THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

INSTABILITY OF A LAYER OF LIQUID FLOWING DOWN AN INCLINED PLANE AT LARGE REYNOLDS NUMBERS

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I. INTRODUCTION

The problem of instability of a layer of liquid flowing down an inclined plane was first studied by Kapitza[4] and others. The first correct formulation was given by Yih,[11] who showed that the problem can be solved by a method of regular perturbation. The governing differential equation is the well known Orr-Sommerfeld equation. The boundary conditions are the non-slip condition at the bottom plane and the stress condition at the free surface. The Orr-Sommerfeld equation and the boundary conditions constitute an eigen-value problem. Yih[12] and Benjamin[2] have obtained solutions to the problem with different methods. It was found that for the case of large angle of inclination, surface waves govern the instability of the problem, which occurs at small Reynolds numbers. It was also found for the case of long waves that the imaginary part of the complex wave speed is zero at α(wave number) = 0. And this imaginary part of the complex wave speed will increase or decrease when α increases from zero, according as

$$R > (5\cot\beta)/6 \quad \text{or} \quad R < (5\cot\beta)/6 ,$$

where R is the Reynolds number and β is the angle of inclination. That is to say the neutral stability curve has a bifircation point at $\alpha = 0$, $R = (5\cot\beta)/6$. The above relation indicates that the neutral stability curve will be shifted to the right in the α - R plane if β is decreased. However the above relation was found under the assumption that αR remains small compared with unity. The question arises naturally as to whether the instability does occur at small αR for a given small β. The answer depends on whether the critical Reynolds number (for a given β and $S = 0$), obtained by assuming αR larger than unity in the problem is greater or smaller than $(5 \cot \beta)/6$. This thesis aims to give a definite answer to the above question, by studying the stability of a mode of disturbances for which αR is much larger than unity. A neutral stability curve for $\beta = 1$ min. and zero surface tension is obtained by use of a method which closely follows the method used by C. C. Lin[8] for the problem of plane Poiseuille flow. A slightly different method is used to check the above neutral stability curve. This slightly different method consists of obtaining asymptotic series solutions of the Orr-Sommerfeld equation by solving the inviscid equation and a related differential equation by use of Frobenius method. The neutral stability curves obtained by these two methods check closely with each other for small wave numbers and differ slightly for large wave numbers, as expected. By use of the second method several neutral stability curves for different angles of inclination and zero surface tension have been obtained. A set of neutral curves for different surface tension and $\beta = 1°$ are also obtained. These numerical results indicate that the surface tension as well as the reduction of the angle of inclination are all stabilizing factors. The main difficulty in the computation for this problem arises from the fact that the flow parameters as well as eigen-values appear in the boundary conditions and the imaginary part of the secular equation is not a function of the wave speed c alone even for small values of α.

By comparing the results obtained by Yih[12] and Benjamin[2] and the results obtained in this thesis, it is shown that a layer of liquid flowing down an inclined plane does occur at small αR for any β, where R is given by $R = (5 \cot \beta)/6$. That is to say, the "soft" waves which grow or decay more slowly than the shear waves do govern the instability of the problem for all values of β.

II. FORMULATION OF THE PROBLEM

A. Two Dimensional and Three Dimensional Disturbances

Squire[9] has shown that the solution of hydrodynamic stability with respect to three-dimensional disturbances in undirectional flows between rigid boundaries is related to the solution for two-dimensional disturbances by a simple transformation. Yih[13] has extended this result to flows with free surfaces, interfaces, or density stratification. Therefore, it is sufficient to study the case of two-dimensional disturbances only.

B. The Governing Differential Equation

Consider a layer of liquid flowing down an inclined plane, under the action of gravity. The plane is of infinite length and the flow is assumed to be parallel to the plate, so that the pressure gradient in the X-direction is zero, and the velocity component parallel to the X-axis does not change along this axis. For the primary flow, the Navier-Stokes equations are simply (c.f. Figure 1)

$$\rho g \sin\beta + \mu \frac{d^2\bar{u}}{dY^2} = 0 \qquad (\beta \neq 0) \tag{1}$$

and

$$-\frac{d\bar{p}}{dY} = \rho g \cos\beta \quad , \tag{2}$$

in which ρ is the constant density, g the gravitational acceleration, μ the viscosity, and $\bar{p}$ is the pressure in this primary flow. All other quantities in Equations (1) and (2) are defined in Figure 1. The boundary conditions are

$$\bar{u} = 0 \quad \text{at } Y = -d, \quad \text{and} \quad (d\bar{u}/dY) = 0 \quad \text{at } Y = 0, \tag{3}$$

Equation (1) can be integrated with the boundary conditions in Equation (3). The result is

$$\bar{u} = (g \sin\beta / 2\nu)(d^2 - Y^2) ,$$

or

$$U(y) = (1 - y^2) ,$$

in which

$$y = Y/d \qquad \text{and} \qquad U = \bar{u}/\bar{u}_m ,$$

$\bar{u}_m$ being the maximum velocity of the primary flow, i.e.,

$$\bar{u}_m = g d^2 \sin\beta / 2\nu \tag{4}$$

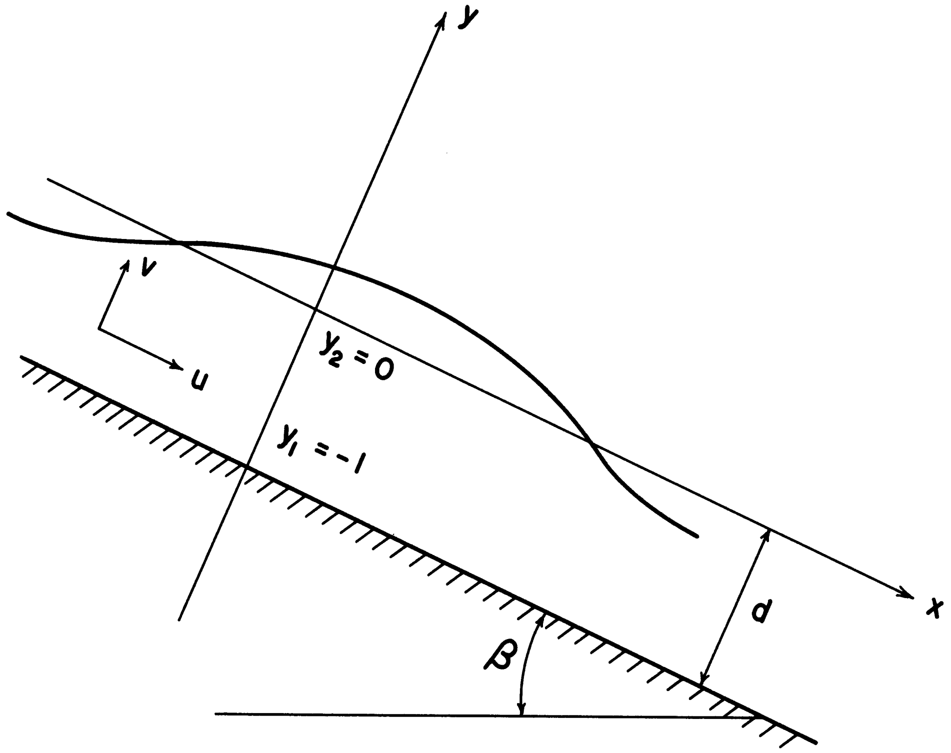


Figure 1. Definition Sketch.

The disturbed flow is not steady. The Navier-Stokes equations and the continuity equation for this unsteady flow are

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial p}{\partial X} + g\sin\beta + \nu\Delta U \ ,$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial p}{\partial Y} - g\cos\beta + \nu\Delta V \ ,$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \ ,$$

in which t is time, p the pressure and Δ is the Laplacian.

Making the following substitutions

$$(u_1, v_1) = (U, V)/\bar{U}_m \ , \qquad (x, y) = (X, Y)/d \ ,$$

$$p_1 = p/\rho\,\bar{U}_m^2 \ , \qquad \tau = t\,\bar{U}_m/d \ ,$$

one can write the equations of motion and continuity in the following dimensionless form:

$$\frac{\partial u_1}{\partial \tau} + u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} = -\frac{\partial p_1}{\partial x} + \frac{\sin\beta}{F^2} + \frac{1}{R}\Delta u_1 \ , \tag{5}$$

$$\frac{\partial v_1}{\partial \tau} + u_1\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} - \frac{\cos\beta}{F^2} + \frac{1}{R}\Delta v_1 \ , \tag{6}$$

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y} = 0 \quad , \tag{7}$$

where R is Reynolds number and F is Froude number, i.e.,

$$R = \bar{u}_m d / \nu \quad ,$$

$$(8)$$

$$F = \bar{u}_m / (g d)^{1/2} \quad .$$

Combining Equation (8) and Equation (4), one obtains the relation between R and F

$$2 F^2 = R \sin \beta \quad .$$

Assume that the velocity and the pressure of the unsteady flow consist of two parts in the following manner:

$$u_1 = U + u' \quad , \qquad v_1 = v' \quad , \qquad p_1 = P + p' \quad , \tag{9}$$

in which accented quantities are velocity and pressure perturbations, and U and P are velocity and pressure in the primary flow. Then, Equations (5), (6), and (7) can be written as

$$u'_t + U u'_x + U_y v' = - p'_x + \frac{1}{R} \Delta u' \quad , \tag{10}$$

$$v'_\tau + U v'_x = - p'_y + \frac{1}{R} \Delta v' , \tag{11}$$

$$u'_x + v'_y = 0 , \tag{12}$$

in which high-order terms in perturbation quantities are neglected. Subscripts in the above equations denote partial differentiations.

Equation (12) is a necessary and sufficient condition for the existence of a stream function ψ in terms of which u' and v' can be expressed as

$$u' = \psi_y \quad , \quad v' = -\psi_x .$$

In terms of stream function, Equations (10) and (11) become

$$\psi_{y\tau} + U \psi_{xy} - U_y \psi_x = - p'_x + \frac{1}{R} \Delta \psi_y \tag{13}$$

$$\psi_{x\tau} + U \psi_{xx} = + p'_y + \frac{1}{R} \Delta \psi_x \tag{14}$$

Since the flow extends to infinity, one can assume that all disturbances propagate to infinity. Furthermore, since any disturbances can be constructed from sinusoidal disturbances by use of Fourier transformation, it is sufficient to study the case of sinusoidal disturbances. Thus, it is natural to assume

$$\psi = \phi(y) \exp[i\alpha(x - c\tau)], \tag{15}$$

$$p' = f(y) \exp[i\alpha(x - c\tau)], \tag{16}$$

where

$$\alpha \equiv \text{wave number} \equiv 2\pi d/\lambda \ ,$$

$$\lambda \equiv \text{wave length} \ ,$$

$$c \equiv \text{complex wave speed} \equiv c_r + i c_i \ .$$

It can be seen from (15) and (16) that c_r is the wave velocity and $c_i\alpha$ is the rate of amplification if positive, or the rate of damping if negative. Substitution of (15) and (16) into (13) and (14) gives

$$-i\alpha c\phi' + i\alpha U\phi' - i\alpha U'\phi = -i\alpha f + \frac{1}{R}(\phi''' - \alpha^2\phi'), \tag{17}$$

$$\alpha^2 c\phi - \alpha^2 U\phi = f' + \frac{1}{R}(i\alpha\phi'' - i\alpha^3\phi) \ , \tag{18}$$

in which the primes denote differentiation with respect to y.

Elimination of f between (17) and (18) by cross-differentiation gives a vorticity equation which is known as the Orr-Sommerfeld equation.

$$\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha R[(U-c)(\phi'' - \alpha^2\phi) - U''\phi]. \qquad (19)$$

Equation (19) is the governing differential equation of hydrodynamic stability of parallel flows.

C. Boundary Conditions

The boundary conditions as formulated by Yih[11] consists of the non-slip conditions

$$u' = \psi_y = 0 \qquad (i)$$

and

$$v' = -\psi_x = 0 \qquad (ii)$$

at the bottom, and the stress boundary conditions

$$\tau_{xy} = 0 \qquad (iii)$$

and

$$\tau_{yy} + T \cdot K = 0 \qquad (iv)$$

at the free surface. In (iii) and (iv) τ_{xy} is the shearing stress, τ_{yy} the normal stress, T the surface tension and K is the curvature of the free surface. Condition (iii) is true because the viscosity of the air is very small compared with that of the liquid. Condition (iv) must be satisfied, otherwise the fluid particles at the free surface will experience an infinite acceleration because of the finite jump in τ_{yy} at free surface.

More explicitly (iii) and (iv) can be written as

$$\frac{\partial V_1}{\partial x} + \frac{\partial U_1}{\partial y} = 0 \quad , \tag{iii}$$

$$+ p_1 - \frac{2}{R}\frac{\partial V_1}{\partial y} + S\frac{\partial^2 \eta}{\partial x^2} = 0, \qquad S = \frac{T}{\rho d \bar{U}_m^2} \quad , \tag{iv}$$

in which η_d is the displacement of the free surface from its mean positions. Using the Taylor series expansions of U and P up to the first order term in η, one can write

$$U_1 = (U + \frac{\partial U}{\partial y}\eta) + u' \quad ,$$

$$p_1 = (P + \frac{\partial P}{\partial y}\eta) + p' \quad ,$$

and the free surface conditions can be written as

$$\frac{d^2 U}{dy^2}\eta + \psi_{yy} - \psi_{xx} = 0 \quad , \tag{iii}$$

$$+ P + P_y \cdot \eta + p' - \frac{2}{R}\psi_{xy} + S\eta_{xx} = 0, \tag{iv}$$

if terms of higher order than $O(\eta^2)$ are neglected. In the above two equations, all quantities except η are to be evaluated at $y = 0$.

Noticing $P(0) = 0$ and $P_y(0) = \cos\beta/F$, one can simplify (iv) to the form

$$(\cos\beta/F^2)\eta - p' - \frac{2}{R}\psi_{xy} - S\eta_{xx} = 0 . \qquad \text{(iv)}$$

The dimensionless displacement η is related to ψ by the kinematic condition at free surface, i.e.,

$$V' = \frac{D\eta}{D\tau} = \eta_t + (U + U_y\eta + u')\eta_x + V\eta_y = 0 .$$

Since $U_y|_{y=0} = 0$, $\eta_y = 0$ and $u'\eta_x$ is very small, the solution of the above equation is readily obtained by putting

$$\eta = \text{Const.}\ \exp[i\alpha(x - c\tau)] .$$

Thus,

$$\eta = \frac{\phi(0)}{c'} \exp[i\alpha(x - c\tau)] , \qquad (20)$$

where $c' = c - 1$.

