THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

PREDICTION OF PERCENT FAILURES WITH STRESS/STRENGTH INTERFERENCE

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November, 1967

IP-792

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

INTRODUCTION

Apparently the factor of safety is meant to account for all the variables which are known to affect the stress and strength of the member. The utilization of a factor of safety of this kind has justification only when its value is based on considerable experience, with parts not too different from the one under consideration. However, when substantial changes in the geometry, the processing, or the function of the part are contemplated, a major error may result if the old factor of safety is projected to the new set of conditions.

This can be illustrated by the problem of automotive axle shafts which, in the past, have been failing in service in large numbers. These shafts have been fabricated from a steel with a tensile strength of 240,000 psi, and yet, the operating stresses as measured in actual service were found to be only 13,000 psi. This produced an apparent factor of safety of 240,000/13,000 = 18.5 This is obviously a fictitious value, since the shafts were failing in service, and the true factor of safety was less than one. The explanation lies in the fact that axle strength to be compared with the 13,000 psi operating stress should not have been the ultimate strength of the material (240,000 psi) but the fatigue strength corresponding to the surface finish of the shafts, the mode of loading to which the shaft was subjected, etc. When the ultimate strength was reduced by these derating factors the resultant value was found to be 12,000 psi. This strength, when compared with the 13,000 psi stress produced the realistic factor of safety of 0.9.

Examples such as this lead to the next phase in the relationship between stress and strength, namely to the concepts of a significant stress and a significant strength. By significant stress is meant the actual stress imposed on the part and it may include the effect of stress raisers, magnification due to impact loading, residual stresses, etc. By significant strength is meant the actual strength of the part in its fabricated form, under actual operating conditions. A rational approach to significant strength still employs ultimate strength as the basis. However, instead of an indiscriminate grouping of all the factors affecting the ultimate strength into one index, it attempts to evaluate quantitatively the effect of each individual factor pertaining to the part and the conditions under consideration. The result is a value which is strictly applicable to the part under consideration and to the set of loading conditions to which the part is subjected in service.

These concepts of significant stress and significant strength represent a major step toward a more realistic prediction of the probability of failure and, as such, they have been included in the present investigation. By themselves, however, they are not sufficient. This is because the prevailing practice is to use the mean values of the calculated strength and stress, ignoring the natural scatter that stresses and strengths may have.

The variability in these two factors results in the existence of a statistical distribution function of stress and strength and is the heart of the Interference Theory. Thus, for proper prediction of the probability of failure, an estimate must be made of both the mean value and the dispersion characteristics of both the strength distribution and the stress distribution.

The strength of the part, as all properties of non-homogeneous materials, varies from specimen to specimen, in view of the variation in hardness, surface finish, degree of stress concentration, etc. The operating stress imposed on the part varies too. These stresses vary from time to time in a particular part, from part to part in a particular design, and from environment to environment. Therefore, both the mean value and the dispersion characteristics of stress and strength must be determined.

Once these distributions are found, probability of failures can be computed from the interference area. Means of determining these distributions and the resultant interference represent the objectives of the present investigation.

STRESS/STRENGTH INTERFERENCE

Suppose there are two barrels containing slips of paper, each having a number printed on it. The numbers in barrel Y are distributed according to distribution Y , as in Figure 1, and the numbers in barrel X are distributed according to distribution X . If, at random, slips of paper from each barrel are selected and paired, they may be classified into successes and failures. A success is constituted by a strength value exceeding a stress value, as for example, when $x_1 > y_1$. Failure will occur if $x_2 < y_2$ as shown. It will be noted that, although the shaded area is a measure of interference, it is not interference itself: a pair of points x_3 and y_3 , although in the shaded area, will not produce failure. By continued pairing of stresses and strengths, at random, pairs will be found where the stress will exceed the strength. By continued experimentation a good estimate of the probability of interference can be found.

Most studies have assumed both the stress and the strength distribution to be normal. This is a natural assumption to make in order to solve a practical problem, as no work was found in literature dealing with an analytical expression for two interference distributions when they are not normal.

When the stress and strength distributions are assumed to be normal the probability of interference (and, therefore, failures) is the area under the standardized normal curve corresponding to the value of standardized normal variate, z , determined from the equation: (1)*

$$z = \frac{|\mu_y - \mu_x|}{\sqrt{\sigma_x^2 + \sigma_y^2}} \qquad (1)$$

* Numbers in parentheses designate References at end of paper.

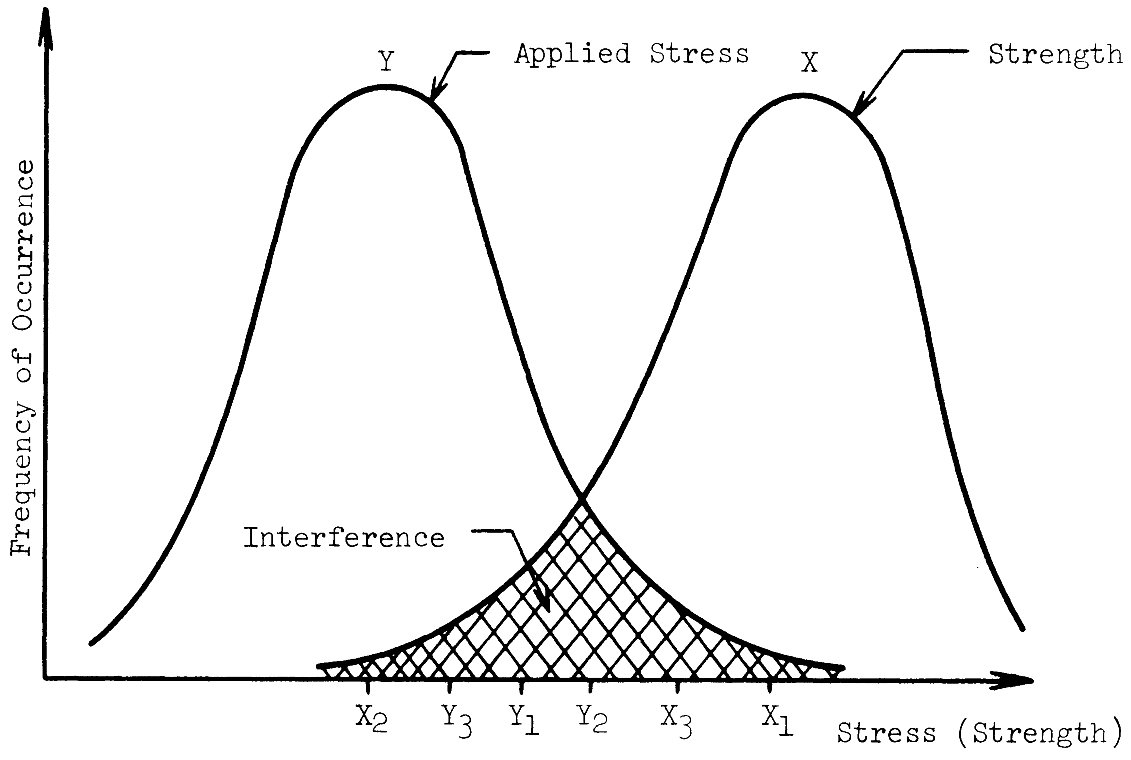


Figure 1. Interference of Stress and Strength Distributions.

where

$$\mu_y = \text{mean stress}$$

$$\mu_x = \text{mean strength}$$

$$\sigma_y^2 = \text{stress variance}$$

$$\sigma_x^2 = \text{strength variance}$$

The value of interference corresponding to the value of z (computed from Equation (1)) can be obtained from Table I of α vs K_α :

where

$$\alpha = \text{probability of interference (failures)}$$

$$K_\alpha = z$$

For example if $z = 1.96$ then from Table I, for $K_\alpha = 1.9$, the interference = .025, that is percent failure = 2.5%.

So far the discussion has been limited to the cases when both the stress and strength distributions can be assumed to be normal. In cases when either one or both are not normal the problem is much more involved. For example, the intersection of a normal and a log-normal distribution produces a distribution of an unknown origin.

In the past, problems such as this were solved largely through a "brute force", by a Monte-Carlo technique. Essentially, Monte-Carlo method consists of a sophisticated means of randomly selecting a sample from one distribution and comparing it with a sample taken from a different distribution. This is accomplished with the aid of tables of Random Numbers. The resultant paired data are plotted as a cumulative density function on a normal probability, Weibull, etc. paper and percent interference is read from the graph.

TABLE I

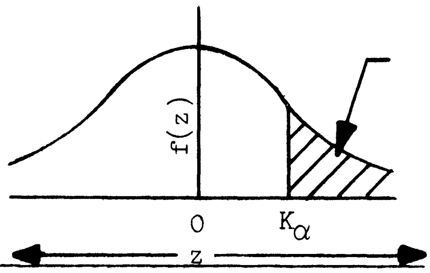
Normal Distribution[1]

Tabulation of the values of α versus K_α for the Standardized Normal Curve.

$$\alpha = P(z > K_\alpha) = \int_{K_\alpha}^{\infty} \frac{1}{2\pi} e^{-z^2/2} dz$$

= Area under the Standardized Normal Curve

from $z = K_\alpha$ to $z = \infty$



K_α	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139

In the present investigation a method of Integrals was used in preference to the Monte-Carlo technique. This method involves determining the expression resulting from the interference of the two distributions under consideration and establishing percent interference from this integral.

The advantages of the Integral method are:

1. For some distributions the integrals have been already tabulated and percent interference can be read directly from the table.
2. In those cases where the integrals have not been already tabulated, they can be evaluated by numerical analysis as done in the present investigation
3. The major shortcoming of the Monte-Carlo technique is that it requires a very large sample size for any accuracy. This shortcoming is avoided when the Integrals are used.
4. One of the objectives of the present study was to develop and evaluate an analytical expression for interference of any two distributions. Such expression is possible when the method of Integrals is used, but not when Monte-Carlo technique is employed.

When the two distributions are normal the interference can be simply expressed by a z-distribution, as described before. From an extensive survey of literature no work was found dealing with the analytical expression when the two interfering distributions are not normal. The purpose of this phase of the investigation, then, was to develop and to evaluate the complex integral resulting from the two non-normal distributions.

