SOME THEORETICAL ISSUES AND OBSERVATIONS ABOUT THE CLASSICAL TREATMENT IN REPLACEMENT ECONOMICS

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ABSTRACT

The classical treatment of replacement problems is a widely known engineering economy subject. The treatment simplifies the computations in a replacement analysis significantly. Unfortunately, the simplifications mask subtly some important theoretical issues about more general (non-classical) replacement problems. This paper highlights several theoretical issues about replacement economics that do not appear to be fully understood by some in practice and in the literature because of the simplifications of the classical treatment.

SOME THEORETICAL ISSUES AND OBSERVATIONS ABOUT THE CLASSICAL TREATMENT IN REPLACEMENT ECONOMICS

Introduction

The classical treatment of replacement problems, where current challengers (new assets) are assumed to repeat identically to the infinite future, is a traditional engineering economy subject [3][4][6][7][8][10][14]. Although the classical treatment may not approximate well many practical replacement problems, and there are many treatments that do not require these assumptions [1][]2][4][5][9][11][13][15], it is nonetheless useful to learn and know. The classical treatment is a convenient point of departure in an introductory study of replacement problems because it simplifies the computations significantly, and understanding the classical treatment facilitates understanding the treatment of other replacement problems when the assumptions are relaxed; afterall, the classical treatment is simply a special case. Unfortunately, the simplifications mask subtly several important theoretical issues about replacement problems in general; and some of these issues do not appear to be fully understood by some in practice and in the literature. For example, the calculations typically performed in the classical treatment seem to lead some to the impression that the economic life of a current challenger is a function of only the economic consequences attributable to the current challenger since the calculations appear to involve only the cash flows that describe the current challenger, and similarly for the defender (old existing asset). The economic life of an asset is a function of not only the economic consequences attributable to the asset itself but also to assets to be available in the future.

This paper highlights some theoretical issues that appear to be easily misunderstood because of the computational simplifications of the classical treatment. This paper will show more fully:

(1) why the equivalent uniform cash flow (EUCF) is the most common measure of worth used in the classical treatment,

(2) why the economic lives of the defender and current challengers are not necessarily the lives that maximize the EUCF for each asset in the classical treatment,

(3) why the popular (commonly used) decision rule to select the current asset with the maximum EUCF in the classical treatment is really an equivalent decision rule (shortcut) to a more theoretically exact but computationally cumbersome decision rule, and

(4) why the salvage value of the defender is typically treated as an initial disbursement to the defender rather than as an initial receipt to the current challenger in the classical treatment.

To illustrate these points it is important to digress briefly and begin with a brief description of the nature of replacement problems in general. We will then address the points above mathematically and with numerical examples.

The Nature of Replacement Problems

A replacement problem involves a situation where a service (or product) is to be provided for a period of time (horizon time) and the assets that provide the service are expected to wear out and/or become obsolete and therefore must be replaced periodically. A replacement decision involves determining the sequence of assets that promises to provide the most economical service from the decision time to the horizon time. If the sign convention that disbursements are negative and receipts are positive is used, then the most economical sequence is the one that maximizes the net present value (NPV) of the sequence. If the reverse sign convention was used then the objective would be to minimize NPV. A sequence of assets is determined when for each asset in the sequence the type of asset and its installation (or replacement) time are specified.

If at the time the service was to first begin (startup), the decision maker (1) knew completely the identity (existence) of all the current and future assets that could provide the service, (2) knew with certainty the cash flows that describe these assets, and (3) knew with certainty the horizon time, then a single decision could be made to determine the most economical sequence of assets, except when the horizon is infinite. An infinite sequence of assets of course could never be identified completely, except in the special case where at some time in the future assets eventually repeat [12]. However, it often can be partially identified [2].

Few (if any) practitioners have such complete and certain knowledge. Typically the existence of only some of the assets available both currently and in the near future will be known at the startup time and the cash flows and horizon time are generally uncertain. Consequently, the decision at the startup time is based on incomplete and uncertain information and therefore the sequence of assets determined would be most economical only if the forecasts proved to be complete and accurate. Seldom (if ever) would this occur. In practice, sometime after the startup time the asset currently in service (the defender) would be analyzed to determine if it should be replaced (by a current challenger). This decision would be based on revised but still incomplete and uncertain information and would involve a choice among alternative sequences of assets from the current decision time to the horizon time, namely, either the defender and its succession of future challengers or one of the alternative current challengers and its succession of future challengers. Although the decision implemented will be to either keep the defender or replace it with a current challenger, the decision must take into account the succession of future challengers because changes in the productivity of those challengers affects the economy of the defender and current challengers. This sequential decision process would be repeated periodically until the horizon time is reached, if the horizon is finite. Thus, replacement problems in practice typically involve a sequence of decisions where each decision is to determine the most economical alternative sequence of assets from the time of the decision to the horizon time.

Many presentations of replacement problems describe the situation of a single replacement decision that is assumed to occur some time after the startup time but before the horizon time. The decision presented typically involves a choice between either a defender or one of several alternative current challengers. The decision of course is only one in the unfolding sequence of decisions and the choice is really between the defender and its sequence of future challengers and one of the alternative current challengers and its sequence of future challengers. Taking into account these future challengers, their changing productivity, and their cash flows is what makes replacement problems difficult. It is precisely this aspect that classical assumptions simplify and make the problem less difficult to solve. Given this background we will now focus on some theoretical issues and the classical treatment.

