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AN EXISTENCE ANALYSIS FOR NONLINEAR NON-SELF-ADJOINT BOUNDARY VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this dissertation we study the equation $Lx = Nx$ under the assumptions that L is a linear operator with domain and range in a real Hilbert space S, and N is an operator (not necessarily linear) whose domain and range also lie in S. Our purpose is to present an existence theory which guarantees that this equation has at least one solution. We apply this theory to the study of nonlinear boundary value problems of ordinary differential equations.

The existence theory presented is closely related to an existence theory recently developed by L. Cesari for boundary value problems of ordinary and partial differential equations. Our theory reduces to the one of Cesari when L is a self-adjoint differential operator. Cesari chooses a system of axioms concerning the existence of certain linear operators H and P possessing special properties. Using these two operators, the problem of determining a solution to the equation $Lx = Nx$ is reduced to solving a finite system of equations in finitely many unknowns.

For our existence theory we assume that L is a closed linear operator satisfying the properties: (a) the domain of L is dense in S; (b) the range of L is closed in S; (c) the null spaces for L and the adjoint L^* have finite dimensions p and q, respectively. It follows that there exist linear operators H, P, and Q with properties analogous to the properties of Cesari's operators H and P. The operator H is a continuous right inverse for L, while the operators P and Q are projection operators which project into the domains of L^* and L, respectively. These three operators depend only on the linear operator L and are independent of the operator N.

Using these three operators, we establish the existence of at least one solution $\hat{x} \in S$ to the equation $Lx = Nx$ provided a certain set of inequalities are satisfied and provided a particular finite system of equations in finitely many unknowns is solvable. At the same time we obtain estimates on the norm of such a solution $\hat{x}$. If $p \geq q$, then the finite system of equations is a system having more unknowns than equations or the same number. For this case we present two existence theorems for the equation $Lx = Nx$.

These existence theorems are used to study the equation $Lx = Nx$ when L is a differential operator on a finite interval $[a,b]$, S is the real Hilbert space $L_2[a,b]$, and N is a nonlinear operator in S. We prove that L is a closed linear operator which satisfies conditions

(a), (b), and (c), and hence, we obtain the existence of the operators H, P, and Q. The operator H is shown to have an integral representation.

Our existence theorems then yield existence theorems for a solution of the differential equation Lx = Nx in the real Hilbert space $S = L_2[a,b]$. In case L is a self-adjoint differential operator, we can construct H, P, and Q in such a way that P = Q, and such that H and P satisfy the axioms of Cesari. Thus, we show that the Cesari theory is applicable to the study of self-adjoint differential operators. Our theory is a generalization of the Cesari theory which is especially useful for the study of non-self-adjoint differential operators.

If the differential equation Lx = Nx has a solution $\hat{x}$, then we prove that all the hypotheses of our existence theorems are satisfied, and that $\hat{x}$ is one of the solutions guaranteed by the existence theorems.

As an application of our theory we show that the nonlinear boundary value problem

$$\begin{cases} x'' + x + \alpha x^2 = \beta t, \; 0 \leq t \leq 2\pi, \\ x(0) = 0 \end{cases}$$

has a solution provided the constants α, β satisfy the conditions $|\alpha| \leq 1$, $|\beta| \leq .001$, and we obtain estimates on the norm of such a solution.

CHAPTER 1

INTRODUCTION

In this thesis we discuss the existence of solutions to the equation

$$Lx = Nx, \tag{1.1}$$

where L is a linear operator with domain and range in a real Hilbert space S, and N is an operator (not necessarily linear) whose domain and range also lie in S. We present an existence theory which is applicable when L is an unbounded operator defined on a proper subset of S. This theory has applications to the study of nonlinear boundary value problems in ordinary differential equations.

Our theory is closely related to an existence theory recently developed by Cesari [3]. When L is a self-adjoint differential operator, our theory reduces to the one of Cesari. In his study of Equation (1.1) Cesari presents a system of axioms concerning the existence of linear operators H and P with convenient properties. Using these two operators, the problem of determining a solution to Equation (1.1) is reduced to the problem of solving a finite system of equations in finitely many unknowns. We examine the Cesari axioms and their implications in Appendix I.

Bartle [2], Cronin [7], and Nirenberg [19] have developed existence theories for an equation which is similar to Equation (1.1). They consider the equation

$$Lx + F(x,y) = 0 \qquad (1.2)$$

in Banach spaces S and Y. In this equation $L: S \to S$ is a bounded linear operator and $F(x,y)$ is a function which maps a neighborhood of the origin in S x Y into S with $F(0,0) = 0$. For y near the origin in Y they seek solutions $x \epsilon S$ of Equation (1.2). They also reduce the problem to one of solving a finite system of equations in finitely many unknowns. In Appendix II we relate these three theories to the theory presented in this thesis, and we determine the relationships which exist between the three of them.

We present an existence theory for Equation (1.1) in Chapter 2. By assuming that the linear operator L satisfies reasonable conditions, we show that there exist linear operators H, P, and Q with properties analogous to the properties of Cesari's operators H and P. Using these three operators, the study of Equation (1.1) is reduced to solving a finite system of equations in finitely many unknowns. The conditions on L are satisfied by all differential operators on a finite interval [a,b], so our existence theory may be used to study nonlinear boundary value problems of ordinary differential equations.

If the operator L is a self-adjoint differential operator, then we can construct H, P, and Q in such a way that P = Q, and such that H and P satisfy the axioms of Cesari. Thus, we show that the Cesari theory is applicable to the study of self-adjoint differential operators. Our theory is a generalization of the Cesari theory and is especially useful for the study of non-self-adjoint differential operators.

Let us summarize our existence theory as it appears in Chapter 2. We consider Equation (1.1),

$$Lx = Nx,$$

in a real Hilbert space S. The operator L is a closed linear operator with domain $\mathcal{D}(L)$ and range $\mathcal{R}(L)$ in S, and the operator N is an operator in S with domain $\mathcal{D}(N)$ such that $\mathcal{D}(L) \cap \mathcal{D}(N) \neq \phi$. We assume that L satisfies the following conditions:

(a) The domain $\mathcal{D}(L)$ is dense in S;

(b) The range $\mathcal{R}(L)$ is closed in S;

(c) The null space for L has dimension $p < \infty$, and the null space for the adjoint operator L^* has dimension $q < \infty$.

When these conditions are satisfied, there exist operators H, P, and Q with convenient properties (see Theorem 2.4). The operator H is a continuous right inverse for L, while the operators P and Q are projection operators which project into $\mathcal{D}(L^*)$ and $\mathcal{D}(L)$, respectively. These three operators depend only on the linear operator L and are independent of the operator N.

Using these three operators, we obtain the existence of at least one solution $\hat{x} \in S$ to Equation (1.1) provided a certain set of inequalities are satisfied and provided a particular finite system of equations in finitely many unknowns is solvable (see Theorem 2.7). At the same time we obtain estimates on the norm of such a solution $\hat{x}$. If $p \geq q$, then the finite system of equations is a system having more unknowns than equations or the same number. For this case we present two existence theorems for Equation (1.1) (see Theorems 2.9 and 2.10).

In Chapter 3 we introduce the notion of a differential operator L in the real Hilbert space $S = L_2[a,b]$ and summarize some of the familiar properties of these operators. In particular, we show that each differential operator in S is a closed linear operator satisfying the above conditions (a), (b), and (c). For a given differential operator L we determine its adjoint L^* using classical methods and construct a continuous right inverse H using the results of Chapter 2. The operator H is represented as an integral operator.

The results of Chapters 2 and 3 are combined in Chapter 4 to yield existence theorems for nonlinear ordinary differential equations of the form $Lx = Nx$ (see Theorems 4.1 and 4.2). As a special case we examine self-adjoint differential operators L, showing that our existence theory reduces to the Cesari theory when L is a self-adjoint differential operator.

In Chapter 5 we assume that there exists an exact solution $\hat{x}$ to the ordinary differential equation $Lx = Nx$, and then show that all the hypotheses of our existence theorems are satisfied. It follows that $\hat{x}$ is one of the solutions guaranteed by our existence theorems.

As an application of our theory we study in Chapter 6 the non-linear boundary value problem

$$\begin{cases} x'' + x + \alpha x^2 = \beta t, \ 0 \leq t \leq 2\pi, \\ x(0) = 0, \end{cases}$$

where α and β are real constants. The linear part L of this equation is a non-self-adjoint differential operator with $p \geq q$ ($p = 1$, $q = 0$). We show that this equation has a solution provided the constants α, β satisfy the conditions $|\alpha| \leq 1$, $|\beta| \leq .001$, and we obtain estimates on the norm of such a solution.

CHAPTER 2

A GENERAL THEORY FOR THE EQUATION $Lx = Nx$

2.1 THE RIGHT INVERSE OPERATOR H

In this chapter we study the functional equation $Lx = Nx$ where L and N are operators defined in a real Hilbert space S. The symbols $\mathcal{D}(L)$ and $\mathcal{R}(L)$ will denote the domain and range, respectively, of any operator L defined in S. In case L is a linear operator, the symbol $\mathcal{N}(L)$ will denote the null space or kernel of L. Most of the linear operators we shall study will be <u>closed</u> operators.

<u>Definition 2.1</u> An operator L is <u>closed</u> if the relations $x_n \in \mathcal{D}(L)$,

$$x_n \to x, \text{ and } Lx_n \to y \text{ imply that } x \in \mathcal{D}(L) \text{ and } Lx = y.$$

This definition is equivalent to the statement that the graph of L is a closed subset of the Hilbert space $S \oplus S$. If L is a linear operator in S, then the adjoint operator L^* is defined iff $\mathcal{D}(L)$ is dense in S. When L^* is defined, it is a closed operator whether L is closed or not. If L is a closed linear operator with $\mathcal{D}(L)$ dense in S, then $\mathcal{D}(L^*)$ is also dense in S and $L^{**} = L$. In the next chapter we shall show that all ordinary differential operators are closed linear operators with dense domains.

Let S be a real Hilbert space with inner product (x,y) and norm $\|x\|$, and let L be a closed linear operator in S with the following properties:

(Ia) The domain $\mathcal{D}(L)$ is dense in S,

(Ib) The range $\mathcal{R}(L)$ is closed in S,

(Ic) The null spaces $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$ are finite-dimensional linear subspaces in S.

Choose elements $\phi_1, \ldots, \phi_p$ in $\mathcal{D}(L)$ to form an orthonormal base for $\mathcal{N}(L)$, and choose elements $\omega_1, \ldots, \omega_q$ in $\mathcal{D}(L^*)$ to form an orthonormal base for $\mathcal{N}(L^*)$. Then

$$p = \dim \mathcal{N}(L), \quad q = \dim \mathcal{N}(L^*). \tag{2.1}$$

Let $\mathcal{N}(L)^\perp$ denote the orthogonal complement of $\mathcal{N}(L)$ in S, i.e.,

$$\mathcal{N}(L)^\perp = \{x \in S \mid (x, \phi_i) = 0 \text{ for } i = 1, \ldots, p\}.$$

We introduce a new linear operator L_1 by taking the restriction of L to the linear subspace $\mathcal{D}(L) \cap \mathcal{N}(L)^\perp$:

$$L_1 = L \mid \mathcal{D}(L) \cap \mathcal{N}(L)^\perp . \tag{2.2}$$

<u>Lemma 2.1</u> The linear operator L_1 is 1-1 and its range is precisely the range of L.

<u>Proof</u>. First, suppose $x \in \mathcal{D}(L_1)$ and $L_1 x = 0$. Then $x \in \mathcal{D}(L) \cap \mathcal{N}(L)^\perp$ and $Lx = 0$, so $x \in \mathcal{N}(L) \cap \mathcal{N}(L)^\perp$. Hence, $x = 0$ and it follows that L_1 is 1-1.

Next, we note that $\mathcal{R}(L_1) \subseteq \mathcal{R}(L)$. Take $y \in \mathcal{R}(L)$. Then there exists $x \in \mathcal{D}(L)$ such that $Lx = y$. Let

$$z = x - \sum_{i=1}^{p} (x,\phi_i)\phi_i.$$

We see that $z \in \mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$ and $L_1 z = Lz = Lx = y$. Thus, $y \in \mathcal{R}(L_1)$ and $\mathcal{R}(L) \subseteq \mathcal{R}(L_1)$.

Q.E.D.

Definition 2.2 We define a linear operator H in S by

$$H = L_1^{-1} = [L|\mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}]^{-1}.$$

Theorem 2.1 The linear operator H has the following properties:

(a) The domain of H is $\mathcal{R}(L)$ and the range of H is $\mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$.

(b) H is a 1-1 continuous linear operator.

(c) $LHy = y$ for all $y \in \mathcal{R}(L)$.

(d) $HLx = x - \sum_{i=1}^{p} (x,\phi_i)\phi_i$ for all $x \in \mathcal{D}(L)$.

Proof. (a) This follows from the last lemma.

(b) Because L_1 is 1-1, we get that H is 1-1. Now $\mathcal{R}(L)$ and $\mathcal{N}(L)^{\perp}$ are both closed linear subspaces in S, so we can consider them as Hilbert spaces under the induced inner products. We consider H as a linear operator defined on all of the Hilbert space $\mathcal{R}(L)$ and having values in the Hilbert space $\mathcal{N}(L)^{\perp}$. Take $x_n \in \mathcal{R}(L)$, $x \in \mathcal{R}(L)$, and

$y \in \mathcal{N}(L)^{\perp}$ with $x_n \to x$ and $Hx_n \to y$. Assert that $x \in \mathcal{R}(L)$ and $Hx = y$. Let $y_n = Hx_n$. Then $y_n \in \mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$ and

$$Ly_n = L_1 y_n = L_1 H x_n = x_n,$$

so

$$y_n \in \mathcal{D}(L),\ y_n \to y,\ \text{and } Ly_n \to x.$$

Since L is a closed operator, we conclude that $y \in \mathcal{D}(L)$ and $Ly = x$. Thus, $y \in \mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$ and $L_1 y = x$. This implies that $x \in \mathcal{R}(L_1) = \mathcal{R}(L)$ and

$$Hx = HL_1 y = y,$$

which establishes the assertion. By the Closed Graph Theorem [9, p. 57] we conclude that H is continuous.

(c) This follows from the definition of H.

(d) Take $x \in \mathcal{D}(L)$ and let $y = x - \sum_{i=1}^{p} (x,\phi_i)\phi_i$. Then $y \in \mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$, so $HL_1 y = y = HLy$. Thus,

$$\begin{aligned} HLx &= HL\left(y + \sum_{i=1}^{p}(x,\phi_i)\phi_i\right) \\ &= HLy \\ &= y. \end{aligned}$$

Q.E.D.

In this theorem we have constructed a continuous right inverse operator H for the given operator L. This operator will be very important in our subsequent work.

2.2 THE PROJECTION OPERATORS P AND Q

Let N be an operator in S with $\mathcal{D}(L) \cap \mathcal{D}(N) \neq \phi$. In general, the operator N will be nonlinear, but later in this chapter we shall assume certain continuity conditions on N. We want to examine the equation

$$Lx = Nx, \tag{2.3}$$

establishing existence theorems for it. We shall introduce two more operators P and Q which, when utilized with H, will yield an existence theory for Equation (2.3).

Let m be an integer with $m \geq q$ where $q = \dim \mathcal{N}(L^*)$. Choose elements $\omega_{q+1}, \ldots, \omega_m$ in $\mathcal{D}(L^*)$ such that the elements $\omega_1, \ldots, \omega_m$ form an orthonormal set in S. Thus, $\omega_i \in \mathcal{D}(L^*)$ for $i = 1, \ldots, m$ and

$$(\omega_i, \omega_j) = \delta_{ij} \text{ for } i,j = 1, \ldots, m. \tag{2.4}$$

Clearly

$$\mathcal{N}(L^*) \subseteq \langle \omega_1, \ldots, \omega_m \rangle$$

where the bracket symbol is used to denote the linear subspace spanned by $\omega_1, \ldots, \omega_m$. We need a lemma.

<u>Lemma 2.2</u> $\mathcal{R}(L) = \mathcal{N}(L^*)^{\perp}$ and $\mathcal{R}(L)^{\perp} = \mathcal{N}(L^*)$.

Proof. First, take $y \in \mathcal{R}(L)$. Then there exists $x \in \mathcal{D}(L)$ such that $Lx = y$. For each $z \in \mathcal{N}(L^*)$ we have

$$(y,z) = (Lx,z) = (x,L^*z) = 0,$$

or $y \in \mathcal{N}(L^*)^{\perp}$. Thus, $\mathcal{R}(L) \subseteq \mathcal{N}(L^*)^{\perp}$.

Next, we take $y \in \mathcal{N}(L^*)^{\perp}$. Then $(y,z) = 0$ for all $z \in \mathcal{N}(L^*)$. Because $\mathcal{R}(L)$ is closed, we can write

$$y = y_1 + y_2$$

where $y_1 \in \mathcal{R}(L)$ and y_2 is orthogonal to $\mathcal{R}(L)$, i.e., $(Lx,y_2) = 0 = (x,0)$ for all $x \in \mathcal{D}(L)$. From the definition of L^* we see that $y_2 \in \mathcal{D}(L^*)$ and $L^*y_2 = 0$. Hence, $y_2 \in \mathcal{N}(L^*)$ and $(y,y_2) = 0$. But

$$\|y_2\|^2 = (y_2,y_2-y) = -(y_2,y_1) = 0$$

or $y_2 = 0$. Thus, $y = y_1 \in \mathcal{R}(L)$ and $\mathcal{N}(L^*)^{\perp} \subseteq \mathcal{R}(L)$. This proves that $\mathcal{R}(L) = \mathcal{N}(L^*)^{\perp}$, and from this we get that

$$\mathcal{R}(L)^{\perp} = \mathcal{N}(L^*)^{\perp\perp} = \mathcal{N}(L^*).$$

Q.E.D.

From this lemma we note that the elements $\omega_{q+1},\ldots,\omega_m$ belong to $\mathcal{R}(L)$, and hence, we can form the elements $H\omega_{q+1},\ldots,H\omega_m$. Clearly $H\omega_i \in \mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$ for $i = q+1,\ldots,m$. Let S_o be the linear subspace in S spanned by the elements $\phi_1,\ldots,\phi_p$, which were chosen to form an orthonormal base for $\mathcal{N}(L)$, and by the elements $H\omega_{q+1},\ldots,H\omega_m$:

$$S_o = \langle \phi_1,\ldots,\phi_p,H\omega_{q+1},\ldots,H\omega_m \rangle. \tag{2.5}$$

We claim these elements are linearly independent. Suppose

$$\sum_{i=1}^{p} b_i \phi_i + \sum_{i=q+1}^{m} c_i H\omega_i = 0.$$

Since $H\omega_i \in \mathcal{N}(L)^\perp$ for $i = q+1,\ldots,m$, we have $(\phi_j, H\omega_i) = 0$ for $j = 1,\ldots,p$ and $i = q+1,\ldots,m$. Thus, $b_j = 0$ for $j = 1,\ldots,p$ and

$$H\left(\sum_{i=q+1}^{m} c_i\omega_i\right) = 0 .$$

But H is 1-1, so $\sum_{i=q+1}^{m} c_i\omega_i = 0$ and $c_i = 0$ for $i = q+1,\ldots,m$. This establishes the independence. We shall assume throughout the rest of this chapter that S_o is a subset of $\mathcal{D}(N)$.

<u>Lemma 2.3</u> The linear subspace S_o has dimension $p+m-q$, S_o is a subset of $\mathcal{D}(L) \cap \mathcal{D}(N)$, and the elements $\phi_1,\ldots,\phi_p,H\omega_{q+1},\ldots,H\omega_m$ form a base for S_o.

The proof of the lemma is clear. We are now ready to define the operators P and Q.

<u>Definition 2.3</u> The operator P: $S \to S$ is defined by

$$Px = \sum_{i=1}^{m} (x,\omega_i)\omega_i \qquad \text{for all } x \in S.$$

The operator Q: $S \to S$ is defined by

$$Qx = \sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=q+1}^{m} (x,L^*\omega_i)H\omega_i \quad \text{for all } x \in S.$$

Clearly P and Q are continuous linear operators in S. The next few theorems will establish their properties.

Theorem 2.2 The operator P is a continuous linear operator in S with the following properties:

(a) $P\omega_i = \omega_i$ for $i = 1,\ldots,m$.

(b) The range of P is the linear subspace $\langle\omega_1,\ldots,\omega_m\rangle$ which is a subset of $\mathcal{D}(L^*)$; the range of the operator I - P is a subset of $\mathcal{R}(L)$.

(c) $P^2 = P$.

(d) $Px = x$ for all $x \in \langle\omega_1,\ldots,\omega_m\rangle$.

Proof. (a) This follows from (2.4) and the definition of P.

(b) The first part follows from (a). For the second part take $x \in S$ and let $y = x - Px$. We want to show that $y \in \mathcal{R}(L)$. For $1 \leq j \leq q$ we have

$$(y,\omega_j) = (x,\omega_j) - \sum_{i=1}^{m} (x,\omega_i)(\omega_i,\omega_j) = 0$$

or

$$y \in \mathcal{N}(L^*)^{\perp} = \mathcal{R}(L).$$

(c) For $x \in S$ we have from (a) that

$$\begin{aligned} P^2x &= P\left[\sum_{i=1}^{m} (x,\omega_i)\omega_i\right] \\ &= \sum_{i=1}^{m} (x,\omega_i)\omega_i \\ &= Px. \end{aligned}$$

(d) This is trivial to check using (a).

Q.E.D.

<u>Theorem 2.3</u> The operator Q is a continuous linear operator in S with the following properties:

(a) $Q\phi_i = \phi_i$ for $i = 1,\ldots,p$ and $QH\omega_i = H\omega_i$ for $i = q+1,\ldots,m$.

(b) The range of Q is the linear subspace S_0 which is a subset of $\mathcal{D}(L)$.

(c) $Q^2 = Q$.

(d) $Qx = x$ for all $x \in S_0$.

<u>Proof</u>. (a) Since $\phi_i \in \mathcal{D}(L)$ and $\omega_j \in \mathcal{D}(L^*)$, we have

$$(\phi_i, L^*\omega_j) = (L\phi_i, \omega_j) = 0 \quad \text{for } j = 1,\ldots,m,$$

so

$$Q\phi_i = \sum_{j=1}^{p} (\phi_i, \phi_j)\phi_j + 0 = \phi_i \quad \text{for } i = 1,\ldots,p.$$

Also, we have $H\omega_i \in \mathcal{D}(L) \cap \mathcal{N}(L)^{\perp}$ and $LH\omega_i = \omega_i$ for $i = q+1,\ldots,m$, and hence,

$$(H\omega_i, \phi_j) = 0 \quad \text{for } j = 1,\ldots,p$$

and

$$(H\omega_i, L^*\omega_j) = (LH\omega_i, \omega_j)$$

$$= (\omega_i, \omega_j) \quad \text{for } i,j = q+1,\ldots,m.$$

Thus,

$$QH\omega_i = H\omega_i \quad \text{for } i = q+1,\ldots,m.$$

(b) and (d) follow from (a). To show (c) take $x \in S$. Using (a) we have

$$Q^2 x = Q\left[\sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i\right]$$

$$= \sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i$$

$$= Qx.$$

Q.E.D.

Note that P and Q are projection operators. The operator P is an orthogonal projection. In general, the operator Q is not an orthogonal projection. The next theorem relates the four operators L, H, P, and Q.

