-}\varphi)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial^n}{\partial x_1 ... \partial x_n} \int_{x_1}^{b_1} ... \int_{x_n}^{b_n} \frac{\varphi(t)dt}{(t-x)^{\alpha}} \quad 0 \prec \alpha \prec 1$$

where $(t-x)^{\alpha} = (t_1 - x_1)^{\alpha_1}....(t_n - x_n)^{\alpha_n}$, $\Gamma(1-\alpha) = \Gamma(1-\alpha_1)\cdots\Gamma(1-\alpha_n)$ and $dt = dt_1...dt_n$, provided φ is such that the derivative on the right side of (4.2) exists.

If a solution exists to the multidimensional Abel equation

$$(4.3)\qquad \frac{1}{\Gamma(\alpha)} \int_{x_1}^{b_1} ... \int_{x_n}^{b_n} \frac{f(t)dt}{(t-x)^{1-\alpha}} = \varphi(x), \quad x \prec b,$$

then clearly $f(x) = (\mathcal{D}^{\alpha,n}_{b-}\varphi)(x)$ is the unique solution. Hence $\mathcal{D}^{\alpha,n}_{b-}(I^{\alpha,n}_{b-}f)(x) = f(x)$. However, in general it is not the case that $I^{\alpha,n}_{b-}(\mathcal{D}^{\alpha,n}_{b-}f)(x) = f(x)$. If there exists a function ϕ satisfying $I^{\alpha,n}_{b-}\phi(x) = f(x)\ \forall x \in [a,b]$, then clearly $I^{\alpha,n}_{b-}(\mathcal{D}^{\alpha,n}_{b-}f)(x) = f(x)$. In order to investigate existence of such a ϕ, we need to introduce the (mixed) finite difference operator $\Delta^{k,n}_h$ acting on functions $f : \mathbb{R}^n \to \mathbb{R}$ parameterized by order vector k and step vector h, defined by

$$(4.4)\qquad (\Delta^{k,n}_h f)(x) = \sum_{0 \preceq j \preceq k} (-1)^{|j|} \binom{k}{j} f(x - j \bullet h)$$

where $j = (j_1, ..., j_n), k = (k_1, ..., k_n)$ are vectors of integers in $\mathbb{R}^n$, and we use of the following notation,

$$\binom{k}{j} := \prod_{i=1}^{n} \binom{k_i}{j_i},\ \ j \bullet t := (j_1 t_1, ..., j_n t_n),\ \text{and}\ |j| := \sum_{i=1}^{n} j_i.$$

For notational simplicity, let us define the function

$$\theta_n(x;t,\alpha,k) = \frac{\prod_{i=1}^n \alpha_i^{k_i}}{\prod_{i=1}^n t_i^{(1+\alpha_i)k_i}(b_i - x_i)^{\alpha_i(1-k_i)}}, \quad x \in \mathbb{R}^n \tag{4.5}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \in (0,1)$ $\forall i \in \{1,...,n\}$. We also introduce the integral operator

(4.6)

$$\left(\Phi^{b,n}_{\alpha,k} f\right)(x) = \left(\prod_{i=1}^n (b_i - x_i)^{k_i - 1}\right) \int_{t_1=0}^{b_1-x_1} \cdots \int_{t_n=0}^{b_n-x_n} \theta_n(x;t,\alpha,k)(\Delta^{k,n}_{-t} f)(x) dt_n ... dt_1.$$

Similarly, define the operator

(4.7)

$$\left(\Phi^{b,n}_{\epsilon,\alpha,k} f\right)(x) = \left(\prod_{i=1}^n (b_i - x_i - \epsilon_i)^{k_i - 1}\right) \int_{t_1=\epsilon_1}^{b_1-x_1} \cdot\cdot \int_{t_n=\epsilon_n}^{b_n-x_n} \theta_n(x;t,\alpha,k)(\Delta^{k,n}_{-t} f)(x) dt_n ... dt_1$$

where $\epsilon = (\epsilon_1, ... \epsilon_n)$.

Definition IV.3. For function $f : [a,b] \to \mathbb{R}$,

$$\mathbf{D}^{\alpha,n}_{b-,\epsilon} f(x) \equiv \left(\prod_{i=1}^n \frac{1}{\Gamma(1-\alpha_i)}\right) \sum_{0 \preceq k \preceq 1} \left(\Phi^{b,n}_{\epsilon,\alpha,k} f\right)(x), \tag{4.8}$$

is called the **truncated Marchaud fractional derivative**, and, where the L^p limit exists

$$\mathbf{D}^{\alpha,n}_{b-} f(x) \equiv \lim_{\epsilon \to 0} \mathbf{D}^{\alpha,n}_{b-,\epsilon} f(x), \tag{4.9}$$

is called the **Marchaud fractional derivative**, where the sum over $0 \preceq k \preceq 1$ represents the sum over all vectors $k \in \{0,1\}^n$.

