THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

A STUDY OF FORCED OSCILLATION SYSTEMS WITH NONLINEAR RESTORING AND NONLINEAR DAMPING FORCES

Yen San Lim

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1960

December, 1960

IP-479

ACKNOWLEDGMENTS

The author wishes to express his gratitude to the chairman of his Doctoral Committee, Professor Louis F. Kazda, for his encouragement, assistance, and interest throughout the study. The author is especially indebted to Professor Wilfred Kaplan for his constant constructive suggestions and criticism during the preparation of the manuscript. Thanks are due to Professors Alan B. Macnee, Gunnar Hok, and C. Bruce Sharpe for their comments and suggestions.

The author wishes to acknowledge the assistance of Mr. V. L. Larrowe and Miss M. M. Spencer, who helped in the preparation of the computer study, and to the Willow Run Laboratories for the use of their analog computer facilities.

The author is grateful to the Rackham Graduate School of The University of Michigan for the fellowships granted to him to do graduate study.

Thanks are also due to the Industry Program of the College of Engineering at The University of Michigan for the printing of the dissertation.

To his wife, the author expresses his deep appreciation for her encouragement and help in typing the draft of the manuscript.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF FIGURES CONT'D

NOMENCLATURE

x	Variable
t	Time variable
$y = \dot{x} = \frac{dx}{dt}$	
$f(y)$	Nonlinear "damping force" function
$g(x)$	Nonlinear "restoring force" function
I	Bound of x
J	Bound of y
E	Maximum value of the input function $e(t)$
V	Lyapunov function
$b, c, b_1, c_1, c_2, \epsilon$	Constants
ω	Angular frequency
u	Normalized frequency
A	Amplitude of the fundamental component of the output
N	Describing function
φ	Phase angle
d	Dead-space length
J_1	Moment of inertia
M	Coulomb friction
ξ	Damping factor
K	Amplification

CHAPTER I

INTRODUCTION

In the field of mechanics and mathematical physics, practically all problems are nonlinear. Linear treatments are possible only if the systems operate in linear regions or they can be approximated by linear equations. If a dynamical system is described by a nonlinear differential equation, the behavior of the motions of the system in general seems unpredictable. Those standard techniques used for analyzing linear systems are no longer applicable for nonlinear systems. New methods must be found which will provide an insight into the operation of the system. Although a great amount of work has been done in the analysis of nonlinear systems, still little information is available. Since up-to the present time no general method has been found which can be applied to investigate all nonlinear problems, each type of problem must be treated by the particular method adapted to it. In nonlinear oscillation problems having one degree of freedom, few methods are available for analyzing all the physical aspects of a system, and such knowledge is naturally confined for the most part to the discussion of a variety of special cases. This study too, is necessarily limited to a particular class of systems, that is describable by a second order differential equation having the form:*

$$\ddot{x} + f(\dot{x}) + g(x) = e(t) \qquad (1\text{-}1)$$

where $e(t)$ is a periodic forcing function, the function $f(x)$ represents the "damping force" and $g(x)$ represents the "restoring force." It is the purpose of this dissertation to investigate the class of systems which is describable

* $\dot{x}$ denotes the derivative of x with respect to t.

by Equation (1-1) with particular attention being given to the boundedness of the solutions and to the establishment of conditions which must be imposed on the nonlinear characteristics $f(\dot{x})$ and $g(x)$ in order to assure the uniqueness and stability of the periodic solutions (steady state). In addition, a useful procedure is also presented to obtain the first order approximation to the periodic solutions. The theory developed can be readily applied to many physical systems including mechanical, electrical and control systems.

In 1943 Levinson(1) introduced an interesting scheme of constructing a closed curve on the phase-plane for the following fairly general system (excluding the case with coulomb friction)

$$\ddot{x} + f(\dot{x}, x)\dot{x} + g(x) = e(t) \qquad (1\text{-}2)$$

He showed that under certain conditions, every solution path* arriving at the closed curve from outside must cross to the inside, and via the Brouwer's fixed point theorem, there must exist at least one periodic solution having the same period as that of the forcing function. Later Reuther(6,8) showed that all solutions $x(t)$ and $\dot{x}(t)$ of system (1-1) are ultimately bounded by a closed curve of the same nature. However, no explicit bounded region was given. Question may arise as to how large the region will be. In 1957, Loud(9) obtained the explicit bounds for the equation:

$$\ddot{z} + f(z)\dot{z} + g(z) = e(t) \qquad (1\text{-}3)$$

---

* Equivalent to phase trajectory on the phase plane.

But in general it is impossible to reduce Equation (1-1) to (1-3) except in some special cases. In Chapter II, using a technique similar to Loud's, explicit bounds for (1-1) are given as part of this study.

The determination of conditions for the uniqueness and stability of the periodic solutions has been studied in several papers. In 1947, Cartwright and Littlewood of England[4] established a remarkable theorem for system (1-3). In Russia, Zelezcov[11] developed a complete analysis for the system (1-1) but with $g(x) = kx$. A detailed study was made of the periodic motions of the forced oscillation system with combined frictions (viscous and coulomb). But his method of analysis can not be used for the more general case when $g(x)$ is a nonlinear function. In the previously mentioned paper by Loud, he presented a result for the stability and uniqueness for the system defined by

$$\ddot{x} + b\dot{x} + g(x) = e(t) \tag{1-4}$$

Unfortunately his method is limited to the case when the "damping" term is linear. In Chapter III of this study, the famous Lyapunov Second Method is applied to the study of the stability of the class of forced oscillation systems given by (1-1). A Lyapunov function is constructed in order to answer the stability question for the following type of second order linear differential equation with variable coefficients:

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0 \tag{1-5}$$

which is related to system (1-1). Via the Lyapunov Second Method, a theorem of uniqueness and stability is established and the conditions which must be imposed on the nonlinear terms are determined.

Chapter IV introduces a method for obtaining a first order approximation to the periodic solution of the Equation (1-1). The essentials of this method were found several decades ago by Ritz and Galerkin and extended by Klotter[34]. However, the interpretation of this method in this study is different from that of Klotter. Also an explicit response function (usually called the describing function) is derived in a convenient complex form. To serve as an example, the describing function of the system with combined friction and dead-space restoring force is obtained in an analytic form.

Chapter V discusses the particular system mentioned above. The bounded region is also illustrated. An analog computer study is made to check the result of Chapter IV.

Chapter VI gives the conclusion.

CHAPTER II

THE BOUNDEDNESS AND EXISTENCE OF THE PERIODIC SOLUTIONS

For the differential equation

$$\ddot{x} + f(\dot{x}) + g(x) = e(t) \qquad (1\text{-}1)$$

it is very important to know whether the solutions are ultimately bounded or not. If the system has negative damping, the solutions may become unbounded. If the solution is not bounded, any attempt to make an analytic approximation to the solution seems to be unjustified. In order for all the solutions of Equation (1-1) to be ultimately bounded the functions $f(\dot{x})$, $g(x)$ and $e(t)$ must have certain characteristics. The objective of this chapter is to present a theorem for the boundedness of solutions of Equation (1-1). The conditions which must be imposed on the functions $f(\dot{x})$, $g(x)$, and $e(t)$ are also determined. In addition, the bounds are determined explictly as a result of this qualitative study. In a later section of this chapter, with the aid of Brouwer's Fixed Point Theorem[28] the existence of the periodic solution is established when the forcing function is periodic.

Theorem I (The Boundedness Theorem):

Given the differential equation

$$\ddot{x} + f(\dot{x}) + g(x) = e(t) \qquad (1\text{-}1)$$

let

(1) $f(y)$, $g(x)$, and $e(t)$ be continuous.

(2) $g(x)$ be a monotonically increasing function and satisfy Lipschitz condition.

(3) $g(0) = 0, \quad f(0) = 0$

and if there exist positive constants b, c, and E such that

(4) $\frac{f(y)}{y} \geq b, \qquad \frac{g(x)}{x} \geq c, \qquad |e(t)| \leq E$

then for any solution of (1-1) when $t > t_o$. (t_o depending on the particular solution),

$$|x(t)| \leq I$$

$$I = \frac{E}{c} \qquad \text{for } \frac{c}{b^2} \leq \frac{3}{16}$$

$$= \frac{16}{3}\frac{E}{b^2} \qquad \text{for } \frac{3}{16} \leq \frac{c}{b^2} \leq \frac{3}{4} \qquad (2\text{-}1)$$

$$= \frac{E}{c} + \frac{4E}{c}\sqrt{\frac{c}{b^2} - \frac{3}{16}} \qquad \text{for } \frac{c}{b^2} \geq \frac{3}{4}$$

$$|\dot{x}(t)| \leq J = \frac{4E}{b} \qquad (2\text{-}2)$$

Remarks: The technique used to prove this theorem is similar to the method used by Loud.[9] For some special cases, namely, when $g(x) = x$, $f(y)$ and $e(t)$ possess first derivatives, then Equation (1-1) can be reduced to (1-3) by letting $z = \dot{x}$. In general, however such a reduction is impossible. The condition on $g(x)$ in this theorem requires that $g(x)$ be continuous together with the continuity of $f(y)$ and $e(t)$ in order that the existence of solutions is guaranteed. This does not imply the solution is necessarily unique with each initial value. The objective of this theorem is to show that every solution ultimately remains in a bounded region.

The functions $g(x)$ and $f(y)$ may be represented as shown respectively in Figure 2-1 and Figure 2-2.

If a system is described by

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g_1(x) - f_1(y) + e_1(t) \end{aligned}\right\}$$

in which $g_1(0) \neq 0$ and $f_1(0) \neq 0$, these difficulties can be removed by letting $g(x) = g_1(x) - g_1(0)$ and $f(y) = f_1(y) - f_1(0)$. Then $g_1(0)$ and

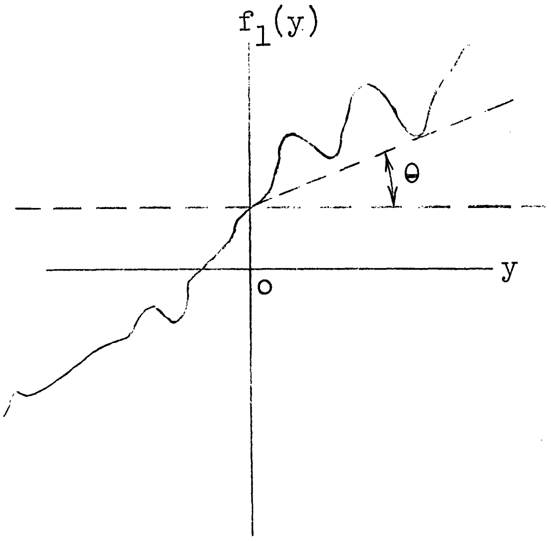

Figure 2-1 Damping Force Characteristics.

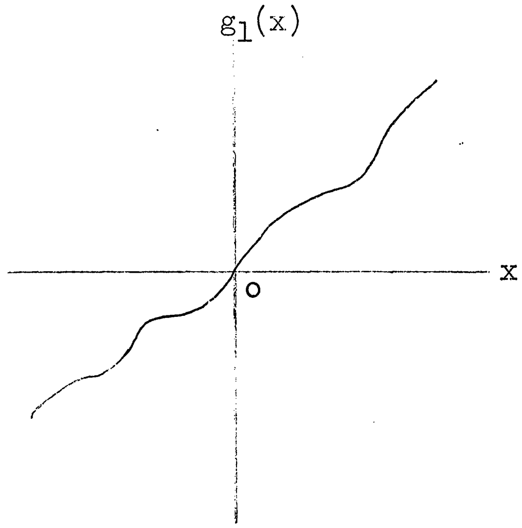

Figure 2-2 Restoring Force Characteristics.

$f_1(0)$ can be combined with the forcing function, i.e., let $e(t) = e_1(t) - g_1(0) - f_1(0)$. The original equation becomes

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) - f(y) + e(t) \end{aligned}\right\} \qquad (2\text{-}3)$$

which satisfies assumption (3).

For a given function $f_1(y)$, the constant b in assumption (4) can be determined as shown in Figure 2-1 by drawing through the new origin a straight line just tangent to the curve $f_1(y)$. The angle $\theta = \tan^{-1}b$ is a minimum angle which $f(y)$ can make from the y-axis. The procedure of finding the constant c is the same as that of obtaining b. The monotonic behavior of $g(x)$ assures that for each $g(x)$ there corresponds a unique value of x. The use of this property will be seen later in the proof.

Proof: From Equation (2-3) one has by division:

$$\frac{dy}{dx} = -\frac{f(y)}{y} - \frac{g(x)}{y} + \frac{e(t)}{y} \tag{2-4}$$

Now consider the following equations:

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(b - \frac{E}{|y|})y - g(x) \end{aligned}\right\} \tag{2-5}$$

then

$$\frac{dy}{dx} = -b + \frac{E}{|y|} - \frac{g(x)}{y} \tag{2-6}$$

(2-6) is obtained from the set of differential equations (2-5). In view of the fact that $f(y)/y \geq b$ and $|e(t)| \leq E$, the slope dy/dx in (2-4) does not exceed dy/dx in (2-6). Then, if a region can be found in which the solutions of (2-5) will be ultimately bounded, the boundedness of the solutions of (2-3) will follow easily.

Let $\alpha(\alpha < 0)$ and $\beta(\beta > 0)$ be such that $g(\alpha) = -E$, and $g(\beta) = E$. α and β are uniquely determined because of the monotonic property of g(x). Then $g(\beta) = E \geq c\beta$ so that $0 < \beta \leq \frac{E}{c}$, and $g(\alpha) = -E \leq c\alpha$, so that $-\frac{E}{c} \leq \alpha < 0$.

Now let dy/dx = 0 in Equation (2-6). A curve can then be drawn in the x-y plane such that

$$-by = g(x) - E \qquad \text{for } y > 0 \tag{2-7}$$

$$-by = g(x) + E \qquad \text{for } y < 0 \tag{2-8}$$

as shown in Figure 2-3. The curve passes through the x-axis at β for (2-7), and α for (2-8). Then every solution of (2-5) in the half plane $y > 0$ has the behavior described below:

To the left of the curve (2-7), $\dot{y} > 0$, so it must move upward and to the right until it reaches a maximum of y when crossing ($\dot{y} = 0$). After crossing, it moves downward until it reaches the x-axis.

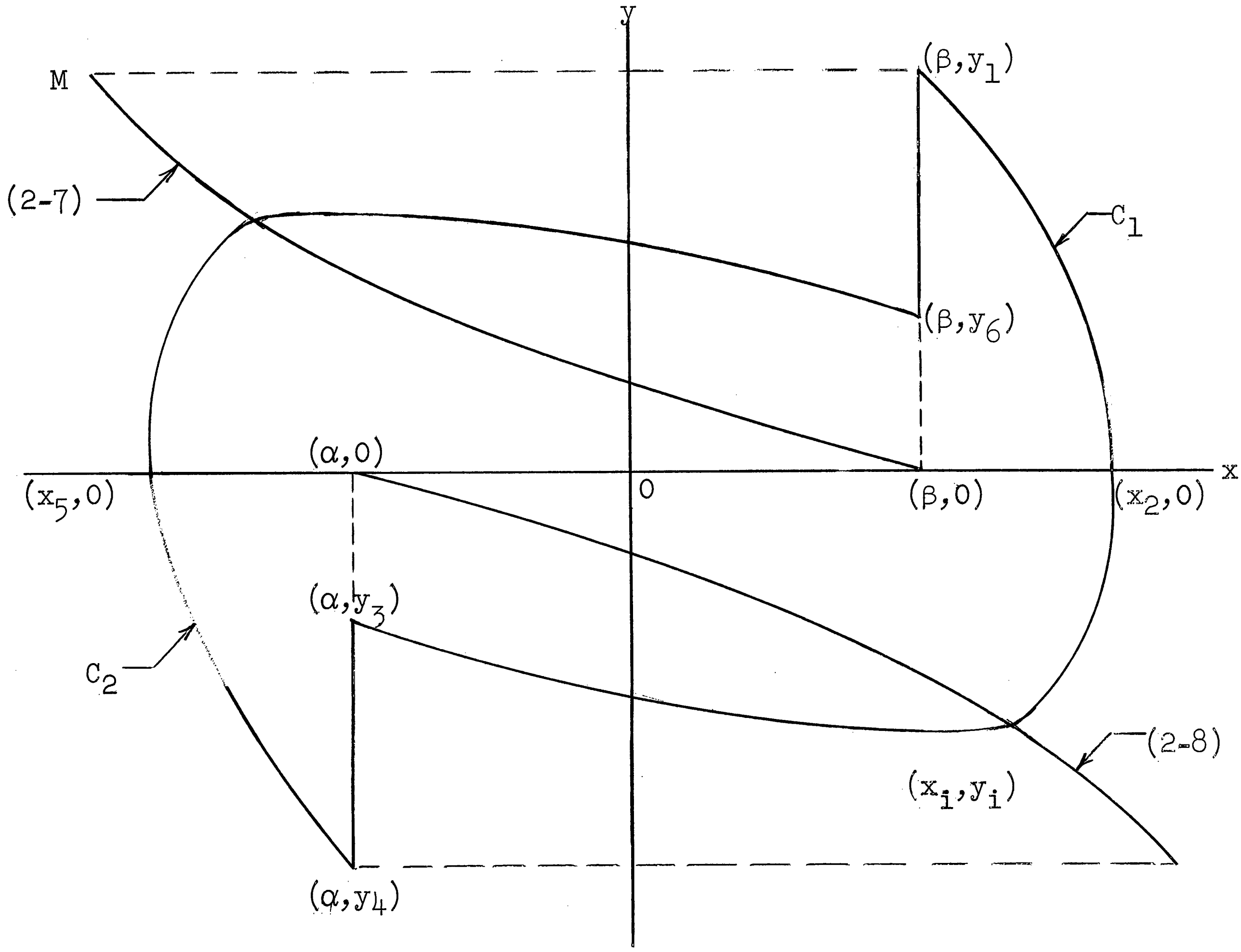

Figure 2-3 Closed Curve on the Phase Plane for Equation (1-1).

Since $\dot{x} = y > 0$, every solution moves from left to right, together with the fact the curve (2-7) intersects the x-axis at β, evidently any solution starting with $x < \beta$ must cross the line $x = \beta$ before it reaches the x-axis at $x > \beta$. Furthermore, for $x > \beta$ as y decreases and x increases, dy/dx becomes increasingly negative; so that the solution must move concave downward toward the x-axis in the region $x > \beta$ and $y > 0$.

A similar statement holds for the solution of (2-5) in the half plane $y < 0$.

Now let (β, y_1) be the initial point of a solution of (2-5) with y_1 not less than $4E/b$. Let the arc C_1 be this solution passing through the x-axis at $(x_2, 0)$ to its next intersection with the line $x = \alpha$ at (α, y_3) in the lower half plane. All intersections mentioned are unique by the statements of the last two paragraphs.

It remains to show that at all points on curve C_1, the inequality $|y| < y_1$ is true.

Let $G(x) = \int_0^x g(x)dx$. Integrating equation (2-6) from (β, y_1) to $(x_2, 0)$ one has

$$\frac{1}{2} y_1^2 = b \int_{\beta}^{x_2} ydx + G(x_2) - G(\beta) - E(x_2 - \beta) \qquad (2\text{-}9)$$

The integral in this equation represents the area in the phase plane under the solution curve and above the x-axis. Since the solution curve is concave downward, this area is greater than the area of the triangle formed by replacing the solution curve by a straight line. Therefore

$$\frac{1}{2} y_1^2 \geq \frac{b}{2} y_1 (x_2 - \beta) + G(x_2) - G(\beta) - E(x_2 - \beta) \qquad (2\text{-}10)$$

Let (x_i, y_i) denote the minimum point of C_1 in the lower half plane which is at the intersection of C_1 with the curve (2-8). Since $x_i > \alpha$, from (2-8), $y_i = -\frac{g(x_i) + E}{b}$. If $x_i < \beta$, then $g(x_i) < g(\beta) = E$; so that $y_i > -\frac{2E}{b}$ and $|y_i| < \frac{4E}{b}$. If $x_i > \beta$, integrating Equation (2-6) from $(x_2, 0)$ to (x_i, y_i), one has

$$-\frac{1}{2} y_i^2 = b \int_{x_2}^{x_i} ydx + G(x_i) - G(x_2) + E(x_i - x_2) \qquad (2\text{-}11)$$

The integral in (2-11) is positive because $y < 0$ and $x_i < x_2$. Hence

$$-\frac{1}{2} y_i^2 > G(x_i) - G(x_2) + E(x_i - x_2) \tag{2-12}$$

Combining (2-10) and (2-12) one has

$$\frac{1}{2}(y_1^2 - y_i^2) \geq \frac{b}{2} y_1 (x_2 - \beta) + G(x_i) - G(\beta) - E(x_2 - \beta) - E(x_2 - x_i)$$

$$> \frac{b}{2} y_1 (x_2 - \beta) - 2E(x_2 - \beta)$$

$$= \frac{b}{2}(x_2 - \beta)(y_1 - \frac{4E}{b}) \geq 0 \tag{2-13}$$

Here the following relations have been used

$$y_1 \geq \frac{4E}{b}, \quad x_i > \beta, \quad G(x_i) > G(\beta) \text{ and } (x_2 - \beta) > (x_2 - x_i)$$

Remark: A similar statement can be made for the corresponding arc C_2 which denotes the solution of (2-5) beginning at (α, y_4) with $y_4 \leq \frac{-4E}{b}$ to (β, y_6) as shown in Figure 2-3.

