A BAYESIAN APPROACH FOR PROCESS CAPABILITY ANALYSIS

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Abstract

A Bayesian approach is developed to perform capability analysis for a one-dimensional normal process with symmetrical bilateral design specifications. The performance of the process is evaluated based on three separate criteria: actual proportion nonconforming, potential proportion nonconforming and process centering. Capability indices based on these criteria are defined and discussed. The probability that the process is capable given an observed sample is derived. The minimum values required for capability indices estimated from a sample to ensure that the process is capable are also presented in both tabular and graphical formats.

1 Introduction

Integrated manufacturing systems require that accurate information be exchanged and analyzed freely between many different departments. Specifically, a quantitative measure of production quality is needed that can be well understood by people throughout the organization. It is this need for improved communication that motivated the development of process capability indices. According to Kane (1986), process capability indices are intended to provide a common, easily understood language for quantifying the performance of a process. For this reason, these indices have become the predominate measure of process performance used in industry today.

Intuitively, process performance can be defined by how well the process produces parts which conform to the engineering design specifications. Unfortunately, the capability indices currently used are frequently inconsistent and misleading measures of conformance even for simple one-dimensional process characteristics. When one considers the expanding use of multivariate tolerances to improve quality, the deficiencies with the current indices become even more alarming. Because of these limitations, we have introduced new capability indices which solve these problems and yet complement the current system in order to minimize disruption to industry standards and procedures.

In this paper, we introduce a system of three indices for measuring process performance and provide a brief discussion and justification for their use. We then employ a Bayesian approach for the case of a one-dimensional normal process with symmetric bilateral design specifications. Using a noninformative prior, the probability that the process is capable given an observed sample is derived and can be computed numerically. We define a capable process as a process where the three indices simultaneously satisfy certain prespecified criteria. Numerical results summarized in tables and figures are given to demonstrate the method and its usefulness in providing sound statistically based inferences for process capability analysis.

2 Process Capability Indices

Process capability indices have become widely used as an attempt to convey the process performance in a dimensionless and easily interpreted manner. The earliest form of this type of capability index is

$$C_p = \frac{U - L}{6\sigma}$$

where U and L are the upper and lower specification limits and σ is the standard deviation of the characteristic. In the standard case of a one-dimensional process with symmetric bilateral design specifications, the nominal target value is centered between the specifications. The process characteristic is generally assumed to be normally distributed and thus the process performance is a function of both the process mean and standard deviation. As explained in Kane (1986), it is apparent that C_p strictly measures the potential process performance since only the standard deviation is related to the design specifications. Since the location of the process mean is not considered, it is possible to have any proportion of parts outside the specification limits for a given C_p by merely locating the process mean sufficiently close to, or outside, the specification limit. Thus C_p only quantifies the *potential* performance of a process which is attained when the process mean is equal to the design target $T = (U + L)/2$. The C_p index has become a fixture in industry for measuring the potential capability of a process with perfect centering. A benchmark of 1.0 was chosen to relate C_p to the standard six sigma spread used on control charts. Therefore, a process with $C_p = 1.0$ with an underlying stable normal distribution will produce nonconforming parts at a rate of 0.27 percent (or 2700 parts per million) assuming the process mean is centered at the target value. Here, we use the ANSI (1982) definition of a nonconforming part defined as *any part that falls outside the tolerance zones.*

Because of this one-to-one relationship between C_p and the proportion nonconforming, capability indices are commonly used as a minimum benchmarking criteria for manufacturing processes. In fact, both GM and Ford Motor Company specifically document the use of process capability indices for assessing process performance. For example, Ford's Q-101 manual states the requirement that all manufacturing processes must achieve minimum process capability indices C_p and C_{pk} of 1.33 or better (1.0 for previously tooled parts). This standardized interpretation has spread and most engineers and managers responsible for quality gauge their processes with process capability indices in comparison to this 1.33 (or 1.0) benchmark.