D. Hydrodynamic Stability Problem as an Eigen-Value Problem

Equation (19) and the boundary conditions (i), (ii), (iii), (iv) constitute an eigen-value problem. A non-trivial solution exists if there exists a relation between R, F, c, S and x. In this problem, R and F are related by (9) for given values of β. Therefore the task is to obtain relation of the form

$$c = c(R, \alpha)$$

for given values of β and S, so that a nontrivial solution exists.

In general, c is complex i.e., $c = c_r = ic_i$, in which

$$c_r = c_r(R, \alpha),$$

(for given S and β)

$$c_i = c_i(R, \alpha).$$

As can be seen from (15) and (16), the flow is stable if $c_i < 0$, and unstable if $c_i > 0$. If $c_i = 0$, there is a sustained oscillation. The relation

$$c_i(R, \alpha) = 0$$

gives the neutral-stability curve in the $R - \alpha$ plane. The wave velocity c on the curve of neutral-stability is then a function of α only.

Now, the complete solution of the Orr-Sommerfeld equation can be written as

$$\phi = C_1\phi_1 + C_2\phi_2 + C_3\phi_3 + C_4\phi_4 ,$$

where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are four independent particular solutions of the equation.

Substituting this into the boundary conditions, one has a system of simultaneous equations in C_1, C_2, C_3 and C_4. Non-trivial solution of this system exists if

$$\begin{vmatrix} \phi_{11} & \phi_{21} & \phi_{31} & \phi_{41} \\ \phi_{11}' & \phi_{21}' & \phi_{31}' & \phi_{41}' \\ (\phi_{12}''+l\phi_{12}) & (\phi_{22}''+l\phi_{22}) & (\phi_{32}''+l\phi_{32}) & (\phi_{42}''+l\phi_{42}) \\ (m\phi_{12}+j\phi_{12}'-i\phi_{12}''') & (m\phi_{22}+j\phi_{22}'-i\phi_{22}''') & (m\phi_{32}+j\phi_{32}'-i\phi_{32}''') & (m\phi_{42}+j\phi_{42}'-i\phi_{42}''') \end{vmatrix} = 0 , \tag{21}$$

in which

$$\phi_{ij} = \phi_i(y_j) , \qquad (i = 1, 2, 3, 4; \; j = 1, 2)$$

$$y_1 = -1, \qquad y_2 = 0 ,$$

$$l = \alpha^2 - 2/c' ,$$

$$m = -\frac{\alpha}{c'}(2\cot\beta + \alpha^2 S R) ,$$

$$j = \alpha(Rc' + 3i\alpha) .$$

Equation (21) is the so-called secular equation. In order to solve it, one has to solve the Orr-Sommerfeld equation first. Langer's[6] method readily yields four particular asymptotic solutions which are, in fact, the same asymptotic solutions Heisenberg obtained with a less systematic method in 1924.[5] In order to use Langer's method, one writes the Orr-Sommerfeld equation in the following form

$$L\phi \equiv \phi^{iv} + \lambda P_1 \phi''' + \lambda^2 P_2 \phi'' + \lambda^3 P_3 \phi' + \lambda^4 P_4 \phi = 0 , \tag{22}$$

where λ is a large parameter $(\alpha \cdot R)^{1/2}$, and

$$P_1 = 0 ,$$

$$P_2 = -i(U-c) - \frac{2\alpha^2}{\lambda^2} ,$$

$$P_3 = 0 ,$$

$$P_4 = \frac{1}{\lambda^2}[-2i + i\alpha^2(U-c)] + \frac{\alpha^4}{\lambda^4} .$$

Assume a solution of the form

$$\phi = e^{\lambda \int \chi_j dy} A_j(y, \lambda) \qquad (j = 1, 2, 3, 4) , \tag{23}$$

where X_j are determined by

$$\chi^4 - i(U-c)\chi^2 = 0 . \tag{24}$$

This equation is actually obtained by substituting (23) into (22) and then letting λ approaches infinity. (A_j in (23) being kept constant.) Solving (24), one has

$$\chi_1, \chi_2, \chi_3, \chi_4 = 0, 0, -\sqrt{i(U-c)} , +\sqrt{i(U-c)} .$$

It follows from (23) that

$$\phi_1,\ \phi_2,\ \phi_3,\ \phi_4 = A_1(y,\lambda),\quad A_2(y,\lambda),$$

$$A_3(y,\lambda)\,e^{\lambda\int \chi_3 dy},\qquad A_4(y,\lambda)\,e^{-\lambda\int \chi_4 dy},$$

where $A_i(y,\lambda)$, $(i = 1,2,3,4)$ are to be so chosen that the Orr-Sommerfeld equation is asymptotically satisfied. Following Langer, let

$$A_j(y,\lambda) = \sum_{k=0}^{r} \frac{a_{jk}(\lambda)}{\lambda^k} \qquad (j = 1,2,3,4)$$

Then substituting ϕ_1, ϕ_2 into (22), one has

$$S_{j,0} + \frac{1}{\lambda} S_{j,1} + \frac{1}{\lambda^2} S_{j,2} + \frac{1}{\lambda^3} S_{j,3} + \cdots = 0, \quad (j=1,2) \qquad (25)$$

in which

$$S_{j,0} = (D^2-\alpha^2)a_{j,0} - \frac{U''}{U-c}\,a_{j,0}\,,$$

$$(j = 1,2) \qquad (26)$$

$$S_{j,1} = (D^2-\alpha^2)a_{j,1} - \frac{U''}{U-c}\,a_{j,1}\,,$$

$$S_{j,2} = \cdots\,,$$

etc.

where $D = \frac{d}{dy}$ in (26).

Letting the first two equations in (26) equal to zero, one obtains two inviscid equations of the same form. This is due to the fact that (24) has so-called simple multiplicities. They are called inviscid equations because they are of the same differential equation as those which are obtained from the Orr-Sommerfeld equation by letting R go to infinity.

Obviously

$$a_{j,1} = \text{Const} \cdot a_{j,0} \quad ,$$

and

$$\phi_j(y, \lambda) = a_{j,0} \left(1 + \frac{\text{const}}{\lambda}\right) + O\left(\frac{1}{\alpha R}\right) , \tag{27}$$

terms smaller than $O(\frac{1}{\lambda^2})$ in (25) being neglected.

One will lose no information by letting the constant in (27) be equal to zero, as far as the eigen-value problem is concerned, since the factor $(1 + \text{const.}/\lambda)$ in the expression of $\phi_j (j = 1, 2)$ can be factored out of the secular determinant (21). To solve the inviscid equation for a_{jo} one can expand the s ution in a uniformly convergent series of α^2, since the solution must be analytic in α^2 as can be seen from the inviscid equation. Substituting

$$a_{j,0} = (U - c)[f_0(y, c, U) + \alpha^2 f_1(y, c, U) + \cdots]$$

into the inviscid equation and equating all coefficients of $\alpha^{2n}(n = 0, 1, 2, ..)$ to zero, one has

$$D^2(U-c)f_0 - U''f_0 = 0 \, ,$$

$$D^2(U-c)f_1 - (U-c)f_0 - U''f_1 = 0 \, ,$$

$$- - - - - - -$$

$$D^2(U-c)f_n - (U-c)f_{n-1} - U''f_n = 0 \, .$$

The solution of (29) is readily obtained

$$f_0 = C_1 \int_1^y \frac{dy}{(U-c)^2} + C_2 \, ,$$

$$f_n = \int_{-1}^y \frac{1}{(U-c)^2} \int (U-c)^2 f_{n-1} \, dy \, dy \, , \quad (n = 1, 2, ---) \, ,$$

where $U'' = -2$ has been used. Thus, two independent solutions are obtained by properly assigning values to C_1 and C_2. Choose C_1 and C_2 so that

for $j = 1, \quad C_2 = 1, \quad C_1 = 0$;

for $j = 2, \quad C_2 = 0, \quad C_1 = 1$.

Then

$$\phi_1 = A_1(y,\lambda) = (U-c)[1+\alpha^2\int_{-1}^{y}\frac{1}{(U-c)^2}\int(U-c)^2 dy\,dy + \cdots] + O(\frac{1}{\alpha R}),$$

$$\phi_2 = A_2(y,\lambda) = (U-c)[\int_{-1}^{y}\frac{dy}{(U-c)^2} + \alpha^2\int_{-1}^{y}\frac{1}{(U-c)^2}\int(U-c)^2\int\frac{dy\,dy\,dy}{(U-c)^2}]$$

$$+ O(\frac{1}{\alpha R}) \quad . \qquad (30)$$

The above solutions are good asymptotic solutions even near y_c where $U = c$ if the path of integration is properly chosen. Since y_c is not a singular point of the Orr-Sommerfeld equation although it is a singular point of the inviscid equation. C. C. Lin[8] showed in a heuristic manner that this proper path of integration must be such that it passes through y_c from below. Lin's conclusion was later confirmed by the rigorous mathematical analyses of Wasow.[10] To obtain ϕ_3 and ϕ_4, one substitute

$$\phi_j = A_j\, e^{\lambda\int\chi_j dy} \quad , \qquad (j = 3,4)$$

into (22) and thus has

$$L A_j e^{\lambda\int\chi_j dy} = \lambda^4[0] + \lambda^3[4\chi_j^3 DA_j + 6\chi^2 A_j D\chi_j$$

$$-i(U-c)(2\chi_j DA_j + A_j D\chi_j)]$$

$$+ \cdots\cdots \quad , \qquad (31)$$

in which X_j $(j = 3, 4)$ are the other two roots of (24). Letting the coefficient of λ^3 to zero, one has

$$4 \chi_j \, i(U-c) D a_{j,0} + 6 i (U-c) a_{j,0} D \chi_j ,$$

$$-i(U-c)(2\chi_j D a_{j,0} + a_{j,0} D \chi_j) = 0 .$$

Thus

$$2 \chi_j D a_{j,0} + 5 a_{j,0} D \chi_j = 0 , \tag{32}$$

if

$$U \neq c . \tag{33}$$

The solution of (32) is

$$a_{j,0} = \chi_j^{-5/2}$$

Hence

$$A_3, A_4 = \text{const.}/(U-c)^{5/4} + O\left(\frac{1}{\sqrt{\alpha R}}\right) ,$$

terms smaller than $O(\lambda^2)$ being neglected in (31), in comparison with terms of $O(\lambda^3)$. Therefore

$$\phi_3 = e^{\lambda\int_{-1}^{y}\sqrt{i(U-c)}\,dy}\left[(U-c)^{-5/4} + O\left(\frac{1}{\sqrt{\alpha R}}\right)\right] ,$$

$$\phi_4 = e^{-\lambda\int_{-1}^{y}\sqrt{i(U-c)}\,dy}\left[(U-c)^{-5/4} + O\left(\frac{1}{\sqrt{\alpha R}}\right)\right] . \qquad (34)$$

Condition (33) indicates that the above solutions are not valid at $y = y_c$. They are also not good asymptotic solutions near y_c. Since at those points $U - c$ is nearly zero and the coefficient of λ^3 in (31), i.e.,

$$i(U-c)(2\chi_j D_{j,0} + 5a_{j,0} D\chi_j) ,$$

may be so small that the λ^3-term is even smaller than the λ^2-term in (31). In such cases the error terms involved in (34) is larger than $O(\frac{1}{\alpha R})$. In the secular Equation (34), ϕ_3, ϕ_4 and their derivatives are evaluated at $y = -1$ which is close to $y_c = y_{cr} + iy_{ci}$ in this problem. (Since c are found to be much smaller than unity in this problem as well as in the problem of plane Poiseullie flow, and $c = 1 - y_c^2$.) Consequently, it is necessary to find solutions which are more accurate for small $y - y_c$. To do this, one introduces a change of independent variables

$$y - y_c = \varepsilon\eta , \qquad (35)$$

in which

$$\varepsilon^n = \frac{1}{\alpha R} \ll 1 , \qquad n > 0 .$$

Taylor's series expansions of U and D^2U around y_c are

$$(U-c) = (U_c - c) + U_c'(\varepsilon\eta) + \frac{1}{2} U_c''(\varepsilon\eta)^2 + \cdots$$

and

$$D^2U = U_c'' + \frac{1}{2} U_c''(\varepsilon\eta) + \cdots\cdots , \tag{36}$$

in which

$$U_c = U\Big|_{y=y_c} , \qquad U_c' = U'\Big|_{y=y_c} , \qquad \text{etc.,}$$

and

$$D = \frac{d}{dy} = \frac{1}{\varepsilon}\frac{d}{d\eta} .$$

Substitution of (36) into Equation (19) gives

$$\left(\frac{d^4}{d\eta^4} - 2\alpha^2\varepsilon^2\frac{d^2}{d\eta^2} + \alpha^4\varepsilon^4\right)\phi \tag{37}$$

$$= i\varepsilon^{2-n}\left\{\left[U_c'\varepsilon\eta + \frac{U_c''}{2}(\varepsilon\eta)^2 + \cdots\right]\left[\left(\frac{d^2}{d\eta^2} - \alpha^2\varepsilon^2\right) - \left(\varepsilon^2 U_c'' + \frac{1}{2}U_c''\varepsilon^3\eta + \cdots\right)\right]\phi\right\} .$$

Now, n in (35) is to be so chosen that both sides of (37) are of the same order of magnitude. Obviously, the proper choice is n = 3. Thus, neglecting terms which are smaller than $O(\epsilon)$, one has from (37)

$$\frac{d^4\Phi}{d\eta^4} = i U_c' \eta \frac{d^2\Phi}{d\eta^2} \quad . \tag{38}$$

To solve this equation, one puts

$$T = \frac{d^2\Phi}{d\eta^2} = \sqrt{\eta}\; F(\xi), \qquad \xi = \frac{2}{3}(i\eta)^{3/2}\sqrt{U_c'} \quad . \tag{39}$$

Thus, (38) is transformed to the well-known Bessel equation

$$\frac{d^2F}{d\xi^2} + \frac{1}{\xi}\frac{dF}{d\xi} + [1 - (\frac{\frac{1}{3}}{\xi})^2]\, F = 0 \quad . \tag{40}$$

Two particular solutions of (40) are

$$H_{1/3}^{(1)}(\xi) \quad \text{and} \quad H_{1/3}^{(2)}(\xi) \quad .$$

Hence

$$\Phi_3 = \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} d\eta \sqrt{\eta}\, H_{1/3}^{(1)}[\frac{2}{3}(i\alpha_0\eta)^{3/2}], \tag{41}$$

$$\Phi_4 = \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} d\eta \sqrt{\eta}\, H_{1/3}^{(2)}[\frac{2}{3}(i\alpha_0\eta)^{3/2}], \tag{42}$$

where

$$\alpha_0 = (U_c')^{1/3} \tag{43}$$

The above solutions were first used by C. C. Lin in his computation for the problem of plane Poiseuille flow. Substituting asymptotic expansions of Hankel functions into the above solutions and then compare the result thus obtained with solutions (34), C. C. Lin also determined the proper path of integration for obtaining ϕ_1 and ϕ_2 to be below the critical point $y = y_c$.