Remark For example, when $n = 1$, $\mathbf{D}^{\alpha,n}_{b-,\epsilon} f(x)$ is the sum of two terms,

$$\mathbf{D}^{\alpha,n}_{b-,\epsilon} f(x) = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_{t=\epsilon}^{b-x} \frac{f(x) - f(t+x)}{t^{1+\alpha}} dt \tag{4.10}$$

In the case $n = 2$,

$$\Big(\prod_{i=1}^{2}\Gamma(1-\alpha_i)\Big)\mathbf{D}^{\alpha,n}_{b-,\epsilon}f(x) = \frac{f(x_1,x_2)}{(b_1-x_1)^{\alpha_1}(b_2-x_2)^{\alpha_2}} + \alpha_1\int_{t_1=\epsilon_1}^{b_1-x_1}\frac{f(x_1,x_2)-f(t_1+x_1,x_2)}{t_1^{1+\alpha_1}(b_2-x_2)^{\alpha_2}}dt_1$$

$$+\alpha_2\int_{t_2=\epsilon_2}^{b_2-x_2}\frac{f(x_1,x_2)-f(x_1,t_2+x_2)}{t_2^{1+\alpha_2}(b_1-x_1)^{\alpha_1}}dt_2$$

$$+\alpha_1\alpha_2\int_{t_1=\epsilon_1}^{b_1-x_1}\int_{t_2=\epsilon_2}^{b_2-x_2}\frac{f(x_1,x_2)-f(t_1+x_1,x_2)-f(x_1,t_2+x_2)+f(t_1+x_1,t_2+x_2)}{t_1^{1+\alpha_1}t_2^{1+\alpha_2}}dt_2dt_1, \tag{4.11}$$

the sum of four terms. It is easy to see that for general n, $\mathbf{D}^{\alpha,n}_{b-,\epsilon}f(x)$ is given by the sum of 2^n such terms where the numerator of the integrand in each term is given by the variation of f along the axes with respect to which the integral is taken.

4.3 Representation of fractional integrals of many variables

In this section we give conditions on functions $f : [a,b] \to \mathbb{R}$ which ensure that there exists a function $\phi \in L^p([a,b])$ such that $f(x) = (I^{\alpha,n}_{b-}\phi)(x)$ for all $x \in [a,b]$, where $[a,b] = [a_1,b_1] \times \cdots \times [a_n,b_n]$. When looking at the truncated Marchaud fractional derivative, the form of $\left(\Phi^{b,n}_{\epsilon,\alpha,k}f\right)(x)$ depends on the sub-region x falls in. We adopt the very important convention of [24], and assume that the function $f : [a,b] \to \mathbb{R}$ vanishes outside the region $[a,b]$.

Definition IV.4. For vectors $a = (a_1,...,a_n)$, $b = (b_1,...,b_n)$ and $\epsilon = (\epsilon_1,...,\epsilon_n)$, where $0 < \epsilon_i < b_i - a_i$ $\forall i = 1,...,n$, define

$$\mathcal{R}(\epsilon) = \mathcal{R}_{a,b}(\epsilon) = [a,b] - \big([a_1,b_1-\epsilon_1) \times \cdots \times [a_n,b_n-\epsilon_n)\big).$$

We will partiion $\mathcal{R}(\epsilon)$ into $2^n - 1$ regions. Consider all rectangles of the form $I_1^{(\epsilon)} \times \cdots \times I_n^{(\epsilon)}$ where exactly k of the $I_j^{(\epsilon)}$'s are of the form $I_j^{(\epsilon)} = [b_j-\epsilon_j, b_j]$, and the remaining $n-k$ of the $I_j^{(\epsilon)}$'s are of the form $I_j^{(\epsilon)} = [a_j, b_j-\epsilon_j)$. There are $\binom{n}{k}$ such rectangles, let us call them $\mathcal{R}_1^k, ..., \mathcal{R}_{\binom{n}{k}}^k$, so that we have

$$\mathcal{R}(\epsilon) = \bigcup_{k=1}^{n} \bigcup_{r=1}^{\binom{n}{k}} R_r^k.$$

To see how these regions affect the form of Marchaud fractional derivative let us look at an example with $n = 2$. Take the last term in (4.11),

$$\int_{t_1=x_1+\epsilon_1}^{b_1} \int_{t_2=x_2+\epsilon_2}^{b_2} \frac{f(x_1,x_2) - f(t_1,x_2) - f(x_1,t_2) + f(t_1,t_2)}{(t_1-x_1)^{1+\alpha_1}(t_2-x_2)^{1+\alpha_2}} dt_2 dt_1. \tag{4.12}$$

If $a_1 \le x_1 < b_1 - \epsilon_1,\ b_2 - \epsilon_2 \le x_2 \le b_2$, that is $x \in \mathcal{R}_2^1$, since f vanishes outside $[a,b]$, the last two terms in the numerator of the integrand become zero and we are left with

$$\left(\frac{1}{\epsilon_2^{\alpha_2}} - \frac{1}{(b_2-x_2)^{\alpha_2}}\right) \int_{t_1=x_1+\epsilon_1}^{b_1} \frac{f(x_1,x_2) - f(t_1,x_2)}{(t_1-x_1)^{1+\alpha_1}} dt_1$$

and if $b_1 - \epsilon_1 \le x_1 \le b_1,\ b_2 - \epsilon_2 \le x_2 \le b_2$ ($x \in \mathcal{R}_1^2$) then (4.12) becomes

$$f(x_1,x_2)\left(\frac{1}{\epsilon_1^{\alpha_1}} - \frac{1}{(b_1-x_1)^{\alpha_1}}\right)\left(\frac{1}{\epsilon_2^{\alpha_2}} - \frac{1}{(b_2-x_2)^{\alpha_2}}\right).$$