Theorem 2.4 The following properties are valid:

(a) $H(I - P)LX = (I - Q)x$ for all $x \in \mathcal{D}(L)$.

(b) $LH(I - P)x = (I - P)x$ for all $x \in S$.

(c) $LQx = PLx$ for all $x \in \mathcal{D}(L)$.

(d) $QH(I - P)x = 0$ for all $x \in S$.

Proof. The proof of (b) is clear since $I - P$ has its range in $\mathcal{R}(L)$ and H is a right inverse for L. To show (a) and (c), take $x \in \mathcal{D}(L)$. Then

$$(I - P)Lx = Lx - \sum_{i=1}^{m} (Lx,\omega_i)\omega_i$$

$$= Lx - \sum_{i=1}^{m} (x, L^*\omega_i)\omega_i$$

$$= Lx - \sum_{i=q+1}^{m} (x, L^*\omega_i)\omega_i ,$$

and hence, by Theorem 2.1 (d) we have

$$H(I - P)Lx = HLx - \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i$$

$$= x - \sum_{i=1}^{p} (x,\phi_i)\phi_i - \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i$$

$$= (I - Q)x.$$

Also, we have

$$LQx = \sum_{i=1}^{p} (x,\phi_i)L\phi_i + \sum_{i=q+1}^{m} (x, L^*\omega_i)LH\omega_i$$

$$= \sum_{i=q+1}^{m} (x, L^*\omega_i)\omega_i$$

and

$$PLx = \sum_{i=1}^{m} (Lx,\omega_i)\omega_i$$

$$= \sum_{i=1}^{m} (x, L^*\omega_i)\omega_i$$

$$= \sum_{i=q+1}^{m} (x, L^*\omega_i)\omega_i .$$

Finally, for each $x \in S$ we have

$$QH(I - P)x = \sum_{i=1}^{p} (H(I - P)x, \phi_i)\phi_i + \sum_{i=q+1}^{m} (H(I - P)x, L^*\omega_i)H\omega_i$$

$$= \sum_{i=q+1}^{m} (LH(I - P)x, \omega_i)H\omega_i$$

$$= \sum_{i=q+1}^{m} (x - Px, \omega_i)H\omega_i$$

where

$$(x - Px, \omega_i) = (x, \omega_i) - \sum_{j=1}^{m} (x, \omega_j)(\omega_j, \omega_i)$$

$$= 0 \quad \text{for } i = q+1, \ldots, m.$$

Thus,

$$QH(I - P)x = 0.$$

Q.E.D.

The properties listed in the last theorem are analogous to the properties satisfied by Cesari's operators in [3]. We shall use these properties to develop an existence theory for Equation (2.3):

$$Lx = Nx.$$

Suppose there exists an element $x \in \mathcal{D}(L) \cap \mathcal{D}(N)$ with $Lx = Nx$. Using part (a) of the last theorem, we have

$$H(I - P)Nx = H(I - P)Lx$$

$$= x - Qx.$$

Thus, there exists an element $x^* \in S_0$ such that

$$\begin{cases} Qx = x^* \\ x = x^* + H(I - P)Nx. \end{cases} \tag{2.6}$$

Let us try to reverse this argument. Take $x^* \in S_0$ and suppose there exists $x \in \mathcal{D}(N)$ such that

$$x = x^* + H(I - P)Nx. \tag{2.7}$$

Clearly $x \in \mathcal{D}(L)$. By part (d) of the last theorem we have $Qx = Qx^*$ or

$$Qx = x^*. \tag{2.8}$$

Thus,

$$x = Qx + H(I - P)Nx$$

and

$$Lx = LQx + LH(I - P)Nx.$$

Using parts (b) and (c) of the last theorem, we get

$$Lx = PLx + Nx - PNx$$

or

$$Lx - Nx = P(Lx-Nx). \tag{2.9}$$

Therefore, x is a solution of Equation (2.3) provided

$$P(Lx-Nx) = 0. \tag{2.10}$$

We have shown that if $x \in \mathcal{D}(N)$ is a solution of Equation (2.7) corresponding to $x^* \in S_0$ and if x is also a solution of Equation (2.10), then x is a solution of the original Equation (2.3). Equation (2.7) will be called the <u>auxiliary equation</u>.

In the next section we shall introduce sufficient conditions for the existence of a unique solution x to the auxiliary Equation (2.7) corresponding to each x^* belonging to a subset V of S_0. In the last section of this chapter we will give sufficient conditions that there exist $x^* \epsilon V$, such that the corresponding element x also satisfies Equation (2.10), and hence, yields a solution to the original Equation (2.3).

2.3 SOLVING THE AUXILIARY EQUATION

Let S' be a linear subspace in S and let μ be a seminorm defined in S'. In most applications μ is actually a norm on S'. We assume the following condition is satisfied:

(IIa) The linear subspace $\mathcal{D}(L)$ is a subset of S'.

In our applications S is the Hilbert space of square-integrable functions f(t) on a finite interval [a,b], L is a differential operator in S whose domain $\mathcal{D}(L)$ consists of functions which are at least continuous, and S' is the set of functions in S which are bounded almost everywhere; for this case condition (IIa) will certainly be satisfied. Note that

$$S_0 \subseteq \mathcal{D}(L) \subseteq S'.$$

We assume that the following condition is satisfied:

(IIb) There exist constants $\mathcal{k} \geq 0$ and $\mathcal{k}' \geq 0$ such that

$$\begin{cases} \|H(I - P)x\| \leq \mathcal{k}\|x\| \\ \mu(H(I - P)x) \leq \mathcal{k}'\|x\| \end{cases} \text{for all } x \epsilon S. \qquad (2.11)$$

Choose an element $x_0 \epsilon S_0$. Noting that $x_0 \epsilon \mathcal{D}(L) \cap \mathcal{D}(N)$, we let

$$\gamma = H(I - P)Nx_0. \tag{2.12}$$

Choose constants e and e' such that

$$\|\gamma\| \leq e, \ \mu(\gamma) \leq e'. \tag{2.13}$$

Let c, d, r, and R_0 be real numbers with $0 < c < d$ and $0 < r < R_0$. We define sets V and $\tilde{S}_0$ in S by

$$V = \{x \epsilon S_0 \mid \|x-x_0\| \leq c, \ \mu(x-x_0) \leq r\} \tag{2.14}$$

and

$$\tilde{S}_0 = \{x \epsilon S' \mid \|x-x_0\| \leq d, \ \mu(x-x_0) \leq R_0\}. \tag{2.15}$$

Clearly $x_0 \epsilon V \subseteq \tilde{S}_0$, so these two sets are nonempty. For each $x^* \epsilon V$ we define

$$S(x^*) = \{x \epsilon S' \mid Qx = x^*, \ \|x-x_0\| \leq d, \ \mu(x-x_0) \leq R_0\}. \tag{2.16}$$

Clearly $x^* \epsilon S(x^*)$, so each of the sets $S(x^*)$ is nonempty. Note that

$$x^* \ \epsilon \ S(x^*) \subseteq \tilde{S}_0 \quad \text{for all } x^* \epsilon V.$$

Assume the following condition is satisfied:

(IIc) The set $\tilde{S}_0$ is a subset of $\mathcal{D}(N)$, and there exists a constant $\ell \geq 0$ such that

$$\|Nx-Ny\| \leq \ell\|x-y\| \quad \text{for all } x,y \epsilon \tilde{S}_0. \tag{2.17}$$

This is the continuity condition on N which we mentioned earlier.

Definition 2.4 The operator $T : \mathcal{D}(N) \to S$ is defined by

$$Tx = Qx + H(I - P)Nx \quad \text{for all } x \epsilon \mathcal{D}(N).$$

Observe that $Tx \in \mathcal{D}(L)$ for each $x \in \mathcal{D}(N)$, and hence,

$$T \ : \ \mathcal{D}(N) \to S'.$$

For each $x^* \in V$ let $T(x^*)$ denote the restriction of T to $S(x^*)$:

$$T(x^*) \ = \ T|S(x^*). \qquad (2.18)$$

Then for each $x^* \in V$ we have $T(x^*) \ : \ S(x^*) \to S'$ with

$$T(x^*) \cdot x \ = \ Qx + H(I - P)Nx$$

or

$$T(x^*) \cdot x \ = \ x^* + H(I - P)Nx \quad \text{for all } x \in S(x^*). \qquad (2.19)$$

Theorem 2.5 If conditions (IIabc) are satisfied and if

$$k\ell < 1, \ c+e \leq (1-k\ell)d, \ r+e' \leq R_0-k'\ell d, \qquad (2.20)$$

then for each $x^* \in V$ the operator $T(x^*)$ is a contraction and maps $S(x^*)$ into $S(x^*)$.

Proof. Fix an element $x^* \in V$. Take $x \in S(x^*)$ and let $y = T(x^*) \cdot x$. We want to show that $y \in S(x^*)$. Now

$$y \ = \ x^* + H(I - P)Nx,$$

so $y \in S'$ and

$$\begin{aligned} Qy \ &= \ Qx^* + QH(I - P)Nx \\ &= \ Qx^* \\ &= \ x^*. \end{aligned}$$

Also, we have

$$\|y - x_0\| \ = \ \|x^* + H(I - P)Nx - x_0\|$$

$$= \|x^*-x_o + H(I - P)Nx - H(I - P)Nx_o + \gamma\|$$

$$\leq c + k\|Nx-Nx_o\| + e$$

$$\leq c + k\ell\|x-x_o\| + e$$

$$\leq c + k\ell d + e$$

$$\leq d$$

and

$$\mu(y-x_o) = \mu(x^*-x_o + H(I - P)Nx - H(I - P)Nx_o + \gamma)$$

$$\leq r + k'\|Nx-Nx_o\| + e'$$

$$\leq r + k'\ell d + e'$$

$$\leq R_o.$$

Thus, $y \epsilon S(x^*)$ and $T(x^*) : S(x^*) \to S(x^*)$. For $x_1, x_2 \in S(x^*)$ we have

$$\|T(x^*)\cdot x_1 - T(x^*)\cdot x_2\| = \|H(I - P)Nx_1 - H(I - P)Nx_2\|$$

$$\leq k\|Nx_1 - Nx_2\|$$

$$\leq k\ell\|x_1 - x_2\|.$$

Hence, $T(x^*)$ is a contraction.

Q.E.D.

We assume that one more condition is satisfied:

(IId) For each $x^* \epsilon V$ the set $S(x^*)$ is closed in S.

<u>Theorem 2.6</u> If conditions (IIabcd) are satisfied and if relations (2.20) are valid, then for each $x^* \epsilon V$ there exists a unique element $x \epsilon S(x^*)$ which is a solution to the auxiliary Equation (2.7) corresponding to x^*:

$$x = x^* + H(I - P)Nx.$$

Furthermore, $x \in \mathscr{D}(L) \cap \mathscr{D}(N)$, $Qx = x^*$, and

$$Lx - Nx = P(Lx - Nx).$$

Also, the solutions x vary continuously with the x^*:

$$\|x-y\| \leq (1-k\ell)^{-1}\|x^*-y^*\|. \tag{2.21}$$

Proof. The first part of the theorem follows from the Banach Fixed Point Theorem applied to the contraction $T(x^*)$ in the complete metric space $S(x^*)$. All the other parts of the theorem have been shown except the continuity (2.21). Take $x^* \epsilon V$ and $y^* \epsilon V$, and let $x \epsilon S(x^*)$ and $y \epsilon S(y^*)$ be the elements with

$$x = x^* + H(I - P)Nx$$

and

$$y = y^* + H(I - P)Ny.$$

Then

$$\begin{aligned} \|x-y\| &\leq \|x^*-y^*\| + \|H(I - P)(Nx-Ny)\| \\ &\leq \|x^*-y^*\| + k\ell\|x-y\| \end{aligned}$$

or

$$(1-k\ell)\|x-y\| \leq \|x^*-y^*\|.$$

Q.E.D.

This last theorem guarantees that the auxiliary Equation (2.7) can be solved for each $x^* \epsilon V$. In fact, it permits us to set up a correspondence between each $x^* \epsilon V$ and the solution $x \epsilon S(x^*)$ of the auxiliary equation.

Definition 2.5 Let conditions (IIabcd) be satisfied and let relations (2.20) be valid. The continuous operator $\mathfrak{X}$: $V \to \mathcal{D}(L) \cap \tilde{S}_0$ is defined by $\mathfrak{X}(x^*) = x$ for $x^* \epsilon V$ where x is the unique element in $S(x^*)$ which is a solution to the auxiliary Equation (2.7) corresponding to x^*.

Note that for each $x^* \epsilon V$ we have

$$\mathfrak{X}(x^*) \in \mathcal{D}(L)$$

and

$$\mathfrak{X}(x^*) \in S(x^*) \subseteq \tilde{S}_0 \subseteq \mathcal{D}(N).$$

Hence, the expression $P(L\mathfrak{X}x^* - N\mathfrak{X}x^*)$ defines an operator mapping V into the linear subspace $< \omega_1, \ldots, \omega_m >$. The next theorem is really a corollary of Theorem 2.6.

Theorem 2.7 Let conditions (IIabcd) be satisfied and let relations (2.20) be valid. If there exists an element $x^* \epsilon V$ such that

$$P(L\mathfrak{X}x^* - N\mathfrak{X}x^*) = 0, \tag{2.22}$$

then the element $x = \mathfrak{X}(x^*)$ is a solution of the original Equation (2.3), $Qx = x^*$, and

$$\|x - x_0\| \leq d, \; \mu(x - x_0) \leq R_0. \tag{2.23}$$

In Theorem 2.7 the problem of solving the original Equation (2.3) has been reduced to the problem of solving Equation (2.22). This is quite a simplification since Equation (2.22) is really a system of m equations in p+m-q unknowns. Equation (2.22) is called the bifurcation equation or the determining equation. We examine it in more detail in the next section.

2.4 SOLVING THE BIFURCATION EQUATION

In this section we introduce sufficient conditions for the existence of a solution $x^*\epsilon V$ to the bifurcation Equation (2.22). We begin by writing Equation (2.22) in a simpler form and showing that it can be defined by means of a continuous operator.

Definition 2.6 The operator ψ: $\mathcal{D}(L)\cap\mathcal{D}(N) \to \langle\omega_1,\ldots,\omega_m\rangle$ is defined by

$$\psi x \quad = \quad P(Lx-Nx) \quad \text{for all } x\epsilon\,\mathcal{D}(L)\cap\mathcal{D}(N).$$

Note that if conditions (IIac) are satisfied, then the sets V and $\mathcal{D}(L)\cap\tilde{S}_o$ are both subsets of $\mathcal{D}(L)\cap\mathcal{D}(N)$, and hence, we can form the operators $\psi|V$ and $\psi|\mathcal{D}(L)\cap\tilde{S}_o$.

Lemma 2.4 Let conditions (IIac) be satisfied. Then ψ is a continuous operator when restricted to the set $\mathcal{D}(L)\cap\tilde{S}_o$.

Proof. Take $x,y\epsilon\mathcal{D}(L)\cap\tilde{S}_o$. Then

$$\begin{aligned}\psi x-\psi y &= P(Lx-Ly) - P(Nx-Ny)\\ &= \sum_{i=1}^{m}(Lx-Ly,\omega_i)\omega_i - \sum_{i=1}^{m}(Nx-Ny,\omega_i)\omega_i,\end{aligned}$$

so

$$\|\psi x-\psi y\| \leq \sum_{i=1}^{m}|(x-y,L^*\omega_i)| + \sum_{i=1}^{m}|(Nx-Ny,\omega_i)|$$

$$\leq \left\{ \sum_{i=q+1}^{m} \|L^* \omega_i\| + m\ell \right\} \|x-y\|.$$

Q.E.D.

Throughout the remainder of this section we assume that conditions (IIabcd) are satisfied and relations (2.20) are valid. Thus, the continuous operators $\mathcal{Z}$, $\Psi|\mathcal{D}(L)\cap\tilde{S}_o$, and $\Psi|V$ exist with

$$V \xrightarrow{\mathcal{Z}} \mathcal{D}(L)\cap\tilde{S}_o \xrightarrow{\Psi} \langle\omega_1,\ldots,\omega_m\rangle$$

and

$$V \xrightarrow{\Psi} \langle\omega_1\ldots,\omega_m\rangle.$$

Note that

$$\Psi\mathcal{Z}x^* = P(L\mathcal{Z}x^* - N\mathcal{Z}x^*) \quad \text{for all } x^*\epsilon V. \tag{2.24}$$

Therefore, the continuous operator $\Psi\mathcal{Z}$ maps the "ball" V, which is a subset of the p+m-q dimensional Euclidean space S_o, into the m dimensional Euclidean space $\langle\omega_1,\ldots,\omega_m\rangle$. The bifurcation Equation (2.22) can be rewritten as

$$\Psi\mathcal{Z}x^* = 0.$$

According to Theorem 2.7 if 0 belongs to the range of the operator $\Psi\mathcal{Z}$, then there exists a solution x to the original Equation (2.3) with

$$\|x-x_o\| \leq d,\ \mu(x-x_o) \leq R_o.$$

In other words, if we can establish sufficient conditions for 0 to belong to the range of the operator $\Psi\mathcal{Z}$, then we will obtain an existence theory for the equation Lx = Nx.

The operator $\psi\mathcal{J}$ is a difficult operator to work with since $\mathcal{J}$ is defined using the Banach Fixed Point Theorem, i.e., $\mathcal{J}$ is defined by an iteration process. On the other hand, the continuous operator $\psi|V$ is an operator which is easily obtained. These two operators can be compared.

<u>Theorem 2.8</u> Let conditions (IIabcd) be satisfied and let relations (2.20) be valid. Then

$$\|\psi\mathcal{J}x^*-\psi x^*\| \leq (k\ell d+e)\ell \quad \text{for all } x^*\epsilon V. \qquad (2.25)$$

<u>Proof</u>. Take $x^*\epsilon V$ and let $x = \mathcal{J}x^*$. Then $x\epsilon S(x^*)$, $Qx = x^* = Qx^*$, and

$$\begin{aligned} PLx &= LQx = LQx^* \\ &= PLx^*, \end{aligned}$$

so

$$\begin{aligned} \psi\mathcal{J}x^* - \psi x^* &= P(Lx-Nx) - P(Lx^*-Nx^*) \\ &= P(Nx^*-Nx). \end{aligned}$$

By Bessel's Inequality we have

$$\|\psi\mathcal{J}x^* - \psi x^*\| \leq \|Nx-Nx^*\|.$$

Now $x^*\epsilon V \subseteq \tilde{S}_0$ and $x\epsilon\tilde{S}_0$ since $\mathcal{J}$ maps V into $\mathcal{D}(L)\cap\tilde{S}_0$, and hence,

$$\|\psi\mathcal{J}x^* - \psi x^*\| \leq \ell\|x-x^*\|.$$

But

$$\begin{aligned} x - x^* &= H(I-P)Nx \\ &= H(I-P)Nx - H(I-P)Nx_0 + \gamma, \end{aligned}$$

so

$$\| \psi \mathcal{T} x^* - \psi x^* \| \leq \ell \left[k\ell \| x - x_0 \| + e \right]$$

$$\leq (k\ell d + e)\ell .$$

Q.E.D.

Using this theorem we shall determine conditions on $\psi|V$ which will guarantee that 0 belongs to the range of the operator $\psi\mathcal{T}$.

Apply the Gram-Schmidt process to the linearly independent elements $H\omega_{q+1}, \ldots, H\omega_m$ to obtain orthonormal elements $\eta_{q+1}, \ldots, \eta_m$. Let

$$\eta_i = \sum_{j=q+1}^{m} a_{ij} H\omega_j \quad \text{for } i = q+1, \ldots, m. \tag{2.26}$$

Then each η_i is an element in $\mathcal{D}(L) \cap \mathcal{R}(L)^{\perp}$,

$$S_o = \langle \phi_1, \ldots, \phi_p, \eta_{q+1}, \ldots, \eta_m \rangle,$$

and each $x \in S_o$ can be written as

$$x = \sum_{i=1}^{p} b_i \phi_i + \sum_{i=q+1}^{m} c_i \eta_i \tag{2.27}$$

where $b_i = (x, \phi_i)$, $c_i = (x, \eta_i)$, and

$$\|x\|^2 = \sum_{i=1}^{p} b_i^2 + \sum_{i=q+1}^{m} c_i^2 . \tag{2.28}$$

Let $M = p+m-q$; let E^M be a copy of Euclidean M-space where we represent each point $\xi \in E^M$ as an M-tuple:

$$\xi = (b_1, \ldots, b_p, c_{q+1}, \ldots, c_m);$$

also, let E^m be a copy of Euclidean m-space where we represent each point $u \in E^m$ as an m-tuple:

$$u = (u_1, \ldots, u_m).$$

We define two operators $\Gamma_1 : E^M \to S_o$ and $\Gamma_2 : \langle \omega_1, \ldots, \omega_m \rangle \to E^m$ by

$$\Gamma_1(b_1, \ldots, b_p, c_{q+1}, \ldots, c_m) = \sum_{i=1}^{p} b_i \phi_i + \sum_{i=q+1}^{m} c_i \eta_i \tag{2.29}$$

and

$$\Gamma_2\left(\sum_{i=1}^{m} u_i \omega_i\right) = (u_1, \ldots, u_m) . \tag{2.30}$$

Clearly Γ_1 and Γ_2 are continuous linear operators with $\|\Gamma_1(\xi)\| = \|\xi\|$ for all $\xi \in E^M$ and $\|\Gamma_2(x)\| = \|x\|$ for all $x \in \langle \omega_1, \ldots, \omega_m \rangle$. Thus, these two maps are isomorphisms. Let

$$x_o = \sum_{i=1}^{p} b_{oi} \phi_i + \sum_{i=q+1}^{m} c_{oi} \eta_i \in S_o, \tag{2.31}$$

and then we define

$$\xi_o = (b_{o1}, \ldots, b_{op}, c_{oq+1}, \ldots, c_{om}) \in E^M. \tag{2.32}$$

Clearly $\Gamma_1(\xi_o) = x_o$.

<u>Lemma 2.5</u> Let conditions (IIabcd) be satisfied and let relations (2.20) be valid. Then there exists a number $\epsilon > 0$ such that the set

$$\{\xi \in E^M \mid \|\xi - \xi_o\| \leq \epsilon\}$$

is mapped by Γ_1 into the set V.

Proof. Choose a constant $M_1 > 0$ such that $\mu(\phi_i) \leq M_1$ for $i = 1,\ldots,p$ and $\mu(\eta_i) \leq M_1$ for $i = q+1,\ldots,m$. Let

$$\epsilon = \min \{ c, \frac{r}{MM_1} \}.$$

If $\xi = (b_1,\ldots,b_p,c_{q+1},\ldots,c_m) \in E^M$ with $\|\xi-\xi_o\| \leq \epsilon$, then we have $\Gamma_1(\xi) \in S_o$ with

$$\|\Gamma_1(\xi) - x_o\| = \|\Gamma_1(\xi-\xi_o)\| = \|\xi-\xi_o\| \leq c$$

and

$$\begin{aligned}\mu(\Gamma_1(\xi)-x_o) &= \mu\left(\sum_{i=1}^{p}(b_i-b_{o_i})\phi_i + \sum_{i=q+1}^{m}(c_i-c_{o_i})\eta_i\right) \\ &\leq \sum_{i=1}^{p} |b_i-b_{o_i}|M_1 + \sum_{i=q+1}^{m} |c_i-c_{o_i}|M_1 \\ &\leq MM_1\|\xi-\xi_o\| \\ &\leq r.\end{aligned}$$

Thus, $\Gamma_1(\xi)\epsilon V$.

