ENGINEERING RESEARCH INSTITUTE
THE UNIVERSITY OF MICHIGAN
ANN ARBOR

ON A PITOT TUBE METHOD OF UPPER ATMOSPHERE MEASUREMENTS

Vi Cheng Liu

Project 2387

LABORATORY PROCUREMENT OFFICE
SIGNAL CORPS SUPPLY AGENCY
CONTRACT NO. DA-36-039 sc-64659
PROJECT NO. 273-PH-55-91-(2218)

December 1955

This manuscript is to be published in the
June, 1956, issue of Journal of Geophysical Research.

TABLE OF CONTENTS

ON A PITOT TUBE METHOD OF UPPER ATMOSPHERE MEASUREMENTS

by

Vi-Cheng Liu

Engineering Research Institute, University of Michigan
Ann Arbor, Michigan

ABSTRACT

The aerodynamic basis of using Pitot tube pressure at supersonic speeds as an indirect measurement of the upper atmospheric density, pressure and temperature, up to 80 km, is discussed. A special means of deduction of these ambient variables from corresponding Pitot tube pressure and other relevant parameters is presented. A unique feature of the method is that it needs only the Pitot tube pressure and the corresponding forward velocity of the Pitot tube (assuming negligible atmospheric wind velocity) to determine the ambient variables. This simplification is made possible by invoking laws which describe the characteristic properties of the atmosphere. This simplification, in the physical sense, results in an integral equation for the ambient density. A special and effective method is developed for the solution of this equation.

I. INTRODUCTION

In the measurement of upper atmospheric density, temperature and pressure by rocket techniques, aerodynamic methods (Rocket Panel 1952) are commonly used because of the necessity of carrying the instruments on a high speed vehicle. These measurements of density, temperature or pressure of the undisturbed (or ambient) atmosphere are inherently indirect. If there exists an aerodynamic theory which gives a unique relation between one of the ambient quantities and a measurable parameter, the result of the latter, in essence, reveals the value of the former. The success of an aerodynamic method depends upon, among other factors, the exactness of the particular aerodynamic theory involved. This, in fact, constitutes a serious drawback of the aerodynamic methods in general because existing knowledge concerning dynamics of the rarefied gases, such as the upper atmospheric air, is very meager (Schaaf 1953).

In the following, the aerodynamic basis of applying Rayleigh's supersonic Pitot tube equation to rocket sounding measurements is discussed. The use of a Pitot tube for this purpose is not new (Havens, etc., 1952; Lagow 1953; Sicinski, etc., 1954). In these experiments Pitot tube pressure was used jointly with either a surface pressure on the sounding rocket or a "nominal" ambient pressure and the relative air velocity to determine ambient temperature, etc. As a consequence of using the consistency conditions of the atmosphere in the present method, it is not required to know or measure a surface pressure or "nominal" ambient pressure. These conditions include the equation of state and the hydrostatic equation for an atmosphere in equilibrium.

II. RAYLEIGH'S SUPERSONIC PITOT TUBE EQUATION

By considering the pressure rise across a plane shock wave and the pressure increase through isentropic deceleration of the flow to zero velocity, Rayleigh (Taylor, etc., 1935) showed that the resultant pressure in a Pitot tube, which is placed in a supersonic stream, may be expressed as

$$\frac{P}{p} = \frac{\gamma + 1}{2} M^2 \left[\frac{(\gamma + 1)^2 M^2}{4\gamma M^2 - 2\gamma + 2} \right]^{1/(\gamma - 1)} \qquad (1)$$

where P represents the Pitot tube pressure; p, the free stream (ambient) pressure; M, the free stream Mach number which is defined as the ratio of the relative fluid velocity to speed of sound both of which refer to the free stream condition; γ, the ratio of specific heat at constant pressure to that at constant volume.

Measurements in wind tunnels with flows at high Reynolds numbers (R.N. = $Vd\rho/\mu$ where d = a characteristic length of the test model, for instance, the diameter of a tube, V = relative air velocity, ρ = air density, μ = air viscosity, all refer to test section condition) indicate that Rayleigh's theoretical value (1) checks remarkably well with experimental results (Hasel, etc., 1948; Gracey, etc., 1951). It has been further shown that with a Pitot tube of optimum design, the measured value of Pitot tube pressure in a supersonic stream remains constant until the angle of yaw increases to 25 deg (Gracey, etc., 1951).