Having obtained four particular solutions ϕ_1, ϕ_2 and Φ_3, Φ_4, one is in a position to use them to solve the secular equation. Before doing so, one can simplify (21) by order of magnitude analysis. To start with, one divides the third column by Φ_{31} and the fourth column by Φ_{42}. Thus (21) is rewritten as

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 & \dfrac{\Phi_{41}}{\Phi_{42}} \\ \phi_{11}' & \phi_{21}' & \dfrac{\Phi_{31}'}{\Phi_{31}} & \dfrac{\Phi_{41}'}{\Phi_{42}} \\ \left(\phi_{12}'' + \ell\phi_{12}\right) & \left(\phi_{22}'' + \ell\phi_{22}\right) & \left(\dfrac{\Phi_{32}''}{\Phi_{31}} + \ell\dfrac{\Phi_{32}}{\Phi_{31}}\right) & \left(\dfrac{\Phi_{42}''}{\Phi_{42}} + \ell\right) \\ \left(\phi_{12} + \dfrac{j}{m}\phi_{12}' - \dfrac{i}{m}\phi_{12}'''\right) & \left(\phi_{22} + \dfrac{j}{m}\phi_{22}' - \dfrac{i}{m}\phi_{22}'''\right) & \left(\dfrac{\Phi_{32}}{\Phi_{31}} + \dfrac{j\Phi_{32}'}{m\Phi_{31}} - \dfrac{i\Phi_{32}'''}{m\Phi_{31}}\right) & \left(1 + \dfrac{j\Phi_{42}'}{m\Phi_{42}} - \dfrac{i}{m}\dfrac{\Phi_{42}'''}{\Phi_{42}}\right) \end{vmatrix} = 0 \tag{44}$$

The last row in (45) has been divided through by m . Each element in the first two columns are of order unity if $j > m$, then j should be factored out instead of m, and the same is true). Therefore it is legitimate to simplify the determinant by comparison of order of magnitude of each element in the last two columns before expanding the determinant. The order of magnitude of relevant quantities are listed below.

$$\left.\begin{aligned}
\frac{\Phi_{31}'}{\Phi_{31}} &= -\sqrt{i\alpha R(y-y_c)} + \frac{B_1'}{B_1}, \\
\frac{\Phi_{32}}{\Phi_{31}} &= \frac{B_2}{B_1} e^{-p}, \\
\frac{\Phi_{32}'}{\Phi_{31}} &= \left(-\sqrt{i\alpha R(y_2-y_c)}\,\frac{B_2}{B_1} + \frac{B_2'}{B_1}\right) e^{-p}, \\
\frac{\Phi_{32}''}{\Phi_{31}} &= (i\alpha R(y_2-y_c)\frac{B_2}{B_1} + O(\sqrt{\alpha R}\,)^{-p}, \\
\frac{\Phi_{32}'''}{\Phi_{31}} &= \left[-\{i\alpha R(y_2-y_c)\}^{3/2}\frac{B_2}{B_1} + O(\alpha R)\right] e^{-p}, \\
\frac{\Phi_{41}}{\Phi_{42}} &= \frac{B_1}{B_2} e^{-p}, \\
\frac{\Phi_{41}'}{\Phi_{42}} &= \left(\frac{B_1'}{B_2} + \frac{B_1}{B_2}\sqrt{i\alpha R(y_1-y_c)}\right) e^{-p},
\end{aligned}\right\} \qquad (45)$$

$$\left.\begin{aligned}
\frac{\Phi_{42}'}{\Phi_{42}} &= \sqrt{i\alpha R(y_2 - y_c)} + \frac{B_2'}{B_2} \quad , \\
\frac{\Phi_{42}''}{\Phi_{42}} &= i\alpha R(y_2 - y_c) + O(\sqrt{\alpha R}) \quad , \\
\frac{\Phi_{42}'''}{\Phi_{42}} &= [i\alpha R(y_2 - y_c)]^{3/2} + O(\sqrt{\alpha R}) \quad ,
\end{aligned}\right\}$$

where

$$P = \int_{y_1}^{y_2} \sqrt{i\alpha R(y - y_c)}\, dy \quad , \tag{46}$$

$$B_i = (y_i - y_c)^{-5/4} \quad , \qquad (i = 1, 2) \quad . \tag{47}$$

For very small c, real part of P is positive. Therefore for large value of αR, e^{-P} is much smaller than unity. Thus, neglecting quantities of order e^{-P}, one can reduce (44) to

$$\begin{vmatrix}
\phi_{11} & \phi_{21} & 1 & 0 \\
\phi_{11}' & \phi_{21}' & \dfrac{\Phi_{31}'}{\Phi_{31}} & 0 \\
(\phi_{12}'' + l\phi_{12}) & (\phi_{22}'' + l\phi_{22}) & 0 & \left(\dfrac{\Phi_{42}''}{\Phi_{42}} + l\right) \\
\left(\phi_{12} + \dfrac{j}{m}\phi_{12}' - \dfrac{i}{m}\phi_{12}'''\right) & \left(\phi_{22} + \dfrac{j}{m}\phi_{22}' - \dfrac{i}{m}\phi_{22}'''\right) & 0 & \left(1 + \dfrac{j}{m}\dfrac{\Phi_{42}'}{\Phi_{42}} - \dfrac{i}{m}\dfrac{\Phi_{42}'''}{\Phi_{42}}\right)
\end{vmatrix} = 0 \, . \tag{48}$$

Expansion of the above determinant with respect to its last column gives

$$\left(1+\frac{d}{m}\frac{\Phi_{42}'}{\Phi_{42}}-\frac{i}{m}\frac{\Phi_{42}'''}{\Phi_{42}}\right)\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ (\phi_{12}'+\ell\phi_{12}) & (\phi_{22}''+\ell\phi_{22}) & 0 \end{vmatrix}$$

$$-\left(\frac{\Phi_{42}''}{\Phi_{42}}+\ell\right)\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ (\phi_{12}+\frac{d}{m}\phi_{12}'-\frac{i}{m}\phi_{12}''') & (\phi_{22}+\frac{d}{m}\phi_{22}'-\frac{i}{m}\phi_{22}'') & 0 \end{vmatrix}=0 \quad (49)$$

The first term of the above equation is of the order $(\alpha R)^{1/2}$ whereas the second term is of order $(\alpha R)^{3/2}$. Hence, for αR so large that (αR) can be neglected compared with unity, one can approximate (49) by

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ \left(\phi_{12}+\frac{j}{m}\phi_{12}'-\frac{i}{m}\phi_{12}'''\right) & \left(\phi_{22}+\frac{j}{m}\phi_{22}'-\frac{i}{m}\phi_{22}'''\right) & 0 \end{vmatrix} = 0, \quad (50)$$

since

$$\Phi_{42}''/\Phi_{42} + \ell \neq 0 .$$

For the case of infinite surface tension, i.e., $S = \infty$, one has $m = \infty$. Thus (50) can be further reduced to

$$\begin{vmatrix} \phi_{11} & \phi_{21} & 1 \\ \phi_{11}' & \phi_{21}' & \Phi_{31}'/\Phi_{31} \\ \phi_{12} & \phi_{22} & 0 \end{vmatrix} = 0 . \quad (51)$$

This is the secular equation for the stability of plane Poiseulle flow, with respect to symmetric disturbances (ϕ odd in y). Thus the vestige of the shear waves is completely traced in the course of the order of magnitude analysis of each element in the secular equation.

It should be pointed out that termwise differentiation of the asymptotic series expansion of a function does not give the derivative of the function, in general. It can be shown easily that termwise differentiation is legitimate if the derivatives of the function also possess asymptotic series expansions. Fortunately, this is the case in this problem. Since it is known that the exact solution of the Orr-Sommerfeld equation must be analytic in y; i.e., derivatives of the solutions of the Orr-Sommerfeld equation possess asymptotic series expansions. Therefore the termwise differentiation in the above analysis is legitimate.

III. METHODS OF COMPUTATION AND THE RESULTS

A. C. C. Lin's Method

It follows from (50) that the secular equation to be solved for this problem is

$$\frac{\Phi_{31}}{\Phi'_{31}} = \frac{\begin{vmatrix} \phi_{11} & \phi_{21} \\ (m\phi_{12} + j\phi'_{12} - i\phi'''_{12}) & (m\phi_{22} + j\phi'_{22} - i\phi'''_{22}) \end{vmatrix}}{\begin{vmatrix} \phi'_{11} & \phi'_{21} \\ (m\phi_{12} + j\phi'_{12} - i\phi'''_{12}) & (m\phi_{22} + j\phi'_{22} - i\phi'''_{22}) \end{vmatrix}} \qquad (52)$$

If one makes use of the explicit solutions of ϕ_1 in (30) to reduce the right-hand side of the above equation, one has, after some manipulation

$$\frac{\Phi_{31}}{\Phi_{31}'} = \frac{(U_1 - c)(m\phi_{22} + j\phi_{22}' - i\phi_{22}''')}{U_1'(m\phi_{22} + j\phi_{22}' - i\phi_{22}''') - (U_1 - c)^{-1}(m\phi_{12} + j\phi_{12}' - i\phi_{12}''')} ,$$

To simplify the computation, the above equation can be rewritten as

$$\left(1 + \frac{U_1'}{c}\,\frac{\Phi_{31}}{\Phi_{31}'}\right)^{-1} = 1 + U_1' c\,\frac{m\phi_{22} + j\phi_{22}' - i\phi_{22}'''}{m\phi_{12} + j\phi_{12}' - i\phi_{12}'''} \qquad (53)$$

In the above expressions

$$\phi_{12} = (1-c)[h_{02} + \alpha^2 h_{12} + \cdots\cdots \alpha^{2n} h_{n2} + \cdots] ,$$

$$\phi_{12}' = (1-c)^{-1} \sum_{n=1}^{\infty} \alpha^{2n} H_{2n-1} , \qquad (54)$$

$$\phi_{12}''' = [\alpha^2 - 2(1-c)^{-1}]\,\phi_{12} ,$$

$$\phi_{22} = (1-c)[k_{02} + \alpha^2 k_{12} + \cdots\cdots \alpha^{2n} k_{n2} + \cdots]$$

$$\Phi_{22}' = (1-c)^{-1}[1+\sum_{n=1}^{\infty}\alpha^{2n}K_{2n}] \quad ,$$

$$\Phi_{22}''' = [\alpha^2 - z(1-c)^{-1}]\Phi_{22}' \quad ,$$

where

$$h_0(y) = 1 \quad ,$$
$$h_{n+1}(y) = \int_{y_1}^{y} dy\,(U-c)^{-2}\int dy\,(U-c)^2 h_n(y) \qquad (55)$$

$$k_0(y) = \int_{y_1}^{y}(U-c)^{-2}dy \quad ,$$
$$k_{n+1}(y) = \int_{y_1}^{y} dy\,(U-c)^{-2}\int_{y_1}^{y} dy\,(U-c)^2 k_n(y) \quad , \qquad (56)$$

$$H_{2n-1} \int_{y_1}^{y}(U-c)^2 h_{n-1}(y)\;dy \quad ,$$
$$K_{2n} = \int_{y_1}^{y_2}(U-c)^2 k_{n-1}(y)\,dy \quad . \qquad (57)$$

Since the computation is for obtaining neutral-stability curves, c is real in the above equation. Substituting (40) into the left hand side of (53) for Φ_3, one has

$$1 + \frac{U_1'}{c}\frac{\Phi_{31}}{\Phi_{31}'} = 1-(1+L)\,F(z) \quad ,$$

where

$$F(z) = \frac{-\int_{\infty}^{-z} d\zeta \int_{\infty}^{\zeta} \zeta^{\frac{1}{2}} d\zeta \, H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right]}{z\left[\int_{\infty}^{-z} \zeta^{1/2} d\zeta \, H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right]\right]} , \tag{58}$$

$$\zeta = (U_c')^{1/3}\eta ,$$

$$z = -\zeta_1 = (\alpha R U_c')^{1/3}(y_c - y_1) ,$$

$$c = U_1'(y_c - y_1)/(1+L) .$$

Thus the secular equation (53), can be rewritten as

$$\{1-(1+\lambda)F(z)\}^{-1} = u + iv , \tag{59}$$

where

$$u + iv = 1 + U_1' c \, \frac{m\phi_{22} + j\phi_{22}' - i\phi_{22}'''}{m\phi_{12} + j\phi_{12}' - i\phi_{12}'''} , \tag{60}$$

is the right-hand side of (53).

Defining

$$\mathcal{F}(z) = \{1 - F(z)\}^{-1} ,$$

one has from (59)

$$\left.\begin{aligned} Q &= \mathcal{F}_r(z) - (1+\lambda)\{u(1+\lambda u) + \lambda v^2\}\{(1+\lambda u)^2 + (\lambda v)^2\}^{-1} = 0 , \\ G &= \mathcal{F}_i(z) - (1+\lambda)\{(1+\lambda u)^2 + (\lambda v)^2\}^{-1} = 0 . \end{aligned}\right\} \tag{61}$$

$F_r(z)$ or $F_i(z)$ is the real or the imaginary part of $F(z)$ respectively. They are tabulated in Table II.

