ENGINEERING RESEARCH INSTITUTE
UNIVERSITY OF MICHIGAN
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LAPLACE'S EQUATION FOR THE STRIP

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Project 2137

DETROIT ORDNANCE DISTRICT
DEPARTMENT OF THE ARMY
CONTRACT NO. DA-20-018-ORD-12916

October, 1954

ABSTRACT

A set of key functions that are harmonic in the semi-infinite strip are defined by their Fourier series. Closed forms are obtained for these functions. The solutions to Laplace's equation for the strip with boundary conditions of Dirichlet, Neumann, or mixed type are expressed in terms of these functions by using a suitable finite Fourier transformation.

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TABLE OF CONTENTS

	Page
ABSTRACT	ii
CHAPTER I	
INTRODUCTION	1
1.1 General Introduction	1
1.2 Finite Transforms Defined	1
1.3 Convolution for Finite Transforms	2
CHAPTER II	
SET OF KEY FUNCTIONS	6
2.1 Definition of the Set	6
2.2 Closed Forms for the Key Functions	8
2.3 Order Properties of Key Functions	11
CHAPTER III	
HARMONIC FUNCTIONS FOR THE SEMI-INFINITE STRIP WITH VARIOUS BOUNDARY CONDITIONS	14
3.1 Nonhomogeneous Condition on Bottom	14
3.2 Verification of Solutions	18
3.3 Infinite Sine and Cosine Transforms	23
3.4 Nonhomogeneous Condition on Side, Case I	25
3.5 Nonhomogeneous Condition on Side, Case II	30
3.6 Remarks	34
BIBLIOGRAPHY	36
TABLES	
I. Finite Sine Transforms	7
II. Finite Cosine Transforms	7

LAPLACE'S EQUATION FOR THE STRIP

CHAPTER I

INTRODUCTION

1.1. General Introduction. This report is concerned with the solution of special problems in elliptic partial differential equations. The equations are solved for the semi-infinite strip in a manner similar to that used by Jacobson [6]* when the region was a rectangle. The solutions to Laplace's equation with boundary conditions of the three types, Dirichlet, Neumann, or mixed, are obtained by making use of the finite Fourier transformations in closed forms in terms of a limited number of prescribed functions.

1.2. Finite Transforms Defined. The finite Fourier transforms may be defined for a bounded integrable function by the following formulae, as is done by Churchill in [2].

$$S\{F(x)\} = f_s(n) = \int_0^\pi F(x)\sin nx\,dx \qquad \text{for } n = 1, 2, \cdots \tag{1}$$

$$C\{F(x)\} = f_c(n) = \int_0^\pi F(x)\cos nx\,dx \qquad \text{for } n = 0, 1, \cdots \tag{2}$$

Since the transforms are, except for constant factors, just the coefficients in the Fourier sine or cosine series respectively, the inverse transforms are easily written in the series forms

$$F(x) = (2/\pi)\sum_{n=1}^{\infty} f_s(n)\sin nx \tag{3}$$

$$F(x) = (1/\pi)f_c(0) + (2/\pi)\sum_{n=1}^{\infty} f_c(n)\cos nx \quad . \tag{4}$$

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* Numbers in square brackets refer to the bibliography.

The conditions under which Equation 3 or 4 converge are numerous (see [6]). One that will be used on occasions in the following work may be stated: If the series $\sum|f_s(n)|$ or $\sum|f_c(n)|$ converge, then Equation 3 or 4 converges uniformly to a continuous function in the interval $(0,\pi)$. It should be noted that the conditions that must be placed on a function in order that the Fourier series formed from its transform represent the function are much more restrictive than those that the function must satisfy to have a finite transform. This will be illustrated in Chapter II, Section 2.3.

The operational properties involving the transforms are easily found by integration by parts. They are stated here as given in [2]. If $F(x)$ is continuous and $F'(x)$ sectionally continuous for $0 \leq x \leq \pi$, then

$$\begin{aligned} S\{F'(x)\} &= -n\,C\{F(x)\} \quad \text{for } n = 1, 2, \cdots \\ C\{F'(x)\} &= n\,S\{F(x)\} - F(0) + (-1)^n F(\pi) \quad \text{for } n = 0, 1, \cdots \end{aligned} \tag{5}$$

If $F(x)$ and $F'(x)$ are continuous and $F''(x)$ is sectionally continuous for $0 \leq x \leq \pi$, then

$$\begin{aligned} S\{F''(x)\} &= -n^2 S\{F(x)\} + n\left[F(0) - (-1)^n F(\pi)\right] \\ C\{F''(x)\} &= -n^2 C\{F(x)\} - F'(0) + (-1)^n F'(\pi) \;. \end{aligned} \tag{6}$$

The properties in Equation 5 are useful in obtaining transforms, as will be seen in Chapter II, Section 2.2, while the formulae in Equation 6 are the ones that make the application of the finite transforms to boundary value problems in partial differential equations useful. In the applications of the finite transforms that will be made, the transforms of two functions, one linear and the other quadratic, will be needed. These may be found in [2], pp. 277-8, and are

$$\begin{aligned} S\left\{\frac{\pi - x}{\pi}\right\} &= 1/n \quad \text{for } n = 1, 2, \cdots \\ C\{q(x)\} &= 0 \quad \text{if } n = 0 \\ &= 1/n^2 \quad \text{for } n = 1, 2 \cdots \;, \end{aligned} \tag{7}$$

where $q(x) = (\pi - x)^2/2\pi - \pi/6$.

1.3. Convolution for Finite Transforms. In the application of any integral transform, the idea of the convolution is an important one. The reason for this is that very frequently it is necessary to determine a function whose transform is the product of known transforms. This function is usually expressed as an integral involving the functions whose transforms make up the product. As in the case of the Laplace transform, the asterisk, $*$, will be used to denote convolution. The fact that there are two different types of finite Fourier transforms makes it necessary to consider four possible forms of the convolution. They are given, together with the notation that will be used to distinguish them, in their generalized form. The notation is slightly different from that used by Jacobson [6] in introducing the generalized convolution, for the sake of simplicity in the applications.

<u>Theorem:</u> Let $F(x,y)$ be a bounded integrable function over the square, $Q = \{(x,y) \mid 0 < x < \pi,\ 0 < y < \pi\}$, and

$$I_1 = \int_0^x F(x - y,y)dy \qquad I_2 = \int_x^\pi F(y - x,y)dy$$

$$I_3 = \int_0^{\pi-x} F(x + y,y)dy \qquad I_4 = \int_{\pi-x}^{\pi} F(2\pi - x - y,y)dy \ .$$

Then, if $S\{C\{F(x,y)\}\}$ denotes the cosine transform with respect to y, followed by the sine transform with respect to x, the following formulae are correct

$$S\{C\{F(x,y)\}\} = (1/2)S\{F*sc(x)\} \ , \ F*sc(x) = I_1 - I_2 + I_3 - I_4 \ .$$

In an analogous manner, the three remaining convolutions are

$$S\{S\{F(x,y)\}\} = (1/2)C\{F*ss(x)\} \ , \ F*ss(x) = -I_1 + I_2 + I_3 - I_4$$

$$C\{C\{F(x,y)\}\} = (1/2)C\{F*cc(x)\} \ , \ F*cc(x) = I_1 + I_2 + I_3 + I_4$$

$$C\{S\{F(x,y)\}\} = (1/2)S\{F*cs(x)\} \ , \ F*cs(x) = I_1 + I_2 - I_3 - I_4 \ .$$

When the function $F(x,y)$ has the special form $F(x)G(y)$, the generalized convolution reduces to the ordinary convolution as given in [2]. The symbolism will now read

$$F(x)*scG(x) = I_1 - I_2 + I_3 - I_4 \ ,$$

where for example

$$I_3 = \int_0^{\pi-x} F(x + y)G(y)dy \ .$$

In the following work, the derivatives of the convolution function will be needed, as many of the solutions will involve the convolution. Thus, the following differentiation formulae will be stated and proved.

Theorem: If $F(x,y)$, $F_x(x,y)$ are continuous in $\overline{Q}$, then

$$\frac{\partial}{\partial x}\left[F*ss(x)\right] = -F_x*cs(x) - 2\,F(0,x) - 2\,F(\pi,\pi - x)\ ,$$

where $F(0,x) = \lim_{u\to 0^+} F(u,x)$, $F(\pi,\pi - x) = \lim_{u\to\pi^-} F(u,\pi - x)$.

(Note: The notation as used above will be employed throughout the work. The bar over a letter denotes the closure of the set and

$$F_x(0,y) = \lim_{x\to 0^+} F_x(x,y)\ ,$$

where $F_x(x,y)$ means the partial derivative with respect to the first position of the function $F(x,y)$.)

Proof: By writing out the convolution, the following formula is obtained.

$$\frac{\partial}{\partial x}\left[F*ss(x)\right] = -\frac{\partial}{\partial x}\int_0^x F(x - y,y)\,dy + \frac{\partial}{\partial x}\int_x^\pi F(y - x,y)\,dy$$
$$+\frac{\partial}{\partial x}\int_0^{\pi-x} F(x + y,y)\,dy - \frac{\partial}{\partial x}\int_{\pi-x}^{\pi} F(2\pi - x - y,y)\,dy\ .$$

Under the assumptions of the theorem, the ordinary rules of differentiation under the integral sign apply. Thus,

$$\frac{\partial}{\partial x}\left[F*ss(x)\right] = -\int_0^x F_x(x-y,y)\,dy - F(0,x) - \int_x^\pi F_x(y-x,y)\,dy - F(0,x)$$
$$+\int_0^{\pi-x} F_x(x+y,y)\,dy - F(\pi,\pi-x) + \int_{\pi-x}^{\pi} F_x(2\pi-x-y,y)\,dy - F(\pi,\pi-x).$$

Collecting the above and making use of the fact that the integrals form a convolution, the formula of the theorem is obtained.