Reynolds number is an index of comparison between the inertia effect ($\sim\rho V^2/d$) and the viscous effect ($\sim\mu V/d^2$) of the flow around an obstacle. Generally speaking, for a flow at high Reynolds number the latter is negligibly small in comparison with the former. This constitutes one of the main hypotheses in Rayleigh's theory of supersonic Pitot pressure (1).

It is expected that the significance of viscous effect increases as Reynolds number decreases. Results of calibration tests of Pitot tubes in a supersonic rarefied gas tunnel (Kane, etc., 1950; Sherman 1953) indicate that equation (1) checks with measurements within experimental error provided Reynolds number is larger than 200 (outer diameter of the Pitot tube is used for "d"). For a flat-nosed Pitot tube, the deviation of measured value from (1) remains less than 2 per cent if Reynolds number is larger than 50. For lower Reynolds numbers the measured Pitot tube pressure appears to increase linearly with (1/R.N.).

It is interesting to note that Sherman's conclusion concerning viscous effect on the Pitot tube pressure agrees very well with earlier work by Barker (1922) who used a small Pitot tube and liquid medium when comparison is made on the basis of subsonic flows.

For convenience of application to sounding measurements, the Reynolds number criterion for equation (1), namely, R.N. > 50, is rewritten. Since $\mu = 1/2\ \rho\bar{c}\lambda$ and $a = 0.742\bar{c}$ (for diatomic molecules) where λ = mean free path

(Maxwell), $\bar{c}$ = mean molecular velocity and a = sound velocity,

$$\frac{\lambda}{d} < 0.03\, M \qquad (2)$$

It is expected that condition (2) would be fulfilled up to 80 km approximately with a Pitot tube of optimum design and a typical sounding rocket trajectory (Newell 1953).

III. DERIVATION OF THE ρ-EQUATION

Consider an atmosphere in equilibrium for which the equation of state and the hydrostatic equation are, respectively,

$$p = n\,k\,T \qquad (3)$$

$$\frac{dp}{dh} = -\rho g \qquad (4)$$

where $\rho = nm$ is the ambient density with n = number density, m = average mass of air molecules; T, the ambient air temperature; g, gravitational acceleration; k, Boltzman's constant; h, altitude coordinate with origin at sea level.

Substituting (3) into (4) gives

$$\frac{1}{p}\frac{dp}{dh} = -\frac{mg}{kT} = -\frac{1}{H} \qquad (5)$$

and also

$$H = \frac{1}{\rho g}\left[\rho_o g_o H_o - \int_{h_o}^{h} \rho\, g\, dh\right] \qquad (6)$$

where $H = kT/(mg)$ is commonly called scale height, subscript "o" indicates the initial condition.

Introduce the abbreviations

$$K = \frac{1}{2}\,(4\gamma)^{\gamma/(\gamma-1)}\,(\gamma+1)^{(\gamma+1)/(1-\gamma)} \qquad (7)$$

$$G(\rho) = \frac{\gamma - 1}{2\,\bar{\rho}\,V^2}\left(\rho_0 g_0 H_0 - \int_{h_0}^{h} \bar{\rho}\, g\, dh\right) \tag{8}$$

$$f(\rho) = \frac{KP}{V^2}\left[1 - G(\rho)\right]^{1/(\gamma-1)} \tag{9}$$

Eliminating p and T from (1), (3) and (6), we obtain an integral equation in ρ.

$$\rho = f(\rho) \tag{10}$$

IV. SOLUTION OF THE ρ-EQUATION

Notice the following properties of the functional operator on the right hand side of (8)

$$\gamma\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \frac{P}{V^2} \leq f(\rho) \leq \frac{KP}{V^2} \tag{11}$$

where the lower bound corresponds to the case when $M = 1$; the upper bound, when $M = \infty$ [See (1)]. Furthermore, it can be shown that for a given velocity function $V(h)$

$$\frac{dG}{d\rho} = \frac{(\gamma-1)^2}{2\gamma^2}\frac{V^2}{M^3}\frac{1}{p} > 0$$

Hence

$$f(\rho^*) \leq f(\rho) \qquad \text{if } \rho^* \geq \rho \tag{12}$$

In view of (11) and (12), one is led to construct a sequence of successive "approximations" $\rho_{(n)}$, starting with

$$\rho_{(1)} = \frac{KP}{V^2} \tag{13}$$

by the recursive equation.

$$\rho_{(n+1)} = f\left[\rho_{(n)}\right] \qquad (n = 1,2,\cdots\cdot) \tag{14}$$

Because of the characteristic properties (11) and (12) of the functional operator, it is expected that the successive sequences of $\rho_{(n)}$ will close in, with an alternating pincer movement, on a uniquely determined limit function ρ. To prove the convergency of the sequences, one should establish that

$$\left| \rho_{(n+1)}(h) = \rho_{(n)}(h) \right| \leq \epsilon\ (n,h) \tag{15}$$

where

$$\lim_{n \to \infty} \epsilon\ (n,h) \to 0$$

It means, in a numerical sense, that after a finite number of repetitions of the iterative process, $\rho_{(n+1)}$ becomes equal to $\rho_{(n)}$ to the accuracy to which the numerical work is taken, then to this accuracy such a function is a solution of (14).