By use of the relations in (54), (60) can be rewritten as

$$U + iV = 1 + U_1' c \, \frac{m\phi_{22} + [j - i\{\alpha^2 - z(1-c)^{-1}\}]\phi_{22}'}{m\phi_{12} + [j - i\{\alpha^2 - z(1-c)^{-1}\}]\phi_{12}'} \quad . \qquad (62)$$

To bring out the dominant terms in the functions ϕ_{12}, ϕ_{12}', ϕ_{22} and ϕ_{22}', the following transformation is used.

$$M_n = H_2 H_{n-2} - H_n \quad , \qquad n \geq 3$$

$$N_n = K_1 H_{n-1} - K_n \quad , \qquad n \geq 2$$

Since $H_n < H_{n-1} \ldots < H_1$ and $K_n < K_{n-1} \ldots < K_1$, as can be seen from (57) and (64), the above transformation does serve the purpose of bringing out the dominant terms in the functions ϕ_{12}, ϕ_{12}', ϕ_{22} and ϕ_{22}'. Using the above transformation, one has from (54)

$$\begin{aligned}
\phi_{12} &= (1-c)(1-\alpha^2 H_2)^{-1}\left(1 - \sum_{n=2}^{\infty} \alpha^{2n} M_{2n}\right) , \\
\phi_{22} &= K_1 \phi_{12} - (1-c)\sum_{n=1}^{\infty} \alpha^{2n} N_{2n+1} , \\
\phi_{12}' &= (1-c)^{-1}(1-\alpha^2 H_2)^{-1}\left(\alpha^2 H_1 - \sum_{n=2}^{\infty} \alpha^{2n} M_{2n-1}\right) , \\
\phi_{22}' &= K_1 \phi_{12}' + (1-c)^{-1}\left(1 - \sum_{n=1}^{\infty} \alpha^{2n} N_{2n}\right) ,
\end{aligned} \qquad (63)$$

where H_1, H_3, etc. and K_2, K_4, etc. are defined in (57). Other functions are defined as follows

$$H_2 = \int_{y_1}^{y_2} (U-c)^{-2} dy \int_{y_1}^{y} (U-c)^2 dy ,$$

$$K_1 = \int_{y_1}^{y} (U-c)^{-2} dy , \tag{64}$$

$$K_3 = \int_{y_1}^{y_2} (U-c)^{-2} dy \int_{y_1}^{y} (U-c)^2 dy \int_{y_1}^{y} (U-c)^{-2} dy ,$$

etc.

The above functions in algebraic forms and their numerical values for different values of c are listed in Appendix A.

Substituting (63) into (62) one has

$$u + iv = 1 + U_1' c K_1$$

$$+ U_1' c \frac{(1-\alpha^2 H_2)\{q(1-\sum_{n=1}^{\infty} \alpha^{2n} N_{2n}) - m(1-c)^2 \sum_{n=1}^{\infty} \alpha^{2n} N_{2n+1}\}}{m(1-c)^2(1-\sum_{n=2}^{\infty} \alpha^{2n} M_{2n}) + q(\alpha^2 H_1 - \sum_{n=2}^{\infty} \alpha^{2n} M_{2n-1})} , \tag{65}$$

in which

$$q = \alpha R (c-1) + 2i\{\alpha^2 + (1-c)^{-1}\} ,$$

$$m = \frac{\alpha}{c-1} (2 \cot\beta + \alpha^2 S R) .$$

Substituting these expressions for q and m in (65), one has

$$u + iv = 1 + U_1' c K_1 + U_1' c (1-\alpha^2 H_2) \cdot$$

$$\cdot \frac{\{-f(1-c)^2 + i[2\alpha^2(1-c)+2]\}(1-\alpha^2 N_2 - \cdots) + \alpha(1-c)^2(2\cot\beta + \alpha S f)(\alpha^2 N_3 + \cdots)}{\{-f(1-c)^2 + i[2\alpha^2(1-c)+2]\}(\alpha^2 H_1 - \alpha^4 M_3 \cdots) - \alpha(1-c)^2(2\cot\beta + \alpha S f)(1-\alpha^4 M_4 + \cdots)} , \tag{66}$$

in which

$$f = \alpha R = \left(\frac{Z}{y_c - y_1}\right)^3 \times \frac{1}{U_c'} = -\left(\frac{Z}{\sqrt{1-c}-1}\right)^3 \times \frac{1}{2\sqrt{1-c}} \quad . \tag{67}$$

If β and S are given in Equation (66), u and v are functions of α, c and Z only. Hence one can solve (61) for α and c for various values of Z. Then corresponding R are obtained from (67).

For actual computation, the explicit expressions of u and v are needed. After a straightforward manipulation they are found to be

$$u = 1 + U_1'c K_{1r} + \frac{U_1'c}{V^2+W^2}\left[(XV + WY)(1-\alpha^2 H_{2r}) + \alpha^2 H_{2i}(VY - XW)\right],$$

$$v = U_1'c K_{1i} + \frac{U_1'c}{V^2+W^2}\left[(VY - XW)(1-\alpha^2 H_{2r}) - \alpha^2 H_{2i}(XV - WY)\right], \tag{68}$$

in which

$$X = -f(1-c)^2(1-\alpha^2 N_{2r}) + z[\alpha^2(1-c)+1]\alpha^2 N_{2i} + (1-c)^2(z\cot\beta + \alpha S f)\alpha^3 N_{3r},$$

$$Y = f(1-c)^2\alpha^2 N_{2i} + z[\alpha^2(1-c)+1][1-\alpha^2 N_{2r}] + (1-c)^2(z\cot\beta + \alpha S f)\alpha^3 N_{3i},$$

$$V = -f(1-c)^2[\alpha^2 H_{1r} - \alpha^4 M_{3r}] + z[\alpha^2(1-c)+1][\alpha^4 M_{3i} - \alpha^2 H_{1i}] - \alpha(1-c)^2\{z\cot\beta + \alpha S f\},$$

$$W = -f(1-c)^2[\alpha^2 H_{1i} - \alpha^4 M_{3i}] + z[\alpha^2(1-c)+1][\alpha^2 H_{1r} - \alpha^4 M_{3r}].$$

Note that only the first two terms in the infinite series in (66) are retained. In the above expressions the subscripts i or r indicate the imaginary or the real part of the function they subscribe.

To solve (61), the so-called Newton's method(3) is used. For given values of Z the corresponding values of c and α are obtained from the following iteration procedure.

$$\left(\frac{\partial Q}{\partial \alpha}\right)_{\alpha_n} \alpha_{n+1} + \left(\frac{\partial Q}{\partial c}\right)_{c_n} c_{n+1} = -Q(\alpha_n, c_n) + \left(\frac{\partial Q}{\partial \alpha}\right)_{\alpha_n} \alpha_n + \left(\frac{\partial Q}{\partial c}\right)_{c_n} c_n \,, \tag{69}$$

$$\left(\frac{\partial G}{\partial \alpha}\right)_{\alpha_n} \alpha_{n+1} + \left(\frac{\partial G}{\partial c}\right)_{c_n} c_{n+1} = -G(\alpha_n, c_n) + \left(\frac{\partial G}{\partial \alpha}\right)_{\alpha_n} \alpha_n + \left(\frac{\partial G}{\partial c}\right)_{c_n} c_n \,,$$

where α_n and c_n are the values of α and c at the n-th iteration, and $\left(\frac{\partial Q}{\partial c}\right)_{c_n}$ etc. are the values of the functions evaluated for c given by c_n etc. The numerical results thus obtained for the case of S = 0 and β = 1 min. are given in Table I. The results are plotted in Figure 2.

B. Method to be Used for Problems in Which α is Larger than Unity

After a change of the independent variable

$$z = y + a \,,$$

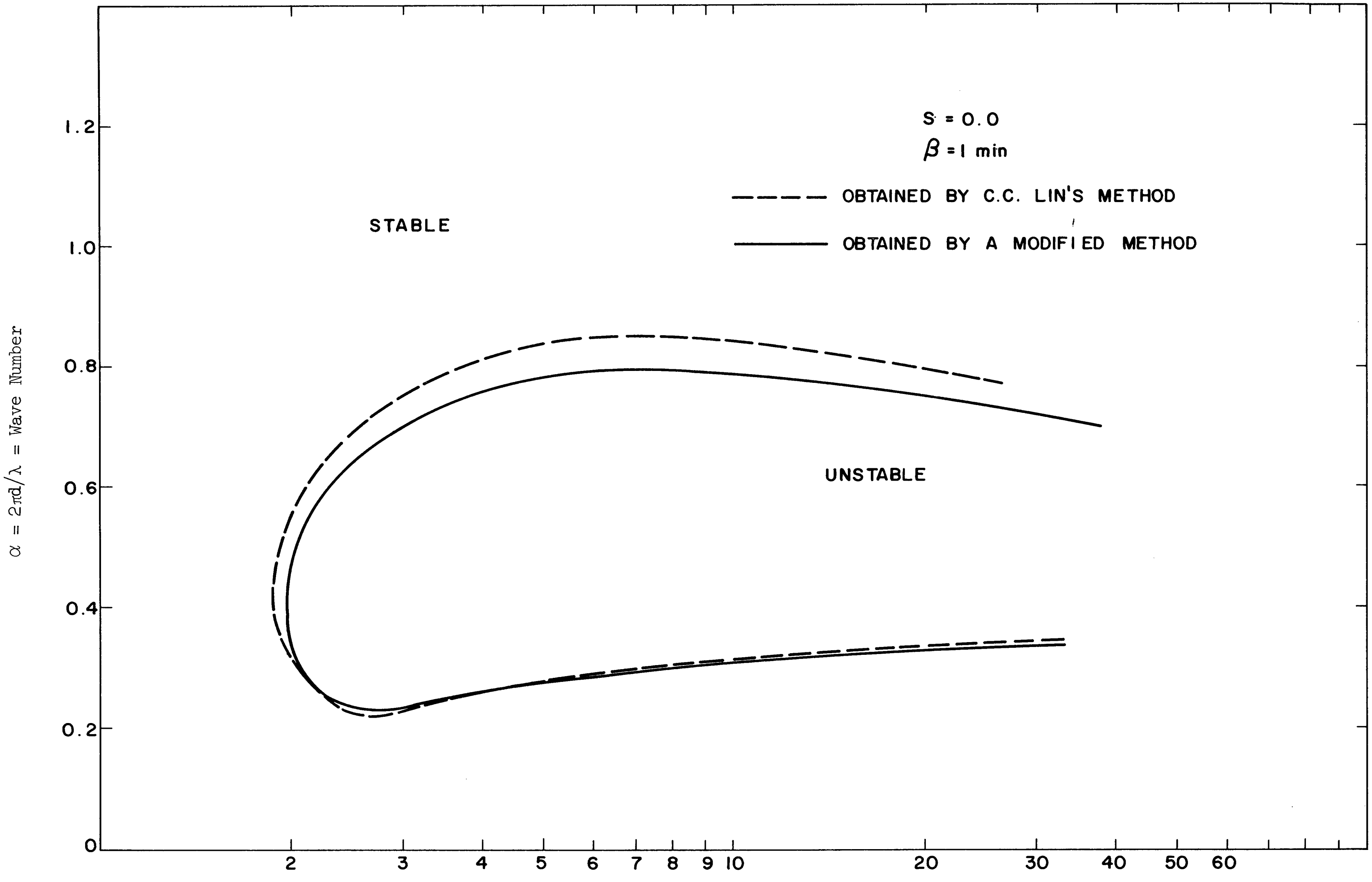
where

$$a = -y_c = (1-c)^{1/2} \,,$$

TABLE I

EIGEN-VALUES OBTAINED BY C. C. LIN'S METHOD FOR S = 0.0 AND β = 1 MIN.

Z	c	α	R
2.414759	.080000	.341187	3.159721×10^5
2.594636	.175000	.265139	4.702408×10^4
2.788177	.236000	.219866	2.823893×10^4
3.051198	.266000	.272340	2.070277×10^4
3.369871	.256000	.452931	1.886246×10^4
3.577772	.235000	.615388	2.159633×10^4
3.767519	.210000	.751831	2.911494×10^4
3.975282	.178500	.838797	5.032895×10^4
4.182047	.145000	.834318	1.108617×10^5



$R(10^4) = U_m d/\nu$ = Reynolds Number

Figure 2. Two Neutral-Stability Curves Obtained with Two Different Methods for a Given Angle of Inclination and a Given Surface Tension.

the inviscid equation can be rewritten as

$$\left(\frac{d^2}{dz^2} - \alpha^2\right)\Phi - \frac{2}{z(z-2a)}\Phi = 0 \quad . \tag{70}$$

It is seen from (70) that $z = 0$ is a regular singular point, and the region of varidity of the power series solution

$$\Phi = \sum_{n=0}^{\infty} a_n z^{n+s} \tag{71}$$

includes the end points $z_1 = a - 1$ and $z_2 = a$; at these end points Φ_1 and Φ_2 are to be evaluated in the secular equation.

Substituting (71) into (70) and equating all coefficients of z^{n+s} to zero, one has

$$n=0: \quad s(s-1)a_0 = 0 \,,$$

$$n=1: \quad s(s-1)a_0 = -2a(s+1)sa_1 - 2a_0 = 0 \,,$$

$$n=2: \quad (s+1)sa_1 - 2a(s+2)(s+1)a_2 + 2a\alpha^2 a_0 - 2a_1 = 0 \,,$$

$$n=3: \quad \sum_{n=3}^{\infty} \Big[\{(n+s-1)(n+s-2)\} a_{n-1} - 2a(n+s)(n+s-1)a_n - \alpha^2 a_{n-3} + 2y_c \alpha^2 a_{n-2} \Big] z^{n+s-1} = 0 \,,$$

Using the exponent $S = 1$, one has

$$a_0 = 1,$$
$$a_1 = -1/2a,$$
$$a_2 = \alpha^2/6,$$
$$a_n = \frac{1}{2n(n+1)a}\left[\{n(n-1)-2\}a_{n-1} + 2a\alpha^2 a_{n-2} - \alpha^2 a_{n-3}\right]. \qquad (n \geq 3) \tag{72}$$

The second solution is of the form

$$\Phi_2 = \Phi_1 \ln z + \psi_3, \tag{73}$$

where

$$\psi_3 = \sum_{n=0}^{\infty} b_n z^{n+\sigma}$$

Substituting (73) into (70), one finds

$$\sigma = 0,$$
$$n = 0 \qquad 0 \cdot b_0 = 0,$$
$$n = 1 \qquad b_0 = -a_0 a = -a,$$
$$n = 2 \qquad b_1 = 1,$$
$$b_2 = \frac{1}{2a} - \frac{a}{2}\alpha^2, \tag{74}$$
$$n \geq 3$$
$$b_n = \frac{1}{2n(n-1)a}\left[n(n-3)b_{n-1} + 2a\alpha^2 b_{n-2} - \alpha^2 b_{n-3} + (2n-3)a_{n-2} + 2a(1-2n)a_{n-1}\right].$$

The proper branch for $\ln z$ has already been determined in the previous method.