Now let R be the region bounded by a closed curve Γ. Γ consists of C_1 and C_2 with $y_1 = |y_4| = \frac{4E}{b}$, and the straight line portions from (β, y_6) to (β, y_1) and from (α, y_3) to (α, y_4).

Consider outside Γ any solution of (2-5). Let this solution first cross $x = \beta$ at $y = y_{10}$. The uniqueness of this solution is guaranteed by $g(x)$ satisfying the Lipschitz condition. Then this solution will cross the line $x = \beta$ at $y = y_{11}, y_{12}, y_{13} \cdots$ after consecutive turns through the lower half plane and back to upper half plane. By (2-13), $y_{10}, y_{11}, y_{12} \cdots$ form a strictly decreasing sequence as long as they are not less than $4E/b$. If, as $t \to \infty$ the solution still could not enter Γ, then y would have a limit and the solution would have a limit cycle outside of Γ. But this contradicts the statement that $|y| < y_1$ for $y_1 > \frac{4E}{b}$. Therefore the solution will spiral into the bounded region R in finite time. It can never leave R. It cannot cross the curved boundary of R since (2-5) has unique solutions through all points. The solutions enter the region only at the straight line portion of Γ.

Now since the slope dy/dx of (2-4) never exceeds dy/dx of (2-5), the solution paths of (2-4) can cross the solution paths of (2-5) only from left to right. Hence all solutions of (2-4) are carried into the bounded region R. Once it enters R or if it starts in R, it will remain permanently in R.

The boundary of R for y is clearly $|y| \leq J = \frac{4E}{b}$. It remains still to define the bounds for x. The maximum $|x|$ is the larger of x_2 and $|x_5|$. From Equation (2-6), the value dy/dx at $(\beta, 4E/b)$ is $-b$. Since the solution curve is concave downward for $y > 0$, and $x > \beta$, one has $x_2 - \beta < \frac{y_1}{b} = \frac{4E}{b^2}$. Hence $x_2 < \beta + \frac{4E}{b^2} < \frac{E}{c} + \frac{4E}{b^2}$. The same reasoning can be applied to the corresponding curve in the lower half plane, so that both x_2 and $|x_5|$ are less than the quantity $\frac{E}{c} + \frac{4E}{b^2}$. However, this quantity may be somewhat large and a smaller quantity for the bound of x is always desirable. Since $|y| \leq \frac{4E}{b}$, the result for the bound of x can be improved by choosing a starting point of the solution curve of (2-6) as close to the point M on the curve (2-7) as possible. Let a solution curve of (2-6) starting at $(0, 4E/b)$ instead at $(\beta, 4E/b)$. The intersection of the curve with the x-axis denoted by x_2' will be considered for the bound of x.

Now consider the system below

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -by - cx + E \end{aligned}\right\} \qquad (2\text{-}14)$$

of which

$$\frac{dy}{dx} = -b - \frac{cx}{y} + \frac{E}{y} \tag{2-15}$$

Since $g(x) \geq cx$, dy/dx in (2-5) never exceeds dy/dx of (2-15). Hence the solution paths of (2-5) can cross the solution paths of (2-15) only from above to below for $x > 0$ and $y > 0$. Let a solution of (2-4) beginning at (0, 4E/b) intersect the x-axis at x_7. Evidently one has $x_2' < x_7$. In determining x_7, cases for different values of the ratio of c to b^2 are considered.

If $\frac{c}{b^2} < \frac{1}{4}$, the solution x(t) of (2-14) with initial condition (0, 4E/b) is

$$x = \frac{E}{c} + D_1 e^{p_1 t} + D_2 e^{p_2 t} \tag{2-16}$$

where p, D_1 and D_2 are easily determined as

$$p = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} < 0$$

$$D_1 = \frac{E}{2c}\left[\frac{8c/b^2 - 1}{(1-4c/b^2)^{1/2}} - 1\right]$$

$$D_2 = \frac{E}{2c}\left[-\frac{8c/b^2 - 1}{(1-4c/b^2)^{1/2}} - 1\right]$$

Then one has $D_1 D_2 = 16\,\frac{E^2}{b^2 c}\,\frac{3/16 - c/b^2}{(1-4c/b^2)} \geq 0$ for $\frac{c}{b^2} \leq \frac{3}{16}$

and $D_1 + D_2 = -\frac{E}{c} < 0.$

These two conditions imply that both D_1 and D_2 are not greater than zero for $\frac{c}{b^2} \leq \frac{3}{16}$. Since $p < 0$, from (2-16) one has $x_7 < \frac{E}{c}$. Hence

$$x_2' \leq x_7 \leq \frac{E}{c} \qquad \text{for} \qquad \frac{c}{b^2} \leq \frac{3}{16}$$

Now consider the case $\frac{c}{b^2} \geq \frac{3}{16}$. At the point $(0, 4E/b)$ Equation (2-15) becomes $\frac{dy}{dx} = \frac{-3}{4} b$. The equation of the straight line passing through the same point $(0, 4E/b)$ with the same slope $-\frac{3}{4}b$ is

$$x = \frac{16}{3}\frac{E}{b^2} - \frac{4}{3b}y \qquad (2\text{-}17)$$

Then dy/dx of (2-15) on this line becomes

$$\frac{dy}{dx} = -\frac{3}{4}b + \left(\frac{4b}{3} - \frac{16E}{3y}\right)\left(\frac{c}{b^2} - \frac{3}{16}\right) \qquad (2\text{-}18)$$

which is obtained by substituting (2-17) in (2-15) and by some rearrangements. Since only $y < \frac{4E}{b}$ is considered, then $\frac{4b}{3} - \frac{16E}{3y} < 0$. If $\frac{c}{b^2} > \frac{3}{16}$, one has in (2-18) $\frac{dy}{dx} < -\frac{3}{4}b$ on this line. This means that the solutions of (2-14) can cross the straight line (2-17) only from above to below for $y > 0$ and $x > 0$. Since the straight line intersects the x-axis at $x = \frac{16E}{3b^2}$, then $x_7 < \frac{16E}{3b^2}$. Hence

$$x_2' \leq x_7 \leq \frac{16E}{3b^2} \quad \text{for} \quad \frac{c}{b^2} \geq \frac{3}{16} \quad .$$

For large value of c/b^2, the quantity $16E/3b^2$ may be too large. The result can still be improved for $\frac{c}{b^2} > \frac{3}{4}$. In the case $\frac{c}{b^2} > \frac{1}{4}$, the solution curve of (2-14) starting at $(0, 4E/b)$ intersects the x-axis at x_7 which is obtained as

$$x_7 = \frac{E}{c} + \frac{4E}{c}\sqrt{\frac{c}{b^2} - \frac{3}{16}}\; e^{-\frac{b}{\sqrt{c-(b/2)^2}}\tan^{-1}\frac{4\sqrt{c-(b/2)^2}}{b}} < \frac{E}{c} + \frac{4E}{c}\sqrt{\frac{c}{b^2} - \frac{3}{16}}$$

Hence $x_2' \leq x_7 < \frac{E}{c} + \frac{4E}{c}\sqrt{\frac{c}{b^2} - \frac{3}{16}}$ for $\frac{c}{b^2} > \frac{3}{4}$

The representation of the bounds for x is summarized as follows:

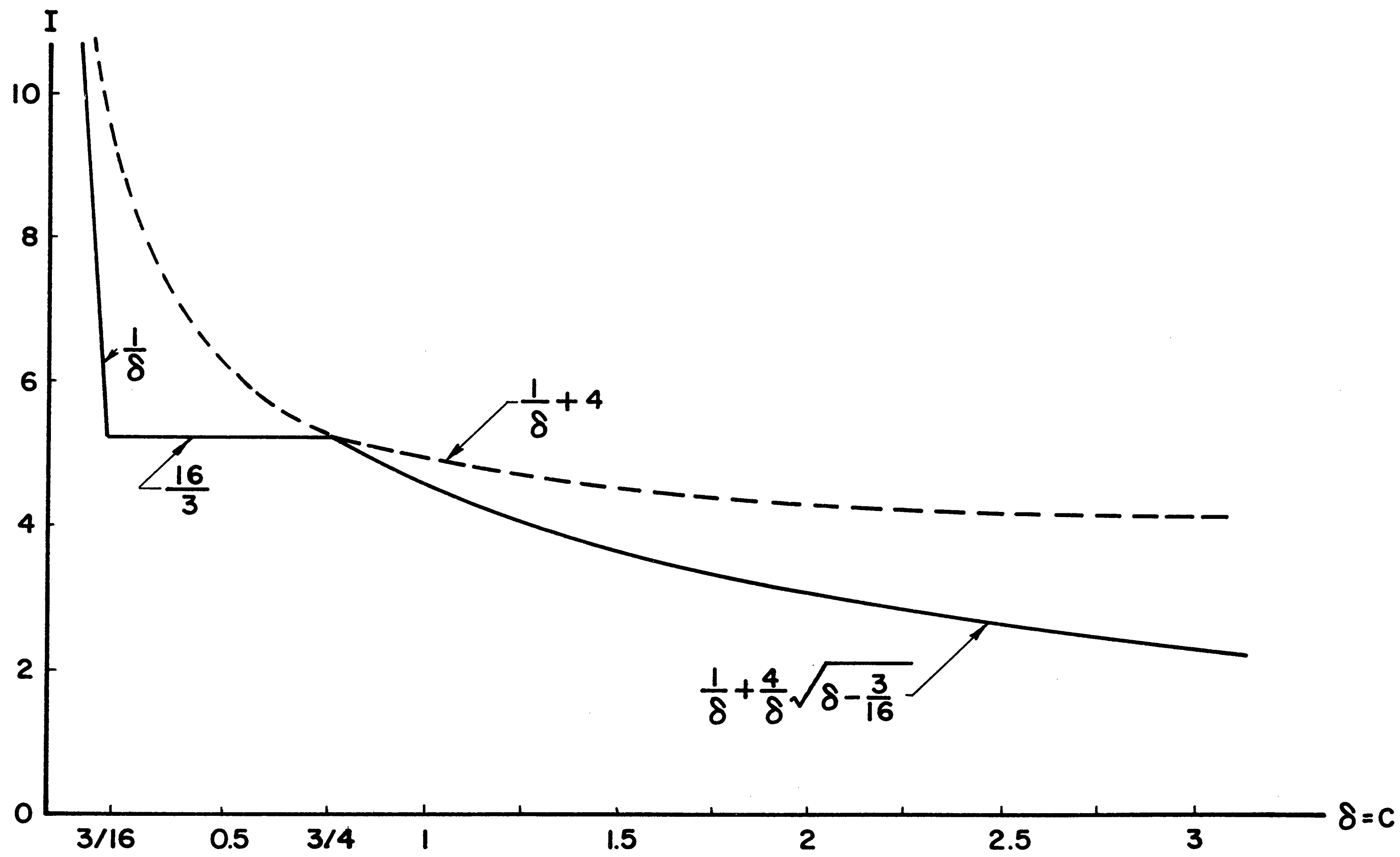

Figure 2-4 Graph of Bound of $|x|$ vs. δ.

$E = b = 1$

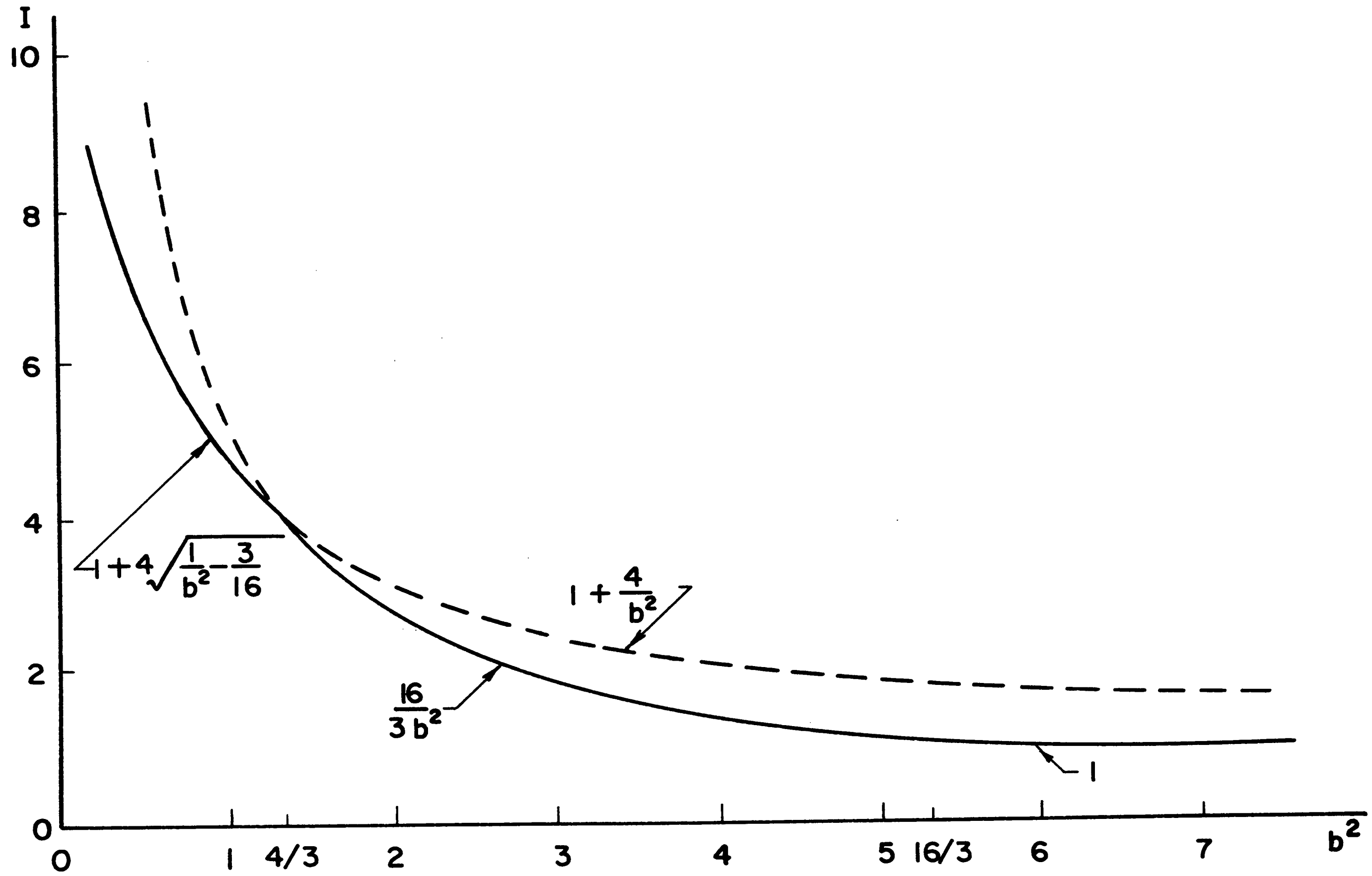

Figure 2-5 Graph of Bound of $|x|$ vs. b^2.

$E = c = 1$

$$
\begin{aligned}
|x(t)| &\leq I \\
I &= \frac{E}{c} && \text{for } \frac{c}{b^2} \leq \frac{3}{16} \\
&= \frac{16E}{3b^2} && \text{for } \frac{3}{16} \leq \frac{c}{b^2} \leq \frac{3}{4} \qquad (2\text{-}1) \\
&= \frac{E}{c} + \frac{4E}{c}\sqrt{\frac{c}{b^2} - \frac{3}{16}} && \text{for } \frac{c}{b^2} \geq \frac{3}{4}
\end{aligned}
$$

The bounds for x are plotted against the argument $\delta = \frac{c}{b^2}$ with E = b = 1 in Figure 2-4. The curve for $I = \frac{E}{c} + \frac{4E}{b^2}$ is also indicated for comparison. Figure 2-5 shows the bounds for x as b^2 varies and with E = c = 1.

This completes the proof of this theorem.

The Existence of a Periodic Solution

Having established a closed region R in which no solution path can leave, it will now be shown that if the forcing function of (1-1) is periodic, there exists at least one periodic solution with the same period T circulating around within this region. Let (x_o, y_o) be any initial point in R of the t = 0 plane. Then $x = x(t, x_o, y_o)$ and $y = y(t, x_o, y_o)$ continue to exist as functions of t, x_o and y_o. In particular, in the plane t = T, the point $x = x(T, x_o, y_o)$, $y = y(T, x_o, y_o)$ defines a transformation of the closed region R into itself. This transformation is continuous since the right members of Equation (2-3) are continuous and the solution paths are all continuous. Therefore by the famous Brouwer's Point Theorem, there exists at least one fixed point (x^*, y^*). If a solution starts at (x^*, y^*), the transformation will carry it back to (x^*, y^*) at T. Since the forcing function is periodic, i.e. $e(t + T) = e(t)$, a second transformation from T to 2T is then identical to the first one. Hence a solution $x(t + T)$, $y(t + T)$ with initial point at (x^*, y^*) will be the same as the solution $x(t)$, $y(t)$ with the same initial point, i.e. $x(t) = x(t + T) = x(t + nT)$ for n = an integer. This shows that the system (1-1) possesses at least one periodic solution with period T.

Remark: Since $f(y)$ has not been assumed to satisfy the Lipschitz condition, the solution is not necessarily unique. Hence the continuous transformation may not have a one to one correspondence. Even in this case, Brouwer's Fixed Point Theorem still can be applied since the theorem does not require one to one correspondence.

In the following chapter the existence of only one fixed point (corresponding to only one periodic solution) is to be shown for the system (1-1) subject to further conditions imposed on $f(y)$ and $g(x)$. Also the behavior of the fixed point is found to be stable.

CHAPTER III

THE STABILITY AND UNIQUENESS OF THE PERIODIC SOLUTIONS

The study of stability in the small of forced oscillations of nonlinear systems is often reduced to a study of linear differential equation with variable coefficients. For example, a second order nonlinear differential equation can be reduced to a variational equation associated with the periodic solution, in which only linear terms need be retained when studying the stability of small variations. Mathieu's equation is one of the equations often used. In this chapter, a linear differential equation with variable coefficients is formed from the nonlinear differential Equation (1-1) and a Lyapunov function is constructed to answer the question of stability of the periodic solution of the nonlinear system. By the second method of Lyapunov not only is it possible to assert that a slight deviation from the periodic solution will reduce to zero, but also to indicate a certain region in the phase space over which the forced oscillation is asymptotically stable. Therefore, the Lyapunov second method is a powerful tool in the study of the stability of nonlinear systems both automomous and nonautonomous.

The Lyapunov Function

Consider a nonlinear or linear system

$$\dot{x}_i = f_i(x_1, x_2, \ldots x_n; t) \qquad i = 1, 2, \ldots n$$

Let f_i be continuous and satisfy Lipschitz condition in an n-dimensional space, so that the existence and uniqueness of solutions are guaranteed; and $f_i(0, 0, \ldots; t) = 0$ for $t > t_o$ so that the origin is the equilibrium point. Let $V(x_1, x_2, \ldots; t)$ be a continuous function both in x_i and t, $V(0, 0, \ldots; t) = 0$, and that V has continuous partial derivatives in x_i and t. The derivative of V is expressed as:

$$\frac{dV}{dt} = \dot{V} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i$$

The equation $V(x_1, x_2, \ldots x_n; t_1) = C$, $C > 0$ for some fixed t_1, represents a certain closed hypersurface enclosing the origin in the n-dimensional space. At $t_2 = t_1 + \Delta t$, the solutions become $x_1(t + \Delta t)\ldots$ If $V(x_1, x_2, \ldots x_n, t_2)$ = some positive quantity but less than C, this means that the closed surface is shrinking and approaching the origin, i.e., $x_i(t) \to 0$ and $C \to 0$ as $t \to \infty$; so that V is decreasing as t increases, the rate of change of V with respect to t must be negative for all $t > t_o$. Then the origin is asymptotically stable. For an autonomous system:

$$\dot{x}_i = f_i(x_1, x_2, \ldots x_n)$$

It can also be imagined that for any $C > 0$, if all solutions $x_i(t)$ outside of the closed surface of $V(x_1, x_2, \ldots x_n) = C$ (no matter how small or how large) will pass through it to the inside. As C decreases the closed surface becomes smaller and approaches the origin as $t \to \infty$; hence $\dot{V} < 0$. Then the system is said to have stability in the large.