Rigorous mathematical proof of (15) has not been obtained. Instead, sample calculations of $\rho(h)$ from measured data of $P(h)$ and $V(h)$, taken in rocket flights, were made which clearly indicate the trend of convergency of the iterative process.

V. A SAMPLE CALCULATION*

Throughout the calculation, atmospheric wind speeds are neglected (Weisner 1954). Since the purpose of this calculation is to demonstrate, in a special case, the property of convergency of the iterative process, the absolute accuracy of the data is not of interest.

To start the iterative process (14), one needs to know H_0 which is the scale height at the initial point of integration [See (8)]. It is not recommended

*This set of data is made available here through the generosity of Mr. N. W. Spencer, Mr. H. S. Sicinski, and Prof. W. G. Dow of the E. E. Department of The University of Michigan.

to start the integration

$$\int_{h_o}^{h} \rho \; g \; dh$$

from the lower end of the range of sounding measurements, because in so doing $G(\rho)$ is made equal to the difference of two large numbers which is a notoriously unsatisfactory process. On the other hand, if the above integral is evaluated starting from the upper end of the measurement range, the value of H_o corresponding to this altitude will have to be supplied from another source. This does not pose a serious problem in view of the fact that the importance of the term $\rho_o g_o H_o$ relative to the integral term in (8) diminishes very rapidly because of the increase of $\rho(h)$, with $(h_o - h)$, in an exponential order.

H_o is estimated with (See Appendix A-4)

$$H_o = (h_o - h) \, \mathrm{Log}_e \, \frac{P_1}{P_o} \frac{V_o^2}{V_1^2} \tag{16}$$

where subscript "1" denotes an altitude, a small distance (say 5 km) below the initial point "o".

This particular set of data on Pitot tube pressure and rocket velocity covers a range extending from 45 km to 90 km approximately. It is expected, according to (2), that actual Pitot tube pressures measured during the range 80 - 90 km may not be closely approximated by (1). In spite of this, (16) may still be a close estimation provided the "viscous effect" correction to equation (1) does not change P_1/P_o appreciably [See Sherman (1953)].

The results of successive approximations for ambient density and temperature are plotted in Figs. 1 and 2 respectively. Corresponding quantities published by the rocket panel (1952) are also shown for comparison purpose. It is to be noted that the rocket panel data (1952) represents a weighted mean of a large number of measurements of the upper atmosphere at different times of the year. Hence

