THE UNIVERSITY OF MICHIGAN

INDUSTRY PROGRAM OF THE COLLEGE OF ENGINEERING

APPLICATION OF ULTRASPHERICAL POLYNOMIALS TO NONLINEAR FORCED OSCILLATIONS

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April, 1966

IP-733

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NOMENCLATURE

$f(x)$	Nonlinear function of the dependent variable
λ	Index of the ultraspherical polynomial, $-1/2 < \lambda < \infty$
$P_n^{(\lambda)}(x)$	Ultraspherical polynomial of order n and index λ
$[\alpha,\beta]$	Interval of minimum and maximum excursion
a,b	Constants of the cubic nonlinear function
$x_c = (\beta+\alpha)/2$	Mean displacement
$A = (\beta-\alpha)/2$	Amplitude about the mean displacement
$x' = (x - x)/A$	Transformed variable of x
$a_n^{(\lambda)}$	Coefficient of the ultraspherical polynomial of order n and index λ
$w(\lambda,x)$	$(1 - x^2)^{\lambda-1/2}$ = Weight function of the ultraspherical approximation
$\Lambda_\lambda(A)$	$\Gamma(\lambda+1)J_\lambda(A)/(A/2)^\lambda$
$\bar{\Lambda}_\lambda(A)$	$\Gamma(\lambda+1)I_\lambda(A)/(A/2)^\lambda$
$g(\dot{x})$	Nonlinear function of velocity
F_o, F_1	Magnitude of the sinusoidal and constant term in the forcing function
$[\gamma,\delta]$	Interval of the minimum and maximum velocity
ω_*, c_*	Equivalent angular frequency and damping coefficient
$v_c = (\delta + \gamma)/2$	Mean velocity
$V = (\delta - \gamma)/2$	Velocity about the mean
μ, ν, ω_o^2	Constants
τ	Period of the oscillation
p	Frequency ratio of the oscillation
$U(x)$	Potential function
$K(k)$	Complete elliptic integral of the first kind

k'	$(1 - k^2)^{1/2}$
η_o	$F_o(b/a^3)^{1/2}$
η_1	$F_1(b/a^3)^{1/2}$
e	$(\omega_o/\omega)^2$
q	ζ_o/L
ζ_o	Amplitude of the support displacement

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I. Introduction

In a series of papers, Denman and Liu [1-3], introduced a procedure for the approximate solution of the nonlinear ordinary differential equation:

$$\ddot{x} + f(x) = 0 , \tag{1}$$

where the dot $\dot{}$ indicates differentiation with respect to time t and $f(x)$ is a nonlinear function. The present paper applies the same procedure and its extensions to discuss certain aspects of the following problems in nonlinear mechanics: (a) the generalized Duffing equation; (b) the generalized Van der Pol (or Rayleigh) equation and finally (c) the generalized Mathieu-type equation. All solutions are compared with those by well-known methods or experimental results.

II. Linear Ultraspherical Polynomial

The ultraspherical polynomials $P_n^{(\lambda)}$ on the interval $[\alpha,\beta]$ are defined as the sets of polynomials orthogonal on this interval with respect to the weight factors $[1-(x-x_c)^2/A^2]^{\lambda-1/2}$, where $A = (\beta-\alpha)/2$, $x_c = (\beta+\alpha)/2$ and $\lambda > -1/2$. Expanding $f(x)$ in these polynomials and truncating after $P_1^{(\lambda)}$,

$$f^*(x) = a_0^{(\lambda)} P_0^{(\lambda)} (x') + a_1^{(\lambda)} P_1^{(\lambda)} (x') ,$$

where $x' = (x-x_c)/A$,

$$a_n^{(\lambda)} = \frac{\int_{-1}^{+1} f(Ax+x_c) P_n^{(\lambda)} (x)\, w(\lambda,x)dx}{\int_{-1}^{+1} [P_n^{(\lambda)} (x)]^2 \, w(\lambda,x)dx} \tag{2}$$

and $w(\lambda,x) = (1-x^2)^{\lambda-1/2}$, $\lambda > -1/2$.

In the usual normalization, $P_0^{(\lambda)} = 1$, $P_1^{(\lambda)}(x) = 2\lambda x$. However, since $a_n^{(\lambda)} P_n^{(\lambda)}(x)$ remains unchanged if $P_n^{(\lambda)}$ is multiplied by a constant, any convenient normalization can be used. Normalization such that $P_1^{(\lambda)}(x) = x$ is chosen in the present paper.

The following are three examples which will be used in the analyses to follow:

1. $f(x) = ax + bx^3$ in $[\alpha,\beta]$ yields the approximation

$$f^*(x) = [ax_c + bx_c^3 + 3bx_c\, A^2/2(\lambda+1)] + [3bA^2/2(2+\lambda) + a + 3bx] [x - x_c] \qquad (3)$$

2. $f(x) = \sin x$ in $[\alpha,\beta]$ yields the approximation

$$f^*(x) = [\sin x_c\ \Lambda_\lambda(A) + \cos x_c\ \Lambda_{\lambda+1}(A)[x-x_c]\ , \qquad (4)$$

where $\Lambda_\lambda(A) = \Gamma(\lambda+1)\, J\, (A)/(A/2)$; Γ is the Gamma function and $J_\lambda(A)$ is an ordinary Bessel function of order λ. The function $\Lambda_\lambda(A)$ is tabulated in [4].

3. $f(x) = \sinh x$ in $[\alpha,\beta]$

$$f\ (x) = [\sinh x\ \overline{\Lambda}_\lambda(A)] + \cosh x_c \overline{\Lambda}_{\lambda+1}(A)(x-x_c)\ , \qquad (5)$$

where $\overline{\Lambda}_\lambda(A) = \Gamma(\lambda+1) I_\lambda(A)/(A/2)^\lambda$, and $I_\lambda(A)$ is the modified Bessel function of order λ.

III. Applications of the Method

Consider the nonlinear differential equation:

$$\ddot{x} + g(\dot{x}) + f(x) = F_0 \sin pt + F_1\ , \qquad (6)$$

where the forcing function consists of a constant plus a sinusoidal part. Assume the existence of a steady-state oscillation and the expandability of the nonlinear functions $f(x)$ and $g(\dot{x})$ into ultraspherical polynomials.