The next formulae may be seen to be true in a similar manner.

$$\frac{\partial}{\partial x}\left[F*cs(x)\right] = -F_x*ss(x)$$

$$\frac{\partial}{\partial x}\left[F*cc(x)\right] = F_x*sc(x)$$

$$\frac{\partial}{\partial x}\left[F*sc(x)\right] = F_x*cc(x) + 2\,F(0,x) + 2\,F(\pi,\pi - x)$$

By a repeated application of some of the formulas just given, the following corollary may be proved.

Corollary: If F, F_x, and F_{xx} are continuous functions of x and y in $\overline{Q}$, then

$$\frac{\partial^2}{\partial x^2}\left[F*ss(x)\right] = F_{xx}*ss(x) - 2\frac{\partial}{\partial x}F(0,x) - 2\frac{\partial}{\partial x}F(\pi,\pi - x)$$

and

$$\frac{\partial^2}{\partial x^2}\left[F*cc(x)\right] = F_{xx}*cc(x) + 2\,F_x(0,x) + 2\,F_x(\pi,\pi - x) \quad .$$

CHAPTER II

SET OF KEY FUNCTIONS

2.1. Definition of the Set. The set of key functions will be defined by means of their Fourier series in the present section. The series all converge uniformly in any closed subset of the semi-infinite strip, $S = \{(x,y) \mid 0 < x < \pi,\ 0 < y\}$, since their coefficients all contain e^{-ny} in the numerator with a power of n in the denominator, and $\sum e^{-ny}$ is absolutely convergent for $y > 0$. The notation that is used for the functions describes what their finite transforms are in the following manner. The letter denotes the transform, while the subscript indicates the power of n in the denominator of the transform. For example,

$$s_1(x,y) = (2/\pi)\sum_{n=1}^{\infty} n^{-1}e^{-ny}\sin nx\ , \quad (x,y) \text{ in } S\ ,$$

and thus, $S\{s_1(x,y)\} = e^{-ny}/n$, while $C\{c_0(x,y)\} = e^{-ny}$ and

$$c_0(x,y) = (2/\pi)\sum_{n=1}^{\infty} e^{-ny}\cos nx.$$

(Note: All the c_i have been taken so that the cosine transform is zero for $n = 0$.)

Since these series may be differentiated term by term in S, the resulting series being uniformly convergent, it may be shown that the following relations hold among the derivatives of c_0 and s_0.

$$\frac{\partial}{\partial x} c_0(x,y) = \frac{\partial}{\partial y} s_0(x,y)\ , \quad \frac{\partial}{\partial x} s_0(x,y) = -\frac{\partial}{\partial y} c_0(x,y) \tag{8}$$

Also by examining the series forms for the key functions, the following relations are seen to be true

$$\begin{aligned} s_0(x,y) &= -\frac{\partial}{\partial y} s_1(x,y) = \frac{\partial^2}{\partial y^2} s_2(x,y) \\ c_0(x,y) &= -\frac{\partial}{\partial y} c_1(x,y) = \frac{\partial^2}{\partial y^2} c_2(x,y) \\ s_0(x,y) &= -\frac{\partial}{\partial x} c_1(x,y) = -\frac{\partial^2}{\partial x^2} s_2(x,y) \end{aligned} \tag{9}$$

$$c_o(x,y) = \frac{\partial}{\partial x} s_1(x,y) = -\frac{\partial^2}{\partial x^2} c_2(x,y) .$$

The work of the next section will be concerned with verifying that the closed forms of the functions given in Tables I and II are correct, as well as pointing out that some of these closed forms are valid on the edge of S where the Fourier series may not even be convergent.

TABLE I

FINITE SINE TRANSFORMS

	$f_s(n)$	$S^{-1} f_s(n)$
1	e^{-ny}	$s_o(x,y) = (\sin x)/[\pi(\cosh y - \cos x)]$, $y > 0$
2	e^{-ny}/n	$s_1(x,y) = (2/\pi) \arctan[(\sin x)/(e^y - \cos x)]$, $y \geq 0$
3	e^{-ny}/n^2	$s_2(x,7) = (-1/\pi)\int_0^x \ln[2e^{-y}(\cosh y - \cos x')] dx'$, $y > 0$

TABLE II

FINITE COSINE TRANSFORMS

	$f_c(n)$ $f_c(0)=0$	$C^{-1} f_c(n)$
1	e^{-ny}	$c_o(x,y) = (\cos x - e^{-y})/[\pi(\cosh y - \cos x)]$, $y > 0$
2	e^{-ny}/n	$c_1(x,y) = (-1/\pi) \ln[2e^{-y}(\cosh y - \cos x)]$, $y > 0$
3	e^{-ny}/n^2	$c_2(x,y) = (-1/\pi)\int_0^{e^{-y}} r^{-1} \ln(1 - 2r\cos x + r^2)dr$, $y \geq 0$

2.2. Closed Forms for the Key Functions. The closed forms given for c_o and s_o in the tables may be obtained by taking the real and imaginary parts of $(1 - z)^{-1}$ respectively, after setting $z = \exp(-y + ix)$ in it and its power series expansion. Jacobson [5] has used the same method on the formula,

$$\ln(1 + z) = -\sum_{n=1}^{\infty} (-z)^n/n \,, \quad |z| < 1 \,,$$

in order to obtain the closed forms for c_1 and s_1 that have been given. This same procedure may be carried out to get the alternate forms for s_2 and c_2, given in Tables I and II, from the equation

$$\int_0^z t^{-1} \ln(1 + t)dt = -\sum_{n=1}^{\infty} (-z)^n/n^2 \,, \quad |z| < 1 \,,$$

although the calculations get more laborious. It may be noted that since the key functions are the real and imaginary parts of analytic functions, $|z| < 1$, they are all harmonic for $y > 0$ since the identification $z = \exp(-y + ix)$ is made in each case.

Two other methods of finding transforms will be illustrated in the following work, that also produces closed forms for the functions c_2 and s_2. In obtaining the closed form for $c_1(x,y)$ by the method mentioned above, Jacobson proved the following identity

$$(-1/\pi)\ln(1 - 2r\cos x + r^2) = (2/\pi)\sum_{n=1}^{\infty} n^{-1}r^n\cos nx \quad \text{for} \quad r \quad 1. \qquad (10)$$

Dividing by r and integrating between 0 and e^{-y} yields

$$(-1/\pi)\int_0^{e^{-y}} r^{-1} \ln(1 - 2r\cos x + r^2)dr = (2/\pi)\int_0^{e^{-y}} \sum_{n=1}^{\infty} n^{-1}r^{n-1}\cos nx \, dr \text{ for } y > 0.$$

It is now permissible to interchange the order of summation and integration on the right side above, since the resulting series is uniformly convergent. Hence, the desired result is obtained for $y > 0$

$$c_2(x,y) = (-1/\pi)\int_0^{e^{-y}} r^{-1} \ln(1 - 2r\cos x + r^2)dr = (2/\pi)\sum_{n=1}^{\infty} n^{-2}e^{-ny}\cos nx \,.$$

It may also be noted at this time that $c_2(x,y)$ is bounded for (x,y) in $\overline{S}$.

The next work shows how the relations regarding derivatives of the transforms may be used to obtain new transforms in addition to pointing out an identity which will be used later regarding the c_1 function. The starting point is the formula used above, Equation 10, which may be written

$$C\left\{(-1/\pi)\ln(1 - 2r\cos x + r^2)dr\right\} = 0 \quad \text{if } n = 0$$

$$= r^n/n \quad \text{for } n \neq 0.$$

By letting $F'(x) = (-1/\pi)\ln(1 - 2r\cos x + r^2)$, and hence,

$$F(x) = (-1/\pi)\int_0^x \ln(1 - 2r\cos x' + r^2)dx' ,$$

in the second formula of Equation 5, which is

$$C\left\{F'(x)\right\} = nS\left\{F(x)\right\} - F(0) + (-1)^n F(\pi) ,$$

the closed form for s_2 may be obtained by replacing r by e^{-y}, $y > 0$, if the identity

$$F(\pi) = (-1/\pi)\int_0^\pi \ln(1 - 2r\cos x + r^2)dx = 0 \qquad (11)$$

may be shown to be true. Note that this is equivalent to showing that $\int_0^\pi c_1(x,y)dx = 0$. This is done as follows. Let

$$G(r) = \int_0^\pi \ln(1 - 2r\cos x + r^2)dx.$$

$G(r)$ is a continuous function of r for $|r| < 1$. $G(0) = \int_0^\pi \ln 1 \, dx = 0.$

The proof is completed by showing that $G'(r)$ for $r > 0$ is zero.

$$G'(r) = \int_0^\pi 2(r - \cos x)/(1 - 2r\cos x + r^2)dx \qquad 0 < r < 1$$

$$= \int_0^\pi r^{-1}dx + \int_0^\pi (r - r^{-1})/\left[(1 + r^2) - 2r\cos x\right]dx$$

$$= \pi/r + (r - r^{-1})\left\{-\left[(1 + r^2)^2\right]^{1/2} \arcsin\left[\frac{-2r + (1 + r^2)\cos x}{(1 + r^2) - 2r\cos x}\right]\right\}_0^\pi$$

$$= \quad \pi/r + (1/r)\left\{\sin^{-1}\left[-\frac{(1+r)^2}{(1+r)^2}\right] - \sin^{-1}\left[\frac{(1-r)^2}{(1-r)^2}\right]\right\}$$

$$= \quad \pi/r + (1/r)(-\pi/2 - \pi/2) \quad = \quad 0 \;.$$

All the closed forms given in Tables I and II may be obtained by setting $p = \exp(-y)$ in those of Fletcher and Thorne [4], page 13, Numbers 47-49, and pages 20-21, Numbers 38-40.