Theorem IV.5. *Let $[a,b]$ be a compact rectangle in $\mathbb{R}^n$. Let $f : [a,b] \to \mathbb{R}$ be a measurable function on $[a,b]$. Suppose $1 \le p < \infty$, and $\alpha_i \in (0, \frac{1}{p})\ \forall i$. If $f \in L^p([a,b])$ and the L^p limit*

$$\lim_{\substack{\epsilon\to 0\\(L^p)}} \left(\Phi_{\epsilon,\alpha,k}^{b,n} f\right)(\cdot) \tag{4.13}$$

exists $\forall\ 0 \preceq k \preceq 1$ (i.e. for all vectors $k \in \{0,1\}^n$) where $\epsilon = (\epsilon_1, ..., \epsilon_n)$ and $\Phi_{\epsilon,\alpha,k}^{b,n}$ is given by (4.7), then there exists function $\varphi \in L^p([a,b])$ such that $f(x) = I_{b-}^{\alpha,n}\varphi(x)$, for all $x \in [a,b]$. Moreover, φ is equal to $\lim_{\substack{\epsilon\to 0\\(L^p)}} \sum_{0\preceq k\preceq 1} \left(\Phi_{\epsilon,\alpha,k}^{b,n} f\right)(\cdot)$.

Proof: Let $f \in L^p([a,b])$ and consider the function

$$\varphi_\epsilon^{(n)}(x) = \left(\prod_{i=1}^{n} \frac{1}{\Gamma(1-\alpha_i)}\right) \sum_{0\preceq k\preceq 1} \left(\Phi_{\epsilon,\alpha,k}^{b,n} f\right)(x). \tag{4.14}$$

It is easy to check directly that $\varphi_\epsilon^{(n)}(x) \in L^p([a,b])$ and when the limit exists, $\lim_{\epsilon\to 0}\varphi_\epsilon^{(n)}(x) := \varphi^{(n)} \in L^p([a,b])$. We will prove that $f = I_{b-}^{\alpha,n}\varphi^{(n)}$. Since the operator $I_{b-}^{\alpha,n}$ is continuous in L^p (see [24]) it is enough to show $f = \lim_{\epsilon\to 0} I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}(x)$, where the limit is taken in L^p.

Step 1 Let us first focus on the region where $x_i < b_i - \epsilon_i\ \forall i$. We first show that over this region,

(4.15)
$$I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}(x) = \int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}} \cdots \int_{t_n=0}^{\frac{b_n-x_n}{\epsilon_n}} \left(\prod_{i=1}^{n} \mathcal{K}_{\alpha_i}(t_i)\right) f(t_1\epsilon_1 + x_1, ..., t_n\epsilon_n + x_n)dt_n...dt_1,$$

where $\mathcal{K}_{\alpha_i}(t_i) = \frac{1}{\Gamma(\alpha_i)\Gamma(1-\alpha_i)}\left(\frac{t_i^{\alpha_i}-(t_i-1)_+^{\alpha_i}}{t_i}\right)$, which has the property that

$$\int_{t_i=0}^{\infty} \mathcal{K}_{\alpha_i}(t_i)dt_i = 1. \tag{4.16}$$

The right hand side of (4.15) can be rewritten as

$$\int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}} \mathcal{K}_{\alpha_1}(t_1)... \int_{t_{n-1}=0}^{\frac{b_{n-1}-x_{n-1}}{\epsilon_{n-1}}} \mathcal{K}_{\alpha_{n-1}}(t_{n-1})\left[\int_{t_n=0}^{\frac{b_n-x_n}{\epsilon_n}} t_n^{\alpha_n-1} f(t_1\epsilon_1 + x_1, ..., t_n\epsilon_n + x_n)dt_n\right.$$
$$\left. - \int_{t_n=1}^{\frac{b_n-x_n}{\epsilon_n}} \frac{(t_n-1)^{\alpha_n}}{t_n} f(t_1\epsilon_1 + x_1, ..., t_n\epsilon_n + x_n)dt_n\right] dt_{n-1}...dt_1 \times \frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_n)}.$$

Making the substitution $y_n = t_n\epsilon_n + x_n$, the term inside brackets becomes

(4.17)
$$\frac{1}{\epsilon_n^{\alpha_n}}\int_{x_n}^{b_n} \frac{f(t_1\epsilon_1 + x_1, ..., y_n)}{(y_n - x_n)^{1-\alpha_n}}dy_n - \frac{1}{\epsilon_n^{\alpha_n}}\int_{x_n+\epsilon_n}^{b_n} \frac{f(t_1\epsilon_1 + x_1, ..., y_n)(y_n - x_n - \epsilon_n)^{\alpha_n}}{(y_n - x_n)}dy_n.$$

Using the fact that

$$\frac{(y-\epsilon-x)^\alpha}{\epsilon^\alpha(y-x)} = \int_{s=x}^{y-\epsilon} \frac{1}{(s-x)^{1-\alpha}(y-s)^{1+\alpha}}ds\ (x < y),$$

which follows from the indefinite integral

$$\int \frac{1}{(s-x)^{1-\alpha}(y-s)^{1+\alpha}}ds = \frac{(y-s)^{-\alpha}(s-x)^\alpha}{\alpha(y-x)},$$

(4.17) becomes

(4.18)
$$\frac{1}{\epsilon_n^{\alpha_n}}\int_{x_n}^{b_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(y_n-x_n)^{1-\alpha_n}}dy_n-\alpha_n\int_{s_n=x_n}^{b_n-\epsilon_n}\int_{y_n=s_n+\epsilon}^{b_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(s_n-x_n)^{1-\alpha_n}(y_n-s_n)^{1+\alpha_n}}ds_n dy_n.$$