Having obtained the recurrence relations for the coefficients of the series solutions Φ_1 and Φ_2, one can write

$$\Phi_1 = z[1-\frac{1}{2a}z + A_{22}\alpha^2 z^2 + A_{23}\alpha^2 z^3 + (A_{24}\alpha^2 + A_{44}\alpha^4)z^4$$

$$+ (A_{25}\alpha^2 + A_{45}\alpha^4 + A_{65}\alpha^6)z^5 + (A_{26}\alpha^2 + A_{46}\alpha^4 + A_{66}\alpha^6)z^6 + \cdots] \quad (75)$$

where

$$\begin{aligned} A_{22} &= \frac{1}{6}, \\ A_{23} &= -1/18a, \\ A_{24} &= -1/720a^2, \\ A_{44} &= 1/120, \\ A_{25} &= 3\cdot A_{24}/10\cdot a, \\ A_{45} &= (18\cdot A_{44} - \frac{1}{6} - \frac{1}{9})/60a, \quad \text{etc.} \end{aligned} \quad (76)$$

Similarly,

$$\Phi_2 = \Phi_1 \ln z + [-a + z + (B_{12} - B_{22}\alpha^2)z^2 + (B_{13} + B_{23}\alpha^2)z^3$$

$$+ (B_{14} + B_{24}\alpha^2 + B_{44}\alpha^4)z^4 + (B_{15} + B_{25}\alpha^2 + B_{45}\alpha^4)z^5 + \cdots] \quad (77)$$

where

$$B_{12} = 1/2a,$$

$$B_{22} = -a/2,$$

$$B_{13} = -1/8a^2,$$

$$B_{23} = 1/9,$$

$$B_{14} = -1/48a^3,$$

$$B_{24} = \frac{23}{24\cdot 18} - \frac{7}{12}\cdot B_{23},$$

$$B_{44} = -a/24,$$

$$B_{15} = B_{14}/4a, \qquad \text{etc.}$$

It follows, after a straightforward manipulation

$$\left.\begin{aligned}
\Phi_{11}(\alpha, c) &= (a-1)\left[\frac{1+a}{2a} + \sum_{N=1}^{\infty} Q_N \alpha^{2N}\right], \\
\Phi_{12}(\alpha, c) &= a\left[\frac{1}{2} + \sum_{N=1}^{\infty} R_N \alpha^{2N}\right], \\
\Psi_{31}(\alpha, c) &= C_1 + \sum_{N=1}^{\infty} S_N \alpha^{2N}, \\
\Psi_{32}(\alpha, c) &= C_2 + \sum_{N=1}^{\infty} T_N \alpha^{2N}, \\
\Phi_{21}(\alpha, c) &= \Phi_{11} \ln(1-a) + \Psi_{31} - i\pi\Phi_{11}, \\
\Phi_{22}(\alpha, c) &= \Phi_{12} \ln a + \Psi_{32}, \\
\Phi_{11}'(\alpha, c) &= \frac{1}{a} + \sum_{N=1}^{\infty} Q_{1N} \alpha^{2N}, \\
\Phi_{12}' &= \sum_{N=1}^{\infty} R_{1N} \alpha^{2N}, \\
\Phi_{12}'' &= -\frac{1}{a} + \sum_{N=1}^{\infty} R_{2N} \alpha^{2N}, \\
\Phi_{12}''' &= \sum_{N=1}^{\infty} R_{3N} \alpha^{2N},
\end{aligned}\right\} \quad (79)$$

$$\Psi_{31}' = C_1' + \sum_{N=1}^{\infty} S_{1N}\,\alpha^{2N} \quad ,$$

$$\Psi_{32}' = C_2' + \sum_{N=1}^{\infty} T_{1N}\,\alpha^{2N} \quad ,$$

$$\Psi_{32}''' = C_2''' + \sum_{N=1}^{\infty} T_{3N}\,\alpha^{2N} \quad ,$$

$$\Phi_{22}' = \Phi_{12}' \ln a + \Phi_{12}\frac{1}{a} + \Psi_{32}' \quad ,$$

$$\Phi_{22}''' = \Phi_{12}''' \ln a + \frac{3}{a}\Phi_{12}'' - \frac{3}{a^2}\Phi_{12}' + \frac{2}{a^3}\Phi_{12} + \Psi_{32}''' \quad ,$$

in which, with $N = 1,2,3,$ etc.,

$$Q_N = \sum_{n=2N}^{\infty} [A_{(2N)n}](a-1)^n \quad ,$$

$$R_N = \sum_{n=2N}^{\infty} [A_{(2N)n}]\,a^n \quad ,$$

$$S_N = \sum_{n=2N}^{\infty} [B_{(2N)n}](a-1)^n \quad ,$$

$$T_N = \sum_{n=2N}^{\infty} [B_{(2N)n}]\,a^n \quad ,$$

$$C_1 = -1 + \sum_{n=2}^{\infty} B_{1n}(a-1)^n \quad ,$$

$$C_2 = \sum_{n=2}^{\infty} B_{1n}\,a^n \quad , \qquad (80)$$

$$Q_{1N} = \sum_{n=2N}^{\infty} (n+1) [A_{(2N)n}] (a-1)^n ,$$

$$R_{1N} = \sum_{n=2N}^{\infty} (n+1) [A_{(2N)n}] a^n ,$$

$$R_{2N} = \sum_{n=2N}^{\infty} (n+1) n [A_{(2N)n}] a^{n-1} ,$$

$$R_{3N} = \sum_{n=2N}^{\infty} (n+1) n (n-1) [A_{(2N)n}] a^{n-2} ,$$

$$c_1' = 1 + \sum_{n=2}^{\infty} n \cdot B_{1n} (a-1)^{n-1} ,$$

$$S_{1N} = \sum_{n=2N}^{\infty} n [B_{(2N)n}] (a-1)^{n-1} ,$$

$$C_2' = 1 + \sum_{n=2}^{\infty} n \cdot B_{1n} a^{n-1} ,$$

$$T_{1N} = \sum_{n=2N}^{\infty} n [B_{(2N)n}] a^{n-1} ,$$

$$C_2''' = \sum_{n=2}^{\infty} n(n-1)(n-2) B_{1n} a^{n-3} ,$$

$$T_{3N} = \sum_{n=2N}^{\infty} n(n-1)(n-2) [B_{(2N)n}] a^{n-3} .$$

The above coefficients i.e., Q_N, R_N etc. for different values of c are listed in the Appendix. In evaluating Φ_{21}, the proper branch of $\ln z$ has been used.

Having obtained each term in the right side of the secular equation, one can separate both numerator and denominator of the right side of (52) into real and imaginary parts. That is to say, one can rewrite (52) in the following form

$$\frac{\Phi_{31}}{\Phi_{31}'} = \frac{X + iY}{U + iV}, \tag{81}$$

where

$$\left.\begin{aligned}
X &= \Phi_{11}(m\Phi_{22} + fc'\Phi_{22}') - (\Phi_{11}\ln(1-a) + \psi_{31})(m\Phi_{12} + fc'\Phi_{12}') \\
&\quad + \pi\Phi_{11}(\Phi_{12}''' - 3\alpha^2\Phi_{12}'), \\
Y &= \Phi_{11}(3\alpha^2\Phi_{22}' - \Phi_{22}''') + \pi\Phi_{11}(m\Phi_{12} + fc'\Phi_{12}') \\
&\quad + (\Phi_{11}\ln(1-a) + \psi_{31})(\Phi_{12}''' - 3\alpha^2\Phi_{12}'), \\
U &= \Phi_{11}'(m\Phi_{22} + fc'\Phi_{22}') - (\Phi_{11}'\ln(1-a) + \frac{\Phi_{11}}{a-1} + \psi_{31})\cdot(m\Phi_{12} \\
&\quad + fc'\Phi_{12}') + \pi\Phi_{11}'(\Phi_{12}''' - 3\alpha^2\Phi_{12}'), \\
V &= \Phi_{11}'(3\alpha^2\Phi_{22}' - \Phi_{22}''') + \pi\Phi_{11}'(m\Phi_{12} + fc'\Phi_{12}') \\
&\quad - (\Phi_{11}'\ln(1-a) + \Phi_{11}\frac{1}{a-1} + \psi_{31})(3\alpha^2\Phi_{12}' - \Phi_{12}''').
\end{aligned}\right\} \tag{82}$$

Now, the left side of the secular equation is

$$(a-1)\,F(Z), \tag{83}$$

where F(Z) is defined by (58).

To evaluate F(Z) for any values of Z, one can do the following. Letting $z = (-iU_c')^{1/3}\eta$ and $T = (-iU_c')^{1/3}\dfrac{d^2\Phi}{d\eta^2}$ in (38), one has

$$\frac{d^2T}{d\zeta^2} + \zeta T = 0 \qquad . \tag{84}$$

It follows directly from the above change of variables and (38), (39) that the solutions of (84) are

$$T_3(\zeta) = (-i U_c')^{1/6} \zeta^{1/2} \times H_{1/3}^{(1)}\left(\frac{2}{3}\zeta^{3/2}\right),$$

$$T_4(\zeta) = (-i U_c')^{1/6} \zeta^{1/2} \times H_{1/3}^{(2)}\left(\frac{2}{3}\zeta^{3/2}\right). \tag{85}$$

On the other hand, the solutions of (84) can also be expressed in terms of power series. They are

$$T_3(\zeta) = g + \frac{i\sqrt{3}}{3}(g - 2f_1),$$

$$T_4(\zeta) = g - \frac{i\sqrt{3}}{3}(g - 2f_1), \tag{86}$$

in which

$$f_1(\zeta) = \frac{2^{1/3}}{\Gamma(\frac{2}{3})}\left[1 + \sum_1^\infty \frac{(-1)^m(3m-2)(3m-5)\cdots 4\cdot 1}{(3m)!}\zeta^{3m}\right],$$

$$g(\zeta) = \frac{2^{1/3}}{3^{2/3}\Gamma(\frac{4}{3})}\left[\zeta + \sum_1^\infty \frac{(-1)^m(3m-1)(3m-4)\cdots 5\cdot 2}{(3m+1)!}\zeta^{3m+1}\right]. \tag{87}$$

With the relations (85), (86) and (87) one can evaluate F(Z) and hence $\mathfrak{F}(Z)$.

The coefficients in the Maclaurin's series (84) are tabulated in Reference 1. The functions F(Z) as well as $\mathcal{F}$(Z) thus obtained are tabulated in the following table.

TABLE II

Z	$F_r(Z)$	$F_i(Z)$	$\mathcal{F}_r(Z)$	$\mathcal{F}_i(Z)$
1.5	.692520	-.185059	2.387441	-1.436897
1.6	.669260	-.159868	2.450892	-1.184677
1.7	.649086	-.135825	2.478400	- .959293
1.8	.631381	-.112566	2.481432	- .757760
1.9	.615631	-.089794	2.467026	- .576332
2.0	.601395	-.067264	2.439287	- .411628
2.1	.588285	-.044772	2.400476	- .261041
2.2	.575945	-.022146	2.351772	- .122821
2.3	.564042	.000755	2.293793	.003973
2.4	.552249	.024045	2.226963	.119594
2.5	.540240	.047809	2.151780	.223759
2.6	.527679	.072099	2.068994	.315830
2.7	.514214	.096928	1.979706	.395009
2.8	.499472	.122261	1.885400	.460535
2.9	.483054	.147997	1.787898	.511861
3.0	.464539	.173955	1.689265	.548791
3.1	.443490	.199843	1.591662	.571568
3.2	.419476	.225235	1.497204	.580894
3.3	.392109	.249535	1.407811	.577896
3.4	.361100	.271957	1.325096	.564045
3.5	.326343	.291518	1.250300	.541053
3.6	.288022	.307068	1.184254	.510756
3.7	.246722	.317381	1.127395	.475009
3.8	.203514	.321309	1.079792	.435597
3.9	.159963	.318012	1.041204	.394168
4.0	.118013	.307201	1.011136	.352185
4.1	.079634	.289320	.988900	.310898
4.2	.046971	.265583	.973673	.271336
4.3	.021011	.237808	.964548	.234300
4.4	.002374	.208107	.960580	.200379
4.5	-.009193	.178515	.960827	.169960
4.6	-.014553	.150703	.964378	.143250
4.7	-.014926	.125822	.970380	.120300
4.8	-.011638	.104493	.978061	.101025
4.9	-.005933	.086894	.986739	.085236
5.0	-.001134	.072882	.995833	.072660

The subscripts r or i in the above table denotes the real or imaginary part of the function it subscribes.

With the left side being given by (83) and the right side by (82), Equation (81) can be rewritten as

$$(a-1)F(z) = U(\alpha, c, z, \beta, S) + i\,V(\alpha, c, z, \beta, S)\,, \tag{88}$$

where

$$U = \frac{X\mathcal{U} + Y\mathcal{V}}{\mathcal{U}^2 + \mathcal{V}^2}\,,$$

$$V = \frac{Y\mathcal{U} - X\mathcal{V}}{\mathcal{U}^2 + \mathcal{V}^2}\,.$$

Or equivalently

$$P = (a-1)F_r(z) - U(\alpha, c, z, \beta, S) = 0\,,$$

$$(89)$$

$$E = (a-1)F_i(z) - V(\alpha, c, z, \beta, S) = 0\,.$$

For given values of β and S, (89) is a system of simultaneous equations in three unknowns α, c, and Z. Thus, for various values of c, one can solve (89) for α and Z. Then the corresponding R can be obtained from

$$\alpha R = -\left(\frac{z}{a-1}\right)^3 \frac{1}{2a}\,. \tag{67}$$

A plot of the relation between α and R, thus obtained is a neutral curve. In solving (89) the following iteration scheme is used.

$$\begin{cases} \left(\frac{\partial P}{\partial \alpha}\right)_{\alpha_n} \alpha_{n+1} + \left(\frac{\partial P}{\partial Z}\right)_{Z_n} Z_{n+1} = -P(\alpha_n, Z_n) + \left(\frac{\partial P}{\partial \alpha}\right)_{\alpha_n} \alpha_n + \left(\frac{\partial P}{\partial Z}\right)_{Z_n} Z_n , \\ \left(\frac{\partial E}{\partial \alpha}\right)_{\alpha_n} \alpha_{n+1} + \left(\frac{\partial E}{\partial Z}\right)_{Z_n} Z_{n+1} = -E(\alpha_n, Z_n) + \left(\frac{\partial E}{\partial \alpha}\right)_{\alpha_n} \alpha_n + \left(\frac{\partial E}{\partial Z}\right)_{Z_n} Z_n . \end{cases} \tag{90}$$

With the above described process the following result is obtained by use of an electronic computer. The results are plotted in Figures 3 and 4.

TABLE III

EIGEN-VALUES OBTAINED BY A MODIFIED METHOD FOR S = 0.0 AND β = 1 MIN.

