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A LINEAR PHASE BAND-PASS FILTER

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

The design and realization of a linear phase band-pass filter for use as a predetection filter in a spinning goniometer radio direction finding system is described. The characteristics of the resulting unit are as follows: (1) center frequency, f_o = 20,000 cps; (2) bandwidth at 3 db-down points, Δf = 500 cps; (3) attenuation for bandwidth of 1,000 cps is greater than 10 db; (4) phase deviation from linearity at the 3 db points is less than one degree.

A LINEAR PHASE BAND-PASS FILTER

1. INTRODUCTION

This report describes the design and realization of a linear-phase band-pass filter. It is to be used as a predetection filter in a spinning-goniometer, radio direction-finding system. The filter has the following specifications:

(1) f_0 = center frequency = 20,000 cps

(2) Δf = bandwidth at 3-db-down points = 500 cps

(3) Attenuation: greater than 10 db when the bandwidth is 1000 cps

(4) Phase deviation from linearity: θ_e, $\leqq 1^\circ$ at the 3-db-down points.

2. LOW-PASS, ALL-POLE FUNCTION

The pole locations for an all-pole linear phase function are given in Table I for orders n = 1, 2, and 3. It is of interest to calculate two sets of curves as a function of the normalized radian frequency, ω, and as a function of the order of the approximation, n, for these all-

TABLE I

POLE LOCATIONS FOR AN ALL-POLE LINEAR PHASE FUNCTION

Order	Pole Location
n = 1	-1.000000
n = 2	-1.500000 ± j0.866025
n = 3	-2.322185 -1.838907 ± j1.754382

pole functions. One is a plot of the number of degrees error from linearity vs. radian frequency; the other, a plot of filter attenuation vs.

radian frequency. These are plotted in Figs. 1 and 2, respectively. The error formula[1] for this case is:

$$\theta_e = \tan^{-1} (-1)^n \frac{J_{(n + \frac{1}{2})}(\omega)}{J_{-(n + \frac{1}{2})}(\omega)} \qquad (1)$$

The attenuation functions for each order are:

$$\underline{n = 1:} \quad \alpha(\omega) = 10 \log_{10} (1 + \omega^2) \qquad (2)$$

$$\underline{n = 2:} \quad \alpha(\omega) = 10 \log_{10}\left(\frac{9 + 3\omega^2 + \omega^4}{9}\right) \qquad (3)$$

$$\underline{n = 3:} \quad \alpha(\omega) = 10 \log_{10}\left(\frac{225 + 45\omega^2 + 6\omega^4 + \omega^6}{225}\right) \qquad (4)$$

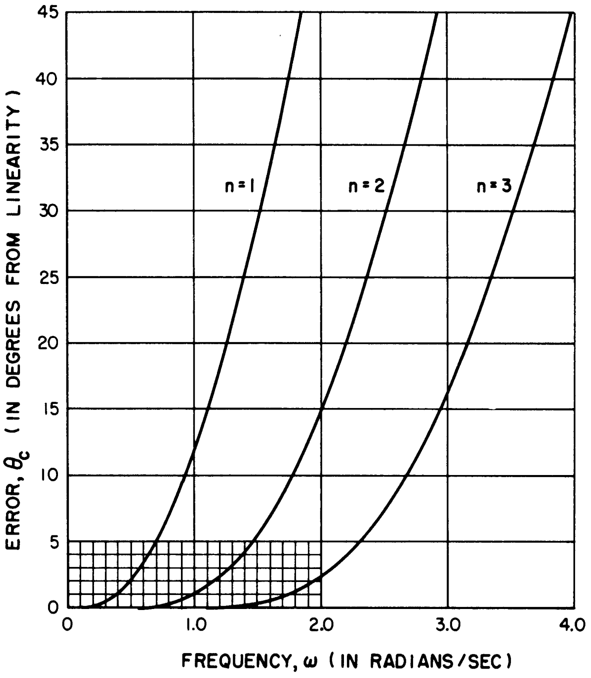

Fig. 1. Plot of numbers of degrees error from linearity vs. radian frequency.

By looking at the attenuation and error curves simultaneously, one can see that the function of order n = 3 will satisfy our requirements. From the attenuation plot (Fig. 2) the 3-db-down points for n = 3 occurs at $\omega \cong 1.75$ radians per second. Calculations show that this frequency is $\omega = 1.75567$ radians per second. From the phase error plot the error in degrees from linearity for n = 3 and $\omega \cong 1.75$ radians per second is approximately one degree.

In order to normalize the trans-

[1]N. Balabanian, Network Synthesis (Eng Inc., 1958).