The second entry in Table I and the third in Table II have been listed as being true even when $y = 0$. This may be shown by noticing that the functions s_1 and c_2 reduce to the linear and quadratic functions given in Equations 7. Thus,

$$s_1(x,0) \quad = \quad (2/\pi)\arctan\frac{\sin x}{1-\cos x} \quad = \quad \frac{2}{\pi}\frac{\pi - x}{2} \quad = \quad \frac{\pi - x}{\pi}$$

is the linear function whose sine transform is $1/n$. While

$$c_2(x,0) \quad = \quad (-1/\pi)\int_0^1 r^{-1}\ln(1 - 2r\cos x + r^2)dr \quad = \quad q(x) \qquad (12)$$

is seen to be correct as follows. The relation 12 does hold when $x = \pi/2$, since

$$q(\pi/2) \quad = \quad (\pi^2/4)(1/2\pi) - \pi/6 \quad = \quad \pi(1/8 - 1/6) \quad = \quad -\pi/24$$

and

$$c_2(\pi/2,0) \quad = (-1/\pi)\int_0^1 r^{-1}\ln(1+r^2)dr \quad = (-1/\pi)\int_0^1\sum_{n=1}^{\infty}(-1)^{n+1}n^{-1}r^{2n-1}dr$$

$$= \quad \lim_{a\to 1}(-1/\pi)\sum_{n=1}^{\infty}(-1)^{n+1}n^{-1}\, r^{2n}/2n\Big|_0^a$$

$$= \quad \lim_{a\to 1}(-1/2\pi)\sum_{n=1}^{\infty}(-1)^{n+1}a^{2n}/n^2 \quad = \quad -(1/2\pi)(\pi^2/12) \;;$$

the limit in the last line above being taken by Abel's theorem. To complete the proof, it is shown that the derivative of both sides of Equation 12 are the same.

$$q'(x) \quad = \quad -(\pi - x)/\pi$$

$$\frac{d}{dx}\left[\frac{-1}{\pi}\int_0^1 r^{-1}\ln(1 - 2r\cos x + r)dr\right] = \frac{-1}{\pi}\int_0^1 \frac{2r\sin x}{1 - 2r\cos x + r^2}\, dr$$

$$= (-2/\pi)\int_0^1 \frac{\partial}{\partial r}\arctan\frac{r\sin x}{1 - r\cos x}\, dr = -\frac{\pi - x}{\pi}\ .$$

The differentiation of the improper integral above is permitted, since the resulting integral is uniformly convergent. The formula that enabled the integration to be carried out above is just another statement of the first formula in the set (Equation 9).

2.3. Order Properties of Key Functions. In showing that the solutions obtained for Laplace's equation in the strip S are valid, it is convenient to know something of the order properties of the functions c_o and s_o, and their derivatives. In this section these properties are found. It is also pointed out that the function s_o presents an unusual behavior on the bottom edge of the strip.

Theorem: The functions c_o and s_o, and their derivatives are bounded by Me^{-y} for $0 < x_o \leq x \quad \pi$, $y \geq 0$, where x_o is an arbitrary positive number, and M is a constant depending on x_o.

Proof: The calculations for s_o and c_o are

$$\begin{aligned} |s_o(x,y)| &= \left|(\sin x)/[\pi(\cosh y - \cos x)]\right| \\ &\leq \left\{\pi(\cosh y)[1 - |\cos x|/(\cosh y)]\right\}^{-1} \\ &\leq 2e^{-y}\left[\pi(1 + e^{-2y})(1 - |\cos x|)\right]^{-1} \\ &< 2\left[\pi(1 - |\cos x_o|)\right]^{-1} e^{-y} \end{aligned}$$

and

$$\begin{aligned} |c_o(x,y)| &= \left|(\cos x - e^{-y})/[\pi(\cosh y - \cos x)]\right| \\ &\leq 2\left[\pi(\cosh y - \cos x)\right]^{-1} \\ &< 4\left[\pi(1 - |\cos x_o|)\right]^{-1} e^{-y} \end{aligned}$$

for $0 < x_o \leq x < \pi$, $y \geq 0$.

For the first partial derivatives the next results are obtained.

$$\frac{\partial}{\partial x} s_o(x,y) = -\frac{\partial}{\partial y} c_o(x,y) = -\left[(\cosh y)(\cos x) - 1\right]/\left[\pi(\cosh y - \cos x)^2\right]$$

$$\frac{\partial}{\partial x} c_o(x,y) \quad = \quad \frac{\partial}{\partial y} s_o(x,y) \quad = \quad -(\sin x)(\sinh y)/\left[\pi(\cosh y - \cos x)^2\right].$$

From which it follows that

$$\left|\frac{\partial}{\partial x} s_o(x,y)\right| \leq 4\left[\pi(1 - |\cos x_o|)^2\right]^{-1} e^{-y}$$

$$\left|\frac{\partial}{\partial x} c_o(x,y)\right| \leq 2\left[\pi(1 - |\cos x_o|)^2\right]^{-1} e^{-y} \quad .$$

As a sample of the estimations employed here, the first result is obtained.

$$\left|\frac{(\cosh y)(\cos x) - 1}{\pi(\cosh y - \cos x)^2}\right| = \frac{|\cos x - (\cosh y)^{-1}|}{(\pi\cosh y)\left[1 - (\cos x)/(\cosh y)\right]^2}$$

$$\leq 4\left[\pi(1 - |\cos x_o|)^2\right]^{-1} e^{-y}$$

for $0 < x_o \leq x < \pi,\ y \geq 0$.

The calculations may be carried out for the higher derivatives and similar results obtained. The reason that all these bounds are valid is that $\cosh y$ occurs to at least one power higher in the denominator than it or $\sinh y$ does in the numerator.

Before leaving the set of key functions and going to some of their applications, it may be noted that for $y = 0$

$$s_o(x,0) \quad = \quad (\sin x)/\left[\pi(1 - \cos x)\right] \quad = \quad (1/\pi)\cot x/2$$

is a function whose sine transform consists entirely of ones. This may be anticipated, since as $y \to 0$, $S\{s_o(x,y)\} = e^{-ny} \to 1$. It should be remarked that the limit function, $(1/\pi)\cot x/2$, is not a function of bounded variation nor is it integrable, since it behaves like x^{-1} as $x \to 0$. Thus, $(1/\pi)\cot x/2$ is a function of a class that is not usually considered in investigations of Fourier series. Its sine transform does exist and is the set of ones as is now shown by induction.

For $n = 1$

$$S\left\{(1/\pi)\cot x/2\right\} = (1/\pi)\int_0^\pi (\sin x)(\sin x)(1 - \cos x)^{-1} dx$$

$$= (1/\pi)\int_0^\pi (1 + \cos x)dx = 1.$$

Assuming it holds for $n = k$, then

$$(1/\pi)\int_0^\pi \cot x/2 \sin kx \, dx = 0$$

and it follows that

$$(1/\pi)\int_0^\pi (\cot x/2)\left[\sin(k+1)x - \sin kx\right]dx =$$

$$(1/\pi)\int_0^\pi (\cot x/2)\left[(\sin kx)(\cos x - 1) - (\cos kx)(\sin x)\right]dx =$$

$$(-1/\pi)\int_0^\pi (\sin x)(\sin kx)dx + (1/\pi)\int_0^\pi 1\cdot\cos kx \, dx + (1/\pi)\int_0^\pi (\cos kx)(\cos x)\, dx .$$

The last line above is shown to yield zero by considering the two cases:

(a) $k = 1$, this is treated by remarking that the middle integral is zero by the orthogonality of the functions $\cos nx$ over 0 to π, while the first and last integrals may be combined to give zero

$$(-1/\pi)\int_0^\pi 2^{-1}(1 - \cos 2x)dx + (1/\pi)\int_0^\pi 2^{-1}(1 + \cos 2x)dx = 0 .$$

(b) $k > 1$, the orthogonality of the sines and cosines respectively over the inerval 0 to π produces the result here. Thus, it is seen that

$$(1/\pi)\int_0^\pi (\cot x/2) \sin\left[(k+1)x\right]dx = 1$$

and the result has been shown.

The Fourier sine series for $(1/\pi)\cot x/2$ does not converge in the usual sense, because the coefficients do not tend to zero, hence an example has been given of a function that has a finite sine transform even though the inverse transform may not be written by means of the Fourier series.

CHAPTER III

HARMONIC FUNCTIONS FOR THE SEMI-INFINITE STRIP WITH VARIOUS BOUNDARY CONDITIONS

In this chapter the solutions to Laplace's equation in the semi-infinite strip, S, with various boundary conditions will be given in terms of the three functions $s_1(x,y)$, $c_0(x,y)$, and $c_1(x,y)$ that were introduced in Chapter II. These problems may all be interpreted as problems in steady temperatures in the semi-infinite slab.