The interval of expansion for $f(x)$ is $[\alpha,\beta]$, where α and β are minimum and maximum amplitudes respectively. Similarly, $g(x^{\cdot})$ is expanded in the interval $[\gamma,\delta]$, the minimum and maximum velocities of the motion. The linearized $f^{*}(x)$ and $g^{*}(x^{\cdot})$ are then substituted into (5) to yield

$$x^{\cdot\cdot} + c_{*}x^{\cdot} + \omega_{*}^{2}x = F_{o} \sin pt + [F_{1} + \omega_{*}^{2} x_{c} - a_{o}^{(\lambda)} + c_{*}v_{c} - \bar{a}_{o}^{(\lambda)}] , \qquad (7)$$

where any starred quantity is associated with the ultraspherical polynomial approximation and the bar over the letter is used to denote quantities associated with the velocity expansion, i.e., $c_{*} = \bar{a}_{1}^{(\lambda)}/V, V = (\delta - \gamma)/2$, $v_{c} = (\delta + \gamma)/2$, and $\omega_{*}^{2} = a_{1}^{(\lambda)}/A$. Equation (7) is a linear differential equation, hence

$$x = [F_{1} + \omega_{*}^{2}x_{c} - a_{o}^{(\lambda)} + c_{*}v_{c} - \bar{a}_{o}^{(\lambda)}]/\omega_{*}^{2} + F_{o} \sin (pt + \phi) [(\omega_{*}^{2} - p^{2})^{2} + (c_{*}p)^{2}]^{-1/2} \qquad (8a)$$

$$\text{where} \quad \phi = \tan^{-1}[c_{*}p/(\omega_{*}^{2} - p^{2})] \qquad (8b)$$

From the definitions of the minimum and maximum excursion and velocity, (8a) yields

$$x_{c} = [F_{1} + \omega_{*}^{2}x_{c} - a_{o}^{(\lambda)} + c_{*}v_{c} - \bar{a}_{o}^{(\lambda)}]/\omega_{*}^{2} ,$$

$$v_{c} = 0 ,$$

$$A = F_{o}[(\omega_{*}^{2} - p^{2})^{2} + (c_{*}p)^{2}]^{-1/2} ,$$

$$V = pA ,$$

which simplifies to

$$F_{1} = a_{o}^{(\lambda)} \qquad (9a)$$

$$F_o^2 = [(\omega_*^2 - p^2)^2 + (c_* p)^2]A^2 \tag{9b}$$

$$v_c = 0 \; ; \qquad V = pA \tag{9c}$$

It is to be noted that for free nonlinear oscillations, i.e., $F_1 = F_o = 0$, Equations (9a) and (9b) are still valid.

Several classes of problems are given below to illustrate the technique.

A. Generalized Duffing Equation

Let the form of the differential equation be

$$x^{\cdot\cdot} + cx^{\cdot} + f(x) = F_o \sin pt + F_1 \, , \tag{10}$$

where c is the damping constant. Immediately, Equations (9a) and (9b) simplify to

$$F_1 = a_o^{(\lambda)} \tag{9a}$$

$$F_o^2 = [(\omega_*^2 - p^2)^2 + (cp)^2]A^2 \tag{9b}$$

Two cases are examined:

Case 1, $f(x) = ax \pm bx^3$, a , $b > 0$ and $F_1 = 0$

This is the original form of Duffing's equation, the response curves in view of (3) are given by (9b), where

$$\omega_*^2 = a \pm 3bA^2/(2 + \lambda)2 \tag{11}$$

Case 2, $f(x) = \sin x$

This case describes the problem of the pull-out torque in synchronous motors [5]. Equation (9a) and (9b) become, in view of (4)

$$F_1 = \sin x_c \quad \Lambda_\lambda(A)$$

$$F_o^2 = \{[\cos x_c \Lambda_{\lambda+1}(A) \quad - p^2]^2 + (cp)^2\} A^2 , \tag{12}$$

which can be combined into

$$F_o^2 = \{[\sqrt{1-(F_1/\Lambda_\lambda (A))^2}\, \Lambda_{\lambda+1} (A) - p^2]^2 + (cp)^2\} A^2 \tag{13}$$

As is common practice with such forced oscillations problems: an amplitude A is first prescribed, and then the amplitude-frequency relationships are shown for different values of the system parameters: c , F_o and F_1 .

In Figure 1 an amplitude-frequency plot for the special case when $F_1 = c = \lambda = 0$, i.e., an undamped sinusoidally forced system, is shown in dotted lines. It is noteworthy to observe that a one-term Chebyshev-polynomial (i.e., $\lambda = 0$) approximation for sin x yields the same qualitative characteristic as a cubic Maclaurin series approximation.

Many other nonlinear $f(x)$'s are given in [1] and [2], which would yield similar results as given above.

B. Self-excited Oscillations

A "self-excited" system is characterized by input as well as dissipation of energy. If more energy is provided than dissipated, the amplitude may increase and conversely. A fixed amplitude, called its "limit cycle", arises only if the two processes balance each other.

1. Free Oscillations

Such systems are typified by Van der Pol's equation:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \quad , \tag{14}$$

where μ is a parameter. It can be written as Rayleigh's equation

$$y^{\cdot\cdot} - \mu(y^{\cdot} - y^{\cdot 3}/3) + y = 0 \quad , \tag{15}$$

through the transformation $y^{\cdot} = x$. Consider the generalization of Equation (15): the velocity term is written as $g(y^{\cdot})$ and the resulting differential equation becomes

$$y^{\cdot\cdot} + g(y^{\cdot}) + y = 0 \ , \tag{16}$$

where $g(y^{\cdot})$ is a nonlinear odd function and $g(0) = 0$.