For reasons stated, the method of Integrals was chosen and an analytical expression was developed for the general case of two interfering distributions.[2] This would include cases such as Weibull-Weibull, Weibull-Normal, Normal-Normal, Exponential-Exponential, etc. It was then necessary to find the way of solving the complex integrals expressing such interferences. Numerical analysis was carried out using IBM 7090 Computer with MAD language to solve these integrals. Tables were then prepared for the interference as a function of the distribution parameters. (See Reference 2)

These tables include the combinations:

Stress Distribution	Strength Distribution
Normal	Weibull
Weibull	Weibull

because Normal and Weibull are the distributions most frequently found in actual engineering practice. The reason for choosing Weibull as the strength distribution for the two cases was that strength data, particularly fatigue strength data, can be more conveniently expressed in terms of the Weibull parameters (X_o, θ, b) than Normal parameters (μ, σ). It was felt that this restriction would not apply to the stress distribution.

STATISTICAL DISTRIBUTION OF STRENGTH - Since fatigue strength represents the major interest in the engineering application of the Interference Theory, this problem was studied in some detail. Statistical distribution of the fatigue strength of a mechanical component is a function of a number of factors, such as type of loading, surface finish, stress concentration, heat treatment, temperature, processes, and time. Each shows variability which is characterized by some form of a distribution. The effects of these factors on the statistical distribution of strength were studied in the present investigation.

CONVERSION OF LIFE DATA TO STRENGTH DATA - Most fatigue testing involves subjecting a number of specimens or parts to the same stress and repeating this process at various stress levels. The data thus obtained, known as life data, are used to construct the conventional S-N diagram. In this case, the scatter obtained is the scatter in life at a given stress. In the present investigation the attention was focused on the nature of the scatter in the fatigue strength at a given life. To obtain such data it is necessary to fatigue test all the specimens with different stresses imposed on them in such a manner that all would fail at a predetermined life. Practically, this is impossible. Another method is Strength Response Test[2] which in the present investigation was found inapplicable because of scarcity of the required data. Therefore, the following method, described below, was chosen.

The fatigue life data were obtained for various materials under various conditions. These data were then plotted on the conventional S-N diagram. Here, it is assumed that to each specimen of the population can be attributed an individual S-N curve, and that there exists for any population of specimens (at fixed test conditions) a family of non-intersecting S-N curves, which can be determined with any desired accuracy, each curve corresponding to a given probability.

Average S-N curve is then fitted to all the test points on the S-N diagram using the least square method. Passing through each test point draw S-N curve parallel to the average S-N curve. These will make a family of S-N curves (Figure 2). Now if the fatigue strength distribution at $N = N_1$ life is required, draw a vertical line at $N = N_1$ intersecting the family of S-N curves. These points of intersection $S_1, S_2, \ldots$ represent a sample from the strength distribution at a desired life.[3] The data are then plotted on the probability paper as a cumulative distribution function to determine the strength distribution. In this study Weibull distribution was adopted.

The Weibull distribution is of great usefulness in the analysis of fatigue data. The utility and value of the Weibull distribution results from the fact that it covers a considerable variety of distribution patterns and data which fit any of these patterns plots as a straight line on special graph paper, known as Weibull probability paper. Although the Weibull distribution provides a versatile means for describing the life characteristics, it can also be used for describing the mechanical properties, such as fatigue, tensile and rupture strengths studied in the present investigation.

The Weibull equation is a three parametric mathematical function having x as a variable. The general expression for the Weibull density function is:

$$f(x) = \frac{b}{\theta - X_o} \left(\frac{x - X_o}{\theta - X_o}\right)^{b-1} e^{-\left(\frac{x - X_o}{\theta - X_o}\right)^{b}} , \qquad (2)$$

$$X_o \leq x \leq \infty$$

and the general expression for the cumulative distribution function is:

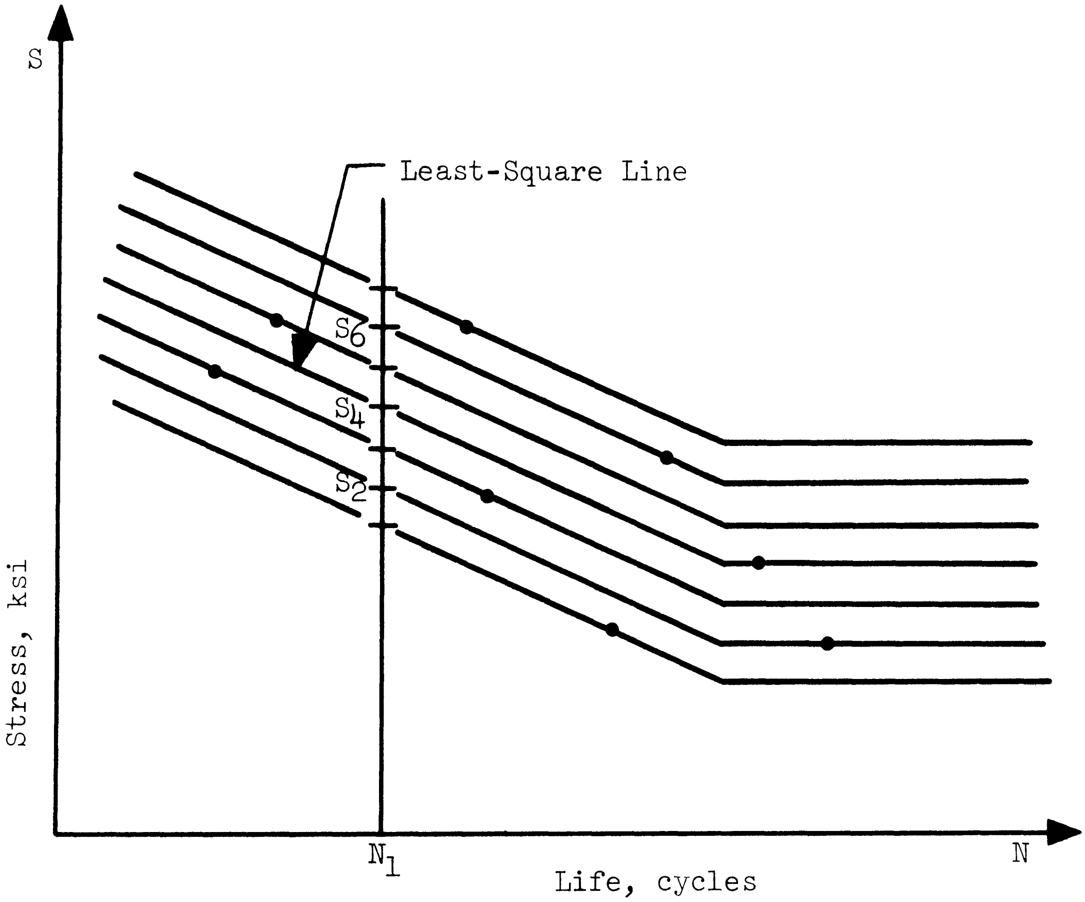


Figure 2. S-N Diagram for Converting Life Data to Strength Data.

$$f(x) = 1 - e^{-\left(\frac{x-X_o}{\Theta-X_o}\right)^b} , \qquad (3)$$

$$X_o \leq x \leq \infty$$

where

X_o is the lower bound of strength

Θ is the characteristic strength, where 63.2% of the population have strengths less than or equal to this value

b is the Weibull slope.

Versatility of the Weibull distribution is illustrated in Figure 3 which shows different forms of the distribution for various values of b. The Weibull slope b defines the shape of the curve, whereas Θ, the characteristic strength, is related to the standard deviation and to the mean strength values. It is therefore possible to have several forms of a particular distribution depending on:

1. The value of b (where Θ and X_o are constant).
2. The value of Θ (where b and X_o are constant).
3. The value of X_o (where Θ and b are constant).

As to special cases of Weibull distribution, it reduces to the truncated normal distribution when b is approximately equal to 3.5 and to the truncated exponential distribution when b is equal to 1.0.

Determination of the Weibull Parameters - In order to determine the Weibull parameters for the strength data the following steps are required:

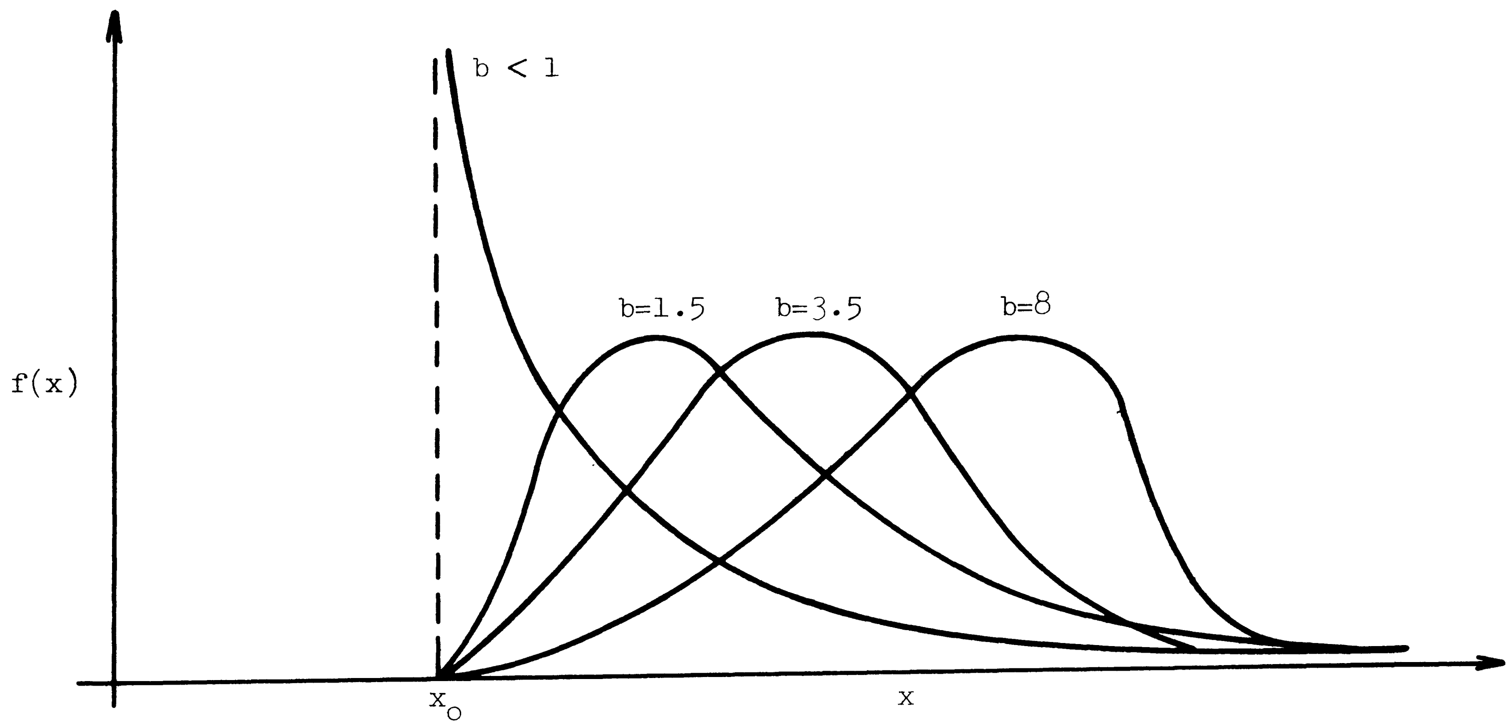


Figure 3. Plot of x vs f(x) in a Weibull Distribution.