Q.E.D.

Remark. In applications the numbers c and r are often related in such a way that $x\epsilon S_o$, $\|x\| \leq c$ implies $\mu(x) \leq r$. For this case we have

$$V = \{x\epsilon S_o \,|\, \|x-x_o\| \leq c\} ,$$

and in the last lemma we can take $\epsilon = c$.

Choose a number $\epsilon > 0$ such that the set

$$\mathcal{V} = \{\xi\epsilon E^M \,|\, \|\xi-\xi_o\| \leq \epsilon\} \tag{2.33}$$

is mapped by Γ_1 into the set V. We have

$$\mathcal{V} \xrightarrow{\Gamma_1} V \xrightarrow{\Psi} \langle \omega_1, \ldots, \omega_m \rangle \xrightarrow{\Gamma_2} E^m$$

and

$$\mathcal{V} \xrightarrow{\Gamma_1} V \xrightarrow{\mathcal{I}} \mathcal{D}(L) \cap \tilde{S}_0 \xrightarrow{\Psi} \langle \omega_1, \ldots, \omega_m \rangle \xrightarrow{\Gamma_2} E^m.$$

Definition 2.7 The operator ψ: $E^M \to E^m$ is defined by

$$\psi(\xi) = \Gamma_2 \Psi \Gamma_1(\xi) \quad \text{for all } \xi \in E^M.$$

The operator $\psi|\mathcal{V}$ is a continuous mapping from the ball $\mathcal{V}$, which is a subset of the Euclidean space E^M, into the Euclidean space E^m. If we write out ψ in coordinate form,

$$\psi = (\psi_1, \ldots, \psi_m),$$

then we have

$$\xi = (b_1, \ldots, b_p, c_{q+1}, \ldots, c_m),$$

$$\Gamma_1(\xi) = \sum_{j=1}^{p} b_j \phi_j + \sum_{j=q+1}^{m} c_j \eta_j,$$

$$\psi \Gamma_1(\xi) = PL\Gamma_1(\xi) - PN\Gamma_1(\xi)$$

$$= P\left(\sum_{j=q+1}^{m} c_j \, L \, \eta_j\right) - PN\Gamma_1(\xi)$$

$$= P\left(\sum_{j=q+1}^{m} c_j \sum_{i=q+1}^{m} a_{ij}\omega_i\right) - PN\Gamma_1(\xi)$$

$$= \sum_{i=q+1}^{m} \left(\sum_{j=q+1}^{m} a_{ij} c_j\right) \omega_i - \sum_{i=1}^{m} (N\Gamma_1(\xi), \omega_i)\omega_i,$$

or

$$\psi_i(b_1,\ldots,b_p,c_{q+1},\ldots,c_m) = \begin{cases} -(N[\sum_{j=1}^{p} b_j\phi_j + \sum_{j=q+1}^{m} c_j\eta_j],\omega_i) & \text{for } i=1,\ldots,q \\ \sum_{j=q+1}^{m} a_{ij}c_j - (N[\sum_{j=1}^{p} b_j\phi_j + \sum_{j=q+1}^{m} c_j\eta_j],\omega_i) & \text{for } i = q+1,\ldots,m. \end{cases} \tag{2.34}$$

The next theorem is an existence theorem for Equation (2.3).

<u>Theorem 2.9</u> Let $m = 1$, let conditions (IIabcd) be satisfied, and let relations (2.20) be valid. If there exists a number $\delta > 0$ such that the interval $[-\delta,\delta]$ is a subset of $\psi(\mathcal{V})$ and if

$$(k\ell d+e)\ell \leq \delta, \tag{2.35}$$

then there exists $x^*\epsilon V$ such that the element $x = \mathcal{I}(x^*)$ is a solution of the original Equation (2.3), $Qx = x^*$, and

$$\|x-x_o\| \leq d,\ \mu(x-x_o) \leq R_o.$$

<u>Proof</u>. Choose $\xi_1,\xi_2\epsilon\mathcal{V}$ such that $\psi(\xi_1) = \delta$ and $\psi(\xi_2) = -\delta$. Let

$$x_1^* = \Gamma_1(\xi_1)\ ,\ x_2^* = \Gamma_1(\xi_2).$$

Clearly x_1^* and x_2^* are elements of V, and by Theorem 2.8

$$\|\Psi\mathcal{I}x_1^* - \Psi x_1^*\| \leq \delta$$

and

$$\|\Psi\mathcal{I}x_2^* - \Psi x_2^*\| \leq \delta.$$

Thus,

$$|\Gamma_2 \Psi \mathcal{J} \Gamma_1(\xi_1) - \delta| = |\Gamma_2 \Psi \mathcal{J} \Gamma_1(\xi_1) - \Gamma_2 \Psi \Gamma_1(\xi_1)|$$

$$= \|\Psi \mathcal{J} x_1^* - \Psi x_1^*\| \leq \delta$$

or

$$\Gamma_2 \Psi \mathcal{J} \Gamma_1(\xi_1) \geq 0.$$

Similarly, we get

$$\Gamma_2 \Psi \mathcal{J} \Gamma_1(\xi_2) \leq 0.$$

Since $\Gamma_2 \Psi \mathcal{J} \Gamma_1(\mathcal{V})$ is connected, there exists $\xi \epsilon \mathcal{V}$ such that

$$\Gamma_2 \Psi \mathcal{J} \Gamma_1(\xi) = 0.$$

If we set $x^* = \Gamma_1(\xi)$, then $x^* \epsilon V$ and

$$P(L \mathcal{J} x^* - N \mathcal{J} x^*) = 0.$$

Applying Theorem 2.7, we obtain the desired result.

Q.E.D.

To conclude this chapter we give another existence theorem which relaxes the condition $m = 1$. Let conditions (IIabcd) be satisfied and let relations (2.20) be valid. In addition we assume that the following conditions are satisfied:

(IIIa) $p \geq q$,

(IIIb) $\psi(\xi_o) = 0$,

(IIIc) The first order partial derivatives of ψ exist and are continuous on $\mathcal{V}$,

(IIId) The Jacobian matrix for ψ has rank m at ξ_o.

The first condition is equivalent to the condition $M \geq m$, which means that ψ maps from a high dimensional space into a lower dimensional space. The second condition says that

$$P(Lx_0 - Nx_0) = 0,$$

which means that x_0 can be thought of as an approximate solution to the equation $Lx = Nx$. In applications this suggests how one should choose x_0. The third condition corresponds to putting certain differentiability conditions on the operator N. The last condition guarantees that the range of ψ covers a neighborhood of the origin in E^m.

Lemma 2.6 Let conditions (IIabcd) be satisfied, let conditions (IIIabcd) be satisfied, and let relations (2.20) be valid. Then there exists a number $\delta > 0$ such that the set

$$W = \{u \in E^m \mid \|u\| \leq \delta\}$$

is a subset of $\psi(\mathcal{V})$. Furthermore, there exists a continuous mapping Λ: $W \to \mathcal{V}$ such that

$$\psi[\Lambda(u)] = u \quad \text{for all } u \in W.$$

Proof. This follows immediately from the Inverse Function Theorem of advanced calculus, the proof of which suggests how one can determine the number δ.

Q.E.D.

Theorem 2.10 Let conditions (IIabcd) be satisfied, let conditions (IIIabcd) be satisfied, and let relations (2.20) be valid. If $\delta > 0$ is a number chosen as in Lemma 2.6 and if

$$(\not{k}\ell d+e)\ell < \delta,$$

then there exists $x^*\epsilon V$ such that the element $x = \mathcal{J} x^*$ is a solution of the original Equation (2.3), $Qx = x^*$, and

$$\|x-x_o\| \leq d, \quad \mu(x-x_o) \leq R_o.$$

Proof. Consider the two continuous maps I: $W \to E^m$ and $\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda$: $W \to E^m$ where $I(u) \equiv u$. Take $u\epsilon W$ and let $x^* = \Gamma_1\Lambda(u)$. Then $x^*\epsilon V$ and

$$\begin{aligned}\|\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda(u) - I(u)\| &= \|\Gamma_2\Psi\mathcal{J}(x^*) - \Gamma_2\Psi\Gamma_1\Lambda(u)\| \\ &= \|\Psi\mathcal{J}(x^*) - \Psi(x^*)\| \\ &\leq (\not{k}\ell d+e)\ell,\end{aligned}$$

or

$$\|\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda(u) - I(u)\| < \delta \quad \text{for all } u\epsilon W.$$

This inequality implies that for each $u\epsilon\partial W$, the line segment joining $I(u)$ and $\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda(u)$ does not contain the origin of E^m. By the Poincaré-Bohl Theorem [7,p.32] we conclude that the local degree of $\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda$ at 0 relative to W is equal to the local degree of I at 0 relative to W:

$$d(\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda,W,0) = d(I,W,0).$$

But $d(I,W,0) = 1$, and hence,

$$d(\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda,W,0) \neq 0.$$

Therefore, there exists an element $u\epsilon W$ such that

$$\Gamma_2\Psi\mathcal{J}\Gamma_1\Lambda(u) = 0.$$

Setting $x^* = \Gamma_1\Lambda(u)$ we have $x^*\epsilon V$ and

$$P(L \mathcal{T} x^* - N \mathcal{T} x^*) = 0.$$

The proof is completed using Theorem 2.7.

Q.E.D.

CHAPTER 3

DIFFERENTIAL OPERATORS

3.1 THE DEFINITION OF THE DIFFERENTIAL OPERATOR L

In the last chapter we developed a general theory for solving the functional equation $Lx = Nx$ in a real Hilbert space S. This theory was designed for applications to nonlinear differential equations. We must define the notion of a differential operator in a Hilbert space and show that these operators have the properties of the operator L in Chapter 2. This is the purpose of Chapter 3. Our main reference for this work is Dunford and Schwartz [9, ch.13].

Throughout this chapter we let I denote the finite interval $[a,b]$ of the real axis. Consider the real Hilbert space $S = L_2(I)$ consisting of all real-valued functions $f(t)$ which are square-integrable on I. We take the usual inner product and norm in S:

$$(f,g) = \int_a^b f(t)g(t)dt$$

and

$$\|f\| = (f,f)^{1/2}.$$

Definition 3.1 A formal differential operator of order n on the interval I is an expression

$$\tau = \sum_{i=0}^{n} a_i(t) \left(\frac{d}{dt}\right)^i$$

where the real-valued functions $a_i(t)$ belong to $C^\infty(I)$, and the function

$a_n(t)$ is not zero at any point of I.

The Hilbert space $S = L_2(I)$ contains many linear subspaces which could serve as a domain for a formal differential operator τ. Corresponding to each such subspace is a different linear operator in S. The next two definitions give two of these linear subspaces which will be very important in our work.

<u>Definition 3.2</u> By $H^n(I)$ we denote the linear subspace of S consisting of all functions f(t) with the following properties: f is continuous on I, the derivatives $f', f'', \ldots, f^{(n-1)}$ exist and are continuous on I, and $f^{(n-1)}$ is absolutely continuous on I with $f^{(n)} \in S$.

<u>Definition 3.3</u> Let $H_o^n(I)$ denote the set of all functions in $H^n(I)$ each of which vanishes outside some compact subset of the interior of I. (The compact subset may vary with the function).

Before examining the linear operators which we get by applying τ to the subspaces $H^n(I)$ and $H_o^n(I)$, let us give a well known existence and uniqueness theorem.

<u>Theorem 3.1</u> Let τ be a formal differential operator of order n on the interval I, let g be a function in S, let $t_o \in I$, and let $c_o, c_1, \ldots, c_{n-1}$ be an arbitrary set of n real numbers. Then there exists a unique function $f \in H^n(I)$ such that

(a) $\tau f = g$,

(b) $\left(\frac{d}{dt}\right)^i f(t_o) = c_i$ for $i = 0, 1, \ldots, n-1$.

Proof. The classical proofs can be altered to prove this theorem. See [9,p.1281].

Corollary 1 If g has k continuous derivatives in I, then f has n+k continuous derivatives in I and $\tau f(t) \equiv g(t)$ for all $t \in I$.

Corollary 2 The set of all functions $f \in H^n(I)$ with $\tau f = 0$ forms an n-dimensional linear subspace of S.

Now we define the linear operators mentioned above.

Definition 3.4 Let τ be a formal differential operator of order n on the interval I. The operators $T_0(\tau)$ and $T_1(\tau)$ are defined in S by

(a) $\mathcal{D}(T_0(\tau)) = H_0^n(I)$, $T_0(\tau)f = \tau f$ for all $f \in \mathcal{D}(T_0(\tau))$,

(b) $\mathcal{D}(T_1(\tau)) = H^n(I)$, $T_1(\tau)f = \tau f$ for all $f \in \mathcal{D}(T_1(\tau))$.

The linear operators $T_0(\tau)$ and $T_1(\tau)$ both have domains which are dense in S, and hence, both operators have adjoints. Clearly

$$T_0(\tau) \subseteq T_1(\tau), \tag{3.1}$$

and by the last corollary the null space for $T_1(\tau)$ has dimension n. We assert that $T_1(\tau)$ is a closed operator. To show this we must introduce the notion of the formal adjoint.

Definition 3.5 Let τ be a formal differential operator of order n on the interval I. The formal differential operator

$$\tau^* = \sum_{j=0}^{n} b_j(t) \left(\frac{d}{dt}\right)^j ,$$

where

$$b_j(t) = \sum_{k=j}^{n} (-1)^k \binom{k}{j} \left(\frac{d}{dt}\right)^{k-j} a_k(t),$$

is called the formal adjoint of τ.

Each of the functions $b_j(t)$ belongs to $C^\infty(I)$ and $b_n(t) = (-1)^n a_n(t)$, so τ^* is indeed a formal differential operator. It can be shown that $\tau = (\tau^*)^*$.

Theorem 3.2 If τ is a formal differential operator of order n on the interval I, then $T_1(\tau) = T_0(\tau^*)^*$.

Proof. See [9,p.1294].

The differential operators which we are going to study are obtained from the operator $T_1(\tau)$ by imposing a set of boundary conditions. The definition of a boundary value will be given in a modern abstract form, which permits us to study differential operators using the methods of functional analysis.

Definition 3.6 Let τ be a formal differential operator of order n on the interval I. Since $T_1(\tau)$ is a closed operator by the last theorem, it is easily seen that its domain $\mathscr{D}(T_1(\tau))$ becomes a Hilbert space under the inner product:

$$[f,g] = (f,g) + (T_1(\tau)f, T_1(\tau)g). \qquad (3.2)$$

A boundary value for τ is a continuous linear functional B on the Hilbert space $\mathscr{D}(T_1(\tau))$ which vanishes on $\mathscr{D}(T_0(\tau))$. In addition, if $B(f) = 0$

for each function f in $\mathcal{D}(T_1(\tau))$ which vanishes in a neighborhood of a, then B is called a <u>boundary value at a</u>. A <u>boundary value at b</u> is defined similarly. An equation $B(f) = 0$, where B is a boundary value for τ, is called a <u>boundary condition</u> for τ.

This concept of a boundary value can be related to the classical notion of a boundary value.

<u>Theorem 3.3</u> Let τ be a formal differential operator of order n on the interval I.

(a) The space of boundary values for τ is a linear space of dimension 2n.

(b) The functionals

$$A^{(i)}(f) = f^{(i)}(a) \quad , \quad i = 0,1,\ldots,n-1, \tag{3.3}$$

and

$$B^{(i)}(f) = f^{(i)}(b) \quad , \quad i = 0,1,\ldots,n-1, \tag{3.4}$$

are boundary values for τ at a and b, respectively. They form a base for the space of boundary values for τ.

(c) If B is a boundary value for τ, then there exist real numbers $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, $\beta_0, \beta_1, \ldots, \beta_{n-1}$ such that

$$B(f) = \sum_{i=0}^{n-1} \alpha_i A^{(i)}(f) + \sum_{i=0}^{n-1} \beta_i B^{(i)}(f) \tag{3.5}$$

for all $f \in \mathcal{D}(T_1(\tau))$. Conversely, each expression (3.5) defines a boundary value for τ.

<u>Proof</u>. See [9, p.1298 and p.1301].

With these preliminaries out of the way, we can now introduce the notion of a differential operator. Let τ be a formal differential operator of order n on the interval I and let $B_i(f) = 0$, $i = 1,\ldots,k$ be a set of k linearly independent boundary conditions for τ (the set may be vacuous).

<u>Definition 3.7</u> The linear operator L, which is the restriction of $T_1(\tau)$ to the linear subspace of $\mathcal{D}(T_1(\tau))$ determined by the boundary conditions $B_i(f) = 0$, $i = 1,\ldots,k$, is called a <u>differential operator</u> of order n on the interval I:

$$L = T_1(\tau) \,|\, \mathcal{D}(L) \tag{3.6}$$

where

$$\mathcal{D}(L) = \{f \in \mathcal{D}(T_1(\tau)) \,|\, B_i(f) = 0 \text{ for } i = 1,\ldots,k\}. \tag{3.7}$$

From this definition we observe that $\mathcal{D}(T_0(\tau)) = \overset{n}{H_0}(I)$ is a subset of $\mathcal{D}(L)$, and hence, $\mathcal{D}(L)$ is a dense linear subspace in S. Thus, the differential operator L has an adjoint. In the study of differential equations the adjoint operator always plays an important role. We shall show how to determine the adjoint operator L^* corresponding to a differential operator L in the Hilbert space $S = L_2(I)$.

3.2 THE ADJOINT OPERATOR L^*

Let τ be a formal differential operator of order n on the interval I and let L be the differential operator obtained from τ by the imposition of a set of k linearly independent boundary conditions $B_i(f) = 0$,

$i = 1,\ldots,k$. We shall determine the adjoint operator L^*, showing that L^* is a differential operator obtained from τ^* by the imposition of a set of $(2n-k)$ linearly independent boundary conditions. The main reference for this section is the paper of Schwartz [21].

For each pair of integers ℓ, j with $0 \leq \ell, j \leq n-1$ we define functions $F_t^{\ell j}(\tau)$ on I by

$$F_t^{\ell j}(\tau) = \begin{cases} \sum_{i=j}^{n-\ell-1} (-1)^i \binom{i}{j} \left(\frac{d}{dt}\right)^{i-j} a_{\ell+i+1}(t), & j+\ell \leq n-1 \\ 0, & j+\ell > n-1. \end{cases} \tag{3.8}$$

These functions appear in the fundamental formula relating τ and τ^*, the well known <u>Green's formula</u>.

<u>Lemma 3.1</u> (Green's formula) If $f(t)$ and $g(t)$ are functions in $H^n(I)$, then

$$(\tau f,g)-(f,\tau^* g) = \sum_{\ell,j=0}^{n-1} F_b^{\ell j}(\tau) f^{(\ell)}(b) g^{(j)}(b) - \sum_{\ell,j=0}^{n-1} F_a^{\ell j}(\tau) f^{(\ell)}(a) g^{(j)}(a). \tag{3.9}$$

<u>Proof</u>. See [9,pp.1285-1288].

Let F_1 and F_2 be the nxn square matrices given by

$$F_1 = [F_a^{\ell j}(\tau)] \quad \text{and} \quad F_2 = [F_b^{\ell j}(\tau)]. \tag{3.10}$$

These two matrices are known to be non-singular [9,p.1287]. Using Theorem 3.3 we write out the boundary values $B_1,\ldots,B_k$ which determine

L:

$$B_i = \sum_{j=0}^{n-1} \alpha_{ij} A^{(j)} + \sum_{j=0}^{n-1} \beta_{ij} B^{(j)} \quad , \quad i = 1,\ldots,k. \qquad (3.11)$$

Let B be the kx2n matrix given by

$$B = \begin{bmatrix} \alpha_{10} \ldots \alpha_{1\ n-1} & \beta_{10} \ldots \beta_{1\ n-1} \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \alpha_{k0} \ldots \alpha_{k\ n-1} & \beta_{k0} \ldots \beta_{k\ n-1} \end{bmatrix} . \qquad (3.12)$$

Because the boundary values $B_1,\ldots,B_k$ were assumed to be linearly independent, the matrix B has rank k. Consider the system of equations

$$\sum_{j=0}^{n-1} \alpha_{ij} x_j + \sum_{j=0}^{n-1} \beta_{ij} y_j = 0, \qquad i = 1,\ldots,k. \qquad (3.13)$$

This is a system of k-equations in the 2n-unknowns $x_0,\ldots,x_{n-1},y_0,\ldots,y_{n-1}$. Since B has rank k, the space of solutions to (3.13) has dimension 2n-k. Let X_i, i = 1,...,2n-k be a set of 2n-tuples of real numbers which form a base for the set of solutions of (3.13). We write

$$X_i = (x_0^i,\ldots,x_{n-1}^i,y_0^i,\ldots,y_{n-1}^i), \quad i = 1,\ldots,2n-k. \qquad (3.14)$$

Finally, let X be the (2n-k)x2n matrix whose rows are the X_i. Since the X_i are independent, the matrix X has rank 2n-k. We define a (2n-k) x2n matrix

$$B^* = \begin{bmatrix} \alpha^*_{10} \ldots \alpha^*_{1\ n-1} & \beta^*_{10} \ldots \beta^*_{1\ n-1} \\ \vdots \quad\quad \vdots & \vdots \quad\quad \vdots \\ \alpha^*_{2n-k0} \quad \alpha^*_{2n-kn-1} & \beta^*_{2n-k0} \quad \beta^*_{2n-kn-1} \end{bmatrix} \qquad (3.15)$$

by

$$\begin{cases} \alpha^*_{ij} = -\sum_{\ell=0}^{n-1} X^i_\ell \, F_a^{\ell j}(\tau) & \text{for } i = 1,\dots,2n-k \\ & \quad j = 0,1,\dots,n-1, \\ \beta^*_{ij} = \sum_{\ell=0}^{n-1} y^i_\ell \, F_b^{\ell j}(\tau) & \end{cases} \tag{3.16}$$

and let B^*_i, $i = 1,\dots,2n-k$ be the boundary values for τ^* defined by

$$B^*_i = \sum_{j=0}^{n-1} \alpha^*_{ij} A^{(j)} + \sum_{j=0}^{n-1} \beta^*_{ij} B^{(j)}, \quad i = 1,\dots,2n-k. \tag{3.17}$$

Since

$$B^* = X \begin{bmatrix} -F_1 & 0 \\ 0 & F_2 \end{bmatrix}, \tag{3.18}$$

we conclude that B^* has rank $2n-k$, and hence, the boundary values B^*_1, $\dots, B^*_{2n-k}$ are linearly independent.

<u>Theorem 3.4</u> The adjoint operator L^* is precisely the differential operator obtained from the formal adjoint τ^* by imposing the $2n-k$ linearly independent boundary conditions $B^*_i(f) = 0$, $i = 1,\dots,2n-k$.

<u>Proof</u>. Let $\tilde{L}$ be the differential operator obtained from τ^* by the imposition of the set of $(2n-k)$ linearly independent boundary conditions $B^*_i(f) = 0$, $i = 1,\dots,2n-k$. We shall show that $\tilde{L} = L^*$.