Adding and subtracting the term

$$\int_{y_n=x_n}^{b_n-\epsilon_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(y_n-x_n)^{1-\alpha_n}(b_n-y_n)^{\alpha_n}}dy_n-\frac{1}{\epsilon^{\alpha_n}}\int_{y_n=x_n}^{b_n-\epsilon_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(y_n-x_n)^{1-\alpha_n}}dy_n$$
$$+\int_{y_n=b_n-\epsilon_n}^{b_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(y_n-x_n)^{1-\alpha_n}(b_n-y_n)^{\alpha_n}}dy_n$$

(4.18) becomes

$$\int_{y_n=x_n}^{b_n}\frac{1}{(y_n-x_n)^{1-\alpha_n}}\left\{\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(b_n-y_n)^{\alpha_n}}+\alpha_n\int_{s_n=y_n+\epsilon_n}^{b_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)-f(t_1\epsilon_1+x_1,...,s_n)}{(s_n-y_n)^{1+\alpha_n}}ds_n\right\}dy_n.$$

So that the right hand side of (4.15) is equal to

$$\frac{1}{\Gamma(\alpha_n)\Gamma(1-\alpha_n)}\int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}}\mathcal{K}_{\alpha_1}(t_1)...\int_{t_{n-1}=0}^{\frac{b_{n-1}-x_{n-1}}{\epsilon_{n-1}}}\mathcal{K}_{\alpha_{n-1}}(t_{n-1})\Bigg[\int_{y_n=x_n}^{b_n}\frac{1}{(y_n-x_n)^{1-\alpha_n}}\Bigg\{\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(b_n-y_n)^{\alpha_n}}$$

$$+\alpha_n\int_{s_n=y_n+\epsilon_n}^{b_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)-f(t_1\epsilon_1+x_1,...,s_n)}{(s_n-y_n)^{1+\alpha_n}}ds_n\Bigg\}dy_n\Bigg]dt_{n-1}...dt_1.$$

Considering $f(t_1\epsilon_1+x_1,...,y_n)$ as a function of y_n only, and letting

$$\begin{aligned}&\varphi_\epsilon^{(1)}(t_1\epsilon_1+x_1,...,y_n)\\&\quad=\frac{1}{\Gamma(1-\alpha_n)}\Bigg[\frac{f(t_1\epsilon_1+x_1,...,y_n)}{(b_n-y_n)^{\alpha_n}}\\&\quad+\alpha_n\int_{s_n=y_n+\epsilon_n}^{b_n}\frac{f(t_1\epsilon_1+x_1,...,y_n)-f(t_1\epsilon_1+x_1,...,s_n)}{(s_n-y_n)^{1+\alpha_n}}ds_n\Bigg],\end{aligned}$$

the right hand side of (4.15) is equal to

$$\left(\prod_{i=1}^{n-1}\frac{1}{\Gamma(\alpha_i)\Gamma(1-\alpha_i)}\right)\int_{t_1=0}^{\frac{b_1-x_1}{\epsilon_1}}\left(\frac{t_1^{\alpha_1}-(t_1-1)_+^{\alpha_1}}{t_1}\right)...$$

(4.19)
$$...\int_{t_{n-1}=0}^{\frac{b_{n-1}-x_{n-1}}{\epsilon_{n-1}}}\left(\frac{t_{n-1}^{\alpha_{n-1}}-(t_{n-1}-1)_+^{\alpha_{n-1}}}{t_{n-1}}\right)I_{b-}^{\alpha,1}\varphi_\epsilon^{(1)}(t_1\epsilon_1+x_1,...,y_n)dt_1...dt_{n-1}dy_n,$$

where $I^{\alpha,1}_{b-}\varphi$ is the mixed fractional integral, that is the single parameter Riemann-Liouville fractional integral taken over only the nth axis. Using the fact that mixed fractional integrals of $L^p([a,b])$ functions are in $L^p([a,b])$ if $\alpha_i < \frac{1}{p}$ $\forall i$ (see [24]), applying Fubini's Theorem and repeating the above arguments in each of the remaining $n-1$ coordinates gives (4.15).

In view of (4.15) and (4.16),

$$I^{\alpha,n}_{b-}\varphi^{(n)}_\epsilon(x) - f(x)$$

(4.20)
$$= \int_{t_1=0}^{\infty}\cdots\int_{t_n=0}^{\infty}\left(\prod_{i=1}^{n}\mathcal{K}_{\alpha_i}(t_i)\right)\Big[f(t_1\epsilon_1+x_1,...,t_n\epsilon_n+x_n)-f(x_1,...,x_n)\Big]dt_n...dt_1$$

for $x \in [a_1, b_1-\epsilon_1)\times\cdots\times[a_n,b_n-\epsilon_n)$.