Let $W(x_1, x_2, \ldots x_n)$ denote a differentiable function associated with V or a function like V but independent of the time. W is then defined as definite positive (or negative) if $W > 0$ (or $W < 0$) and $W = 0$ only for all $x_i = 0$. Similarly, let $U(x_1, x_2, \ldots x_n)$ be such a function associated with $\dot{V}(x_1, x_2, \ldots x_n; t)$.

If a function V satisfies the following conditions:

(1) V is definite positive, i.e. $V(x_1, x_n, \ldots x_n; t) > W(x_1, x_2, \ldots x_n)$ where W is definite positive; V approaches zero with* $|x|$ uniformly in t, i.e., if given ϵ, there exists a $\delta > 0$ such that if $|x| \leq \delta$ and $t \geq t_o$, then $V < \epsilon$.

(2) $\dot{V} = \frac{dV}{dt}$ is definite negative, i.e. $\dot{V}(x_1, x_2, \ldots x_n; t) <$ $U(x_1, x_2, \ldots x_n)$ where U is definite negative.

Then V is called the Lyapunov function which guarantees the asymptotic stability of the system.

For a given system, the main problem is to find a Lyapunov function if such a function exists. It may be very difficult to find one for a rather complicated system. Since any function which satisfies these two conditions is a Lyapunov function, obviously a Lyapunov function is not unique. Hence there is no general rule or method to construct a Lyapunov function; it depends more or less on ingenuity of the user. If more than one Lyapunov function is found, the question of which function is better depends on which one gives simpler and less restrictive conditions on the system parameters for stability and some other system considerations.

Consider a second order differential equation

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0 \qquad (1\text{-}5)$$

which is equivalent to the system:

$$\left.\begin{aligned} \dot{u} &= v \\ \dot{v} &= -q(t)u - p(t)v \end{aligned}\right\} \qquad (3\text{-}1)$$

and assume $0 < a_1 < p(t) < a_2$, $0 < a_3 < q(t) < a_4$, both differentiable functions of t. Evidently the origin is the equilibrium point. In this case, it is always possible to write down a Lyapunov function in a quadratic

* $|x|$ denotes a convenient norm of the vector $(x_1, x_2, \ldots x_n)$, for instance, the sum of the absolute values of its components.

form in u and v and then determine the coefficients of the quadratic form. But it involves three coefficients. One logical way to try to find a function is to write it in the following form:

$$V = A(p,\ q,\ t)u^2 + B(p,\ q,\ t)v^2$$

having only two coefficients which are function p, q and t. Upon substituting $\dot{v} = -qu - pv$ in V, the derivation of V becomes

$$\begin{aligned}\dot{V} &= \dot{A}u^2 + 2Auv + \dot{B}v^2 + 2Bv(-pv - qu)\\ &= \dot{A}u^2 + (2A - 2Bq)uv + (\dot{B} - 2Bp)v^2\end{aligned}$$

Now let $2A - 2Bq = 0$, then $B = \frac{A}{q} > 0$ and $\dot{B} = \frac{\dot{A}q - A\dot{q}}{q^2}$, so that

$$\left.\begin{aligned}V &= Au^2 + \frac{A}{q}v^2 = A(u^2 + \frac{1}{q}v^2)\\ \dot{V} &= \dot{A}u^2 + \frac{\dot{A}q - A\dot{q} - 2Apq}{q^2}v^2\end{aligned}\right\} \qquad (3\text{-}2)$$

Since q is bounded, A must be greater than some fixed positive quantity in order for V to be definite positive ($V > W$). In the following three conditions:

$$\begin{aligned}&(1)\quad -\dot{A} > \epsilon_1 \quad\text{and}\quad -\frac{(\dot{A}q - A\dot{q} - 2Apq)}{q^2} \geq 0\\ &(2)\quad -\dot{A} \geq 0 \quad\text{and}\quad -\frac{(\dot{A}q - A\dot{q} - 2Apq)}{q^2} > \epsilon_2 \qquad (3\text{-}3)\\ &(3)\quad \text{both } -\dot{A} \quad\text{and}\quad -\frac{(\dot{A}q - A\dot{q} - 2Apq)}{q^2} > \epsilon_3\end{aligned}$$

where ϵ_1, ϵ_2 and ϵ_3 are some fixed positive constants, condition (3) assures $\dot{V}$ to be definite negative while (1) and (2) imply $\dot{V}$ being semi-definite negative. One can satisfy the second condition with a simple form for V.

Let A = 1, then

$$V = u^2 + \frac{1}{q} v^2 \tag{3-4}$$

$$\dot{V} = - \frac{\dot{q} + 2pq}{q^2} v^2 \tag{3-5}$$

The condition on p(t) and q(t) becomes

$$\dot{q} + 2pq > \epsilon_2 q^2 \qquad \text{or} \qquad p + \frac{\dot{q}}{2q} > \frac{\epsilon_2}{2} q \tag{3-6}$$

But this condition implies stability only. In Equation (3-5), $\dot{V}$ is independent of u. It is called semi-definite because $\dot{V}$ can be zero with $u \neq 0$. According to Lyapunov, if a definite positive function exists for which $\dot{V} \leq 0$ the solutions are stable. This means that given any $\epsilon > 0$ there exists a $\delta > 0$ such that if $|(u_o, v_o)| \leq \delta$, then $|u(t, u_o, v_o, t_o), v(t, u_o, v_o, t_o)| \leq \epsilon$ for all $t > t_o$. However, this does not imply asymptotic stability which, in addition to the above statement, requires $\lim_{t \to \infty} u(t, u_o, v_o, t_o) = 0$. Nevertheless, in the system being considered, although $\dot{V}$ is semi-definite negative, $\frac{dV}{dt} = 0$ occurs only whenever the solution path intersects the u-axis of the phase plane. The only problem now is to determine whether the solution paths will be approaching the origin or not. This can be answered by examining the system equation itself. When $v = 0$, $\dot{v} = -qu$, the solution path must cross the u-axis downward from the upper half plane to the lower half plane for $u > 0$ because $\dot{v} < 0$ and move upward for $u < 0$ (since $q > 0$). Once the solution path leaves the u-axis, then $\dot{V} < 0$, and V is decreasing. In order words, V is always decreasing except at those isolated points on the u-axis. Now it will be shown by contradiction that as $t \to \infty$, $V \to 0$. Suppose that instead V approaches a positive constant V_o as $t \to \infty$, and let $V_o < V < V_o + \epsilon$ for

$t > t_1$, where ϵ is some positive quantity. Since q is bounded from above and below, i.e., $0 < k_1 < \frac{1}{q} < k_2 < \infty$, then

$$u^2 + k_1 v^2 < u^2 + \frac{1}{q} v^2 < u^2 + k_2 v^2$$

Since V is a non-increasing function, then $V_o < V < u^2 + k_2 v^2$ and $u^2 + k_1 v^2 < V < V_o + \epsilon$. Hence the solution path for all $t > t_1$ must be circulating around in between the two ellipses:

$$L_1: \quad u^2 + k_1 v^2 = V_o + \epsilon$$

$$L_2: \quad u^2 + k_2 v^2 = V_o$$

as shown in Figure 3-1.

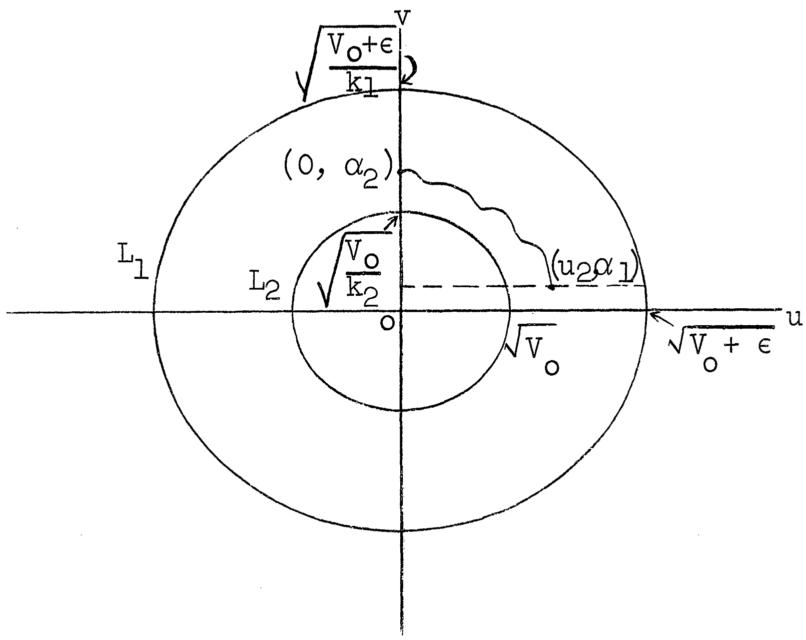

Figure 3-1 Phase Plane Illustration.

Assume a solution path which passes $(0, \alpha_2)$ at t_1 and (u_2, α_1) at t_2, where α_1 can be very small, but $0 < \alpha_1 < \alpha_2$. Integrating (3-1) from t_1 to t_2, one has

$$\int_{t_1}^{t_2} \dot{v}dt = -\int_{t_1}^{t_2} (pv + qu)dt.$$

Since $p < a_2$, $v < \sqrt{\frac{V_o+\epsilon}{k_1}} = \alpha_3$, $q < a_4$ and $u < \sqrt{V_o + \epsilon} = \beta_1$,

$$\alpha_2 - \alpha_1 = \int_{t_1}^{t_2} (pv + qu)dt < (a_2\alpha_3 + a_4\beta_1)(t_2 - t_1)$$

then

$$t_2 - t_1 > \frac{\alpha_2 - \alpha_1}{a_2\alpha_3 + a_4\beta_1} > 0$$

and $\qquad \dot{V} < -\epsilon_2\alpha_1^2 < 0 \quad \text{for} \quad t_1 < t \leq t_2$

Hence $\qquad \Delta V = V_1 - V_2 > \epsilon_2\alpha_1^2 \cdot \dfrac{\alpha_2 - \alpha_1}{a_2\alpha_3 + a_4\beta_1} > 0$

This implies that as long as the solution path lies outside L_2, V decreases at least by ΔV, where ΔV is greater than some positive constant, in each arc such as that of Figure 3-1. Since the solution considered has an infinite number of such arcs, V would have to eventually become negative, which is a contradiction. Hence $V_o = 0$ or $V \to 0$ as $t \to \infty$. This shows that both u and v are approaching the origin as a limit regardless what the initial values of u and v are, and the solution is asymptotically stable.

One conclusion which can be drawn if the system parameters p(t) and q(t) satisfy the condition of (3-6) is that the system possesses stability in the large and that the origin is asymptotically stable. For the special case when q is a positive constant, every solution eventually goes to zero as long as the damping is greater than some fixed positive quantity (no matter how small and how p(t) is fluctuating). Of course, for both p and q constant, the system reduces to the one with constant coefficients, a linear system, for which the origin is asymptotically stable for $p > a_1 > 0$.

Application of Lyapunov Function to the Stability and Uniqueness of the Periodic Solutions:

In the preceding chapter, the solutions of a system with nonlinear friction and nonlinear spring force have been shown to be ultimately bounded in the region $|x| \leq I$, $|\dot{x}| \leq J$, and there exists at least one periodic solution which has the same period as that of the forcing function. The question may arise as to whether more than one periodic solution or subharmonic exists, and whether they are stable or not. The answer to this question is very difficult, even for equations with only one nonlinear term and, unfortunately so far as has been determined from the literature, very little information can be obtained about this aspect. However, this question can be treated in another manner and it is the objective of this section to show that under certain conditions the system (1-1) will have all solutions converging to a unique steady state periodic solution.

Theorem II (the Convergence Theorem): For the second order differential equation

$$\ddot{x} + f(\dot{x}) + g(x) = e(t) \tag{1-1}$$

in addition to the assumptions in Theorem I, if

(1) e(t) is a periodic function with period T.

(2) $g'(x)$ is continuous and $g'(x) \geq c_1$

(3) $g''(x)$ exists and $|g''(x)| \leq c_2$

(4) $f'(y)$ exists and $f'(y) \geq b_1$

where c_1, c_2 and b_1 are some positive constants, then all solutions which ultimately have $|x| \leq I$ and $|\dot{x}| \leq J$ converge to a periodic solution (a unique steady state periodic solution) which is asymptotically stable provided $b_1 > \frac{c_2}{c_1} J$.

Proof: Let $x_1(t)$ and $x_2(t)$ be any two distinct solutions with $|\dot{x}_2| \leq J$ and $|\dot{x}_1| \leq J$. From (1-1) one has

$$\ddot{x}_2 + f(\dot{x}_2) + g(x_2) = e(t) \tag{3-7}$$

$$\ddot{x}_1 + f(\dot{x}_1) + g(x_1) = e(t) \tag{3-8}$$

Let $u = x_2 - x_1$, $v = \dot{u} = \dot{x}_2 - \dot{x}_1 = y_2 - y_1$

Subtracting Equation (3-8) from Equation (3-7), one has

$$\ddot{u} + p(t)\dot{u} + q(t)u = 0 \tag{1-5}$$

$$\text{where} \quad p = \frac{f(y_2) - f(y_1)}{y_2 - y_1} \quad \text{and} \quad q = \frac{g(x_2) - g(x_1)}{x_2 - x_1} \tag{3-9}$$

But p has the following properties:

1. Since $f'(y) \geq b_1$, then $p \geq b_1 > 0$
2. Since $f(y)$ is continuous function of y and $y = y(t)$, then p is continuous function of t.

For similar reasons, one has $q \geq c_1 > 0$ and $q(t)$ a continuous function of t.

Hence Equation (1-5) becomes a second order linear differential equation with time-varying coefficients. To prove the convergence of x in (1-1) is equivalent to showing $u \to 0$ and $t \to \infty$. It has been shown in the preceding section that if the condition (3-6), i.e., $p + \frac{\dot{q}}{2q} > \frac{\epsilon_2 q}{2}$, is satisfied, then $u \to 0$ as $t \to \infty$.

Now $$\frac{\dot{q}}{2q} = \frac{1}{2}\frac{d}{dt}\log q = \frac{1}{2}\frac{d}{dt}\{\log[g(x_2) - g(x_1)] - \log(x_2 - x_1)\}$$

$$= \frac{1}{2}\left[\frac{g'(x_2)y_2 - g'(x_1)y_1}{g(x_2) - g(x_1)} - \frac{y_2 - y_1}{x_2 - x_1}\right] \qquad (3\text{-}10)$$

Let x_3 be some intermediate value between x_2 and x_1, then

$$g(x_2) - g(x_1) = g'(x_3)(x_2 - x_1)$$

$$\frac{\dot{q}}{2q} = \frac{1}{2g'(x_3)}\left[y_2\frac{g'(x_2) - g'(x_3)}{x_2 - x_1} + y_1\frac{g'(x_3) - g'(x_1)}{x_2 - x_1}\right] \qquad (3\text{-}11)$$

Since x_3 is the value between x_2 and x_1, so that

$$\left|\frac{g'(x_2) - g'(x_3)}{x_2 - x_1}\right| \leq \left|\frac{g'(x_2) - g'(x_3)}{x_2 - x_3}\right| = |g''(x_4)|$$

$$\left|\frac{g'(x_3) - g'(x_1)}{x_2 - x_1}\right| \leq \left|\frac{g'(x_3) - g'(x_1)}{x_3 - x_1}\right| = |g''(x_5)|$$

where x_4 is some value between x_2 and x_3, and x_5 is some value between x_3 and x_1. Then

$$\left|\frac{\dot{q}}{2q}\right| \leq \frac{1}{2c_1}\left[|y_2| \cdot \left|\frac{g'(x_2) - g'(x_3)}{x_2 - x_1}\right| + |y_1| \cdot \left|\frac{g'(x_3) - g'(x_1)}{x_2 - x_1}\right|\right]$$

$$\leq \frac{J}{2c_1}\left[|g''(x_4)| + |g''(x_5)|\right] \leq \frac{Jc_2}{c_1} \qquad (3\text{-}12)$$

If $b_1 > \frac{c_2}{c_1}J$, let $b_1 - \frac{c_2}{c_1}J = k$. Since $q \leq \max.\ g'(x)$ and $p \geq b_1$, then an ϵ_2 can be chosen so small such that

$$p - \frac{c_2}{c_1}J \geq b_1 - \frac{c_2}{c_1}J = k > \frac{\epsilon_2 q}{2}$$

then

$$p + \frac{\dot{q}}{2q} \geq p - \left|\frac{\dot{q}}{2q}\right| \geq p - \frac{c_2}{c_1}J \geq b_1 - \frac{c_2}{c_1}J > \frac{\epsilon_2 q}{2} \qquad (3\text{-}13)$$

It follows that the Lyapunov function (3-4) implies that $u = x_2(t) - x_1(t) \to 0$ as $t \to \infty$ and the solution $u = 0$ is asymptotically stable. Any two distinct solutions of (1-1) will converge to a unique solution which is asymptotically stable.* Since it has been proved in Chapter II that there exists at least one periodic solution when the forcing function is periodic, then it follows that the solutions will converge to a unique periodic solution which is asymptotically stable. This completes the proof of this theorem.

A further conclusion which can be drawn is that from the result of Theorem I, $J = \frac{4E}{b}$ and since $b \geq b_1$, Theorem II holds provided

$$b_1 > \frac{4c_2E}{c_1b_1} \qquad \text{or } b_1^2 > \frac{4c_2E}{c_1} \tag{3-14}$$

As a consequence of this theorem, the periodic solution (or the fixed point in the topological equivalent) is also stable in the large.

Remark: If f(y) and g(x) are piecewise continuous, p(t) and g(x) will be undefined at some points of discontinuity. There is difficulty in this case in determining whether the periodic solutions of (1-1) are still stable and unique. In these cases, however, one can always approximate the piecewise continuous curves of f(y) and g(x) by some smooth curves as close as one desires. If the periodic solutions of the approximated system are stable and unique by Theorem II, then the periodic solutions of the original system will have most likely the same type of behavior. Since in most engineering problems, the characteristic curves are smooth, Theorem II is applicable without too much difficulty.

For the special case when g(x) is linear, it just requires a monotonically increasing function of f(y) for the convergence of the

* This is true even when the forcing function is non-periodic because up to this point the periodicity property of e(t) has not been used.

solutions. In this case, let $g(x) = x$, the Lyapunov function $V = u^2 + v^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ has the physical meaning of the square of the distance between any two trajectories. Then $\dot{V} = 2uv + 2v(-pv - u) = -pv^2 < 0$ for $p > b_1 > 0$. This means that the distance is always decreasing and any two solutions will converge to a unique periodic solution.

For the special case when $f(y)$ is linear; let $f(y) = by$, then $p = b > 0$. Suppose $g(x) = x + ax^3$, $a > 0$, the equation with a sinusoidal forcing function becomes the Duffing equation

$$\ddot{x} + b\dot{x} + x + ax^3 = E \sin\omega t \tag{3-15}$$

The solutions of this equation are all bounded according to Theorem I as follows:

$$\begin{aligned} |x(t)| \leq I &= E && \text{for } b^2 \leq \frac{16}{3} \\ &= \frac{16}{3}\frac{E}{b^2} && \text{for } \frac{4}{3} \geq b^2 \geq \frac{16}{3} \\ &= E + 4E\sqrt{\frac{1}{b^2} - \frac{3}{16}} && \text{for } b^2 \geq \frac{4}{3} \end{aligned} \tag{3-16}$$

$$|y(t)| \leq J = \frac{4E}{b} \tag{3-17}$$

But it may have three periodic solutions or subharmonics depending on the relative magnitudes of the parameters and the frequency. This can be explained by the fact $g(x)$ depends on x, i.e.

$$g''(x) = 6ax \tag{3-18}$$

If a, the coefficient of the nonlinear spring term, is not too small, and the assumption (3) of Theorem II is not satisfied, then the convergence of the solutions is not guaranteed, and the jump phenomena can appear as seen from the frequency response curve. According to Stoker[25], more than one periodic solution and subharmonic are possible only when the damping coefficient b is small, namely, about the order of "a". For small b, the largest value of I that sould be taken as the bound of x is

$$I = E + 4E\sqrt{\frac{1}{b^2} - \frac{3}{16}}$$

which is represented by the highest portion of the curve in Figure 2-5. From (3-15) one obtains

$$g'(x) = 1 + 3ax^2 > 1 \qquad c_1 = 1 \tag{3-19}$$

$$f'(y) = b \qquad b_1 = b \tag{3-20}$$

If "a" is sufficiently small such that the following inequality hold:

$$b > \frac{c_2}{c_1} J = 6aIJ = 24aE^2\left(\frac{1}{b} + \frac{4}{b}\sqrt{\frac{1}{b^2} - \frac{3}{16}}\right) \tag{3-21}$$

then the solutions of the Duffing equation will converge to a unique periodic solution with the same period as that of the forcing function regardless of what the value of the driving frequency is.