(a) Take $g \in \mathcal{D}(\tilde{L})$ and let $g^* = \tilde{L}g = \tau^* g$. Then for each $f \in \mathcal{D}(L)$ the 2n-tuple

$$X = (f(a), f'(a), \ldots, f^{(n-1)}(a), f(b), f'(b), \ldots, f^{(n-1)}(b))$$

is a solution of the system (3.13), and hence, there exist real numbers $c_1, \ldots, c_{2n-k}$ such that

$$X = \sum_{i=1}^{2n-k} c_i X_i ,$$

that is,

$$\begin{cases} f^{(\ell)}(a) = \sum_{i=1}^{2n-k} c_i X_\ell^i \\ f^{(\ell)}(b) = \sum_{i=1}^{2n-k} c_i y_\ell^i \end{cases} \quad \ell = 0, 1, \ldots, n-1.$$

Thus, using Green's formula we have

$$\begin{aligned}
(Lf,g)-(f,g^*) &= (\tau f,g)-(f,\tau^* g) \\
&= \sum_{\ell,j=0}^{n-1} \sum_{i=1}^{2n-k} F_b^{\ell j}(\tau) c_i y_\ell^i g^{(j)}(b) \\
&\quad - \sum_{\ell,j=0}^{n-1} \sum_{i=1}^{2n-k} F_a^{\ell j}(\tau)\, c_i x_\ell^i g^{(j)}(a) \\
&= \sum_{i=1}^{2n-k} c_i \sum_{j=0}^{n-1} \beta^*_{ij}\, B^{(j)}(g) \\
&\quad + \sum_{i=1}^{2n-k} c_i \sum_{j=0}^{n-1} \alpha^*_{ij}\, A^{(j)}(g) \\
&= \sum_{i=1}^{2n-k} c_i B_i^*(g)
\end{aligned}$$

$$= 0$$

or

$$(Lf,g) = (f,g^*) \quad \text{for all } f\in\mathcal{D}(L).$$

This implies that $g\in\mathcal{D}(L^*)$ and $L^*g = g^* = \tilde{L}g$. Therefore, $\tilde{L} \subseteq L^*$.

(b) To complete the proof it is sufficient to show that $\mathcal{D}(L^*) \subseteq \mathcal{D}(\tilde{L})$. Take any function $g\in\mathcal{D}(L^*)$. Now $T_0(\tau) \subseteq L$, so by Theorem 3.2 we have

$$L^* \subseteq T_0(\tau)^* = T_1(\tau^*),$$

and hence, $g\in\mathcal{D}(T_1(\tau^*)) = H^n(I)$. Choose functions $\sigma_i(t)$, $i = 1,\ldots,2n-k$ in $C^\infty(I)$ such that

$$\sigma_i^{(\ell)}(a) = x_\ell^i, \quad \sigma_i^{(\ell)}(b) = y_\ell^i \quad \text{for } \ell = 0,1,\ldots,n-1.$$

Clearly $\sigma_i(t)\in\mathcal{D}(L)$ for $i = 1,\ldots,2n-k$. Again using Green's formula we get

$$\begin{aligned} 0 &= (L\sigma_i,g)-(\sigma_i,L^*g) \\ &= (\tau\sigma_i,g)-(\sigma_i,\tau^*g) \\ &= \sum_{\ell,j=0}^{n-1} F_b^{\ell j}(\tau)y_\ell^i g^{(j)}(b) - \sum_{\ell,j=0}^{n-1} F_a^{\ell j}(\tau)x_\ell^i g^{(j)}(a) \\ &= \sum_{j=0}^{n-1} \beta_{ij}^* B^{(j)}(g) + \sum_{j=0}^{n-1} \alpha_{ij}^* A^{(j)}(g) \\ &= B_i^*(g) \quad \text{for } i = 1,\ldots,2n-k. \end{aligned}$$

Hence, $g \in \mathcal{D}(\tilde{L})$.

Q.E.D.

Let us consider several examples.

<u>Example 1</u> Let I = [0,1], let $\tau f = f'' + f$, and let L be the differential operator obtained from τ by imposing the two boundary conditions

$$\begin{cases} B_1(f) = f(0) = 0 \\ B_2(f) = f(1) + f'(1) = 0. \end{cases}$$

Then $n = 2$, $k = 2$, $a_2(t) \equiv 1$, $a_1(t) \equiv 0$, and $a_0(t) \equiv 1$. By Definition 3.5 the formal adjoint τ^* is given by

$$\tau^* f = b_2(t)f'' + b_1(t)f' + b_0(t)f$$

where

$$b_2(t) = (-1)^2 \binom{2}{2} a_2(t) = 1,$$

$$b_1(t) = (-1)^1 \binom{1}{1} a_1(t) + (-1)^2 \binom{2}{1} a_2'(t) = 0,$$

$$b_0(t) = (-1)^0 \binom{0}{0} a_0(t) = 1,$$

or

$$\tau^* f = f'' + f.$$

Thus, τ is formally self-adjoint. From (3.8) we have

$$F_t^{00}(\tau) = (-1)^0 \binom{0}{0} a_1(t) + (-1)^1 \binom{1}{0} a_2'(t) = 0,$$

$$F_t^{01}(\tau) = (-1)^1 \binom{1}{1} a_2(t) = -1,$$

$$F_t^{10}(\tau) = (-1)^0 \binom{0}{0} a_2(t) = 1,$$

$$F_t^{11}(\tau) = 0,$$

so

$$F_1 = F_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Now

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} ,$$

and from this we see that we can take

$$X = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} .$$

Therefore, by (3.18) we have

$$B^* = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

or

$$\begin{cases} B_1^*(f) = f(0) = 0 \\ B_2^*(f) = f(1) + f'(1) = 0. \end{cases}$$

Hence, L and L* are determined from $\tau = \tau^*$ by the same set of boundary conditions, that is, $L = L^*$. This self-adjoint differential operator was studied by Cesari in [3].

<u>Example 2</u> Let $I = [0,2\pi]$, let $\tau f = f'' + f$, and let L be the differential operator obtained from τ by imposing the single boundary condition

$$B_1(f) = f(0) = 0.$$

In this example $n = 2$ and $k = 1$. As in the last example the formal adjoint is given by

$$\tau^* f = f'' + f,$$

and also

$$F_1 = F_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Now

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} ,$$

so we can take

$$X = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Thus,

$$B^* = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} ,$$

or

$$\begin{cases} B_1^*(f) = f(0) = 0 \\ B_2^*(f) = f'(2\pi) = 0 \\ B_3^*(f) = f(2\pi) = 0. \end{cases}$$

The adjoint operator L^* is obtained from $\tau = \tau^*$ by imposing the three boundary conditions $B_i^*(f) = 0$, $i = 1,2,3$. We use the differential operator L of this example in Chapter 6.

The next theorem gives a characterization of the self-adjoint differential operators.

Theorem 3.5 Let τ be a formal differential operator of order n on the interval I and let L be the differential operator obtained from τ by the imposition of a set of k linearly independent boundary conditions $B_i(f) = 0$, $i = 1,\ldots,k$. The operator L is self-adjoint, $L = L^*$, iff the following conditions are satisfied:

(a) $k = n$;

(b) $\tau = \tau^*$, i.e., τ is formally self-adjoint;

(c) The matrices B and B^* are row equivalent.

Proof. Suppose L is self-adjoint. Consider the two systems of equations

$$(*) \qquad \sum_{j=0}^{n-1} \alpha_{ij} x_j + \sum_{j=0}^{n-1} \beta_{ij} y_j = 0, \qquad i = 1,\ldots,k$$

and

$$(**) \qquad \sum_{j=0}^{n-1} \alpha^*_{ij} x_j + \sum_{j=0}^{n-1} \beta^*_{ij} y_j = 0, \qquad i = 1,\ldots,2n-k;$$

the coefficients are just the elements in the matrices B and B^*.

(a) Let W_1 denote the solution space of (*) and let W_2 denote the solution space of (**). We know that

$$\dim W_1 = 2n-k \text{ and } \dim W_2 = 2n - (2n-k) = k.$$

Assert $W_1 = W_2$. Take a 2n-tuple $(x_0,\ldots,x_{n-1},y_0,\ldots,y_{n-1}) \epsilon W_1$. Then we can choose a function $f(t)$ in $C^\infty(I)$ such that

$$f^{(j)}(a) = x_j\ ,\ f^{(j)}(b) = y_j \qquad \text{for } j = 0,1,\ldots,n-1.$$

Thus,

$$B_i(f) = \sum_{j=0}^{n-1} \alpha_{ij} f^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{ij} f^{(j)}(b)$$

$$= 0 \quad \text{for } i = 1,\ldots,k,$$

or $f \in \mathcal{D}(L) = \mathcal{D}(L^*)$. Hence,

$$B_i^*(f) = 0 = \sum_{j=0}^{n-1} \alpha_{ij}^* x_j + \sum_{j=0}^{n-1} \beta_{ij}^* y_j, \quad i = 1,\ldots,2n\text{-}k,$$

or $(x_0,\ldots,x_{n-1},y_0,\ldots,y_{n-1}) \in W_2$ and $W_1 \subseteq W_2$. Similarly we can show that $W_2 \subseteq W_1$. We conclude that $W_1 = W_2$ and $2n\text{-}k = k$ or $k = n$.

(b) We have

$$\tau = \sum_{i=0}^{n-1} a_i(t) \left(\frac{d}{dt}\right)^i$$

and

$$\tau^* = \sum_{i=0}^{n-1} b_i(t) \left(\frac{d}{dt}\right)^i .$$

We must show that $a_i(t) \equiv b_i(t)$ for $i = 0,1,\ldots,n\text{-}1$. Fix the integer i with $0 \leq i \leq n$ and take any t_0 with $a < t_0 < b$. Choose a function $\phi(t)$ in $C^\infty(I)$ which is identically 1 in a neighborhood of t_0 and is identically zero in neighborhoods of a and b. Consider the function

$$f(t) = \frac{1}{i!} \phi(t) (t-t_0)^i \text{ on } I.$$

Clearly $f \in \mathcal{D}(L) = \mathcal{D}(L^*)$, so $Lf = L^*f$ or $\tau f = \tau^* f$. But in a neighborhood of t_0 we have

$$f(t) = \frac{1}{i!} (t-t_0)^i,$$

and hence,

$$\tau f(t_o) = a_i(t_o) = \tau^* f(t_o) = b_i(t_o).$$

Thus, $a_i(t) = b_i(t)$ for $a < t < b$, which shows that $a_i(t) \equiv b_i(t)$ on I. Since i was arbitrary, we have $\tau = \tau^*$.

(c) We showed that the matrices B and B* were row equivalent in part (a) of the proof.

The reverse implication is trivial to prove.

Q.E.D.

3.3 AN INTEGRAL REPRESENTATION FOR THE OPERATOR H

To conclude this chapter we show that each differential operator L in the real Hilbert space $S = L_2(I)$ is a closed linear operator and satisfies conditions (Iabc) of Chapter 2. This establishes the existence of the right inverse operator H, which can be represented as an integral operator in $S = L_2(I)$.

Let $\tau = \sum_{i=0}^{n} a_i(t)\left(\frac{d}{dt}\right)^i$ be a formal differential operator of order n on the interval I and let L be the differential operator in $S = L_2(I)$ obtained from τ by the imposition of a set of k linearly independent boundary conditions $B_i(f) = 0$, $i = 1,\ldots,k$. We assert that L is a closed operator.

<u>Theorem 3.6</u> The differential operator L is a <u>closed</u> linear operator in S.

Proof. Take a sequence of functions $f_i \in \mathcal{D}(L)$ with $f_i \to f$ and $Lf_i \to g$. We must show that $f \in \mathcal{D}(L)$ and $Lf = g$. Since $L \subseteq T_1(\tau)$, we have $f_i \in \mathcal{D}(T_1(\tau))$ and $Lf_i = T_1(\tau)f_i$. But $T_1(\tau)$ is a closed operator by Theorem 3.2, so $f \in \mathcal{D}(T_1(\tau))$ and $T_1(\tau)f = g$. Now in the Hilbert space $\mathcal{D}(T_1(\tau))$ we have

$$\begin{aligned}[f_i-f, f_i-f] &= \|f_i-f\|^2 + \|T_1(\tau)f_i - T_1(\tau)f\|^2 \\ &= \|f_i-f\|^2 + \|Lf_i-g\|^2,\end{aligned}$$

and hence, the functions f_i converge to the function f in the Hilbert space $\mathcal{D}(T_1(\tau))$. Since $B_j(f_i) = 0$, by the continuity of B_j on $\mathcal{D}(T_1(\tau))$ we get

$$B_j(f) = \lim_i B_j(f_i) = 0$$

for $j = 1,\ldots,k$. Thus, $f \in \mathcal{D}(L)$ and $Lf = T_1(\tau)f = g$.

Q.E.D.

We observed earlier that the domain $\mathcal{D}(L)$ is dense in S. Since

$$T_0(\tau) \subseteq L \subseteq T_1(\tau) \tag{3.19}$$

and

$$T_0(\tau^*) \subseteq L^* \subseteq T_1(\tau^*), \tag{3.20}$$

from Corollary 2 of Theorem 3.1 it follows that the null spaces $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$ are finite-dimensional linear subspaces in S of dimension $\leq n$. We assert that the range of L is closed in S.

Choose functions $\phi_1,\ldots,\phi_n$ in $\mathcal{D}(T_1(\tau))$ to form an orthonormal base for the null space $\mathcal{N}(T_1(\tau))$. With no loss of generality we can assume that the ϕ_i are chosen so that the functions $\phi_1,\ldots,\phi_p$ are in $\mathcal{D}(L)$ and form a base for the null space $\mathcal{N}(L)$. Clearly each ϕ_i belongs to $C^\infty(I)$. Consider the nxn matrix

$$\Phi(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) & \ldots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & & \phi_n'(t) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \phi_1^{(n-1)}(t) & \ldots & & \phi_n^{(n-1)}(t) \end{bmatrix} . \tag{3.21}$$

It is well known that this matrix is non-singular for each $t \in I$ [6,ch.3]. If we compute Φ^{-1} by forming the adjoint matrix of Φ, then we find that the element in the j-th row, n-th column of Φ^{-1} is just

$$\frac{W_j(t)}{\det\Phi(t)}$$

where $W_j(t)$ is the determinant of the matrix obtained from $\Phi(t)$ by replacing the j-th column by $(0,\ldots,0,1)$. Thus,

$$\sum_{j=1}^{n} \phi_j^{(i)}(t)\,\frac{W_j(t)}{\det\Phi(t)} = \begin{cases} 0 & \text{for } i = 0,1,\ldots,n-2 \\ 1 & \text{for } i = n-1. \end{cases} \tag{3.22}$$

We define a function $G(t,s)$ on the square IxI by

$$G(t,s) = \sum_{j=1}^{n} \frac{\phi_j(t)\,W_j(s)}{a_n(s)\det\Phi(s)}, \quad a \le t,\ s \le b. \tag{3.23}$$

Clearly $G(t,s)$ is continuous on IxI. This function is frequently used in the study of differential equations, e.g., see [6,pp.87-88]. We examine its properties in the next two lemmas.

Lemma 3.2 If $u(t)$ is an absolutely continuous function on I and if $u'(t) = g(t)$ a.e. where $g(t)$ is a continuous function on I, then $u \in C^1(I)$ and $u'(t) = g(t)$ for all $t \in I$.

Proof. We have

$$u(t) = u(a) + \int_a^t u'(s)ds = u(a) + \int_a^t g(s)ds.$$

The fundamental theorem of calculus tells us that $u(t)$ is differentiable on I and $u'(t) = g(t)$ for all $t \in I$.

Q.E.D.

Lemma 3.3 Let $f(t)$ be any function in $S = L_2(I)$ and let

$$u(t) = \int_a^t G(t,s)f(s)ds \qquad \text{for } t \in I.$$

Then the function $u(t)$ belongs to $\mathscr{D}(T_1(\tau))$ and

$$T_1(\tau)u = f.$$

Proof. By definition we have

$$u(t) = \sum_{j=1}^{n} \phi_j(t) \int_a^t \left[\frac{W_j(s)}{a_n(s)\ \det\Phi(s)}\right] f(s)ds$$

for $a \leq t \leq b$. This representation shows that $u(t)$ is continuous on I and, in fact, is absolutely continuous on I. Using (3.22) we get

$$u'(t) = \sum_{j=1}^{n} \phi_j'(t) \int_a^t \frac{W_j f}{a_n \det\Phi} ds + \frac{f(t)}{a_n(t)} \sum_{j=1}^{n} \phi_j(t) \frac{W_j(t)}{\det\Phi(t)}$$

$$= \sum_{j=1}^{n} \phi_j'(t) \int_a^t \left[\frac{W_j(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds \quad \text{a.e..}$$

By Lemma 3.2 we conclude that $u'(t)$ exists everywhere on I and is a continuous function on I. By repeated use of (3.22) and Lemma 3.2 we conclude that the derivatives $u', u'', \ldots, u^{(n-1)}$ exist and are continuous on I with

$$u^{(i)}(t) = \sum_{j=1}^{n} \phi_j^{(i)}(t) \int_a^t \left[\frac{W_j(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds, \quad a \leq t \leq b,$$

for $i = 0,1,\ldots,n-1$. Thus, $u^{(n-1)}$ is absolutely continuous on I, and again by (3.22) we have

$$u^{(n)}(t) = \sum_{j=1}^{n} \phi_j^{(n)}(t) \int_a^t \frac{W_j f}{a_n \det\Phi} ds + \frac{f(t)}{a_n(t)} \sum_{j=1}^{n} \phi_j^{(n-1)}(t) \frac{W_j(t)}{\det\Phi(t)}$$

$$= \sum_{j=1}^{n} \phi_j^{(n)}(t) \int_a^t \left[\frac{W_j(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds + \frac{f(t)}{a_n(t)} \text{ a.e.}.$$

Hence, $u^{(n)} \epsilon S$ and we have shown that $u(t)$ belongs to $\mathcal{D}(T_1(\tau))$.

Finally we note that

$$T_1(\tau)u = a_n(t)u^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t)u^{(i)}(t)$$

$$= f(t) + \sum_{i=0}^{n} a_i(t) \sum_{j=1}^{n} \phi_j^{(i)}(t) \int_a^t \left[\frac{W_j(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds$$

$$= f(t) + \sum_{j=1}^{n} T_1(\tau)\phi_j(t) \cdot \int_a^t \left[\frac{W_j(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds$$

$$= f.$$

Q.E.D.

<u>Theorem 3.7</u> The range of the differential operator L is closed in S.

<u>Proof</u>. Let $f_i(t)$ be a sequence of functions in $\mathcal{R}(L)$, let $f(t)\epsilon S$, and let $f_i \to f$. We must show that $f\epsilon\mathcal{R}(L)$. Let

$$u_i(t) = \int_a^t G(t,s)f_i(s)ds \quad \text{for } t\epsilon I$$

and

$$u(t) = \int_a^t G(t,s)f(s)ds \quad \text{for } t\epsilon I.$$

By Lemma 3.3 the functions $u_i(t)$, $i = 1,2,\ldots$ and the function $u(t)$ belong to $\mathcal{D}(T_1(\tau))$ and $T_1(\tau)u_i = f_i$, $T_1(\tau)u = f$. Assert $u_i \to u$. Choose $M_1 > 0$ such that $|G(t,s)| \leq M_1$ on th square IxI. Then

$$|u_i(t)-u(t)| \leq \int_a^t |G(t,s)||f_i(s)-f(s)|ds$$

$$\leq M_1 \int_a^b 1\cdot|f_i(s)-f(s)|ds$$

$$\leq M_1 (b-a)^{1/2} \|f_i-f\|,$$

and hence,

$$\|u_i-u\|^2 \leq M_1^2(b-a)^2\|f_i-f\|^2.$$

Thus, $u_i\epsilon\mathcal{D}(T_1(\tau))$, $u\epsilon\mathcal{D}(T_1(\tau))$, $u_i \to u$, and $T_1(\tau)u_i \to T_1(\tau)u$. This means that the u_i converge to u in the Hilbert space $\mathcal{D}(T_1(\tau))$. By the continuity of B_j on $\mathcal{D}(T_1(\tau))$ we have

$$\lim_{i\to\infty} B_j(u_i) = B_j(u) \quad \text{for } j = 1,\ldots,k.$$

Now choose functions $v_i(t)$ in $\mathcal{D}(L)$ such that $Lv_i = f_i$. Then

$v_i - u_i \in \mathcal{D}(T_1(\tau))$ and $T_1(\tau)(v_i - u_i) = 0$. Hence, there exist real numbers $c_1^{(i)}, \ldots, c_n^{(i)}$ such that

$$v_i(t) = \sum_{\ell=1}^{n} c_\ell^{(i)} \phi_\ell(t) + u_i(t), \; i = 1,2,\ldots \; .$$

But $B_j(v_i) = 0$ for $j = 1,\ldots,k$, so

$$\sum_{\ell=1}^{n} c_\ell^{(i)} B_j(\phi_\ell) = - B_j(u_i), \; j = 1,\ldots,k$$

or

$$(*) \qquad \lim_{i\to\infty} \sum_{\ell=1}^{n} c_\ell^{(i)} B_j(\phi_\ell) = - B_j(u), \; j = 1,\ldots,k.$$

If we let T be the linear transformation from Euclidean n-space into Euclidean k-space induced by the matrix with coefficients $B_j(\phi_\ell)$, then (*) says that the k-tuple $(-B_1(u),\ldots,-B_k(u))$ is a limit point of the range of T. But the range of T is certainly a closed linear subspace in Euclidean k-space, and hence, this k-tuple belongs to the range of T. Thus, there exist real numbers $c_1,\ldots,c_n$ such that

$$(**) \qquad \sum_{\ell=1}^{n} c_\ell B_j(\phi_\ell) = - B_j(u), \; j = 1,\ldots,k.$$

Let

$$v(t) = \sum_{\ell=1}^{n} c_\ell \phi_\ell(t) + u(t).$$

From (**) it follows that $v \in \mathcal{D}(L)$, and also,

$$Lv = T_1(\tau)v = T_1(\tau)u = f.$$

Q.E.D.

We have shown that the differential operator L is a closed linear operator in the Hilbert space $S = L_2(I)$ and satisfies conditions (Iabc) of Chapter 2. Therefore, the linear operator

$$H = [L|\mathcal{D}(L)\cap\mathcal{N}(L)^{\perp}]^{-1}$$

exists and is a continuous right inverse for L. The basic properties of H are given in Theorem 2.1. We show next that H can be represented as an integral operator.

<u>Theorem 3.8</u> Let $\hat{B}$ be the kx(n-p) matrix

$$\hat{B} = \begin{bmatrix} B_1(\phi_{p+1}) & B_1(\phi_{p+2}) & \cdots & B_1(\phi_n) \\ B_2(\phi_{p+1}) & B_2(\phi_{p+2}) & \cdots & \cdot \\ \vdots & \vdots & & \vdots \\ B_k(\phi_{p+1}) & \cdots & \cdots & B_k(\phi_n) \end{bmatrix}$$

and let T be the linear transformation from Euclidean (n-p) - space into Euclidean k-space induced by T:

$$T(c_{p+1},\ldots,c_n) = \left(\sum_{i=p+1}^{n} B_1(\phi_i)c_i,\ \ldots,\sum_{i=p+1}^{n} B_k(\phi_i)c_i\right).$$

Then T is 1-1.