The Classical Treatment

Assume we have a choice between a defender, j = 0, and a current challenger, j = 1. The relevant operation and maintenance expenses are summarized by a cash flow matrix $\underline{M}$ in which the j-th column is the operation and maintenance cash flow vector $\underline{m}_j$ for asset j = 0,1, and the n-th row contains operation and maintenance the cash flows m[j,n] after $n = 0,1,2,\ldots,\bar{n}_j$ periods of service of the asset, as shown below. Similarly, the salvage values are summarized in a matrix $\underline{V}$ in which the j-th column is the salvage value vector $\underline{v}_j$ for asset j, and the n-th row contains the salvage value v[j,n] after n periods of service of the asset.

	$\underline{M}$		$\underline{V}$	
	$\underline{m}_0$	$\underline{m}_1$	$\underline{v}_0$	$\underline{v}_1$
n	m[0,n]	m[1,n]	v[0,n]	v[1,n]
0	0	0	+45,000	+70,000
1	-24,250	-18,250	+22,500	+47,250
2	-24,625	-21,875	+11,250	+32,150
3	-30,970	-28,980	0	+21,000
$\bar{n}_j$ =4	-45,660	-40,000	0	+13,960

The figures for the defender represent n more periods of service and the salvage value for the current challenger for n = 0 represents its first cost, i.e., FC = -v[1,0]. We assume both assets have physical lives of $\bar{n}_j$ = 4 periods. The physical life is the time beyond which the asset does not provide satisfactory service and either cannot be restored or cannot be restored without major repair such that the rebuilt asset is for practical purposes a new asset with a new physical life. Several alternative current challengers could be considered here, j = 2,3,...,$\bar{j}$, as well as many other cash flow consequences affected by the choice (such as taxes); they would only increase the calculations and would not enhance the illustration.

The classical treatment can be summarized as follows and as shown in Table 1 [4][8][10][14]:

(1) Compute the net cash flows that describe each asset j for n = 1,2,...,$\bar{n}_j$ periods of service. The net cash flows for n periods of service for each current asset are summarized by a matrix $\underline{S}^n$ in which the j-th column is the net cash flow vector $\underline{s}_j^n$ for asset j, and the t'-th row contains the cash flows at time t'. Time t' represents the relative time in the life of asset j where t' = 0 is the time of installation and t' = n is the time of replacement. An element (cash flow) in the matrix is $s_n[j,t']$. We assume $s_n[j,t'] < 0$ for net disbursements and $s_n[j,t'] > 0$ for net receipts. For this example:

$$s_n[j,0] = -v[j,0] \qquad \text{for } t' = 0 \qquad (1)$$

$$s_n[j,t'] = m[j,t'] \qquad \text{for } t' = 1,\ldots,n-1 \qquad (2)$$

$$s_n[j,t'] = m[j,t'] + v[j,t'] \qquad \text{for } t' = n \qquad (3)$$

(2) Compute the equivalent uniform periodic cash flow EUCF($\underline{s}_j^n$) for $n = 1,2,\ldots,\bar{n}_j$ periods of service for each asset $j = 0,1$. An asset's economic life, n_j^*, is often defined as equal to the life associated with the asset's maximum EUCF value, $\hat{n}_j$. This definition is imprecise at best and inaccurate at worst. A more precise and accurate definition would be that it is the asset's service life in the most economical sequence of assets. Only under special circumstances will $n_j^* = \hat{n}_j$. Let $\underline{s}_j^{n*}$ represent the cash flow vector associated with n_j^* and $\underline{s}_j^{\hat{n}}$ represent the vector associated with $\hat{n}_j$. In our example, $\hat{n}_0 = \hat{n}_1 = 3$ periods using a discount rate of $r = 0.10$ per period as can be seen in Table 1.

(3) Keep the defender if EUCF($\underline{s}_0^{\hat{n}}$) > EUCF($\underline{s}_1^{\hat{n}}$), otherwise replace it with the current challenger if EUCF($\underline{s}_1^{\hat{n}}$) > EUCF($\underline{s}_0^{\hat{n}}$). In this example, both assets are equally economical because EUCF($\underline{s}_1^3$) = EUCF($\underline{s}_0^3$). The equal values of $\hat{n}_j$ was unintentional; unequal values would not affect the comparison or the decision.

Why Use EUCF? When Does n_j^* Occur at the Maximum EUCF?

The EUCF is perhaps the most common measure of worth used in the classical treatment because it requires the least amount of computation. Other equivalent but computationally more cumbersome measures of worth could be used and they would lead to the same decision if used correctly. The economic life of a current asset will occur at the maximum EUCF with the assumptions of repeatability and an infinite horizon time. Change either one or both of these assumptions and the economic life of the current asset will not necessarily correspond to the maximum EUCF for the current asset. These issues can be illustrated both mathematically and through our example.

Recall, the monetary objective is to choose the sequence of assets that promises to provide the most economical service, or equivalently, the sequence that maximizes NPV. We now define the following additional notation.

$\underline{A}^n$ is a matrix summarizing the net cash flows of infinite sequences of assets in which the j-th column is the net cash flow vector $\underline{a}_j^n$ for an infinite sequence beginning with current asset j placed in service n periods, and the t-th row contains the cash flows at time t = 0,1,2,...,∞ .