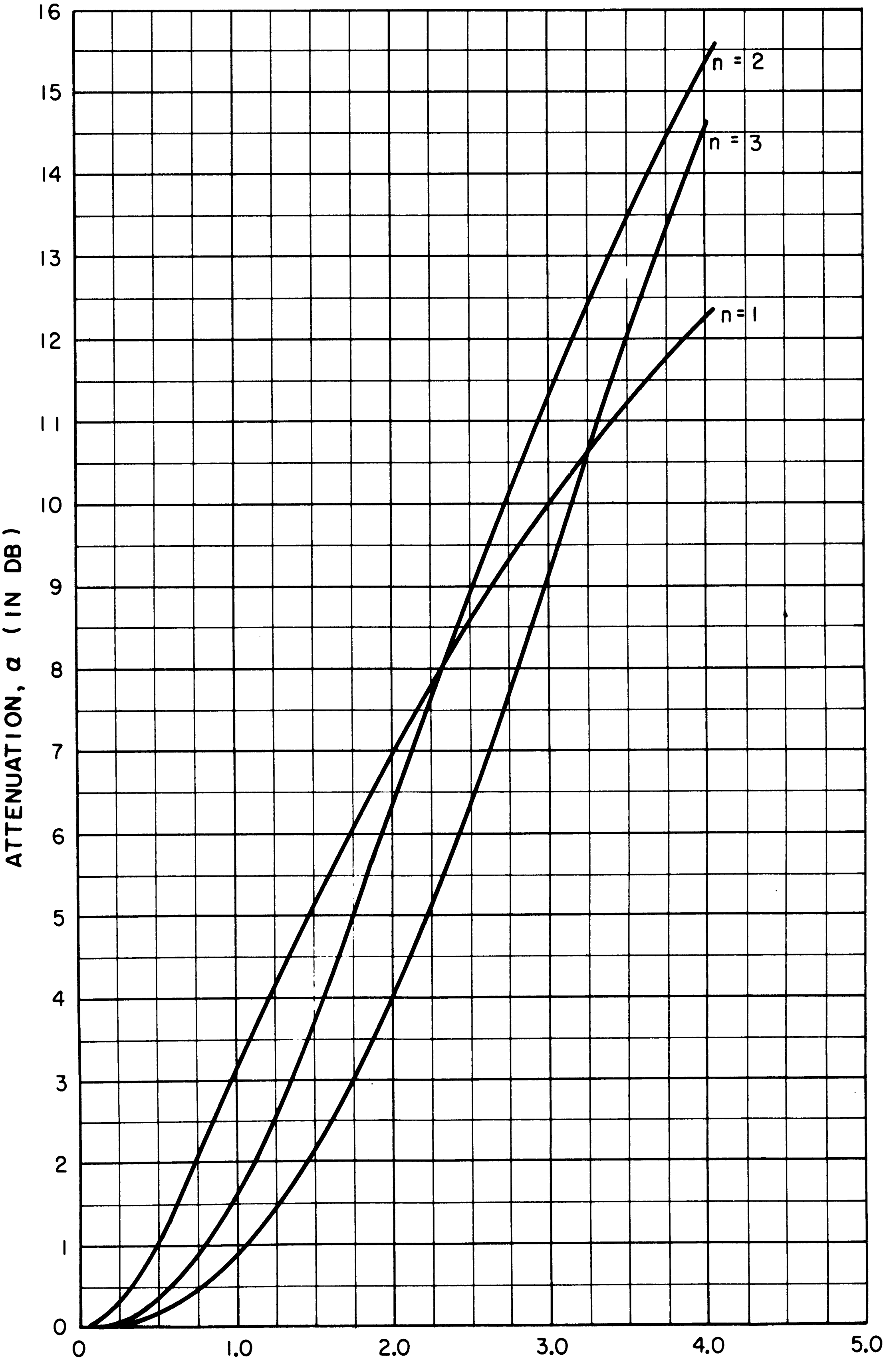

Fig. 2. Plot of filter attenuation vs. radian frequency.

fer function (n = 3) so that the 3-db-down point occurs at $\omega \cong 1.0$ radian per second, we can divide the pole positions by 1.75567. We then obtain as the pole positions for n = 3: p = - 1.32267, p = - 1.04741 ± j0.99927. The transfer function to be synthesized is therefore

$$F(p) = \frac{2.7718}{p^3 + 3.4175\, p^2 + 4.8664\, p + 2.7718} \tag{5}$$

Before proceeding further it is of interest to examine some aspects of direction finding which are pertinent to this problem.

3. DIRECTION FINDING CONSIDERATIONS

One of the important factors affecting the ability of the human operator in using a **spinning-goniometer** radio **direction-finder** with accuracy is the shape of the propeller pattern on the cathode ray tube. If the ends of the pattern are rounded off, it becomes increasingly difficult to make an accurate bearing determination. Such rounding off of the pattern is called "blur" and may be caused by a number of factors. Before second detection a quantitative measure for blur has been derived.[1] The expression is

$$\text{Percent blur} = 100 \left(\frac{1 - \gamma_r}{1 + \gamma_r}\right) \quad 0 \leqq \gamma_r \leqq 1 \tag{6}$$

where:

γ_r is the ratio of the magnitudes of the two sideband voltages which were originally of equal magnitude when obtained from the spinning goniometer.

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[1] S. F. George, "Direction Finder Bandwidth Requirements," NRL Rpt. No. R-3182.

Let us now determine the blur that would occur if the signal were tuned to the 3-db-down point of the proposed linear phase filter. It will be assumed that the goniometer spin rate is 30 revolutions per second and the filter bandwidth is 500 cycles per second. This spin rate causes a separation of 60 cycles per second between the two sidebands. For n = 3 of the normalized curve of attenuation vs. frequency (Fig. 2), the 3-db-down point falls at $\omega = 1.756$ radians per second which corresponds to 250 cycles per second of the filter. Hence, a frequency separation of 60 cycles per second corresponds to $60/250 \times 1.756 = 0.421$ radians per second of the normalized frequency scale used. Assuming that the signal is tuned to the 3-db-down point, it can be seen from the curve that the difference in sideband amplitudes originally equal in magnitude will now be approximately 1.55 db. The quantity γ_r can be calculated as follows:

$$\left\{\begin{array}{l}\text{Difference in Sideband} \\ \text{Voltage Magnitudes (in db)}\end{array}\right\} = \Delta E_s = 20 \log_{10} \frac{1}{\gamma_r} \qquad (7)$$

With $\Delta E_s = 1.55$ db, $\gamma_r = 0.837$. From Eq. (6) it is found that the percent blur is 8.87.

When the signal is properly tuned it can be shown that the blur is less than 0.5%. Hence, it is rather easy to determine whether or not the signal is properly tuned in by watching the shape of the propeller pattern. As a consequence of the above the blur caused by the linear phase filter will not affect the bearing accuracy, when the signal is properly tuned.