Z	c	α	R
2.414964	.080000	.336732	3.202343×10^5
2.598376	.175000	.264869	4.727604×10^4
2.802864	.235000	.227307	2.811159×10^4
2.999574	.258500	.263662	2.218076×10^4
3.195362	.261000	.346283	1.982209×10^4
3.397784	.250000	.469903	2.004248×10^4
3.599316	.230000	.608539	2.374924×10^4
3.795285	.204500	.722210	3.360005×10^4
4.003972	.173000	.783976	6.052522×10^4
4.200095	.141500	.773417	1.304778×10^5
4.386976	.112000	.712072	3.281373×10^5
4.598310	.082000	.610088	1.132477×10^6

TABLE IV

EIGEN-VALUES FOR S = 0.0 AND β = 1°

Z	c	α	R
2.400196	.070600	.371858	4.152194×10^5
2.500516	.129500	.539594	5.164133×10^4
2.591080	.172500	.656387	1.976407×10^4
2.799279	.236700	.849766	7327.3423
2.995686	.261900	.966309	5791.7020
3.212561	.264000	1.039897	6476.6970
3.368392	.255000	1.062555	8126.8064
3.552588	.237000	1.060654	1.195319×10^4
3.800499	.204400	1.014059	2.406453×10^4
3.992671	.175000	.946671	4.799069×10^4
4.191120	.143000	.852935	1.138520×10^5
4.400189	.110000	.736120	3.382574×10^5
4.602193	.081500	.618260	1.141233×10^6

TABLE V

EIGEN-VALUES FOR S = 0.2, β = 1°

Z	c	α	R
2.400041	.070500	.310212	4.997700×10^5
2.507280	.130000	.449408	5.765097×10^4
2.600485	.176200	.538139	2.284523×10^4
2.800388	.236000	.672941	9348.09155
2.996595	.260500	.760146	7491.3995
3.197549	.260000	.817137	8222.5165
3.599313	.230700	.847652	1.689200×10^4
3.800190	.200000	.823235	2.981317×10^4
4.000727	.173500	.773467	6.066319×10^4
4.201071	.141300	.701191	1.446378×10^5
4.403286	.109500	.611922	4.134361×10^5
4.594385	.082500	.521939	1.296325×10^6

TABLE VI

EIGEN-VALUES FOR S = 0.5, $\beta = 1°$

Z	c	α	R
2.399266	.070000	.256467	6.170290×10^5
2.493119	.125500	.356033	8.531772×10^4
2.578898	.167000	.423036	3.336996×10^4
2.761200	.227000	.525043	1.293781×10^4
2.995937	.259600	.611403	9407.138916
3.202144	.262000	.660228	1.034078×10^4
3.399280	.250600	.686155	1.364330×10^4
3.599472	.230200	.691719	2.084060×10^4
3.798191	.204000	.675956	3.625166×10^4
3.999799	.173500	.638265	7.346216×10^4
4.171495	.146000	.590360	1.522784×10^5
4.399831	.110000	.510017	4.880966×10^5
4.662390	.074000	.409423	2.398347×10^6

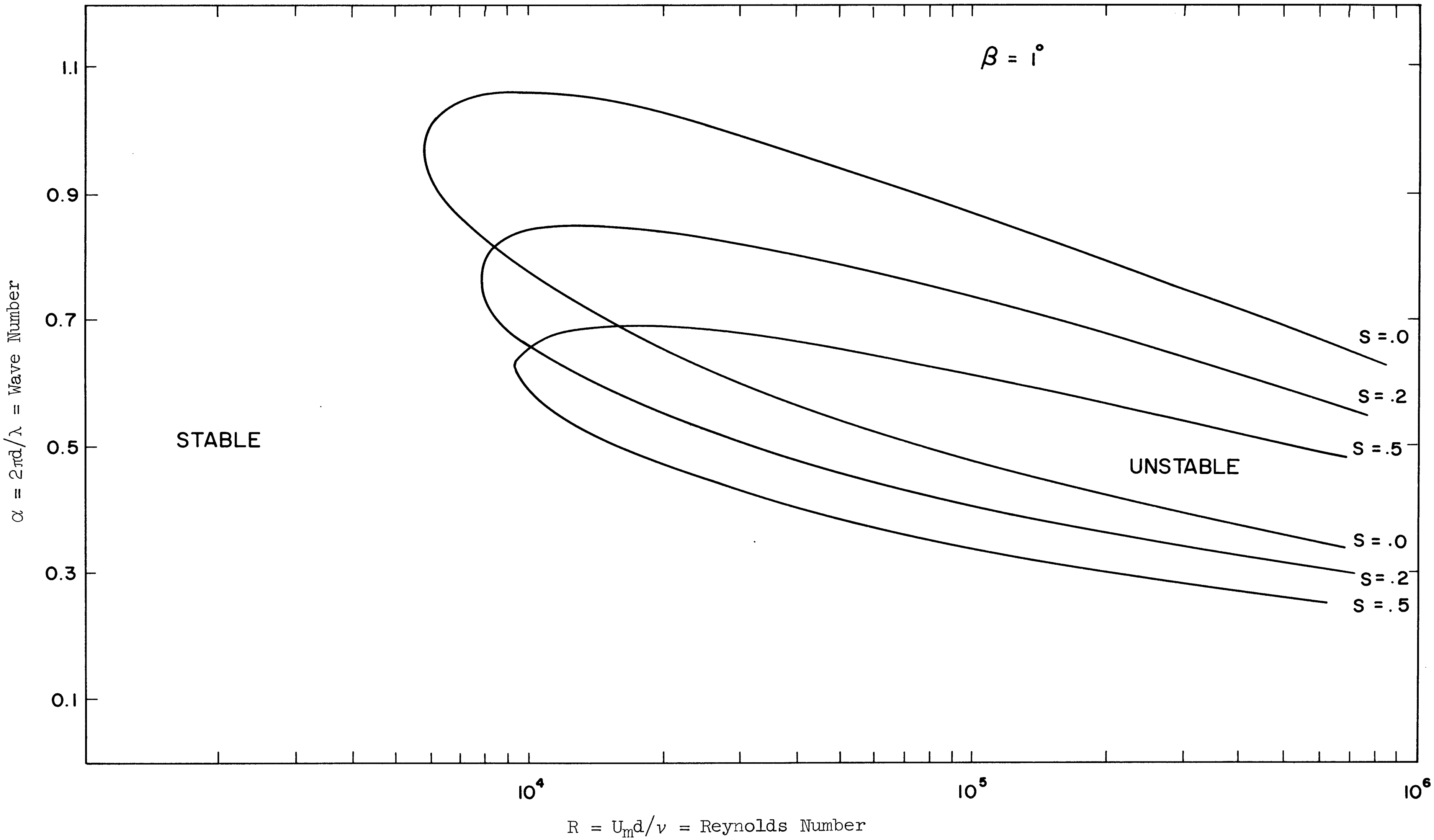


Figure 3. Neutral-Stability Curves Showing That Surface Tension is a Stabilizing Factor.

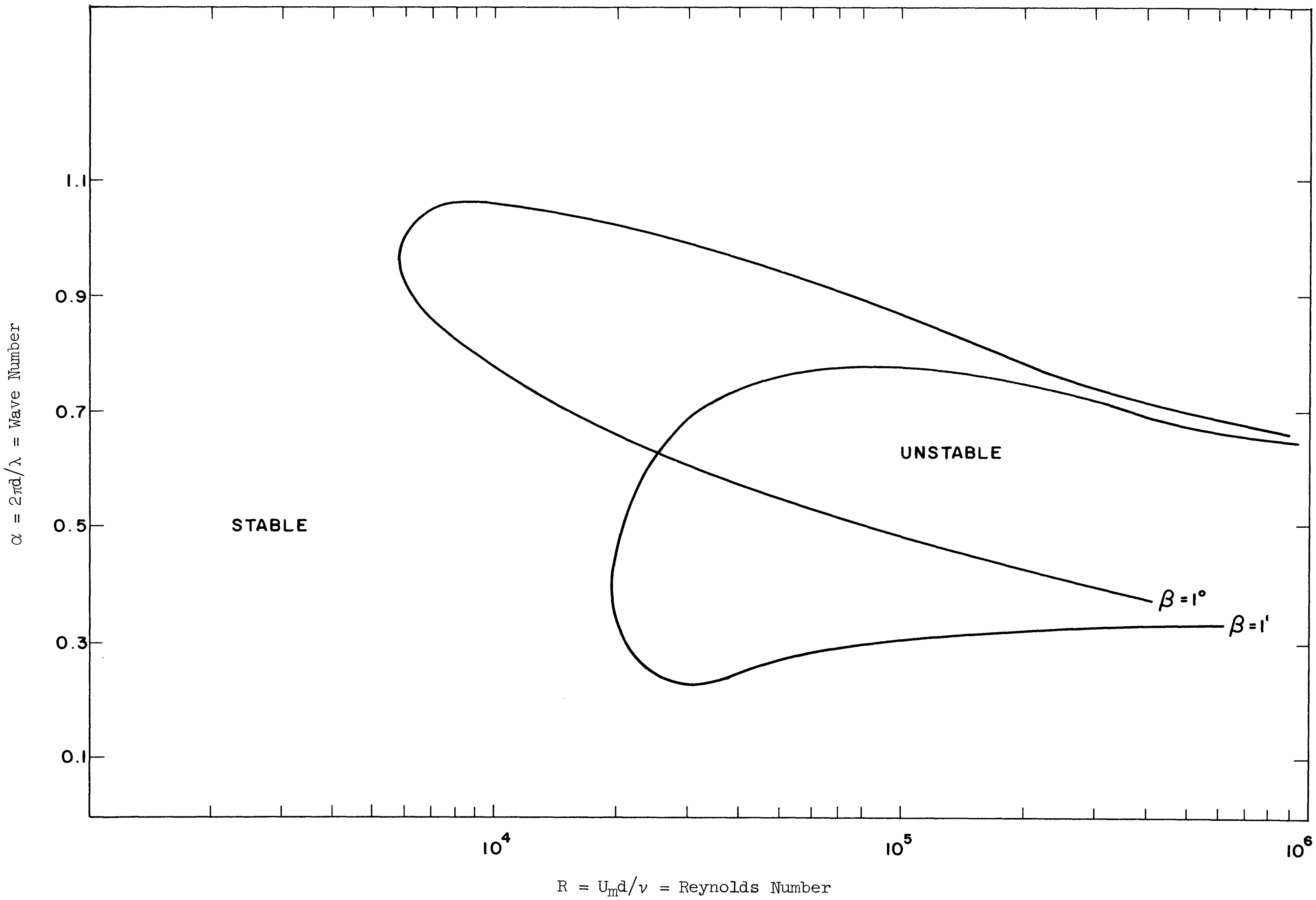


Figure 4. Neutral-Stability Curves Showing That the Reduction of Angle of Inclination is a Stabilizing Factor.

IV. DISCUSSION ON THE RESULTS AND ON THE METHOD OF COMPUTATION

A study of the neutral curves for different values of S in Figure 3 shows that surface tension is a stabilizing factor. It is also observed in Figure 3 that the space between neutral curves for different values of S decreases as α decreases. Therefore the stabilizing effect of the surface tension decreases as the wave length becomes larger. It is easily seen from Figure 4 that the reduction of the angle of inclination is also a stabilizing factor. As a limiting case, one naturally expects that the neutral curve will be shifted to $R \to \infty$ when $\beta \to 0$. Since when $\beta = 0$ there will be no flow, the liquid layer lying flat will be always stable with respect to any infinitesimal disturbances. The two neutral curves shown in Figure 2 are obtained from the two different methods developed in the preceding chapter. These two neutral curves for the same β and S check with each other when α is small and differ slightly when α is large. This result is quite to be expected since only two terms are retained in all power series in the first method, whereas more than twenty terms are retained in all power series in the second method.

For all values of α, the power series in α^2 in the secular equation which appears in the first method converge faster than the series which appear in the second method. Unfortunately, in the first method the coefficients of the series, i.e., K_1, N_2, N_3, ...N_9, etc., cannot be obtained by the computer alone. This is due to the fact that these constants are actually a series of multiple integrals, the path of integration of which do not lie entirely on the real axes. Consequently, one has to obtain a series of multiple integrals by hand. The evaluation of which requires considerable amount of algebraic work. The amount of work becomes formidable for the problems in which α are greater than one. Since for this value of α, the series converges very slowly. Although the series obtained from the Frobenius method converge even slower, their coefficients are obtainable from some recurrence relations, one can rely upon the computer to carry out the lengthy computation. Thus, it is very likely that the second method is more desirable for the problems in which $\alpha > 1$. For the problems in which the primary flow is of higher order than two, one may have more than one singularity within the domain of flow. If the first method is used for these problems, one has to decide which path to take around the singularities. As a result the evaluation of the multiple integrals mentioned above will become even more complicated. On the other hand, if the second method is to be used, one has to obtain two different power series, one of which is to be used for the bottom boundary conditions and the other for the top boundary conditions.

The difficulty of the computation in this problem arises from the following two facts. First of all, the imaginary part of the secular equation is not a function of c alone, therefore a fairly simple computational scheme used by C. C. Lin(8) in solving the problem of plane Poiseuille flow cannot be applied. Another fact is that the eigen-values as well as flow parameters appear in the boundary conditions. As a result the secular equation becomes considerably more complicated.

V. CONCLUSIONS

For various values of β and S, several neutral stability curves have been obtained in Chapter III. These neutral curves, however, are neutral curves with respect to the disturbance of a mode in which αR is much larger than unity. The instability of the same problem with respect to the disturbances of another mode in which αR is much less than unity has been studied by Yih[12] and Benjamin[2]. In their study, $R = (5 \cot \beta)/6$ is given as a critical Reynolds number for very long wave and zero surface tension. The critical Reynolds number (denote by R_s in Table VII) from this relation for three values of β and the corresponding critical Reynolds number (denoted by R_h in Table VII) obtained in Chapter III for $S = 0$ are listed in the following table for comparison.

TABLE VII

CRITICAL REYNOLDS NUMBERS

β	30"	1'	1°	90°
R_s	6,875.5	2,865	57.3	0
R_h	-------	19,600	5,790	---

Although the neutral curves for β smaller than 1 min. have not been obtained, the general trend of the neutral curves for different β shows that probably $R_s < R_h$ also for $\beta < 1$ min. Thus, the results obtained in this thesis together with the results obtained by Yih[12] and Benjamin[2], show that $R_s < R_h$ for all β. The general features of the neutral curves for the soft and hard modes are shown in the following figure. The two curves never intersect with each other since in their ranges of validity. $(\alpha R)_s < (\alpha R)_h$ along the neutral-stability curves.