$NPV(\underline{a}_j^n)$ = $\underline{a}_j^n\underline{x}$ is the net present value of infinite sequence $\underline{a}_j^n$ where $\underline{x}$ is a (row) vector of discount factors $x[t] = 1/(1+r)^t$. We will let $\underline{a}_j^{n*}$ denote the sequence that maximizes NPV.

$NPV(\underline{s}_j^n)$ = $\underline{s}_j^n\underline{x}$ is the net present value of current asset j for n_j periods of service.

T=t-t' is the time an asset is installed, T=0, n_j, $2n_j$,...,∞ .

The NPV of an infinite sequence beginning with the current challenger placed in service n_1 periods followed by repeating identical future challengers, $\underline{a}_1^n$, can be written as follows.

$$NPV(\underline{a}_1^n) = \underline{a}_1^n\underline{x} \tag{4}$$

$$= \sum_{T=0}^{\infty} \underline{s}_1^n\underline{x}\cdot x[T] \tag{5}$$

$$= \sum_{T=0}^{\infty} NPV(\underline{s}_1^n)x[T] \tag{6}$$

$$= NPV(\underline{s}_1^n)[1+(1+r)^{-n_1}+(1+r)^{-2n_1}...+(1+r)^{-\infty n_1}]\frac{[(1+r)^{n_1}-1]}{[(1+r)^{n_1}-1]} \tag{7}$$

$$= NPV(\underline{s}_1^n)[(1+r)^{n_1}+1+(1+r)^{-n_1}+(1+r)^{-2n_1}...+(1+r)^{-(\infty-1)n_1} \tag{8}$$

$$- 1 -(1+r)^{-n_1}-(1+r)^{-2n_1}...-(1+r)^{-(\infty-1)n_1}-(1+r)^{-\infty n_1}]/[(1+r)^{n_1}-1]$$

$$= \quad NPV(\underline{s}_1^n)[(1+r)^{n_1}-(1+r)^{-\infty n_1}]/[(1+r)^{n_1} - 1] \tag{9}$$

$$= \quad EUCF(\underline{s}_1^n) \cdot \frac{[(1+r)^{n_1}-1]}{[r(1+r)^{n_1}]} \cdot \frac{[(1+r)^{n_1}-(1+r)^{-\infty n_1}]}{[(1+r)^{n_1}-1]} \tag{10}$$

$$= \quad EUCF(\underline{s}_1^n)/r \tag{11}$$

Equation (4) is the NPV of the infinite net cash flow series $\underline{a}_1^n$ and (5) expresses (4) as an infinite sequence of repeating challengers each described by identical cash flows. Only cyclic sequences of challengers that repeat every n_1 periods need to be considered when future challengers are assumed to be identical to the current challenger because if n_1 periods of service of the current challenger proved to be the most economical then it would be the most economical period of service for the future challengers too. The quantity in the brackets in (5) is the NPV of the current challenger for n_1 periods of service as shown in (6). Equations (7)-(11) are the result of algebraic manipulations. Equation (7) expands the summation in (6) and multiplies by unity in the form $[(1+r)^{n_1}-1]/[(1+r)^{n_1}-1]$. The numerator of the unity term is multiplied through (7) to produce (8) which is simplified to (9) because all except two terms in the bracketed numerator cancel. The NPV is replaced in (9) with the equivalent term $EUCF(\underline{s}_1^n)[(1+r)^{n_1}-1]/[r(1+r)^{n_1}]$ in (10) where the formula in the brackets here is the well known interest equivalence factor $(P/A,r,n_1)$, i.e., $EUCF(\underline{s}_1^n)(P/A,r,n_1)$. Equation (10) is simplified to (11) where $(1+r)^{-\infty n_1} = 0$.

It is apparent from equation (11) that the maximum value of $EUCF(\underline{s}_1^n)$, namely, $EUCF(\underline{s}_1^{\hat{n}})$, also maximizes $NPV(\underline{a}_1^n,)$, namely, $NPV(\underline{a}_1^{n^*})$. Consequently, the life associated with the maximum EUCF value, $\hat{n}_1$, is also the economic life of the current challenger, n_1^*, because that life and its EUCF value identifies the most economical <u>infinite</u> <u>sequence of challengers</u>. Thus, in general, whenever any current asset j is also the asset used to form the infinite sequence of identical repeating future assets then $n_j^* = \hat{n}_j$ is guaranteed. Consequently, a defender's economic life n_0^* is not necessarily equal to $\hat{n}_0$ because defenders are usually not assumed to repeat identically in the future. Thus, when multiple alternative current challengers are involved, $j = 2,3,\ldots,\bar{j}$, then $n_j^* = \hat{n}_j$ is guaranteed only for the alternative current challenger that proves to be most economical but not for the other less economical alternative current challengers since the current asset used to form the infinite sequence of identical repeating future assets that follows each current challenger should be the most economical current challenger. This can be illustrated easily for the defender as follows.