<u>Proof</u>. Suppose $T(c_{p+1},\ldots,c_n) = 0$. Let

$$f(t) = \sum_{i=p+1}^{n} c_i\phi_i(t) \qquad \text{for } t\in I.$$

Since the ϕ_i are orthonormal, we must have $f\epsilon\mathcal{N}(L)^{\perp}$. Also, we have $f\epsilon\mathcal{D}(T_1(\tau))$, $T_1(\tau)f = 0$, and

$$B_j(f) = \sum_{i=p+1}^{n} c_i B_j(\phi_i) = 0 \quad \text{for } j = 1,\dots,k.$$

Thus, $f\epsilon\mathcal{D}(L)$ and $Lf = 0$, or $f\epsilon\mathcal{N}(L)\cap\mathcal{N}(L)^{\perp}$. This implies that f = C and $c_i = 0$ for $i = p+1,\dots,n$.

Q.E.D.

This theorem has three corollaries which are very useful in establishing the integral representation for H.

<u>Corollary 1</u> $n-p \leq k$ where n is the order of the formal differential operator τ, p is the dimension of the null space $\mathcal{N}(L)$, and k is the number of linearly independent boundary conditions defining the differential operator L.

<u>Corollary 2</u> The matrix $\hat{B}$ has rank n-p.

<u>Corollary 3</u> There exists a (n-p)xk matrix

$$A = \begin{bmatrix} A_{p+1\,1} & A_{p+1\,2} & \cdots & A_{p+1\,k} \\ A_{p+2\,1} & A_{p+2\,2} & \cdots & \vdots \\ \vdots & & & \vdots \\ A_{n1} & \cdots & \cdots & A_{nk} \end{bmatrix}$$

such that $A\hat{B} = I_{n-p}$, i.e.,

$$\sum_{\ell=1}^{k} A_{i\ell}B_{\ell}(\phi_j) = \delta_{ij} \quad \text{for } i,j = p+1,\ldots,n. \tag{3.24}$$

Proof. Since $\hat{B}$ has rank (n-p), there exists a finite sequence of kxk elementary matrices $E_1,\ldots,E_\rho$ which transform $\hat{B}$ to row-echelon form:

$$E_\rho \ldots E_1 B = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \\ & & \\ 0 & & \end{bmatrix}}_{n-p} k \quad .$$

Let A be the (n-p)xk matrix obtained from $E_\rho \ldots E_1$ by retaining only the first (n-p) rows.

Q.E.D.

Remark Given the matrix $\hat{B}$ we can easily compute the matrix A by using elementary row operations as in the proof of Corollary 3.

We define functions $\psi_i(s)$, i = 1,...,n on the interval I by

$$\psi_i(s) = \begin{cases} -\displaystyle\int_s^b \phi_i(t)G(t,s)dt & i = 1,\ldots,p \\ -\displaystyle\sum_{\ell=1}^{k}\sum_{j=0}^{n-1}\sum_{\rho=1}^{n} A_{i\ell}\beta_{\ell j}\phi_\rho^{(j)}(b)\,\frac{W_\rho(s)}{a_n(s)\det\Phi(s)} & i = p+1,\ldots,n. \end{cases} \tag{3.25}$$

The numbers $\beta_{\ell j}$ are just the coefficients of the boundary values $B_1,\ldots,B_k$ as given in (3.11). Clearly each function $\psi_i(s)$ is continuous on I.

Theorem 3.9 The right inverse operator H for the differential operator L has an integral representation given by

$$Hf(t) = \sum_{j=1}^{n} \phi_j(t) \cdot \int_a^b \psi_j(s) f(s) ds + \int_a^t G(t,s) f(s) ds, \quad t \in I, \tag{3.26}$$

for all functions f(t) in the range of L.

Proof. Take any function f(t) in the range of L and let g = Hf. Let

$$u(t) = \int_a^t G(t,s) f(s) ds \quad \text{for } t \in I.$$

Both g and u belong to $\mathcal{D}(T_1(\tau))$ and $T_1(\tau)(g-u) = 0$. Thus, there exist real numbers $c_1, \ldots, c_n$ such that

$$(*) \qquad g(t) = \sum_{j=1}^{n} c_j \phi_j(t) + u(t).$$

Now $g \in \mathcal{N}(L)^{\perp}$, and hence, $(g, \phi_i) = 0$ for $i = 1, \ldots, p$ or

$$c_i = -\int_a^b u(t) \phi_i(t) dt$$

$$= -\int_a^b \left\{ \int_a^t G(t,s) f(s) ds \right\} \phi_i(t) dt \quad \text{for } i = 1, \ldots, p.$$

This integral can be rewritten using Fubini's Theorem as

$$c_i = -\int_a^b \left\{ \int_s^b \phi_i(s)\ G(t,s) dt \right\} f(s) ds$$

or

$$c_i = \int_a^b \psi_i(s) f(s) ds \qquad \text{for } i = 1, \ldots, p.$$

Also, we have $g \in \mathcal{D}(L)$, so $B_\ell(g) = 0$ for $\ell = 1, \ldots, k$ or

$$\sum_{j=p+1}^{n} c_j B_\ell(\phi_j) = - B_\ell(u) \quad \text{for } \ell = 1,\ldots,k.$$

Using (3.24) we get

$$\sum_{\ell=1}^{k} \sum_{j=p+1}^{n} c_j A_{i\ell} B_\ell(\phi_j) = -\sum_{\ell=1}^{k} A_{i\ell} B_\ell(u) \quad \text{for } i = p+1,\ldots,n$$

or

$$c_i = -\sum_{\ell=1}^{k} A_{i\ell} B_\ell(u) \quad \text{for } i = p+1,\ldots,n.$$

We shall write out the expression $B_\ell(u)$ in detail. In the proof of Lemma 3.3 we showed that

$$u^{(j)}(t) = \sum_{\rho=1}^{n} \phi_\rho^{(j)}(t) \int_a^t \left[\frac{W_\rho(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds, \quad j = 0,\ldots,n-1,$$

and hence,

$$B_\ell(u) = \sum_{j=0}^{n-1} \alpha_{\ell j} u^{(j)}(a) + \sum_{j=0}^{n-1} \beta_{\ell j} u^{(j)}(b)$$

$$= \sum_{j=0}^{n-1} \beta_{\ell j} \cdot \sum_{\rho=1}^{n} \phi_\rho^{(j)}(b) \int_b^a \left[\frac{W_\rho(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds$$

for $\ell = 1,\ldots,k$. Thus,

$$c_i = - \sum_{\ell=1}^{k} \sum_{j=0}^{n-1} \sum_{\rho=1}^{n} A_{i\ell} \beta_{\ell j} \phi_\rho^{(j)}(b) \int_a^b \left[\frac{W_\rho(s)}{a_n(s)\det\Phi(s)}\right] f(s)ds$$

or

$$c_i = \int_a^b \Psi_i(s) f(s)ds \quad \text{for} \quad i = p+1,\ldots,n.$$

Substituting these expressions for the c_i into (*), we get the desired result.

Q.E.D.

We define a function $K(t,s)$ on the square IxI by

$$K(t,s) = \sum_{j=1}^{n} \phi_j(t)\psi_j(s) + G^*(t,s), \quad a \leq t,s \leq b, \tag{3.27}$$

where

$$G^*(t,s) = \begin{cases} G(t,s) & \text{for } a \leq s \leq t \leq b \\ 0 & \text{for } a \leq t < s \leq b. \end{cases} \tag{3.28}$$

Clearly the function $K(t,s)$ is square-integrable on the square IxI.

Corollary The operator H is a completely continuous operator from $\mathcal{R}(L)$ into $\mathcal{N}(L)^{\perp}$, and has an integral representation given by

$$Hf(t) = \int_a^b K(t,s)f(s)ds, \quad t\epsilon I, \tag{3.29}$$

for all functions $f(t)$ in the range of L.

Proof. The integral representation follows from the theorem. For the complete continuity we must show that if $\{f_i\}$ is a bounded sequence in $\mathcal{R}(L)$, then the sequence $\{Hf_i\}$ contains a subsequence converging to some limit in $\mathcal{N}(L)^{\perp}$. Let $\{f_i\}$ be a bounded sequence in $\mathcal{R}(L)$. The linear operator H_1: $S \rightarrow S$ defined by

$$H_1f(t) = \int_a^b K(t,s)f(s)ds \quad \text{for all } f\epsilon S$$

is known to be completely continuous in S. Hence, there exists a subsequence $\{H_1 f_{i_j}\}$ which converges to a function $g \in S$. But $H = H_1 | \mathcal{R}(L)$, so

$$Hf_{i_j} = H_1 f_{i_j} \in \mathcal{N}(L)^{\perp}$$

and

$$Hf_{i_j} \longrightarrow g \quad \text{as} \quad j \longrightarrow \infty.$$

Since $\mathcal{N}(L)^{\perp}$ is closed in S, we conclude that $g \in \mathcal{N}(L)^{\perp}$. This completes the proof.

Q.E.D.

By means of this theorem and its corollary we can determine two important bounds on the operator H. As in Chapter 2 we let $q = \dim \mathcal{N}(L^*)$ and then choose an integer $m \geq q$ and functions $\omega_1(t),\ldots,\omega_m(t)$ in $\mathcal{D}(L^*)$ such that the ω_i form an orthonormal set in S and $\omega_1(t),\ldots,\omega_q(t)$ form a base for the null space $\mathcal{N}(L^*)$. Let P be the projection operator given in Definition 2.3, i.e.,

$$Pf = \sum_{i=1}^{m} (f,\omega_i)\omega_i \quad \text{for all } f \in S.$$

Fix $t \in I$ and consider $K(t,s)$ as a function of s on the interval I. Clearly this function is square-integrable on I, and

$$\int_a^b K(t,\xi)\omega_i(\xi)d\xi = \sum_{j=1}^{n} \phi_j(t)\int_a^b \psi_j(\xi)\omega_i(\xi)d\xi + \int_a^t G(t,\xi)\omega_i(\xi)d\xi$$

for $i = 1,\ldots,m$ while

$$\int_a^b [K(t,\xi)]^2 d\xi = \sum_{i,j=1}^{n} \phi_i(t)\phi_j(t)\int_a^b \psi_i(\xi)\psi_j(\xi)d\xi + 2\sum_{j=1}^{n}\phi_j(t)\int_a^t G(t,\xi)\psi_j(\xi)d\xi$$

$$+ \int_a^t [G(t,\xi)]^2 d\xi.$$

Thus, these two integrals give continuous functions of t on the interval I. We define a function $K_m(t,s)$ on the square IxI by

$$K_m(t,s) = K(t,s) - \sum_{i=1}^{m}\left(\int_a^b K(t,\xi)\omega_i(\xi)d\xi\right)\omega_i(s), \quad a\le t,s\le b. \tag{3.30}$$

Clearly this function is square-integrable on the square IxI. If we fix $t\in I$ and consider $K_m(t,s)$ as a function of s on the interval I, then this function is square-integrable on I with

$$\int_a^b [K_m(t,\xi)]^2 d\xi = \int_a^b [K(t,\xi)]^2 d\xi - \sum_{i=1}^{m}\left(\int_a^b K(t,\xi)\omega_i(\xi)d\xi\right)^2 . \tag{3.31}$$

Thus, this last integral defines a continuous function of t on the interval I. Choose constants k and k' such that

$$\left(\int_a^b\int_a^b [K_m(t,s)]^2 ds dt\right)^{1/2} \le k \tag{3.32}$$

and

$$\left(\max_{t\in I}\int_a^b [K_m(t,\xi)]^2 d\xi\right)^{1/2} \le k'. \tag{3.33}$$

<u>Theorem 3.10</u> The linear operator H(I - P) has an integral representation given by

$$H(I - P)f(t) = \int_a^b K_m(t,s)f(s)ds, \quad t\epsilon I, \qquad (3.34)$$

for all functions f(t) in S. Also, for each f in S:

$$\|H(I - P)f\| \leq \mathscr{k}\|f\| \qquad (3.35)$$

and

$$|H(I - P)f(t)| \leq \mathscr{k}'\|f\| \quad \text{for all } t\epsilon I. \qquad (3.36)$$

Proof. Take any function f(t) in S. Then

$$H(I - P)f(t) = \int_a^b K(t,s)\left[f(s) - \sum_{i=1}^m\left(\int_a^b f(\xi)\omega_i(\xi)d\xi\right)\omega_i(s)\right] ds$$

$$= \int_a^b K(t,s)f(s)ds - \sum_{i=1}^m\left(\int_a^b K(t,s)\omega_i(s)ds\right)\left(\int_a^b f(\xi)\omega_i(\xi)d\xi\right)$$

$$= \int_a^b K(t,s)f(s)ds - \sum_{i=1}^m \int_a^b\left[\left(\int_a^b K(t,\xi)\omega_i(\xi)d\xi\right)\omega_i(s)\right] f(s)ds$$

$$= \int_a^b\left[K(t,s) - \sum_{i=1}^m \left(\int_a^b K(t,\xi)\omega_i(\xi)d\xi\right)\omega_i(s)\right] f(s)ds$$

$$= \int_a^b K_m(t,s)f(s)ds \quad , \quad t\epsilon I,$$

and by the Schwartz inequality we have

$$|H(I - P)f(t)| \leq \left(\int_a^b [K_m(t,s)]^2 ds\right)^{1/2}\left(\int_a^b [f(s)]^2 ds\right)^{1/2}$$

$$\leq \mathscr{k}'\|f\| \quad \text{for all } t\epsilon I.$$

Using this same argument we get

$$\|H(I - P)f\|^2 = \int_a^b |H(I - P)f(t)|^2 \, dt$$

$$\leq \|f\|^2 \int_a^b \int_a^b [K_m(t,s)]^2 \, ds \, dt$$

$$\leq \mathcal{L}^2 \|f\|^2.$$

Q.E.D.

CHAPTER 4

THE NONLINEAR DIFFERENTIAL EQUATION Lx = Nx

4.1 THE EXISTENCE THEOREMS

Let I be the finite interval [a,b] of the real axis and let S = $L_2(I)$ be the real Hilbert space consisting of all real-valued functions f(t) which are square-integrable on I with the usual inner product (f,g) and norm $\|f\|$. Let S' be the linear subspace in S consisting of all functions which are bounded a.e. and let μ be the uniform norm in S', i.e., $\mu(f) = \inf\{c \mid |f| \leq c \text{ a.e.}\}$ for $f \in S'$. The number $\mu(f)$ is frequently called the essential supremum of $|f|$.

Let $\tau = \sum_{i=0}^{n} a_i(t)\left(\frac{d}{dt}\right)^i$ be a formal differential operator of order n on I and let L be the differential operator in S obtained from τ by the imposition of a set of k linearly independent boundary conditions $B_i(f) = 0$, $i = 1,\ldots,k$. Let N be an operator in S with $\mathcal{D}(L)\cap\mathcal{D}(N) \neq \phi$.

We are going to combine the results of Chapters 2 and 3 to obtain existence theorems for the nonlinear differential equation

$$Lx = Nx. \tag{4.1}$$

We proceed by carrying out the following steps:

1. Determine the adjoint operator L* using Chapter 3, Section 2.

2. Let p = dim $\mathcal{N}(L)$ and choose functions $\phi_1(t),\ldots,\phi_n(t)$ in $\mathcal{D}(T_1(\tau)) = H^n(I)$ to form an orthonormal base for the null space

$\mathcal{N}(T_1(\tau))$ in such a way that the functions $\phi_1(t),\ldots,\phi_p(t)$ are in $\mathcal{D}(L)$ and form a base for the null space $\mathcal{N}(L)$.

3. Let $q = \dim \mathcal{N}(L^*)$ and choose functions $\omega_1(t),\ldots,\omega_q(t)$ in $\mathcal{D}(L^*)$ to form an orthonormal base for the null space $\mathcal{N}(L^*)$.

4. Determine the function $G(t,s)$ using (3.23); determine the matrix A using Corollary 3 of Theorem 3.8; determine the functions $\psi_1(t),\ldots,\psi_n(t)$ using (3.25); determine the function $K(t,s)$ using (3.27); and finally, determine the integral representation for the operator H using Theorem 3.9 and its Corollary.

5. Choose an integer $m \geq q$ and choose functions $\omega_{q+1}(t),\ldots,\omega_m(t)$ in $\mathcal{D}(L^*)$ such that the functions $\omega_1(t),\ldots,\omega_m(t)$ form an orthonormal set in S.

6. Compute the functions $L^*\omega_{q+1},\ldots,L^*\omega_m$ and the functions $H\omega_{q+1},\ldots,H\omega_m$.

7. Determine the projection operators P and Q using Definition 2.3.

8. Determine the function $K_m(t,s)$ using (3.30).

9. Choose constants k and k' by means of (3.32) and (3.33), respectively.

10. Let S_o be the linear subspace $\langle\phi_1,\ldots,\phi_p,H\omega_{q+1},\ldots,H\omega_m\rangle$ and check that S_o is a subset of $\mathcal{D}(N)$.

11. Apply the Gram-Schmidt process to the functions $H\omega_{q+1},\ldots,H\omega_m$ to obtain orthornormal functions $\eta_{q+1},\ldots,\eta_m$ with

$$\eta_i = \sum_{j=q+1}^{m} a_{ji} H\omega_j \quad \text{for } i = q+1,\ldots,m.$$

12. Let $M = p+m-q$ and determine the linear operators Γ_1: $E^M \to S_0$ and Γ_2: $\langle\omega_1,\ldots,\omega_m\rangle \to E^m$ which are given by (2.29) and (2.30), respectively.

13. Let Ψ and ψ be the operators given in Definition 2.6 and Definition 2.7, respectively. Determine ψ explicitly using (2.34).

14. Choose a function $x_0 \in S_0$ and let

$$x_0 = \sum_{i=1}^{p} b_{0i}\phi_i + \sum_{i=q+1}^{m} c_{0i}\eta_i.$$

Let $\xi_0 = (b_{01},\ldots,b_{0p},c_{0q+1},\ldots,c_{0m}) \in E^M$. (We usually choose x_0 such that $\Psi(x_0) = 0$, that is, $\psi(\xi_0) = 0$).

15. Determine the function $\gamma(t) = H(I - P)Nx_0(t)$ and choose constants e and e' by means of (2.13).

We have compiled this list in order to facilitate the application of the theory of Chapter 2 to Equation (4.1). Each of the steps listed above can easily be carried out in practice. Most of them are independent of the operator N; only in the tenth step is a restriction placed on N, namely that $S_0 \subseteq \mathcal{D}(N)$. This is a very mild restriction which is satisfied in most applications since N is usually defined for at least all the continuous functions in S. This condition guarantees that x_0 is an element in $\mathcal{D}(N)$, and hence, we can determine the element $\gamma = H(I - P)Nx_0$. We can eliminate this condition completely if we assume that condition (IIc) is satisfied, which automatically guarantees that $x_0 \in \mathcal{D}(N)$.

Using Theorem 2.9 and Theorem 2.10, we obtain the following two existence theorems for Equation (4.1).

Theorem 4.1 Let m = 1. If there exist constants c, d, r, R_0, ℓ, ϵ, and δ with $0<c<d$, $0<r<R_0$, $\ell \geq 0$, $\epsilon>0$, and $\delta>0$ such that

(a) The set $\tilde{S}_0 = \{x \in S' \mid \|x-x_0\| \leq d,\ \mu(x-x_0) \leq R_0\}$ is a subset of $\mathcal{D}(N)$ and

$$\|Nx-Ny\| \leq \ell \|x-y\| \quad \text{for all } x,y \in \tilde{S}_0,$$

(b) $k\ell < 1$, $c+e \leq (1-k\ell)d$, $r+e' \leq R_0 - k'\ell d$,

(c) The set $\mathcal{V} = \{\xi \in E^M \mid \|\xi-\xi_0\| \leq \epsilon\}$ is mapped by Γ_1 into the set $V = \{x \in S_0 \mid \|x-x_0\| \leq c,\ \mu(x-x_0) \leq r\}$,

(d) $[-\delta,\delta] \subseteq \psi(\mathcal{V})$ and $(k\ell d+e)\ell \leq \delta$,

then there exists $x \in \mathcal{D}(L) \cap \mathcal{D}(N)$ which is a solution to Equation (4.1), and

$$\begin{cases} \|x-x_0\| \leq d, & \|Qx-x_0\| \leq c \\ \mu(x-x_0) \leq R_0, & \mu(Qx-x_0) \leq r. \end{cases} \tag{4.2}$$

Proof. Clearly conditions (IIabc) are satisfied and relations (2.20) are valid. Condition (IId) has been shown to hold by Cesari [3,p.404]. We have only to apply Theorem 2.9 to complete the proof.

Q.E.D.

<u>Theorem 4.2</u> Let $p \geq q$ and let $\psi(\xi_0) = 0$. If there exist constants c, d, r, R_0, ℓ, ϵ, and δ with $0 < c < d$, $0 < r < R_0$, $\ell \geq 0$, $\epsilon > 0$, and $\delta > 0$ such that

(a) The set $\tilde{S}_0 = \{x \epsilon S' \mid \|x-x_0\| \leq d,\ (x-x_0) \leq R_0\}$ is a subset of $\mathscr{D}(N)$ and

$$\|Nx-Ny\| \leq \ell\|x-y\| \quad \text{for all } x,y \epsilon \tilde{S}_0;$$

(b) $k\ell < 1$, $c+e \leq (1-k\ell)d$, $r+e' \leq R_0 - k'\ell d$;

(c) The set $\mathscr{V} = \{\xi \epsilon E^M \mid \|\xi-\xi_0\| \leq \epsilon\}$ is mapped by Γ_1 into the set $V = \{x \epsilon S_0 \mid \|x-x_0\| \leq c,\ \mu(x-x_0) \leq r\}$, the first order partial derivatives of ψ exist and are continuous on $\mathscr{V}$, the Jacobian matrix for ψ has rank m at ξ_0;

(d) The set $W = \{u \epsilon E^m \mid \|u\| \leq \delta\}$ is a subset of $\psi(\mathscr{V})$, there exists a continuous mapping Λ: $W \longrightarrow \mathscr{V}$ with

$$\psi\,[\Lambda(u)] = u \quad \text{for all } u \epsilon W,$$

and $(k\ell d+e)\ell < \delta$;

then there exists $x \epsilon \mathscr{D}(L) \cap \mathscr{D}(N)$ which is a solution to Equation (4.1), and

$$\begin{cases} \|x-x_0\| \leq d, & \|Qx-x_0\| \leq c \\ \mu(x-x_0) \leq R_0, & \mu(Qx-x_0) \leq r. \end{cases} \tag{4.3}$$

<u>Proof</u>. This follows immediately from Theorem 2.10.

4.2 THE SELF-ADJOINT CASE

Consider the special case in which the differential operator L is self-adjoint, i.e.,

$$L = L^*. \tag{4.4}$$

In this case we have $\mathcal{D}(L) = \mathcal{D}(L^*)$, and

$$\mathcal{N}(L) = \mathcal{N}(L^*), \ p = q. \tag{4.5}$$

Thus, by Lemma 2.2 the Hilbert space $S = L_2(I)$ is the orthogonal direct sum of $\mathcal{N}(L)$ and $\mathcal{R}(L)$, and hence,

$$H: \mathcal{R}(L) \longrightarrow \mathcal{R}(L).$$

When we choose the functions $\phi_i(t)$, $i = 1,\ldots,p$ and the functions $\omega_i(t)$, $i = 1,\ldots,q$, we can take

$$\phi_i(t) = \omega_i(t) \quad \text{for } i = 1,\ldots,p = q. \tag{4.6}$$

The functions $\omega_{p+1}(t),\ldots,\omega_m(t)$ can be chosen in a manner which will simplify our work.

Theorem 4.3 If the differential operator L is self-adjoint, then the operator H is a completely continuous self-adjoint linear operator on $\mathcal{R}(L)$.