It is a well-known fact that in order for a system described by Equation (16) to execute self-excited oscillations, it is essential that the slope of $g(y^{\cdot})$ be negative for small $y^{\cdot}$. If there exists a steady-state limit cycle about the origin, then the velocity interval is obviously $[-V,V]$ or the linearized form of (16) becomes

$$y^{\cdot\cdot} + c_{*}y^{\cdot} + y = 0 \ . \tag{17}$$

However, a limit-cycle motion demands $c_{*} = 0$, which yields the condition on the maximum velocity in the limit cycle. Again, several examples are given:

<u>Case (a)</u>, $g(y^{\cdot}) = \mu(1 - y^{\cdot 2}/3)y^{\cdot}$ (Rayleigh's equation)

Equation (17) becomes, in view of (3)

$$y^{\cdot\cdot} - \mu[1 - V^2/2(2+\lambda)]y^{\cdot} + y = 0 \ . \tag{18}$$

The condition that c_{*} must vanish gives:

$$V = [2(2 + \lambda)]^{1/2} \tag{19}$$

Hence, for steady-state oscillation, the linear approximation is

$$V = [2(2 + \lambda)]^{1/2} \sin(t + \phi) \tag{20}$$

For Chebyshev polynomials, i.e. $\lambda = 0$, (20) becomes

$$x = y^{\cdot} = 2 \sin(t + \phi) , \tag{21}$$

a well-known approximate result for (14) and (15).

Alternatively, one can examine the linearized version (18) directly: note that if $V > [2(2 + \lambda)]^{1/2}$, the system is dissapative while for $V < [2(2 + \lambda)]^{1/2}$ the system is unstable. At $V = [2(2 + \lambda)]^{1/2}$, a strongly stable limit cycle is established. Whatever the initial velocity and displacement the oscillation will tend toward the stationery value (21).

Case (b), $g(y^{\cdot}) = - \sin y^{\cdot}$

The linearized differential equation becomes

$$y^{\cdot\cdot} - \Lambda_{\lambda+1}(V)y^{\cdot} + y = 0 \tag{22}$$

Limit-cycle velocity occurs at

$$\Lambda_{\lambda+1}(V) = 0 \tag{23a}$$

or

$$J_{\lambda+1}(V) = 0 \tag{23b}$$

For any given λ , there exists an infinity of roots in (23b).

This indicates an infinite number of limit cycles which can be shown to be alternately stable and unstable. For $\lambda = 0$, the limit cycles occur at the zeros of $J_1(V) = 0$, which agrees with the results of Eckweiler [6], who compared his results with the limit cycles obtained from the Liénard construction.

Case (c), $g(y^{\cdot}) = - \sinh y^{\cdot} + cy^{\cdot}, \quad 0 < c < 1$

Similar to the previous case, one obtains an infinity of alternately stable and unstable limit-cycles which occur at the zeros of

$$I_{\lambda+1}(V) = c(V/2)^{\lambda+1}/\Gamma(\lambda+2) \tag{24}$$

2. Forced Oscillations

(a) Van der Pol's Equation with Forcing Term

To begin with one notes the linearization scheme when applied to the equation

$$\ddot{x} - \mu\dot{x} + \nu\dot{x}^3 + \omega_o^2 x = F_o \cos(pt + \phi) , \qquad (25)$$

where μ , ν and ω_o^2 are positive constants, yields the steady-state solution

$$x = F_o[(\omega_o^2 - p^2)^2 - (3\nu V^2/2(2+\lambda) - \mu)^2 p^2]^{-1/2}\cos(pt + \phi)$$

For steady-state oscillation, $V = pA$, hence

$$A^2 = F_o^2[(\omega_o^2 - p^2)^2 - (3\nu A^2p^2/(2+\lambda)2 - \mu)^2p^2]^{-1}$$

Or following [5], one can rewrite the above as

$$\rho[\sigma^2 + (1-\rho)^2] = F^2 , \qquad (26)$$

where $\sigma = (p^2 - \omega_o^2)/\mu p$, $\rho = 3\nu A^2p^2/2\mu(2+\lambda)$ and $F^2 = 3F_o^2\nu/\mu^3(2+\lambda)2$.

Again, Equation (22) compares exactly with the standard equivalent linearization with $\lambda = 0$. If the "spring" is non-linear in (25), only the ω_o^2 term in (26) changes. The procedure is straightforward.

(b) $f(x) = \sinh x$ and $g(\dot{x}) = - \sin \dot{x}$ in Equation (6)

The differential equation is

$$\ddot{x} - c \sin \dot{x} + \sinh x = F_o\sin pt + F_1 , \quad c > 0 , \qquad (27)$$

which typifies the problem with both nonlinear variable damper and spring. To eliminate F_1 , one usually resorts to a substitution of the form

$z = x + F_1$, which fails to achieve a simplication in the present case. The method of ultraspherical approximation, however, yields the result in a straightforward manner. Following (9a), (9b) and (9c) one obtains

$$F_1 = \sinh x_c \, \bar{\Lambda}_\lambda(A)$$

$$F_o^2 = \{[\cosh x_c \, \bar{\Lambda}_{\lambda+1}(A) - p^2]^2 + [cp \, \Lambda_{\lambda+1}(pA)]^2\} A^2 , \qquad (28)$$

which combines into

$$F_o^2 = \{[\sqrt{1 + (F_1/\bar{\Lambda}_\lambda(A))^2}\, \bar{\Lambda}_{\lambda+1}(A) - p^2]^2$$

$$+ [cp \, \Lambda_{\lambda+1}(pA)]^2\}A^2 \qquad (29)$$

As usual, one prescribes amplitude A to obtain the response curves.

IV. Method of Continuous Variation of λ

In [7], a method was introduced to take advantage of the "free" index λ of the ultraspherical polynomials $P_n^{(\lambda)}(x)$. The free oscillation problem governed by (1) has an "exact" solution in terms of quadratures, i.e. its period is a function of amplitude or $\tau = \tau(A)$. The linearized version of (1) also has a period solution of the form $\tau^* = \tau^*(\lambda,A)$. By equating the exact and linearized periods, one obtains a functional relationship between A and λ, i.e. $\lambda = \lambda(A)$, which can be used profitably in a variety of situations to improve the results of (ultraspherical polynomial) equivalent linearization. The present section extends the technique and applies it to forced single-degree-of-freedom non-linear problems governed by the differential equation (6). Suppose $g(\dot{x})$ and forcing terms vanished in (6), i.e.,

$$\ddot{x} + f(x) = 0 \qquad (30)$$

Further, assume (30) has a steady-state period given by

$$\tau = \tau(x_c, A) \ . \tag{31}$$

The corresponding approximate solution of (30) is

$$\tau^* = \tau^*(\lambda, x_c, A) \ , \tag{32}$$

where any starred quantity is associated with the ultraspherical polynomial approximation. By equating (31) and (32), one may obtain the functional relationship

$$\lambda = \lambda(x_c, A) \tag{33}$$

which will be used to improve the solutions to (6). The improvement results through the assumption that the period-amplitude relationship of forced oscillation given by (6) is related as in "free" motion denoted by (30).