3.1. Nonhomogeneous Condition on Bottom. The present section is concerned with the derivation of the harmonic functions in S for the case in which the only nonhomogeneous condition applies to the bottom of the strip. The details of the derivation are worked out in full for the first problem, in order to illustrate the methods used. The function $J(x)$ is assumed to be bounded and continuous in these problems.

PROBLEM 1:

$$V_{xx}(x,y) + V_{yy}(x,y) = 0 \text{ in } S$$

$$V(0,y) = V(\pi,y) = 0 \text{ for } 0 < y$$

$$V(x,0) = J(x), \ |V(x,y)| < M \text{ for } 0 \leq x \leq \pi .$$

The solution to this problem is obtained quickly by letting

$$S\{V(x,y)\} = v(n,y), \quad S\{J(x)\} = j(n) ,$$

thus forming the transformed problem,

$$v''(n,y) - n^2 v(n,y) = 0$$

$$v(n,0) = j(n), \ |v(n,y)| < m .$$

The solution to this problem is

$$v(n,y) = e^{-ny}\, j(n) .$$

Hence, the solution to Problem 1 is now found by taking the inverse sine tranform of $v(n,y)$ with the aid of a convolution and the special function $c_0(x,y)$ that was defined in Chapter II. Thus,

$$V(x,y) = (1/2)c_0(x,y) * csJ(x) \qquad (13a)$$

is a short form for the solution of Problem 1. using the closed form that was given for $c_0(x,y)$ and writing out the convolution, the solution takes the form

$$V(x,y) = (1/2\pi)\int_0^\pi \left[\frac{\cos(x-x') - e^{-y}}{\cosh y - \cos(x-x')} - \frac{\cos(x+x') - e^{-y}}{\cosh y - \cos(x+x')}\right] J(x')dx' \qquad (13b)$$

Before going on to list the solutions to the other problems of this section, it is interesting to note that the form of the solution just obtained reduces to a known result for the case $J(x) = 1$. The problem in this case was solved by conformal mapping by Churchill [1], p. 149. The result given there is

$$T(u,v) = (2/\pi)\arctan\left[(\cos u)/(\sinh v)\right]$$

for the solution of $\Delta T(u,v) = 0$ with the boundary conditions

$$T(u,0) = 1, \quad T(\pi/2,v) = T(-\pi/2,v) = 0 .$$

By setting $x = u + \pi/2$, $y = v$, this solution is transformed to one that fits the domain in the form treated here. The solution becomes

$$T(x,y) = (2/\pi)\arctan\left[(\sin x)/(\sinh y)\right] .$$

Now, recalling $\frac{\partial}{\partial x} s_1(x,y) = c_0(x,y)$ from Chapter II, Section 2.1, and the closed form for s_1 that was given in Table I, Equation 13b becomes

$$V(x,y) = (-1/\pi)\int_0^\pi \left[d \tan^{-1} \frac{\sin(x-x')}{e^y - \cos(x-x')} + d \tan^{-1} \frac{\sin(x+x')}{e^y - \cos(x+x')}\right]$$

for the case $J(x) = 1$. Evaluation of the integrals yields the formula

$$V(x,y) = (2/\pi)\left[\tan^{-1} \frac{\sin x}{e^y - \cos x} + \tan^{-1} \frac{\sin x}{e^y - \cos x}\right].$$

By employing the formula for the tangent of the sum of two angles, the form

$$V(x,y) = (2/\pi)\arctan \frac{2e^y \sin x}{e^{2y} - 1}$$

is obtained, from which it is easily seen that the general case of Problem 1 has reduced to the known result when a constant temperature is maintained along the bottom of the slab.

The following problems are solved by methods exactly analogous to that used to obtain the solution 13a to Problem 1.

PROBLEM 2:
$$V_{xx}(x,y) + V_{yy}(x,y) = 0 \text{ in } S$$
$$V(0,y) = V(\pi,y) = 0 \text{ for } 0 < y$$
$$V_y(x,0) = J(x), \ |V(x,y)| < M \text{ for } 0 \leq x \leq \pi .$$

The solution may be written in the forms

$$V(x,y) = (-1/2)c_1(x,y)*csJ(x)$$
$$V(x,y) = (1/2\pi)\int_0^\pi \ln \frac{\cosh y - \cos(x-x')}{\cosh y - \cos(x+x')} J(x')dx' . \qquad (14)$$

PROBLEM 3:
$$V_{xx}(x,y) + V_{yy}(x,y) = 0 \text{ in } S$$
$$V_x(0,y) = V_x(\pi,y) = 0 \text{ for } 0 < y$$
$$V(x,0) = J(x), \ |V(x,y)| < M \text{ for } 0 \leq x \leq \pi .$$

The solutions may be written in the forms

$$V(x,y) = (1/\pi)\int_0^\pi J(x)dx + (1/2)c_0(x,y)*ccJ(x)$$
$$V(x,y) = (1/\pi)\int_0^\pi J(x)dx + (1/2\pi)\int_0^\pi \frac{\cos(x-x') - e^{-y}}{\cosh y - \cos(x-x')} J(x')dx' \qquad (15)$$
$$+ (1/2\pi)\int_0^\pi \frac{\cos(x+x') - e^{-y}}{\cosh y - \cos(x+x')} J(x')dx' .$$

Again, if $J(x) = 1$, the above reduces to the obvious result $V = 1$.

PROBLEM 4:
$$V_{xx}(x,y) + V_{yy}(x,y) = 0 \text{ in } S$$
$$V_x(0,y) = V_x(\pi,y) = 0 \text{ for } 0 < y$$
$$V_y(x,0) = J(x), \ |V_y(x,y)| < M \text{ for } 0 \leq x \leq \pi .$$

The solution may be written in the forms

$$V(x,y) = k + (1/2)c_1(x,y)*ccJ(x) \qquad (16)$$

$$V(x,y) = k - (1/2\pi)\int_0^{\pi} \ln\left\{4e^{-2y}\left[\cosh y - \cos(x-x')\right]\left[\cosh y - \cos(x-x')\right]\right\}J(x')dx'$$

where $J(x)$ is such that $\int_0^{\pi} J(x)dx = 0.$

The condition imposed on the last problem is apparent from physical considerations, as well as being necessary in the process of obtaining the solution by the methods used in this report. It simply states that if no heat is to pass through the other boundaries of the slab and a steady temperature is desired, then the average flow of heat through the remaining portion of the boundary must be zero, since there are no sources or sinks present in the body.

Before going into the verification of these solutions, the following connections between the problems of the current section may be noted. By partial differentiation with respect to y of the equation and the first two boundary conditions of Problem 2, a problem that is exactly equivalent to Problem 1 is obtained after $V_y(x,y)$ is replaced by $V(x,y)$ in the resulting problem. This relationship is seen to be true for the solutions 13a and 14, since it may be recalled from Chapter II, Section 2.1 that

$$\frac{\partial}{\partial y} c_1(x,y) = -c_0(x,y).$$

A similar relationship may be recognized to exist between Problems 3 and 4.

The connection between Problems 1 and 2 is also pointed out when it is observed that for the special boundary condition, $J(x) = (\pi - x)/\pi$, the solutions to Problems 1 and 2 reduce to the key functions s_1 and $-s_2$, respectively. The calculations involved in showing these reductions involve integration by parts and are quite long, hence, they are omitted. That the key functions are solutions is easily shown.

The closed form in Table I for s_1 is

$$s_1(x,y) = (2/\pi)\arctan\left[(\sin x)/(e^y - \cos x)\right] .$$

It may be seen from this that

$$s_1(x,0) = (\pi - x)/\pi, \; s_1(0,y) = s_1(\pi,y) = 0 \text{ for } y < 0,$$

and thus, s_1 is a solution of the special case of Problem 1 where $J(x)$ is the linear function.

Similarly, from the closed form for s_2 and the relation $-\frac{\partial}{\partial y} s_2(x,y) = s_1(x,y)$ given in Chapter II, Section 2.1, it may be seen that $-s_2(x,y)$ is the solution to Problem 2 when $J(x)$ is the linear function.

3.2. Verification of Solutions. One procedure that may be used to verify the solution obtained for the problems of Section 3.1 makes use of the fact that some of the key functions are the solutions for special boundary conditions as was just pointed out in the case of Problems 1 and 2. The procedure is illustrated now in showing that solution 15 does satisfy all the conditions of Problem 3.