Step 2 Next we show $\|I^{\alpha,n}_{b-}\varphi^{(n)}_\epsilon\|_{L^p(\mathcal{R}(\epsilon))} \to 0$ as $\epsilon \to 0$. Specifically we will examine the norm over individual regions making up $\mathcal{R}(\epsilon)$, as

$$\|I^{\alpha,n}_{b-}\varphi^{(n)}_\epsilon(x)\|_{L^p(\mathcal{R}(\epsilon))} \le \sum_{\ell=1}^{n}\sum_{r=1}^{\binom{n}{\ell}}\|I^{\alpha,n}_{b-}\varphi^{(n)}_\epsilon\|_{L^p(\mathcal{R}^\ell_r(\epsilon))}, \tag{4.21}$$

where each term in the sum satisfies

$$\|I^{\alpha,n}_{b-}\varphi^{(n)}_\epsilon(x)\|_{L^p(\mathcal{R}^\ell_r(\epsilon))} \le \left(\prod_{i=1}^{n}\Gamma(1-\alpha_i)\right)^{-1}\sum_{j=1}^{2^n}\|I^{\alpha,n}_{b-}\Phi^{b,n}_{\epsilon,\alpha,j}f(x)\|_{L^p(\mathcal{R}^\ell_r(\epsilon))} \tag{4.22}$$

where $\Phi^{b,n}_{\epsilon,\alpha,j}f$ are the individual terms (integrals) making up $\varphi^{(n)}_\epsilon(x)$ and each $j \in \{1,...,2^n\}$ represents an element of the set of vectors $0 \preceq k \preceq 1$, that is all vectors $k \in \{0,1\}^n$.

For notational simplicity we will look only at the region $\mathcal{R}^m_1(\epsilon)$, where $b_i - \epsilon_i \le x_i \le b_i$

for $i = 1, ..., m$. All other terms are handled in the same manner. The substitution $q_i = b_i + x_i - t_i,\ \forall i$ yields

$$I_{b-}^{\alpha,n}\Phi_{\epsilon,\alpha,j}^{b,n}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x_1}^{b_1}...\int_{x_n}^{b_n}\frac{\Phi_{\epsilon,\alpha,j}^{b,n}f(b_1+x_1-q_1,...,b_n+x_n-q_n)}{\prod_{i=1}^{n}(b_i-q_i)^{1-\alpha_i}}dq_n...dq_1.$$

Applying the Minkowski inequality for integrals [10],

$$\begin{aligned}
&\|I_{b-}^{\alpha,n}\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p(\mathcal{R}_1^m(\epsilon))}\\
&= \frac{1}{\Gamma(\alpha)}\Bigg(\int_{x_1=b_1-\epsilon_1}^{b_1}\cdots\int_{x_m=b_m-\epsilon_m}^{b_m}\int_{x_{m+1}=0}^{b_{m+1}-\epsilon_{m+1}}\cdots\int_{x_n=0}^{b_n-\epsilon_n}\\
&\left|\int_{q_1=x_1}^{b_1}...\int_{q_n=x_n}^{b_n}\frac{\Phi_{\epsilon,\alpha,j}^{b,n}f(b_1+x_1-q_1,...,b_n+x_n-q_n)}{\prod_{i=1}^{n}(b_i-q_i)^{1-\alpha_i}}dq_n...dq_1\right|^p dx_n...dx_1\Bigg)^{\frac{1}{p}}\\
&\le \frac{1}{\Gamma(\alpha)}\int_{q_1=b_1-\epsilon_1}^{b_1}\cdot\cdot\int_{q_m=b_m-\epsilon_m}^{b_m}\int_{q_{m+1}=0}^{b_{m+1}}\cdot\cdot\int_{q_n=0}^{b_n}\Bigg(\int_{x_1=b_1-\epsilon_1}^{q_1}\cdot\cdot\int_{x_m=b_m-\epsilon_m}^{q_m}\int_{x_{m+1}=0}^{q_{m+1}\wedge(b_{m+1}-\epsilon_{m+1})}\\
&\cdots\int_{x_n=0}^{q_n\wedge(b_n-\epsilon_n)}\left|\frac{\Phi_{\epsilon,\alpha,j}^{b,n}f(b_1+x_1-q_1,...,b_n+x_n-q_n)}{\prod_{i=1}^{n}(b_i-q_i)^{1-\alpha_i}}\right|^p dx_n...dx_1\Bigg)^{\frac{1}{p}}dq_n...dq_1\\
&\le \frac{1}{\Gamma(\alpha)}\int_{q_1=b_{1-\epsilon_1}}^{b_1}\frac{1}{(b_1-q_1)^{1-\alpha_1}}\cdots\int_{q_n=0}^{b_n}\frac{1}{(b_n-q_n)^{1-\alpha_n}}dq_n...dq_1\|\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p(\mathcal{R}_1^m(\epsilon))}\\
&= \frac{1}{\Gamma(\alpha)}\epsilon_1^{\alpha_1}\cdots\epsilon_m^{\alpha_m}b_{m+1}^{\alpha_{m+1}}\cdots b_n^{\alpha_n}\|\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p(\mathcal{R}_1^m(\epsilon))}
\end{aligned}$$

$$\le \frac{1}{\Gamma(\alpha)}\epsilon_1^{\alpha_1}\cdots\epsilon_m^{\alpha_m}b_{m+1}^{\alpha_{m+1}}\cdots b_n^{\alpha_n}\|\Phi_{\epsilon,\alpha,j}^{b,n}f\|_{L^p([a,b])}\to 0\quad \text{as } \epsilon\to 0. \tag{4.23}$$