V. Applications of the Technique

A. The Slightly Damped Problem with Sinusoidal Forcing

The differential equation is

$$x^{\cdot\cdot} + cx^{\cdot} + f(x) = F_o \sin pt \tag{34}$$

where $f(x)$ is assumed to be odd. According to (9b), one obtains

$$F_o^2 = [(\omega_*^2 - p^2)^2 + (cp)^2]A^2 \tag{35}$$

Condition (33) demands, in the present case

$$\tau^* = \tau = 2\pi/\omega_* = 2^{3/2}\int_0^A [U(A) - U(x)]^{-1/2}dx \ , \tag{36}$$

where $U(x) = \int f(x)dx$ = potential function. Hence, a substitution of (36) into (35) yields

$$F_o^2 = \{[(2\pi/\tau)^2 - p^2]^2 + (cp)^2\}A^2 \tag{37}$$

We note that in (37), if c and F_o vanishes, it reduces to the exact solution for period of the free oscillation problem.

The evaluation of $\tau = \tau(A)$ in (37) falls into two categories depending on the way $f(x)$ is given: (a) $f(x)$ is a given mathematical function, then the problem reduces to (36), i.e., one of quadratures; (b) $f(x)$ is given as a set of points on a graph, then the graphical techniques presented in [8] and [9] is preferred over lengthy numerical integration. Using the case of $f(x) = \sin x$ as an example, its exact period in free oscillation is

$$\tau = 4K[\sin(A/2)] , \tag{38}$$

where $K(k)$ is the complete elliptic integral of the first kind. Hence, (37) gives

$$F_o^2 = \{[\pi/2K(\sin(A/2))]^2 - p^2\}^2A^2 + (cpA)^2 \tag{39}$$

Figure 1 compares the results of: (a) when $\lambda = 0$ in (13), (b) when either the numerical method of Runga-Kutta-Gill (fourth order) or an analog computer is used on the governing differential equation. The superiority of the present technique for large amplitudes is quite obvious from the figure.

B. Hyperbolic Sine Spring with Damping and Forcing

For the case of $f(x) = \sinh x$, its exact free period is given in [1], hence choice for λ is:

$$\overline{\Lambda}_{\lambda+1}(A) = \{\pi \cosh(A/2)/2K[\tanh(A/2)]\}^2 \tag{40}$$

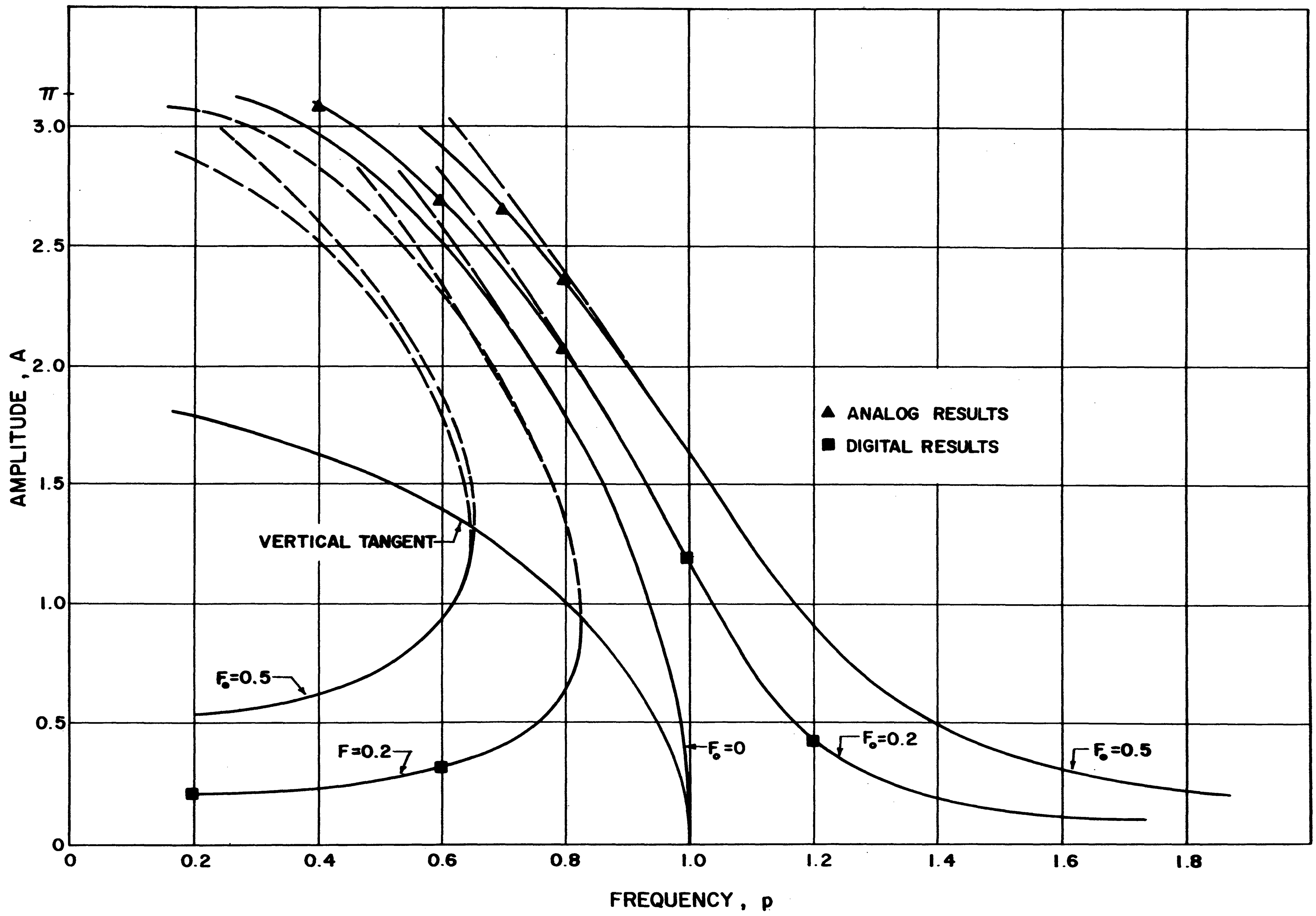

Figure 1. Amplitude-Frequency Curves for $x'' + \sin x = F_o \sin pt$

The new response curves for $g(x^\cdot) = cx^\cdot$ in (6)