One major use of process capability indices is enable decision makers with little statistical training to make informed statistical decisions regarding the purchase or removal of long term capital equipment. An informed decision regarding the capability of a process must then consider both the potential and actual performance of the process. While process potential is well summarized by C_p, there exists some debate on how to effectively measure the actual process performance. The two most common measures that have been introduced are C_{pk} [Kane (1986)] and C_{pm} [Chan, Cheng and Spiring (1988)]. These indices simultaneously account for both the mean and standard deviation of the process in an attempt to reflect the actual process performance. Unfortunately, neither of these two indices have direct physical meaning. It has been shown that both C_{pk} and C_{pm} are inconsistent measures of the actual proportion nonconforming thus making interpretation difficult [Boyles (1991), Lam and Littig (1992)]. Furthermore, these indices are unsuitable for extension to more complex multivariate processes. In order to solve this problem, Lam and Littig (1992) suggest reporting measures that have direct physical meaning and supply information relevant to process description and improvement. In particular, they recommend three separate criteria to judge or evaluate the performance of a manufacturing process.

1. $p \equiv$ *Actual Proportion Nonconforming*

   The actual proportion of nonconforming parts currently being produced by the process is often of primary interest. It is a function of both process mean and process variation and therefore reflects the current state of the process. This measure describes the expected performance assuming no actions are taken to shift the process mean or to reduce the process variation.

2. $p^{\star} \equiv$ *Potential Proportion Nonconforming*

   This measures the minimum possible proportion of nonconforming parts that can be achieved through simple location shifts of the process mean. In the case of a symmetric distribution such as normal considered here, this minimum value will be attained when the process mean is centered within the specification limits.

3. $k \equiv$ *Process Centering*

   This measures the deviation of the process mean from the design target with respect to the allowable tolerance. For symmetric bilateral design specifications, a natural quantitative measure of process centering is the k index defined in Kane (1986). Specifically, if μ is the process mean, Kane (1986) defines

   $$k = \frac{2\,|\,T - \mu\,|}{U - L}$$

   which is the ratio of the deviation of the process mean from the design target with respect to the allowable tolerance.

As discussed in Lam and Littig (1992), while it is informative to report actual and potential proportion nonconforming, it is not directly compatible with the current corporate requirements. It is therefore convenient to transform actual and potential proportion nonconforming into indices that retain their physical meaning and at the same time communicate these values in a numerical format that is familiar to engineers, managers and technicians. In particular, if p^* and p are the potential and actual proportion nonconforming of a process, they define

$$C_{p^*} = \frac{1}{3}\Phi^{-1}\left(1 - \frac{p^*}{2}\right) \qquad \text{and} \qquad C_{pp} = \frac{1}{3}\Phi^{-1}\left(1 - \frac{p}{2}\right) \tag{1}$$

where $\Phi(x)$ is the integral of a standardized normal density from minus infinity to any real number x. Note that C_{p^*} in Equation (1) above is defined such that $C_{p^*} = C_p$ when the underlying process follows a one-dimensional normal distribution. Hence, the interpretation of C_{p^*} is consistent with the traditional C_p. This is of practical importance since C_p has become a well known, understood, and deeply entrenched fixture in industry for measuring potential capability. While C_p is only applicable to one-dimensional processes and used in situations when normality or near-normality is assumed, C_{p^*} and C_{pp} applies to all manufacturing processes provided that the actual and potential proportion nonconforming can be computed with sufficient accuracy. In particular, Littig and Lam (1993) compute C_{p^*} and C_{pp} for nonsymmetric bilateral design specifications, and unilateral design specifications. Littig, Lam and Pollock (1992) apply C_{p^*} and C_{pp} to multi-dimensional processes. In general, the authors recommend that C_{p^*}, C_{pp} and the **k** indices should be reported for any process characteristic in order to consistently and effectively communicate the ability of a process to meet the design specifications.

3 Joint Posterior Density Function of C_{p^*} and CPU

Let X be the one-dimensional characteristic of interest and assume that it is normally distributed with mean μ and standard deviation σ. Define CPU as given in Kane (1986),

$$\text{CPU} = \frac{U - \mu}{3\sigma}.$$

The likelihood function of μ and σ based on a sample $x_1, x_2, \ldots, x_n$ is given by

$$L(\mu,\sigma) = (2\pi\sigma^2)^{-n/2}\exp\left[-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\right].$$