Consider the auxiliary problem

$$W_{xx}(x,y) + W_{yy}(x,y) = 0 \quad \text{in } S$$

$$W_x(0,y) = W_x(\pi,y) = 0 \quad \text{for } 0 < y \tag{17}$$

$$W(x,0) = q(x), \quad |W(x,y)| < M \quad \text{for } 0 \leq x \leq \pi ,$$

where $q(x)$ is the quadratic function defined in Equation 7. The solution to problem 17 is found using the finite cosine transform. Thus, setting $C\{W(x,y)\} = w(n,y)$ and making use of

$$C\{q(x)\} = 0 \quad \text{for } n = 0$$

$$= n^{-2} \quad \text{for } n \neq 0,$$

The equation in problem 17 becomes

$$n = 0, \quad w''(n,y) = 0 \qquad n \neq 0, \quad w''(n,y) - n^2 w(n,y) = 0$$

$$w(n,0) = 0 \qquad w(n,0) = n^{-2} .$$

Since the solution when $n = 0$ is just $w(0,y) = 0$ and for $n \neq 0$ is $w(n,y) = \exp(-ny)/n^2$, the solution to problem 17 may be written using the key function $c_2(x,y)$ as

$$W(x,y) = c_2(x,y) = (-1/\pi)\int_0^{e^{-y}} r^{-1} \ln(1 - 2r\cos x + r^2)dr .$$

It is easy to show that the above satisfies problem 17 by making use of the properties of the special functions. The nonhomogeneous condition is met, since

$$W(x,0) = c_2(x,0) = q(x)$$

making use of Equation 10. Next, the partial derivative of W with respect to x is computed.

$$W_x(x,y) = \frac{\partial}{\partial x} c_2(x,y) = -s_1(x,y) = (-2/\pi) \arctan\left[\frac{\sin x}{e^y - \cos x}\right]$$

From this it is seen that for $y > 0$, $W_x(0,y) = W_x(\pi,y) = 0$. Since $c_2(x,y)$ is a harmonic function in S, the differential equation is satisfied. It was pointed out in Section 2.2 that $c_2(x,y)$ is bounded in $\bar{S}$.

Letting $C\{V(x,y)\} = v(n,y)$, $C\{J(x)\} = j(n)$, Problem 3 transforms to

$$v''(n,y) - n^2 v(n,y) = 0$$

$$v(n,0) = j(n), \ |v(n,y)| < m .$$

For $n = 0$ the solution to the above is

$$v(0,y) = j(0) = \int_0^\pi J(x)dx .$$

For $n \neq 0$, $v(n,y) = \exp(-ny)j(n)$ and this may be written in terms of $w(n,y)$, as $v(n,y) = n^2 w(n,y)j(n)$. The solution to Problem 3 is found on taking the inverse cosine transform to be

$$V(x,y) = (1/\pi)\int_0^\pi J(x)dx - \frac{\partial^2}{\partial x^2}\left[W(x,y)*ccJ(x)\right] .$$

By making use of the differentiation property of the convolution, since $W_x(0,y) = W_x(\pi,y) = 0$ and $W_{xx}(x,y) = -c_0(x,y)$, the solution becomes just that given in the previous section

$$V(x,y) = (1/\pi)\int_0^\pi J(x)dx - (1/2)W_{xx}(x,y)*ccJ(x) .$$

Now using the differentiation property of the convolution once more, it is seen that

$$V_x(x,y) = (-1/2)\frac{\partial}{\partial x}\left[W_{xx}(x,y)*ccJ(x)\right] = (-1/2)W_{xxx}(x,y)*scJ(x) .$$

But

$$W_{xxx}(x,y) = (2/\pi)\sum_{n=1}^{\infty} ne^{-ny}\cos nx$$

and writing out the last convolution above, the boundary conditions

$$V_x(0,y) = V_x(\pi,y) = 0 \quad \text{for } y > 0$$

are seen to be met. The differential equation of Problem 3 is satisfied, since

$$V_{xx}(x,y) = (-1/2)\frac{\partial^2}{\partial x^2}\left[W_{xx}(x,y)*ccJ(x)\right]$$

$$V_{yy}(x,y) = (-1/2)\frac{\partial^2}{\partial x^2}\left[W_{yy}(x,y)*ccJ(x)\right]$$

in S and adding the two above, recalling that $\Delta W = 0$, it is seen that $V(x,y) = 0$ in S.

In showing that the nonhomogeneous condition is met, care must be used in taking the limit as $y \to 0$. A quick formal check is the following:

$$V(x,0) = (1/\pi)\int_0^\pi J(x)dx + (-1/2)\frac{\partial^2}{\partial x^2}\left[W(x,0)*ccJ(x)\right]$$

but

$$W(x,0) = q(x), \quad q'(x) = -(\pi - x)/\pi, \quad q''(x) = 1/\pi$$

thus,

$$V(x,0) = (1/\pi)\int_0^\pi J(x)dx - (1/2)\left[1/\pi*ccJ(x)\right] + \left.\frac{\pi - x}{\pi}\right|_0 J(x) + \left.\frac{\pi - x}{\pi}\right|_\pi J(\pi - x)$$

and hence

$$V(x,0) = (1/\pi)\int_0^\pi J(x)dx - (1/\pi)\int_0^\pi J(x)dx + J(x) ,$$

where use has been made of the differentiation property of the convolution. The work necessary to justify the interchange of limit operations, taking the second derivative and letting $y \to 0$, involves the actual evaluation of

$$\lim_{y\to 0} c_0(x,y)*ccJ(x) = \lim_{y\to 0}(1/2)\int_0^\pi\left[c_0(x-x',y) + c_0(x+x',y)\right]J(x')dx' \;. \quad (18)$$

But taking this limit produces the rigorous verification of the nonhomogeneous condition. The work proceeds as follows. Since

$$c_0(x,y) = \frac{\cos x - \exp(-y)}{(\cosh y - \cos x)}$$

for $0 < x < \pi$, the limit of the second integral on the right-hand side of expression 18 above may be taken by letting $y \to 0$ inside the integral, and is found to be $(-1/2\pi)\int_0^{\pi} J(x')dx$.

It is in taking the limit of the first integral that care must be used. Hence, given $\eta > 0$, it may be written

$$(1/2)\int_0^{\pi} c_0(x-x',y)J(x')dx' \quad = \quad (1/2)\int_0^{x-\eta} c_0(x-x',y)J(x')dx'$$

$$+ (1/2)\int_{x-\eta}^{x+\eta} c_0(x-x',y)\left[J(x') - J(x) + J(x)\right]dx' + (1/2)\int_{x+\eta}^{\pi} c_0(x-x',y)J(x')dx' \ .$$

As $y \to 0$ the first and last integrals on the right above tend toward

$$(-1/2\pi)\int_0^{x-\eta} J(x')dx' \ , \qquad (-1/2\pi)\int_{x+\eta}^{\pi} J(x')dx' \ ,$$

respectively.

Now, making use of the fact that $\frac{\partial}{\partial x} c_1(x,y) = c_0(x,y)$ for $y > 0$, the following equations are found to be true.

$$(1/2)\int_{x-\eta}^{x+\eta} c_0(x-x',y)dx' \quad = \quad (1/2)\int_{x-\eta}^{x} c_0(x-x',y)dx' + (1/2)\int_{x}^{x+\eta} c_0(x'-x,y)dx'$$

$$= \quad (-1/\pi)\tan^{-1} \frac{\sin(x-x')}{e^y - \cos(x-x')}\Bigg|_{x-\eta}^{x} + (1/\pi)\tan^{-1} \frac{\sin(x'-x)}{e^y - \cos(x'-x)}\Bigg|_{x}^{x+\eta}$$

$$= \quad (2/\pi)\arctan\left[(\sin\eta)/(e^y - \cos\eta)\right] \ .$$

Hence,

$$V(x,y) - J(x)(2/\pi)\arctan \frac{\sin\eta}{e^y - \cos\eta} \quad = \quad (1/\pi)\int_0^{\pi} J(x)dx$$

$$+ (1/2)\int_0^{x-\eta} c_0(x-x',y)J(x')dx' + (1/2)\int_{x+\eta}^{\pi} c_0(x'-x,y)J(x')dx'$$

$$+ (1/2)\int_0^{\pi} c_0(x+x',y)J(x')dx' + (1/2)\int_{x-\eta}^{x+\eta} c_0(x-x',y)\left[J(x') - J(x)\right]dx' \ .$$

Thus, given $\epsilon > 0$, there exists a $\delta > 0$ such that $|J(x') - J(x)| < \epsilon$ for $|x-x'| < \delta$, taking $\eta \leq \delta$ and such that

$$\left|(1/2\pi)\int_{x-\eta}^{x+\eta} J(x)dx\right| < \epsilon,$$

it is seen that

$$\lim_{y\to 0} V(x,y) - J(x)(2/\pi)\tan^{-1}\frac{\sin\eta}{e^y - \cos\eta} = V(x,0) - J(x)(\pi - \eta)/\pi$$

$$\leq \epsilon + \epsilon\frac{\pi - \eta}{\pi}$$

for all $\epsilon > 0$ and $\eta > 0$; hence the result, $V(x,0) = J(x)$, follows.

To complete the verification of the solution obtained for Problem 3, it is noted that

$$|V(x,y)| \leq \left|(1/\pi)\int_0^\pi J(x)dx\right| + (1/2)\int_0^\pi\Big[|c_o(x-x',y)| + |c_o(x+x',y)|\Big]|J(x')|\,dx' ,$$

since $|c_o(x,y)| \leq 2/[\pi(\cosh y - 1)]$ for $y \geq \delta > 0$, $|V|$ is bounded. But it was just shown that $\lim_{y\to 0} V(x,y) = J(x)$ which is bounded, thus the result follows.

The solution to Problem 4 may not be checked by the above technique, since the linear or quadratic variation may not be substituted for the nonhomogeneous condition because of the added condition, $\int_0^\pi J(x)dx = 0$, which must be applied in this problem. The verification in this case is now carried out.