Inequality (3-21) can be rearranged as:

$$a < \frac{b^2}{24E\left(1 + 4\sqrt{\frac{1}{b^2} - \frac{3}{16}}\right)} \tag{3-22}$$

This inequality can be satisfied by changing either one or two of the three parameters a, E and b if the others are fixed in the system. As "a" becomes smaller, the system approaches a linear forced oscillation of which the frequency curve is a single-valued function for all frequency. As "a" becomes larger the inequality no longer holds, and the peak of the resonant curve will bend to the right. In this case the response will be a multi-valued function for some frequencies. Jumps in the amplitude of the solution may also occur.

It should be pointed out that the effect of changing the driving frequency on the amplitude of $x(t)$ can not be observed here since the methods of Theorem I and Theorem II are not analytic approaches to solving the problem.

CHAPTER IV

FIRST APPROXIMATION OF SINUSOIDAL ANALYSIS

Having established the existence and convergence of periodic solutions of Equation (1-1), it is useful to obtain an appropriate analytical method to approximate the periodic solutions. An exact solution of a nonlinear system is often very difficult to obtain, if not impossible. However, it is often satisfactory for practical purposes to obtain a first approximation to the solution. In this chapter a method is developed which is based on the principle of harmonic balance. It shows a first approximation of the periodic solutions of Equation (1-1) with harmonic inputs. In a nonlinear system with a sinusoidal input, the output is no longer sinusoidal but can contain, in addition to the fundamental component, higher harmonics and even subharmonics. The first approximation is to consider only the fundamental component of the response. The basic principle behind this method is not new, it is due to Ritz-Galerkin and interpreted in the Hamilton sense by Klotter.[34] However, the procedure developed below is different from that of Klotter. Furthermore, it gives a general expression for the describing function in complex form. In some cases the general expression for the describing function can be expressed in an analytical form which contains the linearized-equivalent parameters and is a function of the output amplitude and frequency.

The Amplitude and Frequency-Dependent Describing Function:

Consider a nonlinear system or element whose output-input relation is described by the second order differential equation:

$$\ddot{x} + f(\dot{x}) + g(x) = e(t) \tag{1-1}$$

where $x(t) = \theta_o(t)$ and $e(t) = \theta_i(t)$ in the following block diagram:

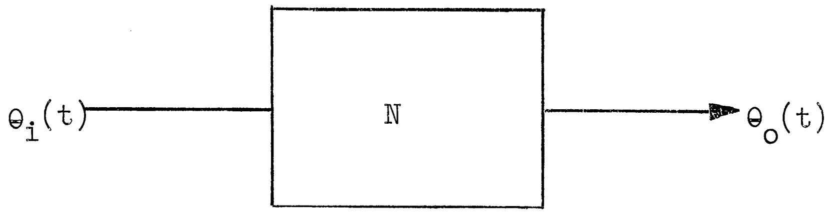

Figure 4-1 Input-Output Block Diagram.

Let the input be a sinusoidal signal $e(t) = E \cos\omega t$ with $\omega = 2\pi f$. Then Equation (1-1) becomes

$$\ddot{x} + f(\dot{x}) + g(x) = E \cos\omega t \tag{4-1}$$

Assume that the periodic solutions of (4-1) are only those having the same period as the forcing function, then one can take the fundamental component of the solution as the first approximation. Let the output fundamental be $\tilde{x} = A \cos(\omega t + \varphi)$ where φ is the phase angle.

In a general sense, the describing function N is the ratio of the complex fundamental amplitude of the output to the complex amplitude of the input, i.e.

$$N = \frac{A}{E} \angle\varphi \tag{4-2}$$

As far as the steady-state periodic solution is concerned, Equation (4-1) can be written with no loss in generality in the following form:

$$\ddot{x} + f(\dot{x}) + g(x) = E \cos(\omega t - \varphi) \tag{4-3}$$

In this case the fundamental output is $\tilde{x} = A \cos \omega t$. For a given E, both A and φ must be determined.

Now a finite integral transformation is applied to Equation (4-3) as follows:

$$\int_0^{2\pi} [\ddot{x} + f(\dot{x}) + g(x)]\, e^{-j\omega t} d(\omega t) = \int_0^{2\pi} E \cos(\omega t - \varphi) e^{-j\omega t} d(\omega t) \tag{4-4}$$

If $x(t)$ is an exact solution having the same period as that of the forcing function, the equal sign holds. If the higher harmonic content is small, the right side of (4-4) is approximately equal to the left side by substituting $\tilde{x} = A \cos\omega t$ in (4-4). The result is evidently only a first approximation. The smallness of higher harmonic contents of course depends on $f(\dot{x})$ and $g(x)$. If $f(\dot{x})$ and $g(x)$ are nearly linear functions, the approximation will be reasonably good. For better accuracy, higher harmonics should be taken into account. But this will require much tedious computation.

In the case when $f(\dot{x})$ and $g(x)$ are not skew symmetrical (non odd) functions, then $x(t)$ will in general not have zero mean value, but rather $\tilde{x} = D + A \cos\omega t$. The integration terms of $f(\dot{x})$ and $g(x)$ in (4-4) will also have a mean value component. The result will be much more complicated. The development below is therefore limited to the case in which both $f(\dot{x})$ and $g(x)$ are odd functions, i.e.

$$f(\dot{x}) = -f(-\dot{x}), \qquad g(x) = -g(-x) \tag{4-5}$$

Let $\omega t = \Theta$. Substituting $x = A\cos\Theta$ and $\dot{x} = -A\omega\sin\Theta$ in (4-4) gives

$$\int_0^{2\pi} f(A\omega\sin\Theta)\cos\Theta\,d\Theta = \int_0^{2\pi} g(A\cos\Theta)\sin\Theta\,d\Theta = 0 \qquad (4\text{-}6)$$

because of the properties of (4-5). Let

$$jF(A,\omega) = j\int_0^{2\pi} f(A\omega\sin\Theta)\sin\Theta\,d\Theta \qquad (4\text{-}7)$$

$$G(A,\omega) = \int_0^{2\pi} g(A\cos\Theta)\cos\Theta\,d\Theta \qquad (4\text{-}8)$$

Then Equation (4-4) becomes

$$-\pi A\omega^2 + G(A,\omega) + jF(A,\omega) = \pi E e^{-j\varphi} \qquad (4\text{-}9)$$

Since the describing function is defined in (4-2), then

$$N = \frac{A}{-A\omega^2 + \frac{1}{\pi}G(A,\omega) + jF(A,\omega)\frac{1}{\pi}}$$

or

$$N = \frac{1}{-\omega^2 + \frac{1}{\pi A}G(A,\omega) + j\frac{1}{\pi A}F(A,\omega)} \qquad (4\text{-}10)$$

The magnitude of N is

$$N = \frac{1}{\left\{[-\omega + \frac{G(A,\omega)}{\pi A}]^2 + [\frac{F(A,\omega)}{\pi A}]^2\right\}^{1/2}} \qquad (4\text{-}11)$$

The phase angle of N is

$$\varphi = -\tan^{-1}\frac{\frac{F(A,\omega)}{\pi A}}{-\omega^2 + \frac{G(A,\omega)}{\pi A}} \qquad (4\text{-}12)$$

where $F(A,\omega)$ and $G(A,\omega)$ can be considered as the linearized equivalent parameters. They are functions of both A and ω. For the linear case, they are functions of ω only. For the usual case, the describing function N is a function of E, the input amplitude. But in (4-11), N is a function of the amplitude of the output fundamental component. For given value of E, it is indeed difficult to solve for A. Nevertheless N can be readily calculated by using A and ω as independent variables. Once N has been calculated, the values of E can be obtained in a straight forward manner by dividing A by N. The advantage of this method is that it gives a rather simple form which can often be applied to yield quick results of adequate accuracy in many applications.

Remarks: Equation (4-10) furnishes only the quantitative relationship between the input and output amplitude. The result of this method is based on the assumption that there exist periodic solutions only with the same period as that of the forcing function. Otherwise, if the solution turns out to be subharmonic, Equation (4-10) will be no longer applicable. Furthermore, the result of this method does not give any information regarding the stability and the behavior of the periodic solutions. In general, the describing function of a nonlinear system can not be used to study the stability problem. Therefore, before applying the result of this method, some knowledge about the system itself and qualitative behavior of the solutions is necessary. For instance: 1) what kind of motions is expected? 2) are the solutions bounded? and if the periodic solution is unstable, it seems meaningless to compute the describing function. To answer these problems, one can resort to the boundedness theorem of Chapter II and the convergence theorem of Chapter III.

As an example, consider the differential system

$$\ddot{x} + 2\xi\dot{x} + \frac{\dot{x}}{|\dot{x}|} + g(x) = E \cos (ut) \qquad (4\text{-}13)$$

which will be studied in detail in the next chapter. The function g(x) is described by

$$\begin{aligned} g(x) &= x - d && \text{for } x > d \\ &= 0 && \text{for } -d < x < d \qquad (4\text{-}14) \\ &= x + d && \text{for } x < -d \end{aligned}$$

The procedure used to obtain $F(A,\omega)$ and $G(A,\omega)$ is given in Appendix I. Both have been formulated by some authors.(35,32) But the study of Equation (4-10) with both nonlinear terms F and G has not been made previously. The describing function for (4-13) has the analytic expression:

$$N = \frac{1}{-u^2 + \left[1 - \frac{2}{\pi} \sin^{-1} \frac{d}{A} - \frac{2d}{\pi A} \cos(\sin^{-1} \frac{d}{A})\right] + j(2\xi u + \frac{4}{\pi A})}$$

(4-15)

The N function is plotted with various values of d and ξ in the next chapter together the experimental result. The calculated values will also be discussed and compared with the experimental results.

CHAPTER V

A PARTICULAR SYSTEM

The purpose of this chapter is to study a particular forced oscillation system, namely, a second order differential system with combined viscous and coulomb frictions, and nonlinear restoring force possessing a dead-space characteristic. As shown in Figure 5-1, the control system under consideration consists of an amplifier, a motor, and gears in the feedback path.

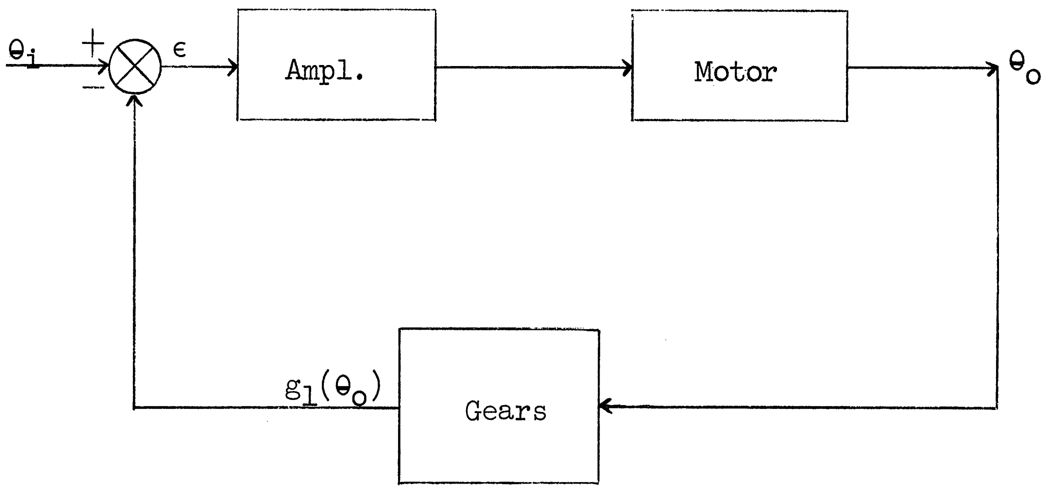

Figure 5-1 Feedback Control System

The equation of motion of this system is

$$J_1\ddot{\theta}_o + f_1(\dot{\theta}_o) = K\epsilon = K\left[\theta_i - g_1(\theta_o)\right] \qquad (5\text{-}1)$$

which can be rearranged to

$$J_1\ddot{\theta}_o + B_1\,\dot{\theta}_o + M\frac{\dot{\theta}_o}{|\dot{\theta}_o|} + Kg_1\,(\theta_o) = K\theta_i \qquad (5\text{-}2)$$

where J_1 is the moment of inertia, B_1 the coefficient of linear damping, M the magnitude of coulomb friction, and K the gain of the amplifier. The combined friction can be represented as shown in Figure 5-2. The gears in the feedback path have the dead-space characteristic shown in Figure 5-3 which defines the function $g_1\ (\theta_o)$ to be

$$g_1(\theta_o) = \begin{cases} \theta_o - d_1 & \theta_o > d_1 \\ 0 & -d_1 < \theta_o < d_1 \\ \theta_o + d_1 & \theta_o < -d_1 \end{cases} \qquad (5\text{-}3)$$

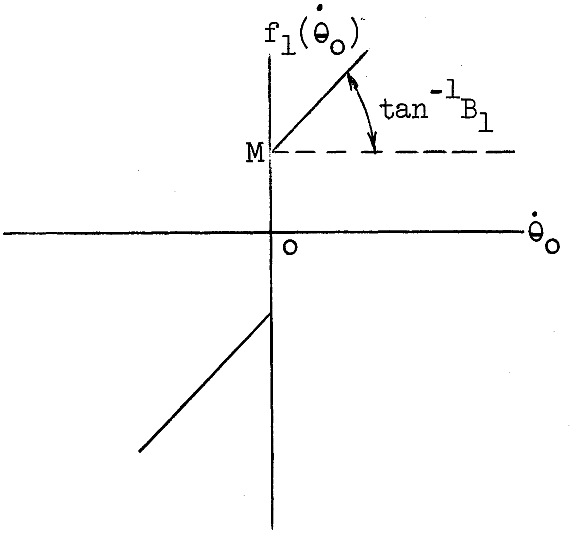

Figure 5-2
Combined Coulomb and Viscous Frictions

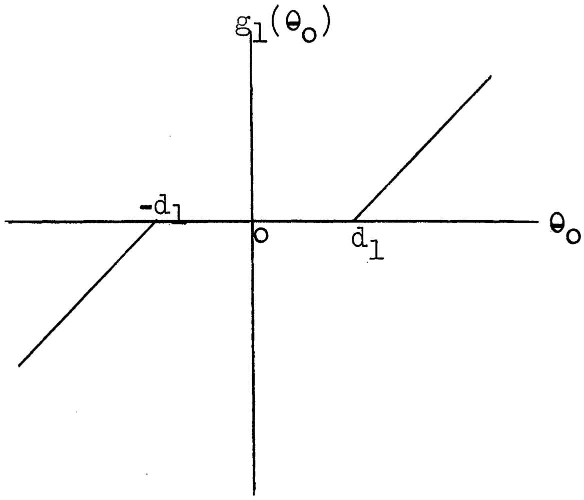

Figure 5-3
Dead-Space Characteristics

Assuming the input to be a sinusoidal signal, the system of (5-2) can be put into the following dimensionless form:

$$\ddot{x} + 2\xi\dot{x} + \frac{\dot{x}}{|\dot{x}|} + g(x) = E \cos ut \qquad (4\text{-}13)$$

The details of normalization are carried out in appendix II. The function g(x) has a dimensionless quantity d for the dead-space as described in (4-14).

First it will be shown that all solutions of (4-13) will ultimately go into a certain bounded region. This region will be explicitly determined. Second, states of rest of the solutions will be found when certain restraints are put on the forcing function, and the stability considerations will be also discussed. Finally, the results of a computer study are presented and compared with the calculated results.

Construction of a Bounded Region:

Due to the discontinuities of $f(y)$ and the dead-space characteristics of $g(x)$, the assumptions of Theorem I are not all satisfied, therefore the result can not be directly applied. However, the technique is essentially the same, only a slight modification is made in the procedure to obtain the bounded region. Equation (4-13) can be replaced by

$$\dot{x} = y$$

$$\dot{y} = -2\xi y - \frac{y}{|y|} - g(x) + E \cos ut. \qquad (5\text{-}4)$$

Since the slope $\frac{dy}{dx}$ of (5-4) never exceeds $\frac{dy}{dx}$ of the following system:

$$\dot{x} = y$$

$$\dot{y} = -2\xi y + E_1\frac{y}{|y|} - g(x) \qquad (5\text{-}5)$$

where $E_1 = E - 1$, if a bounded region R_1 can be found for the system (5-5), all solutions of (5-4) will also be carried into R_1. In Figure 5-4, R_1 is shown bounded by a closed curve Γ_1. The equation of the curve representing $\frac{dy}{dx} = 0$ in (5-5) is:

$$-2\xi y = g(x) - E_1 \qquad \text{for } y > 0 \qquad (5\text{-}5a)$$

which has three straight line portions; one of them is a horizontal line in the interval $-d<x<d$. If a solution path of (5-5) intersects this line, it will move along to the right on this line until $x = d$, because $\frac{dy}{dt}$ (or $\frac{dy}{dx}$) is zero in this interval. The curve (5-5a) now intersects

the x-axis at $x = \beta = E_1 + d$. Hence the statement still holds that every solution of (5-5) starting to the left of the line $x = E_1 + d$ must cross it and reaches the x-axis at $x > \beta$.

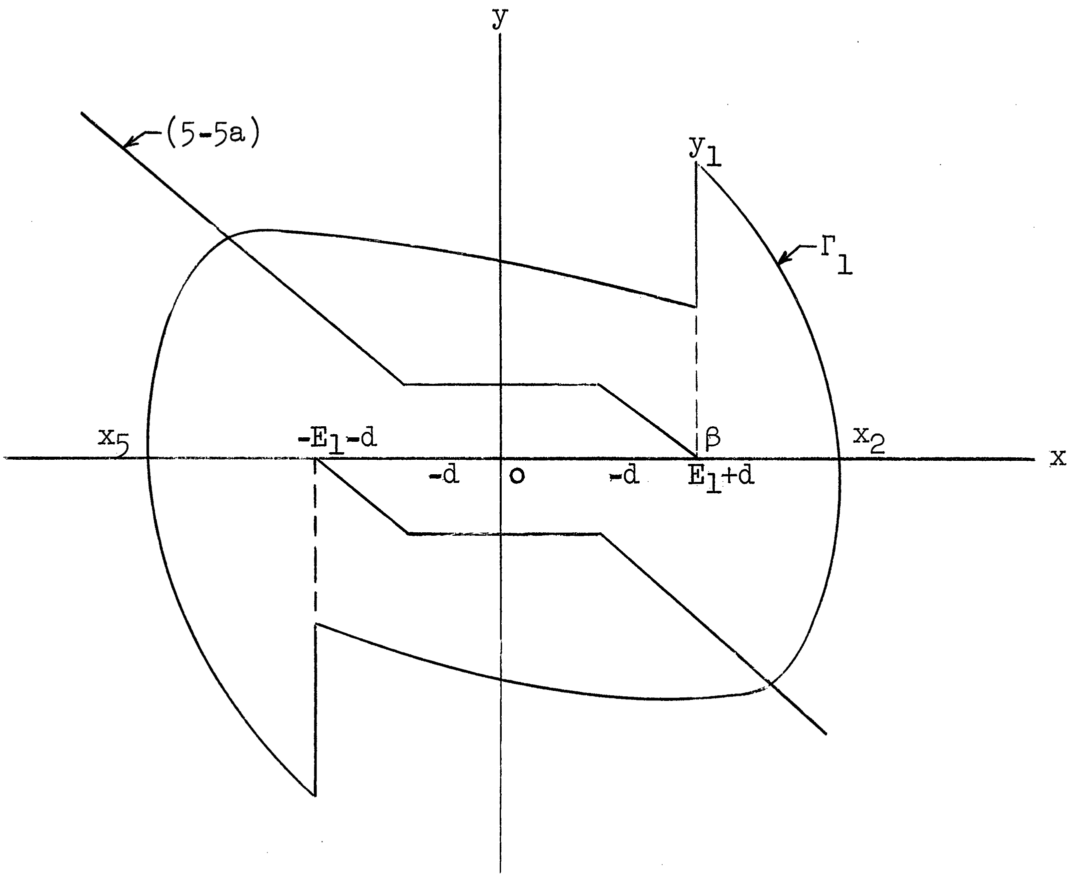

Figure 5-4 Closed Curve on the Phase Plane for Equation (4-13)

Now Γ_1 is constructed in the same way as it was in proving Theorem I where now $y_1 = \frac{4E_1}{2\xi} = \frac{2E_1}{\xi}$. Following the same procedure, one then obtains

$$|y(t)| \leq J = \frac{2E_1}{\xi} \tag{5-6}$$

It remains to define the boundary for x. Since $x_2 - \beta \leq \frac{2E_1}{2\xi^2} = \frac{E_1}{\xi^2}$ and $\beta = E_1 + d = E - 1 + d$, so that

$$|x(t)| \leq I = E - 1 + d + \frac{E - 1}{\xi^2} \qquad \text{if } 2\,\xi > 1$$

$$\text{or} \quad |x(t)| \leq I = E - 1 + d + \frac{2(E - 1)}{\xi} \qquad \text{if } 2\,\xi < 1 \tag{5-7}$$

The last inequality of (5-7) is found from the solution of (5-5) starting at $(\beta, \frac{2E_1}{\xi})$ and intersecting the x-axis at x_2.

$$x_2 = E - 1 + d + \frac{2(E - 1)}{\xi} e^{-\xi(\frac{\pi}{2} - \tan^{-1}\xi)} < E - 1 + d + \frac{2(E - 1)}{\xi}$$

Inspection of (5-6) and (5-7) reveals that if there is linear damping in the system ($\xi > 0$), the motion is always bounded, and the bounded region also depends on d, the dead-space. The greater the value of d, the larger the bounded region.