The above mentioned two different modes represent two different type of disturbances. The mode in which αR is much larger than unity corresponds to hard waves which damp or grow rapidly. The other mode in which αR is less than unity corresponds to soft waves which damp or grow less rapidly than the former mode. This can be seen easily from the following argument. Given c_i, $(\alpha R)_s < (\alpha R)_h$ for all α and R. For a given flow condition one has the same R for both modes. Thus $\alpha_s < \alpha_h$, for given c_i. Consequently, $(\alpha c_i)_s < (\alpha c_i)_h$. That is to say the rate of growth or decay of soft waves is smaller than that of hard waves.

In conclusion, one can state that the soft waves which arises from the presence of the free surface do govern the instability of a layer of liquid flowing down an inclined plane.

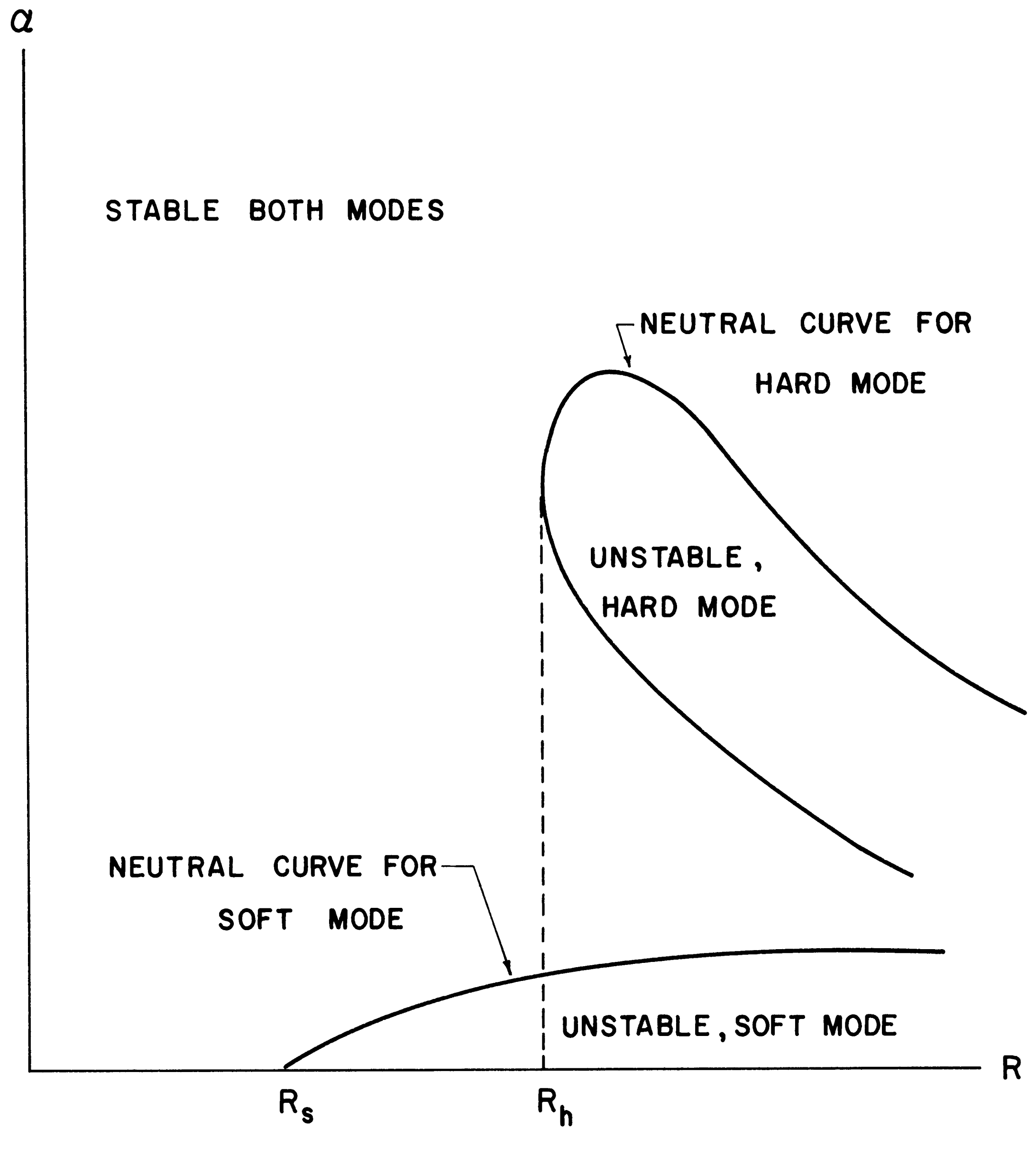


Figure 5. General Features of Neutral-Stability Curves.

VI. APPENDIX A

SOME FUNCTIONS AND THEIR NUMERICAL VALUES USED IN C. C. LIN'S METHOD

$$K_1 = \frac{1}{2a^2(a^2-1)} + \frac{1}{4a^3}\left[\ln\left(\frac{1+a}{1-a}\right) + i\pi\right],$$

$$H_1 = a^4 - \frac{2}{3}a^2 + \frac{1}{5},$$

$$H_2 = \frac{4a^2-3}{30\cdot a^2} + \frac{H_1}{2a^3}\ln(1+a) - \frac{2a^2}{15}\ln a^2 - \frac{B}{4a^3}[\ln(1-a^2) - i\pi],$$

$$N_2 = -\frac{2}{15}a^2\ln a^2 - \frac{B}{4a^3}[\ln(1-a) - i\pi] + \frac{H_1 + \frac{8a^5}{15}}{4a^3}\ln(1+a)$$

$$+ \left(\frac{2}{9} + \frac{1}{50}\right)a^2 - \frac{(a-1)^2}{2} + \frac{7}{18}\frac{(a-1)^3}{a} - \frac{1}{16}\frac{(a-1)^4}{a^2}$$

$$- \frac{1}{100}\frac{(a-1)^5}{a^3} - \frac{(a+1)^3}{4a^3}\left[\frac{(a+1)^2}{25} - \frac{a(a+1)}{4} + \frac{4a^2}{9}\right],$$

$$N_3 = S\cdot Re(K_1)$$

$$- B\cdot[Im(K_1)]^2 - \frac{47}{900}\left(\frac{1}{a+1} - \frac{1}{a-1}\right) + \frac{1}{10a^2} + \frac{47}{900a}\ln\frac{1+a}{1-a}$$

$$- \frac{1}{20a^2}[(a+1)\ln(a+1) - (a-1)\ln(1-a)] - \frac{1}{20a^2}$$

$$- \frac{1}{40a^3}\left\{[(a-1)^2 - 4a^2]\ln(a+1) - [(a+1)^2 - 4a^2]\ln(1-a)\right\}$$

$$+ \frac{1}{15a}[\ln 2a\cdot\ln(1-a)] - \frac{1}{30a}\left\{[\ln(a+1)]^2 - \pi^2\right\}$$

$$-\frac{1}{15a}\left[\sum_{n=1}^{\infty}\frac{(a-1)^n}{n^2(2a)^n}+\sum_{n=1}^{\infty}\left(\frac{2a}{a+1}\right)^n\Big/n^2\right]$$

$$+\frac{1}{15}\left[\frac{a-1}{2a(a+1)}\ln(1-a)-\frac{a+1}{2a(a+1)}\ln(1+a)\right]$$

$$+i\Big\{[B\cdot Re(K_1)+S]\cdot Im(K_1)$$

$$-\pi\left[\frac{-99.5}{900a}+\frac{1}{15a}[\ln 2a-\ln(a+1)]+\frac{1}{40a^3}+\frac{1}{15(a+1)}\right]\Big\},$$

$$M_3=-\frac{8a^2}{15}H_1\cdot\ln a+\frac{a^2}{6}\cdot H_1+\left(H_1+\frac{8a^5}{15}\right)^2\cdot\frac{\ln(1+a)}{4a^3}$$

$$-\frac{B^2}{4a^3}\ln(1-a)+\frac{1}{4a^3}\left(H_1+\frac{8a^5}{15}\right)\Big\{\frac{211}{900}a^5+(a+1)^3\Big[-\frac{(a+1)^2}{25}$$

$$+\frac{a}{4}(a+1)-\frac{4}{9}a^2\Big]+\frac{2a^2}{3}[(a-1)^3-a^3]+[a^4-(a-1)^4]\frac{a}{4}\Big\}$$

$$-\frac{a}{5}\left\{\frac{11}{30}a^6+(a+1)^4\left[-\frac{(a+1)^2}{6}+\frac{4a}{5}(a+1)-a^2\right]\right\}$$

$$+\frac{1}{10}\left\{\frac{29}{105}a^7+(a+1)^5\cdot\left[-\frac{(a+1)^2}{7}+\frac{2}{3}a(a+1)-\frac{4}{5}a^2\right]\right\}$$

$$+\frac{B}{4a^2}\left\{\frac{11}{12}a^4+(a-1)^2\left[-2a^2+\frac{4}{3}a(a-1)-\frac{(a-1)^2}{4}\right]\right\}$$

$$-\frac{B}{4a^3}\left\{-\frac{211}{900}a^5+(a-1)^3\left[\frac{4a^2}{9}-\frac{a}{4}(a-1)+\frac{(a-1)^2}{25}\right]\right\}$$

$$+i\left\{\frac{\pi}{4a^3}B^2\right\},$$

where

$$a = \sqrt{1-c} \quad ,$$

$$B = H_1 - \frac{8a^5}{15} \quad ,$$

$$S = -\frac{(a^2-1)^2}{8a^2} - \frac{B}{4a^3} \ln(1-a) + \frac{1}{4a^3}\left(H_1 + \frac{8a^5}{15}\right)\ln(1+a)$$

$$+ (a-1)^3\left[\frac{-1}{9a} + \frac{a-1}{16a^2} - \frac{(a-1)^2}{100a^3}\right]$$

$$+ (a+1)^3\left[\frac{-1}{9a} + \frac{a+1}{16a^2} - \frac{(a+1)^2}{100a^3}\right] \quad .$$

c	K_{1r}	K_{1i}	H_1	H_{2r}	H_{2i}
.05	-9.35007	.84821	.46917	.20696	.00002
.10	-4.49066	.91987	.41000	.19516	.00155
.15	-2.89951	1.00222	.35583	.18278	.00573
.20	-2.11623	1.09763	.30667	.16982	.01503
.25	-1.65287	1.20920	.26250	.15626	.03256
.30	-1.34799	1.34104	.22333	.14209	.06284
.35	-1.13287	1.49872	.18917	.12729	.11236
.40	- .97338	1.68991	.16000	.11182	.19058
.45	- .85069	1.92551	.13583	.09566	.31165
.50	- .75355	2.22144	.11667	.07875	.49729

c	N_{2r}	N_{2i}	N_{3r}	N_{3i}	M_{3r}	M_{3i}
.05	.20696	.00002	.20401	-.00051	.05045	.000000
.10	.19516	.00155	.21521	-.00220	.04181	.000000
.15	.18278	.00573	.22668	-.00534	.03439	.000000
.20	.16982	.01503	.23826	-.01036	.02811	.000002
.25	.15626	.03256	.24957	-.01782	.02291	.000009
.30	.14209	.06284	.25984	-.02851	.01869	.000030
.35	.12729	.11236	.26761	-.04354	.01535	.000084
.40	.11182	.19058	.27016	-.06449	.01278	.000215
.45	.09566	.31165	.26243	-.09372	.01083	.000504
.50	.07875	.49729	.23474	-.13477	.00935	.001113

The subscript r or i denotes the real or imaginary part of the function they subscribe.

VII. APPENDIX B

COEFFICIENTS OF THE POWER SERIES SOLUTION OF THE INVISCID EQUATION

c	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
.05	1.078×10^{-4}	3.448×10^{-9}	3.257×10^{-14}	4.678×10^{-19}	2.725×10^{-24}	1.148×10^{-29}
.10	4.468×10^{-4}	5.859×10^{-8}	3.664×10^{-12}	1.338×10^{-16}	3.199×10^{-21}	5.679×10^{-26}
.15	1.044×10^{-3}	3.158×10^{-7}	4.563×10^{-11}	3.851×10^{-15}	2.128×10^{-19}	8.968×10^{-24}
.20	1.931×10^{-3}	1.066×10^{-6}	2.815×10^{-10}	4.342×10^{-14}	4.388×10^{-18}	3.475×10^{-22}
.25	3.145×10^{-3}	2.790×10^{-6}	1.184×10^{-9}	2.939×10^{-13}	4.779×10^{-17}	6.265×10^{-21}
.30	4.735×10^{-3}	6.226×10^{-6}	3.921×10^{-9}	1.445×10^{-12}	3.489×10^{-16}	6.993×10^{-20}
.35	6.757×10^{-3}	1.246×10^{-5}	1.103×10^{-8}	5.710×10^{-12}	1.939×10^{-15}	5.632×10^{-19}
.40	9.284×10^{-3}	2.309×10^{-5}	2.759×10^{-8}	1.930×10^{-11}	8.862×10^{-15}	3.588×10^{-18}
.45	.012409	4.041×10^{-5}	6.328×10^{-8}	5.809×10^{-11}	3.500×10^{-14}	1.922×10^{-17}
.50	.016254	6.773×10^{-5}	1.360×10^{-7}	1.601×10^{-10}	1.238×10^{-13}	9.020×10^{-17}

c	R_1	R_2	R_3	R_4	R_5	R_6
.05	1.036×10^{-1}	5.492×10^{-3}	1.320×10^{-4}	1.813×10^{-6}	1.610×10^{-8}	1.055×10^{-10}
.10	9.819×10^{-2}	4.929×10^{-3}	1.122×10^{-4}	1.460×10^{-6}	1.229×10^{-8}	8.046×10^{-11}
.15	9.273×10^{-2}	4.397×10^{-3}	9.452×10^{-5}	1.162×10^{-6}	9.235×10^{-9}	6.033×10^{-11}
.20	8.728×10^{-2}	3.895×10^{-3}	7.880×10^{-5}	9.116×10^{-7}	6.820×10^{-9}	4.438×10^{-11}
.25	8.182×10^{-2}	3.423×10^{-3}	6.492×10^{-5}	7.042×10^{-7}	4.939×10^{-9}	3.196×10^{-11}
.30	7.637×10^{-2}	2.982×10^{-3}	5.278×10^{-5}	5.343×10^{-7}	3.498×10^{-9}	2.246×10^{-11}
.35	7.091×10^{-2}	2.571×10^{-3}	4.226×10^{-5}	3.973×10^{-7}	2.415×10^{-9}	1.535×10^{-11}
.40	6.546×10^{-2}	2.191×10^{-3}	3.323×10^{-5}	2.884×10^{-7}	1.618×10^{-9}	1.016×10^{-11}
.45	6.000×10^{-2}	1.841×10^{-3}	2.559×10^{-5}	2.036×10^{-7}	1.047×10^{-9}	6.472×10^{-12}
.50	5.455×10^{-2}	1.521×10^{-3}	1.922×10^{-5}	1.391×10^{-7}	6.504×10^{-10}	3.941×10^{-12}