By differentiating the form of the solution given in Equation 16 and making use of the differentiation formulas given in Chapter I, Section 1.3, the following relations are obtained, for which the results are indicated.

$$V_x(x,y) = (1/2)s_o(x,y)*scJ(x)$$

from which it is seen by writing out the convolution that

$$V_x(0,y) = V_x(\pi,y) = 0 \quad \text{if } y > 0 .$$

$$V_{xx}(x,y) = (-1/2)\left[\frac{\partial}{\partial x}s_o(x,y)\right]*ccJ(x) + 2s_o(0,y)J(x) + 2s_o(\pi,y)J(\pi - x)$$

$$= (-1/2)\left[\frac{\partial}{\partial x}s_o(x,y)\right]*ccJ(x) \quad \text{for } (x,y) \text{ in } S ,$$

since $s_0(x,y) = (\sin x)/[\pi(\cosh y - \cos x)]$ is a continuous function of x in $0 \leq x \leq \pi$ for $y > 0$.

$$V_y(x,y) = (1/2)\ c_0(x,y) * ccJ(x),$$

and from the verification of Problem 3, the limit of the right-hand side of the above as $y \to 0$ is simply $J(x)$, using the condition that $\int_0^\pi J(x)dx = 0$ which holds in this problem. $|V_y(x,y)| < M$ by the result demonstrated in showing the similar condition for Problem 3.

$$V_{yy}(x,y) = (-1/2)\left[\frac{\partial}{\partial y} c_0(x,y)\right] * ccJ(x) \quad \text{for } (x,y) \text{ in } S$$

and since, from Section 2.1

$$\frac{\partial}{\partial y} c_0(x,y) = -\frac{\partial}{\partial x} s_0(x,y) \quad \text{for } (x,y) \text{ in } S$$

the differential equation is seen to be satisfied on addition of the expressions that have been obtained for V_{xx} and V_{yy}.

3.3. Infinite Sine and Cosine Transforms. In obtaining the solutions to Laplace's equation in the case where the nonhomogeneous condition occurs on a side of the strip, the infinite sine and cosine transforms may be used advantageously. Their definitions are given together with a few of the important properties. The operational calculus based on them is so similar to that of the finite transforms, that only the results are listed here. Additional information as well as the proofs may be found in the standard books such as Sneddon's [7].

DEFINITION: Let $F(x)$ be absolutely integrable $0 \leq y < \infty$ and $\int_0^c F(x)dx$ exist for each $c > 0$, then

$$S_\infty\{F(x)\} = f_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) \sin \alpha x\ dx \tag{19}$$

$$C_\infty\{F(x)\} = f_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) \cos \alpha x\ dx\ . \tag{20}$$

With the transforms defined as above, the inverse transforms are exactly the same integrals with the position of the functions and their transforms interchanged, if the transforms meet the conditions of the definition.

The important operational formulae are:

$$S_\infty\{F''(x)\} = -\alpha^2 f_s(\alpha) + \sqrt{\frac{2}{\pi}}\, \alpha F(0) \tag{21}$$

$$C_{\infty}\left\{F''(x)\right\} = -\alpha^2 f_c(\alpha) - \sqrt{\frac{2}{\pi}}\, F'(0) \;, \qquad (22)$$

where $F(x)$ and $F'(x)$ are continuous $x \geq 0$, $F''(x)$ is sectionally continuous $0 \leq x \leq x_0$ for every $x_0 > 0$ and F and F' satisfy the conditions in the definition of the infinite transforms. As in the case of the finite transforms, there are four possible types of convolution, but only two are given below, as they are all that are needed in the present applications. The formulae are valid if F and G satisfy the conditions given in the definition.

$$C_{\infty}^{-1}\left\{f_c(\alpha)g_c(\alpha)\right\} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[F(|x-x'|) + F(x+x')\right] G(x')dx' \qquad (23)$$

$$S_{\infty}^{-1}\left\{f_c(\alpha)g_s(\alpha)\right\} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left[F(|x-x'|) - F(x+x')\right] G(x')dx' \qquad (24)$$

The following transform that may also be found in the first chapter of [7] will be needed in the subsequent work.

$$C_{\infty}\left\{\sqrt{\frac{\pi}{2}}\, e^{-ny}\right\} = n/(\alpha^2 + n^2)$$

The continuation of the treatment of Laplace's equation for the semi-infinite strip by the methods that have been used thus far requires that the solutions to the following boundary value problems in ordinary differential equations be known. The problems may be stated as

$$\begin{aligned} u''(n,y) - n^2u(n,y) &= f(n,y), \quad y > 0 \\ u(n,0) &= 0, \quad |u(n,y)| < m, \quad y \geq 0 \end{aligned} \qquad (25)$$

$$\begin{aligned} u''(n,y) - n^2u(n,y) &= f(n,y), \quad y > 0 \\ u'(n,0) &= 0, \quad |u(n,y)| < m, \; y \geq 0 \;, \end{aligned} \qquad (26)$$

where the function $f(n,y)$ is assumed to be bounded and continuous $y \geq 0$. The solutions to these problems are easily found formally by using the infinite transforms. The method is illustrated in getting the solution to Equation 25. Setting $S_{\infty}\left\{u(n,y)\right\} = w(n,\alpha)$, $S_{\infty}\left\{f(n,y)\right\} = g(n,\alpha)$, and employing Equation 21, the problem becomes

$$-\alpha^2 w(n,\alpha) - n^2 w(n,\alpha) = g(n,\alpha) \;.$$

The solution to this equation may be written

$$w(n,\alpha) \quad = \quad -1/(\alpha^2 + n^2)g(n,\alpha) \ .$$

The inverse transform is now found by making use of the special transform given above and the second convolution that was given. Thus

$$u(n,y) \quad = \quad \int_0^\infty \left[(2n)^{-1}e^{-n(y+y')} + (-2n)^{-1}e^{-n|y-y'|}\right]f(n,y')dx' \qquad (27)$$

has been obtained as a possible solution to Equation 25. In a similar manner, the next formula is a formal solution to Equation 26.

$$u(n,y) \quad = \quad \int_0^\infty \left[(-2n)^{-1}e^{-n(y+y')} + (-2n)^{-1}e^{-n|y-y'|}\right] f(n,y')dy' \qquad (28)$$

These solutions must be verified, as the methods used to derive them were quite formal. For example, in order that the infinite transforms and inverses exist, it is usually assumed that $f(n,y)$ be absolutely integrable, while the conditions placed on $f(n,y)$ here are boundedness and continuity. These conditions may be shown to be sufficient by showing that the solutions 27 and 28 do satisfy their equations by performing the differentiation under the integral sign.

3.4. Nonhomogeneous Condition on Side, Case I. The two problems solved in this section correspond to Numbers 1 and 2 of Section 3.1. The difference is that now the nonhomogeneous boundary condition is on the side instead of the bottom of the strip. In this section the first problem is written together with its solution, while the method used to obtain it is illustrated in obtaining and verifying the solution to the other problem. The function $H(y)$ is assumed to be bounded and continuous for $0 < y$ in each case.

PROBLEM 5:

$$V_{xx}(x,y) + V_{yy}(x,y) \quad = \quad 0 \quad \text{in } S$$

$$V(0,y) \quad = \quad H(y), \quad V(\pi,y) \quad = \quad 0, \ 0 < y$$

$$V(x,0) \quad = \quad 0, \quad 0 < x < \pi; \ |V(x,y)| < M \quad \text{for } (x,y) \text{ in } \overline{S}.$$

The solution may be written

$$V(x,y) \quad = \quad (1/2)\int_0^\infty \left[s_0(x,|y-y'|) - s_0(x,y+y')\right] H(y')dy' \ . \qquad (29)$$

In the special case that $H(y) = 1$, this reduces to

$$V(x,y) = \frac{\pi - x}{\pi} - (2/\pi)\arctan\frac{\sin x}{e^y - \cos x} = (2/\pi)\arctan\frac{\tanh(y/2)}{\tan(x/2)} ;$$

making use of the fact that $\frac{\partial}{\partial y} s_1(x,y) = -s_0(x,y)$. This last result is the same as that obtained by conformal mapping, [1] p. 162, after a transformation of axes is made.

PROBLEM 6:

$$V_{xx}(x,y) + V_{yy}(x,y) = 0 \text{ in } S$$

$$V(0,y) = H(y), \quad V(\pi,y) = 0, \quad 0 < y$$

$$V_y(x,0) = 0, \quad 0 < x < \pi; \quad |V(x,y)| < M, \quad (x,y) \text{ in } \overline{S} .$$

This problem is solved by setting $S\{V(x,y)\} = v(n,y)$, thus obtaining the transformed problem

$$v''(n,y) - n^2v(n,y) = -nH(y)$$

$$v'(n,0) = 0, \quad |v(n,y)| < m .$$

The above is a special case of the problem given in Equation 26 of the previous section, and using its solution, Equation 28, it is seen that

$$v(n,y) = (1/2)\int_0^{\infty}\left[e^{-n(y+y')} + e^{-n|y-y'|}\right]H(y')dy' .$$

The inverse transform now gives

$$V(x,y) = (1/2)\int_0^{\infty}\left[s_0(x,|y-y'|) + s_0(x,y+y')\right]H(y')dy' \qquad (30a)$$

as a form of the solution to Problem 6. The solution reduces to $(\pi - x)/\pi$ in case $H(y) = 1$. This last result is just what is expected, since the bottom of the slab is insulated, the left-hand side maintained at the constant temperature one, while the right side is held at zero.