Using a noninformative prior for μ and σ meaning that μ and $\log\sigma$ are approximately locally uniform, i.e.,

$$p(\mu,\sigma) \propto \frac{1}{\sigma},$$

the joint posterior density function is given in Box and Tiao (1973)

$$f_1(\mu,\sigma \mid \bar{x},s) = c\sigma^{-n+1}\left[\frac{(n-1)s^2}{2}\right]^{(n-1)/2}\exp\left\{-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{x}-\mu)^2\right]\right\} \qquad (2)$$

where $\bar{x}$ and s are the sample mean and standard deviation respectively,

$$c = 2\left(\frac{n}{2\pi}\right)^{1/2}\Gamma\left(\frac{n-1}{2}\right)^{-1},$$

and $\Gamma(x)$ is the gamma function evaluated at x. Observe that C_{p^*} and CPU are functions of μ and σ and we can uniquely solve for μ and σ in terms of C_{p^*} and CPU. In particular,

$$\sigma = \frac{U-L}{6C_{p^*}} \qquad \text{and} \qquad \mu = U - \left(\frac{U-L}{2C_{p^*}}\right)\text{CPU}. \qquad (3)$$

If $\widehat{C_{p^*}}$ and $\widehat{\text{CPU}}$ are point estimates of C_{p^*} and CPU when $\bar{x}$ and s are used to estimate μ and σ respectively, then

$$s = \frac{U-L}{6\widehat{C_{p^*}}} \qquad \text{and} \qquad \bar{x} = U - \left(\frac{U-L}{2\widehat{C_{p^*}}}\right)\widehat{\text{CPU}}. \qquad (4)$$

Using the change of variables given in Equation (3) above with Jacobian equal to $(U-L)^2/(12\,C_{p^*}^3)$ and substituting Equation (4) into (2), it is readily verified that the joint posterior density function of C_{p^*} and CPU is given by

$$f_2(C_{p^*}, \text{CPU} \mid \widehat{C_{p^*}}, \widehat{\text{CPU}})$$
$$= \frac{3c}{C_{p^*}}\left(\sqrt{\frac{n-1}{2}}\frac{C_{p^*}}{\widehat{C_{p^*}}}\right)^{n-1}\exp\left\{-\frac{1}{2}\left[(n-1)\left(\frac{C_{p^*}}{\widehat{C_{p^*}}}\right)^2 + 9nC_{p^*}^2\left(\frac{\text{CPU}}{C_{p^*}} - \frac{\widehat{\text{CPU}}}{\widehat{C_{p^*}}}\right)^2\right]\right\}.$$

The marginal posterior density function of C_{p^*} is therefore given by

$$h(C_{p^*} \mid \widehat{C_{p^*}}) = \frac{c}{C_{p^*}} \left(\frac{2\pi}{n}\right)^{1/2} \left(\sqrt{\frac{n-1}{2}}\frac{C_{p^*}}{\widehat{C_{p^*}}}\right)^{n-1} \exp\left\{-\frac{1}{2}\left[(n-1)\left(\frac{C_{p^*}}{\widehat{C_{p^*}}}\right)^2\right]\right\}.$$

4 A Bayesian Approach to Analyze $C_{p^\star}$, C_{pp} and k

Suppose that a process is considered capable whenever $C_{p^*} > c_1$, $C_{pp} > c_2$ and $k < k_0$ where c_1, c_2 and k_0 are prespecified constants. Observe that $k < k_0$ is equivalent to

$$C_{p^*}(1 - k_0) < CPU < C_{p^*}(1 + k_0).$$

Also using Equation (1), $C_{pp} > c_2$ is equivalent to the actual proportion nonconforming

$$p < 2\left[1 - \Phi\left(3c_2\right)\right].$$

Since we assume that the underlying one-dimensional process is normal, it follows that

$$1 - p = \Phi(3CPU) - \Phi(3CPU - 6C_{p^*})$$

and we require that

$$\Phi(3CPU) - \Phi(3CPU - 6C_{p^*}) > 2\Phi(3c_2) - 1$$

for the process to be considered capable. Given any positive real number x, define a function g(x) to be the smallest real number such that

$$\Phi(3g(x)) - \Phi(3g(x) - 6x) = 2\Phi(3c_2) - 1.$$

Since the normal density is symmetric, it is clear that $C_{pp} > c_2$ is now equivalent to

$$g(C_{p^*}) < CPU < 2C_{p^*} - g(C_{p^*}).$$

We can now evaluate

$$P(\text{process is capable} \mid \text{an observed sample})$$

to be equal to

$$q(c_1,c_2,k_0) \equiv P(C_{p*} > c_1, C_{pp} > c_2, k < k_0 \mid \widehat{C_{p*}}, \widehat{CPU})$$

$$= \int_{c_1}^{\infty} \int_{\max\{x(1-k_0),g(x)\}}^{\min\{x(1+k_0),2x-g(x)\}} f_2(x,y \mid \widehat{C_{p*}}, \widehat{CPU})\, dy\, dx.$$