$$F_o^2 = \{[1 + (F_1/\bar{\Lambda}_\lambda(A))^2]^{1/2}[\pi \cosh(A/2)/2K(\tanh(A/2))]^2 - p^2\}^2 A^2 + (cpA)^2 \tag{41}$$

The process of determining the value of λ for which Equation (40) holds is rather tedius. Fortunately, in the present case, it was shown in [1] that the range of λ varied at most from $0 \to 1/2$ for A from $0 \to \infty$. The choice of $\lambda = 0$ in (41) is indicated by the results in [1]. On Figure 2 are also shown typical amplitude-frequency plots with F_o, F_1 and c as parameters. Similarly, the new response curves for (27) are easily shown to be given by:

$$F_o^2 = \{[1 + (F_1/\bar{\Lambda}_\lambda(A))^2]^{1/2}[\pi \cosh(A/2)/2K[\tanh(A/2))]^2 - p^2\}^2 A^2 + [cp\,\Lambda_{\lambda+1}(pA)A]^2 \tag{42}$$

VI. Alternate Procedure

If one were to rewrite (6) as

$$x^{\cdot\cdot} + g(x^\cdot) + h(x) = F_o \sin pt, \tag{6}$$

where $h(x) = f(x) - F_1$, then the exact and approximate period solutions of

$$x^{\cdot\cdot} + h(x) = 0 \tag{43}$$

will play similar roles as before, i.e., the assumptions contained in Equations (30) to (33) are equally valid for (43).

One example will suffice to illustrate the procedure. Consider the generalized Duffing equation:

$$x^{\cdot\cdot} + ax + bx^3 = F_1 + F_o \sin pt, \quad a, b > 0 \tag{44}$$

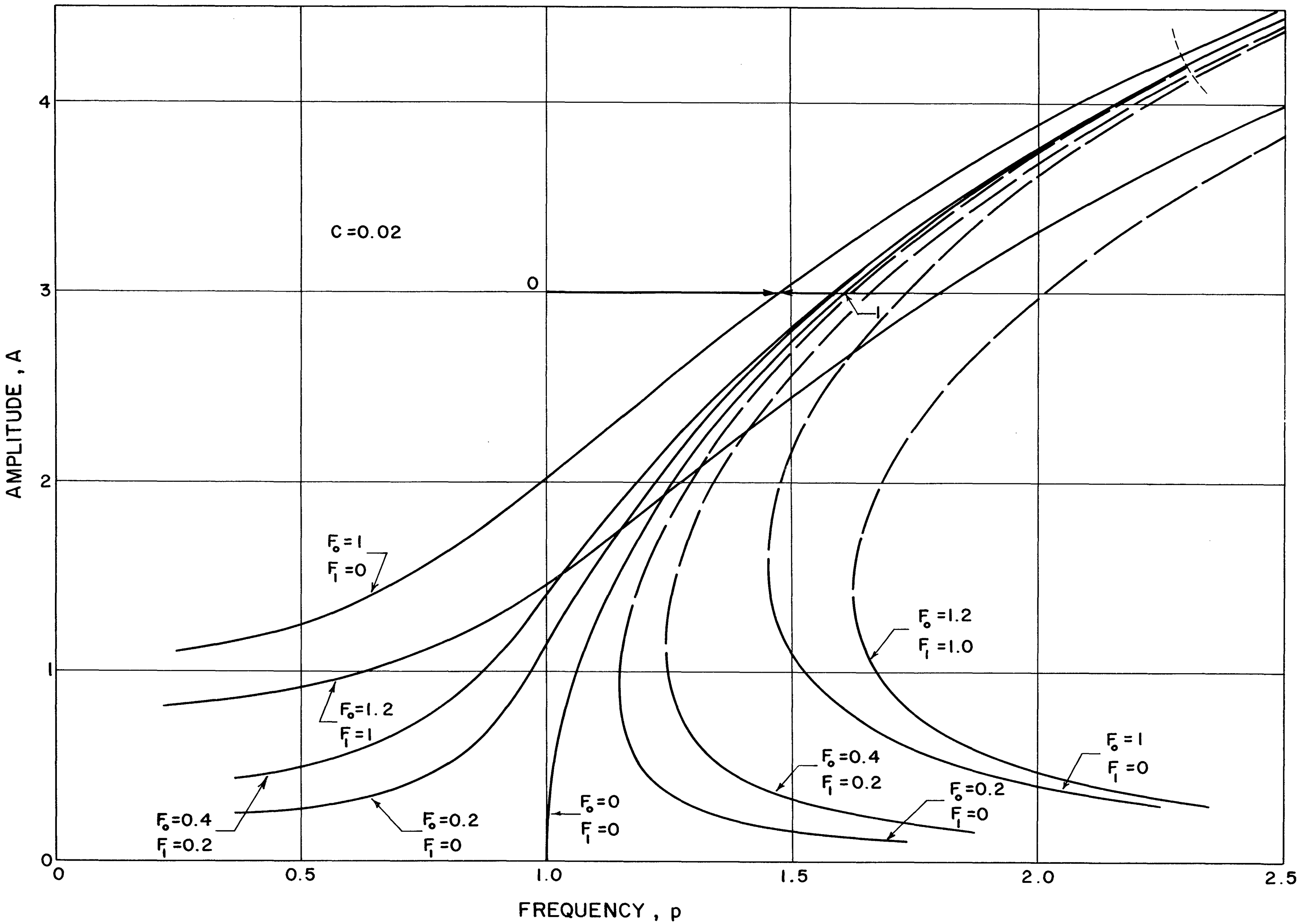

Figure 2. Amplitude-Frequency Curves for $\ddot{x} + c\dot{x} + \sinh x = F_o \sin pt + F_1$