The NPV for n_0 periods of service of the defender and its subsequent infinite sequence of challengers can be computed as follows.

$$NPV(\underline{a}_0^n) = \underline{a}_0^n\underline{x} \tag{12}$$

$$= \underline{s}_0^n\underline{x} + \underline{a}_1^n\underline{x}\cdot x[n_0] \tag{13}$$

$$= NPV(\underline{s}_0^n) + NPV(\underline{a}_1^n)x[n_0] \tag{14}$$

$$= EUCF(\underline{s}_0^n)(P/A,r,n_0) + EUCF(\underline{s}_1^n)/r(1+r)^{n_0} \tag{15}$$

Although $EUCF(\underline{s}_1^{\hat{n}})$ would maximize the second term in (15), it is not necessarily true that the maximum value $EUCF(\underline{s}_0^{\hat{n}})$ maximizes $NPV(\underline{a}_0^n)$. Further calculations would be necessary to determine the EUCF value and economic life for the defender that maximize $NPV(\underline{a}_0^n)$. The calculations, however, are unnecessary if the decision maker wishes to know only whether the defender or current challenger is more economical. If $EUCF(\underline{s}_0^{\hat{n}}) > EUCF(\underline{s}_1^{\hat{n}})$, then without having to calculate any NPVs, it is apparent that $NPV(\underline{a}_0^{n*}) > NPV(\underline{a}_1^{n*})$ because $NPV(\underline{a}_0^{n*}) \geq NPV(\underline{a}_0^{\hat{n}}) > NPV(\underline{a}_1^{n*}) = NPV(\underline{a}_1^{\hat{n}})$ and the defender is the more economical asset. If $EUCF(\underline{s}_1^{\hat{n}}) > EUCF(\underline{s}_0^{\hat{n}})$, then $NPV(\underline{a}_1^{\hat{n}}) = NPV(\underline{a}_1^{n*}) > NPV(\underline{a}_0^{n*}) \geq NPV(\underline{a}_0^{\hat{n}})$ and the current challenger is the more economical choice. That is, it is clear from equations (11) and (15) that if $EUCF(\hat{\underline{s}}_0^n) > EUCF(\hat{\underline{s}}_1^n)$, then $NPV(\underline{a}_0^{n*}) > NPV(\underline{a}_1^{n*})$ and vice versa. The additional calculations would be necessary if the decision maker wanted to know the defender's economic life, or in the case of multiple alternative current challengers, the economic life of any of the current challengers except the most economical challenger. These calculations are illustrated in the next section.

Notice that the popular decision rule to select the current challenger when $EUCF(\hat{\underline{s}}_1^n) > EUCF(\hat{\underline{s}}_0^n)$ or the defender when $EUCF(\hat{\underline{s}}_0^n) > EUCF(\hat{\underline{s}}_1^n)$, is really an equivalent decision rule. It is equivalent to the more theoretically exact but computationally cumbersome decision rule to select the current challenger when $NPV(\underline{a}_1^{n*}) > NPV(\underline{a}_0^{n*})$ or the defender when $NPV(\underline{a}_0^{n*}) > NPV(\underline{a}_1^{n*})$. Other more theoretically exact decision rules are also possible, such as select the current challenger when $EUCF(\underline{a}_1^{n*}) > EUCF(\underline{a}_0^{n*})$. The measure of worth EUCF with its equivalent decision rule is easier to use because (1) the net cash flows that describe the alternative infinite sequences, $\underline{a}_0^n$ and $\underline{a}_1^n$, do not have to be computed explicitly (eqs. (4) and (12)), they can be computed implicitly with the cash flows that describe the defender and the current challenger (eqs. (5) and (13)), and (2) only alternative sequences involving cyclic replacement need to be considered (eqs. (6) and (14)).

What is n_j^* for Any Current Asset?

The economic life of a current asset, either the defender or an alternative current challenger, can be determined from the incremental net cash flows $\hat{\underline{a}}_j^n$ that describe the differences between the $\bar{n}_j$ alternative infinite sequences of assets that begin with current asset j in service $n = 1,2,3,\ldots,\bar{n}_j$ periods. For example, the incremental net cash flow for keeping the defender in service 2 periods rather than 1 is illustrated in Table 2. The top third of the table identifies the net cash flows $\underline{a}_0^1$ for 1 period of service of the defender and the middle third identifies the net cash flows $\hat{\underline{a}}_0^2$ for 2 periods of service. The lower third shows the incremental net cash flows $\hat{\underline{a}}_0^2 = \underline{a}_0^2 - \underline{a}_0^1$. Notice, in general, the incremental net cash flows $\hat{\underline{a}}_j^n$ can be viewed as comprised of two component incremental net cash flows, $\hat{\underline{a}}_j^n = \hat{\underline{s}}_j^n + \hat{\underline{s}}_i^{n*}$, where $\hat{\underline{s}}_j^n$ represents the incremental (marginal) cash flows of keeping the current asset j in service one more period from n-1 to n and $\hat{\underline{s}}_i^{n*}$ represents the incremental cash flows between most economical challenger repeating identically (every n_i^* periods) to the infinite future beginning at time n rather than time n-1. Because $NPV(\hat{a}_0^2) = +5,268 > 0$, keeping the defender in service two periods is more economical than one. Similar computations for other service lives for the defender yield, $NPV(\hat{a}_0^1) = -6,136$, $NPV(\hat{a}_0^3) = +868$ and $NPV(\hat{a}_0^4) = -792$, therefore, the economic life of the defender is 3 periods. Formally, then, the longest life n for which $NPV(\hat{\underline{a}}_j^n) \geq 0$ is the economic life $n_j^* = n$ for current asset j in the classical treatment.