c	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}
.05	2.534×10^{-1}	2.527×10^{-2}	8.815×10^{-4}	1.584×10^{-5}	1.735×10^{-7}	1.356×10^{-9}
.10	2.401×10^{-1}	2.268×10^{-2}	7.494×10^{-4}	1.276×10^{-5}	1.324×10^{-7}	1.038×10^{-9}
.15	2.268×10^{-1}	2.023×10^{-2}	6.313×10^{-4}	1.015×10^{-5}	9.949×10^{-8}	7.807×10^{-10}
.20	2.134×10^{-1}	1.792×10^{-2}	5.262×10^{-4}	7.963×10^{-6}	7.347×10^{-8}	5.764×10^{-10}
.25	2.001×10^{-1}	1.575×10^{-2}	4.335×10^{-4}	6.151×10^{-6}	5.321×10^{-8}	4.164×10^{-10}
.30	1.867×10^{-1}	1.372×10^{-2}	3.524×10^{-4}	4.668×10^{-6}	3.768×10^{-8}	2.936×10^{-10}
.35	1.734×10^{-1}	1.183×10^{-2}	2.821×10^{-4}	3.470×10^{-6}	2.602×10^{-8}	2.013×10^{-10}
.40	1.601×10^{-1}	1.008×10^{-2}	2.218×10^{-4}	2.519×10^{-6}	1.743×10^{-8}	1.337×10^{-10}
.45	1.467×10^{-1}	8.471×10^{-3}	1.708×10^{-4}	1.779×10^{-6}	1.128×10^{-8}	8.541×10^{-11}
.50	1.334×10^{-1}	7.000×10^{-3}	1.283×10^{-4}	1.215×10^{-6}	7.006×10^{-9}	5.217×10^{-11}

c	R_{21}	R_{22}	R_{23}	R_{24}	R_{25}	R_{26}
.05	.27581	8.975×10^{-2}	5.063×10^{-3}	1.249×10^{-4}	1.734×10^{-6}	1.646×10^{-8}
.10	.26846	8.276×10^{-2}	4.421×10^{-3}	1.034×10^{-4}	1.359×10^{-6}	1.301×10^{-8}
.15	.26089	7.596×10^{-2}	3.832×10^{-3}	8.463×10^{-5}	1.051×10^{-6}	1.012×10^{-8}
.20	.25310	6.936×10^{-2}	3.292×10^{-3}	6.844×10^{-5}	8.001×10^{-7}	7.731×10^{-9}
.25	.24507	6.2956×10^{-2}	2.800×10^{-3}	5.460×10^{-5}	5.984×10^{-7}	5.794×10^{-9}
.30	.23676	5.677×10^{-2}	2.356×10^{-3}	4.288×10^{-5}	4.387×10^{-7}	4.246×10^{-9}
.35	.22814	5.079×10^{-2}	1.956×10^{-3}	3.308×10^{-5}	3.143×10^{-7}	3.033×10^{-9}
.40	.21919	4.505×10^{-2}	1.601×10^{-3}	2.500×10^{-5}	2.192×10^{-7}	2.104×10^{-9}
.45	.20986	3.954×10^{-2}	1.287×10^{-3}	1.843×10^{-5}	1.482×10^{-7}	1.410×10^{-9}
.50	.20010	3.427×10^{-2}	1.013×10^{-3}	1.320×10^{-5}	9.651×10^{-8}	9.065×10^{-10}

c	R_{31}	R_{32}	R_{33}	R_{34}	R_{35}	R_{36}
.05	-.52302	.20024	2.316×10^{-2}	8.482×10^{-4}	1.547×10^{-5}	1.827×10^{-7}
.10	-.52302	.18971	2.077×10^{-2}	7.211×10^{-4}	1.246×10^{-5}	1.493×10^{-7}
.15	-.52302	.17917	1.851×10^{-2}	6.074×10^{-4}	9.915×10^{-6}	1.202×10^{-7}
.20	-.52302	.16863	1.638×10^{-2}	5.063×10^{-4}	7.780×10^{-6}	9.522×10^{-8}
.25	0.52302	.15809	1.438×10^{-2}	4.171×10^{-4}	6.010×10^{-6}	7.411×10^{-8}
.30	-.52302	.14755	1.251×10^{-2}	3.391×10^{-4}	4.560×10^{-6}	5.651×10^{-8}
.35	-.52302	.13701	1.077×10^{-2}	2.714×10^{-4}	3.390×10^{-6}	4.211×10^{-8}
.40	-.52302	.12647	9.166×10^{-3}	2.135×10^{-4}	2.461×10^{-6}	3.055×10^{-8}
.45	-.52302	.11593	7.688×10^{-3}	1.644×10^{-4}	1.738×10^{-6}	2.148×10^{-8}
.50	-.52302	.10539	6.339×10^{-3}	1.235×10^{-4}	1.187×10^{-6}	1.456×10^{-8}

c	S_1	S_2	S_3	S_4	S_5	S_6
.05	-3.142×10^{-4}	-1.673×10^{-8}	-3.572×10^{-13}	-4.087×10^{-18}	-2.910×10^{-23}	-1.413×10^{-28}
.10	-1.264×10^{-3}	-2.754×10^{-7}	-2.412×10^{-11}	-1.133×10^{-15}	-3.313×10^{-20}	-6.606×10^{-25}
.15	-2.857×10^{-3}	-1.436×10^{-6}	-2.904×10^{-10}	-3.153×10^{-14}	-2.132×10^{-18}	-9.830×10^{-23}
.20	-5.103×10^{-3}	-4.674×10^{-6}	-1.728×10^{-9}	-3.430×10^{-13}	-4.242×10^{-17}	-3.579×10^{-21}
.25	-8.007×10^{-3}	-1.177×10^{-5}	-6.994×10^{-9}	-2.235×10^{-12}	-4.449×10^{-16}	-6.043×10^{-20}
.30	-1.157×10^{-2}	-2.517×10^{-5}	-2.221×10^{-8}	-1.054×10^{-11}	-3.119×10^{-15}	-6.293×10^{-19}
.35	-1.579×10^{-2}	-4.815×10^{-5}	-5.970×10^{-8}	-3.985×10^{-11}	-1.659×10^{-14}	-4.709×10^{-18}
.40	-2.066×10^{-2}	-8.486×10^{-5}	-1.422×10^{-7}	-1.283×10^{-10}	-7.224×10^{-14}	-2.774×10^{-17}
.45	-2.614×10^{-2}	-1.405×10^{-4}	-3.088×10^{-7}	-3.661×10^{-10}	-2.707×10^{-13}	-1.347×10^{-16}
.50	-3.220×10^{-2}	-2.214×10^{-4}	-6.245×10^{-7}	-9.506×10^{-10}	-9.031×10^{-13}	-5.847×10^{-16}

c	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
.05	2.080×10^{-4}	2.645×10^{-6}	8.465×10^{-11}	1.291×10^{-15}	1.149×10^{-20}	6.698×10^{-26}
.10	8.288×10^{-4}	2.149×10^{-5}	2.821×10^{-9}	1.766×10^{-13}	6.455×10^{-18}	1.545×10^{-22}
.15	1.852×10^{-3}	7.369×10^{-5}	2.233×10^{-8}	3.232×10^{-12}	2.731×10^{-16}	1.511×10^{-20}
.20	3.260×10^{-3}	1.775×10^{-4}	9.825×10^{-8}	2.600×10^{-11}	4.019×10^{-15}	4.068×10^{-19}
.25	5.018×10^{-3}	3.521×10^{-4}	3.134×10^{-7}	1.335×10^{-10}	3.321×10^{-14}	5.412×10^{-18}
.30	7.077×10^{-3}	6.180×10^{-4}	8.163×10^{-7}	5.163×10^{-10}	1.909×10^{-13}	4.622×10^{-17}
.35	9.356×10^{-3}	9.965×10^{-4}	1.849×10^{-6}	1.645×10^{-9}	8.558×10^{-13}	2.915×10^{-16}
.40	1.174×10^{-2}	1.510×10^{-3}	3.785×10^{-6}	4.553×10^{-9}	3.204×10^{-12}	1.476×10^{-15}
.45	1.403×10^{-2}	2.180×10^{-3}	7.171×10^{-6}	1.133×10^{-8}	1.047×10^{-11}	6.335×10^{-15}
.50	1.597×10^{-2}	3.027×10^{-3}	1.278×10^{-5}	2.594×10^{-8}	3.080×10^{-11}	2.393×10^{-14}

c	T_1	T_2	T_3	T_4	T_5	T_6
.05	-.287395	-2.951×10^{-2}	-1.021×10^{-3}	-1.808×10^{-5}	-1.957×10^{-7}	-1.432×10^{-9}
.10	-.265010	-2.578×10^{-2}	-8.447×10^{-4}	-1.418×10^{-5}	-1.453×10^{-7}	-1.008×10^{-9}
.15	-.243233	-2.234×10^{-2}	-6.915×10^{-4}	-1.096×10^{-5}	-1.061×10^{-7}	-6.957×10^{-10}
.20	-.222090	-1.920×10^{-2}	-5.593×10^{-4}	-8.343×10^{-6}	-7.604×10^{-8}	-4.694×10^{-10}
.25	-.201598	-1.634×10^{-2}	-4.462×10^{-4}	-6.240×10^{-6}	-5.332×10^{-8}	-3.088×10^{-10}
.30	-.181778	-1.375×10^{-2}	-3.505×10^{-4}	-4.575×10^{-6}	-3.648×10^{-8}	-1.974×10^{-10}
.35	-.162654	-1.143×10^{-2}	-2.704×10^{-4}	-3.277×10^{-6}	-2.427×10^{-8}	-1.221×10^{-10}
.40	-.144252	-9.354×10^{-3}	-2.044×10^{-4}	-2.286×10^{-6}	-1.563×10^{-8}	-7.262×10^{-11}
.45	-.126601	-7.525×10^{-3}	-1.507×10^{-4}	-1.545×10^{-6}	-9.684×10^{-9}	-4.130×10^{-11}
.50	-.109736	-5.930×10^{-3}	-1.080×10^{-4}	-1.01×10^{-6}	-5.733×10^{-9}	-2.226×10^{-11}

c	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}
.05	-.342584	-.109817	-6.042×10^{-3}	-1.454×10^{-4}	-1.983×10^{-6}	-1.749×10^{-8}
.10	-.324553	-.098561	-5.137×10^{-3}	-1.171×10^{-4}	-1.513×10^{-6}	-1.265×10^{-8}
.15	-.306522	-.087914	-4.328×10^{-3}	-9.318×10^{-5}	-1.137×10^{-6}	-8.985×10^{-9}
.20	-.288492	-.077876	-3.608×10^{-3}	-7.312×10^{-5}	-8.398×10^{-7}	-6.251×10^{-9}
.25	-.270461	-.068445	-2.973×10^{-3}	-5.648×10^{-5}	-6.082×10^{-7}	-4.247×10^{-9}
.30	-.252430	-.059624	-2.417×10^{-3}	-4.286×10^{-5}	-4.308×10^{-7}	-2.810×10^{-9}
.35	-.234400	-.051401	-1.935×10^{-3}	-3.187×10^{-5}	-2.974×10^{-7}	-1.804×10^{-9}
.40	-.216369	-.043805	-1.522×10^{-3}	-2.314×10^{-5}	-1.993×10^{-7}	-1.117×10^{-9}
.45	-.198338	-.036808	-1.172×10^{-3}	-1.634×10^{-5}	-1.290×10^{-7}	-6.637×10^{-10}
.50	-.180307	-.030420	-8.809×10^{-4}	-1.116×10^{-5}	-8.009×10^{-8}	-3.752×10^{-10}

c	T_{31}	T_{32}	T_{33}	T_{34}	T_{35}	T_{36}
.05	2.237432	-.227318	-.104981	-5.946×10^{-3}	-1.443×10^{-4}	-1.976×10^{-6}
.10	2.237432	-.215354	-.094219	-5.056×10^{-3}	-1.162×10^{-4}	-1.510×10^{-6}
.15	2.237432	-.203390	-.084039	-4.259×10^{-3}	-9.256×10^{-5}	-1.136×10^{-6}
.20	2.237432	-.191426	-.074441	-3.551×10^{-3}	-7.255×10^{-5}	-8.395×10^{-7}
.25	2.237432	-.179462	-.065424	-2.926×10^{-3}	-5.605×10^{-5}	-6.088×10^{-7}
.30	2.237432	-.167498	-.056990	-2.379×10^{-3}	-4.253×10^{-5}	-4.317×10^{-7}
.35	2.237432	-.155533	-.049137	-1.905×10^{-3}	-3.162×10^{-5}	-2.985×10^{-7}
.40	2.237432	-.143570	-.041866	-1.498×10^{-3}	-2.296×10^{-5}	-2.004×10^{-7}
.45	2.237432	-.131605	-.0351774	-1.154×10^{-3}	-1.621×10^{-5}	-1.299×10^{-7}
.50	2.237432	-.119641	-.029070	-8.669×10^{-4}	-1.107×10^{-5}	-8.087×10^{-8}

c	c_1	c_2	c_1'	c_2'	c_2'''
.05	-.999791	.339571	.999841	1.500000	-2.083289
.10	-.998593	.328789	.998916	1.500000	-2.222175
.15	-.996628	.319525	.997362	1.500000	-2.352892
.20	-.993589	.309985	.994908	1.500000	-2.499947
.25	-.989246	.300142	.991320	1.500000	-2.666610
.30	-.983302	.289964	.986292	1.500000	-2.857083
.35	-.975367	.279417	.979414	1.500000	-3.076858
.40	-.964927	.268455	.970130	1.500000	-3.333263
.45	-.951279	.257026	.957664	1.500000	-3.636287
.50	-.933452	.245065	.940914	1.500000	-3.999916

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