Let $y = x(b/a)^{1/2}$, $T = ta^{1/2}$, $\eta_1 = F_1(b/a^3)^{1/2}$ and $\eta_o = F_o(b/a^3)^{1/2}$, then (44) has the dimensionless form

$$y'' + y + y^3 - \eta_1 = \eta_o \sin \Omega T \tag{45}$$

where $\Omega = p/a^{1/2}$ and $' = d/dT$. The exact solution to

$$y'' + y + y^3 - \eta_1 = 0 \tag{46}$$

is given in [2] as

$$\tau/\tau_o = (4/\pi)\,[k_1 k'_1/s(Y_2-Y_1)]^{1/2}\,K(k_1) \ , \tag{47}$$

where Y_1 and Y_2 corresponds to the turning points α, β, i.e., $Y_1 = \alpha(b/a)^{1/2}$, $Y_2 = \beta(b/a)^{1/2}$. The other constants are: $r = (Y_1 + Y_2)/2$, $s = r^2 + 2 + (Y^2 + Y_2^2)/2$, $k_1^2 = (1/2) - (1 + 3r^2)[s^2 + (Y_2 + r)^2][s^2 + (Y_1 + r)^2]$, ${k'_1}^2 = 1 - k_1^2$ and $\tau_o = 2\pi a^{1/2}$. Hence, by (9a) and (9b), one obtains

$$\eta_1 = r + r^3 + 3rA^3/2(\lambda+1)$$

and

$$\eta_o = A\{s(Y_2-Y_1)/k_1\, k'[(4/\pi)\, K(k_1)]^2 - \Omega^2\} \ ,$$

which combines into

$$\eta_o^2 = \frac{(\eta_1 - r - r^3)2(\lambda + 1)}{3r}\,\{s(Y_2-Y_1)/k_1\, k_1'\,[(4/\pi)K(k_1)]^2 - \Omega^2\}^2 \ , \tag{48}$$

where the judicious choice for λ is given by

$$[1 + 3r^2 + 3A^2/2(2 + \lambda)] = s(Y_2 - Y_1)/k_1 k'_1[(4/\pi)K(k_1)]^2 \tag{49}$$

It is to be noted that if the external sinusoidal forcing $\eta_o = 0$, (48) yields the exact solution of the step function excitation problem. In contrast, if $\eta_o = \eta_1 = 0$, (48) does not yield the correct solution to the free oscillation problem. A natural question to ask is: which one of the two procedures is better? If $\eta_1 >> \eta_o$, the technique of the present Section is obviously superior, while the method of Section V would be the choice if $\eta_o >> \eta_1$. In case the two η's are of the same order of magnitude, the preference would be decided on the basis of analytical simplicity, which is decidedly in favor of Section V.

VII. Parametric Excitations of a Pendulum

In recent papers, Skalak and Yarymovych [10] and Struble [11] have examined the problem of a pendulum executing large amplitudes which are sustained by a harmonic oscillation of the pendulum support in the vertical direction. The motion of the pendulum is governed by the nonlinear equation

$$\ddot{\theta} + 2K\dot{\theta} + (\omega_o^2 + \ddot{\zeta}/L) \sin\theta = 0 , \tag{50}$$

where K = damping coefficient, $\omega_o = (g/L)^{1/2}$ = natural frequency of small-amplitude oscillation and ζ = vertical displacement of the pendulum support. Equation (50) is first transformed into the dimensionless form

$$\theta'' + 2c\theta' + (e - n^2 q \cos nz) \sin\theta = 0 , \tag{51}$$

where $z = \omega t$, $c = K/\omega^2$, $e = \omega_o^2/\omega^2$, $q = \zeta_o/L$ and $\zeta = \zeta_o \cos n\omega t$. The assumption of $n\omega$ as the support frequency indicates that the solutions anticipated are subharmonics of order $1/n$, where n are the integers greater than one. If $\sin\theta$ were replaced by the first term of its Maclaurin series expansion, one obtains the classical Mathieu equation

$$\Theta'' + 2c\Theta' + (e - n^2 q \cos nz)\,\Theta = 0 \tag{52}$$

The form of the solution and its stability is completely delineated by the Strutt diagram, which separates the (e,q) plane into stable and unstable regions with c as a parameter.[5] Suppose one were to choose the system parameters such that the solution is unstable with respect to (52), in what way does the nonlinearity in (51) induces stable finite steady-states? In the examinations of [10] and [11], the $\sin\Theta$ term was expanded into a Maclaurin series, which was then truncated to suit the order of the analysis attempted. It is not convenient to give the solution to (52) immediately. Treating c and q as small quantities, one can, following [11], express a general perturbational solution of Equation (51) in the form of an asymptotic series:

$$\Theta = A\cos(z - \phi) + \sum_{i=1}^{n} q^i\,\Theta_i \tag{53}$$

For $c = q = o$, Equation (52) yields the elementary solution $\Theta = A\cos(z - \phi)$, where A and ϕ are arbitrary constants. For $q \neq 0$, but small, one permits slow time-variations in A and ϕ as in the variation of parameter approach. The additive terms in powers of the parameter q follow an expansion-type perturbational procedure. The assumed solution (53) is substituted into (51) with $\sin\Theta$ represented by a polynomial, one obtains:

$$\begin{aligned}&[2A\phi' + A'' - A\phi'^2 - A]\cos(z - \phi) - [2A' - 2A'\phi' - A\phi'']\sin(z - \phi) + q\Theta_1''\\ &+ q^2\Theta_2'' + \ldots\ldots + 2c[A'\cos(z - \phi) - A(1 - \phi')\sin(z - \phi) + q\Theta_1' + q^2\Theta_2' + \ldots]\\ &+ (e - n^2 q\cos nz)\{a_1[A\cos(z - \phi) + q\Theta_1 + q^2\Theta_2 + \ldots] + a_3[A^3\cos^3(z - \phi)\\ &+ 3A^2\cos^2(z - \phi)q\Theta_1 + 3A\cos(z - \phi)q^2\,\Theta_2^2 + \ldots\ O(q^3)]\} = 0\end{aligned} \tag{54}$$

where a_n's are constants which are functions of the amplitude A. Using only the first term of the ultraspherical polynomial expansion for $\sin\Theta$ and the standard justification for the variation of amplitude and phase, i.e., the first derivative terms are of $O(q)$ and the second derivatives are of $O(q^2)$, one obtains, to the first order in q, the variational equations:

$$2A\phi' + [\Lambda_{\lambda+1}(A)e - 1]\, A = 0 \,, \tag{55}$$

$$2A' + cA = 0 \,, \tag{56}$$

where $c = O(q)$. Equations (55) and (56) have well-known solutions.