Obviously, computing $\bar{n}_j$-1 incremental vectors $\underline{a}_j^n$ to determine an asset's economic life is rather cumbersome. One shortcut would be to use the $\bar{n}_j$ vectors $\underline{a}_j^n$ and compute $NPV(\underline{a}_j^n)$ for each. The life n associated with the maximum NPV would be the current asset's economic life, n_j^*, and of course it would be the same life as determined by the more theoretically exact but computationally cumbersome incremental method because $NPV(\hat{\underline{a}}_j^n) = NPV(\underline{a}_j^n) - NPV(\underline{a}_j^{n-1})$.

Another and simpler shortcut is to compute the current asset's marginal equivalent uniform cash flow [4][6][7]. This approach involves the incremental component cash flows $\hat{\underline{s}}_j^n$ and $\hat{\underline{s}}_i^{n*}$. The incremental NPV can be written as follows.

$$NPV(\hat{\underline{a}}_j^n) = \hat{\underline{a}}_j^n \underline{x} \tag{16}$$

$$= \hat{\underline{s}}_j^n \underline{x} + \hat{\underline{s}}_i^{n*} \underline{x} \tag{17}$$

$$= \hat{\underline{s}}_j^n \underline{x} + \underline{a}_i^{n*} \underline{x} \cdot x[n] - \underline{a}_i^{n*} \underline{x} \cdot x[n-1] \tag{18}$$

$$= \{\hat{s}_n[j,n-1]/x[1] + \hat{s}_n[j,n] + \underline{a}_i^{n*} - \underline{a}_i^{n*} \underline{x}/x[1]\}\ x[n] \tag{19}$$

$$= \{MEUCF(\hat{\underline{s}}_j^n) - \underline{a}_i^{n*} \underline{x} r\} \cdot x[n] \tag{20}$$

$$= \{MEUCF(\hat{\underline{s}}_j^n) - EUCF(\underline{s}_i^{n*})\} x[n] \tag{21}$$

Equation (16) is the NPV of the incremental cash flows $\hat{\underline{a}}_j^n$ and (17) expresses $\hat{\underline{a}}_j^n$ in terms of its incremental component cash flows. The second term in (17) is expanded in (18) in the form of the difference in the NPV of the infinite sequence of repeating challengers beginning at time n rather than n-1. The discount factor x[n] is factored out of (18) in (19), and equation (19) also shows the two non-zero cash flows that comprise $\hat{\underline{s}}_j^n$. Notice that the vector $\hat{\underline{s}}_j^n$ will always consist of only two non-zero elements (cash flows), which summarize the marginal economy of keeping the current asset j in service one more period from n-1 to n. The first two terms in (19) compute marginal (incremental) equivalent uniform cash flow at time n, MEUCF($\hat{s}_j^n$), for one more period of service of current asset j from n-1 to n as shown in (20). The second term in (20) is a simplification of the last two terms in (19) and can be replaced in (21) with EUCF($\underline{s}_i^{n*}$) because EUCF($\underline{s}_i^{n*}$) = EUCF($\underline{a}_i^{n*}$) = $\underline{a}_i^{n*}\underline{x}$r. Clearly from equation (21), if MEUCF($\hat{\underline{s}}_j^n$) > EUCF($\underline{s}_i^{n*}$) then NPV($\hat{\underline{a}}_j^n$) > 0. Therefore, an equivalent definition for a current asset's economic life would be that it is the longest life such that the asset's marginal equivalent uniform cash flow is greater than the maximum equivalent uniform cash flow of the most economical challenger. That is, the current asset should be kept in service as long as the marginal economy gained by keeping it in service one more period, MEUCF($\hat{\underline{s}}_j^n$), is greater than the marginal economy foregone, EUCF($\underline{s}_i^{n*}$), by not implementing the most economical sequence of future challengers one period sooner. This is the most computationally convenient method to determine n_j^* since only the net cash flows that describe each current asset are needed. It is important to remember though that this is an equivalent (shortcut) method to the more theoretically exact but computationally cumbersome method to compute NPV($\hat{\underline{a}}_j^n$).

It can be shown for the most economical current challenger j' that MEUCF($\hat{s}_{j'}^n$) $\geq$ EUCF($\underline{s}_{i=j'}^{n*}$) for $n_j \leq n_{j'}^*$ and MEUCF($\hat{\underline{s}}_{j'}^n$) < EUCF($\underline{s}_{i=j'}^{n*}$) for $n_j > n_{j'}^*$. Consequently, $n_{j'}^*$ will always occur at $\hat{n}_{j'}$, $n_{j'}^* = \hat{n}_{j'}$, the life that maximizes its

EUCF. However, for the defender, j = 0, and other alternative current challengers, $j \neq j'$, the current asset's economic life $n_{j'}^*$ will not necessarily be equal to $\hat{n}_j$. It is noteworthy to mention that if $EUCF(\underline{s}_j^{\hat{n}}) \geq EUCF(\underline{s}_{j'}^{n^*})$ then current asset j has an economic life between $\hat{n}_j \leq n_j^* \leq \bar{n}_j$, and if $EUCF(\underline{s}_j^{\hat{n}}) \leq EUCF(\underline{s}_{j'}^{\hat{n}^*})$ then $1 \leq n_j^* \leq \hat{n}_j$.