1. The scatter of fatigue life at a given stress level, as obtained from the literature or other sources, is converted to the scatter of fatigue strengths at a given life in the manner discussed in Section: CONVERSION OF LIFE DATA TO STRENGTH DATA.
2. The fatigue strengths obtained from above are then arranged in the increasing order of value and median rank is assigned to each value as described in the example that follows.
3. The strengths are then plotted on the modified Weibull probability paper on the abscissa against the median ranks on the ordinate. (Figure 4)
4. In plotting the data an assumption was made that the lower bound of strength X_o (i.e. the minimum strength that can be expected in the whole population) is not zero, which is an obvious case, and therefore the next step was to determine the probable value of X_o. This value should be somewhere between the lowest value of the sample and zero.
5. X_o was determined by trial and error. This was done first by subtracting the assumed value of X_o from the original set of data, and then plotting on the Weibull probability paper. This was repeated until the straight line plot resulted.
6. As shown in Figure 4, the Weibull slope b was read directly and the value of θ was found equal to $(\theta_1 + X_o)$.

D_{6AC} Steel, Life of 10^6 Cycles, Fatigue Strength at 80°F

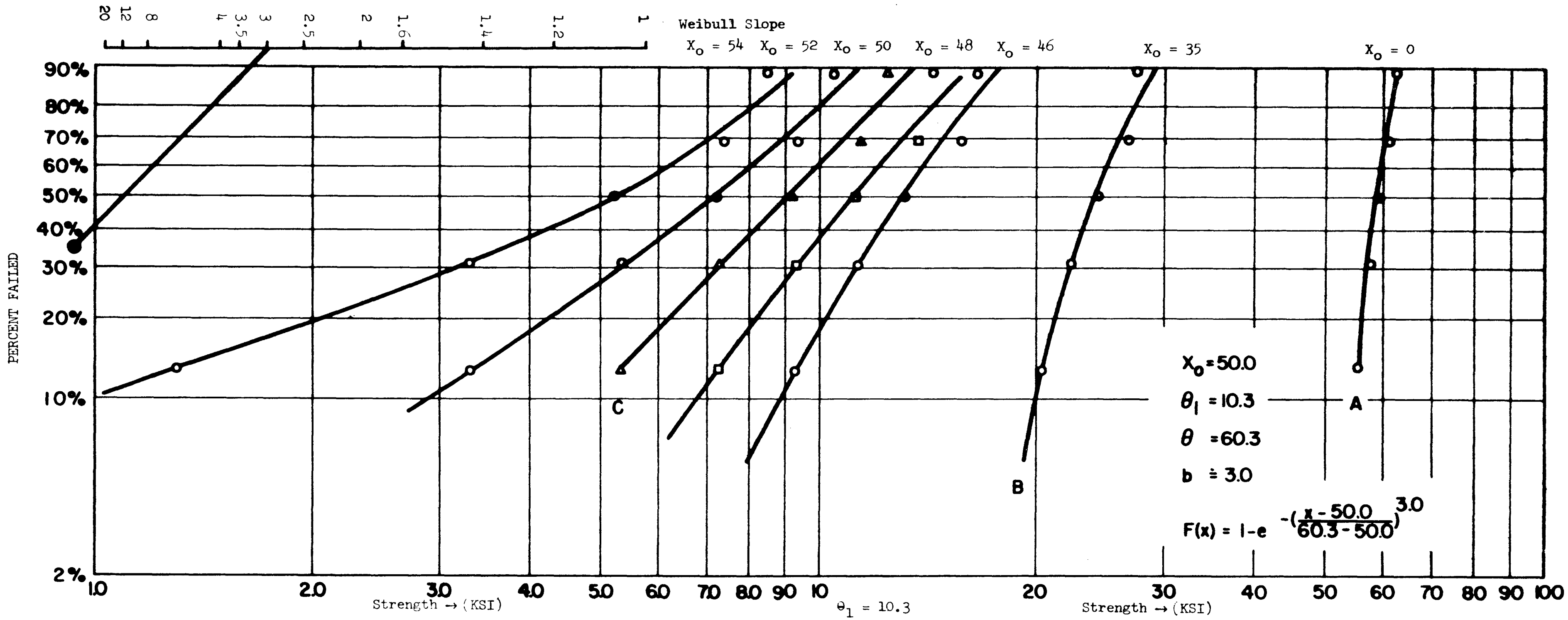


Figure 4. Modified Weibull Plot for Determination of the Weibull Parameters for the Above Conditions.

This method is illustrated by the following example:

Material: D_{6AC} Steel, $S_u = 270$ ksi

Conditions: Type of Load - Completely Reversed Bending

Surface Finish - Mechanically Polished

Stress Concentration Factor, $K_t = 1.0$

Test Temperature, 80°F

Fatigue Strength distribution data at 10^6 cycles are (in ksi):

57.3, 59.2, 62.5, 55.3, 61.4

In order to make Weibull cumulative plot, it becomes necessary to decide what rank is to be assigned to each particular strength value. The lowest strength in a group tested will have a definite percentage of the total population having strength lower than this, if the entire population were tested. If we knew exactly the percentage of the population having strength lower than the lowest in the sample, that percentage would be the true rank of the lowest strength in the sample. However, since we do not know the true rank, we make an estimate of it. We use an estimate such that in the long run the positive and negative errors of the estimate cancel each other. That is, half the time we would give the lowest strength a rank that is too high and the other half of the time a rank too low. A rank with this property is called median rank. A table of median ranks is given in Table II. The test data are then arranged in an increasing order of value and the appropriate median ranks for sample size $n = 5$ are read from Table II as follows:

TABLE II

Table of Median Ranks(1)

Sample Size = n

r	1	2	3	4	5	6	7	8	9	10
1	.5000	.2929	.2063	.1591	.1294	.1091	.0943	.0830	.0741	.0670
2		.7071	.5000	.3864	.3147	.2655	.2295	.2021	.1806	.1632
3			.7937	.6136	.5000	.4218	.3648	.3213	.2871	.2594
4				.8409	.6853	.5782	.5000	.4404	.3935	.3557
5					.8706	.7345	.6352	.5596	.5000	.4519
6						.8909	.7705	.6787	.6065	.5481
7							.9057	.7979	.7129	.6443
8								.9170	.8194	.7406
9									.9259	.8368
10										.9330

Sample Size = n

r	11	12	13	14	15	16	17	18	19	20
1	.0611	.0561	.0519	.0483	.0452	.0424	.0400	.0378	.0358	.0341
2	.1489	.1368	.1266	.1178	.1101	.1034	.0975	.0922	.0874	.0831
3	.2366	.2175	.2013	.1873	.1751	.1644	.1550	.1465	.1390	.1322
4	.3244	.2982	.2760	.2568	.2401	.2254	.2125	.2009	.1905	.1812
5	.4122	.3789	.3506	.3263	.3051	.2865	.2700	.2553	.2421	.2302
6	.5000	.4596	.4253	.3958	.3700	.3475	.3275	.3097	.2937	.2793
7	.5878	.5404	.5000	.4653	.4350	.4085	.3850	.3641	.3453	.3283
8	.6756	.6211	.5747	.5347	.5000	.4695	.4425	.4184	.3968	.3774
9	.7634	.7018	.6494	.6042	.5650	.5305	.5000	.4728	.4484	.4264
10	.8511	.7825	.7240	.6737	.6300	.5915	.5575	.5272	.5000	.4755
11	.9389	.8632	.7987	.7432	.6949	.6525	.6150	.5816	.5516	.5245
12		.9439	.8734	.8127	.7599	.7135	.6725	.6359	.6032	.5736
13			.9481	.8822	.8249	.7746	.7300	.6903	.6547	.6226
14				.9517	.8899	.8356	.7875	.7447	.7063	.6717
15					.9548	.8966	.8450	.7991	.7579	.7207
16						.9576	.9025	.8535	.8095	.7698
17							.9600	.9078	.8610	.8188
18								.9622	.9126	.8678
19									.9642	.9169
20										.9659