The perturbational equation is, to $O(q)$:

$$\Theta_1'' + 2c\Theta_1' + [e\,\Lambda_{\lambda+1}(A)]\Theta_1 = n^2 \cos nz\ \Lambda_{\lambda+1}(A)\ A\cos(z-\phi) \tag{57}$$

However, it is very easy to verify that,

$$2\cos nz\cos(z-\phi) = \cos[(1+n)z-\phi] + \cos[(1-n)z-\phi] \tag{58}$$

Hence, Equation (57) becomes

$$\Theta_1'' + 2c\Theta_1' + [e\Lambda_{\lambda+1}(A)]\Theta_1 = (n^2A/2), \Lambda_{\lambda+1}(A)\{\cos[(1+n)z-\phi] + \cos[(1-n)z-\phi]\} \tag{59}$$

The complete non-resonant solution, is, therefore,

$$\Theta = A\cos(z-\phi) + q\Theta_1 \,, \tag{60}$$

where A and ϕ are the solutions given by (55) and (56) and Θ_1 is given by the solution of (59).

For nearly-resonant conditions of the 1/2 subharmonic, i.e., $n \to 2$, expand the second term of (59) as

$$\cos[(1 - n)z - \phi] = \cos[(n - 2)z + 2\phi] \cos(z - \phi) - \sin[(n - 2)z + 2\phi] \sin(z - \phi) \quad (61)$$

Substituting (61) back into (54) and equating the coefficients of $\sin(z - \phi)$ and $\cos(z - \phi)$, one obtains the nearly resonant variational equations

$$2A\phi' + A'' - A\phi'^2 + 2cA' - A(1 - ea_1) = (n^2 q a_1 A/2)\cos[(n - 2)z + 2\phi] \quad (62)$$

$$[2A' - 2A'\phi' - A\phi''] + 2cA(1 - \phi') = (n^2 q a_1 A/2)\sin[(n - 2)z + 2\phi] \quad (63)$$

As before, to the first order in q,

$$\phi' = [(1 - a_1 e)/2] + (n^2 q a_1/4)\cos\gamma , \quad (64)$$

$$A' = -cA + (n^2 q A a_1/4)\sin\gamma \quad (65)$$

where $\gamma = (n - 2)z + 2\phi$. Equation (64) can be rewritten as

$$\gamma' = (n - 1 - a_1 e) + (n^2 q a_1 \cos\gamma)/2 \quad (66)$$

Solving for dz in Equations (65) and (66) yields

$$\frac{dA}{d\gamma} = \frac{-cA + n^2 q A a_1 \sin\gamma/4}{(n - 1 - a_1 e) + (n^2 q a_1 \cos\gamma)/2} = \frac{P(A,\gamma)}{Q(A,\gamma)} \quad (67)$$

Singular points correspond to P and Q approaching zero simultaneously or

$$\sin\gamma = 4c/n^2 q \; \Lambda_{\lambda+1}(A) , \quad (68)$$

where use was made of $a_1 = \Lambda_{\lambda+1}(A)$ and accordingly,

$$\left(\frac{\omega}{\omega_0}\right)^* = \left(\frac{1}{e}\right)^{1/2} = \left\{\frac{\Lambda_{\lambda+1}(A)}{1 \pm 2\sqrt{[q\Lambda_{\lambda+1}(A)]^2 - c^2}}\right\}^{1/2} \tag{69}$$

Equation (69) is the bifurcation equation of the first ultraspherical polynomial approximation.

From the method of continuous variation of λ, Equation (38) is substituted into the above Equation (69) to yield

$$\left(\frac{\omega}{\omega_0}\right)^* = \left\{\frac{[\pi/2K(k_1)]}{1 + 2\sqrt{q^2[\pi/2K(k_1)]^4 - c^2}}\right\}^{1/2} \tag{70}$$

Figure 3 compares the three results obtained with (a) reference [10], (b) when $\lambda = 0$ in (69) and (c) Equation (70). The superiority of the linear ultraspherical approximation over a second-order Maclaurin expansion is indicated.

In order to study the effect of initial conditions, one must obtain the integral curves of motion given by (67). For simplicity in discussion, let $c = 0$, Equation (67) can be rewritten as

$$\left\{(n - 1)/K[(\sin A/2)] - [e - (n^2 q \cos \gamma)/2]\right\} A dA$$
$$-[(n^2 q A^2 \sin \gamma)/4]\, d\gamma = 0 , \tag{71}$$

where use was made of (38). Equation (71) is an exact differential equation whose integral is:

$$(n - 1) \int [A/K(\sin A/2)]\, dA - [(e/2) - (n^2 q \cos \gamma)/4]A^2 = M = \text{const} \tag{72}$$

For each value of M, (72) defines a trajectory in the $A - \phi$ phase plane, where A is the polar distance from the origin and γ the polar angle. The singular points for the undamped case occur at either