In our example, the values of $NPV(\hat{\underline{a}}_0^n)$, $NPV(\underline{a}_0^n)$ and $MEUCF(\hat{\underline{s}}_0^n)$ for the defender are as follows.

n	$NPV(\hat{\underline{a}}_0^n)$	$NPV(\underline{a}_0^n)$	$MEUCF(\hat{\underline{s}}_0^n)$
1	-6,136 < 0	-451,136	-51,250 < -44,500
2	+5,268 > 0	-445,868	-38,125 > -44,500
3	+ 868 > 0	-445,000	-43,345 > -44,500
4	- 792 < 0	-445,792	-45,660 < -44,500

Thus, $n_0^* = 3$. It is only a coincidence in our example that $n_0^* = \hat{n}_0$.

What if the Horizon is Finite? What if Challengers Don't Repeat?

If either the horizon time, H, is finite or if challengers do not repeat, or both, then alternative sequences of assets involving non-cyclic replacement must also be considered and (1) the equivalent decision rule for the EUCF cannot be used and (2) the economic life for any current asset does not necessarily occur at the maximum EUCF value. However, there is one exception. If challengers repeat and the economic life of the most economical current challenger $n_{j'}^*$ is a multiple of H as well as $H-n_0^*$ and $H-n_j^*$ for $j \neq j'$ then the solution will involve only cyclic replacement and the equivalent decision rule can be used.

Consider first the effect of only a finite horizon where H = 4. Table 3 displays all 16 alternative sequences of assets, eight begin with the defender and eight begin with the current challenger where we redefine $\underline{a}_j^n$ to $\underline{a}_j^k$ as an alternative net cash flow vector for a finite sequence k that begins with current asset j placed in service for at least one period, and the t-th row contains the cash flow $a_k[j,t]$. The most economical sequence beginning with the defender, $\underline{a}_0^1$, keeps the defender

4 periods and the most economical sequence beginning with the current challenger, $\underline{a}_1^3$, keeps the current challenger 2 periods. Notice that the economic lives of the defender and current challenger are now $n_0^* = 4$ and $n_1^* = 2$ respectively, whereas when $H = \infty$ they were $n_0^* = 3$ and $n_1^* = 3$.

Now assume the horizon is still H = 4 and that we expect changes in the productivity of the future challengers. Assume the relevant operation and maintenance expenses and salvage values for the future challengers j = 2, 3, 4 available at time T = 1,2,3 for $n = 0,1,\ldots,\bar{n}_j$ periods of service are as follows.

n	m[2,n]	m[3,n]	m[4,n]	v[2,n]	v[3,n]	v[4,n]
0	0	0	0	+60,000	+55,000	+50,000
1	-14,000	-11,500	-11,500	+36,000	+28,000	+23,000
2	-19,300	-20,000	-21,400	+17,000	+11,000	+9,500
3	-29,000	-31,555	-33,360	+9,665	0	0
$\bar{n}_j = 4$	-42,854	-44,820	-46,141	+9,665	0	0

The net cash flows that describe these challengers $\underline{s}_j^n$ are shown in Table 4, and the 16 alternative sequences are shown in Table 5. The most economical sequence beginning with the defender, $\underline{a}_0^3$, indicates the economic life of the defender has changed again and is now $n_0^* = 2$ periods, and the most economical sequence beginning with the current challenger, $\underline{a}_1^5$, indicates its economic life has also changed and is now $n_1^* = 1$ period.

In both examples above the most economical sequence could have been found using dynamic programming thereby obviating the need to identify all the alternative sequences explicitly [11]. All the alternative sequences were identified only for the sake of illustration.

What About the Salvage Value of the Defender?

The salvage value of the defender can be treated as either an initial disbursement to the defender (the conventional approach [3][4]) or as an initial receipt to the current challenger (the cash flow approach [3][4]) provided the subsequent economic analysis is performed correctly. Of course, to do both would be incorrect because it would double count the defender's salvage value. The classical treatment typically assumes the conventional approach, as was used in Table 1 previously, because it is computationally easier and the equivalent decision rule can be used. To illustrate the difference in the approaches, Table 6 is identical to Table 1 except that the cash flow approach has been used. If the equivalent decision rule was used it would suggest that the current challenger be installed for 1 period and replaced identically every year thereafter because $EUCF(\underline{s}_1^{\hat{n}=1}) > EUCF(\underline{s}_0^{\hat{n}=1})$. This would be incorrect because the comparison of EUCF values has the subtle implication that the salvage value of the defender today would be received each time a challenger is installed in the future. This of course will not occur since if the defender is sold today the salvage value will be received only today and never again in the future. This subtle implication can be seen easily by refering back to either equations (11) and (15) or to Table 2 and replacing the cash flows $\underline{s}_j^n$ shown by the cash flows in Table 6. With the cash flow approach, additional calculations would be required to explicitly counteract this implicit assumption. This can be avoided easily by simply using the computationally simplier conventional approach.

Summary

This paper showed mathematically and by example that: (1) the EUCF is the most common measure of worth used because it is the most convenient computationally for the classical treatment, (2) the economic life of a defender or an alternative current challenger is not necessarily the life that maximizes the asset's EUCF but rather the life of the asset in the sequence of assets that maximizes the NPV of the sequence, (3) the popular decision rule to select the current asset with the maximum EUCF is an equivalent decision rule to the more exact but cumbersome NPV decision rule, (4) computing the MEUCF of a current asset is an equivalent method to computing incremental NPV for the sequence to determine the asset's economic life, and (5) the conventional approach to the salvage value of the defender is computationally easier than the cash flow approach.