r = Failure number

Median Ranks

Sample Size = n

r	21	22	23	24	25	26	27	28	29	30
1	.0330	.0315	.0301	.0288	.0277	.0266	.0256	.0247	.0239	.0231
2	.0797	.0761	.0728	.0698	.0670	.0645	.0621	.0599	.0579	.0559
3	.1264	.1207	.1155	.1108	.1064	.1023	.0986	.0951	.0919	.0888
4	.1731	.1653	.1582	.1717	.1457	.1402	.1351	.1303	.1259	.1217
5	.2198	.2099	.2009	.1927	.1851	.1781	.1716	.1655	.1599	.1546
6	.2665	.2545	.2437	.2337	.2245	.2159	.2081	.2007	.1939	.1875
7	.3132	.2992	.2864	.2746	.2638	.2538	.2445	.2359	.2279	.2204
8	.3599	.3438	.3291	.3156	.3032	.2917	.2810	.2711	.2619	.2533
9	.4066	.3884	.3718	.3566	.3425	.3295	.3175	.3063	.2959	.2862
10	.4533	.4330	.4145	.3975	.3819	.3674	.3540	.3415	.3299	.3191
11	.5000	.4776	.4572	.4385	.4212	.4053	.3905	.5767	.3639	.3519
12	.5466	.5223	.5000	.4795	.4606	.4431	.4270	.4119	.3979	.3848
13	.5933	.5669	.5427	.5204	.5000	.4810	.4635	.4471	.4319	.4177
14	.6400	.6115	.5854	.5614	.5393	.5189	.5000	.4823	.4659	.4506
15	.6867	.6561	.6281	.6024	.5787	.5568	.5364	.5176	.5000	.4835
16	.7334	.7007	.6708	.6433	.6180	.5946	.5729	.5528	.5340	.5164
17	.7801	.7454	.7135	.6843	.6574	.6325	.6094	.5880	.5680	.5493
18	.8268	.7900	.7562	.7253	.6967	.6704	.6459	.6232	.6020	.5822
19	.8735	.8346	.7990	.7662	.7361	.7082	.6824	.6584	.6360	.6151
20	.9202	.8792	.8417	.8072	.7754	.7461	.7189	.6936	.6700	.6480
21	.9669	.9238	.8844	.8482	.8148	.7840	.7554	.7288	.7040	.6808
22		.9684	.9271	.8891	.8542	.8218	.7918	.7640	.7380	.7137
23			.9698	.9301	.8935	.8597	.8283	.7992	.7720	.7466
24				.9711	.9329	.8976	.8648	.8344	.8060	.7795
25					.9722	.9354	.9013	.8696	.8400	.8124
26						.9733	.9378	.9048	.8740	.8453
27							.9743	.9400	.9080	.8782
28								.9752	.9420	.9111
29									.9760	.9440
30										.9768

x, ksi	Median Ranks, %
55.3	12.94
57.3	31.47
59.2	50.00
61.4	68.53
62.5	87.06

These data are then plotted on the modified Weibull probability paper as shown in Figure 4, curve A.

In plotting these data an assumption was made that the lower-bound of strength X_o (i.e. the minimum strength that can be expected in the whole population) is zero. This is obviously not the case, and the next step was to determine the probable value of X_o. This value should be somewhere between the lowest value of the sample (55.3 ksi) and zero. As the first trial therefore assume that X_o is 35 ksi.

By subtracting X_o from the original set of data, the following is obtained:

$(x - X_o)$ ksi	Median Ranks, %
20.3	12.94
22.3	31.47
24.2	50.00
26.4	68.53
27.5	87.06

When these are plotted (Figure 4, curve B) the resultant curve is not a straight line. Therefore, other values of X_o are assumed,(3) and the same procedure is repeated until, for a certain assumed X_o one can best linearize all the test points. In this case the best line nearest to a straight line is for $X_o = 50$ ksi, curve C. Through these points, then, a straight line is fitted using the Least Square Method.

The value of $(x - X_o)$ at 63.2% is read off to determine the characteristic strength θ :

$$\theta = x \text{ at } 63.2\%$$

$$(x-x_o)_{63.2\%} = \theta_1 = 10.3 \text{ ksi}$$

$$\theta = (x)_{63.2\%} = \theta_1 + x_o = 10.3 + 50$$

$$= 60.3 \text{ ksi}$$

The Weibull slope b is determined by drawing a line parallel to the straight line of $X_o = 50$ and passing it through the pivot point. The point where this line intersects the Weibull slope scale is the value of the Weibull slope. In this case, $b = 3.0$. Hence, the Weibull parameters for the given set of fatigue strength data are:

$$X_o = 50 \text{ ksi}$$

$$\theta = 60.3 \text{ ksi}$$

$$b = 3.0$$

The analytical form for the corresponding Weibull equation is:

$$F(x) = 1 - e^{-\left(\frac{x-50}{60.3-50}\right)^{3.0}}$$

The parameters X_o , b and θ were determined for various materials, under various conditions, on the basis of all the available test data obtained. These were tabulated and the most representative parameters were then plotted.[2] A sample of the tables and the plots is shown in Figure 10 and Table III.

TABLE III

Weibull Parameters

Ti-6Al-4V

S_u = 177 ksi
S_y = 166 ksi

ROTARY BENDING

Composition[1]
Specimen Condition[3]
Meaning of Symbols[4]

T °F	H.T.[2]	S_m ksi	X_o ksi			b			θ ksi		
Life, cycles			10^4	10^5	10^6	10^4	10^5	10^6	10^4	10^5	10^6
Effect of Temperature											
80	A	0	67.0	57.0	50.0	2.8	2.85	3.35	89.2	78.0	68.0
400	A	0	75.0	56.0	41.0	2.75	2.95	3.1	96.1	72.7	55.3
600	A	0	50.0	40.0	33.0	2.65	2.7	3.0	77.1	60.5	47.0
800	A	0	43.0	34.0	26.0	2.9	3.1	3.25	71.0	55.0	44.0
900	A	0	45.0	34.0	25.0	3.5	4.0	4.1	65.0	49.7	38.0
80	B	0	70.0	55.0	46.0	3.05	3.35	3.48	97.7	80.2	66.1
400	B	0	45.0	35.0	25.0	1.9	2.2	2.7	78.8	57.5	44.0
600	B	0	42.0	32.0	22.0	3.15	3.2	3.6	79.2	59.9	44.5
800	B	0	40.0	30.0	20.0	3.25	3.5	4.1	71.7	50.4	36.6
Effect of Heat Treatment											
80	A	0	70.0	40.0	26.0	2.9	3.8	4.1	118.6	77.2	50.2
80	B	0	70.0	55.0	46.0	3.05	3.35	3.48	97.7	80.2	66.1

STATISTICAL DISTRIBUTION OF STRESS

STRESS SPECTRUM VS STRESS DISTRIBUTION - The problem of stress distribution, in the Interference Theory, is much more involved than the problem of strength distribtution. Consider, for example, the problem of a connecting rod in a reciprocating engine. Because of the variation in hardness, surface finish, etc. the fatigue strength will vary from one rod to another. This will result in a distribution curve, in which the strength will be plotted on the abscissa and the number of rods having a given strength (i.e. frequency of occurrence) on the ordinate.

Consider now the stress distribution in the connecting rods. The stresses in the rod result from the combined effect of gas pressure loading and inertia loading. If the attention is now focused on a single rod, then the variation in the two types of loading will produce a distribution of stresses in this particular rod. The resultant curve will be a plot of the stresses in the rod on the abscissa and the number of times that this stress occurs in this particular rod on the ordinate (Figure 5(a)).

This, however, is not what is wanted in the application of the Interference Theory, because this distribution of stresses cannot be matched with the distribution of strength. In the strength distribution the ordinate gives number of rods having given strength. Therefore in stress distribution the ordinate must read number of rods having given stress (and not number of times given stress occurs in a single rod). This can be obtained by considering the fact that different engines will be subjected in service to different operating conditions and, therefore,

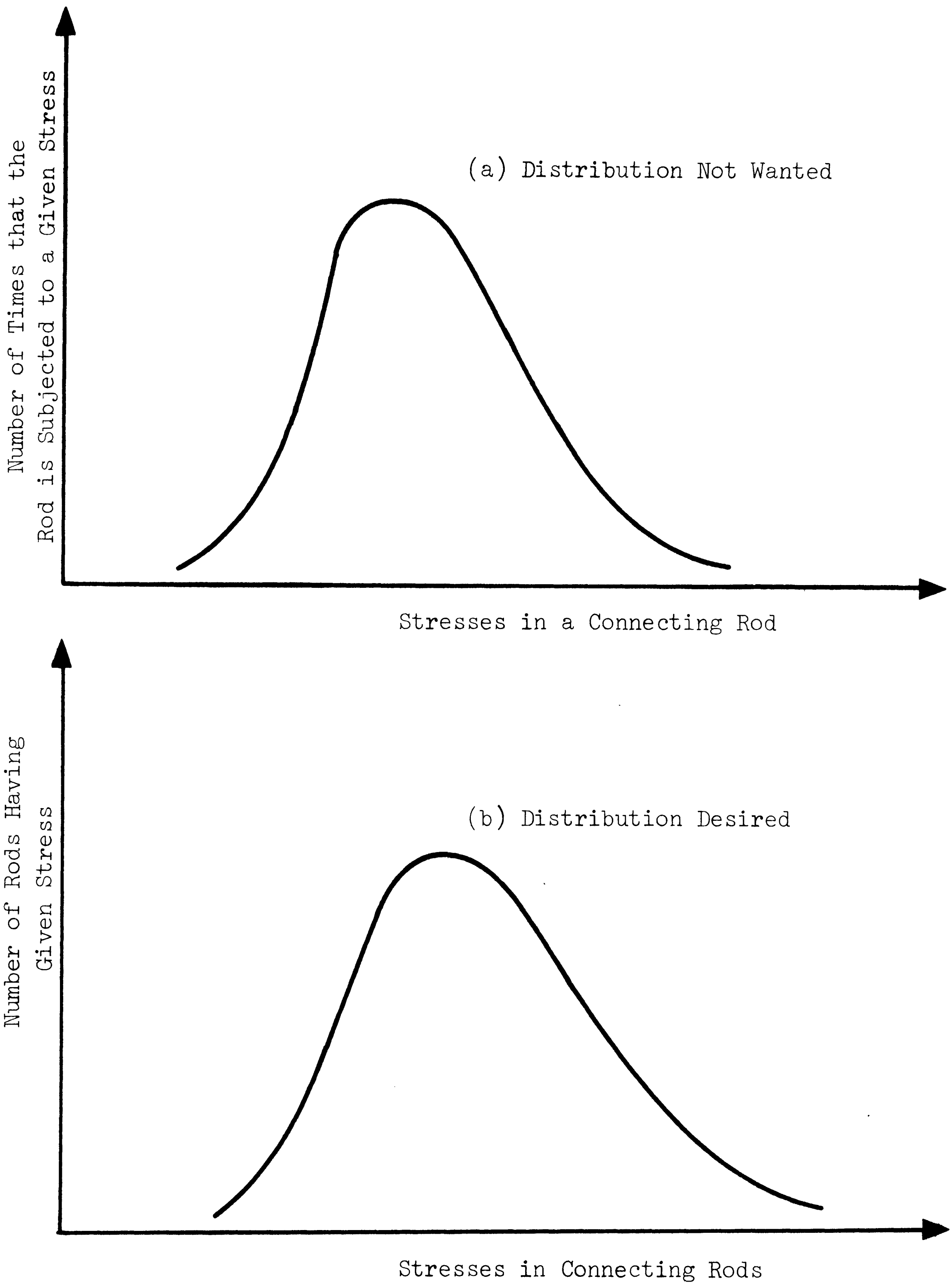


Figure 5. Stress Distribution for the Interference Theory.

the distribution of gas pressure loading and inertia loading will vary from engine to engine. As pointed out in the following section, a spectrum of stresses must be converted to an equivalent stress for the purpose of Interference Theory. Therefore, if a spectrum of loading due to different service conditions varies from engine to engine, in a population of connecting rods the equivalent stress will vary from rod to rod. Thus the statistical stress distribution desired for the Interference Theory may be obtained (Figure 5(b)). In this distribution the equivalent stress will be plotted on the abscissa and the number of rods (frequency of occurrence) having that stress on the ordinate. The distribution then can be compared with the strength distribution to obtain the probability of interference.