In the special case when $c_2 = 0$ and $k_0 = \infty$,

$$q(c_1,0,\infty) = P(C_{p*} > c_1 \mid \widehat{C_{p*}}) = \int_{c_1}^{\infty} h(x \mid \widehat{C_{p*}})\, dx.$$

The Bayesian approach to analyze the capability of a normal process with symmetric bilateral design specifications has been studied before by Cheng and Spiring (1989) for C_p and Chan, Cheng and Spiring (1988) for C_{pm} with $\mu = T$. The minimum values of $\widehat{C_{p*}}$ required to ensure that $q(c_1,0,\infty) = 0.9$, 0.95 and 0.99 with $c_1 = 1$ are tabulated in Cheng and Spiring (1989). Hence, their results are a special case of the approach considered here.

In Figure 1, the curves represent the minimum required values for $\widehat{C_{p*}}$ and $\widehat{C_{pp}}$ such that $q(1.0, 1.0, 0.33) = 0.90$, 0.95 and 0.99 for sample size n = 25, 50, 75 ,100, 150 and 200. Suppose the process is considered to be capable whenever $C_{p*} > 1.0$, $C_{pp} > 1.0$ and the process mean lies within the middle third of specification zone (i.e., $k < k_0 = 0.33$). For example if n = 50, $\widehat{C_{p*}} = 1.5$ and $\widehat{C_{pp}} = 1.25$, then the probability that the process is capable is greater than 0.95 but less than 0.99. This can be seen from Figure 1 since the point for the the sample values of $\widehat{C_{p*}}$ and $\widehat{C_{pp}}$ lies above the curve for n=50 at 0.95 but below the curve for n=50 at 0.99.

Given an observed value of $\widehat{C_{p*}}$, Tables 1 and 2 provide the minimum required $\widehat{C_{pp}}$ values such that $q(c_1, c_2, k_0) = 0.90$, 0.95 and 0.99 for sample size n = 25, 50, 75 ,100, 150 and 200, and various combinations of c_1, c_2 and k_0. Suppose now that the process is considered to be capable whenever $C_{p*} > 1.33$ and $C_{pp} > 1.33$. For example, if n = 100 and we observe that $\widehat{C_{p*}} = 2.0$, then the process is considered capable with probability 0.95 whenever $\widehat{C_{pp}} \geq 1.508$ from Table 2 with $q(1.33, 1.33, \infty) = 0.95$.

Note that in Figure 1, all the curves decrease and then increase sharply for large values of $\widehat{C_{p*}}$. This is because for large $\widehat{C_{p*}}$ values, $k < k_0 = 0.33$ is the dominant constraint whereas for smaller values of $\widehat{C_{p*}}$, the constraint $C_{pp} > 1.0$ dominates. In the case when $k_0 > 1.0$, all the curves decrease to a constant value of $\widehat{C_{pp}}$ as $\widehat{C_{p*}}$ increases. This can be seen directly from Table 1 with $q(1.0, 1.0, \infty)$. For example when n = 50 at 0.99, as $\widehat{C_{p*}}$ increases from 1.33 to 2.0, the minimum required $\widehat{C_{pp}}$ decreases from 1.309 to the constant value of 1.292.