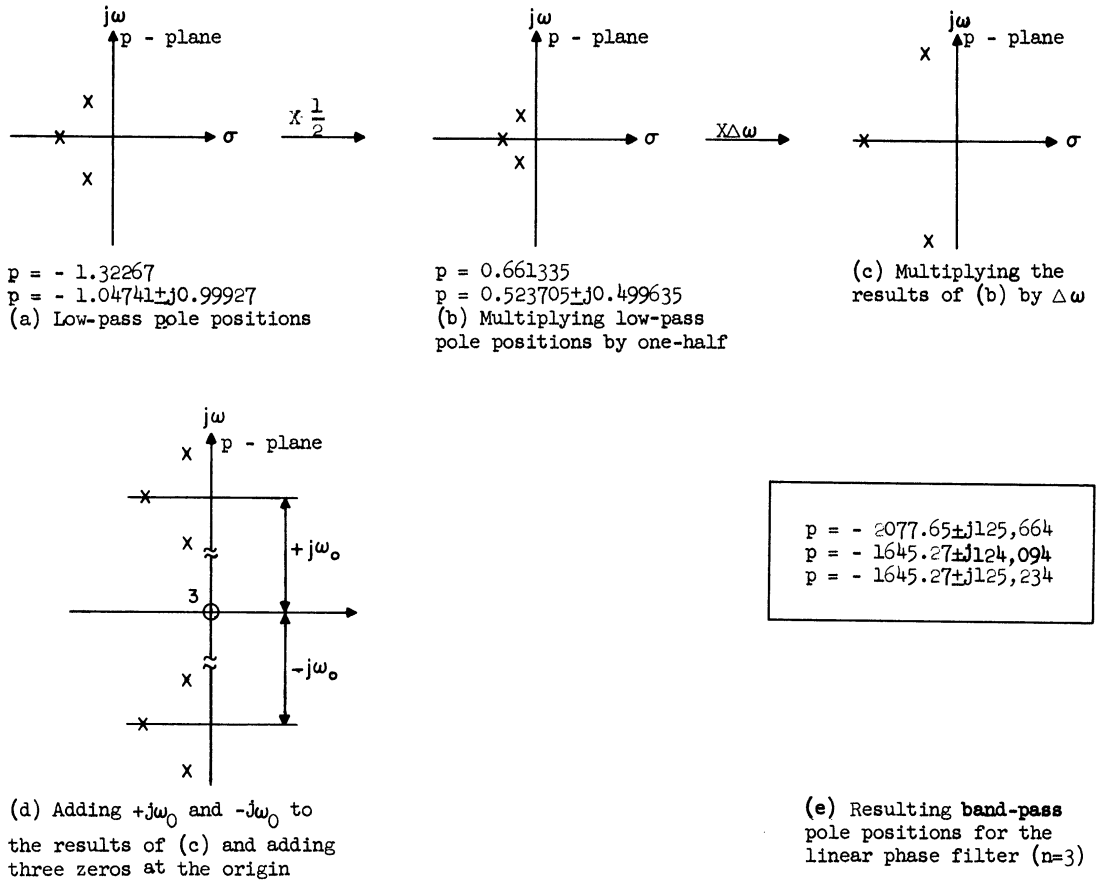
4. TRANSFORMATION OF A LOW-PASS FUNCTION TO A BAND-PASS FUNCTION

In the preceding section a low-pass function was discussed; however, a bandpass function is desired. The usual transformation used for this purpose is obtained by replacing the complex variable p of the transfer function by $\frac{p^2 + \omega_o^2}{\Delta\omega p}$, where ω_o is the midband radian frequency of the filter and $\Delta\omega$ is its radian bandwidth. If one knows each pole position (as we do in this case), he can transform each pole individually; however, in each case a quadratic equation must be solved. Examples of this are shown in Appendix A. It is shown there that if the ratio of the bandwidth to the center frequency is small enough, a simplification can be made. If the required conditions are met (see Appendix A), the following approximations are valid for transforming low-pass pole positions to the band-pass case:

Low-pass Pole Positions		Band-pass Pole Positions (Approximate)	
$p = -a$	$\rightarrow$	$p = -\frac{(\Delta\omega)a}{2} \pm j\omega_o$	(8)
$p = -a + jb$	$\rightarrow$	$p = -\Delta\omega(\frac{a + jb}{2}) \pm j\omega_o$	(9)
$p = -a - jb$	$\rightarrow$	$p = -\Delta\omega(\frac{a - jb}{2}) \pm j\omega_o$	(10)

For each pole transformed to the band-pass case a zero must be added at the origin of the complex plane. The process can be divided into four steps, which are illustrated in Fig. 3.

Formulas (8), (9), and (10) are implemented as shown above in order to calculate the pole positions of the band-pass filter. However, a word of caution is in order. Had the ratio of $\frac{\Delta\omega}{\omega_o}$ been greater than approximately 0.05, the effects of the complex conjugate poles and the



(a) Low-pass pole positions

(b) Multiplying low-pass pole positions by one-half

(c) Multiplying the results of (b) by $\Delta\omega$

(d) Adding $+j\omega_0$ and $-j\omega_0$ to the results of (c) and adding three zeros at the origin

(e) Resulting band-pass pole positions for the linear phase filter (n=3)

Fig. 3. Low-pass to band-pass transformation procedure.

three zeros at the origin would have caused the phase characteristics to be adversely affected. In other words, the phase characteristics of the low-pass function are <u>not</u> preserved under the given low-pass to band-pass transformation; however, in the narrowband case they are sufficiently close to the desired characteristics that the transformation can be applied.

5. REALIZATION OF THE BAND-PASS FUNCTION

For realization of the transfer function it was decided to realize each complex conjugate pole pair along with an associated zero at the origin of the complex plane as a separately tuned interstage between pentode tubes acting as current sources. As is well known, the overall transfer function for such a combination is the product of the transfer functions of each stage. The circuit used for each plate circuit is shown in Fig. 4(a). The transfer function, (e/i), for this circuit is:

$$\left(\frac{e}{i}\right) = \frac{\left(p + \frac{R_s}{L}\right)}{C\left[p^2 + \left(\frac{1}{RC} + \frac{R_s}{L}\right)p + \frac{1}{LC}\left(\frac{R_s}{R_p} + 1\right)\right]} \tag{11}$$

It is possible to derive expressions for the element values in

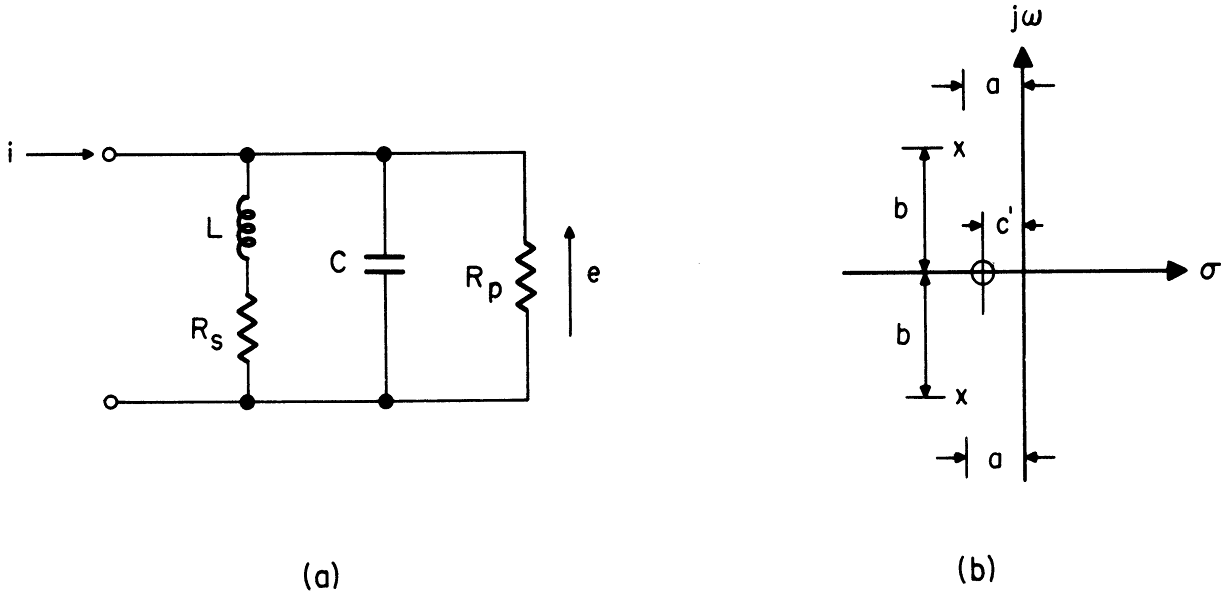


Fig. 4. Pentode plate circuit configuration.