As mentioned previously, there are a variety of non-classical treatments that deal with replacement problems involving non-repeating challengers for either finite or infinite horizon problems. Although the classical assumptions occur infrequently in practice, the classical treatment nonetheless will remain an important methodology. The classical treatment is frequently the point of departure in an introductory study of replacement problems and it can be viewed as the foundation for non-classical treatments. Consequently, a thorough understanding of the classical treatment and its subtlties is important to understanding non-classical treatments and for developing new methodologies.

Table 1: Cash flows and EUCFs for the defender j = 0 and current challenger j = 1 for n = 1,2,3,4 periods of service.

	n	$s_n[0,0]$	$s_n[0,1]$	$s_n[0,2]$	$s_n[0,3]$	$s_n[0,4]$	$EUCF(\underline{s}_0^n)$
	1	-45,000	-1,750				-51,250
	2	-45,000	-24,250	-13,375			-45,000
$\hat{n}_0 =$	3	-45,000	-24,250	-24,625	-30,970		-44,500
	4	-45,000	-24,250	-24,625	-30,970	-45,660	-44,750

	n	$s_n[1,0]$	$s_n[1,1]$	$s_n[1,2]$	$s_n[1,3]$	$s_n[1,4]$	$EUCF(\underline{s}_1^n)$
	1	-70,000	+29,000				-48,000
	2	-70,000	-18,250	+10,275			-45,000
$\hat{n}_1 =$	3	-70,000	-18,250	-21,875	-7,980		-44,500
	4	-70,000	-18,250	-21,875	-28,980	-26,040	-45,500

Table 2: Net cash flows and incremental (marginal) cash flows for 2 alternative sequences involving the defender in service either 1 or 2 periods.

Cash Flows for 1 Period

t	0	1	2	3	4	5	6	7	8
n_0	$s_1[0,0]$	$s_1[0,1]$			$s_3[1,0]$	$s_3[1,1]$	$s_3[1,2]$	$s_3[1,3]$	
1	-45,000	-1,750			-70,000	-18,250	-21,875	-7,980	
		$s_3[1,0]$	$s_3[1,1]$	$s_3[1,2]$	$s_3[1,3]$			$s_3[1,0]$	$s_3[1,1]$...
	______	-70,000	-18,250	-21,875	-7,980	______	______	-70,000	-18,250...
	$a_1[0,0]$	$a_1[0,1]$	$a_1[0,2]$	$a_1[0,3]$	$a_1[0,4]$	$a_1[0,5]$	$a_1[0,6]$	$a_1[0,7]$	$a_1[0,8]$...
	-45,000	-71,750	-18,250	-21,875	-77,980	-18,250	-21,875	-77,980	-18,250...

Cash Flows for 2 Periods

n_0	$s_2[0,0]$	$s_2[0,1]$	$s_2[0,2]$			$s_3[1,0]$	$s_3[1,1]$	$s_3[1,2]$	$s_3[1,3]$...
2	-45,000	-24,250	-13,375		-70,000	-18,250	-21,875	-7,980	
			$s_3[0,1]$	$s_3[1,1]$	$s_3[1,2]$	$s_3[1,3]$			$s_3[1,0]$...
	______	______	-70,000	-18,250	-21,875	-7,980	______	______	-70,000...
	$a_2[0,0]$	$a_2[0,1]$	$a_2[0,2]$	$a_2[0,3]$	$a_2[0,4]$	$a_2[0,5]$	$a_2[0,6]$	$a_2[0,7]$	$a_2[0,8]$...
	-45,000	-24,250	-83,375	-18,250	-21,875	-77,980	-18,250	-21,875	-77,980...

Incremental Cash Flows

$s_2[0,0]$	$s_2[0,1]$	$s_2[0,2]$						
0	-22,500	-13,375						
	$s_3[1,1]$	$s_3[1,2]$	$s_3[1,3]$	$s_3[1,4]$	$s_3[1,5]$	$s_3[1,6]$	$s_3[1,7]$	$s_3[1,8]$...
______	+70,000	-51,750	+3,625	+56,105	-59,730	+3,625	+56,105	-59,730...
$a_2[0,0]$	$a_2[0,1]$	$a_2[0,2]$	$a_2[0,3]$	$a_2[0,4]$	$a_2[0,5]$	$a_2[0,6]$	$a_2[0,7]$	$a_2[0,8]$...
0	+47,500	-65,125	+3,625	+56,105	-59,730	+3,625	+56,105	-59,730...

Table 3: 16 Alternative sequences when H = 4 and current challenger repeats.

Seq. k	$a_k[0,0]$	$a_k[0,1]$	$a_k[0,2]$	$a_k[0,3]$	$a_k[0,4]$	NPV($\underline{a}_0^k$)
1	-45,000	-24,250	-24,625	-30,970	-45,660	-141,851
2	-45,000	-24,250	-24,625	-30,970		
				-70,000	+29,000	-143,450
3	-45,000	-24,250	-13,375			
			-70,000	-18,250	+10,275	-142,644
4	-45,000	-24,250	-13,375	-70,000	+29,000	
			-70,000	+29,000		-146,947
5	-45,000	-1,750				
		-70,000	-18,250	-21,875	-7,980	-147,195
6	-45,000	-1,750		-70,000	+29,000	
		-70,000	-18,250	+10,275		-150,375
7	-45,000	-1,750	-70,000	-18,250	+10,275	
		-70,000	+29,000			-150,805
8	-45,000	-1,750	-70,000	+29,000		
		-70,000	+29,000	-70,000	+29,000	-155,108