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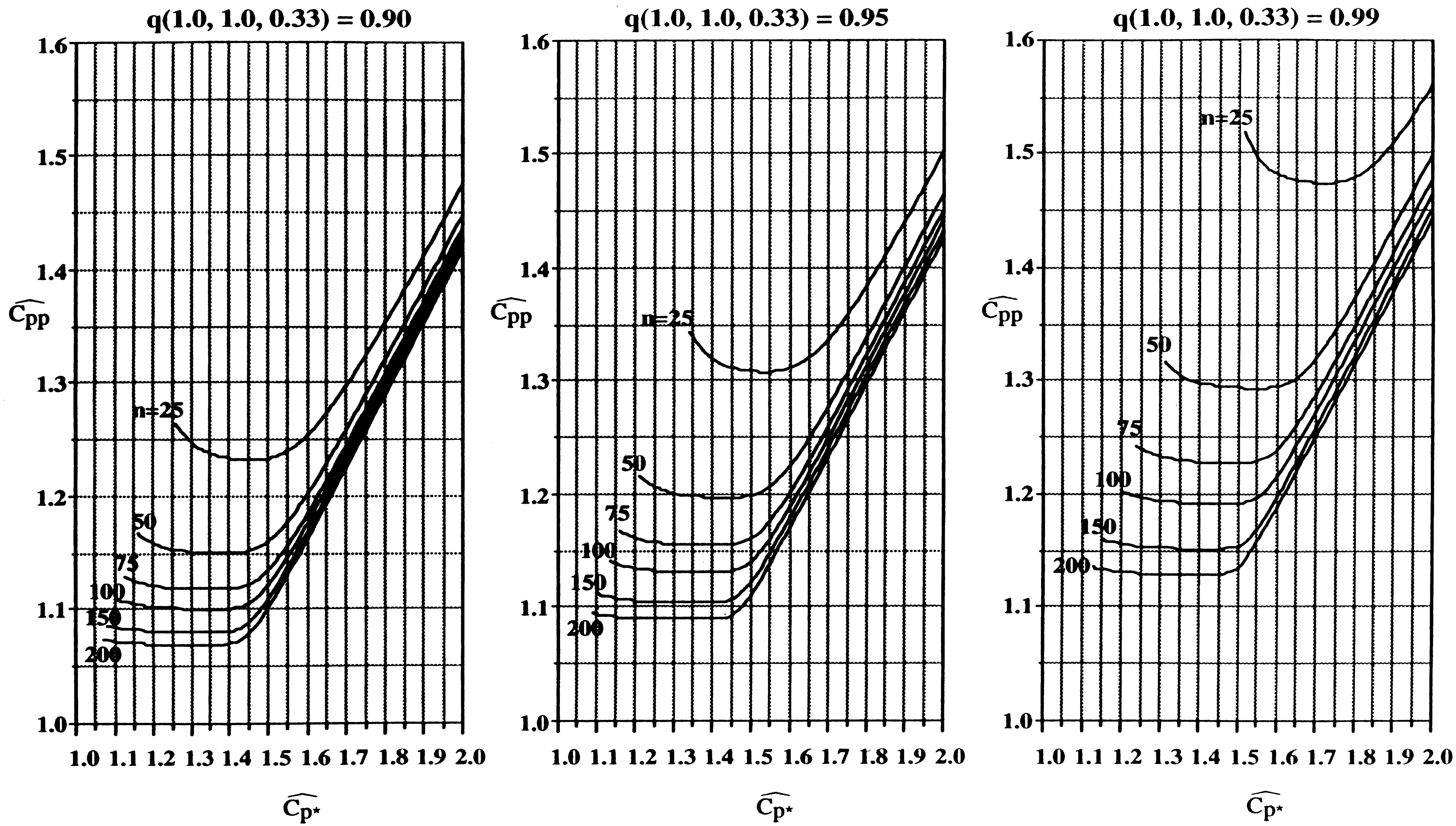

Figure 1: Minimum Required Values for $\widehat{C_{p^*}}$ and $\widehat{C_{pp}}$ such that q(1.0, 1.0, 0.33) = 0.90, 0.95 and 0.99