terms of the pole-zero positions shown in Fig. 4(b) and the inductance, L (see Appendix B). Certain realizability conditions have to be observed:

$$\begin{aligned} &(1) \quad 2a > c' \\ &(2) \quad a^2 + b^2 > c'(2a - c') \end{aligned} \tag{12}$$

If the inequalities of Eqs. 12 are satisfied, then the desired relations are:

$$R_s = c' L \tag{13}$$

$$C = \frac{1}{L(a^2 + b^2 + c'^2 - 2ac')} \tag{14}$$

$$R_p = \frac{L(a^2 + b^2 + c'^2 - 2ac')}{2a - c'} \tag{15}$$

It must be noted here that R_s is not merely an ohmic resistance, but it includes any loss component which the inductor, L, itself may have. Hence, it is possible that the zero location indicated in Eq. 11 cannot be realized because the Q of the inductor is not high enough, since the closer the zero is to the origin the higher the Q required. It was for this reason that in the design equations (13), (14), and (15) emphasis was placed upon picking L first instead of, say, R_p or C.

The proper choice of zero locations, $p = -c'$, in each circuit can compensate for the adverse effect on the phase of the complex conjugate poles clustered around $p = -j\omega_o$. The desired phase for the filter (n = 3) can be calculated from $\theta = -K\omega$, where K = 1; at the 3-db-down point, $\theta = -(1)(1.75567) = -1.75567$ radians $= -100.5925^\circ$. From the pole-zero plot, the phase-shift of the function was calculated for 19,750, 20,000, and 20,250 cycles per second and was found to be 100.688°, 1.224°, and -98.231°, respectively. Hence, the error, θ_e, in degrees from the ideal was 0.096°, 1.224°, and 2.2618°, respectively. Moving the zero location of each plate circuit to the left by the proper amount, exactly compensates for the error, θ_e, at the midband frequency. We shall let the zero location of each circuit be the same. It turns out that for exact compensation at 20,000 cycles per second, $c' = 894.71$. The calculated error at 19,750 and 20,250 cycles per second is now 1.14° and 1.15°, respectively. With this value of c' the Q of the inductance must be 140.45.

The circuit used is shown in Fig. 5. The first stage is a converter (6BE6) which changes the input frequency from 455 kc to 20 kc. The plate circuit of the 6BE6 is tuned to 20,000 cps. The voltage divider following the tuned circuit was inserted to prevent the overall gain from being too high and causing saturation effects in the final stages.

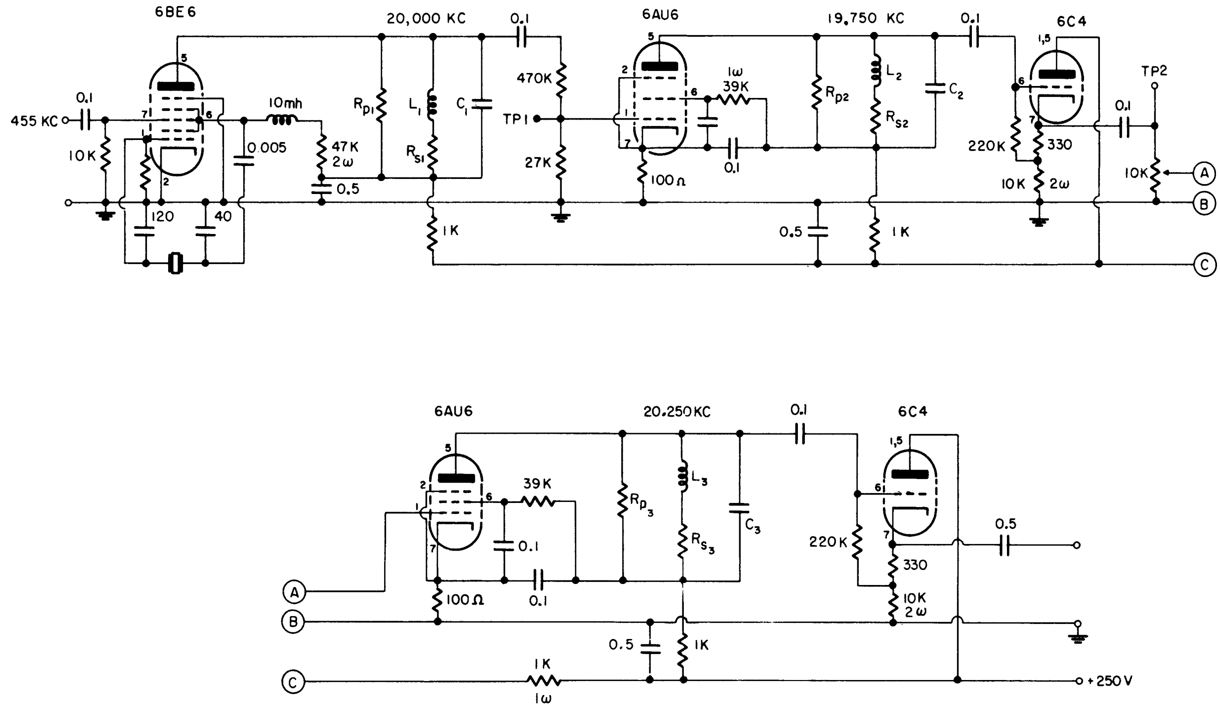


Fig. 5. Circuit diagram of linear phase **band-pass filter.**

TABLE II

SUMMARY OF ELEMENT VALUES FOR

EACH TUNED PLATE CIRCUIT

f_1 = 20,000 cps	f_2 = 19,750 cps	f_3 = 20,250 cps
L_1 = 10mh @ Q = 140.45	L_2 = 10mh @ Q = 140.45	L_3 = 10mh @ Q = 140.45
C_1 = 0.0063320 μf	C_2 = 0.0064936 μf	C_3 = 0.0061770 μf
R_{p_1} = 48,436 Ω	R_{p_2} = 64,278 Ω	R_{p_3} = 67,572 Ω

The plate circuit of the second stage (6AU6) was tuned to 19,750 cps. This circuit was followed by a cathode follower (6C4) in order to minimize the capacitance effects of the potentiometer. If the potentiometer had been placed directly across the tank circuit, its resonant frequency would have been changed from approximately 0 to 25 cps, depending upon the position of the potentiometer. The fourth stage (6AU6) was tuned to 20,250 cps. Isolation between the output and the last tuned circuit was provided by a cathode follower (6C4). Greater than normal decoupling was used in the plate voltage supply line to prevent regeneration effects.