States of Rest and Periodic Solutions:

The system (4-13) may have two kinds of solutions depending on the amplitude E of the input function. It will be shown first that the solutions have states of rest if E is smaller than 1, and then the solutions are periodic if $E > 1$. By the state of rest is meant that $y = 0$ and x = constant for all time after some instant. The physical property of the coulomb friction in the $y = 0$ plane can be expressed as follows:

$$\begin{aligned} \dot{y} &> 0 \qquad \text{for } -g(x) + e(t) \geq f(0+) = 1 \\ \dot{y} &< 0 \qquad \text{for } -g(x) + e(t) \leq f(0-) = -1 \\ \dot{y} &= 0 \qquad \text{for } -1 \leq -g(x) + e(t) \leq 1 \end{aligned} \tag{5-8}$$

The last expression of (5-8) shows that if the value of the net force $-g(x) + e(t)$ is in between $f(0-)$ and $f(0+)$ when the velocity is zero, the coulomb friction adjusts itself to such a value to just compensate for this force. Then the system remains in rest until this force goes out of this interval. Relation (5-8) can be plotted in the $y = 0$ plane. There are three regions for each expression of (5-8).

$$\dot{y} > 0 \quad \begin{array}{llll} (1) & x \geq d & \text{and} & x \leq e(t) + d - 1 \\ (2) & -d \leq x \leq d & \text{and} & e(t) \geq 1 \\ (3) & x \leq -d & \text{and} & x \leq e(t) - d - 1 \end{array} \qquad (5\text{-}9)$$

$$\dot{y} < 0 \quad \begin{array}{llll} (4) & x \geq d & \text{and} & x \geq e(t) + d + 1 \\ (5) & -d \leq x \leq d & \text{and} & e(t) \leq -1 \\ (6) & x \leq -d & \text{and} & x \geq e(t) - d + 1 \end{array} \qquad (5\text{-}10)$$

$$\dot{y} = 0 \quad \begin{array}{llll} (7) & x \geq d & \text{and} & e(t) + d - 1 \leq x \leq e(t) + d + 1 \\ (8) & -d \leq x \leq d & \text{and} & -1 \leq e(t) \leq 1 \\ (9) & x \leq -d & \text{and} & e(t) - d - 1 \leq x \leq e(t) - d + 1 \end{array} \qquad (5\text{-}11)$$

The case for the amplitude of the normalized forcing function $E \leq 1$ is shown in Figure 5-5, and d is assumed less than 1.

There are three regions in the $y = 0$ plane. The portion for $\dot{y} = 0$ is shown by the shaded area. The area to the right is for $\dot{y} < 0$, and the area to the left is for $\dot{y} > 0$. Phase trajectories arriving at the $y = 0$ plane outside of the shaded area must pass through the half space $y > 0$ into the half space $y < 0$ in case $x \geq e(t) + d + 1$ (corresponding to $\dot{y} < 0$). Phase trajectories falling in the shaded area from the space $y \neq 0$ will remain in the $y = 0$ plane but move along the lines $x =$ constants until they arrive at the boundary of the shaded area. And then, they will pass into the half space $y>0$ ($y<0$), if they arrive at the left hand (right hand) boundary. It is clear that if a phase trajectory falls in the interval D: $-(1 + d - E) < x < 1 + d - E$, it will move along the line $x =$ constant in the $y = 0$ plane permanently.

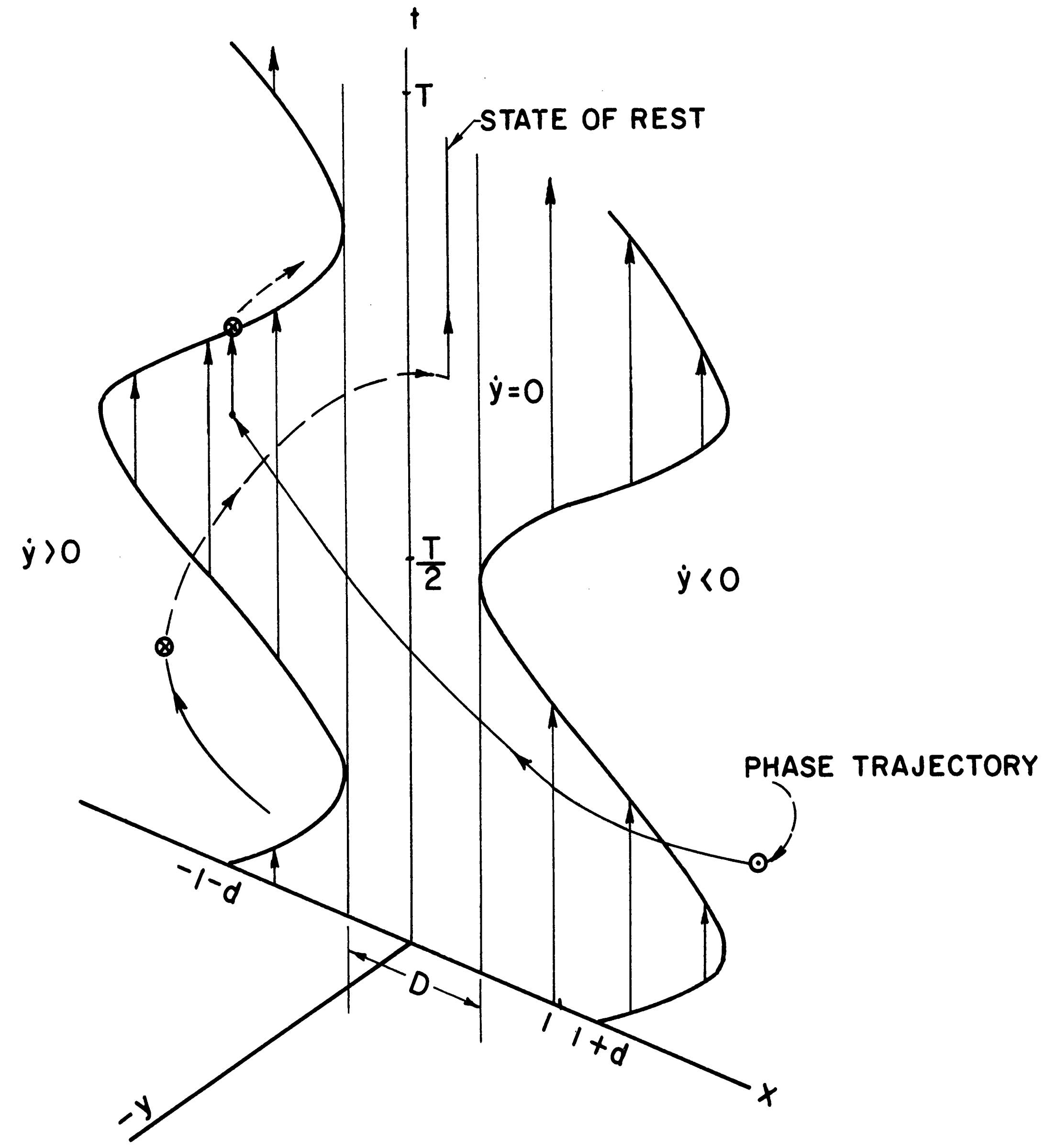

Figure 5-5 Phase Space Portrait, Where States of Rest are Possible.

This corresponds to a state of rest of the system. Therefore all points in the interval are fixed points.

For $E < 1$, it can be shown that phase trajectories starting outside of the rest interval D will finally turn into this interval. Recalling that the slope $\frac{dy}{dx}$ of (5-4) never exceeds $\frac{dy}{dx}$ of (5-5), of all solution paths of (5-5) converge to this rest interval, every solution path of (5-4) will be carried into this same interval. Let $E_2 = -E_1 = 1 - E > 0$, then the system represented by (5-5) becomes an autonomous system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -2\xi y - E_2\frac{y}{|y|} - g(x) \end{aligned} \tag{5-12}$$

which has an effective coulomb friction of a magnitude equal to E_2. The phase vectors near the x-axis is indicated in Figure 5-6.

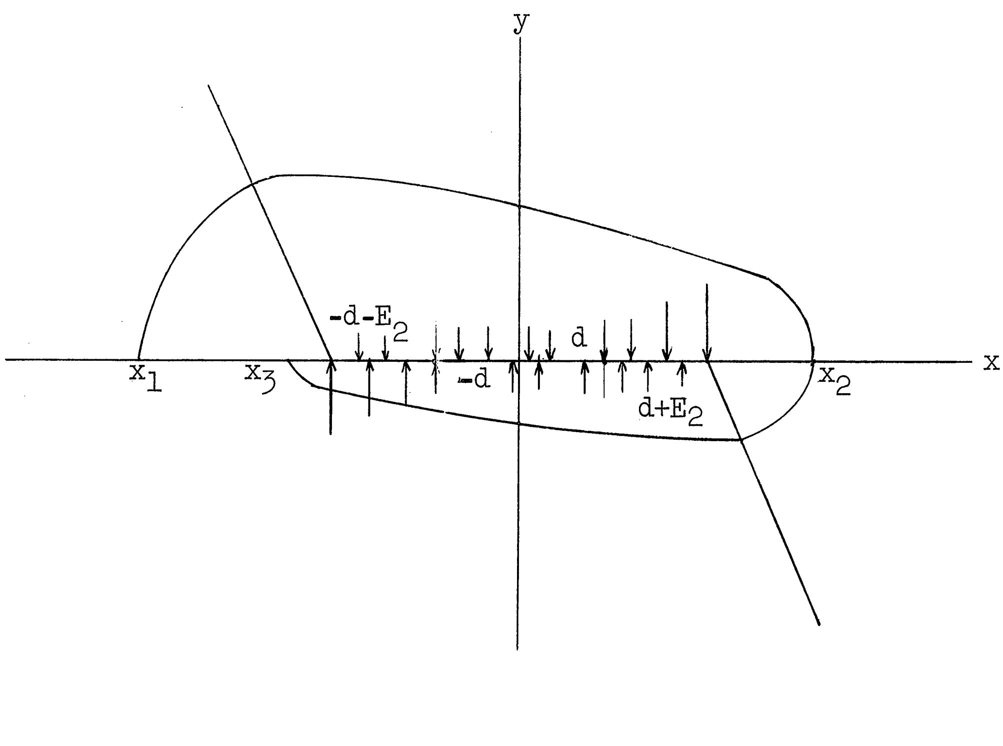

Figure 5-6 Phase Plane Portrait $E < 1$.

In the interval D, these vectors are all pointing toward the x-axis. The relative magnitudes of these phase vectors are sketched as linearly varying in the interval D. Evidently once a solution path falls on the x-axis in the interval D, it will remain there. For a solution path beginning at some initial point in the phase plane, it is also easy to show that this solution path will finally move to the rest interval D. The lines for which the slope $\frac{dy}{dx} = 0$ are also shown in Figure 5-6. Any solution path starting to the left of the line in $y > 0$ half plane must move upward from left to right to a maximum point at the intersection with the line, and then move downward toward the x-axis. Suppose that a solution path cuts the negative x-axis at x_1 for the first time and then moves to the right cutting the positive x-axis at x_1. Integrating (5-12) from $(x_1, 0)$ to $(x_2, 0)$ one obtains

$$-2\xi \int_{x_1}^{x_2} y dx - E_2(x_2 - x_1) - G(x_2) + G(x_1) = 0 \qquad (5\text{-}13)$$

$$G(x_1) - G(x_2) = 2\xi \int_{x_1}^{x_2} y dx + E_2(x_2 - x_1) > 0 \qquad (5\text{-}14)$$

because the integration term is positive and $x_2 > x_1$. From the characteristic of $g(x)$, one concludes that $x_2 < |x_1|$. When the solution path continues to move in $y < 0$ half plane and cuts the negative axis at x_3, then by the same procedure one has $|x_3| < x_2$. Then $|x_1|, x_2, |x_3| \ldots$ form a decreasing sequence. The solution path will finally fall in the rest interval D. Therefore every solution path of equation (5-5) and thus of equation (5-4), will be carried into the same rest interval on the x-axis.

For the case $E > 1$, according to (5-9), (5-10) and (5-11) the phase trajectories passing through the $y = 0$ plane are shown in

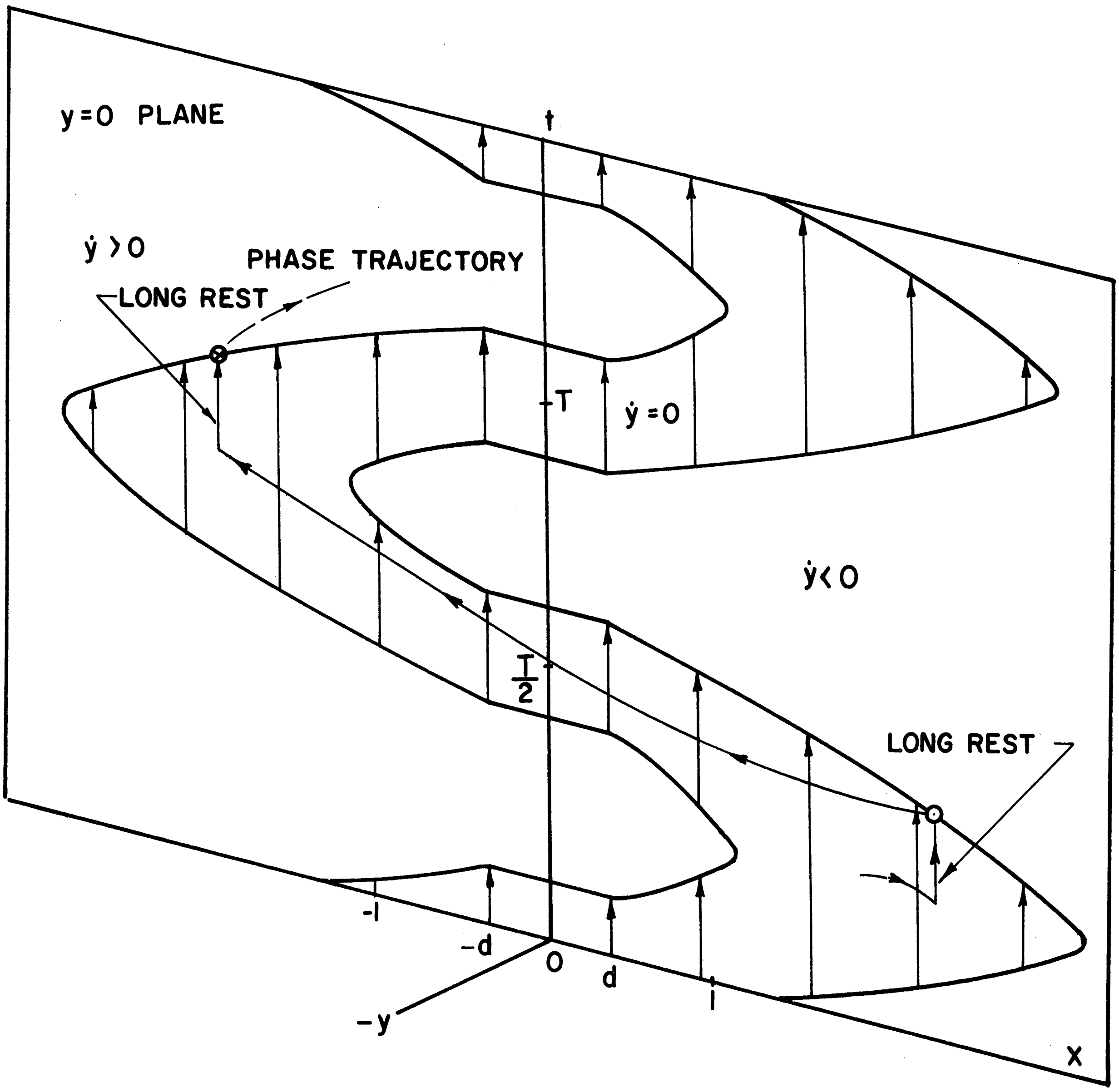

Figure 5-7 Phase Space Portrait, Where States of Rest are Possible.

Figure 5-7. Phase trajectories arriving at the region $\dot{y} = 0$ move along the lines x = constants until they reach the boundary of the region $\dot{y} > 0$ (or $\dot{y} < 0$), and then they pass into the half space $y > 0$ (or $y < 0$). Clearly there is no state of rest although a solution x(t), y(t) may have long rests*.

For $y \neq 0$, the function g(x) and f(y) are well defined, so the solutions x(t) and y(t) are continuous and unique in the space $y \neq 0$. Furthermore, the behavior of a phase trajectory moving on the $y = 0$ plane is uniquely determined by the moment and the point of the plane at which the trajectory arrives. Hence the phase trajectories of the system (5-4) are continuous curves for $t > t_o$, and depend on the initial points (x_o, y_o) continuously. Therefore the Fixed Point Theorem can be applied and thus there exists a periodic solution having the period of the forcing function.

In the system (5-4) the function g(x) and $f(\dot{x})$ are odd functions of their arguments, and the forcing function is periodic. Hence system (5-4) is a symmetric oscillator. The periodic solutions are necessarily symmetric, i.e., if $x^*(t)$ is a periodic solution then $x^*(t + \frac{T}{2}) = -x^*(t)$. This can be shown as follows: Let $x' = -x$, $\dot{x}' = -\dot{x}$ and $\tau = t - \frac{T}{2}$. By changing the variables the new system has the same form as in system (5-4). Therefore they have identical periodic solutions, i.e., $x^*(t) = x'^*(\tau)$. But $x'^*(\tau) = -x^*(\tau) = -x^*(t + \frac{T}{2})$. Hence $x^*(t) = -x^*(t + \frac{T}{2})$.

The phase trajectories of the periodic solutions of system (5-4) on the x-y plane have different types of patterns depending on

*Long rest means $y=0$ and x = constant for some definite interval of time.

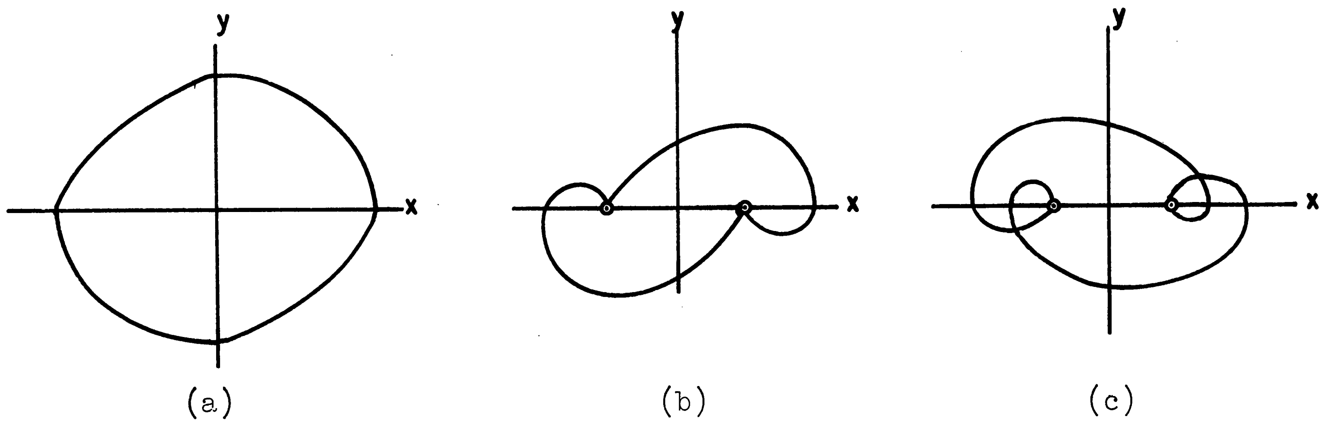

Figure 5-8 Possible Patterns of Limit Cycles - Trajectories Arriving at the Region $\dot{y} < 0$ from the Half Space $x < 0$.