The rigorous verification of the solution to Problems 5 and 6 may be carried out as follows. The absolute value signs are removed by writing Equation 30a in the form

$$V(x,y) = (1/2)\int_0^y s_0(x,y-y')H(y')dy' + (1/2)\int_y^\infty s_0(x,y'-y)H(y')dy' + (1/2)\int_0^\infty s_0(x,y+y')H(y')dy' \; . \tag{30b}$$

Since $|s_0(x,y)| < Me^{-y}$ for $x \geq x_0 > 0$, these integrals converge at each point of S, for H(y) was assumed to be bounded. The boundedness of the solution follows from the fact that each of the three integrals that make up Equation 30b may be shown to be bounded. Consider, for example, the second one which may be written

$$\int_y^\infty s_0(x,y'-y)H(y')dy' = \int_0^\infty s_0(x,y'')H(y''+y)dy''$$

by setting $y'-y = y''$. But

$$\left| \int_0^\infty s_0(x,y'')H(y''+y)dy'' \right| \leq \int_0^\infty |s_0(x,y'')| \, |H(y''+y)| \, dy'' \; .$$

Now letting M' be the bound on $|H(y)|$, $y \geq 0$, it follows from the fact that $s_0(x,y)$ is positive for $0 \leq x \leq \pi$, and

$$\int_0^\infty s_0(x,y)dy = \int_0^\infty -\frac{\partial}{\partial y} s_1(x,y)dy = s_1(x,0) = (\pi - x)/\pi$$

that

$$\left| \int_0^\infty s_0(x,y'')H(y''+y)dy'' \right| < M'(\pi - x)/\pi \leq M' \quad \text{for } (x,y) \text{ in } \bar{S} \; .$$

In a similar manner, the remaining two integrals in Equation 30b may be shown to have the same bound; thus the condition of boundedness is met.

The boundary condition $V(\pi,y) = 0$ is satisfied, since $s_0(x,y)$ is continuous at $x = \pi$ and $s_0(\pi,y) = 0$. The condition $V(0,y) = H(y)$ is seen to be met by the next argument. By adding and subtracting H(y), V(x,y) may be written as follows

$$V(x,y) = (1/2)\int_0^\infty \left[s_0(x,|y-y'|) + s_0(x,y+y')\right]\left[H(y') - H(y)\right] dy' + H(y)(1/2)\int_0^\infty \left[s_0(x,|y-y'|) + s_0(x,y+y')\right] dy' \; . \tag{31}$$

Since the last integral above is just that obtained by setting $H(y) = 1$ in Equation 30b and using the fact that it reduces to simply $(\pi - x)/\pi$ as $x \to 0$, the last summand in Equation 31 yields the desired result, $H(y)$. The first integral is seen to approach 0 as $x \to 0$ by the next work. Assuming $H(y)$ is continuous for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|H(y) - H(y')| < \epsilon \quad \text{if } |y-y'| < \delta ,$$

hence

$$\left|(1/2)\int_0^\infty \left[s_o(x,|y-y'|) + s_o(x,y+y')\right]\left[H(y') - H(y)\right] dy'\right|$$

$$\leq (1/2)\int_0^{y-\delta} \left[s_o(x,y-y') + s_o(x,y+y')\right]\left|H(y') - H(y)\right| dy'$$

$$+ (\epsilon/2)\int_{y-\delta}^{y+\delta} \left[s_o(x,|y-y'|) + s_o(x,y+y')\right] dy'$$

$$+ (1/2)\int_{y+\delta}^\infty \left[s_o(x,y'-y) + s_o(x,y+y')\right]\left|H(y') - H(y)\right| dy' .$$

The first and last integrals on the right side of the above equation tend to 0 as $x \to 0$, since $s_1(x,y)$ is well behaved for the ranges of the second argument that are involved. The middle integral is $\leq \epsilon$, since the integrand is positive over the extended range of integration from 0 to ∞, and this integral equals $(\pi - x)/\pi$. But ϵ was arbitrary, hence, the first integral in Equation 31 does approach 0, and the nonhomogeneous condition in Problem 6 is met.

To show that the last boundary condition and the differential equation are satisfied, it is necessary to differentiate expression 30b for $V(x,y)$. In showing that the usual process is allowed, the order properties of the functions s_o and c_o and their derivatives given in Chapter II, Section 2.3 will be used. Ordinary differentiation yields

$$\begin{aligned} V_y(x,y) = {} & (1/2)\int_0^y \frac{\partial}{\partial y} s_o(x,y-y')H(y')dy' + s_o(x,0)H(y) \\ & - (1/2)\int_y^\infty \frac{\partial}{\partial y} s_o(x,y'-y)H(y')dy' - s_o(x,0)H(y) \qquad (32) \\ & + (1/2)\int_0^\infty \frac{\partial}{\partial y} s_o(x,y+y')H(y')dy' \end{aligned}$$

and this will be correct if the infinite integrals converge uniformly with respect to y for each fixed x, $0 < x_0 \leq x$. That this is the case is now shown for the first infinite integral,

$$\int_y^\infty \frac{\partial}{\partial y} s_0(x,y'-y)H(y')dy' = -\int_y^\infty \frac{\partial}{\partial(y'-y)} s_0(y'-y)H(y')dy'$$

$$= -\int_0^\infty \frac{\partial}{\partial w} s_0(x,w)H(w+y)dw \ ,$$

by setting $y'-y = w$. But this last integral is uniformly convergent for all $y \geq 0$, since

$$\left| \int_0^\infty s_0(x,w)H(w+y)dw \right| < M' \int_0^\infty e^{-w}dw = M'$$

because $|H(y)| < M$, and $\left|\frac{\partial}{\partial w} s_0(x,w)\right| < 2e^{-w}(1 - |\cos x_0|)^{-2}$.
The other infinite integral may be treated in a similar manner.

Now from Equation 32 as $y \to 0$, it is seen that $V_y(x,y) \to 0$, since

$$\left.\frac{\partial}{\partial y} s_0(x,y'-y)\right|_{y=0} = -\left.\frac{\partial}{\partial(y'-y)} s_0(x,y'-y)\right|_{y=0} = -\frac{\partial}{\partial y'} s_0(x,y')$$

and similarly

$$\left.\frac{\partial}{\partial y} s_0(x,y+y')\right|_{y=0} = \frac{\partial}{\partial y'} s_0(x,y') \ .$$

The differentiation of the expression in Equation 32 may be continued by the usual process, since the resulting infinite integrals may be shown to be uniformly convergent, making use of the order property,

$$\left|\frac{\partial^2}{\partial y^2} s_0(x,y)\right| < M''e^{-y} \quad \text{for} \quad 0 < x_0 \leq x \leq x_1 < \pi \ ,$$

that was noted earlier.

$$V_{yy}(x,y) = (1/2)\int_0^y \frac{\partial^2}{\partial y^2} s_0(x,y-y')H(y')dy' + \left.\frac{\partial}{\partial y} s_0(x,y-y')H(y')\right|_{y'=y}$$

$$- (1/2)\int_y^\infty \frac{\partial^2}{\partial y^2} s_0(x,y'-y)H(y')dy' + \left.\frac{\partial}{\partial y} s_0(x,y'-y)H(y')\right|_{y'=y}$$

$$+ (1/2)\int_0^\infty \frac{\partial^2}{\partial y^2} s_o(x,y+y')H(y')dy' .$$

The terms not in the integrals annihilate each other, since

$$\frac{\partial}{\partial y} s_o(x,y-y')\bigg|_{y'=y} = \frac{\partial}{\partial(y-y')} s_o(x,y-y')\bigg|_{y'=y} = -\frac{\partial}{\partial y} s_o(x,y'-y)\bigg|_{y'=y}$$

The form 30b for $V(x,y)$ may be differentiated twice with respect to x for $0 < x_o \leq x \leq x_1 < \pi$ by the usual processes, since the resulting infinite integrals may be shown to be uniformly convergent by making use of the order properties as was done before. Thus,

$$\begin{aligned} V_{xx}(x,y) \quad &= \quad (1/2)\int_0^y \frac{\partial^2}{\partial x^2} s_o(x,y-y')H(y')dy' \\ &+ (1/2)\int_y^\infty \frac{\partial^2}{\partial x^2} s_o(x,y'-y)H(y')dy' \\ &+ (1/2)\int_0^\infty \frac{\partial^2}{\partial x^2} s_o(x,y+y')H(y')dy' . \end{aligned}$$

Now $\Delta V = 0$ since $s_o(x,y)$ is a harmonic function in the strip S as was mentioned in Chapter II, Section 2.1.

This completes the verification of Problem 6. The solution given to Problem 5 may be checked in an analogous manner. The connection noted in Section 3.1 between Problems 1 and 2 is also applicable here, although it is not as obvious. The reason for this is that an integration by parts on the solution is required to show that the result of taking the partial derivative with respect to y of the equation and first two boundary conditions of Problem 6 together with the remaining conditions, gives a problem equivalent to Number 5. This may be seen by integrating expression 32 for $V_y(x,y)$ by parts and then replacing $H(y)$ in Problem 5 by $H'(y)$.

3.5. Nonhomogeneous Condition on Side, Case II. In all the problems dealt with so far, some condition of boundedness has been applied throughout the strip in order to obtain a definite solution. The two problems to be considered now do not have bounded solutions. Their solutions involve the same key functions, but there are additional terms that must be added. To illustrate how the condition of boundedness needs to be replaced in order to obtain a definite solution, the following problem, a special case of Problem 7, is now treated.

$$W_{xx}(x,y) + W_{yy}(x,y) = 0 \text{ in } S$$

$$W_x(0,y) = 1, \quad W_x(\pi,y) = 0 \quad \text{for } 0 < y \tag{33}$$

$$W(x,0) = 0, \; 0 < x < \pi; \; W(x,y) \text{ bounded on each compact subset of } S .$$

The finite cosine transform is applied to obtain a solution here. Thus, letting $C\{W(x,y)\} = w(n,y)$, the problem is transformed to

$$w''(n,y) - n^2 w(n,y) = 1$$

$$w(n,0) = 0, \quad w(n,y) \text{ bounded for finite } y .$$

If $n = 0$, then $w''(0,y) = 1$ and

$$w(0,y) = y^2/2 + k_1 y + k_2 ,$$

but $w(0,0) = 0$ implies that $k_2 = 0$.