A word is now in order concerning alignment procedures. Each of the inductors obtained had a $Q > 140.45$; hence, it was necessary to lower the Q of each to the proper value by the addition of a small series resistance, R_s. This resistance consisted of a few turns of No. 30 B. & S., constantan wire wound on a 3/16" diameter phenolic coil form. Through the use of a Boonton Type 260-A Q-meter each inductor was individually adjusted to the proper Q at its operating frequency. The next step was to adjust each tuned circuit to the proper center frequency and bandwidth. Throughout the alignment procedure an HP-225B electronic counter was used to measure frequency. The test setup is shown in Fig. 6.

With the converter operating, a test signal of 20,000 cps was applied to the signal grid of the 6BE6. The tuned circuits of the second (6AU6) and third tuned stages were shunted by 560-ohm, one-watt resistors. An oscilloscope was connected to the filter output for monitoring purposes. The capacitor C_1 was placed across L_1 and adjusted until a maximum deflection was obtained on the oscilloscope at exactly 20,000 kc. Then R_{p_1} was adjusted to give the proper bandwidth. This was accomplished in the following way. First, the 3-db-down frequencies of each tuned circuit were calculated. These are given in Table III for convenience.

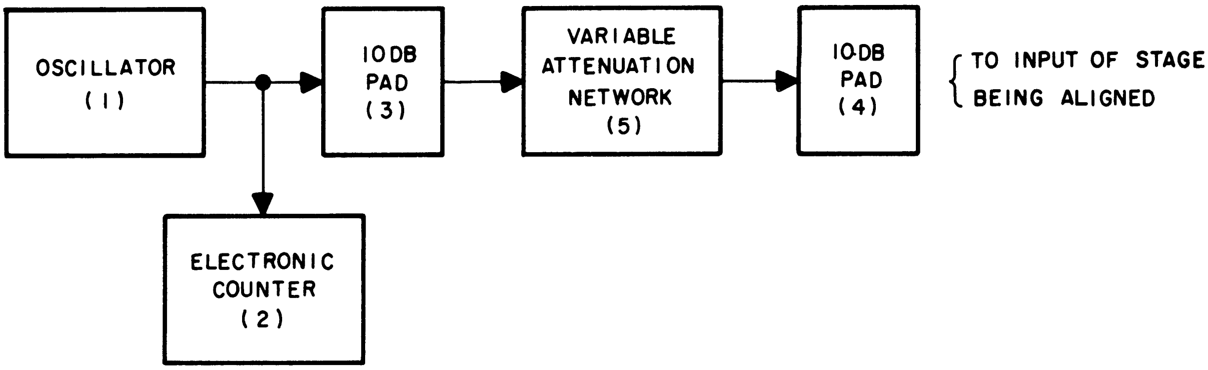


Fig. 6. Test setup for the alignment of each tuned plate.

The method of calculation of these frequencies is shown in Appendix C. After introducing 3 db of attenuution into the input signal channel by means of the variable attenuator, the oscillator is tuned to the resonant frequency of the circuit and the deflection on the oscilloscope noted; then the 3 db of attenuation is removed and the oscillator frequency is changed until the same oscilloscope deflection as noted before is obtained.

TABLE III

TABULATION OF 3-db-DOWN FREQUENCIES,

FOR EACH TUNED CIRCUIT

f_o = 20,000 cps	f_o = 19,750 cps	f_o = 20,250 cps
20,333 cps	20,014 cps	20,514 cps
19,672 cps	19,490 cps	19,990 cps

The frequencies at which this occurs are the 3 db down frequencies. The shunt resistor, R_{p_1}, is changed until the frequencies listed in Table III are obtained. The first circuit is then properly tuned. The next step is to remove the 560-ohm shunting resistor from the second tuned stage, shunt the first tuned circuit with this same resistor, and proceed in the same manner as described. Finally, we align the last stage. By carefully making these measurements and adjustments, we obtain an overall transfer function which will be correct without any further adjustment.

6. MEASUREMENTS AND CONCLUSIONS

The phase shift of the filter was measured and the results are shown in Fig. 7. A Hewlett-Packard Model 524C electronic counter with a plug-in Model 526B time-interval unit was used to measure this phase shift. By measuring the time interval between the positive-going input voltage and the positive-going output voltage and by knowing the frequency of the input wave, we can calculate the phase shift of the filter from: $\theta = 360Tf$; where θ is the phase shift in degrees, T the measured time interval in seconds, and f the frequency in cycles per second. With the electronic counter an accuracy of $\pm$ 1 degree can be obtained. The measured curve clearly shows that the original phase specification has been met. Figure 8 shows the measured relative voltage gain of the filter.

An interesting observation can be made regarding whether or not the direction-finding receiver has been properly tuned to the desired signal. If the output of the filter is observed on an oscilloscope, the null of the double-sideband signal is distinct when the signal is properly tuned; it is not distinct if the receiver has been mistuned.

In conclusion it can be stated that a linear-phase band-pass filter meeting the desired specifications can be designed and constructed using the methods contained in this report.