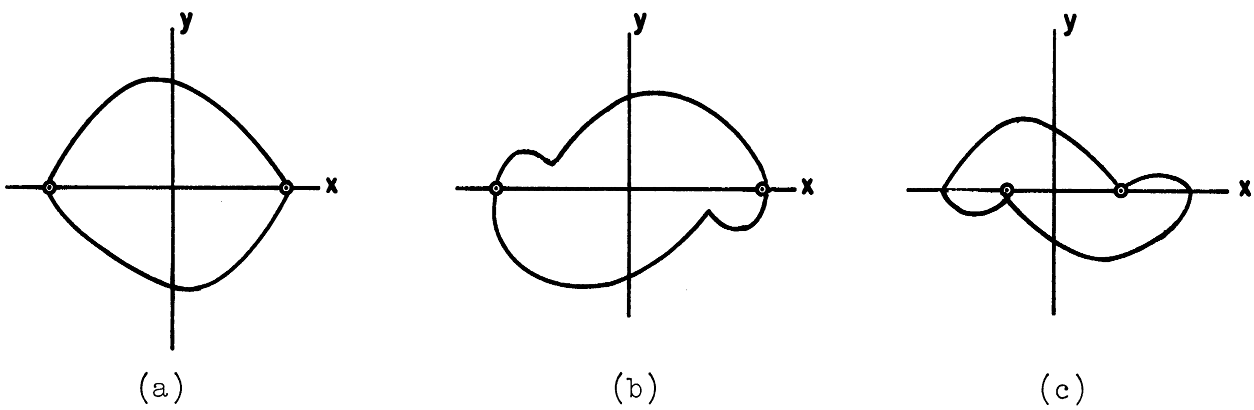

Figure 5-9 Possible Patterns of Limit Cycles - Trajectories Arriving at the Region $\dot{y} = 0$ from the Half Space $x < 0$.

the system parameters. Figure 5-8 and Figure 5-9 illustrate the possible types of the phase trajectories* of the periodic solutions as visualized from Figure 5-7. The patterns in Figure 5-8 correspond to those trajectories arriving at the region $\dot{y} < 0$ from the half space $x < 0$. In (a), the trajectory passes through the $y = 0$ plane and back to half space $x < 0$ without rest. In (b), the trajectory after crossing the $y = 0$ plane moves to the region $\dot{y} = 0$ back in the $y = 0$ plane with x and y unchanged for a while, then moves back to the half space $x < 0$. In (c), the phase trajectory after crossing the $y = 0$ plane moves to the region $\dot{y} = 0$ which is under the boundary of the region $\dot{y} > 0$; this trajectory makes a small loop before turning back to the $x < 0$ space. Figure 5-9 shows those trajectories arriving at the region $\dot{y} = 0$ from the half space $x < 0$. These trajectories have long rest for some interval of time after they arrive at the region $\dot{y} = 0$.

The patterns of the phase trajectories of the periodic solutions depend on the system parameters as well as the forcing function. At this stage it should be pointed out that due to the complexity of the problem, it is difficult to obtain an exact solution of system (5-4). Thus no attempt is made as to determine the dependence on the forcing function and the conditions on the system parameters under which a particular pattern of phase trajectory may occur. However, if the nonlinearities are relatively small, i.e., for small values of dead-space and coulomb friction, and if the amplitudes of the forcing function are relatively large, the effect of these nonlinearities on the periodic solutions becomes insignificant, and the system approaches a linear one. The pattern of the phase trajectories of the periodic solutions will be

* The mark "o" on the x-axis indicates the long rest.

close to an ellipse. In order to obtain an approximate solution, the method stated in Chapter IV can be used to give satisfactory results.

Remarks: Because of the dead-space characteristics of the function $g(x)$, Theorem II of chapter III can not be applied. This makes it very difficult to test whether the periodic solution is stable and unique or not. For instance, suppose $g(x)$ is zero in a very large interval d. The system (5-4) may become $\ddot{x} + f(\dot{x}) = E \cos ut$. There can be an infinite number of solutions for x. When d is small the periodic solution is likely to be stable and unique as it is for the case when $g(x)$ is linear. On the other hand, if $g(x)$ is a piecewise linear function, and if in the interval $-d < x < d$, $g'(x)$ is very small but not zero (with the dead-space as the limiting case), then by Theorem II a periodic solution that is stable and unique is possible, since in this case the corners in the piecewise linear curve of $g(x)$ may be unimportant, and can often be approximated by a continuous curve in practical cases.

Computer Study:

An analog computer study was made to check the result of the analysis of chapter IV. The system being simulated is rewritten here:

$$\dot{x} + 2\xi\dot{x} + \frac{\dot{x}}{|\dot{x}|} + g(x) = E \cos ut \qquad (4\text{-}13)$$

The computer circuit diagram and the illustration of the set up are given in Appendix III. The experimental work was performed at the Willow Run Research Laboratory using the RACK Analog Computer. To measure the describing function of the system, the scheme by Larrowe and Spencer [39] was used and briefly mentioned in Appendix III.

The damping coefficient ξ was taken for 0.25 and 0.7. For each value of ξ, the dead-space d was taken for 0.5 and 1. The input amplitude E in the following equation:

$$N = \frac{1}{-u^2 + \left[1 - \frac{2}{\pi}\sin^{-1}\frac{d}{A} - \frac{2d}{\pi A}\cos(\sin^{-1}\frac{d}{A})\right] + j\,(2\xi u + \frac{4}{\pi A})} \qquad (4\text{-}15)$$

was calculated for each assumed value of A, as tabulated in Appendix IV. Since the experiment data were to be compared with the calculated results, the value of the input amplitude E used in the computer was taken to be the same as that calculated from equation (4-15). The frequency range used was from 0.05 to 1.3 rad./sec. The data are also presented in Appendix IV. Both the measured and calculated describing functions are plotted in the complex plane and are shown in Figure 3-10. The time waveforms of the systems whose phase plane plots are shown in Figure 5-12 (a), (b), (c) and (d) are given in Figure 5-11 (a), (b), (c) and (d) respectively. For $\xi = 0.7$ and $d = 0.5$, the difference between the calculated and experimental results is only about 0.5 percent for amplitude and 1° for phase angle. The largest difference between results appears when the input amplitude is less than 2 and for $\xi = 0.25$, $d = 1$; and it is about 20% for amplitude and 15° for phase angle.

Obviously these curves show that the larger the input amplitude, the smaller the error to be expected. The difference becomes as large as 24% at smaller input amplitudes when $u = 0.3$ and 0.5. This is due to the fact that for such cases the periodic solution has a long rest in every half period which can be observed from the waveforms. These waveforms are quite different from sine wave upon which the formulation of F(a, u) and G(A, u) are based, and hence a large error is possible between the observed and calculated values. In short, for input amplitudes greater than 4, the calculated values agree very well with the experimental results. The results obtained from this comparison validate the first approximation analysis given chapter IV. The inaccuracy of the computer components may account for part of the errors, but this was observed to be small.

In Figure 5-11, (d) shows that the velocity waveform is very nearly sinusoidal when E is large, whereas (b) shows that a long rest exists for some interval of time in every half period when E is small.

Figure 5-12 was obtained from an x-y plotter showing the phase trajectories of different types of patterns, the patterns were in good agreement with the predicted ones as presented in Figure 5-9.

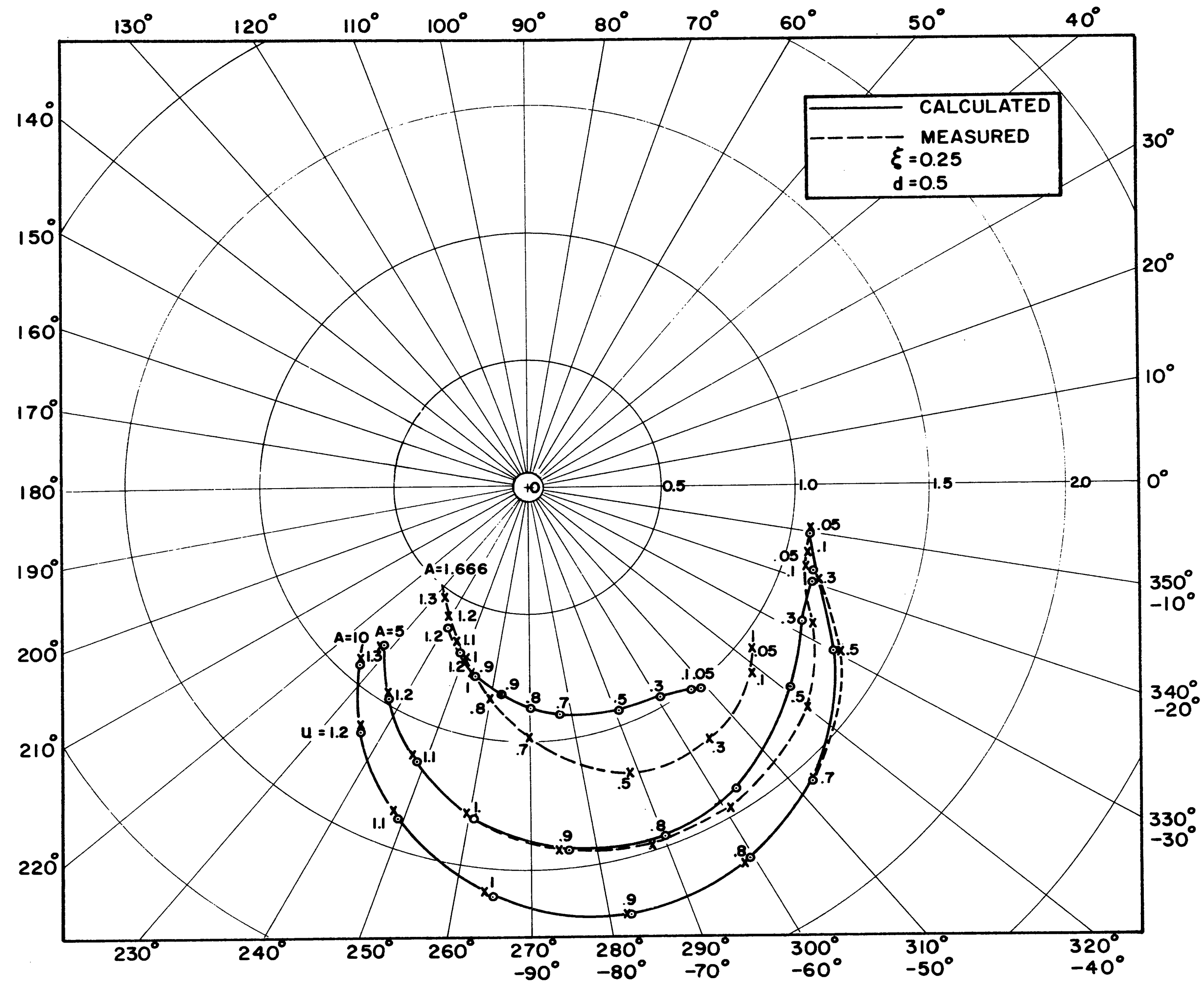

Figure 5-10(a) The Describing Functions of Equation (4-13) in Polar Plot.

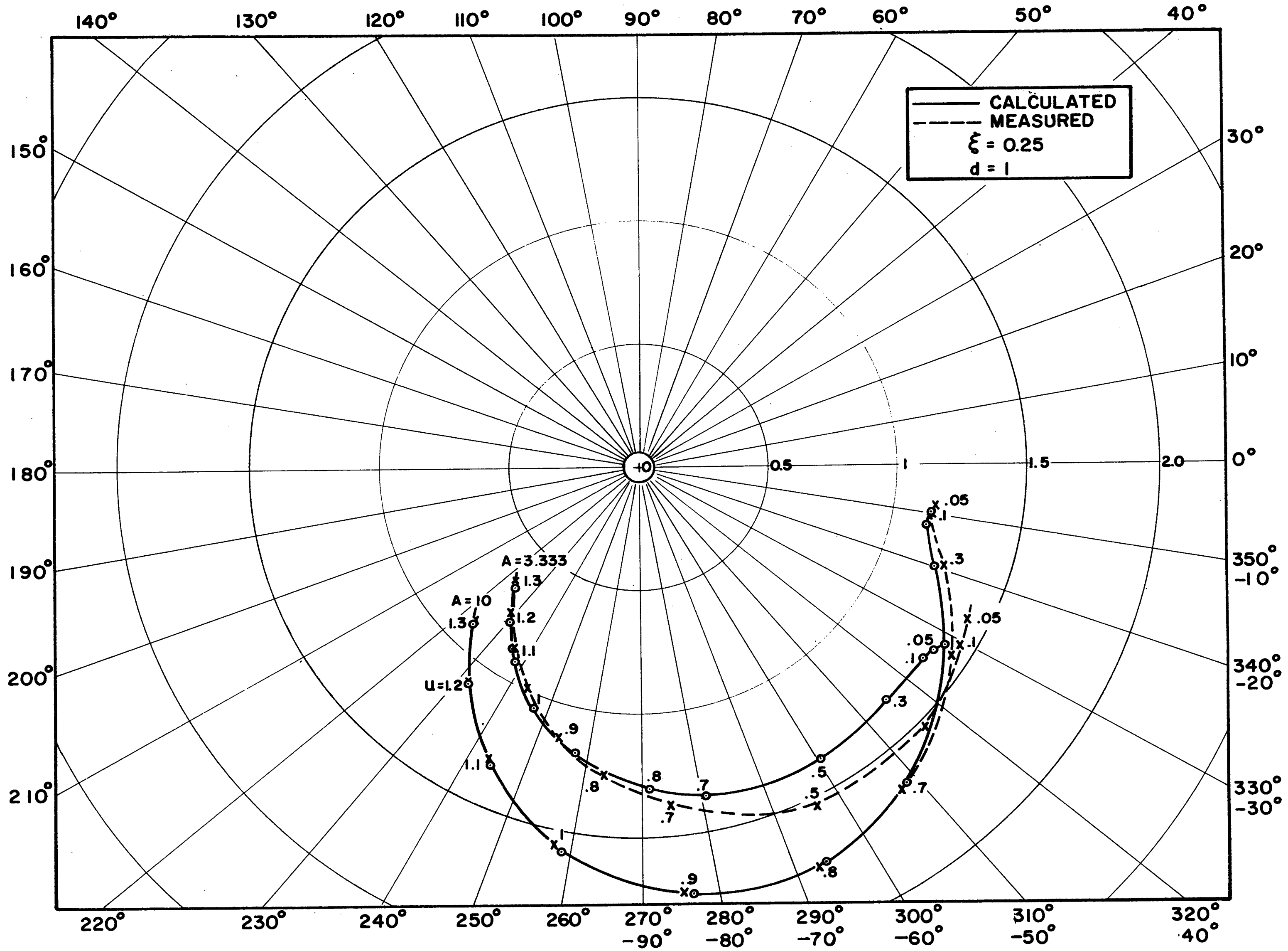

Figure 5-10(b) The Describing Functions of Equation (4-13) in Polar Plot.

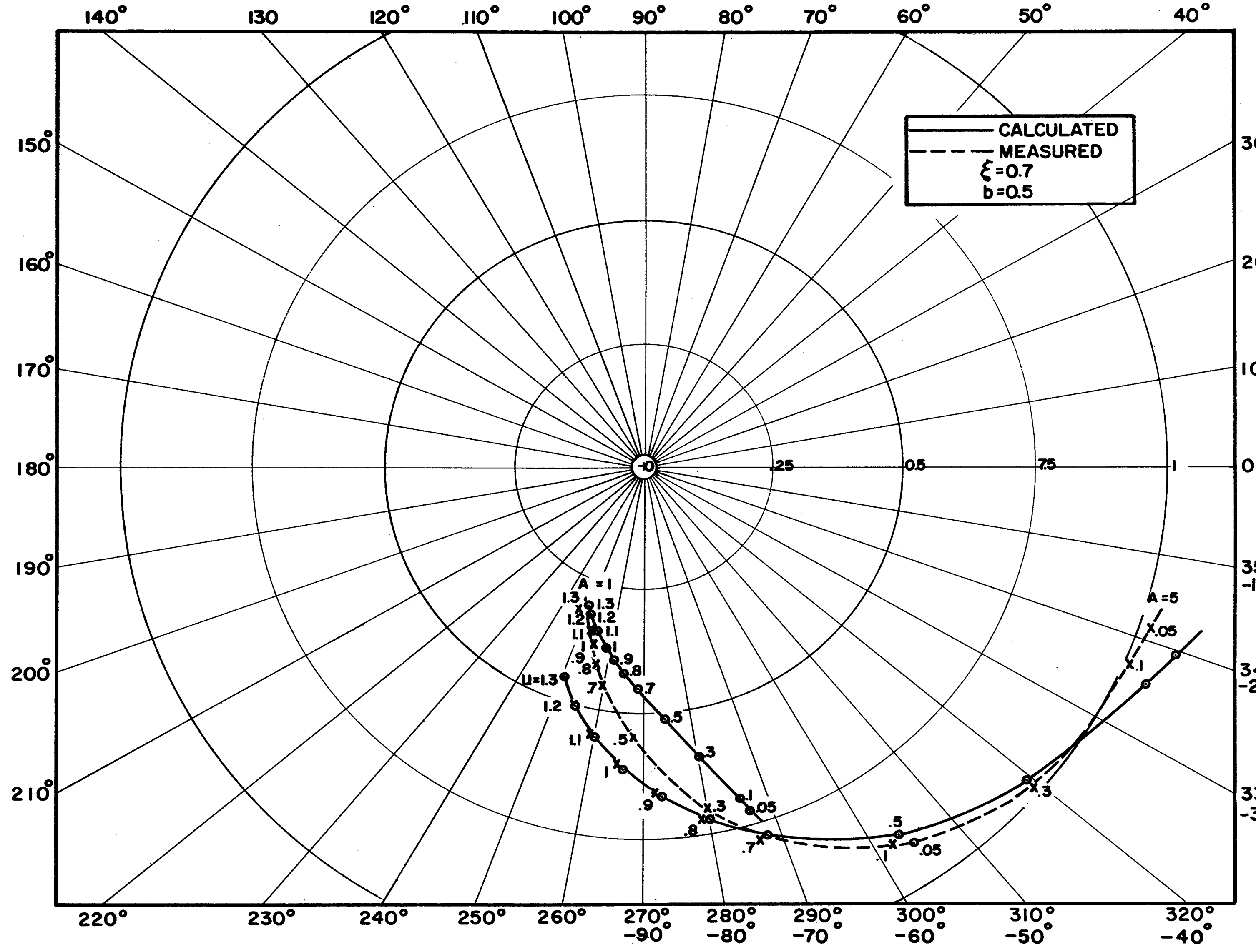

Figure 5-10(c) The Describing Functions of Equation (4-13) in Polar Plot.