Similar arguments show that for any $0 \preceq k \preceq 1$, $\|I_{b-}^{\alpha,n}\Phi_{\epsilon,\alpha,k}^{b,n}f\|_{L^p(\mathcal{R}_r^\ell(\epsilon))} \to 0$ for all $\mathcal{R}_r^\ell(\epsilon)$ making up $\mathcal{R}(\epsilon)$. This result along with (4.21) and (4.22) gives

$$\|I_{b-}^{\alpha,n}\varphi_\epsilon^{(n)}\|_{L^p(\mathcal{R}(\epsilon))}\to 0\quad as\ \epsilon\to 0.$$

Step 3 As $f \in L^p(a,b)$,

$$\|f\|_{L^p(\mathcal{R}(\epsilon))}\to 0\quad as\ \epsilon\to 0.$$

Upon combining this with (4.20) and using the properties of mean continuity of L^p functions and the dominated convergence theorem, we obtain that

$$\|I^{\alpha,n}_{b-}\varphi^{(n)}_{\epsilon} - f\|_{L^p(a,b)} \leq \|I^{\alpha,n}_{b-}\varphi^{(n)}_{\epsilon} - f\|_{L^p(\mathcal{R}(\epsilon))} + \int_{t_1=0}^{\infty} \cdots \int_{t_n=0}^{\infty} \left(\prod_{i=1}^{n} \mathcal{K}_{\alpha_i}(t_i) \right)$$
$$\times \Big\| f(t_1\epsilon_1 + x_1, ..., t_n\epsilon_n + x_n) - f(x_1, ..., x_n) \Big\|_{L^p(a,b)} dt_n ... dt_1 \longrightarrow 0$$

as $\epsilon \to 0$ and the desired result follows.

□

4.4 Inverse stochastic transfer principle

In this section we describe the inverse transfer principle and the operator for the class of functions $L^2_{\Phi,H}([0,T]^n)$, which allow us to represent multiple stochastic integrals with respect to Brownian motion in terms of multiple stochastic integrals with respect to fractional Brownian motion.

First, for Hurst parameter $H \in (\frac{1}{2}, 1)$ and for $(t_1, ..., t_n) \in [0,T]^n$, let us recall from [22], the following continuous, isometric operator $\Gamma^{(n)}_{H,T} : L^2_{H,n}([0,T]^n) \to L^2([0,T]^n)$ defined by

$$\begin{aligned} \left(\Gamma^{(n)}_{H,T}(f)\right)(t_1, ..., t_n) \quad &:= \quad (C^*_H)^n \left(\prod_{j=1}^{n} (t_j)^{\frac{1}{2}-H} \right) \\ &\times \left(I^{H-\frac{1}{2},n}_{T-} \left(\prod_{k=1}^{n} s_k^{H-\frac{1}{2}} \right) f(s_1, ...s_n) \right)(t_1, ...t_n) \end{aligned} \tag{4.24}$$

where

$$C^*_H = \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \right)^{\frac{1}{2}}.$$

Next, let us recall the following (direct) stochastic transfer principle of Perez-Abreu and Tudor.

Theorem IV.6. (Stochastic Transfer Principle) ([22]) *For arbitrary* $f \in L^2_{H,n}([0.T]^n)$,

$$\mathcal{I}_n^{B^H}(f) = I_n^{W^B}(\Gamma^{(n)}_{H,T} f) \tag{4.25}$$

where $\mathcal{I}_n^{B^H}(\cdot)$ *is the nth order Wiener integral with respect to* B^H*, taken over* $[0,T]^n$ *and* $I_n(\cdot)$ *is the multiple stochastic integral taken over* $[0,T]^n$ *with respect to the standard Brownian motion* $W_t^{B^H} = \int_{[0,t]} K_H^{-1}(t,s)dB_s^H$.

In order to define the *inverse* stochastic transfer principle, let us introduce the operator $\Gamma^{(n)(-1)}_{H,T} : L^2([0,T]^n) \to L^2_{H,n}([0,T]^n)$,

$$\begin{aligned}\left(\Gamma^{(n)(-1)}_{H,T} f\right)(t_1,...,t_n) \;&:=\; \left(\frac{1}{C_H^*}\right)^n \left(\prod_{j=1}^n (t_j)^{\frac{1}{2}-H}\right) \\ &\times \left(\mathcal{D}_{T-}^{H-\frac{1}{2},n}\left(\prod_{k=1}^n s_k^{H-\frac{1}{2}}\right) f(s_1,...s_n)\right)(t_1,...t_n).\end{aligned} \tag{4.26}$$

We also make use of the following definition:

Definition IV.7. Define $L^2_{\Phi,H}([0,T]^n)$ to be the class of functions $f : [0,T]^n \to \mathbb{R}$ such that $f \in L^2([0,T]^n)$ and

$$\lim_{\substack{\epsilon\to 0\\(L^2)}} \left(\prod_{i=1}^n x_i\right)^{\frac{1}{2}-H} \left(\Phi^{T,n}_{\epsilon,\underline{H}-\frac{1}{2},k} f_H^*\right)(x) \tag{4.27}$$

exists $\forall\ 0 \preceq k \preceq 1$ where Φ is given by (4.7). Here $\underline{H}$ denotes the vector $\underline{H} = (H,...,H)$ so that $\underline{H} - \frac{1}{2} = (H-\frac{1}{2},...,H-\frac{1}{2})$, $\epsilon = (\epsilon_1,...,\epsilon_n)$ and $f_H^*(x_1,...,x_n) := (x_1\cdots x_n)^{H-\frac{1}{2}} f(x_1,...,x_n)$.

Note A trivial example of a function f such that limit (4.27) exists is given by $f(x_1,...,x_n) = C(x_1\cdots x_n)^{\frac{1}{2}-H}$, where $C \in \mathbb{R}$.