CONVERSION OF STRESS SPECTRUM TO AN EQUIVALENT STRESS (S_{equ}) - By definition, equivalent stress is a completely reversed stress of constant amplitude which, when imposed on a part should cause failure at the same life as if the stress spectrum was imposed instead. Thus, the damage accumulated at any given life, due to this equivalent stress, will be the same as if due to the spectrum of stress.

The first step is to convert the operating stresses, which may have some mean stress associated with them, to zero mean stresses, that is, to the completely reversed stress. This can be done by means of the Modified Goodman Diagram. Draw the Goodman diagram on a cartesian paper. (Figure 6) The point X or point Y on this diagram is the endurance strength (S_n) of the part subjected to the loads. S_n can be computed from the equation:

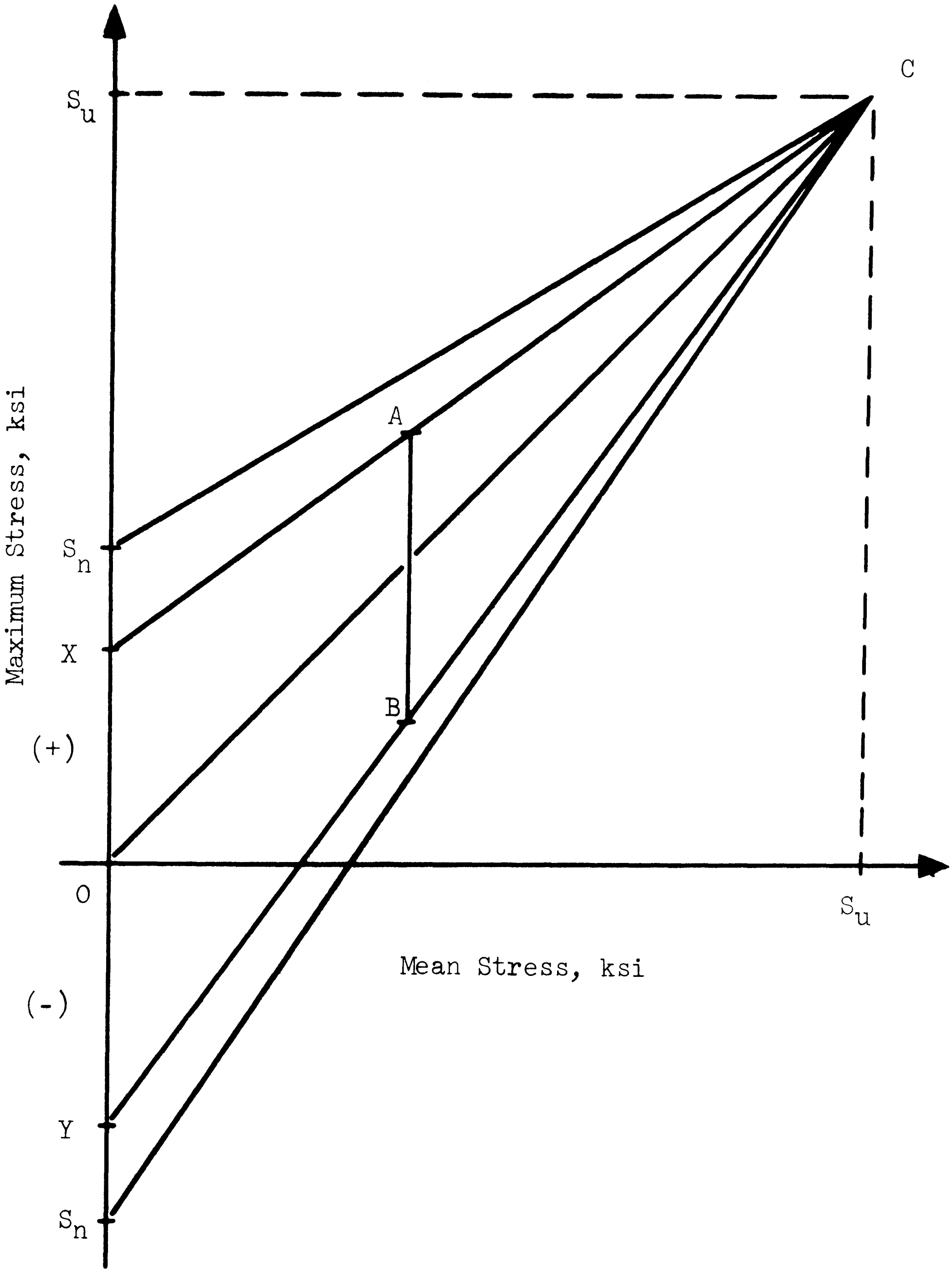


Figure 6. Modified Goodman Diagram.

$$S_n = S_n' \times K_1 \times K_2 \times K_3 \times K_4 \times K_5 \times K_6 \times K_7 \tag{4}$$

Where S_n' is endurance limit of the material

K_1 is the size factor

K_2 is the loading factor

K_3 is the surface finish factor

K_4 is the surface treatment factor

K_5 is the stress concentration factor

K_6 is the temperature factor

K_7 is the life factor .

S_u is the ultimate tensile strength of the material.

Now, from the spectrum of operating stresses plot each stress cycle on this diagram as shown (for example, line AB). Connect CA and CB and extend to the vertical line where mean stress is equal to zero. Hence, XY is the zero mean stress equivalent to AB . After reducing all such mean stresses, the stress spectrum will then have all stress cycles completely reversed. This spectrum can then be reduced to a single equivalent stress (S_{equ}) of constant amplitude, by means of Miner's Rule which implies linear damage accumulation. For the procedure see the solved example in Section: APPLICATION TO DESIGN PROBLEMS.

INTERFERENCE OF STRESS DISTRIBUTION WITH STRENGTH DISTRIBUTION

After strength distribution and stress distribution are determined the two are compared and percent interference is determined. For a given strength distribution the percent interference will depend on the distribution of the equivalent stress S_{equ}. A search through literature and other sources produced considerable amount of data leading to strength distribution but very little information on stress distribution.

In some engineering applications there is very little scatter in stresses. This leads to a stress distribution with standard deviation equal to zero. This distribution can be represented by a straight line, as in Figure 7, and the interference can be determined as shown.

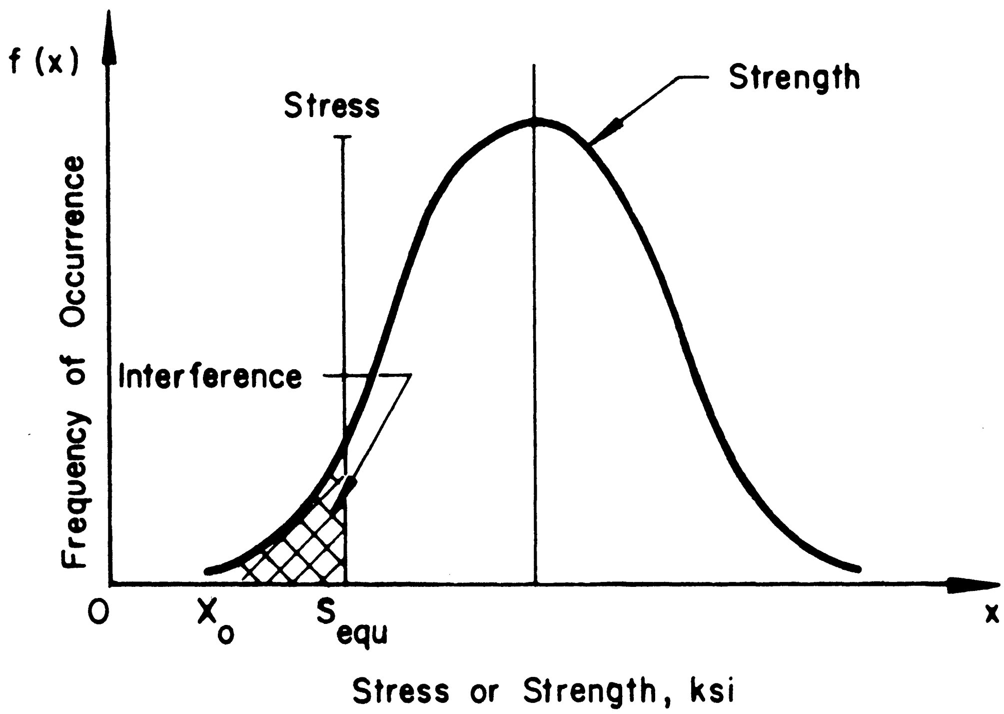


Figure 7. Interference with Standard Deviation of Stress equal to Zero.

For a given S_{equ} , interference may increase or decrease, if the life to which the components are designed is changed. This is shown in terms of S-N diagram in Figure 8.

In those engineering applications where the scatter in stresses is appreciable the above approach will obviously not apply. On the basis of past experience, in the present investigation the stress distribution (S_{equ}) was assumed to be normal and the range of standard deviations to be not less than $.01\mu$ and not more than $.10\mu$ where μ is equal to S_{equ} . The resulting interference is represented qualitatively in Figure 9.

A design problem employing this method is illustrated in section that follows.