Proof. By the Corollary to Theorem 3.9 the operator H is a completely continuous operator from $\mathcal{R}(L)$ into $\mathcal{N}(L)^{\perp} = \mathcal{R}(L)$. Take $f \in \mathcal{R}(L)$, $g \in \mathcal{R}(L)$ and let $u = Hf$, $v = Hg$. Then $u \in \mathcal{D}(L)$, $v \in \mathcal{D}(L)$, and $Lu = f$, $Lv = g$. Hence,

$$(Hf,g) = (u,Lv) = (Lu,v) = (f,Hg)$$

or

$$(Hf,g) = (f,Hg) \quad \text{for all } f,g \in \mathcal{R}(L).$$

Therefore, H is self-adjoint.

Q.E.D.

Theorem 4.4 If the differential operator L is self-adjoint, then there exists an orthonormal sequence of functions $\omega_i(t)$, $i = p+1, p+2, \dots$ in $\mathcal{R}(L)$ and a sequence of non zero real numbers h_i, $i = p+1, p+2, \dots$ such that:

(a) $H\omega_i = h_i\omega_i$ for $i = p+1, p+2, \dots$;

(b) Each eigenvalue for H occurs in the sequence $\{h_i\}$;

(c) $Hf = \sum_{i=p+1}^{\infty} h_i(f,\omega_i)\omega_i$ for all $f \in \mathcal{R}(L)$.

Proof. This follows from the fact that H is a completely continuous self-adjoint linear operator in the Hilbert space $\mathcal{R}(L)$. See [23, p.336].

Q.E.D.

Choose functions $\omega_i(t)$, $i = p+1, p+2, \dots$ as in the last theorem. Then for any integer $m \geq p$ the functions $\omega_1(t), \dots, \omega_m(t)$ have all the required properties of the last section. Note that if $f \in S$ and $(f,\omega_i) = 0$ for $i = 1,2,\dots$, then $f \in \mathcal{N}(L)^{\perp} = \mathcal{R}(L)$ and

$$Hf = \sum_{i=p+1}^{\infty} h_i(f,\omega_i)\omega_i =' 0,$$

or $f = 0$ since H is 1-1. Therefore, the functions $\omega_i(t)$, $i = 1,2,\dots$ form a complete orthonormal sequence in the Hilbert space $S = L_2(I)$.

Let

$$\phi_i(t) = \omega_i(t) \quad \text{for } i = p+1, p+2, \dots , \tag{4.7}$$

and let

$$\lambda_i = \begin{cases} 0 & \text{for } i = 1,\ldots,p \\ \dfrac{1}{h_i} & \text{for } i = p+1, p+2, \ldots \end{cases} \qquad (4.8)$$

The functions $\phi_i(t)$, $i = 1,2,\ldots$ are functions in $\mathcal{D}(L)$, they form a complete orthonormal sequence in S, and

$$L\phi_i = \lambda_i \phi_i \quad \text{for } i = 1,2,\ldots \; . \qquad (4.9)$$

The projection operators P and Q are given by

$$Px = \sum_{i=1}^{m} (x,\phi_i)\phi_i$$

and

$$Qx = \sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=p+1}^{m} (x,L\phi_i)H\phi_i$$

$$= \sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=p+1}^{m} (x,\phi_i)\lambda_i h_i \phi_i$$

$$= \sum_{i=1}^{m} (x,\phi_i)\phi_i \, ,$$

or

$$P = Q. \qquad (4.10)$$

In case the differential operator L is self-adjoint and the ϕ_i and ω_i are chosen as eigenfunctions as above, the general theory given in the last section reduces to the theory of Cesari [3].

CHAPTER 5

THE BASIC CONDITIONS OF THE EXISTENCE THEOREMS

In Chapter 2 we have developed a theory for solving the equation $Lx = Nx$, which has been applied in Chapter 4 to the differential equation $Lx = Nx$ where L is a differential operator of order n in the Hilbert space $S = L_2(I)$ and N is some operator in S. Our basic existence theorem is Theorem 2.7. The existence Theorems 2.9, 2.10, 4.1, and 4.2 are all special cases of this theorem. To apply Theorem 2.7 we have to show that conditions (IIabcd) are satisfied, that the inequalities

$$\mathcal{K}\ell < 1, \quad c+e \le (1-\mathcal{K}\ell)d \;, \quad r+e' \le R_0 - \mathcal{K}'\ell d \tag{5.1}$$

are valid, and that there exists $x^* \epsilon V$ such that

$$P(L\mathcal{J}x^* - N\mathcal{J}x^*) = 0. \tag{5.2}$$

In the event that the equation $Lx = Nx$ is known to have a solution $\hat{x}$, it is reasonable to ask whether the above conditions are satisfied and if $\hat{x}$ is one of the solutions given by Theorem 2.7? In this chapter we examine this question in detail for the case of differential operators.

As in Chapter 4 let I be the finite interval $[a,b]$ of the real axis and let $S = L_2(I)$. Let S' be the linear subspace in S consisting of all functions which are bounded a.e. and let μ be the uniform norm in S'. Let L be a differential operator of order n in S obtained from a formal differential operator τ by imposing a set of k linearly independent boundary conditions and let N be an operator in S with $\mathcal{D}(L) \cap \mathcal{D}(N) \neq \phi$. To simplify our calculations we assume that $\mathcal{D}(L)$ is a subset of $\mathcal{D}(N)$. This is usually the case in applications.

Suppose there exists a function $\hat{x}(t)$ in $\mathcal{D}(L) \cap \mathcal{D}(N)$ such that

$$L\hat{x} = N\hat{x}. \tag{5.3}$$

We shall show that under reasonable conditions it is possible to choose the integer m, the projection operators P and Q, the constants $\mathcal{k}$ and $\mathcal{k}'$, the function $x_0(t)$, the constants e and e', and the constants c, d, r, R_0, and ℓ such that the hypothesis of Theorem 2.7 is satisfied, and that $\hat{x}$ is one of the solutions guaranteed by Theorem 2.7.

We begin by carrying out part of the program of Chapter 4:

1. Determine the adjoint operator L^*.

2. Let $p = \dim \mathcal{N}(L)$ and choose functions $\phi_1(t),\ldots,\phi_n(t)$ in $\mathcal{D}(T_1(\tau)) = H^n(I)$ to form an orthonormal base for the null space $\mathcal{N}(T_1(\tau))$ in such a way that the functions $\phi_1(t),\ldots,\phi_p(t)$ are in $\mathcal{D}(L)$ and form a base for the null space $\mathcal{N}(L)$.

3. Determine the function G(t,s), the functions $\psi_1(t),\ldots,\psi_n(t)$, and the function K(t,s) so as to obtain the integral representation for the operator H:

$$Hf(t) = \int_a^b K(t,s)f(s)ds = \sum_{j=1}^{n} \phi_j(t) \int_a^b \psi_j(s)f(s)ds + \int_a^t G(t,s)f(s)ds. \tag{5.4}$$

Let $q = \dim \mathcal{N}(L^*)$ and choose a sequence of functions $\omega_i(t)$, $i = 1,2,\ldots$ in $\mathcal{D}(L^*)$ such that the ω_i form a complete orthonormal sequence in S and the functions $\omega_1(t),\ldots,\omega_q(t)$ form a base for the null space $\mathcal{N}(L^*)$. We can find such a sequence because $\mathcal{D}(L^*)$ is dense in S. For each integer $m \geq q$ we define projection operators P_m and Q_m in S by

$$P_m x = \sum_{i=1}^{m} (x,\omega_i)\omega_i, \quad x\epsilon S, \tag{5.5}$$

and

$$Q_m x = \sum_{i=1}^{p} (x,\phi_i)\phi_i + \sum_{i=q+1}^{m} (x, L^*\omega_i)H\omega_i, \quad x\epsilon S. \tag{5.6}$$

These are the same projection operators we have been working with in the last three chapters. For each integer $m \geq q$ let $K_m(t,s)$ be the function on IxI defined by

$$K_m(t,s) = K(t,s) - \sum_{i=1}^{m} \left(\int_a^b K(t,\xi)\omega_i(\xi)d\xi \right) \omega_i(s), \tag{5.7}$$

and let k_m and k'_m be the constants given by

$$k_m = \left(\int_a^b \int_a^b \left[K_m(t,s)\right]^2 ds\, dt \right)^{1/2} \tag{5.8}$$

and

$$k'_m = \left(\max_{t\epsilon I} \int_a^b \left[K_m(t,\xi)\right]^2 d\xi \right)^{1/2}. \tag{5.9}$$

By Theorem 3.10 we have that

$$H(I - P_m)f(t) = \int_a^b K_m(t,s)f(s)ds, \quad t\epsilon I, \qquad (5.10)$$

for all functions f(t) in S, and

$$\begin{cases} \|H(I - P_m)f\| \le k_m\|f\| \\ \mu(H(I - P_m)f) \le k'_m\|f\| \end{cases} \quad \text{for all } f\epsilon S. \qquad (5.11)$$

This is true for each integer $m \ge q$.

<u>Theorem 5.1</u> (a) $\lim_{m\to\infty} k_m = 0$.

(b) $\lim_{m\to\infty} k'_m = 0$.

<u>Proof</u>. In Chapter 3 we observed that the integrals

$$\int_a^b [K(t,\xi)]^2 d\xi \quad \text{and} \quad \int_a^b K(t,\xi)\omega_i(\xi)d\xi \ , \quad i = 1,2,\ldots$$

define continuous functions of t on the interval I. Let

$$f(t) = \int_a^b [K(t,\xi)]^2 d\xi$$

and

$$\alpha_i(t) = \int_a^b K(t,\xi)\omega_i(\xi)d\xi \quad \text{for } i = 1,2,\ldots .$$

These are continuous functions on I and $f(t) \ge 0$ on I. Fix $t\epsilon I$ and consider $K(t,s)$ as a function of s on I. Since this function is square-integrable on I, by Parseval's equality we have

$$\int_a^b [K(t,\xi)]^2 d\xi = \lim_{m\to\infty}\sum_{i=1}^m \left(\int_a^b K(t,\xi)\omega_i(\xi)d\xi\right)^2$$

or

$$f(t) = \lim_{m\to\infty} \sum_{i=1}^{m} |\alpha_i(t)|^2 \quad \text{for each } t\epsilon I.$$

Using Dini's Theorem, we conclude that the sequence of continuous functions

$$f(t) - \sum_{i=1}^{m} |\alpha_i(t)|^2 , \quad m = 1,2,\ldots$$

converges uniformly to 0 on I. By this last result and Equation (3.31) we have

$$\begin{aligned}
0 &= \lim_{m\to\infty} \left\{ \int_a^b \left(f(t) - \sum_{i=1}^{m} |\alpha_i(t)|^2 \right) dt \right\} \\
&= \lim_{m\to\infty} \left\{ \int_a^b \left(\int_a^b [K(t,\xi)]^2 d\xi - \sum_{i=1}^{m} [\int_a^b K(t,\xi)\omega_i(\xi)d\xi]^2 \right) dt \right\} \\
&= \lim_{m\to\infty} \left\{ \int_a^b \int_a^b [K_m(t,\xi)]^2 d\xi \, dt \right\} \\
&= \lim_{m\to\infty} \{k_m\}^2
\end{aligned}$$

and

$$\begin{aligned}
0 &= \lim_{m\to\infty} \mu \left(f(t) - \sum_{i=1}^{m} |\alpha_i(t)|^2 \right) \\
&= \lim_{m\to\infty} \left\{ \max_{t\epsilon I} \left(\int_a^b [K(t,\xi)]^2 d\xi - \sum_{i=1}^{m} [\int_a^b K(t,\xi)\omega_i(\xi)d\xi]^2 \right) \right\} \\
&= \lim_{m\to\infty} \left\{ \max_{t\epsilon I} \int_a^b [K_m(t,\xi)]^2 d\xi \right\} \\
&= \lim_{m\to\infty} \{k'_m\}^2 .
\end{aligned}$$

Q.E.D.

Theorem 5.2 Let x be any function in $\mathcal{D}(L)$. If $x_m = Q_m x$ for $m = q, q+1, \ldots$, then $x_m \to x$, $Lx_m \to Lx$, and the functions $x_m(t)$ converge uniformly on I to the function x(t). The convergence is determined by:

$$\begin{cases} \|x-x_m\| \leq 2\,k_m\|Lx\|, \\ \mu(x-x_m) \leq 2\,k'_m\|Lx\|, \\ \|Lx-Lx_m\| = \left(\|Lx\|^2 - \sum_{i=1}^{m} |(Lx,\omega_i)|^2\right)^{1/2}. \end{cases} \tag{5.12}$$

Proof. By Theorem 2.4 we have $Lx_m = LQ_m x = P_m Lx$ or

$$Lx_m = \sum_{i=1}^{m} (Lx,\omega_i)\omega_i \ .$$

Since the ω_i are complete, we get the desired convergence $Lx_m \to Lx$. Note that $\|Lx_m\| \leq \|Lx\|$ by Bessel's inequality. Also by Theorem 2.4 we have

$$H(I - P_m)L(x-x_m) = (I - Q_m)(x-x_m)$$

$$= x-x_m,$$

so

$$\|x-x_m\| \leq k_m\|Lx-Lx_m\|$$
$$\leq 2\,k_m\|Lx\|$$

and

$$\mu(x-x_m) \leq 2\,k'_m\|Lx\|.$$

Q.E.D.

For each pair of real numbers $\epsilon > 0$, $\delta > 0$ let

$$W(\epsilon,\delta) = \{x \epsilon S' \mid \|x-\hat{x}\| \leq \epsilon,\ \mu(x-\hat{x}) \leq \delta\}. \qquad (5.13)$$

We assume that the following condition is satisfied:

(IV) There exist constants $\epsilon > 0$, $\delta > 0$, and $\ell \geq 0$ such that the set $W(\epsilon,\delta)$ is a subset of $\mathcal{D}(N)$ and

$$\|Nx-Ny\| \leq \ell\|x-y\| \quad \text{for all } x,y \epsilon W(\epsilon,\delta).$$

Note that $\hat{x} \epsilon W(\epsilon,\delta)$, so this set is nonempty. Let

$$x_m = Q_m \hat{x} \quad \text{for } m = q, q+1, \ldots \ . \qquad (5.14)$$

Each function x_m belongs to S' and by the last theorem

$$\|x_m-\hat{x}\| \longrightarrow 0, \qquad \mu(x_m-\hat{x}) \longrightarrow 0.$$

Choose an integer $m_o \geq q$ such that the following conditions are satisfied:

(a) $x_m \epsilon W(\epsilon/2,\delta/2)$ for all $m \geq m_o$,

(b) $k_m \ell < 1/3$ for all $m \geq m_o$,

(c) $k_m(\ell\epsilon + \|N\hat{x}\|) \leq \epsilon/6$ for all $m \geq m_o$,

(d) $k'_m(\ell\epsilon + \|N\hat{x}\|) \leq \delta/6$ for all $m \geq m_o$,

(e) $k'_m \ell\epsilon \leq \delta/3$ for all $m \geq m_o$.

Now we choose all the quantities which are needed in order to apply Theorem 2.7. Fix any integer $m \geq m_o$ and let P and Q be the projection operators

$$P = P_m, \quad Q = Q_m.$$

We choose the constants k and k' by taking

$$\mathcal{k} = \mathcal{k}_m, \mathcal{k}' = \mathcal{k}'_m.$$

Let S_o be the linear subspace in S spanned by the functions $\phi_1, \ldots,$ $\phi_p, H\omega_{q+1}, \ldots, H\omega_m$ and let

$$x_o = x_m = Q_m \hat{x} \in S_o.$$

Corresponding to the function $\gamma = H(I - P)Nx_o$, we choose

$$e = \mathcal{k}_m(\ell\epsilon + \|N\hat{x}\|), \quad e' = \mathcal{k}'_m(\ell\epsilon + \|N\hat{x}\|).$$

Then

$$\begin{aligned} \|\gamma\| &= \|H(I - P)Nx_o\| \\ &\leq \mathcal{k}_m(\|Nx_o - N\hat{x}\| + \|N\hat{x}\|) \\ &\leq \mathcal{k}_m(\ell\epsilon + \|N\hat{x}\|) \\ &\leq e, \end{aligned}$$

and similarly,

$$\mu(\gamma) \leq e'.$$

Let c, d, r, and R_o be the constants given by

$$c = \epsilon/6, \quad d = \epsilon/2, \quad r = \delta/6, \quad R_o = \delta/2.$$

Clearly $0 < c < d$ and $0 < r < R_o$. As in Chapter 2 we let V and $\tilde{S}_o$ be the sets given by

$$V = \{x \in S_o \mid \|x - x_o\| \leq c, \ \mu(x - x_o) \leq r\}$$

and

$$\tilde{S}_o = \{x \in S' \mid \|x - x_o\| \leq d, \ \mu(x - x_o) \leq R_o\}.$$

Also, for each $x^* \in V$ let

$$S(x^*) = \{x \in S' \mid Qx = x^*, \ \|x - x_o\| \leq d, \ \mu(x - x_o) \leq R_o\}.$$

Note that if $x \in \tilde{S}_o$, then $x \in S'$,

$$\|x-\hat{x}\| \leq \|x-x_0\| + \|x_0-\hat{x}\|$$

$$\leq d + \epsilon/2 = \epsilon,$$

and

$$\mu(x-\hat{x}) \leq \mu(x-x_0) + \mu(x_0-\hat{x})$$

$$\leq R_0 + \delta/2 = \delta,$$

and hence, $\tilde{S}_0 \subseteq W(\epsilon,\delta) \subseteq \mathcal{D}(N)$. Therefore, conditions (IIabcd) are satisfied. Also, we have

$$\mathcal{k}\ell = \mathcal{k}_m \ell < 1/3,$$

$$c + e + \mathcal{k}\ell d \leq \epsilon/6 + \epsilon/6 + 1/3 \cdot \epsilon/2 = d,$$

and

$$r + e' + \mathcal{k}'\ell d \leq \delta/6 + \delta/6 + \mathcal{k}'\ell\epsilon/2 \leq R_0.$$

Therefore, the relations (2.20) are valid. From Definition 2.5 we obtain the continuous operator $\mathcal{J}: V \longrightarrow \mathcal{D}(L) \cap \tilde{S}_0$ which assigns to each $x^* \epsilon V$ the unique element $x \epsilon S(x^*)$ which satisfies the equation

$$x = x^* + H(I - P)Nx.$$

Note that $x_0 \epsilon V$, $\hat{x} \epsilon S(x_0)$, and by Theorem 2.4 we have

$$x_0 + H(I - P)N\hat{x} = x_0 + H(I - P)L\hat{x}$$

$$= Q\hat{x} + (I - Q)\hat{x}$$

$$= \hat{x},$$

and hence, $\mathcal{J} x_0 = \hat{x}$ and $P(L\mathcal{J}x_0 - N\mathcal{J}x_0) = 0$. Therefore, the hypothesis of Theorem 2.7 is satisfied, and we see that $\hat{x}$ is one of the solutions which is guaranteed by this theorem.

CHAPTER 6

AN EXAMPLE

In this chapter we use the results of Chapter 4 to study the non-linear boundary value problem

$$\begin{cases} x''(t) + x(t) + \alpha x^2(t) = \beta t, \; 0 \leq t \leq 2\pi \\ x(0) = 0 \end{cases} \tag{6.1}$$

where α and β are real constants. From Theorem 4.1 it will follow that this equation has a solution for all (α,β) with $|\alpha| \leq 1$, $|\beta| \leq .001$. We shall obtain estimates on the norms of these solutions.

Let I be the finite interval $[0,2\pi]$ and let $S = L_2(I)$ be the real Hilbert space consisting of all real-valued functions $f(t)$ which are square-integrable on $[0,2\pi]$. The inner product and norm in S will be denoted by (f,g) and $\|f\|$, respectively. Let S' be the linear subspace in S consisting of all functions which are bounded a.e. and let μ be the uniform norm in S'.

Let L be the differential operator of order 2 in S obtained from the formal differential operator $\tau x = x'' + x$ by imposing the single boundary condition

$$B_1(f) = f(0) = 0. \tag{6.2}$$

We have $n = 2$ and $k = 1$. Let $N\colon S' \longrightarrow S$ be the operator given by

$$Nx = -\alpha x^2(t) + \beta t, \; 0 \leq t \leq 2\pi, \tag{6.3}$$

where α and β are real constants. Clearly $\mathcal{D}(N) = S'$ and $\mathcal{D}(L) \subseteq \mathcal{D}(N)$.

We proceed to carry out the steps listed in Chapter 4.

1. In Example 2, Chapter 3 we showed that τ is formally self-adjoint and that the adjoint operator L^* is obtained from $\tau = \tau^*$ by imposing the three linearly independent boundary conditions

$$\begin{cases} B_1^*(f) = f(0) = 0 \\ B_2^*(f) = f'(2\pi) = 0 \\ B_3^*(f) = f(2\pi) = 0. \end{cases} \tag{6.4}$$

2. Let

$$\phi_1(t) = \frac{1}{\sqrt{\pi}} \sin t, \quad \phi_2(t) = \frac{1}{\sqrt{\pi}} \cos t. \tag{6.5}$$

Clearly $\phi_1(t)$ and $\phi_2(t)$ are functions in $\mathcal{D}(T_1(\tau)) = H^2(I)$ which form an orthonormal base for the null space $\mathcal{N}(T_1(\tau))$. If $x(t)$ is any function in the null space $\mathcal{N}(L)$, then $x = b_1\phi_1 + b_2\phi_2$ and $x(0) = 0$, i.e., $b_2 = 0$ and $x(t) = b_1\phi_1(t)$. Thus,

$$\mathcal{N}(L) = \langle\phi_1(t)\rangle, \; p = 1. \tag{6.6}$$

3. Note that if $x(t)$ is any function in the null space $\mathcal{N}(L^*)$, then $x = b_1\phi_1 + b_2\phi_2$ and

$$x(0) = \frac{b_2}{\sqrt{\pi}} = 0,$$

$$x'(2\pi) = \frac{b_1}{\sqrt{\pi}} = 0,$$

$$x(2\pi) = \frac{b_2}{\sqrt{\pi}} = 0,$$

or $b_1 = b_2 = 0$. Thus,

$$\mathcal{N}(L^*) = \langle 0 \rangle, \quad \mathcal{R}(L) = S, \quad q = 0. \tag{6.7}$$

4. We have

$$\Phi(t) = \frac{1}{\sqrt{\pi}} \begin{bmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{bmatrix},$$

$$\det\Phi(t) = -\frac{1}{\pi},$$

$$W_1(t) = \det \begin{bmatrix} 0 & \frac{1}{\sqrt{\pi}} \cos t \\ 1 & -\frac{1}{\sqrt{\pi}} \sin t \end{bmatrix} = -\frac{1}{\sqrt{\pi}} \cos t,$$

$$W_2(t) = \det \begin{bmatrix} \frac{1}{\sqrt{\pi}} \sin t & 0 \\ \frac{1}{\sqrt{\pi}} \cos t & 1 \end{bmatrix} = \frac{1}{\sqrt{\pi}} \sin t,$$

and hence,

$$G(t,s) = -\pi\,[\phi_1(t)W_1(s) + \phi_2(t)W_2(s)]$$

or

$$G(t,s) = \sin t \cos s - \cos t \sin s, \; 0 \leq t,s \leq 2\pi. \tag{6.8}$$

Now

$$\hat{B} = [B_1(\phi_2)] = \left[\frac{1}{\sqrt{\pi}}\right],$$

so we can take

$$A = [\sqrt{\pi}\,].$$

Thus,

$$\psi_1(s) = -\frac{1}{\sqrt{\pi}} \int_s^{2\pi} \sin t \,(\sin t \cos s - \cos t \sin s)dt$$

$$= \sqrt{\pi}\,(-\cos s + \frac{s}{2\pi} \cos s - \frac{1}{2\pi} \sin s)$$

and

$$\psi_2(s) = 0 \quad \text{since} \quad \beta_{10} = \beta_{11} = 0,$$

or

$$\begin{cases} \psi_1(s) = \sqrt{\pi}\,(-\cos s + \frac{s}{2\pi} \cos s - \frac{1}{2\pi} \sin s) & 0 \leq s \leq 2\pi \\ \psi_2(s) = 0 . \end{cases} \tag{6.9}$$

Therefore,

$$Hf(t) = \sin t \cdot \int_0^{2\pi} (-\cos s + \frac{s}{2\pi} \cos s - \frac{1}{2\pi} \sin s) f(s)ds$$

$$+ \sin t \cdot \int_0^t \cos s\, f(s)ds - \cos t \cdot \int_0^t \sin s\, f(s)ds, \tag{6.10}$$

$0 \leq t \leq 2\pi$, for all functions $f(t)$ in $\mathcal{R}(L) = S$.