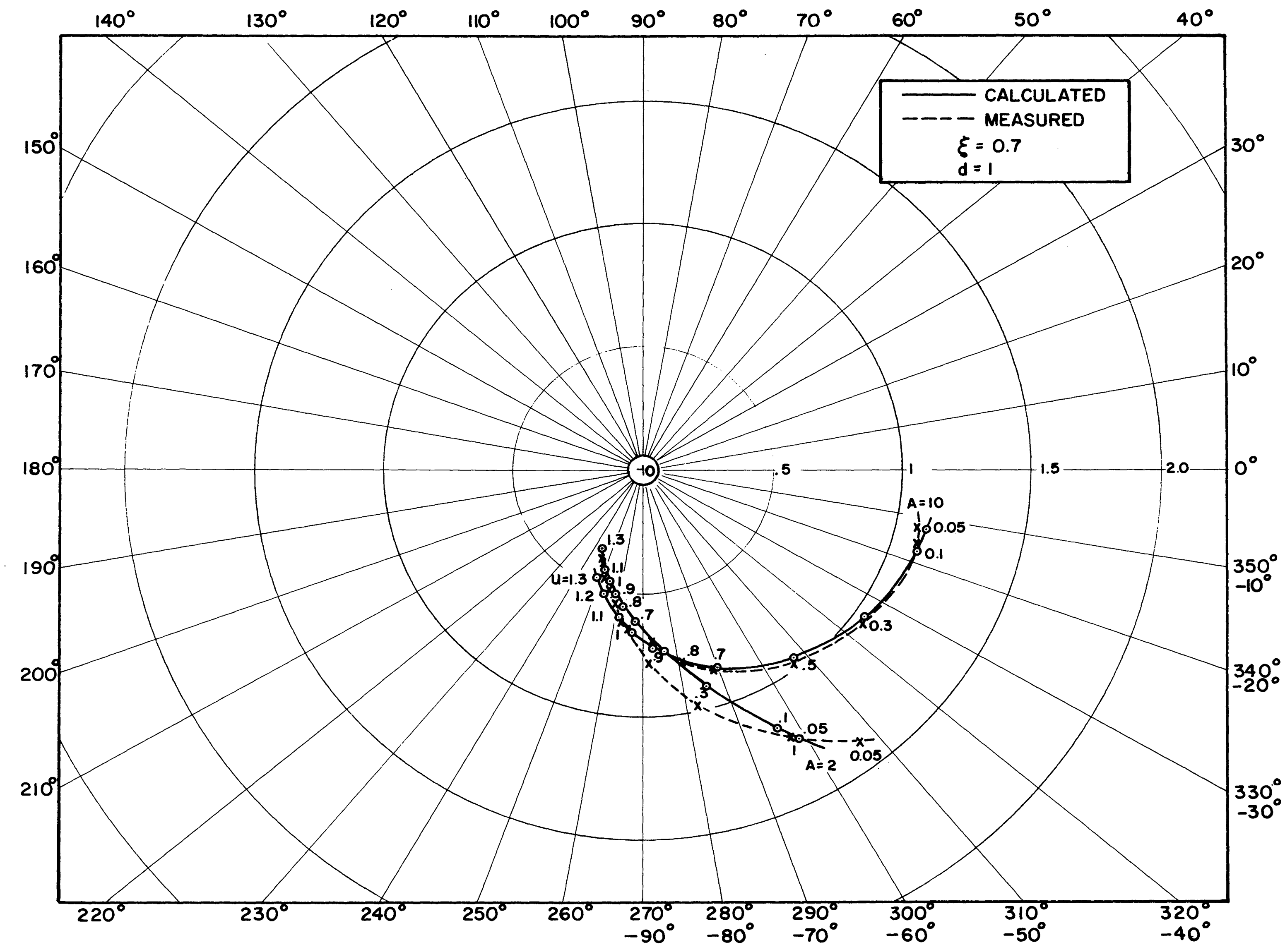

Figure 5-10(d) The Describing Functions of Equation (4-13) in Polar Plot.

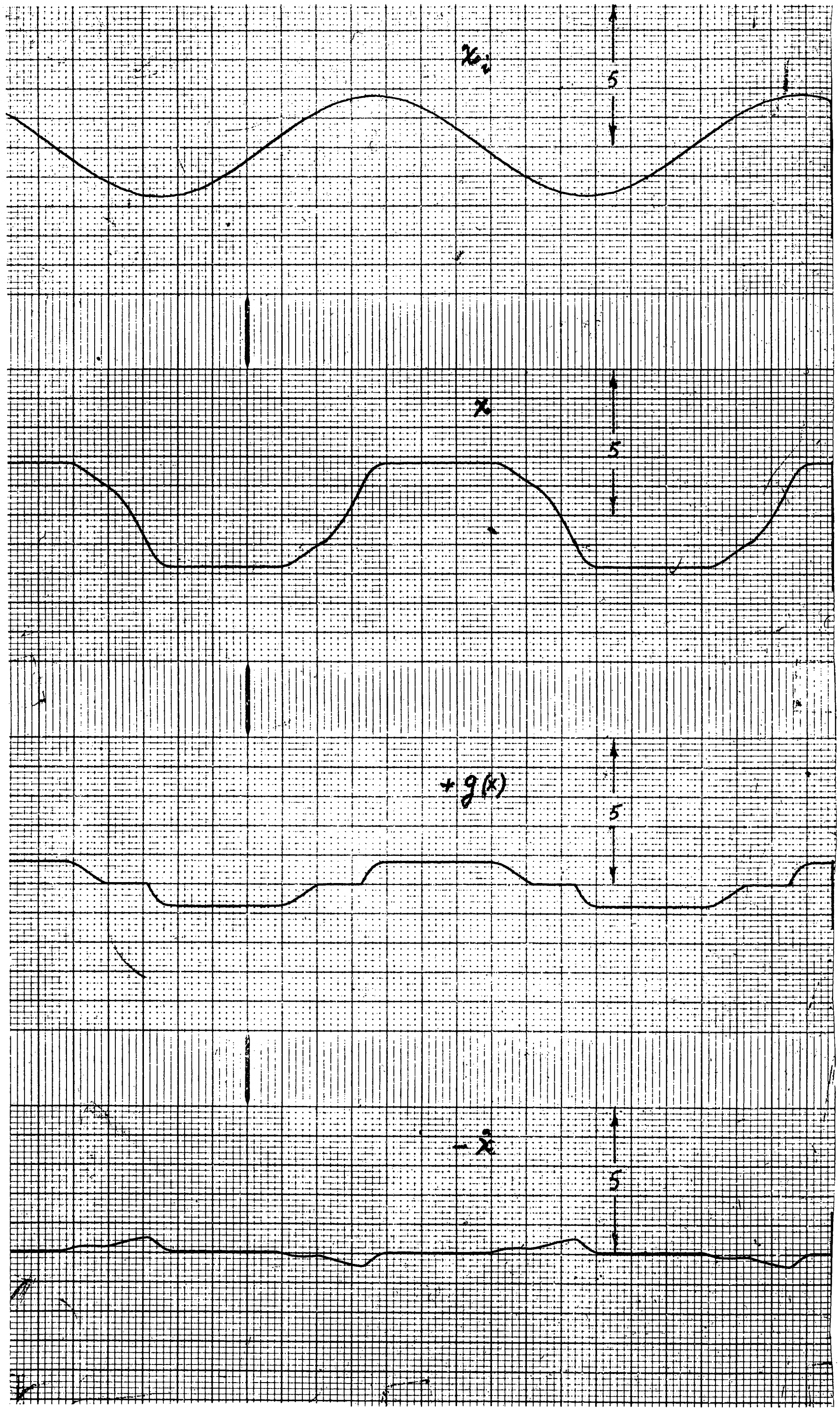

Figure 5-11(a) Waveform Plot

u = 0.1 ξ = 0.7 d = 1

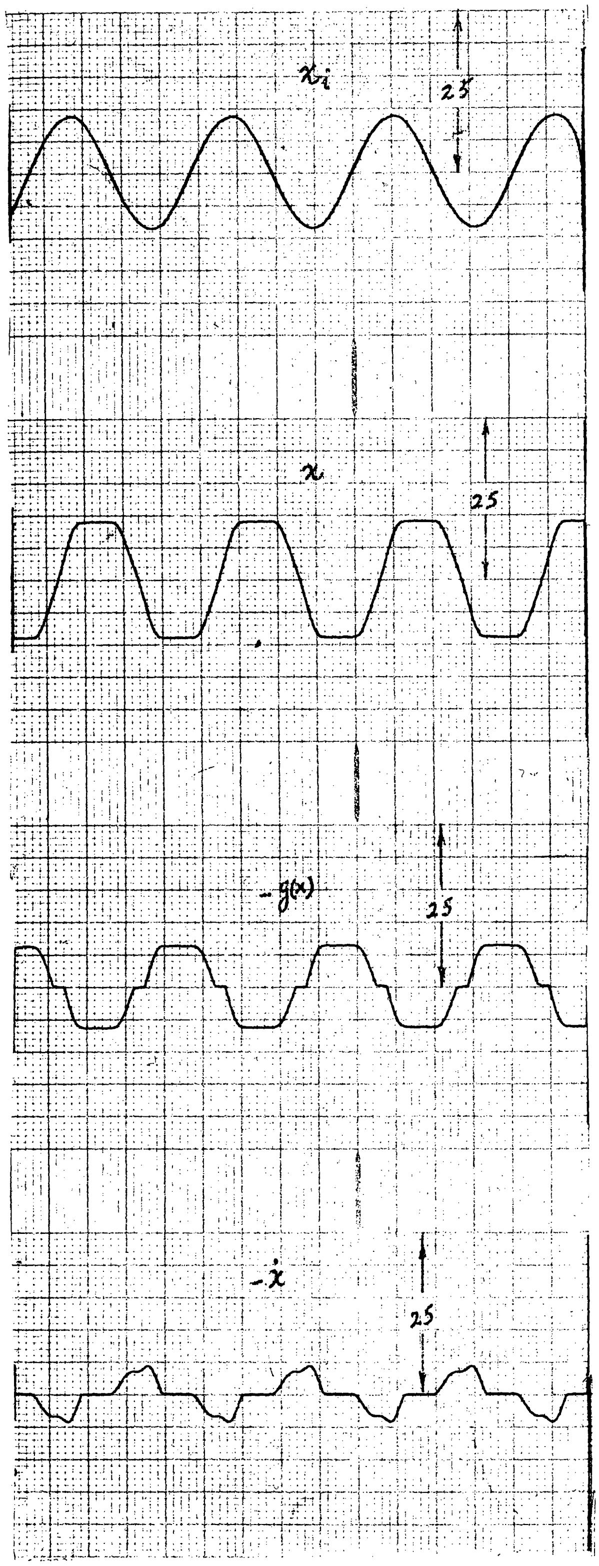

Figure 5-11(b) Waveform Plot

u = 0.3 ξ = 0.25 d = 0.5

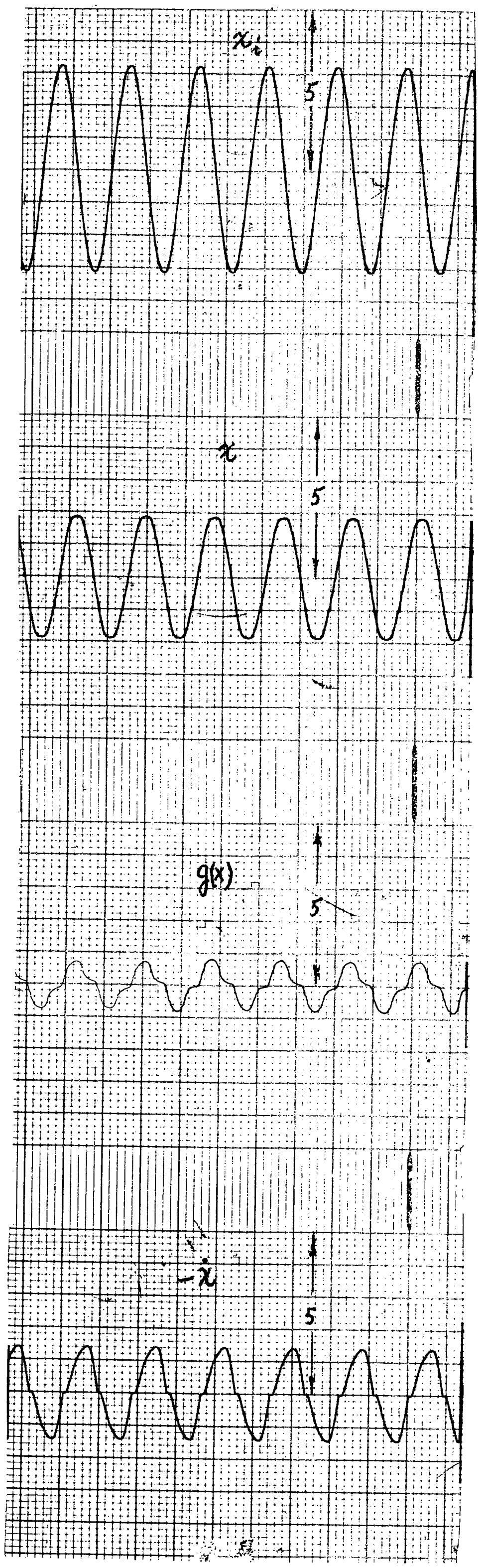

Figure 5-11(c) Waveform Plot

u = 0.7 ξ = 0.7 d = 1

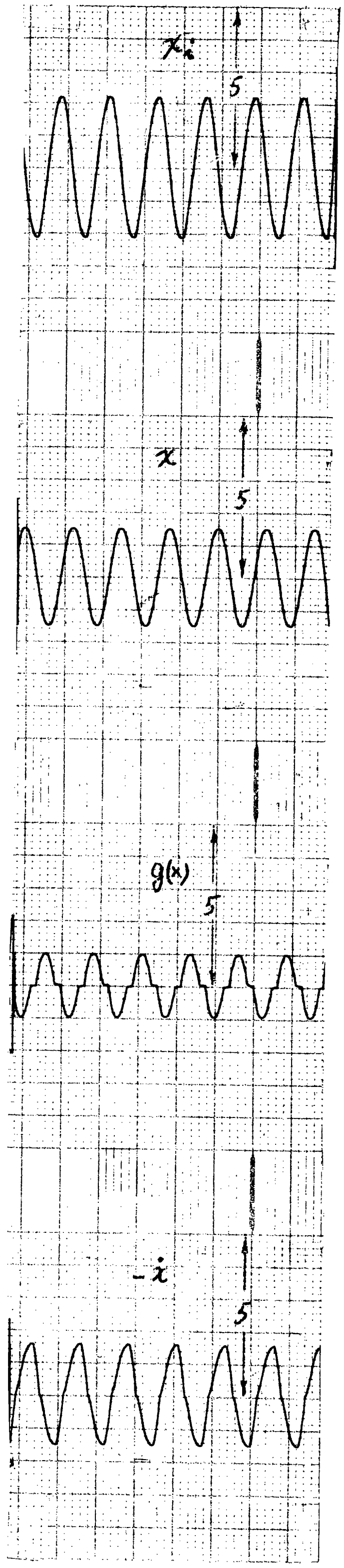

Figure 5-11(d) Waveform Plot

u = 1 ξ = 0.25 d = 0.5

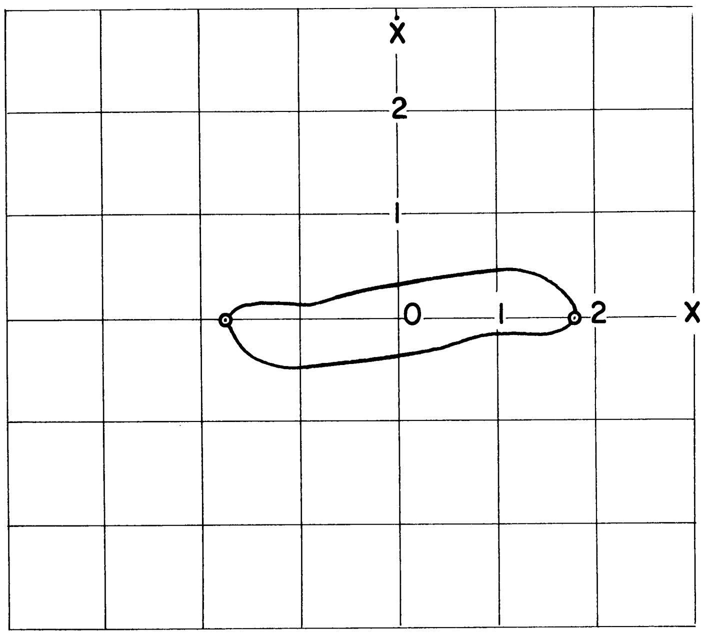

Figure 5-12(a) Phase Plane Plot

u = 0.1 ξ = 0.7 d = 1

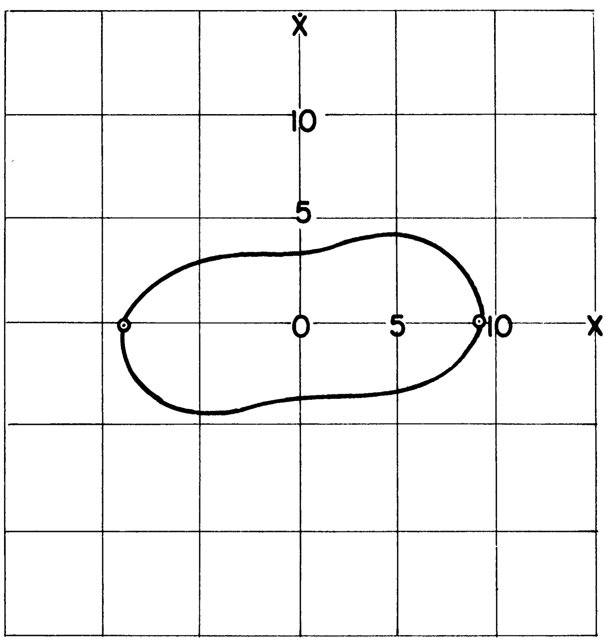

Figure 5-12(b) Phase Plane Plot

u = 0.3 ξ = 0.25 d = 0.5

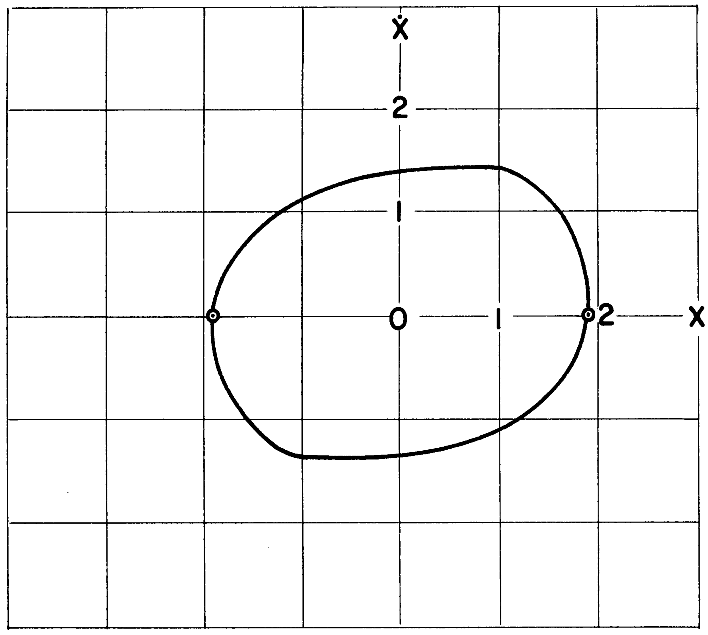

Figure 5-12(c) Phase Plane Plot

u = 0.7 ξ = 0.7 d = 1

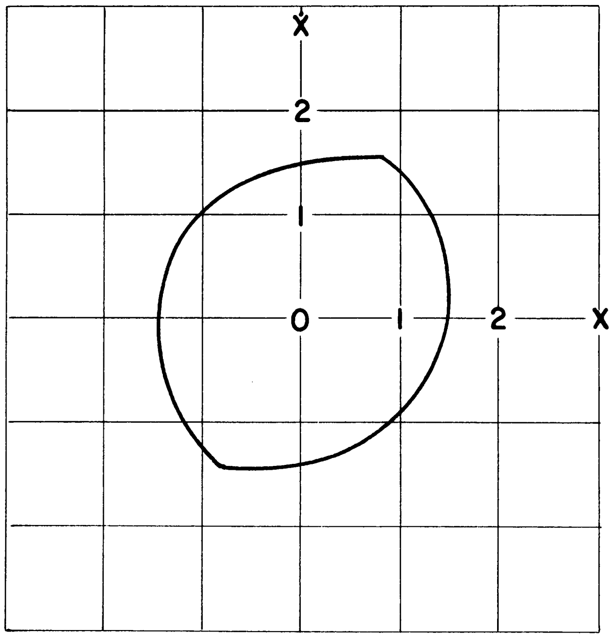

Figure 5-12(d) Phase Plane Plot

u = 1 ξ = 0.25 d = 0.5

CHAPTER VI

CONCLUSION

This dissertation has studied a class of second order forced oscillation systems with nonlinear "damping force" and nonlinear "restoring force". Particularly, the boundedness of the solutions and the conditions which suffice the stability and uniqueness of the periodic solutions have been determined. The results obtained provide some insight to the behaviors of the solutions which were previously unknown. A method for calculating the describing function of the second equation has been presented which is proved to be useful in the computer simulation of a system with combined coulomb and viscous frictions and dead-space restoring characteristics.

It is the belief of the author that the contribution of this dissertation includes the following items: First, by using Loud's method to prove the Boundedness Theorem, the result is new, especially the technique to obtain the bounds for x is original. This theorem gives an explicit region in which the solutions are finally confined. Second, considered an important part of this study is the establishment of the Convergence Theorem, and the determination of the conditions imposed on system parameters for the stability and uniqueness of the periodic solutions by using the Lyapunov Second Method are also original. Furthermore, the finding of a Lyapunov function for equation (1-5) is interesting in that it is not only useful in proving the Convergence Theorem, but also applicable to a class of time-varying systems. Third, the describing function of equation (4-13) has been given in a convenient complex form and has been derived in a different sense from what was previously known. A computer study has been made to corroborate the calculated results.