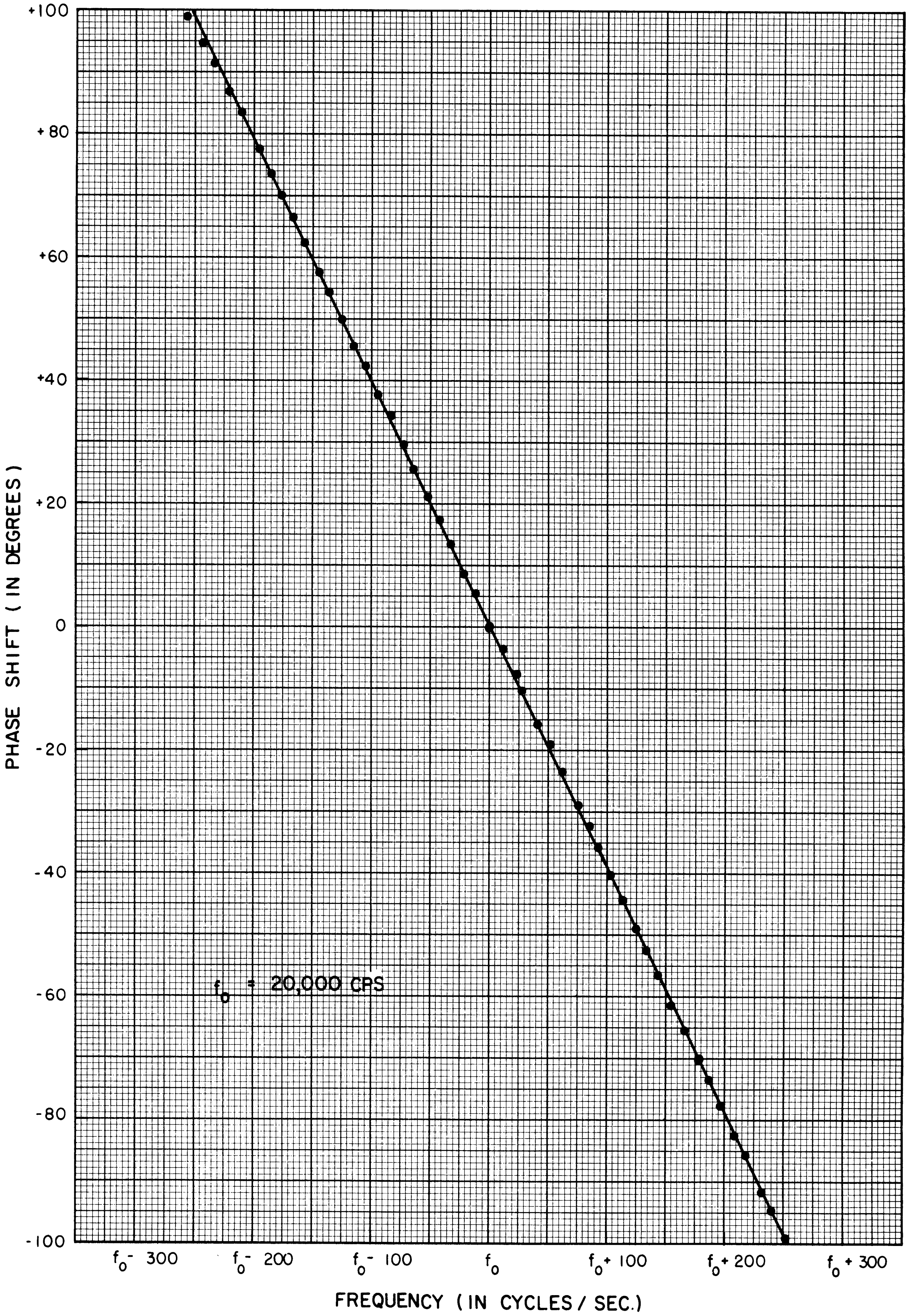


Fig. 7. Measured phase shift of linear phase filter.

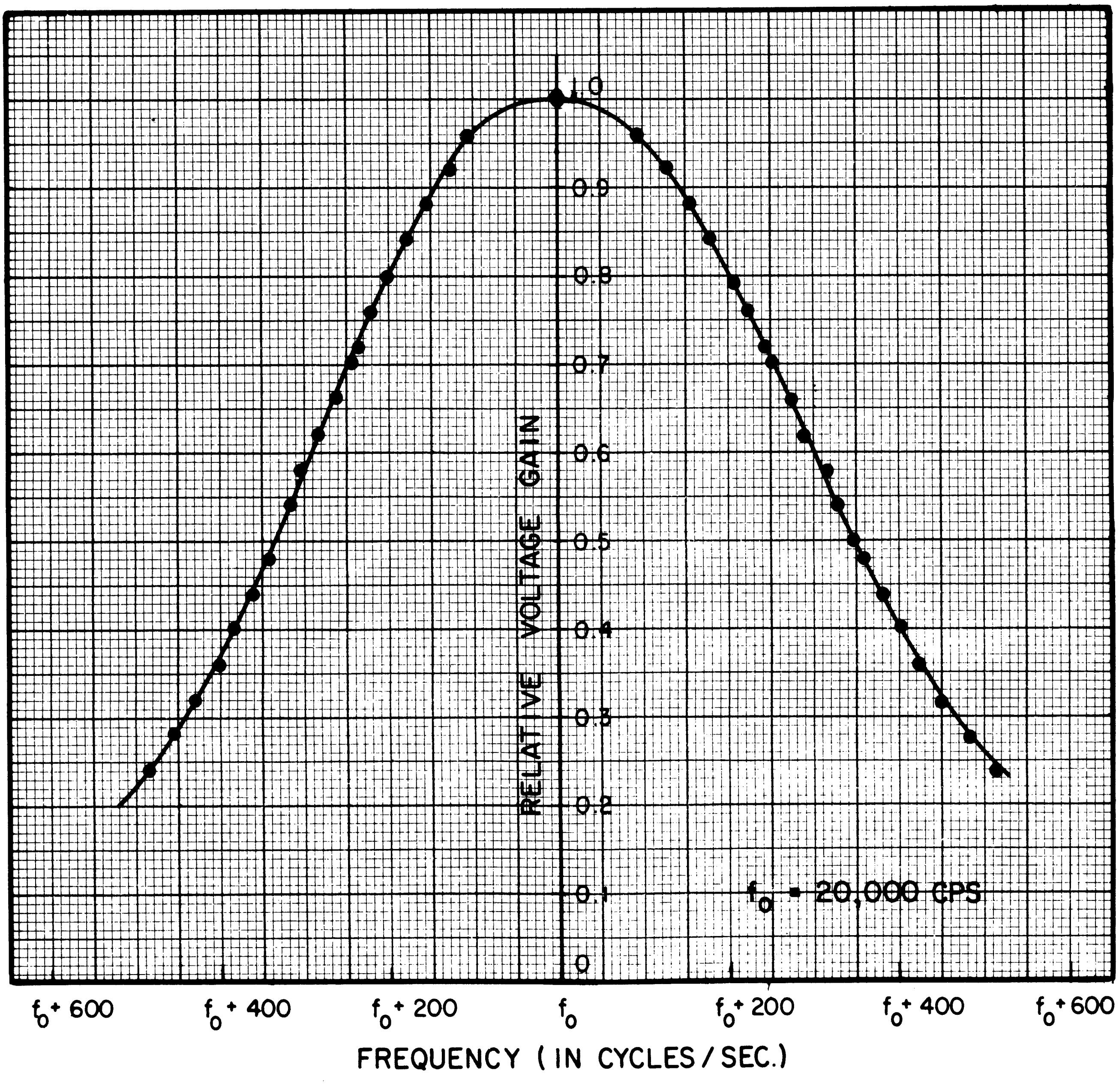


Fig. 8. Measured relative voltage gain of linear phase filter.