5. Consider the function $\omega(t) = (t-2\pi)\sin t$. Clearly $\omega(0) = \omega(2\pi) = 0$ and

$$\omega'(2\pi) = [(t-2\pi)\cos t + \sin t]_{2\pi} = 0,$$

and hence, $\omega \in \mathcal{D}(L^*)$. We choose $m = 1$ and $\omega_1(t) = \frac{1}{\|\omega\|} \omega(t)$. Now

$$\|\omega\|^2 = \int_0^{2\pi} (t-2\pi)^2 \sin^2 t \, dt$$

$$= \frac{8\pi^3 - 3\pi}{6},$$

so

$$m = 1,\ \omega_1(t) = \left[\frac{6}{8\pi^3 - 3\pi}\right]^{1/2} (t-2\pi)\sin t,\ 0 \le t \le 2\pi\ . \tag{6.11}$$

6. To compute $L^*\omega_1$ we have

$$\omega(t) = (t-2\pi)\sin t,$$

$$\omega'(t) = (t-2\pi)\cos t + \sin t,$$

$$\omega''(t) = -(t-2\pi)\sin t + 2\cos t,$$

and hence,

$$L^*\omega_1(t) = 2\left[\frac{6}{8\pi^3 - 3\pi}\right]^{1/2} \cos t,\ 0 \le t \le 2\pi. \tag{6.12}$$

To compute $H\omega_1$ we first compute the functions $H(\sin t)$ and $H(t \sin t)$. By (6.10) we have

$$H(\sin t) = \sin t \cdot \int_0^{2\pi} (-\cos s + \frac{s}{2\pi}\cos s - \frac{1}{2\pi}\sin s)\sin s\, ds$$

$$+ \sin t \cdot \int_0^t \cos s \sin s\, ds - \cos t \cdot \int_0^t \sin^2 s\, ds$$

$$= -\frac{1}{4}\sin t - \frac{t}{2}\cos t$$

and

$$H(t \sin t) = \sin t \cdot \int_0^{2\pi} (-\cos s + \frac{s}{2\pi}\cos s - \frac{1}{2\pi}\sin s)s \sin s\, ds$$

$$+ \sin t \cdot \int_0^t s \sin s \cos s\, ds - \cos t \cdot \int_0^t s \sin^2 s\, ds$$

$$= -\frac{\pi}{2}\sin t + \frac{t}{4}\sin t - \frac{t^2}{4}\cos t.$$

Therefore,

$$H\omega_1(t) = \left[\frac{6}{8\pi^3 - 3\pi}\right]^{1/2} \left[\frac{t}{4}\sin t + \pi t \cos t - \frac{t^2}{4}\cos t\right], \quad 0 \le t \le 2\pi. \tag{6.13}$$

7. The projection operators P and Q are given by

$$Px(t) = \left[\frac{6}{8\pi^3 - 3\pi}\right] (t-2\pi)\sin t \cdot \int_0^{2\pi} x(\xi)\,(\xi-2\pi)\sin\xi\, d\xi, \quad 0 \le t \le 2\pi, \tag{6.14}$$

and

$$Qx(t) = \frac{1}{\pi}\sin t \cdot \int_0^{2\pi} x(\xi)\sin\xi\, d\xi + 2\left[\frac{6}{8\pi^3 - 3\pi}\right]\left[\frac{t}{4}\sin t + \pi t \cos t - \frac{t^2}{4}\cos t\right] \cdot \int_0^{2\pi} x(\xi)\cos\xi\, d\xi, \quad 0 \le t \le 2\pi, \tag{6.15}$$

for all functions x(t) in S.

8. We have

$$K(t,s) = \sin t\left(-\cos s + \frac{s}{2\pi}\cos s - \frac{1}{2\pi}\sin s\right) + G^*(t,s), \quad 0 \le t,s \le 2\pi \tag{6.16}$$

where

$$G^*(t,s) = \begin{cases} G(t,s) & \text{for } 0 \le s \le t \le 2\pi \\ 0 & \text{for } 0 \le t < s \le 2\pi \end{cases}.$$

Also,

$$\int_0^{2\pi} K(t,\xi)\omega_1(\xi)\,d\xi = H\omega_1(t),$$

and hence,

$$K_1(t,s) = K(t,s) - H\omega_1(t)\cdot\omega_1(s), \quad 0 \le t,s \le 2\pi. \tag{6.17}$$

9. To determine the constants k and k', we must compute several integrals. We have

$$\int_0^{2\pi} [\psi_1(s)]^2 ds = \pi \int_0^{2\pi} (-\cos s + \frac{s}{2\pi}\cos s - \frac{1}{2\pi}\sin s)^2 ds$$

$$= \frac{\pi^2}{3} + \frac{5}{8},$$

$$\int_0^t G(t,s)\psi_1(s)ds = \sqrt{\pi}\int_0^t [\sin t \cos s - \cos t \sin s] [-\cos s + \frac{s}{2\pi}\cos s - \frac{1}{2\pi}\sin s]ds$$

$$= \frac{1}{8\sqrt{\pi}} t^2 \sin t - \frac{\sqrt{\pi}}{2} t \sin t + \frac{3}{8\sqrt{\pi}} t \cos t - \frac{3}{8\sqrt{\pi}} \sin t,$$

and

$$\int_0^t [G(t,s)]^2 ds = \int_0^t [\sin t \cos s - \cos t \sin s]^2 ds$$

$$= \frac{t}{2} - \frac{1}{2}\sin t \cos t.$$

Therefore,

$$\int_0^{2\pi} [K(t,s)]^2 ds = \int_0^{2\pi} [\phi_1(t)\psi_1(s) + G^*(t,s)]^2 ds$$

$$= [\phi_1(t)]^2 \cdot \int_0^{2\pi} [\psi_1(s)]^2 ds + 2\phi_1(t) \cdot \int_0^t G(t,s)\psi_1(s)ds + \int_0^t [G(t,s)]^2 ds$$

or

$$\int_0^{2\pi} [K(t,s)]^2 ds = \frac{t}{2} + (\frac{\pi}{3} - \frac{1}{8\pi})\sin^2 t - \frac{1}{2}\sin t \cos t - t \sin^2 t$$

$$+ \frac{3}{4\pi} t \sin t \cos t + \frac{1}{4\pi} t^2 \sin^2 t. \tag{6.18}$$

By (3.31) we have

$$\int_0^{2\pi} [K_1(t,s)]^2 ds = \int_0^{2\pi} [K(t,s)]^2 ds - [H\omega_1(t)]^2, \tag{6.19}$$

and hence,

$$\int_0^{2\pi}\int_0^{2\pi} [K_1(t,s)]^2 dsdt = \int_0^{2\pi}\int_0^{2\pi} [K(t,s)]^2 dsdt - \|H\omega_1\|^2$$

where

$$\int_0^{2\pi}\int_0^{2\pi} [K(t,s)]^2 dsdt = \frac{2\pi^2}{3} - \frac{5}{8}$$

and

$$\|H\omega_1\|^2 = \left[\frac{6}{8\pi^3 - 3\pi}\right] \int_0^{2\pi} \left(\frac{t}{4} \sin t + \pi t \cos t - \frac{t^2}{4} \cos t\right)^2 dt,$$

$$\|H\omega_1\|^2 = \left[\frac{6}{8\pi^3 - 3\pi}\right] \left[\frac{8\pi^5}{15} - \frac{\pi^3}{6} \quad \frac{7\pi}{32}\right]. \tag{6.20}$$

Therefore,

$$\int_0^{2\pi}\int_0^{2\pi} [K_1(t,s)]^2 dsdt = \frac{2\pi^2}{3} - \frac{5}{8} - \frac{6}{8\pi^3 - 3\pi} \left(\frac{8\pi^5}{15} - \frac{\pi^3}{6} - \frac{7\pi}{32}\right)$$

$$= 1.998188.$$

Since $[1.998188]^{1/2} = 1.413573$, we choose

$$k = 1.414. \tag{6.21}$$

If we write out (6.19), we get

$$\int_0^{2\pi} [K_1(t,\xi)]^2 d\xi = .5\ t + \sin^2 t\ [1.007409 - t + .078005t^2]$$

$$+ \sin 2t\ [-.25 + .119366t - .019748t^2 + .001572t^3]$$

$$+ \cos^2 t[-.248161t^2 + .039496t^3 - .001572t^4].$$

Using a computer to carry out the calculations, we get the estimates

$$.691546 \leq \max_{0 \leq t \leq 2\pi} \int_0^{2\pi} [K_1(t,\xi)]^2 d\xi \leq .692546.$$

Since $[.692546]^{1/2} = .832194$, we choose

$$\mathcal{k}' = .833 . \qquad (6.22)$$

10. Let S_o be the linear subspace $\langle \phi_1, H\omega_1 \rangle$. We note that

$$S_o \subseteq \mathcal{D}(L) \subseteq \mathcal{D}(N).$$

11. Using (6.20) we have

$$\eta_1(t) = a_{11} H\omega_1(t), \quad 0 \leq t \leq 2\pi, \qquad (6.23)$$

where

$$a_{11} = \frac{1}{\|H\omega_1\|}$$

$$= \left[\left(\frac{6}{8\pi^3 - 3\pi}\right)\left(\frac{8\pi^5}{15} - \frac{\pi^3}{6} - \frac{7\pi}{32}\right)\right]^{-1/2}$$

$$= .502738.$$

Thus,

$$\eta_1(t) = \left[\frac{8\pi^5}{15} - \frac{\pi^3}{6} - \frac{7\pi}{32}\right]^{-1/2} \left[\frac{t}{4} \sin t + \pi\ t \cos t - \frac{t^2}{4} \cos t\right] \qquad (6.24)$$

for $0 \leq t \leq 2\pi$.

12. We have

$$M = p+m-q = 2. \qquad (6.25)$$

Let E^2 be a copy of Euclidean 2-space where we denote the points by $\xi = (b_1, c_1)$ and let E^1 be a copy of Euclidean 1-space where we denote the points by $u = (u_1)$. Then Γ_1: $E^2 \longrightarrow S_0$ and Γ_2: $\langle\omega_1\rangle \longrightarrow E^1$ are given by

$$\Gamma_1(b_1, c_1) = b_1\phi_1 + c_1\,\eta_1 \qquad (6.26)$$

and

$$\Gamma_2(u_1\omega_1) = u_1. \qquad (6.27)$$

13. The function ψ can be computed using (2.34). We have

$$\begin{aligned}
\psi(b_1,c_1) &= a_{11}c_1 - \int_0^{2\pi} N[b_1\phi_1 + c_1\,\eta_1]\omega_1(t)dt \\
&= a_{11}c_1 - \int_0^{2\pi} \omega_1(t)\cdot[-\alpha\,(b_1\phi_1(t) + c_1\,\eta_1(t))^2 + \beta t]dt \\
&= a_{11}c_1 + \alpha b_1^2 \cdot \int_0^{2\pi} \omega_1(t)[\phi_1(t)]^2 dt + 2\alpha b_1 c_1 \cdot \int_0^{2\pi} \omega_1(t)\phi_1(t)\,\eta_1(t)dt \\
&\quad + \alpha c_1^2 \cdot \int_0^{2\pi} \omega_1(t)[\,\eta_1(t)]^2 dt - \beta \int_0^{2\pi} t\,\omega_1(t)dt.
\end{aligned}$$

These four integrals are given by

$$\begin{aligned}
\int_0^{2\pi} \omega_1(t)[\phi_1(t)]^2 dt &= \frac{1}{\pi}\left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2} \cdot \int_0^{2\pi} (t-2\pi)\sin^3 t\,dt \\
&= -\frac{4}{3}\left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2},
\end{aligned}$$

$$\int_0^{2\pi} \omega_1(t)\phi_1(t)\,\eta_1(t)dt = \frac{a_{11}}{\sqrt{\pi}}\left(\frac{6}{8\pi^3 - 3\pi}\right)\cdot\int_0^{2\pi}(t-2\pi)\sin^2 t\left(\frac{t}{4}\sin t + \pi t\cos t - \frac{t^2}{4}\cos t\right)dt$$

$$= \frac{2\pi^2 a_{11}}{3\sqrt{\pi}}\left(\frac{6}{8\pi^3 - 3\pi}\right),$$

$$\int_0^{2\pi}\omega_1(t)[\eta_1(t)]^2 dt = a_{11}^2\left(\frac{6}{8\pi^3 - 3\pi}\right)^{3/2}\int_0^{2\pi}(t-2\pi)\sin t\left(\frac{t}{4}\sin t + \pi t\cos t - \frac{t^2}{4}\cos t\right)^2 dt$$

$$= a_{11}^2\left(\frac{6}{8\pi^3 - 3\pi}\right)^{3/2}\left(\frac{34\pi^3}{27} - \frac{140\pi}{81}\right),$$

and

$$\int_0^{2\pi} t\,\omega_1(t)dt = \left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2}\int_0^{2\pi} t(t-2\pi)\sin t\,dt$$

$$= 0.$$

Therefore,

$$\psi(b_1,c_1) = a_{11}c_1 - \frac{4}{3}\left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2}\alpha b_1^2 + \frac{4\pi^2 a_{11}}{3\sqrt{\pi}}\left(\frac{6}{8\pi^3 - 3\pi}\right)\alpha b_1 c_1 + a_{11}^2\left(\frac{6}{8\pi^3 - 3\pi}\right)^{3/2}\left(\frac{34\pi^3}{27} - \frac{140\pi}{81}\right)\alpha c_1^2$$

or

$$\psi(b_1,c_1) = .502738c_1 - .211427\alpha b_1^2 + .093851\alpha b_1 c_1 + .033884\alpha c_1^2. \qquad (6.28)$$

14. Let $x_o(t) \equiv 0$, i.e., we take $x_o = b_{o1}\phi_1 + c_{o1}\,\eta_1$ where $b_{o1} = c_{o1} = 0$. Clearly $x_o \in S_o$,

$$\xi_o = (0,0) \in E^2,$$

and

$$\psi(\xi_o) = 0.$$

15. We must determine the function $\gamma(t) = H(I - P)Nx_o(t)$. We have

$$Nx_o(t) = \beta t,$$

$$PNx_o(t) = \beta\left(\frac{6}{8\pi^3 - 3\pi}\right)(t-2\pi)\sin t \cdot \int_0^{2\pi} \xi(\xi-2\pi)\sin\xi\, d\xi$$

$$= 0,$$

$$(I - P)Nx_o(t) = \beta t,$$

and

$$\begin{aligned} H(I - P)Nx_o(t) &= \beta H(t) \\ &= \beta \sin t \cdot \int_0^{2\pi} \left(-\cos s + \frac{s}{2\pi}\cos s - \frac{1}{2\pi}\sin s\right) s\, ds \\ &\quad + \beta \sin t \cdot \int_0^t s \cos s\, ds - \beta \cos t \cdot \int_0^t s \sin s\, ds \\ &= 2\beta \sin t + \beta t. \end{aligned}$$

Thus,

$$\gamma(t) = 2\beta \sin t + \beta t,\ 0 \leq t \leq 2\pi. \tag{6.29}$$

Now

$$\|\gamma\|^2 = |\beta|^2 \cdot \int_0^{2\pi} (2 \sin t + t)^2 dt$$

$$= |\beta|^2 \left(\frac{8\pi^3}{3} - 4\pi\right)$$

or

$$\|\gamma\| = |\beta| \left(\frac{8\pi^3}{3} - 4\pi\right)^{1/2} = 8.373592\ |\beta|.$$

We choose

$$e = 8.374|\beta|. \tag{6.30}$$

Let $f(t) = 2 \sin t + t$, $0 \leq t \leq 2\pi$. Then $\gamma(t) = \beta f(t)$, $f(t) \geq 0$ on the interval $[0,2\pi]$, and $|\gamma(t)| = |\beta| \cdot f(t)$. Thus,

$$\max_{0 \leq t \leq 2\pi} |\gamma(t)| = |\beta| \cdot \max_{0 \leq t \leq 2\pi} f(t).$$

To maximize $f(t)$ on $[0,2\pi]$ we have

$$f'(t) = 2 \cos t + 1 = 0,$$

$$\cos t = -1/2,$$

or

$$t = \frac{2\pi}{3}, \frac{4\pi}{3}.$$

Thus, the maximum for $f(t)$ occurs at one of the four points 0, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, or 2π:

$$\begin{cases} f(0) = 0 \\ f(\frac{2\pi}{3}) = \sqrt{3} + \frac{2\pi}{3} \\ f(\frac{4\pi}{3}) = -\sqrt{3} + \frac{4\pi}{3} \\ f(2\pi) = 2\pi. \end{cases}$$

We see that the maximum occurs at 2π, and hence,

$$\max_{0 \leq t \leq 2\pi} |\gamma(t)| = 2\pi|\beta|.$$

We choose

$$e' = 6.284|\beta|. \tag{6.31}$$

Having chosen $x_0(t) \equiv 0$, we must choose the constants c, d, r, R_0, ℓ, ϵ, and δ such that the hypothesis of Theorem 4.1 is satisfied. For any number $c > 0$ and any function $x(t)$ in S_0 with $\|x\| \le c$, we have

$$x(t) = b_1\phi_1(t) + c_1\eta_1(t) \quad \text{with } b_1^2 + c_1^2 \le c^2,$$

so

$$|x(t)| \le c|\phi_1(t)| + c|\eta_1(t)|.$$

But

$$|\phi_1(t)| = \frac{1}{\sqrt{\pi}}|\sin t| \le \frac{1}{\sqrt{\pi}}$$

and

$$\begin{aligned} |\eta_1(t)| &= a_{11}|H\omega_1(t)| \\ &= a_{11}\left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2} \left|\frac{t}{4} \sin t + \pi t \cos t - \frac{t^2}{4} \cos t\right| \\ &\le a_{11}\left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2} \left(\frac{\pi}{2} + 2\pi^2 + \pi^2\right), \end{aligned}$$

and hence,

$$|x(t)| \le c\left[\frac{1}{\sqrt{\pi}} + a_{11}\left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2}\left(3\pi^2 + \frac{\pi}{2}\right)\right].$$

Let

$$r = c\left[\frac{1}{\sqrt{\pi}} + a_{11}\left(\frac{6}{8\pi^3 - 3\pi}\right)^{1/2}\left(3\pi^2 + \frac{\pi}{2}\right)\right] = 3.049797c \tag{6.32}$$

and

$$\epsilon = c. \tag{6.33}$$

Then the set

$$V = \{x \in S_0 \mid \|x-x_0\| \leq c,\ \mu(x-x_0) \leq r\}$$

reduces to the set

$$V = \{x \in S_0 \mid \|x\| \leq c\},$$

and the set

$$\mathcal{V} = \{\xi \in E^2 \mid \|\xi\| \leq c\}$$

is mapped by Γ_1 into the set V. Also, if d and R_0 are numbers with $0 < c < d$ and $0 < r < R_0$, and if

$$\tilde{S}_0 = \{x \in S' \mid \|x\| \leq d,\ \mu(x) \leq R_0\},$$

then $\tilde{S}_0$ is a subset of $\mathcal{D}(N)$ and for $x \in \tilde{S}_0$, $y \in \tilde{S}_0$ we have

$$\begin{aligned} Nx(t) - Ny(t) &= -\alpha x^2(t) + \alpha y^2(t) \\ &= \alpha[y(t) + x(t)]\,[y(t) - x(t)], \end{aligned}$$

so

$$|Nx(t) - Ny(t)| \leq 2R_0|\alpha|\ |x(t) - y(t)|$$

and

$$\|Nx - Ny\|^2 \leq 4R_0^{\,2}|\alpha|^2\ \|x - y\|^2,$$

or

$$\|Nx - Ny\| \leq 2R_0|\alpha|\ \|x - y\|.$$

Let

$$\ell = 2R_0|\alpha|. \tag{6.34}$$

We assume that $|\alpha| \leq 1$. The principal inequality that we must satisfy is $k\ell < 1$, i.e.,

$$(1.414)(2R_o|\alpha|) < 1.$$

This inequality is satisfied if $(1.414)(2R_o) < 1$ or

$$R_o < .353607. \qquad (6.35)$$

We shall replace the inequality $k\ell < 1$ by the Inequality (6.35) and the assumption $|\alpha| \leq 1$ in the work that follows. Now we want to choose c, d, and R_o such that $r < R_o$, i.e., $3.049797c < R_o$. Thus, we must have

$$c < \frac{.353607}{3.049797} = .115944.$$

If c satisfies this inequality, then

$$\begin{aligned} \psi(0,c) &= .502738c + .033884\alpha c^2 \\ &\geq c(.502738 - .033884c) \end{aligned}$$

and

$$\begin{aligned} \psi(0,-c) &= -.502738c + .033884\alpha c^2 \\ &\leq -c(.502738 - .033884c), \end{aligned}$$

and hence, the interval

$$[-c(.502738 - .033884c), c(.502738 - .033884c)]$$

is a subset of $\psi(\mathcal{V})$. Let

$$\delta = .502738c - .033884c^2. \qquad (6.36)$$

We must determine a bound on the parameter β and choose constants c, d, and R_o such that

$$\begin{cases} 0 < c < d \\ 3.049797c < R_o \\ R_o < .353607 \\ c + 8.374\ |\beta| + (1.414)(2R_o)d \le d \\ 3.049797c + 6.284\ |\beta| + (.833)(2R_o)d \le R_o \\ [(1.414)(2R_o)d + 8.374\ |\beta|]\ (2R_o) \le .502738c - .033884c^2 \end{cases}$$

or

$$\begin{cases} 0 < c < d \\ 3.049797c < R_o \\ R_o < .353607 \\ c + 8.374\ |\beta| + 2.828\ R_o d \le d \\ 3.049797c + 6.284\ |\beta| + 1.666R_o d \le R_o \\ 5.656R_o^2 d + 16.748\ |\beta| R_o \le .502738c - .033884c^2. \end{cases} \tag{6.37}$$

We assume that the parameters α and β satisfy the conditions

$$|\alpha| \le 1,\ |\beta| \le .001, \tag{6.38}$$

and choose the constants c, d, and R_o as follows:

$$\begin{cases} c = .01 \\ d = .03 \\ R_o = .1 \end{cases} . \tag{6.39}$$

Then

$$\begin{cases} r = .030498 \\ \ell = .2|\alpha| \\ \epsilon = .01 \\ \delta = .005024, \end{cases} \tag{6.40}$$

and the inequalities (6.37) are satisfied. From Theorem 4.1 we conclude that for each pair of real numbers (α,β) with $|\alpha| \le 1$, $|\beta| \le .001$ there exists a real-valued function $x(t)$ which is twice continuously differentiable on the interval $[0,2\pi]$ with

$$\begin{cases} x''(t) + x(t) + \alpha x^2(t) = \beta t, \ 0 \leq t \leq 2\pi \\ x(0) = 0 \end{cases}$$

and

$$\|x\| \leq .03, \ |x(t)| \leq .1 \quad .$$

APPENDIX I

THE THEORY OF CESARI

The existence theory for the equation Lx = Nx described in Chapter 2 is closely related to the existence theory of Cesari [3], where the same equation is considered. Cesari gives a system of axioms concerning the existence of linear operators H and P with special properties. These two operators play crucial roles in the development of his theory.