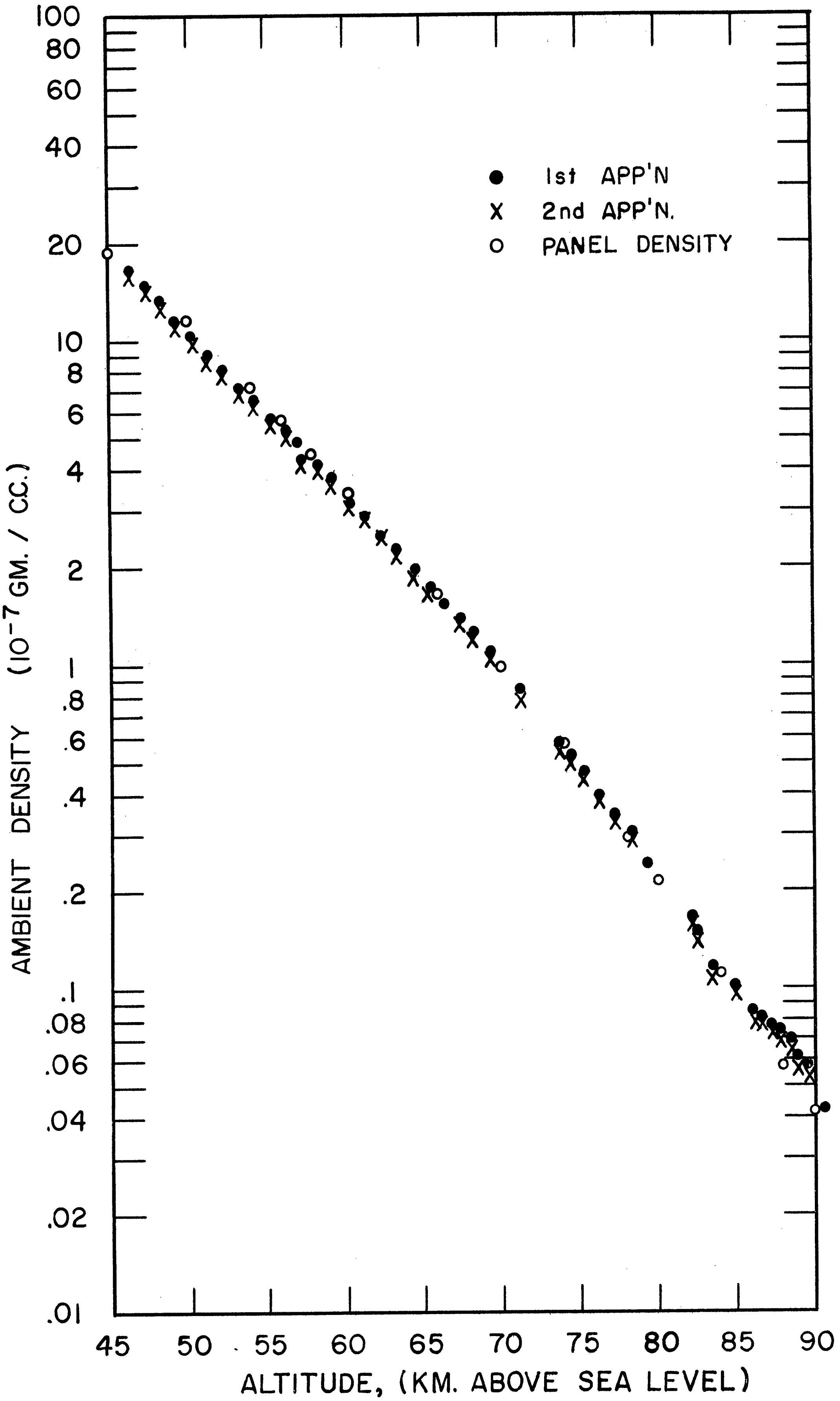
100
80
60
40
20
10
8
6
4
2
1
.8
.6
.4
.2
.1
.08
.06
.04
.02
.01
AMBIENT DENSITY (10^{-7} GM. / CC.)
● 1st APP'N
X 2nd APP'N.
O PANEL DENSITY
45
50
55
60
65
70
75
80
85
90
ALTITUDE, (KM. ABOVE SEA LEVEL)