		q(1.0, 1.0, ∞)			q(1.0, 1.0, 0.33)			q(1.0, 1.0, 0.2)		
$\widehat{C_{p*}}$	n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
1.33	25	1.239			1.239			1.259		
	50	1.149	1.198	1.309	1.149	1.198	1.309	1.200	1.224	1.309
	75	1.117	1.154	1.230	1.117	1.154	1.230	1.183	1.198	1.238
	100	1.100	1.130	1.192	1.100	1.130	1.192	1.174	1.186	1.214
	150	1.079	1.103	1.151	1.079	1.103	1.151	1.165	1.175	1.194
	200	1.068	1.088	1.128	1.068	1.088	1.128	1.160	1.169	1.184
1.50	25	1.230	1.308		1.232	1.308		1.352	1.384	
	50	1.148	1.195	1.293	1.158	1.197	1.293	1.318	1.336	1.372
	75	1.117	1.153	1.227	1.133	1.158	1.227	1.307	1.320	1.347
	100	1.099	1.130	1.190	1.120	1.138	1.190	1.300	1.312	1.334
	150	1.079	1.103	1.150	1.107	1.119	1.152	1.292	1.302	1.320
	200	1.068	1.088	1.128	1.100	1.110	1.132	1.288	1.296	1.312
1.67	25	1.229	1.305	1.475	1.280	1.323	1.475	1.474	1.501	1.561
	50	1.148	1.195	1.292	1.237	1.256	1.305	1.446	1.463	1.497
	75	1.117	1.153	1.226	1.223	1.237	1.267	1.435	1.449	1.476
	100	1.099	1.130	1.190	1.215	1.227	1.251	1.429	1.441	1.463
	150	1.079	1.103	1.150	1.207	1.217	1.235	1.421	1.431	1.449
	200	1.068	1.088	1.128	1.203	1.211	1.227	1.416	1.425	1.440
2.00	25	1.229	1.305	1.470	1.474	1.501	1.559	1.731	1.757	1.809
	50	1.148	1.195	1.292	1.446	1.464	1.497	1.705	1.723	1.756
	75	1.117	1.153	1.226	1.435	1.449	1.476	1.694	1.708	1.735
	100	1.099	1.130	1.190	1.429	1.441	1.463	1.688	1.700	1.723
	150	1.079	1.103	1.150	1.421	1.431	1.449	1.680	1.690	1.708
	200	1.068	1.088	1.128	1.416	1.425	1.440	1.675	1.684	1.700

Table 1: Minimum Required Values for $\widehat{C_{p*}}$ and $\widehat{C_{pp}}$ such that $q(1.0, 1.0, k_0) = 0.90, 0.95$ and 0.99 for Various k_0 and n

		q(1.33, 1.33, ∞)			q(1.33, 1.33, 0.2)			q(1.33, 1.0, 0.25)		
$\widehat{C_{p^*}}$	n	0.90	0.95	0.99	0.90	0.95	0.99	0.90	0.95	0.99
	25							1.458		
	50	1.535	1.604		1.536	1.604		1.372	1.396	
1.67	75	1.491	1.541	1.651	1.492	1.541	1.651	1.356	1.371	1.410
	100	1.467	1.508	1.593	1.469	1.508	1.593	1.349	1.361	1.385
	150	1.440	1.472	1.536	1.445	1.472	1.536	1.341	1.350	1.369
	200	1.425	1.452	1.505	1.432	1.453	1.505	1.336	1.344	1.360
	25	1.643	1.746		1.748	1.790		1.637	1.655	1.764
	50	1.534	1.597	1.727	1.708	1.726	1.770	1.608	1.625	1.659
2.00	75	1.491	1.540	1.638	1.695	1.709	1.737	1.597	1.611	1.638
	100	1.467	1.508	1.589	1.688	1.700	1.723	1.590	1.602	1.625
	150	1.440	1.472	1.535	1.680	1.690	1.708	1.582	1.592	1.611
	200	1.425	1.452	1.505	1.675	1.684	1.700	1.578	1.586	1.602
	25	1.643	1.745	1.969	1.994	2.022	2.083	1.878	1.904	1.957
	50	1.534	1.597	1.727	1.966	1.984	2.018	1.852	1.870	1.903
2.33	75	1.491	1.540	1.638	1.955	1.969	1.996	1.841	1.855	1.882
	100	1.467	1.508	1.589	1.949	1.961	1.984	1.834	1.846	1.869
	150	1.440	1.472	1.535	1.941	1.951	1.969	1.826	1.836	1.855
	200	1.425	1.452	1.505	1.936	1.945	1.961	1.822	1.830	1.846

Table 2: Minimum Required Values for $\widehat{C_{p^*}}$ and $\widehat{C_{pp}}$ such that $q(1.33, c_2, k_0) = 0.90$, 0.95 and 0.99 for Various c_2, k_0 and n