Theorem IV.8. *(***Inverse Stochastic Transfer Principle)** *For functions* $f \in L^2_{\Phi,H}([0,T]^n)$*, the following equality holds:*

$$I_n^{W^B}(f) = \mathcal{I}_n^{B^H}(\Gamma^{(n)(-1)}_{H,T} f), \tag{4.28}$$

where $I_n^{WB}(\cdot)$ *and* $\mathcal{I}_n^{BH}(\cdot)$ *are as described in Theorem IV.6.*

Proof First note that $f \in L^2([0,T]^n)$ implies $f_H^* \in L^2([0,T]^n)$ since $x_i^{H-\frac{1}{2}}$ is bounded on the compact interval $[0,T]$ for each i. The existence of (4.27) and boundedness of $x_i^{H-\frac{1}{2}}$ on $[0,T]$ imply the existence of

$$\lim_{\substack{\epsilon\to 0\\(L^2)}} \left(\Phi^{T,n}_{\epsilon,\underline{H}-\frac{1}{2},k} f_H^*\right)(x),$$

for all $0 \preceq k \preceq 1$. Let us therefore define

$$\varphi_{*,\epsilon}^{(n)}(x) := \left(\frac{1}{\Gamma(\frac{3}{2}-H)}\right)^n \sum_{0\preceq k\preceq 1} \left(\Phi^{T,n}_{\epsilon,\underline{H}-\frac{1}{2},k} f_H^*\right)(x).$$

From Theorem (IV.5) it follows that $f_H^* = \lim_{\substack{\epsilon\to 0\\(L^2)}} I_{T-}^{H-\frac{1}{2},n}\varphi_{*.\epsilon}^{(n)}$. By continuity of the fractional integral in L^2 it follows that $f_H^* = \left(I_{T-}^{H-\frac{1}{2},n} \lim_{\substack{\epsilon\to 0\\(L^2)}} \varphi_{*.\epsilon}^{(n)}\right)$.
Next we define the term

(4.29)
$$\phi_{*,\epsilon,H,T}^{(n)}(x) := \left(\frac{1}{\Gamma(\frac{3}{2}-H)}\right)^n \sum_{0\preceq k\preceq 1} \left(\prod_{i=1}^n x_i\right)^{\frac{1}{2}-H} \left(\Phi^{T,n}_{\epsilon\underline{H}-\frac{1}{2},k} f_H^*\right)(x),\ \forall x \in [0,T]^n.$$

Note that the L^2 limit of $\phi_{*,\epsilon,H,T}^{(n)}(x)$ exists as $\epsilon \to 0$, as a consequence of (4.27). We can write f_H^* in terms of $\phi_{*,\epsilon}^{(n)} := \phi_{*,\epsilon,H,b}^{(n)}$ as

$$f_H^*(x) = \left(I_{T-}^{H-\frac{1}{2},n} \lim_{\substack{\epsilon\to 0\\(L^2)}} \left(\prod_{i=1}^n u_i\right)^{H-\frac{1}{2}} \phi_{*.\epsilon}^{(n)}(u)\right)(x)$$

$$= \left(I_{T-}^{H-\frac{1}{2},n} \left(\prod_{i=1}^n u_i\right)^{H-\frac{1}{2}} \lim_{\substack{\epsilon\to 0\\(L^2)}} \phi_{*.\epsilon}^{(n)}(u)\right)(x).$$

And hence

$$f(x) = (x_1 \cdots x_n)^{\frac{1}{2}-H} f_H^*(x_1, ..., x_n)$$
$$= \left(\prod_{i=1}^n x_i\right)^{\frac{1}{2}-H} \left(I_{T-}^{H-\frac{1}{2},n} \left(\prod_{i=1}^n u_i\right)^{H-\frac{1}{2}} \lim_{\substack{\epsilon\to 0\\(L^2)}} \phi_{*.\epsilon}^{(n)}(u)\right)(x)$$

$$(4.30) \qquad = (C_H^*)^n \left(\prod_{i=1}^n x_i\right)^{\frac{1}{2}-H} \left(I_{T-}^{H-\frac{1}{2},n} \left(\prod_{i=1}^n u_i\right)^{H-\frac{1}{2}} \left(\lim_{\substack{\epsilon\to 0\\(L^2)}} (C_H^*)^{-n}\, \phi_{*.\epsilon}^{(n)} \right)(u) \right)(x)$$

So the existence of $\lim_{\substack{\epsilon\to 0\\(L^2)}} \phi_{*,\epsilon}^{(n)}(x)$ implies that for any $f \in L^2([0,T]^n)$, there exists $\phi \in L^2([0,T]^n)$ such that

$$(4.31) \qquad f(x) = (C_H^*)^n \left(\prod_{i=1}^n x_i\right)^{\frac{1}{2}-H} \left(I_{T-}^{H-\frac{1}{2},n} \left(\prod_{i=1}^n u_i\right)^{H-\frac{1}{2}} \phi(u) \right)(x).$$