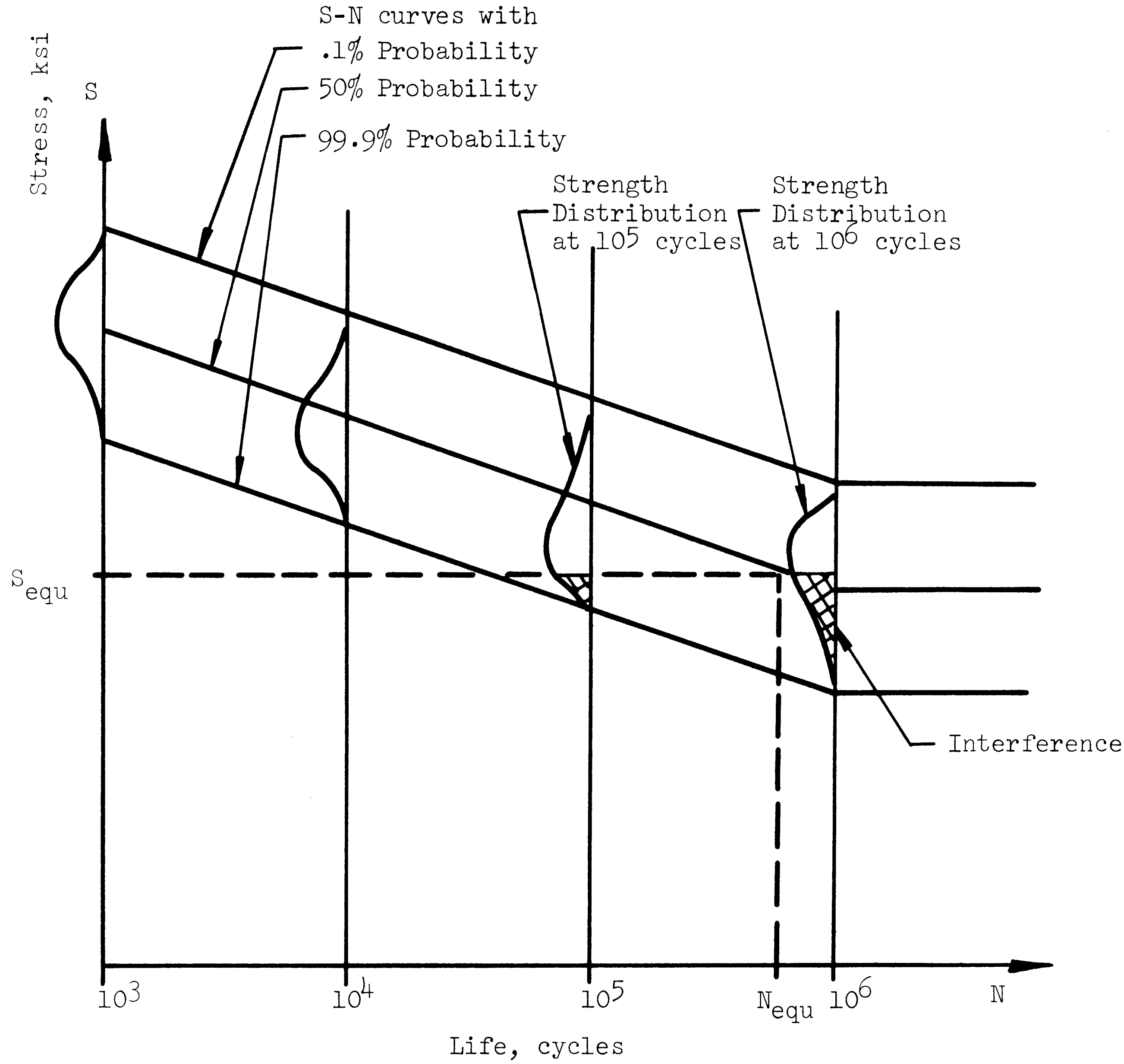


Figure 8. S-N Diagram Representing the Dependence of Interference on Life.

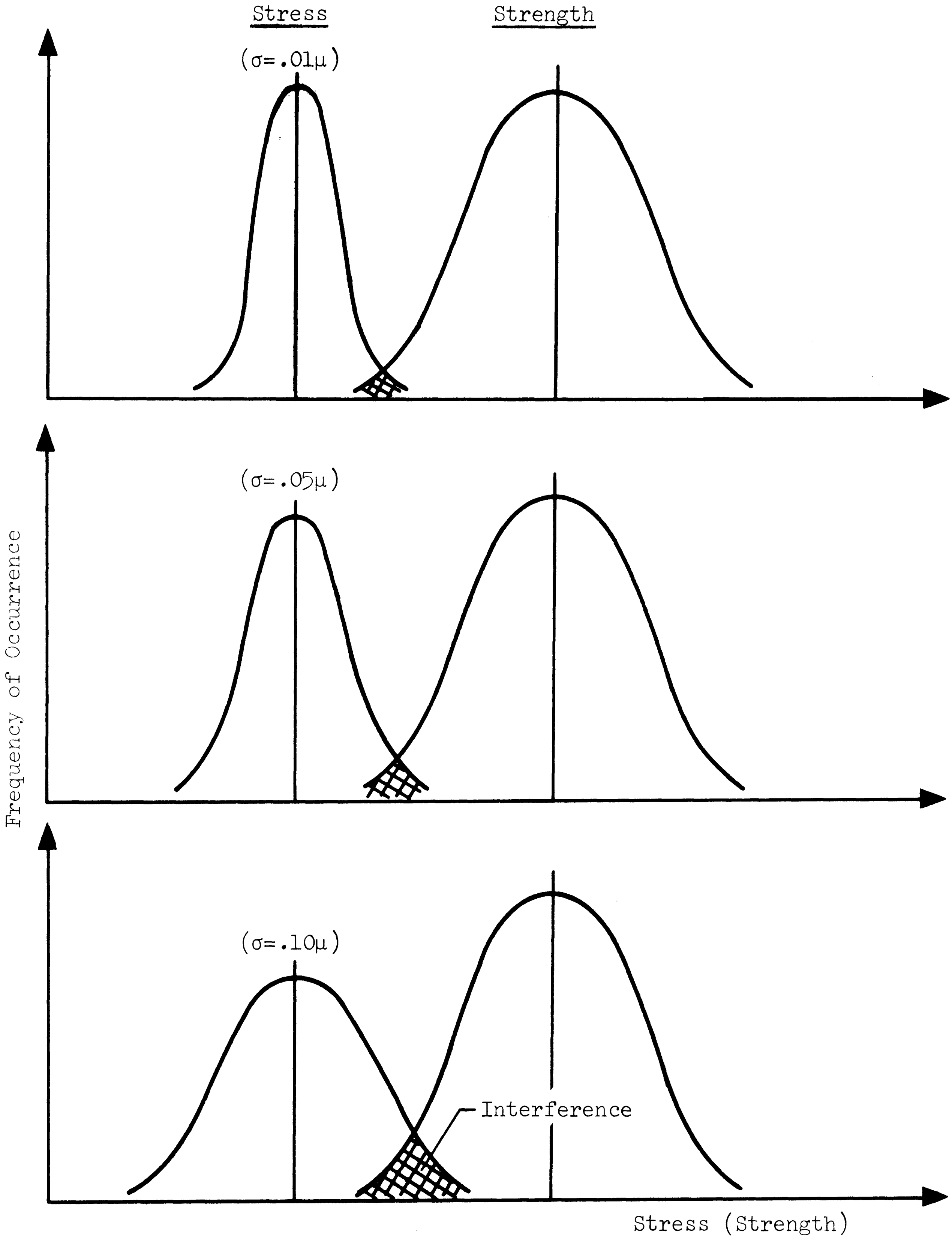


Figure 9. Interference of S_{equ} with Strength Distribution for Various Values of Standard Deviation.

APPLICATION TO DESIGN PROBLEM

Once the parameters of the strength distribution $(X_o, b\ \Theta)$ and stress distribution ($\mu = S_{equ}$ and $\sigma = k\mu$, where k represents fraction of the average stress) are determined, the percent interference, that is percent failures, can be computed with the aid of Tables VII and VIII. (Tables of interference values corresponding to a wide range of values of stress and strength distribution parameters are given in Reference 2.) Specific steps to be taken are illustrated by the following example.

A certain machine part was designed to withstand in service 10,000 load cycles. The problem was to predict its reliability under the following conditions:

Material: T_1-6Al-4V, S_u = 177 ksi, S_y = 166 ksi

Design Life: 10^4 cycles

Type of Loading: Bending, completely reversed

Size: 0.25 in.

Surface Finish: Hot rolled

Theoretical Stress Concentration Factor: $k_t = 1.0$

Operating Temperature: 600°F

Weibull Parameters

The first step was to determine strength distribution in terms of the Weibull parameters. From Table III or Figure 10. Weibull parameters corresponding to the above conditions were found to be:

X_o = 50 ksi

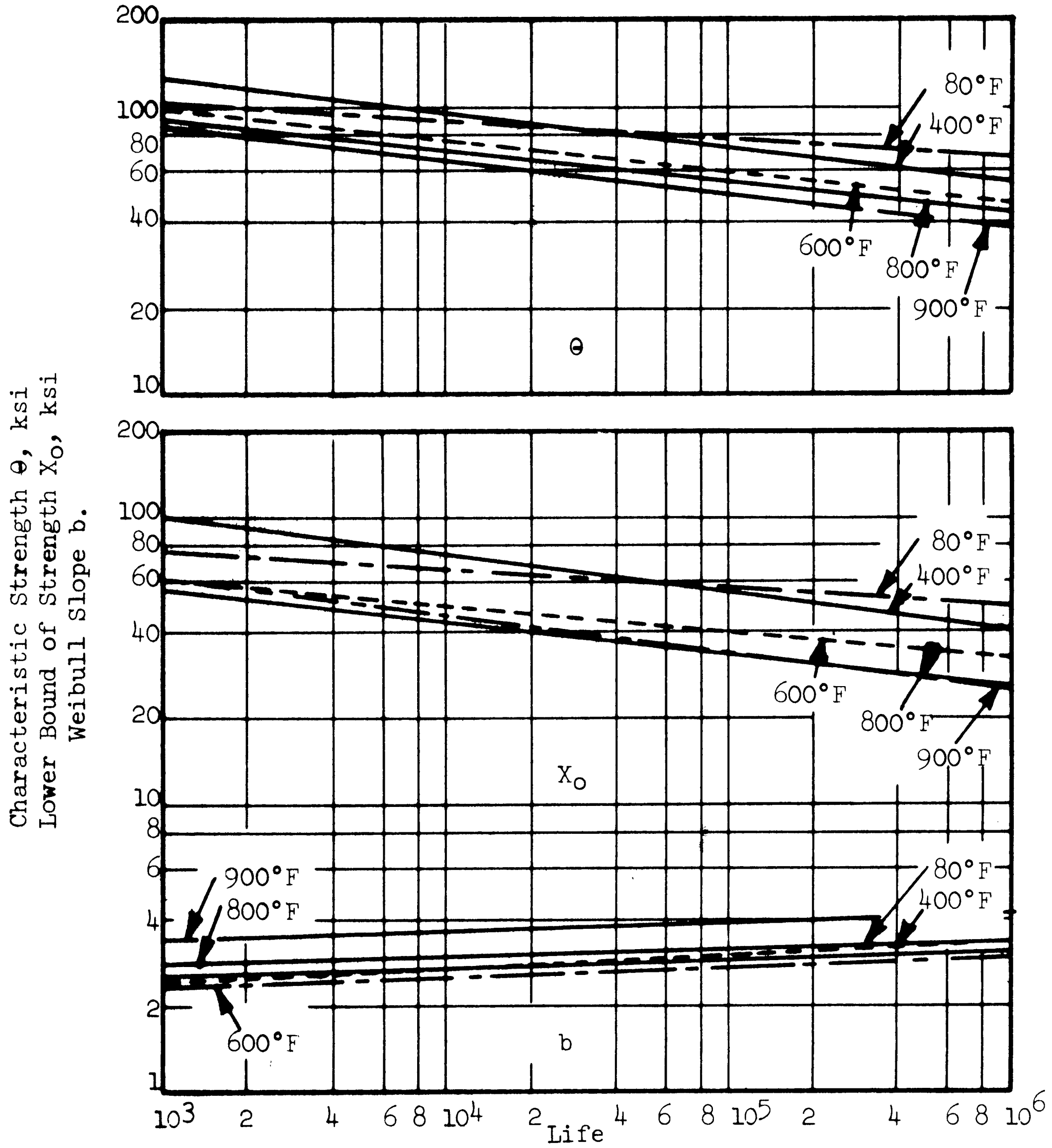
b = 2.65

Θ = 77.1 ksi.

FATIGUE STRENGTH

Effect of Temperature

S_u = 177 ksi, S_y = 166 ksi



Rotary Beam Bending

Hot Rolled

Composition:
6% Al, 4%V, Max .07% N_i , max .10% C,
max .015% H, max .40% Fe, max .30% O

Mean Stress = 0
Heat Treatment:
(A: sol. treated 1690°F, 12 min.
WQ, aged 900°F, 4 hrs. air cooled)

Figure 10. Weibull Parameters.

The Equivalent Stress

As to the stress distribution, the part was instrumented and stress spectrum was recorded as shown in columns 1 and 2 of Table IV. In order to determine the parameters of the stress distribution ($S_{equ} = \mu$, and σ) Miner's rule was used. From the S-N curve of the material (Figure 11), the number of cycles to failure, N , corresponding to stresses in Column 1, Table IV were determined. This is shown in Column 3, Table IV. Using Miner's rule and tabulated data in Table IV, N_{equ} was determined:

$$N_{equ} = 1 \times \frac{\sum n_i}{\sum \frac{n_i}{N_i}}$$

$$N_{equ} = 1 \times \frac{425}{21.845 \times 10^{-4}} = 1.945 \times 10^5 \text{ cycles.}$$

From the S-N curve (Figure 11), the stress corresponding to $N_{equ} = 1.945 \times 10^5$ cycles was found to be $S_{equ} = 55$ ksi . Hence, a completely reversed stress application of 55 ksi can be substituted for the recorded stress spectrum (Columns 1 and 2, Table IV).

PREDICTION OF PERCENT FAILURES - Once the strength and stress distribution parameters are established, percent failures can be determined.

In some engineering applications the scatter in the operating stresses is very small and, therefore, the standard deviation of the stress can be assumed to be zero. In those cases the percent failures can be determined as follows:

TABLE IV

Stress and Life Data for Miner's Rule

Completely* reversed stress S , ksi	Occurrences n , cycles	Number of cycles to failure, N	$\frac{n}{N}$
1	2	3	4
52.0	200	3.5×10^5	5.710×10^{-4}
54.1	80	2.4×10^5	3.333×10^{-4}
56.5	50	1.6×10^5	3.125×10^{-4}
58.0	60	1.2×10^5	5.000×10^{-4}
59.3	20	1.0×10^5	2.000×10^{-4}
62.0	10	6.6×10^4	1.515×10^{-4}
64.8	5	4.3×10^4	1.162×10^{-4}
	$\Sigma n_i = 425$		$\Sigma \frac{n_i}{N_i} = 21.845 \times 10^{-4}$

* Actually, stress was not completely reversed. It was reduced with the aid of the Goodman diagram to a completely reversed stress using the procedure given in Section: CONVERSION OF STRESS SPECTRUM TO AN EQUIVALENT STRESS (S_{equ}).

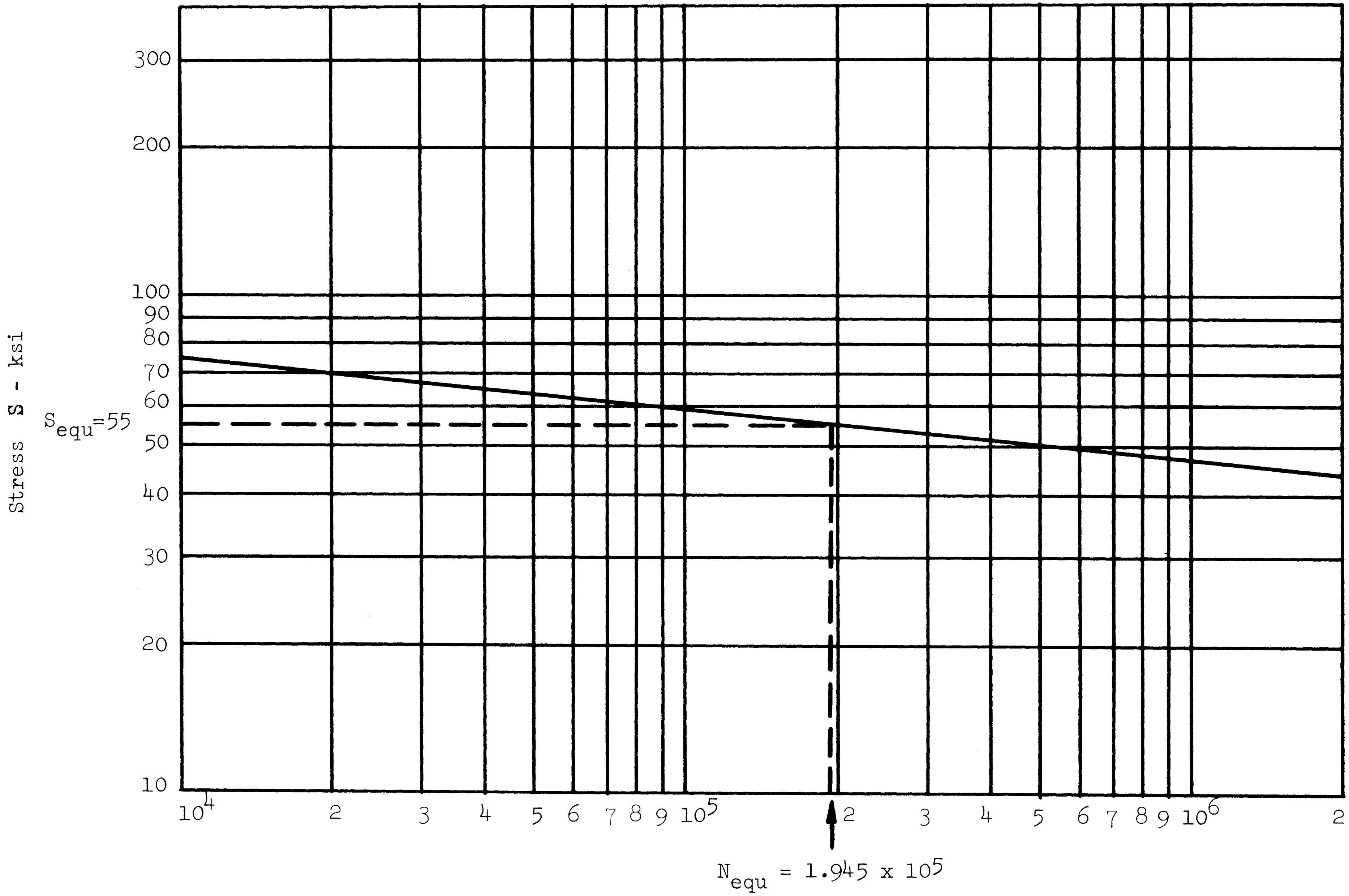


Figure 11. S-N Relationship.

Interference (Failures) = $F(x) = 1 - e^{-\left(\frac{x - X_o}{\Theta - X_o}\right)^b}$ = shaded area under the curve in Figure 7 where

$$x = S_{equ} = 55 \text{ ksi}$$

$$X_o = 50 \text{ ksi}$$

$$b = 2.65$$

$$\Theta = 77.1 \text{ ksi} .$$

$$F(x) = 1 - e^{-\left(\frac{55 - 50}{77.1 - 50}\right)^{2.65}}$$

$$= 1 - e^{-.0114}$$

$$= .0113$$

Percent Failures = 1.13%

In those engineering applications where the scatter of stress is appreciable percent failures may be found as follows. In actual engineering practice, standard deviation lies in the range

$$0.01 \leq \frac{\sigma}{\mu} \leq 0.10$$

In the absence of any specific information, an average value of $\frac{\sigma}{\mu} = 0.05$ can probably be assumed. Using this value, percent failures were determined:

Strength	Stress
$X_o = 50$ ksi	$\mu = S_{equ} = 55$ ksi
$b = 2.65$	$\sigma = 0.05\mu$
$\Theta = 77.1$ ksi	$= 0.05 \times 55$ ksi
	$= 2.75$ ksi

From the above data, parameters C, A and B(x), to be used in the tables of Interference values (Tables VII and VIII), were computed:

$$C = \frac{\Theta - X_o}{\sigma} = \frac{77.1 - 50}{2.75} \simeq 10$$

$$A = \frac{X_o - \mu}{\sigma} \quad \frac{50 - 55}{2.75} = -1.82$$

$$B = b = 2.65$$

The Interference value corresponding these parameters was determined by interpolation between Table VII (for B(x) = 2.0) and Table VIII (for B(x) = 3.0):

Interference $\simeq$ .0245

or, Percent Failures = 2.45%.