Fig. 1

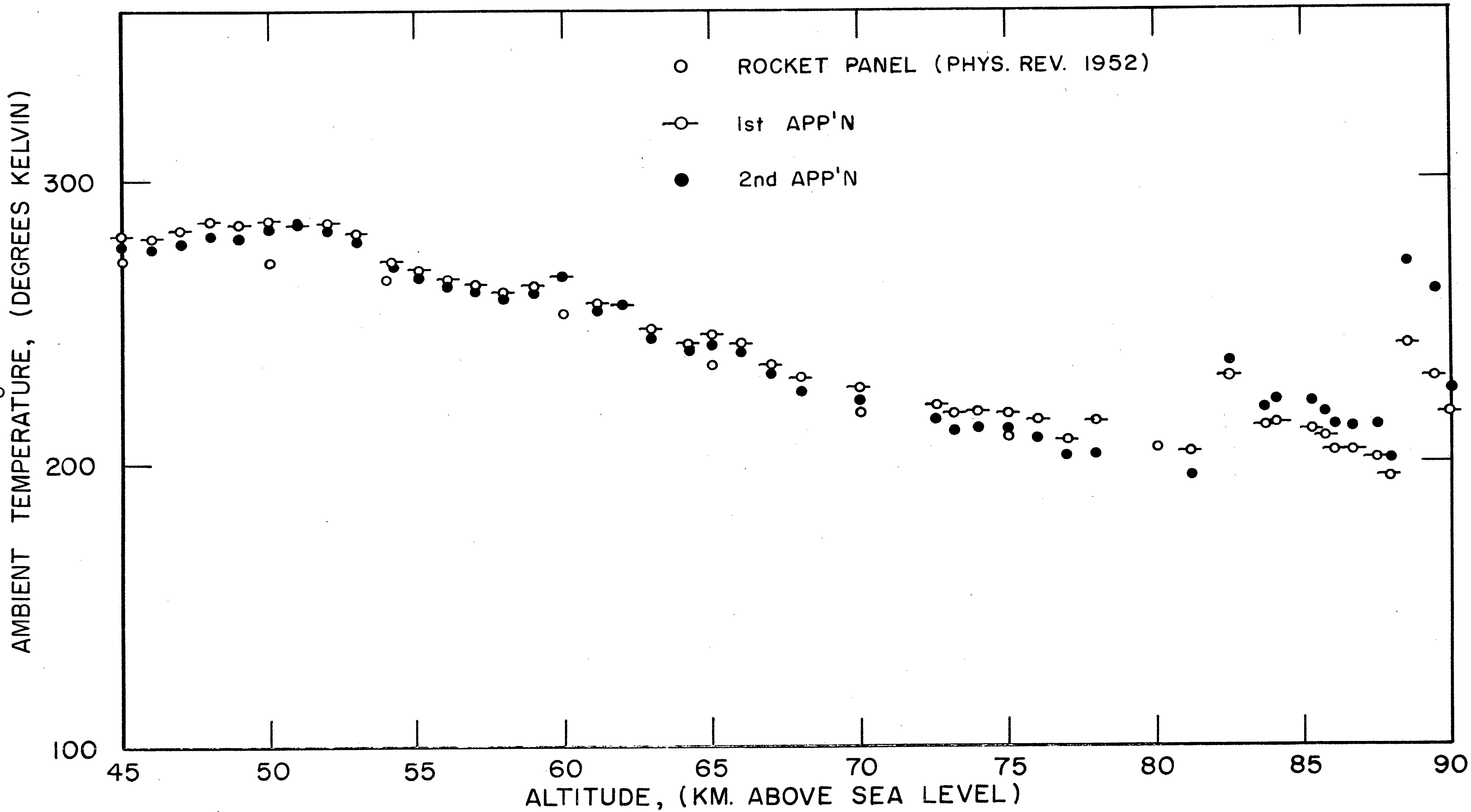
ROCKET PANEL (PHYS. REV. 1952)
1st APP'N
2nd APP'N
AMBIENT TEMPERATURE, (DEGREES KELVIN)
300
200
100
45
50
55
60
65
70
75
80
85
90
ALTITUDE, (KM. ABOVE SEA LEVEL)

Fig. 2

it does not necessarily give the most accurate ambient temperature at a specific instant.

For a rocket trajectory with lower speed, more iterations are needed to attain the same approximation (Liu, 1953).

VI. AN APPROXIMATE AMBIENT DENSITY FORMULA

In view of (10) and (13), an approximate formula for ρ, which is equivalent to a second approximation (14), can be written

$$\rho = \frac{KP}{V^2}\left[1 - \frac{\gamma-1}{2P}\left(\frac{P_o}{V_o^2} g_o H_o - \int_{h_o}^{h} \frac{P}{V^2} g\, dh\right)\right]^{1/(\gamma-1)} . \qquad (17)$$

Equation (17) can be considered as a modified form of Rayleigh's supersonic Pitot tube equation (1) for the determination of ambient density from Pitot tube pressure and the relative air velocity of the Pitot tube. The significance of the initial values corresponding to "o" has been discussed in Section V.

VII. CONCLUSION

The foregoing method, which constitutes a self-sufficient means of determining the ambient density, temperature and pressure from measured Pitot tube pressure and relative air velocity has been shown to be valid provided that condition (2) is fulfilled. An asset of the method is its simplicity which makes it feasible for applications in various configurations (see Appendix B).

APPENDIX A

An alternative form of (7) is

$$\frac{1}{P}\frac{dP}{dh} + \frac{1}{H} = \left[1 - \frac{gH}{2V^2 - (\gamma-1)gH}\right]\frac{2}{V}\frac{dV}{dh} - \left[1 - \frac{gH}{2V^2 - (\gamma-1)gH}\right]\frac{1}{H}\frac{dH}{dh} \tag{A-1}$$

which can be used to construct a sequence of approximations for H because of the relation

$$\frac{gH}{2V^2 - (\gamma-1)gH} \ll 1 \tag{A-2}$$

where V is assumed to be larger than the speed of sound. Hence, with approximation H of the first order, (A-1) becomes

$$\frac{1}{P}\frac{dP}{dh} + \frac{1}{H} = \frac{2}{V}\frac{dV}{dh} - \frac{1}{H}\frac{dH}{dh} \tag{A-3}$$