It follows that $\Gamma_{H,T}^{(n)(-1)} f(x) = \phi(x)$ and $f(x) = \Gamma_{H,T}^{(n)}\phi(x)$ $\forall x \in [0,T]^n$. Using the fact that

$$(4.32) \qquad \widetilde{\Gamma_H^{(n)}\phi(x)} = \Gamma_H^{(n)}\widetilde{\phi(x)},$$

we obtain that

(4.33)

$$I_n^{W^B}(f) = I_n^{W^B}(\Gamma_{H,T}^{(n)}\phi) = I_n^{W^B}(\widetilde{\Gamma_{H,T}^{(n)}\phi}) = I_n^{W^B}(\Gamma_{H,T}^{(n)}\widetilde{\phi}) = \mathcal{I}_n^{B^H}(\widetilde{\phi}) = \mathcal{I}_n^{B^H}(\widetilde{\Gamma_{H,T}^{(n)(-1)} f}),$$

where the third equality follows from (4.32) and the fourth follows from direct application of (4.25). Since $\phi \in L^2([0,T]^n)$, by Theorem III.18, $\phi \in L^2_{H,n}([0,T]^n)$ so that $\mathcal{I}_n^{B^H}(\widetilde{\phi})$ and $I_n^{W^B}(\widetilde{\Gamma_{H,T}^{(n)}\phi})$ are well defined.

□

4.5 An example in the context of nonlinear filtering

As an application of Theorem IV.8, we look at the nonlinear filtering problem described in[5]. Consider the following nonlinear filtering problem for a random process $X = (X_t, t \in [0,T])$, where $T < \infty$, and X is not observed directly. Suppose that a noisy process Y is observed instead and that the observation model, which is of interest in many engineering and economics applications, is given by:

$$(4.34) \qquad Y_t = \int_0^t h(X_s)ds + B_t^H, \quad t \in [0,T]$$

where h is a suitably regular and integrable nonlinear function, $B^H = (B^H_t)$ is a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and the signal process X is assumed to be independent of the observation noise B^H. The goal is to characterize the optimal filter, or, in other words, to study the best mean-square estimate of the signal, conditioning on $\mathcal{F}^Y_t := \sigma\{Y_s : 0 \le s \le t\}$, the σ-field generated by the observation process.

Recall from Chapter III, the following representaion of the optimal filter holds:

$$\mathbb{E}\big[f(X_t)|\mathcal{F}^Y_t\big] \stackrel{a.s.}{=} \frac{\sum_{p=0}^{\infty} \frac{1}{p!} I_p^{W^Y,t}\left(\mathbb{E}\Big[f(X_t)(\delta_{\cdot}(X))^{\otimes p}\Big]\right)}{\sum_{p=0}^{\infty} \frac{1}{p!} I_p^{W^Y,t}\left(\mathbb{E}\Big[(\delta_{\cdot}(X))^{\otimes p}\Big]\right)}, \tag{4.35}$$

where the series converge in L^2, $(\cdot)^{\otimes p}$ denotes the pth order tensor product and $I_p^{W^Y,t}$ denotes the pth order multiple stochastic fractional integral of the Itô type with respect to the Brownian motion W^Y, with the integral taken over $[0, t]$.

By truncating the sum in (4.35) it is possible approximate the optimal filter by

$$\mathbb{E}\big[f(X_t)|\mathcal{F}^Y_t\big] \approx \frac{\sum_{p=0}^{N} \frac{1}{p!} I_p^{W^Y,t}\left(\mathbb{E}\Big[f(X_t)(\delta_{\cdot}(X))^{\otimes p}\Big]\right)}{\sum_{p=0}^{N} \frac{1}{p!} I_p^{W^Y,t}\left(\mathbb{E}\Big[(\delta_{\cdot}(X))^{\otimes p}\Big]\right)}. \tag{4.36}$$

From a practical point of view however, this representation is not easy to implement, since it requires that we compute the trajectory of $(W^Y_t,\ t \in [0.T])$ using the kernel K_H^{-1}, whose form changes at each point t in the parameter space $[0, T]$.

We can use the Inverse Stochastic Transfer Principle to represent the integrals in

(4.36) as multiple fractional stochastic integrals with respect to observation process Y. Namely, provided

$$\mathbb{E}\left(f(X_t)(\delta_{\cdot}(X))^{\otimes p}\right)(\cdot) \in L^2_{\Phi,H}([0,T]^p) \quad \forall p = 1, ..., N, \ \forall t \in [0,T]$$

we can approximate (4.35) by

$$\mathbb{E}\big[f(X_t)|\mathcal{F}^Y_t\big] \approx \frac{\sum_{p=0}^{N} \frac{1}{p!} I_p^{Y,t}\left(\Gamma^{(p)(-1)}_{H,t}\mathbb{E}\left[f(X_t)(\delta_{\cdot}(X))^{\otimes p}\right]\right)}{\sum_{p=0}^{N} \frac{1}{p!} I_p^{Y,t}\left(\Gamma^{(p)(-1)}_{H,t}\mathbb{E}\left[(\delta_{\cdot}(X))^{\otimes p}\right]\right)}, \tag{4.37}$$

where $I_p^{Y,t}$ denotes the pth order multiple stochastic integral of the Itô type with respect to the observation process Y, with the integral taken over $[0,t]$.

The approximation (4.37) allows for numerical implementation of the representations given by (4.35). The Inverse Stochastic Transfer Principle gives multiple integrals with respect to the observation process so we can directly use the observation process to obtain the integrals as opposed to those in (4.35) which require continuously updating the kernel K_H^{-1} to obtain the process W^Y. We should note however, that in this case we will need to continuously update the operator $\Gamma^{(p)(-1)}_{H,t}$ depending on the spatial parameter t.

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