APPENDIX A

CALCULATION OF BAND-PASS POLE POSITIONS FROM LOW-PASS POLE POSITIONS

Two cases are to be considered: (1) real low-pass poles; and (2) pairs of low-pass, complex conjugate poles. In order to make the transformation $\frac{p^2 + \omega_o^2}{\Delta\omega p}$ is substituted for p in the low-pass transfer. function. This will be carried out below for cases (1) and (2), and an approximate transformation will be derived.

Case 1:

Transfer function of the form $F(p) = \frac{K}{p + a}$ (real pole). Substituting $\frac{p^2 + \omega_o^2}{(\Delta\omega)p}$ for p, we obtain:[1]

$$F(p) = \frac{K}{\frac{p^2 + \omega_o^2}{(\Delta\omega)p} + a} = \frac{(\Delta\omega)K\,p}{p^2 + (\Delta\omega)ap + \omega_o^2} \qquad (A.1)$$

The solution for the pole positions is obtained by setting the denominator equal to zero.

Therefore,

$$p^2 + (\Delta\omega)ap + \omega_o^2 = 0$$

$$p = \frac{-(\Delta\omega)a}{2} \pm \sqrt{\left(\frac{\Delta\omega a}{2}\right)^2 - \omega_o^2} = \frac{-(\Delta\omega)a}{2} \pm j\omega_o\sqrt{1 - \left(\frac{\Delta\omega a}{2\omega_o}\right)^2} \qquad (A.2)$$

If now $\frac{(\Delta\omega)a}{2\omega_o} << 1$, then

[1]Note the zero introduced at the origin of the complex plane by the transformation.

$$p = \frac{-(\Delta\omega)a}{2} \pm j\omega_o \quad \text{(Approximation)} \qquad (A.3)$$

For this approximation to be true within 1%:

$$1.00 \geq \sqrt{1 - \left(\frac{\Delta\omega}{\omega_o}\right)^2 \left(\frac{a}{2}\right)^2} \geq 0.99$$

$$1.00 \geq 1 - \left(\frac{\Delta\omega}{\omega_o}\right)^2 \left(\frac{a}{2}\right)^2 \geq 0.9801$$

$$0 \leq \left(\frac{\Delta\omega}{\omega_o}\right)^2 \left(\frac{a}{2}\right)^2 \leq 0.0199$$

$$\left(\frac{\Delta\omega}{\omega_o}\right) \left(\frac{a}{2}\right) \leq 0.1411$$

or

$$\frac{\Delta\omega}{\omega_o} = \frac{1}{3.544a} \quad \text{for 1\% accuracy of the approximation} \qquad (A.4)$$

Case 2:

Transfer function of the form, $F(p) = \dfrac{K}{(p - p_1)(p - \bar{p}_1)}$

where:

$p_1 = -a + jb$, and

$\bar{p}_1 = -a - jb$.

Substitution of $\dfrac{p^2 + \omega_o^2}{(\Delta\omega)p}$ for p gives:[1]

[1]Again note the zero introduced at the origin of the complex plane for each of the poles by the transformation.

$$F(p) = \frac{K}{\left[\frac{p^2 + \omega_o^2}{(\Delta\omega)p} - p_1\right]\left[\frac{p^2 + \omega_o^2}{(\Delta\omega)p} - \bar{p}_1\right]}$$

$$= \frac{(\Delta\omega)^2 p^2 K}{[p^2 - (\Delta\omega)p_1 + \omega_o^2][p^2 - (\Delta\omega)\bar{p}_1 + \omega_o^2]} \qquad (A.5)$$

Finding the zeros of each factor of the denominator, the following is obtained:

$$p = -\Delta\omega\left(\frac{a + jb}{2}\right) \pm j\omega_o \sqrt{1 - \left[\frac{\Delta\omega}{\omega_o} \cdot \frac{a + jb}{2}\right]^2} \qquad (A.6)$$

$$p = -\Delta\omega\left(\frac{a - jb}{2}\right) \pm j\omega_o \sqrt{1 - \left[\frac{\Delta\omega}{\omega_o} \cdot \frac{a - jb}{2}\right]^2} \qquad (A.7)$$

If $\left|\frac{\Delta\omega}{\omega_o} \cdot \frac{2 \pm jb}{2}\right|^2 << 1$, then (A.6) and (A.7) become:

$$p = -\Delta\omega\left(\frac{a + jb}{2}\right) \pm j\omega_o, \quad p = -\Delta\omega\left(\frac{a - jb}{2}\right) \pm j\omega_o \qquad (A.8), (A.9)$$

Approximations

For these approximations to be true within 1%:

$$1.00 \geqq \left|\sqrt{1 - \left[\frac{\Delta\omega}{\omega_o} \cdot \frac{a \pm jb}{2}\right]^2}\right| \geqq 0.99$$

$$1.00 \geqq \left|1 - \left(\frac{\Delta\omega}{\omega_o}\right)^2 \left(\frac{a \pm jb}{2}\right)^2\right| \geqq 0.980$$

Since $\left|1 - \left(\frac{\Delta\omega}{\omega_o}\right)^2 \left(\frac{a \pm jb}{2}\right)^2\right| \geqq \left|1 - \left(\frac{\Delta\omega}{\omega_o}\right)^2 \left|\left(\frac{a \pm jb}{2}\right)^2\right|\right|$,

the above inequalities will be true if

$$1 - \left(\frac{\Delta\omega}{\omega_o}\right)^2 \left|\frac{a \pm jb}{2}\right|^2 \geqq 0.9801$$

$$\left(\frac{\Delta\omega}{\omega_o}\right)^2 \left|\frac{a \pm jb}{2}\right|^2 \geqq 0.0199$$