It can be shown that (A-3) is equivalent to (11). A complete solution cannot be determined unless the initial value for H is given (See Section V). This difficulty can be avoided by starting the iteration with approximation H of the zeroth order, namely, the use of step functions, for small intervals (δh) to represent H. It means, in the physical sense, the use of equivalent isothermal atmosphere for small intervals (δh) to satisfy the hydrostatic equation (4). Since $\delta H/\delta h$ vanishes,

$$H = (\delta h) \operatorname{Log}_e \frac{P_2}{P_1}\frac{V_1^2}{V_2^2} \tag{A-4}$$

where $\delta h = h_2 - h_1$, a small interval.

Equation (A-1), an exact equation for H, is not given here for the purpose of deriving (A-4), which can be easily obtained by substituting (13) into (4) assuming an isothermal atmosphere. It is intended here, rather, to point out that (A-1) can be used as the fundamental governing equation of an iteration process starting with (A-4). This, however, is not pursued here.

APPENDIX B

A NEW SCHEME FOR VERTICAL WIND MEASUREMENT

In view of the simplicity of the present Pitot tube method and the insensitiveness of Pitot pressure yaw, a new scheme for vertical wind measurement by means of a Pitot tube appears feasible.

Consider a sounding rocket with a detachable nose unit that carries a Pitot tube. This nose unit which separates from the main body of the rocket after it reaches the peak of the trajectory is supposed to probe the atmosphere with its Pitot tube during the downward leg of the trajectory. Useful Pitot tube pressures can be obtained if the nose unit can be oriented (by some mechanical means) so that the axis of the Pitot tube will point toward the direction of the center of the earth within 25°. The Pitot tube orientation problem might also be solved by the use of multiple pressure holes in several directions. Free fall velocity for the nose unit with nominal drag correction can be used with acceptable accuracy for a range of 50 km from the peak of the trajectory of a 100 km sounding rocket.

Comparison of the ambient densities, or their related variables, determined from the two independent sets of measurements, taken during the upward and downward leg of the trajectory respectively, should throw some light on the problem of vertical component of atmospheric wind. The latter has remained a mystery in spite of its significance to the aerodynamic methods of upper atmosphere measurement. (Liu, 1951)

ACKNOWLEDGEMENT

This work was carried out in a research project on atmospheric phenomena at high altitudes in the Department of Aeronautical Engineering of The University of Michigan under the sponsorship of the U.S. Army Signal Corps in contract with the Engineering Research Institute. The author is indebted to Professor M. H. Nichols and Mr. L. M. Jones, the supervisors of this project, for their constant encouragements and discussions. He is very grateful to Mr. N. W. Spencer, Mr. H. S. Sicinski and Professor W. G. Dow for making available to him sounding measurement data as one of the specimens for testing the method.

REFERENCES

1. Rocket Panel, Phys. Rev., 88:1027 (1952).
2. Schaaf, S. A., Heat Transfer, Univ. of Mich. Press, 1953, p. 237.
3. Havens, R. J., Koll, R. T., and Lagow, H. E., J. Geophys. Res., 57:59 (1952).
4. Lagow, H. E., J. Atmos. Terr. Phys., Special Supplement 1, 1954, p. 73.
5. Sicinski, H. S., Spencer, N. W., and Dow, W. G., J. App. Phys., 25:161 (1954).
6. Taylor, G. I., and Maccoll, J. W., (Durand ed.), Aerodynamic Theory III, Berlin; Springer, 1935.
7. Hasel, L. E., and Coletti, D. E., NACA RM L8-102, 1948.
8. Gracey, W., Coletti, D. E., and Russel, W. R., NACA TN-2261, 1951.
9. Barker, M., Proc. Roy. Soc. (London), Series A, p. 435, (1922).
10. Kane, E. D., and Maslach, G. J., NACA TN-2210, 1950.
11. Sherman, F. S., NACA TN-2995, 1953.
12. Liu, V. C., "Atmospheric Phenomena at High Altitudes," (Signal Corps Proj. No. 172 B), Memo No. 6, 1953.
13. Weisner, A., J. Atmos. Terr. Phys., Special Supplement 1, 1954, p. 133.
14. Newell, H. E., High Altitude Rocket Research 39, New York; Acad. Press, 1953.
15. Liu, V. C., "Atmospheric Phenomena at High Altitudes," (Signal Corps Proj. 172 B), Memo No. 3, 1951.