If $n \neq 0$, then the general solution may be written

$$w(n,y) = k_3 e^{ny} + k_4 e^{-ny} - 1/n^2 .$$

The condition that $w(n,y)$ be bounded for finite y implies that $k_3 = 0$. The other boundary condition, $w(n,0) = 0$ gives $k_4 = 1/n^2$. Thus, the solution to the problem in Equation 33 may be written

$$W(x,y) = y^2/2\pi + ky + c_2(x,y) - q(x) . \tag{34}$$

The constant k is actually the average quantity of heat flowing across the bottom of the slab, since

$$W_y(x,y) = y/\pi + k + c_1(x,y) ,$$

and hence

$$\int_0^\pi W_y(x,y)dx = y + \pi k + \int_0^\pi c_1(x,y)dx .$$

Thus $\lim_{y \to 0} \int_0^\pi W_y(x,y)dx = \pi k$, for it was shown in Chapter II, Section 2.2 that

$$\int_0^\pi c_1(x,y)dx = 0 .$$

It may be noticed that solution 34 behaves like a quadratic expression as y increases, and hence is unbounded. From this example it is seen that one condition that may be used to obtain a definite solution to Problem 7 is as given in the statement of the problem.

PROBLEM 7:

$$V_{xx}(x,y) + V_{yy}(x,y) = 0 \quad \text{in } S$$

$$V_x(x,y) = H(y), \quad V_x(x,y) = 0 \quad \text{for } y > 0$$

$$V(x,0) = 0, \quad 0 < x < \pi; \quad V(x,y) \text{ bounded on each compact}$$

$$\text{subset of } S, \ \lim_{y \to 0} \int_0^{\pi} V_y(x,y)dx = k \ ,$$

where $H(y)$ is taken to be bounded and continuous for $0 \leq y$.

This problem is also solved using the finite cosine transformation, with the solution assuming the form

$$\begin{aligned} V(x,y) = {} & (1/\pi)\int_0^y \int_0^{y'} H(y'')dy''dy' + ky/\pi \\ & + (1/2)\int_0^{\infty}\left[c_1(x,y+y') - c_1(x,|y-y'|)\right]H(y')dy' \ . \end{aligned} \tag{35}$$

The conditions $V(x,0) = 0$ and $V(x,y)$ bounded for each compact subset of S are seen to be met by inspection. The first two conditions and the differential equation may be shown to be satisfied by work very similar to that used in the previous section. Thus all that remains in the verification of Equation 35 as the solution to Problem 7 is to show that the additional condition is satisfied. By differentiating Equation 35 with respect to y, the expression

$$\begin{aligned} V_y(x,y) = {} & (1/\pi)\int_0^y H(y')dy' + k/\pi - (1/2)\int_0^{\infty} c_o(x,y+y')H(y')dy' \\ & + (1/2)\int_0^y c_o(x,y-y')H(y')dy' - (1/2)\int_y^{\infty} c_o(x,y'-y)H(y')dy' \end{aligned}$$

is obtained for $0 < x < \pi$. The process is legal, since the resulting integrals are uniformly convergent. Thus,

$$\lim_{y\to 0}\int_0^\pi V_y(x,y)\,dx \;=\; k + \lim_{y\to 0}\lim_{\epsilon\to 0}(-1/2)\int_0^\infty\int_\epsilon^\pi c_0(x,y+y')H(y')\,dx\,dy'$$

$$+\lim_{y\to 0}\lim_{\epsilon\to 0}(1/2)\int_0^y\int_\epsilon^\pi c_0(x,y-y')H(y')\,dx\,dy' + \lim_{y\to 0}\lim_{\epsilon\to 0}(-1/2)\int_y^\infty\int_\epsilon^\pi c_0(x,y'-y)H(y')\,dx\,dy' .$$

The condition will be seen to hold, once it is shown that each of the first limits above is zero. As an example, the second one is considered now. Let

$$L \;=\; \lim_{\epsilon\to 0}\;(-1/2)\int_0^y\int_\epsilon^\pi c_0(x,y-y')H(y')\,dx\,dy' .$$

Since $\frac{\partial}{\partial x}\, s_1(x,y) \;=\; c_0(x,y)$, it is true that

$$L \;=\; \lim_{\epsilon\to 0}\;(-1/2)\int_0^y s_1(x,y-y')\Big|_\epsilon^\pi H(y')\,dy'$$

$$=\; \lim_{\epsilon\to 0}\;(1/2)\int_0^y s_1(\epsilon,y-y')H(y')\,dy' .$$

But $s_1(\epsilon,y-y')$ is positive when its second argument is greater than or equal to zero, thus, since $H(y)$ is bounded, the result will be shown, once it is proved for $H = 1$. This is now done. Since

$$\frac{\partial}{\partial(y-y')}\, s_2(\epsilon,y-y') \;=\; -s_1(\epsilon,y-y')$$

and

$$s_2(\epsilon,y-y') \;=\; (-1/\pi)\int_0^\epsilon \ln\left\{2e^{-(y-y')}\left[\cosh\,(y-y') - \cos\,x\right]\right\}dx ,$$

the limit L may be written

$$L \;=\; \lim_{\epsilon\to 0}(1/2)\left|(-1/\pi)\int_0^\epsilon \ln\left\{2e^{-(y-y')}\left[\cosh\,(y-y') - \cos\,x\right]\right\}\right|_{y'=0}^{y}\,dx$$

$$=\; \lim_{\epsilon\to 0}\left\{(-1/2\pi)\int_0^\epsilon \ln\left[2(1-\cos\,x)\right]dx + (1/2\pi)\int_0^\epsilon \ln\left[2e^{-y}(\cosh\,y - \cos\,x)\right]dx\right\} .$$

For $y > 0$ the last limit above is zero, since the integrand is bounded. The first limit is also zero, since the integral is a convergent one, as may be checked by integration by parts.

$$(1/2\pi)\int_0^\pi \ln\left[2(1 - \cos x)\right] dx = (1/2\pi)x \ln\left[2(1 - \cos x)\right]\Big|_0^\pi - (1/2\pi)\int_0^\pi (x/2)\cot(x/2)dx/2$$

$$= \ln 2 - \lim_{x\to 0} (1/2\pi)x \ln\left[2(1 - \cos x)\right] - (1/2)\ln 2 ,$$

where the last integral in the first line above has been evaluated by means of Formula 1, Table 205 in the DeHaan Tables [3]. The limit in the last equation may be seen to be zero using L'Hospital's rule. Thus, the solution as given in Equation 35 to Problem 7 may be shown to check.

Problem 8 is also interesting, as it may have unbounded solutions for certain boundary conditions.

PROBLEM 8:

$$V_{xx}(x,y) + V_{yy}(x,y) = 0 \text{ in } S$$

$$V_x(0,y) = H(y), \quad V_x(\pi,y) = 0 \text{ for } 0 < y$$

$$V_y(x,0) = 0, \quad 0 < x < \pi; \quad |V_y(x,y)| < M, \ (x,y) \text{ in } \bar{S} .$$

The solution may be written in the form

$$V(x,y) = k + (1/\pi)\int_0^y \int_0^{y'} H(y'')dy'' \, dy'$$

$$- (1/2)\int_0^\infty \left[c_1(x,y+y') + c_1(x,|y-y'|)\right] H(y')dy' , \qquad (36)$$

where k is an arbitrary constant under the assumption that $H(y)$ is continuous and absolutely integrable. This solution may be verified in a manner similar to that used on Problems 6 and 7. Note that in this case, as opposed to that of Problem 4, it is only necessary to assume that $\int_0^\infty |H(y)| dy$ exists in order to obtain a solution, while before it was necessary that the average value of the nonhomogeneous boundary condition be zero.

3.6. Remarks. In the previous work, the solutions that have been obtained for the problems considered have been verified under the assumption that the nonhomogeneous boundary condition is a continuous function. With slight modifications, the same results may be shown to be correct when the boundary condition is a sectionally continuous function. Here the solution will approach the mean value, $(1/2)\left[f(x-0) + f(x+0)\right]$, of the function at every point.

To show that the solutions to Problems 1 through 4 are still valid, it is convenient to know that the differentiation formulas for the convolution given in Chapter I, Section 1.3, hold with slight changes when the function $F(x,y)$ is sectionally continuous in its second argument. These may be shown by simply breaking up the path of integration, as was done before, still further so that the integrand is a continuous function of both arguments on the sub-intervals. The changes that occur are in the form of the additional terms that now take into account the jump in the function at a point of discontinuity.

If the integrals that were used in taking the limits in the demonstration that the nonhomogeneous conditions were met, are broken up in the manner indicated in the previous paragraph, the verification of the solutions may be completed for the case of the sectionally continuous variation.

It should be mentioned that by the addition of some of the solutions to Problems 1 through 8, it is possible to obtain the solutions to problems of a more general type; that is, problems where there is a nonhomogeneous condition on each of the edges of the semi-infinite strip.

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