Thus, probabilities of failure to be expected are:

In the event of no scatter in stresses - 1.13% Failures.

For the scatter of the order of 0.05μ - 2.45% Failures.

The Effect of Design Factors

In this manner, the effect of various design factors on percent failures, can be determined. Table V shows the effect of temperature on percent failures for design conditions states in the above example. Table VI gives the effect of life on interference for a different set of conditions stated below:

Material: M10 Tool Steel, S_u = 330 ksi

Design Life: 10^5 cycles

Type of loading: Bending, completely reversed

TABLE V

Effect of Temperature on Percent Failures

Material	Temperature °F	Equivalent Stress S_{equ}, ksi	Weibull Parameters of Strength X_o ksi	b	θ, ksi	Percent Failures σ = 0	σ = 0.05μ
T_i-6Al-4V	600	55.0	50	2.65	77.1	1.13	2.45
	80	55.7	70	3.2	96.8	0.0	0.0

TABLE VI

Effect of Life on Percent Failures

Material	Life, cycles	Equivalent Stress, S_{equ}, ksi	Weibull Parameters of Strength X_o, ksi	b	θ, ksi	Percent Failures $\sigma = 0$	$\sigma = .028\mu$
M 10 Tool Steel	10^4	122	127	1.89	163.5	0.0	0.0
	10^5		119	1.95	153.2	0.865	2.04
	10^6		111	2.0	143.5	10.80	12.03

TABLE VII

Interference Values

STRESS DISTRIBUTION - Normal
STRENGTH DISTRIBUTION - Weibull

B(X) = 2.00

$C = \frac{\theta - x_o}{\sigma}$, $A = \frac{x_o - \mu}{\sigma}$

A \ C	10	15	20	25	30	35	40	45	50	55
.8	.0011	.0005	.0003	.0002	.0001	.0001	.0001	.0001	.0000	.0000
.6	.0017	.0008	.0004	.0003	.0002	.0001	.0001	.0001	.0001	.0001
.4	.0025	.0011	.0006	.0004	.0003	.0002	.0002	.0001	.0001	.0001
.2	.0035	.0016	.0009	.0006	.0004	.0003	.0002	.0002	.0001	.0001
.0	.0049	.0022	.0012	.0008	.0006	.0004	.0003	.0002	.0002	.0002
-.2	.0067	.0030	.0017	.0011	.0008	.0006	.0004	.0003	.0003	.0002
-.4	.0089	.0040	.0023	.0014	.0010	.0007	.0006	.0004	.0004	.0003
-.6	.0116	.0052	.0030	.0019	.0013	.0010	.0007	.0006	.0005	.0004
-.8	.0149	.0067	.0038	.0024	.0017	.0012	.0010	.0008	.0006	.0005
-1.0	.0188	.0085	.0048	.0031	.0021	.0016	.0012	.0009	.0008	.0006
-1.4	0.284	.0128	.0073	.0047	.0032	.0024	.0018	.0014	.0012	.0010
-1.8	.0407	.0185	.0105	.0067	.0047	.0034	.0026	.0021	.0017	.0014
-2.2	.0557	.0254	.0144	.0093	.0065	.0047	.0036	.0029	.0023	.0019
-2.6	.0733	.0336	.0191	.0123	.0086	.0063	.0048	.0038	.0031	.0026
-3.0	.0935	.0431	.0246	.0158	.0110	.0081	.0062	.0049	.0040	.0033
-3.4	.1159	.0538	.0308	.0198	.0138	.0102	.0078	.0062	.0050	.0041
-3.8	.1406	.0658	.0377	.0243	.0170	.0125	.0096	.0076	.0062	.0051
-4.2	.1671	.0789	.0453	.0293	.0205	.0151	.0116	.0092	.0074	.0061
-4.6	.1954	.0930	.0536	.0347	.0243	.0179	.0137	.0109	.0088	.0073
-5.0	.2251	.1082	.0626	.0406	.0284	.0210	.0161	.0127	.0103	.0086
-5.5	.2640	.1286	.0748	.0486	.0341	.0251	.0193	.0153	.0124	.0103
-6.0	.3043	.1504	.0879	.0573	.0402	.0297	.0228	.0181	.0147	.0121
-6.5	.3457	.1735	.1020	.0667	.0468	.0346	.0266	.0211	.0171	.0142
-7.0	.3876	.1977	.1170	.0767	.0539	.0399	.0307	.0244	.0198	.0164
-8.0	.4713	.2490	.1493	.0985	.0695	.0516	.0398	.0316	.0256	.0212
-9.0	.5525	.3032	.1845	.1226	.0869	.0646	.0499	.0396	.0322	.0267
-10.0	.6285	.3591	.2222	.1488	.1059	.0790	.0611	.0486	.0396	.0328

TABLE VIII

Interference Values

STRESS DISTRIBUTION - Normal
STRENGTH DISTRIBUTION - Weibull

B(X) = 3.00

$C = \frac{\theta - x_o}{\sigma}$, $A = \frac{x_o - u}{\sigma}$

A \ C	10	15	20	25	30	35	40	45	50	55
.8	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.6	.0002	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.4	.0004	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.2	.0005	.0002	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000
.0	.0008	.0002	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000
-.2	.0011	.0003	.0001	.0001	.0000	.0000	.0000	.0000	.0000	.0000
-.4	.0016	.0005	.0002	.0001	.0001	.0000	.0000	.0000	.0000	.0000
-.6	.0022	.0007	.0003	.0001	.0001	.0001	.0000	.0000	.0000	.0000
-.8	.0030	.0009	.0004	.0002	.0001	.0001	.0000	.0000	.0000	.0000
-1.0	.0041	.0012	.0005	.0003	.0002	.0001	.0001	.0000	.0000	.0000
-1.4	.0069	.0021	.0009	.0004	.0003	.0002	.0001	.0001	.0001	.0000
-1.8	.0111	.0033	.0014	.0007	.0004	.0003	.0002	.0001	.0001	.0001
-2.2	.0169	.0051	.0022	.0011	.0006	.0004	.0003	.0002	.0001	.0001
-2.6	.0247	.0075	.0032	.0016	.0009	.0006	.0004	.0003	.0002	.0002
-3.0	.0349	.0106	.0045	.0023	.0013	.0008	.0006	.0004	.0003	.0002
-3.4	.0475	.0145	.0062	.0032	.0018	.0012	.0008	.0005	.0004	.0003
-3.8	.0630	.0193	.0082	.0042	.0024	.0015	.0010	.0007	.0005	.0004
-4.2	.0815	.0252	.0108	.0055	.0032	.0020	.0014	.0010	.0007	.0005
-4.6	.1031	.0322	.0138	.0071	.0041	.0026	.0017	.0012	.0009	.0007
-5.0	.1279	.0404	.0173	.0089	.0052	.0033	.0022	.0015	.0011	.0008
-5.5	.1634	.0524	.0225	.0116	.0067	.0043	.0029	.0020	.0015	.0011
-6.0	.2037	.0665	.0287	.0148	.0086	.0054	.0036	.0026	.0019	.0014
-6.5	.2485	.0828	.0360	.0186	.0108	.0068	.0046	.0032	.0023	.0018
-7.0	.2973	.1013	.0443	.0230	.0134	.0084	.0057	.0040	.0029	.0022
-8.0	.4039	.1454	.0645	.0336	.0196	.0124	.0083	.0059	.0043	.0032
-9.0	.5165	.1985	.0897	.0471	.0276	.0175	.0117	.0083	.0060	.0045
-10.0	.6269	.2600	.1202	.0636	.0374	.0237	.0160	.0112	.0082	.0062

Surface Finish: Mechanically Polished

Theoretical Stress Concentration Factor: $k_t = 1.0$

Heat Treatment: Preheat 1450°F 1/2 hr., harden 1250°F 5 min., OQ until black, AC, Temp. 1100°F 2 hrs., AC, Retemp. 1100°F 2 hrs., AC, after finishing operation, Nitrided 975°F 48 hrs.

CONCLUSIONS AND RECOMMENDATIONS

A method was developed for employing stress-strength Interference Theory as a practical engineering tool to be used for designing and quantitatively predicting the reliability of mechanical parts and components subjected to mechanical loading.

This method is based on the considerable empirical data gathered and it also has sound theoretical basis. This method eliminates the concept of a Factor of Safety and substitutes Percent Failures.

Although a great deal of data were gathered and analyzed in the course of the present study, no data were found to permit the establishment of confidence intervals on the probability of failure.

This method can be used for two cases most commonly encountered in engineering practice:

Stress Distribution	Strength Distribution
Normal	Normal
Normal	Weibull

The effect of type loading, surface finish, surface treatment, temperature, stress concentration, heat treatment etc., on the statistical distribution was also studied. These effects were expressed in terms of Weibull parameters X_o, θ, and b^2.

For most of the materials studied, the lower bound of fatigue strength (X_o) and the characteristic strength (θ) have a linearly decreasing relationship with life, on a log-log scale. In the case of the Weibull slope (b) it decreases or increases linearly with life, on a log-log scale, depending on the material and the loading, surface, etc. conditions.

As to the problem of stress distribution, the data found in literature and other sources were in the spectral form. For use in the Interference Theory they had to be converted into a distribution of equivalent stresses.

The problem of stress distribution demands further work. Means of conversion from stress spectrum to stress distribution should be refined and a more exact form of the statistical distribution of the equivalent stress should be established.

In order to verify the validity of the Interference Technique developed here it should be checked against an actual life situation. That is, percent failures should be computed for an actual engineering problem. These results then should be compared with actual service failures.

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