The original problem for this thesis was to determine conditions on the linear operator L which would be sufficient to guarantee the existence of the corresponding operators H and P. It was hoped that the resulting class of linear operators L would be large enough to include all the differential operators on a finite interval [a,b]. However, this was not the case. As long as L was restricted to being a self-adjoint differential operator, the desired operators could be shown to exist; when L was allowed to be non-self-adjoint, then examples arose where it was impossible to construct H and P with the desired properties. If we desire an existence theory which is applicable to all differential operators, then we need a new theory. It is just such a theory which is presented in Chapter 2. In case L is a self-adjoint differential operator, our theory reduces to the Cesari theory.

We are going to examine the existence theory in [3] and illustrate the difficulties which can arise when one tries to construct the operators H and P. Let S be a real Hilbert space with inner product (x,y) and norm $\|x\|$, let L be a linear operator in S with domain $\mathcal{D}(L)$, and let N be an operator in S with domain $\mathcal{D}(N)$ such that $\mathcal{D}(L) \cap \mathcal{D}(N) \neq \emptyset$. Cesari discusses the existence of solutions $\hat{x} \in S$ to the equation

$$Lx = Nx. \tag{1}$$

He assumes that linear operators H and P exist with the following properties:

(a) The operator P is a projection operator from S into S with finite-dimensional range S_0 and null space S_1; S_0 is a subset of $\mathcal{D}(L)$, and S_0 is spanned by orthonormal elements $\phi_1, \ldots, \phi_m$; the range of the operator $I - P$ is S_1.

(b) $Px = \sum_{i=1}^{m} (x, \phi_i)\phi_i$ for all $x \in S$.

(c) The operator H maps S_1 into S_1, and

$$H(I - P)Lx = (I - P)x \quad \text{for all } x \in \mathcal{D}(L).$$

(d) $PH(I - P)x = 0$ for all $x \in S$.

Choosing an element $x_0 \in S_0$ and a number $c > 0$, and letting V be the subset of S_0 given by

$$V = \{x \in S_0 \mid \|x - x_0\| \leq c\} ,$$

Cesari introduces suitable restrictions to show that it is possible to determine a unique solution $x \in \mathcal{D}(N)$ to the equation

$$x = x^* + H(I - P)Nx \tag{2}$$

corresponding to each element $x^* \in V$. He also makes the following assumption:

(e) For each $x^* \in V$ the corresponding element $x \in \mathcal{D}(N)$ is an element of $\mathcal{D}(L)$, and

$$\begin{cases} LH(I - P)Nx = (I - P)Nx \\ LPx = PLx. \end{cases}$$

Under these restrictions the unique element $x \in \mathcal{D}(N)$ corresponding to $x^* \in V$ is a solution to Equation (1) provided x also satisfies the equation

$$P(Lx - Nx) = 0. \qquad (3)$$

Thus, Cesari proceeds to determine $x^* \in V$ such that the corresponding x satisfies Equation (3), and hence, yields a solution to Equation (1).

We shall make two remarks to illustrate the difficulties which arise when one tries to construct the operators H and P with properties (a) - (e).

Remark 1 It is desirable to construct H and P using only the properties of the linear operator L and independent of the operator N. The resulting theory should be general enough to be applicable to a large class of nonlinear operators N. This class of operators should include all the operators N_y: $S \longrightarrow S$ of the form

$$N_y x = y \quad \text{for all } x \in S$$

where y is an element in S. If the Cesari theory is to be applicable to all the operators N_y, then from (e) we must have

$$H(I - P)y \in \mathcal{D}(L)$$

and

$$LH(I - P)y = (I - P)y \quad \text{for all } y \in S.$$

The last equation says that

$$LHx = x \quad \text{for all } x \epsilon S_1,$$

which implies that S_1 is a subset of the range of L and H: $S_1 \longrightarrow S_1 \cap \mathcal{D}(L)$. Thus, H must be a partial right inverse for L.

Let $\mathcal{R}(L)$ and $\mathcal{N}(L)$ denote the range and null space for the operator L, respectively. Since S_1 is the range of the operator I - P, it follows that

$$P: \; S \longrightarrow \mathcal{D}(L)$$

and

$$I - P: \; S \longrightarrow \mathcal{R}(L).$$

Take any element x belonging to the null space $\mathcal{N}(L^*)$ of the adjoint operator L^*. For each $y \epsilon S$ the element y-Py is an element in $\mathcal{R}(L)$, and hence, there exists $z \epsilon \mathcal{D}(L)$ such that Lz = y-Py. Thus,

$$\begin{aligned}(x,y-Py) &= (x,Lz) \\ &= (L^*x,z) \\ &= 0\end{aligned}$$

or

$$(x,y-Py) = 0 \quad \text{for all } y \epsilon S.$$

But P is self-adjoint, so

$$(x-Px,y) = 0 \quad \text{for all } y \epsilon S$$

or

$$x = Px \in S_o.$$

Thus, the null space $\mathcal{N}(L^*)$ must be a subset of $\mathcal{D}(L)$. This condition is not satisfied for all differential operators.

Example Suppose $Lx = x'' + x$ is the differential operator on $[0,2\pi]$ whose domain $\mathcal{D}(L)$ is determined by the boundary conditions $x(0) = 0$, $x(2\pi) = 0$, and $x'(2\pi) = 0$. It can be shown that $L^*x = x'' + x$ with domain $\mathcal{D}(L^*)$ determined by the single boundary condition $x(0) = 0$. It follows that $\mathcal{N}(L^*)$ is the 1-dimensional subspace generated by the function sin t. But sin t does not belong to $\mathcal{D}(L)$, i.e., $\mathcal{N}(L^*)$ is not a subset of $\mathcal{D}(L)$. We examine these particular differential operators in Chapter 6.

Remark 2 If the theory of Cesari is applied to the special case where N is given by

$$Nx = 0 \quad \text{for all } x \in S,$$

then Equation (2) reduces to finding x such that

$$x = x^*$$

where $x^* \in V$. From (e) we must have

$$Lx^* = PLx^* \in S_0 \quad \text{for each } x^* \in V,$$

or L maps V into S_0. For each element $y \in S_0$ we can choose $\alpha > 0$ such that $\|\alpha y\| \leq c$, and setting $z = x_0 + \alpha y$, we have $z \in V$, $Lz \in S_0$, $x_0 \in V$, $Lx_0 \in S_0$, and

$$\alpha Ly = Lz - Lx_0 \in S_0,$$

or $Ly \in S_0$. Thus, S_0 is a finite-dimensional invariant subspace for the linear operator L. In general, not all differential operators have such invariant subspaces.

Example Consider the differential operator $Lx = -x''$ on $[0,\pi]$ whose domain $\mathcal{D}(L)$ is determined by the boundary conditions $x(0) + 2x(\pi) = 0$, $x'(0) - 2x'(\pi) = 0$. Suppose L has a finite-dimensional invariant subspace S_0. Let $\phi_1(t),\ldots,\phi_m(t)$ be functions which form a base for S_0. Then we can write

$$-\phi_i'' = \sum_{j=1}^{m} a_{ji}\phi_j \quad \text{for } i = 1,\ldots,m$$

where the a_{ji} are real numbers. Choose a complex number λ and complex numbers $c_1,\ldots,c_m$ not all zero such that

$$\sum_{i=1}^{m} a_{ji}c_i = \lambda\, c_j \quad \text{for } j = 1,\ldots,m,$$

and let

$$\phi(t) = \sum_{i=1}^{m} c_i\phi_i(t) \quad \text{for } 0 \leq t \leq \pi.$$

The complex-valued function $\phi(t)$ has the properties that $\phi(0) + 2\phi(\pi) = 0$, $\phi'(0) - 2\phi'(\pi) = 0$, and

$$\begin{aligned} -\phi'' &= -\sum_{i=1}^{m} c_i\phi_i'' \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} a_{ji}c_i\,\phi_j \\ &= \lambda\phi. \end{aligned}$$

But the associated differential operator over the complex numbers is known to have no eigenvalues [6,p.300]. Thus, L has no finite-dimensional invariant subspaces.

The above remarks illustrate the difficulties which arise when one tries to use the Cesari theory for an arbitrary differential operator L.

APPENDIX II

THE THEORIES OF BARTLE, CRONIN, AND NIRENBERG

Bartle [2], Cronin [7], and Nirenberg [19] have each developed existence theories for the equation

$$Lx + F(x,y) = 0 \qquad (1)$$

in Banach spaces S and Y, where L is a bounded linear operator defined on all of S with values in S and F(x,y) is a function which maps a neighborhood of the origin in SxY into S with F(0,0) = 0. The purpose of this Appendix is to examine each of these three theories, determining any relationships which might exist between them and determining any relationship which they might have with the existence theory developed in this thesis.

Before getting into the details of these three theories, let us make several comments. Note that if we fix an element $y \in Y$ and let $Nx = -F(x,y)$, then Equation (1) has the same form as the equation we studied in Chapter 2. Thus, when the theories of Bartle, Cronin, and Nirenberg are applied in a Hilbert space, we would expect them to be related to the special case of our theory when we assume L is everywhere defined and bounded. We will find that such a relationship exists when we examine the Nirenberg theory.

Nirenberg treats Equation (1) is a very general setting, presenting three methods for establishing the existence of a solution. The material for his three methods is based on lectures by K. O. Friedrichs [10,11]. Bartle considers Equation (1) under more specialized circumstances, and his theory is essentially the same as Nirenberg's third method. Cronin discusses a very special form of Equation (1); her work is an abstract version of part of the work of Schmidt [20] on non-linear integral equations, and it is also intimately related to Nirenberg's third method.

Now let us begin our discussion of these three theories. We shall examine the general pattern of development which each one uses: the construction of an inverse operator for L, the determination of an auxiliary equation, and the determination of a bifurcation equation. We shall start with the Nirenberg theory because it is the most general.

In [19] Nirenberg assumes the following hypothesis for the linear operator L: the null space $\mathcal{N}(L)$ admits a projection operator Q and the range $\mathcal{R}(L)$ is a closed subspace in S which also admits a projection operator ρ. Let P denote the projection operator $I - \rho$, so $I - P$ is a projection of S onto $\mathcal{R}(L)$. Under the above hypothesis Nirenberg shows that there exists a bounded linear operator $H: \mathcal{R}(L) \longrightarrow S$ which is a right inverse for L and has the properties:

$$LHx = x \quad \text{for all } x \in \mathcal{R}(L),$$

and

$$HLx = x - Qx \quad \text{for all } x \in S.$$

The operator $H(I - P)$ gives an extension of H to all of S.

<u>Method 1</u> If $x \in S$ is a solution to Equation (1), then applying the operator $H(I - P)$ it follows that

$$x = x^* - H(I - P)F(x,y) \tag{2}$$

where $x^* = Qx \in \mathcal{N}(L)$. Equation (2) is the auxiliary equation used in Method 1. Under suitable restrictions it can be shown that Equation (2) has a unique solution $x = x(y,x^*)$ for $x^* \in \mathcal{N}(L)$ and $y \in Y$ with sufficiently small norms. For such a solution

$$Lx + F(x,y) = PF(x,y),$$

i.e., $x = x(y,x^*)$ is a solution to Equation (1) iff

$$PF(x(y,x^*),y) = 0. \tag{3}$$

This last equation is the bifurcation equation for Method 1. At this point the element y is usually fixed sufficiently small. To obtain a solution to Equation (1) one only needs to determine a solution x^* to the bifurcation Equation (3), which is usually a system of q-equations in p-unknowns where $p = \dim \mathcal{N}(L)$, $q = \operatorname{codim} \mathcal{R}(L)$.

Note that if we let $Nx = - F(x,y)$, then Equations (1), (2), and (3) can be written as

$$Lx = Nx, \tag{1'}$$

$$x = x^* + H(I - P)Nx, \tag{2'}$$

and

$$P[Lx(y,x^*) - Nx(y,x^*)] = 0. \tag{3'}$$

When S is a real Hilbert space, $\operatorname{codim} \mathcal{R}(L) = q < \infty$, and Q and ρ are the orthogonal projections onto $\mathcal{N}(L)$ and $\mathcal{R}(L)$, respectively, then

Method 1 is essentially the same as the special case of our theory when $\mathcal{D}(L) = S$ and $m = q$. Both theories use the same operators H, P, and Q, and both theories have the same auxiliary and bifurcation equations.

<u>Method 2</u> The second method that Nirenberg describes is equivalent to the first method. Since it adds nothing pertinent to our discussion, we shall not discuss it here.

<u>Method 3</u> (due to P. Ungar) The third method differs from the first two methods in that it uses a different type of inverse operator for L. It assumes the additional hypothesis that

$$\dim \mathcal{N}(L) = \operatorname{codim} \mathcal{R}(L) = p < \infty .$$

Let $B: \mathcal{N}(L) \longrightarrow P(S)$ be a 1-1 linear operator from $\mathcal{N}(L)$ onto $P(S)$, let $W = BQ$, and let $L_1: S \longrightarrow S$ be the bounded linear operator given by

$$L_1 = L + W.$$

L_1 is a 1-1 mapping of S onto itself, and hence, L_1 has a bounded inverse. Let

$$T = L_1^{-1} = (L + W)^{-1}.$$

The operators L and T are related by

$$TL = I - Q.$$

For any solution $x \in S$ to Equation (1) it follows that

$$x + TF(x,y) = x^* \tag{4}$$

where $x^* = Qx \in \mathcal{N}(L)$. Equation (4) is the auxiliary equation which is used in this method; it is very similar to the auxiliary equation used in Method 1. If $x = x(y,x^*)$ is a solution to Equation (4), then

$$(L+W)x + F(x,y) = (L+W)x^*,$$

or

$$Lx + F(x,y) = W(x^*-x),$$

i.e., $x = x(y,x^*)$ is a solution to Equation (1) iff

$$W(x^* - x(y,x^*)) = 0. \qquad (5)$$

Equation (5) is the bifurcation equation for this method. For fixed y it reduces to a system of p-equations in p-unknowns.

The one new feature which distinguishes Method 3 from Method 1 is the use of the operator T instead of the operator H. However, T is closely related to H as the following lemma shows.

Lemma The linear operator H is the restriction of the linear operator T to the subspace $\mathcal{R}(L)$, and

$$LT = I - P.$$

Proof. First, take any element $x \in \mathcal{R}(L)$. Then there exists $y \in S$ such that $Ly = x$. Letting $z = y-Qy$, we have $Qz = 0$, $Wz = 0$, and $Lz = Ly = x$. Thus,

$$\begin{aligned} Hx &= HLz = z-Qz \\ &= z \\ &= T(L+W)z \end{aligned}$$

$$= TLz = Tx.$$

Next, take any $x \in S$. Since $Px \in P(S)$, there exists $z \in \mathcal{N}(L)$ such that $Bz = Px$. We have

$$\begin{aligned} z &= T(L+W)z \\ &= TBQz \\ &= TPx, \end{aligned}$$

so

$$\begin{aligned} LTx &= LT(x-Px) + LTPx \\ &= LH(x-Px) + Lz \\ &= x-Px. \end{aligned}$$

Q.E.D.

Let us reexamine Equation (1) under two new sets of hypotheses for the operator L. We shall divide the discussion into two parts. In the first part we shall use the hypothesis of Bartle, establishing the relationship between Bartle and Nirenberg; in the second part we shall use the hypothesis of Cronin, establishing the relationship between Cronin and Nirenberg.

I. Consider Equation (1) when the linear operator L satisfies the following hypothesis: the range $\mathcal{R}(L)$ is a closed subspace in S, the null space $\mathcal{N}(L)$ is finite-dimensional, the null space $\mathcal{N}(L^*)$ for the adjoint operator L^* is finite-dimensional in the dual space S^*, and

$$\dim \mathcal{N}(L) = \dim \mathcal{N}(L^*) = p < \infty.$$

We assert that the linear operator L also satisfies the hypothesis of

Nirenberg's Method 3. In order to see this let $u_1,\ldots,u_p$ be elements in S which form a base for $\mathcal{N}(L)$, and let $f_1,\ldots,f_p$ be functionals in S^* which form a base for $\mathcal{N}(L^*)$. Then there exist functionals $g_1,\ldots,g_p$ in S^* and elements $z_1,\ldots,z_p$ in S such that

$$g_i(u_j) = \delta_{ij} , f_i(z_j) = \delta_{ij}.$$

Define projection operators P and Q by

$$Px = \sum_{i=1}^{p} f_i(x)z_i , x\in S, \tag{6}$$

and

$$Qx = \sum_{i=1}^{p} g_i(x)u_i , x\in S, \tag{7}$$

and let ρ be the projection operator I - P. Clearly $\mathcal{N}(L)$ admits Q as a projection operator. Also, we can show that $\rho x\in\mathcal{R}(L)$ for all $x\in S$, and that S is the direct sum of the subspace $\mathcal{R}(L)$ and the subspace $\langle z_1,\ldots,z_p\rangle$. It follows that $\mathcal{R}(L)$ admits ρ as a projection operator, and

$$\dim \mathcal{N}(L) = \operatorname{codim} \mathcal{R}(L) = p < \infty.$$

Hence, Nirenberg's Method 3 is applicable when the linear operator L satisfies the above hypothesis. Suppose we determine the particular form that Method 3 takes when P and Q are given by (6) and (7), respectively, and the linear operator B: $\mathcal{N}(L) \longrightarrow P(S)$ is defined by

$$B(u_i) = -z_i \quad \text{for } i = 1,\ldots,p. \tag{8}$$

The operator $W = BQ$ is just

$$Wx = -\sum_{i=1}^{p} g_i(x)z_i \quad \text{for all } x \in S, \tag{9}$$

and the operator $L_1 = L + W$ is

$$L_1x = Lx - \sum_{i=1}^{p} g_i(x)z_i \quad \text{for all } x \in S . \tag{10}$$

Again we obtain the inverse operator $T = (L+W)^{-1}$, the auxiliary Equation (4), and the bifurcation Equation (5). Now the bifurcation equation can be rewritten in a different form. Let $x = x(y,x^*)$ be a solution to Equation (4), i.e.,

$$x + TF(x,y) = x^* \in \mathcal{N}(L).$$

Applying the operator L and using the lemma, we get

$$Lx + F(x,y) = PF(x,y).$$

On the otherhand we observed in Method 3 that

$$Lx + F(x,y) = W(x^*-x),$$

and hence,

$$PF(x(y,x^*),y) = W(x^*-x(y,x^*)). \tag{11}$$

Thus, the bifurcation Equation (5) can also be written as

$$PF(x(y,x^*),y) = 0. \tag{5'}$$

The existence theory which we have developed in part I using Nirenberg's Method 3 is identical to the existence theory which Bartle developes in [2]. The hypothesis for the linear operator L is the same as his hypothesis for L; the operators P, Q, L_1, and T are the same as the operators Z, U, L', and R, respectively, which he constructs in his theory; and Equations (4) and (5') are the auxiliary and bifurcation equations which he uses.

II. Consider Equation (1) when the linear operator L and the function F(x,y) satisfy the following hypothesis: the operator L is of the form L = I+C where C is a completely continuous linear operator in S, and the function F(x,y) is of the form F(x,y) = R(x) - y where R is a function which maps a neighborhood of the origin in S into S with R(0) = 0 and $y \in Y = S$. Equation (1) takes on the special form

$$(I + C)x + R(x) = y. \qquad (1'')$$

For such a linear operator L it is well known that the range $\mathcal{R}(L)$ is a closed subspace in S, the null spaces $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$ are finite-dimensional, and

$$\dim \mathcal{N}(L) = \dim \mathcal{N}(L^*) = p < \infty.$$

Thus, we can proceed to apply Method 3 exactly as we did in part I. As before we choose the elements $u_1,\ldots,u_p$, $z_1,\ldots,z_p$ in S and the functionals $f_1,\ldots,f_p$, $g_1,\ldots,g_p$ in S*, and define the projection operators P and Q by means of Equations (6) and (7), respectively. Instead of defining the operator B by means of Equation (8), we define B: $\mathcal{N}(L) \longrightarrow P(S)$ by

$$B(u_i) = z_i \quad \text{for } i = 1,\ldots,p. \qquad (8')$$

For this choice of B the operators W = BQ and $L_1 = L + W$ are given by

$$Wx = \sum_{i=1}^{p} g_i(x)z_i \quad \text{for all } x \in S, \tag{9'}$$

and

$$L_1 x = Lx + \sum_{i=1}^{p} g_i(x)z_i \quad \text{for all } x \in S, \tag{10'}$$

and again the inverse operator $T = (L+W)^{-1}$ exists.

Up to this point the theory described in part II is identical to the existence theory of Cronin [7]. She studies Equation (1") under the same hypothesis that we used in part II, and she constructs the same operators P, Q, L_1, and T. However, at this point she proceeds to derive a different pair of auxiliary and bifurcation equations. Note that if $x \in S$ is a solution to Equation (1"), then applying T we have

$$T(I + C)x + TR(x) = Ty,$$

or

$$x - Qx + TR(x) = Ty.$$

Letting $x^* = Qx \in \mathcal{N}(L)$ and $z = x - x^*$, this last equation can be written as

$$z + TR(x^* + z) = Ty;$$

applying the operator I-Q we obtain the equation

$$z + (I - Q)TR(x^* + z) = (I - Q)Ty. \tag{12}$$

Equation (12) serves as the auxiliary equation in Cronin's theory; it is to be solved for $z = z(y, x^*)$ with $x^* \in \mathcal{N}(L)$, $y \in S$. Observe that if $z = z(y,x^*)$ is a solution to Equation (12), then $Qz = 0$, $(I - Q)z = z$, and

$$\begin{aligned} T[L(x^*+z) + R(x^*+z) - y] &= (x^*+z) - Q(x^*+z) + TR(x^*+z) - Ty \\ &= z + TR(x^*+z) - Ty \\ &= (I - Q)\,[z + TR(x^*+z) - Ty] \\ &\qquad + Q[z + TR(x^*+z) - Ty] \\ &= QTR(x^*+z) - QTy. \end{aligned}$$

Since T is 1-1, it follows that $x = x^*+z$ is a solution to Equation (1") iff $QTR(x^*+z) - QTy = 0$, or

$$QTR(x^* + z(y,x^*)) - QTy = 0. \qquad (13)$$

Equation (13) is the bifurcation equation used by Cronin. For fixed y one only needs to determine a solution $x^* \in \mathcal{N}(L)$ to Equation (13) in order to obtain a solution $x = x^* + z(y,x^*)$ to Equation (1").

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