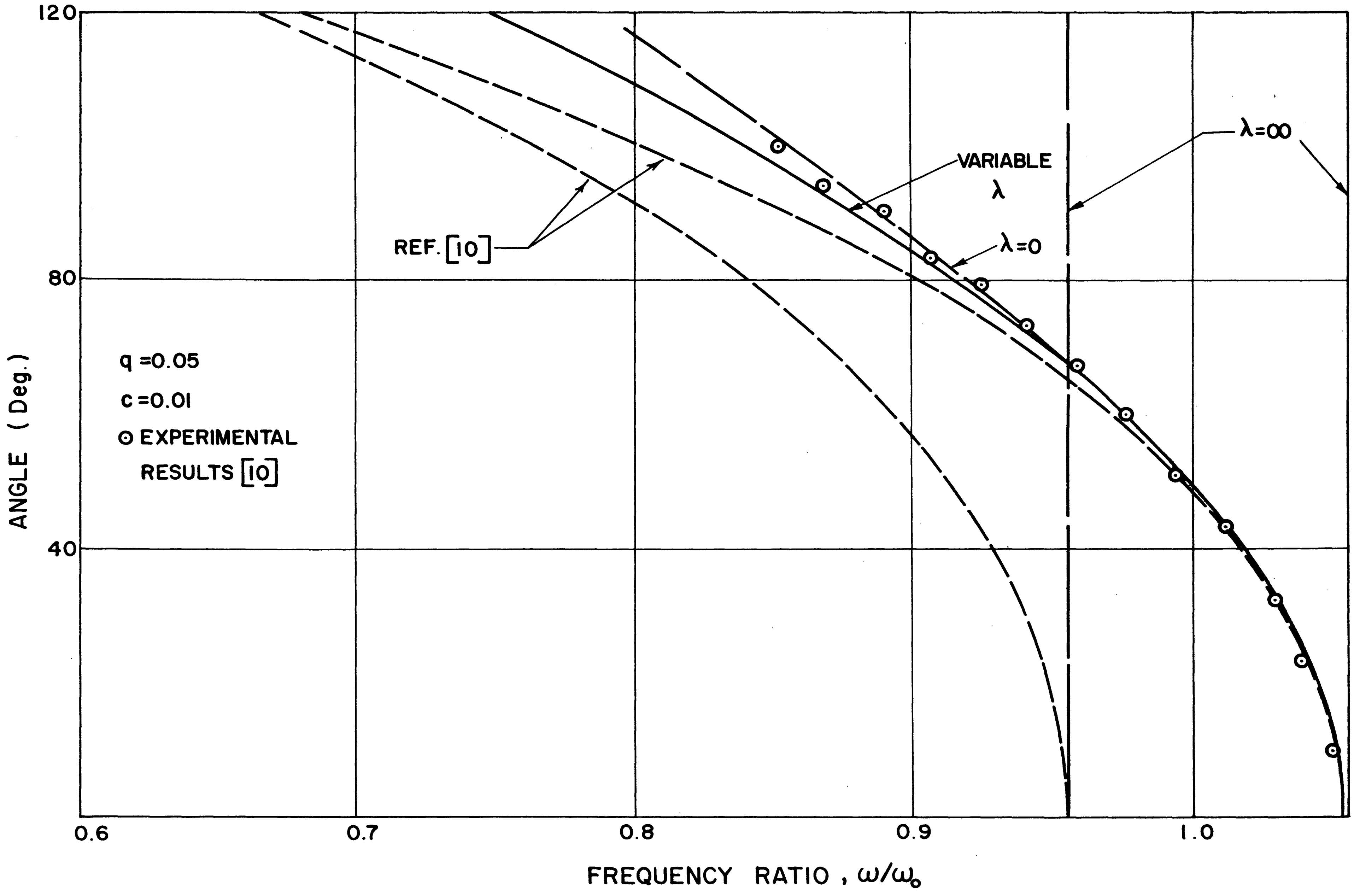

Figure 3. Amplitude-Frequency Curves for 1/2 Subharmonic (Parametric Excitation of A Pendulum)

$$\gamma = 0 \quad \text{or} \quad \pi \,, \quad \text{if} \quad A \neq 0 \tag{73}$$

or

$$A = 0 \tag{74}$$

and

$$K[\sin(A/2)][e \mp 2q] = 1 \,. \tag{75}$$

In order to find the positive roots of A in (75), one notes that the interval $2/\pi \leq e \mp 2q < 0$ corresponds to $0 \leq A < \pi$. The roots give centers and saddles as singular points and depending upon the values of $e \mp 2q$, one gets four phase portraits similar to those shown in [12].

Higher-order subharmonics can be similarly analyzed but since the paper is primarily to illustrate the applicability of the method of ultraspherical polynomial approximation, further examples will be shown in a later paper.

VIII. Discussion and Conclusions

In this paper, the approximate behavior of a single-degree-of-freedom nonlinear oscillator with a forcing function consisting of a constant part and a sinusoidal part is obtained. The linearization of the nonlinear functions in the system by ultraspherical polynomials yield results which are equivalent to standard techniques (e.g. first approximation of the method of Kryloff and Bogoliuboff) when the polynomial is the Chebyshev polynomial of the first kind, i.e., $\lambda = 0$. In this sense, the present method is a generalization of many previous linearization schemes. Next, the free index λ was taken advantage of. It was observed that in undamped free oscillations, the approximate period must equal the exact period for at least one combination of x_c and A for the interval $-1/2 < \lambda < \infty$. The appropriate value of λ for a given x_c and A is noted. When one further assumes that the steady forced oscillations have identical period-amplitude characteristics as in free oscillations, then the appropriate value of λ is used for the forced oscillation problem. The net result of these manipulations is to begin the first approximation with the free nonlinear oscillation. Of course, the period-amplitude relationships in forced motion are not identical to those in free motion, hence the results given are approximations. As pointed out by Rauscher [13], developments which begin at point 1 in Figure 2 will have more rapid convergence to the true solution than the usual perturbations from 0. (the neighborhood of the free linear solution) The method of Rauscher differs from the present one in that he used the solution of the nonlinear free oscillation problem (by quadrature) to convert the forced oscillation equation back into another free oscillation. The process is iterated through a series of quadratures.

The present linearization process allows one to deal with a linear forced differential equation, whose solution generally can be written down by inspection, without recourse to secular term arguments. The nonlinear part of the analyses is only associated with the free-oscillation problem, which is a considerable simplification from most methods.

At no point in these approximation procedures was it necessary to restrict the nonlinear spring parameters to a small value, (the damping coefficient and forcing function are considered small) a usual requirement for analytical methods. The method is very accurate as evidenced by Figure 1.

Given a nonlinear $f(x)$, either analytically or graphically, the period solution of the free oscillation problem furnishes immediately accurate results for the associated nonlinear forced oscillation problem.

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