The author feels that for the future work the following two aspects may be suggested: First, if $f(\dot{x})$ is negative for small $\dot{x}$ and $\frac{f(\dot{x})}{\dot{x}} > b$ for large $\dot{x}$, it might be possible to obtain an explicit region for the solutions of equation (1-1) to be ultimately bounded. Second, it would be interesting if the Convergence Theorem could be proved without requiring $g''(x)$. However these problems would probably need a great deal of study.

APPENDIX I

FORMULATION OF THE FUNCTIONS $\frac{1}{\pi A} F(A,u)$ AND $\frac{1}{\pi A} G(A,u)$ OF EQUATION (4-15)

In Chapter IV, the approximate solution of the following equation

$$\ddot{x} + 2\xi\dot{x} + \frac{\dot{x}}{\dot{x}} + g(x) = E \cos ut \qquad (4\text{-}13)$$

was assumed to be $x = A \cos ut$. Then $\dot{x} = - Au \sin ut$, and $\ddot{x} = - Au^2 \cos ut$. For the combined friction term $f(\dot{x}) = 2\xi\dot{x} + \frac{\dot{x}}{|\dot{x}|}$, the function $\frac{1}{\pi A} F(A,u)$ is obtained from equation (4-7) as follows:

$$\frac{1}{\pi A} F(A,u) = \frac{1}{\pi A}\left[\int_0^{2\pi}\left(-2\xi Au \sin\theta + \frac{-Au \sin\theta}{|Au \sin\theta|}\right)e^{-j\theta}\, d\theta\right]$$

$$= \frac{1}{\pi A}\left[j2\pi\xi Au + j4\right] = j\left(2\xi u + \frac{4}{\pi A}\right)$$

For the term $g(x)$ having the dead-space characteristic, the function $\frac{1}{\pi A} G(A,u)$ is obtained from equation (4-8) as:

$$\frac{1}{\pi A}G(A,u) = \frac{1}{\pi A}\int_0^{2\pi} g(A \cos\theta)e^{-j\theta} d\theta = \frac{4}{\pi A}\int_0^{\theta_1 = \cos^{-1}d/A} (A \cos\theta - d) \cos\theta\, d\theta$$

$$= \frac{4}{\pi A}\left[\frac{A\theta_1}{2} + \frac{A \sin 2\theta_1}{4} - d \sin\theta_1\right] =$$

$$\frac{1}{\pi}\left[2 \cos^{-1}\frac{d}{A} + 2 \cos\theta_1 \sin\theta_1 - 4 \cos\theta_1 \sin\theta_1\right]$$

$$= \frac{1}{\pi}\left[2 \cos^{-1}\frac{d}{A} - 2\frac{d}{A} \sin\left(\cos^{-1}\frac{d}{A}\right)\right] =$$

$$\frac{2}{\pi} \cos^{-1}\frac{d}{A} - \frac{2}{\pi}\frac{d}{A} \sin\left(\cos^{-1}\frac{d}{A}\right)$$

$$= 1 - \frac{2}{\pi} \sin^{-1}\frac{d}{A} - \frac{2d}{\pi A} \cos\left(\sin^{-1}\frac{d}{A}\right)$$

APPENDIX II

REDUCTION TO DIMENSIONLESS FORM OF EQUATION (5-2)

Equation (5-2) is rewritten as:

$$J_1\ddot{\theta}_o + B_1\dot{\theta}_o + M\frac{\dot{\theta}_o}{|\dot{\theta}_o|} + Kg_1\ (\theta_o) = K\ \theta_i(t)$$

Let $x = \frac{K}{M}\theta_o$, $x_i = \frac{K}{M}\theta_i = E \cos u\ t$ and $g(x) = \frac{K}{M}g_1\ (\theta_o)$. Then equation (5-2) becomes

$$\frac{J_1}{K}\ddot{x} + \frac{B_1}{K}\dot{x} + \frac{\dot{x}}{|\dot{x}|} + g(x) = E \cos ut$$

and the "restoring force" term becomes

$$\begin{aligned} g(x) &= x-d & x &> d \\ &= 0 & -d < x &< d \\ &= x+d & x &< -d \end{aligned}$$

where $d = \frac{K}{M}d_1$. Now let $\omega_n^2 = \frac{K}{J_1}$, $\xi = \frac{B_1}{2\sqrt{J_1K}}$, $u = \frac{\omega}{\omega_n}$ and $\tau = \omega_n t$,

then equation (5-2) will finally reduce to the following form:

$$\ddot{x} + 2\xi\dot{x} + \frac{\dot{x}}{|\dot{x}|} + g(x) = E \cos ut \qquad (4\text{-}13)$$

APPENDIX III

COMPUTER CIRCUITS

The computer circuit for equation (4-13) is shown in Figure A-1. Potentiometer P_1 is set at the value of 2ξ. The real time was used for τ in equation (4-13), and the real frequency is used for the normalized frequency u.

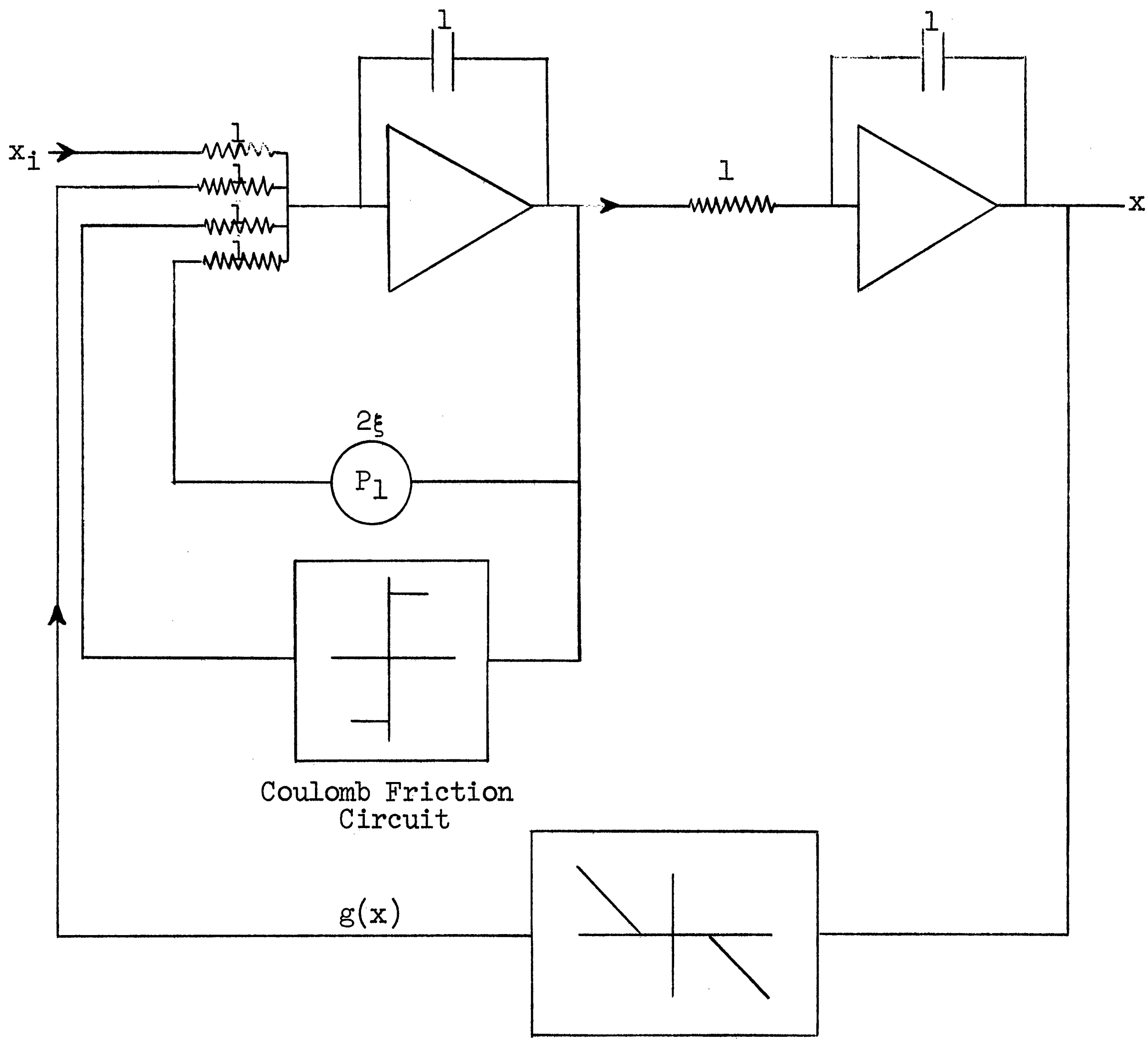

Figure A-1 Computer Circuit of Equation (4-13)

Figure A-2 is the circuit diagram for a method of producing coulomb friction on an analog computer. A signal going through the high gain amplifier is limited at a certain level by the diode limiters; pot. P_2 is used for adjusting the magnitude of the coulomb friction.

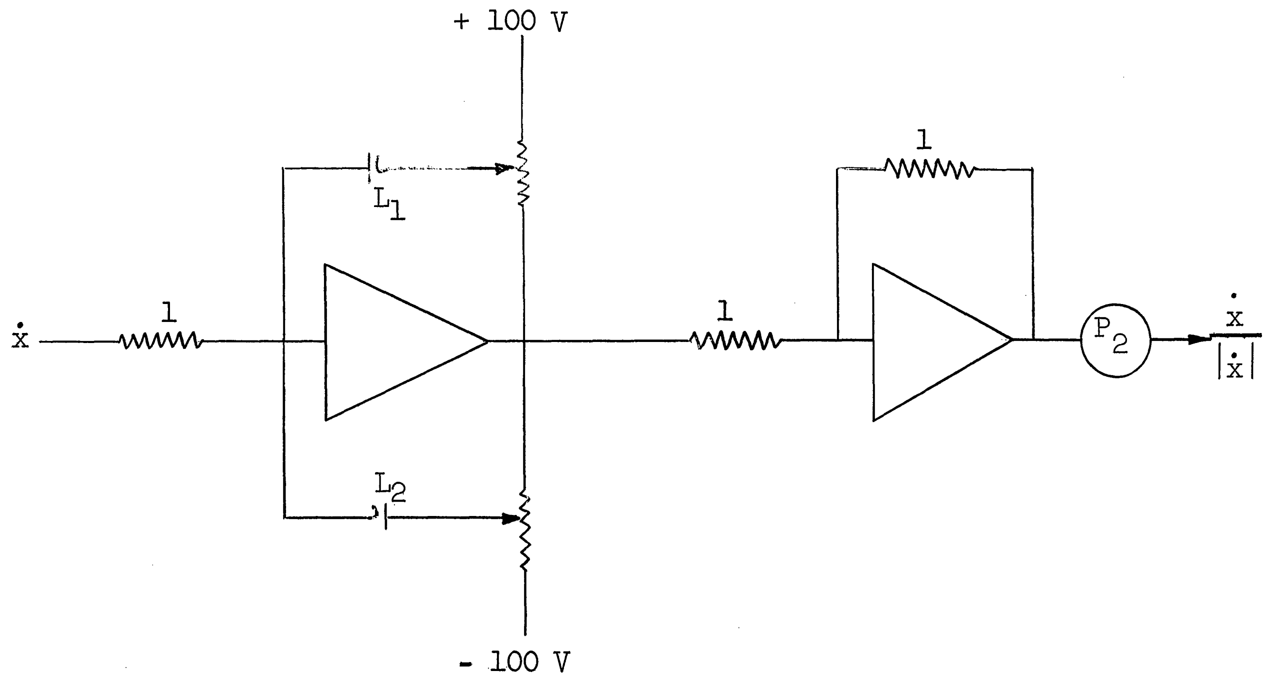

Figure A-2 Circuit Diagram for Simulating Coulomb Friction.

Figure A-3 is the circuit diagram for simulating dead-space characteristic. R_1, R_2 and R_3 are used to adjust the scale factor. When x is below the value d, the voltages at R_2 and R_3 are equal but opposite in sign so that there is no output in g(x). When x is greater then the value d, one of the diodes conducts and the voltage at R_2 is limited; so that the output g(x) has the desired characteristic.

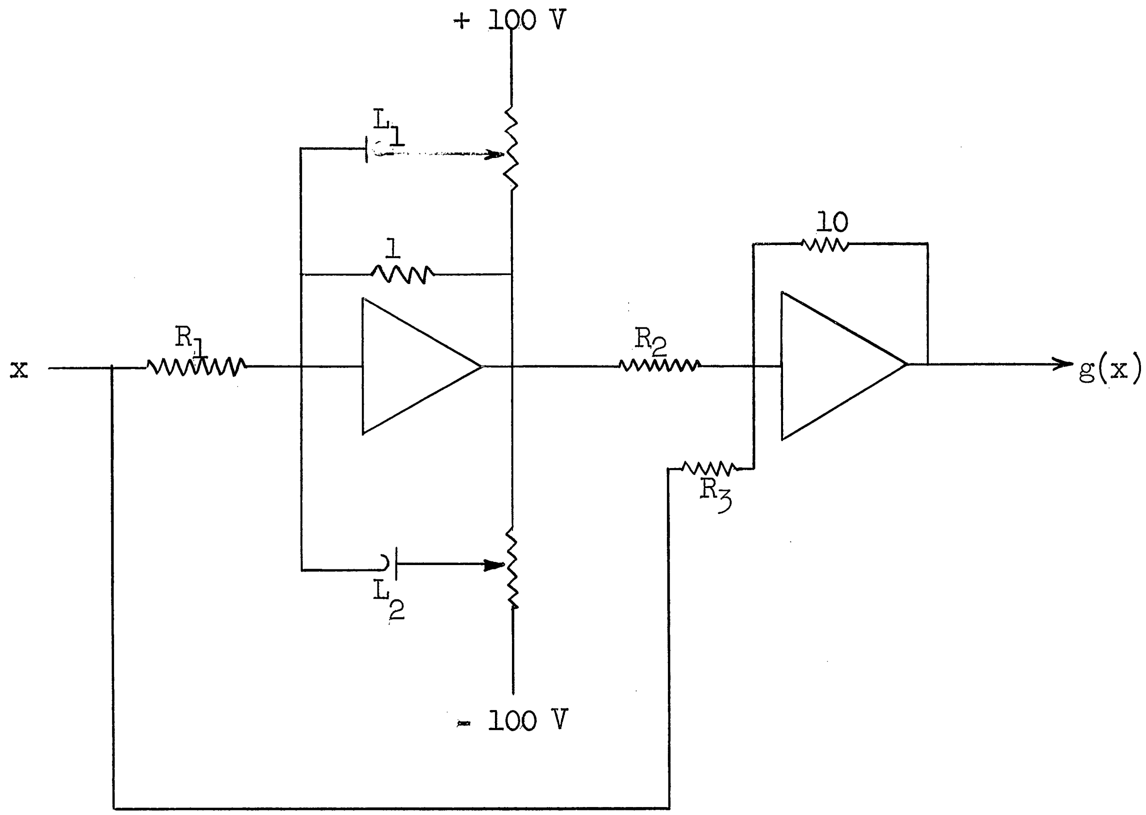

Figure A-3 Circuit Diagram for Simulating Dead-Space Characteristics

Figure A-4 is the computer circuit for measuring the describing function. This circuit is accredited to Larrowe and Spencer [39]. The integrators A and C, and the inverting amplifier, make up a sinusoidal oscillator circuit. Assume that the output of the integrator C is E sin ut which is used as the input for the computer circuit of Figure A-1. As far as the describing function is concerned, it makes no difference whether E cos ut or E sin ut is used as the input. Since the fundamental components of the output of the circuit Figure A-1 will be of the form - (A sin ut + B cos ut), then the signal to amplifier E passing through pot. R_6, where it is multiplied by P, has two fundamental components: -A P sin ut -B P cos ut. A second input to the summing amplifier E is A_1' P sin ut, which is obtained from pot. P_4.

Through the use of two way switch, A_1' may be made either positive or negative. The third input to amplifier E is B_1' P cos ut. The fundamental component of the output of the summing amplifier E will be

$P(A_1 - A_1')\sin ut + P(B_1 - B_1')\cos ut.$

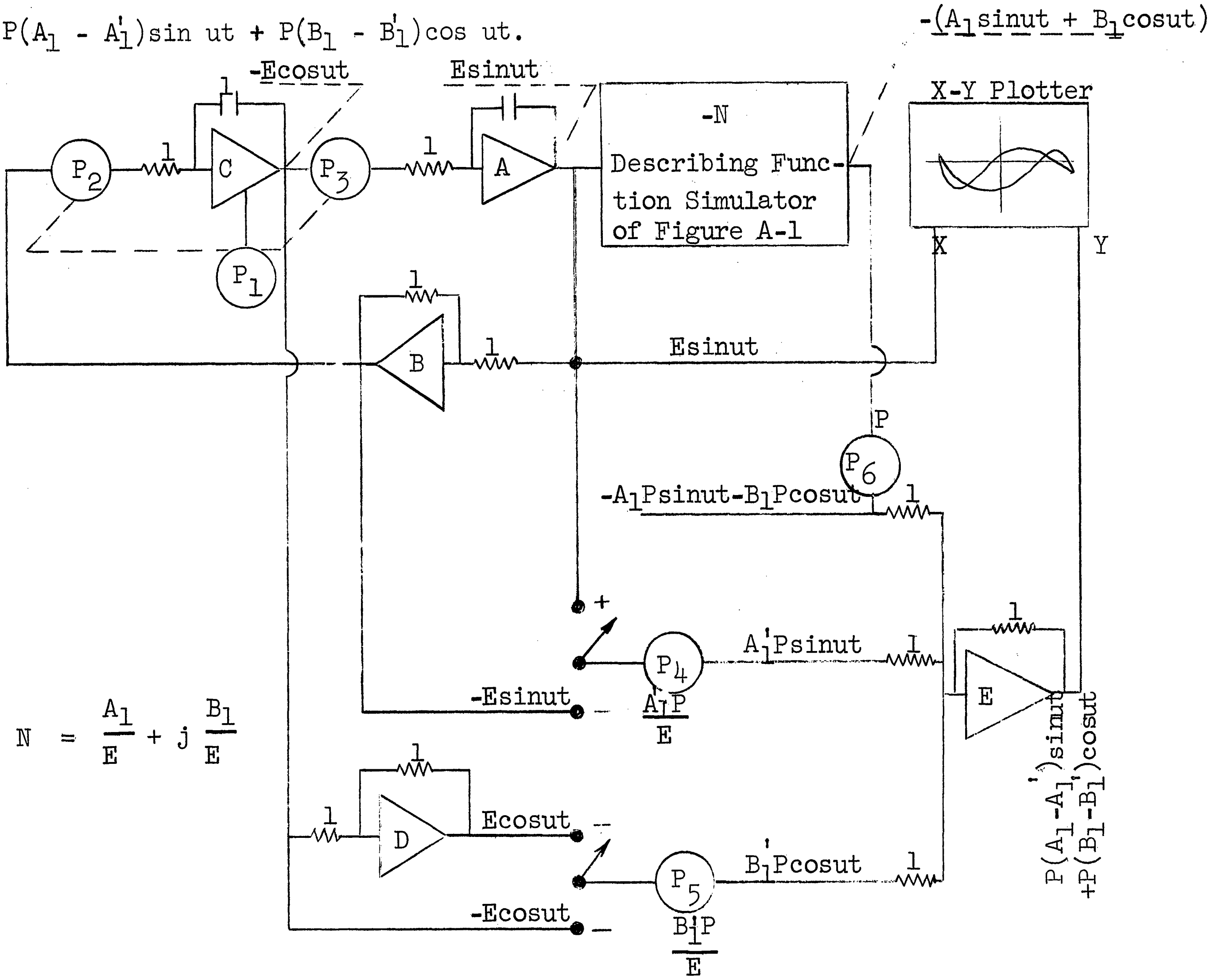

Figure A-4 Circuit Diagram for Measuring the Describing Function.

Thus, when P_4 and P_5 are adjusted to give zero output fundamental component, P_4 will be set to $\frac{A_1 P}{E}$, and since P (the setting of P_6) is known, the ratio $\frac{A_1}{E}$ which corresponds to the real part of the describing function may be easily determined. Similarly the imaginary part of the describing function is determined from the setting of P_5. Nulling the fundamental components is determined by inspection of the lissajous patterns which appear on the x - y plotter.

APPENDIX IV

TABLES OF MEASURED AND CALCULATED DATA

In the tables following, Nc denotes the calculated values of the describing function while Nm denotes the measured values, E and Nc are obtained from equation (4-13).