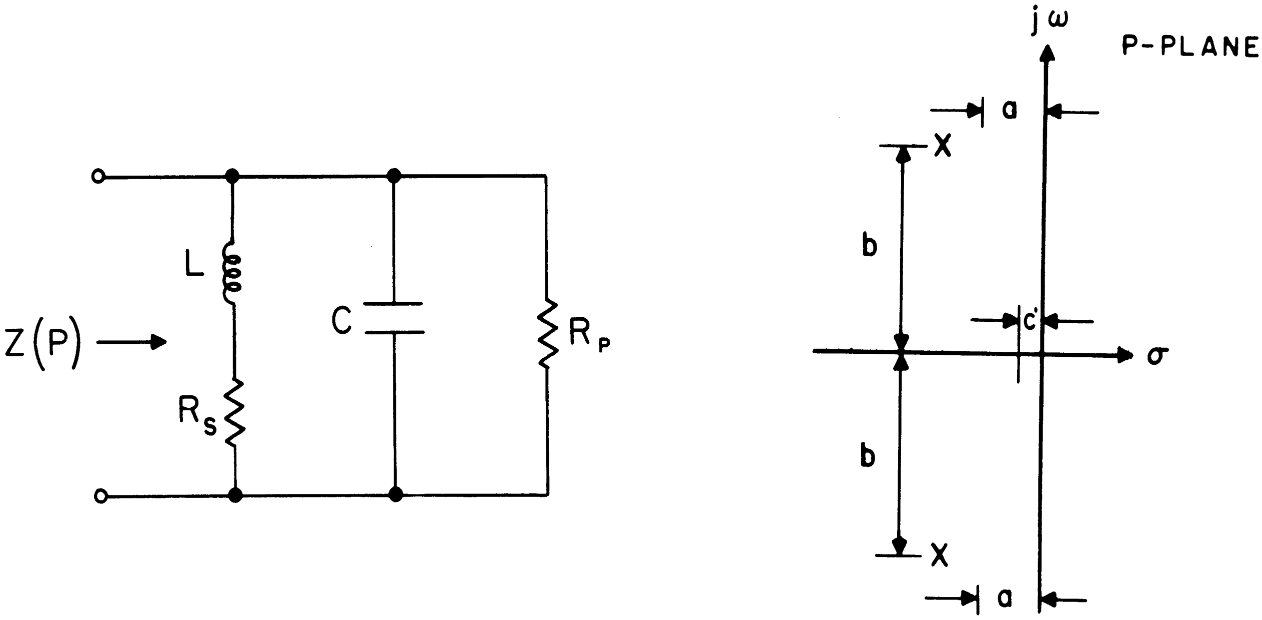
$$\left(\frac{\Delta\omega}{\omega_o}\right) \left|\frac{a \pm jb}{2}\right| \leqq 0.1411$$

or

$$\boxed{\frac{\Delta\omega}{\omega_o} \leqq \frac{1}{3.544\sqrt{a^2 + b^2}} \quad \text{for 1\% accuracy of the approximation}} \qquad (A.10)$$

APPENDIX B

DERIVATION OF ELEMENT VALUES OF PARALLEL RLC NETWORK WITH POLE-ZERO LOCATIONS AND THE INDUCTOR, L, SPECIFIED



(a) Circuit. **(b) Pole-zero locations.**

Fig. B.1. Parallel RLC netowrk.

$$Z(p) = \frac{\left(p + \frac{R_s}{L}\right)}{C\left[p^2 + \left(\frac{1}{R_pC} + \frac{R_s}{L}\right)p + \frac{1}{LC}\left(\frac{R_s}{R_p} + 1\right)\right]} \tag{B.1}$$

This equation can be put into the form:

$$Z(p) = \frac{p + c'}{C[p^2 + 2\alpha p + \omega_o^2]} \tag{B.2}$$

where:

$$2\alpha = \frac{1}{R_pC} + \frac{R_s}{L}, \quad c' = \frac{R_s}{L}, \text{ and}$$

$$\omega_o^2 = \frac{1}{LC}\left(\frac{R_s}{R_p} + 1\right) \tag{B.3}$$

Solving for the pole positions, one obtains:

$$p = -\alpha \pm j\sqrt{\omega_o^2 - \alpha^2} = -\alpha \pm jb \qquad (B.4)$$

Therefore

$$a = \frac{1}{2}\left[\frac{1}{R_pC} + \frac{R_s}{L}\right];\ b = \sqrt{\omega_o^2 - \alpha^2} = \sqrt{\frac{1}{LC}\left(\frac{R_s}{R_p} + 1\right) - \alpha^2} \qquad (B.5)$$

Since

$$c' = \frac{R_s}{L},\ \text{by definition,}\ \boxed{R_s = c'\,L} \qquad (B.6)$$

If Eqs. B.5 and B.6 are used now, R_p and C can be solved for in terms of a, b, c', and L:

$$a^2 + b^2 = \frac{1}{LC}\left(\frac{R_s}{R_p} + 1\right) = c'\left(\frac{1}{R_pC}\right) + \frac{1}{LC} = c'\,(2a - c') + \frac{1}{LC}$$

$$\frac{1}{LC} = a^2 + b^2 + c'^2 - 2a\,c'$$

$$\boxed{C = \frac{1}{L[a^2 + b^2 + c'^2 - 2ac']}} \qquad (B.7)$$

$$2a = \frac{1}{R_pC} + c'$$

$$R_p = \frac{1}{C[2a - c']}$$

$$\boxed{R_p = \frac{L[a^2 + b^2 + c'^2 - 2ac']}{2a - c'}} \qquad (B.8)$$

APPENDIX C

METHOD OF CALCULATION OF 3-db-DOWN FREQUENCIES FOR EACH TUNED CIRCUIT

It is found that

$$Z(p) = \frac{p + \frac{R_s}{L}}{C[p^2 + (\frac{1}{R_pC} + \frac{R_s}{L})p + \frac{1}{LC}(\frac{R_s}{R_p} + 1)]}$$

$$Z(p) = \frac{p + c'}{C[p^2 + 2\alpha\, p + \omega_o^2]}$$

where:

$$c' = \frac{R_s}{L},\ \alpha = \frac{1}{2}[\frac{1}{R_pC} + \frac{R_s}{L}],\ \text{and}\ \omega_o^2 = \frac{1}{LC}\ [\frac{R_s}{R_p} + 1]$$

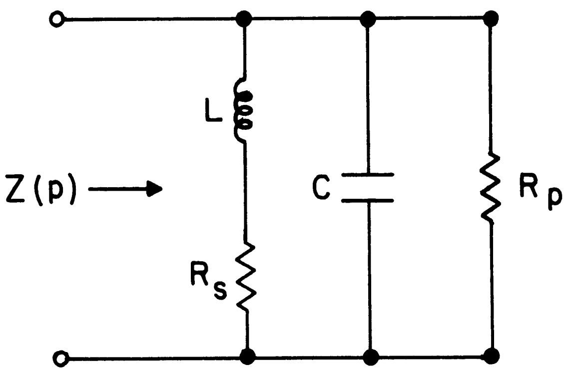


Fig. C.1. Tuned circuit.

For the purposes of this problem it will be assumed that

$$\frac{R_s}{R_p} << 1 \text{ and } \frac{R_s}{L} << \omega_r = \frac{1}{\sqrt{LC}}$$

Therefore

$$Z(p) \cong \frac{p}{C[p^2 + 2\alpha p + \omega_r^2]} \tag{C.1}$$

At resonance ($\omega = \omega_r$):

$$|Z(j\omega_r)| \cong |\frac{j\omega_r}{C[2\alpha\ j\omega_r]}| = \frac{1}{2\alpha C} \tag{C.2}$$