Seq. k	$a_k[1,0]$	$a_k[1,1]$	$a_k[1,2]$	$a_k[1,3]$	$a_k[1,4]$	NPV($\underline{a}_1^k$)
1	-70,000	-18,250	-21,875	-28,980	-26,040	-144,228
2	-70,000	-18,250	-21,875	-7,980		
				-70,000	+29,000	-143,450
3	-70,000	-18,250	+10,275			
			-70,000	-18,250	+10,275	-142,644
4	-70,000	-18,250	+10,275	-70,000	+29,000	
			-70,000	+29,000		-146,947
5	-70,000	+29,000				
		-70,000	-18,250	-21,875	-7,980	-144,241
6	-70,000	+29,000		-70,000	+29,000	
		-70,000	-18,250	+10,275		-147,420
7	-70,000	+29,000	-70,000	-18,250	+10,275	
		-70,000	+29,000			-147,851
8	-70,000	+29,000	-70,000	+29,000		
		-70,000	+29,000	-70,000	+29,000	-152,154

Table 4: Cash flows for future challengers j = 2,3,4 available at times T = 1,2,3, respectively.

n	$s_n[2,0]$	$s_n[2,1]$	$s_n[2,2]$	$s_n[2,3]$	$s_n[2,4]$
1	-60,000	+22,000			
2	-60,000	-14,000	-2,300		
3	-60,000	-14,000	-19,300	-19,335	
4	-60,000	-14,000	-19,300	-29,000	-33,189

n	$s_n[3,0]$	$s_n[3,1]$	$s_n[3,2]$	$s_n[3,3]$	$s_n[3,4]$
1	-55,000	-16,500			
2	-55,000	-11,500	-9,000		
3	-55,000	-11,500	-20,000	-31,555	
4	-55,000	-11,500	-20,000	-31,555	-42,820

n	$s_n[4,0]$	$s_n[4,1]$	$s_n[4,2]$	$s_n[4,3]$	$s_n[4,4]$
1	-50,000	+11,500			
2	-50,000	-11,500	-11,900		
3	-50,000	-11,500	-21,400	-33,360	
4	-50,000	-11,500	-21,400	-33,360	-46,141

Table 5: 16 Alternative sequences when H=4 and current challenger does not repeat.

Seq. k	$a_k[0,0]$	$a_k[0,1]$	$a_k[0,2]$	$a_k[0,3]$	$a_k[0,4]$	NPV($\underline{a}_0^k$)
1	-45,000	-24,250	-24,625	-30,970	-45,660	-141,851
2	-45,000	-24,250	-24,625	-30,970		
				-50,000	+11,500	-140,376
3	-45,000	-24,250	-13,375			
			-55,000	-11,500	-9,000	-138,341
4	-45,000	-24,250	-13,375	-50,000	+11,500	
			-55,000	+16,500		-140,868
5	-45,000	-1,750				
		-60,000	-14,000	-19,300	-19,335	-140,413
6	-45,000	-1,750		-50,000	+11,500	
		-60,000	-14,000	-2,300		-144,146
7	-45,000	-1,750	-55,000	-11,500	-9,000	
		-60,000	+22,000			-143,196
8	-45,000	-1,750	-55,000	+16,500		
		-60,000	+22,000	-50,000	+11,500	-145,723

Seq. k	$a_k[1,0]$	$a_k[1,1]$	$a_k[1,2]$	$a_k[1,3]$	$a_k[1,4]$	NPV($\underline{a}_1^k$)
1	-70,000	-18,250	-21,875	-28,980	-26,040	-144,228
2	-70,000	-18,250	-21,875	-7,980		
				-50,000	+11,500	-140,376
3	-70,000	-18,250	+10,275			
			-55,000	-11,500	-9,000	-138,341
4	-70,000	-18,250	+10,275	-50,000	+11,500	
			-55,000	+16,500		-140,868
5	-70,000	+29,000				
		-60,000	-14,000	-19,300	-19,335	-137,459
6	-70,000	+29,000		-50,000	+11,500	
		-60,000	-14,000	-2,300		-141,191
7	-70,000	+29,000	-55,000	-11,500	-9,000	
		-60,000	+22,000			-140,242
8	-70,000	+29,000	-55,000	+16,500		
		-60,000	+22,000	-50,000	+11,500	-142,769

Table 6: Cash flows and EUCFs for the defender j=0 and current challenger j=1 for n = 1,2,3,4 periods of service using the cash flow approach.

n	$s_n[0,0]$	$s_n[0,1]$	$s_n[0,2]$	$s_n[0,3]$	$s_n[0,4]$	$EUCF(\underline{s}_0^n)$
1	0	-1,750				-1,750
2	0	-24,250	-13,375			-19,071
3	0	-24,250	-24,625	-30,970		-26,405
4	0	-24,250	-24,625	-30,070	-45,660	-30,554

n	$s_n[1,0]$	$s_n[1,1]$	$s_n[1,2]$	$s_n[1,3]$	$s_n[1,4]$	$EUCF(\underline{s}_1^n)$
1	-25,000	+29,000				-1,500
2	-25,000	-18,250	+10,275			-19,071
3	-25,000	-18,250	-21,875	-7,980		-26,405
4	-25,000	-18,250	-21,875	-28,980	-26,040	-31,304

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