FATOU BIEBERBACH DOMAINS AND AUTOMORPHISMS OF $\mathbb{C}^k$ TANGENT TO THE IDENTITY

by

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The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.

Jules Henri Poincaré (1854-1912) French mathematician.

A mis padres

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CHAPTER I

Introduction

In the 1920's Fatou and Bieberbach proved the existence of proper subdomains of $\mathbb{C}^2$ that are biholomorphically equivalent to $\mathbb{C}^2$, nowadays known as Fatou-Bieberbach domains. Their examples were the basins of attraction of automorphisms of $\mathbb{C}^2$ with more than one fixed point. In fact, the basin of attraction of an *attracting* fixed point of an automorphism of $\mathbb{C}^k$ is always biholomorphic to $\mathbb{C}^k$, with the fixed point in the interior of the domain. A complete proof of this fact, as well as a variety of examples was given by Rosay and Rudin [Ro-Ru]. In the same article the authors posed several questions about Fatou-Bieberbach domains. In the recent years a lot of progress has been made towards solving some of these questions (see for example work by Peters and Wold[Pe-Wo], Peters [Pe], Stensønes [St], Wold [Wo1], [Wo2]). One of the most interesting remaining open questions is the following: Does there exist a Fatou-Bieberbach domain in $\mathbb{C}^2$ that avoids the set $\{zw = 0\}$? In the aforementioned paper, Rosay and Rudin proved that it is possible to avoid one complex line. It has also been proved by Green [Gr], that a non-degenerate image of $\mathbb{C}^2$ cannot avoid three complex lines. The answer for the case of two lines still remains open. As a step towards finding (counter)examples for open questions about Fatou-Bieberbach domains, it is natural to look for methods to construct such domains.

In 1997 Weickert [We1], and then Hakim [Hak1] found an entire new class of Fatou-Bieberbach domains. The construction goes as follows: Let F be an automorphism of $\mathbb{C}^k$ tangent to the identity at a fixed point p (i.e. $F(p) = p$ and $DF(p) = \mathrm{Id}$). Under some hypotheses on the 2-jet of the automorphism there will exist a *non-degenerate characteristic direction* $[v] \in \mathbb{P}^{k-1}$ (we should think of v as a vector in $\mathbb{C}^k$ and of $[v]$ as the natural image in $\mathbb{P}^{k-1}$). Define the basin of attraction of p along $[v]$ as follows:

$$\Omega_{p,[v]} = \{z \in \mathbb{C}^k \mid \lim_{n\to\infty} f^n(z) = p; \lim_{n\to\infty} [f^n(z)] = [v]\}.$$

Then depending on a certain index $r_{[v]}$ associated to this direction we have that Ω is biholomorphic to $\mathbb{C}^k$, and p is in the boundary of Ω.

One strategy to find a Fatou-Bieberbach domain that avoids two lines is to consider an automorphism of $\mathbb{C}^2$ for which the origin is a neutral fixed point and restricts to the identity on the coordinate axes. Then, if we could satisfy the extra assumptions needed in Hakim's theorem we could find a Fatou-Bieberbach domain attracted to the origin that avoid both axes. In this thesis we prove that, assuming a well-known conjecture, this will never be the case. In particular we prove that, under the assumptions of Hakim's theorem the number $r_{[v]}$ is always negative and therefore there will not be an open domain attracted to the origin.

Nonetheless, we prove that there exist automorphisms of $\mathbb{C}^2$ tangent to the indentity at a fixed point admitting basins of attraction along *degenerate* characteristic directions. These automorphisms, however, do not preserve both axes.

Using this approach we are able to find a Fatou-Bieberbach domain in $\mathbb{C}^2$ that avoids one axis and one image of $\mathbb{C}$ in $\mathbb{C}^2$ that is locally tangent to an arbitrarily high order to the other axis. We will make these statements more precise in Chapter 2.

The local dynamics of maps tangent to the identity at a fixed point have been studied with a lot of success in the last years by Weickert [We1], Hakim [Hak1], Abate [Ab1] and Abate-Bracci-Tovena [ABT]. Our work is also a contribution in this direction, because it features dynamical behavior that has not been seen before.

The plan of the thesis is as follows: In Chapter 2 we give background material and definitions as well as the precise statement of the theorems mentioned here. In Chapter 3, under the assumption of a well-known conjecture, we prove the non-existence of a Fatou-Bieberbach domain in $\mathbb{C}^* \times \mathbb{C}^*$ along a non-degenerate characteristic direction. In Chapter 4 we construct our example of a Fatou-Bieberbach domain attracted to the origin along a degenerate characteristic direction avoiding one axis and one injective image of the complex plane.

CHAPTER II

Background Information

2.1 Fatou-Bieberbach Domains

We are interested in basins of attractions of automorphisms F with a fixed point p. Let us introduce some standard notation:

Definition II.1. Let F be an automorphism of $\mathbb{C}^k$ with a fixed point p i.e. $F(p) = p$. Then:

- If all eigenvalues of the matrix $DF(p)$ have modulus less than 1, we say that the fixed point p is *attracting*.
- If all eigenvalues of the matrix $DF(p)$ have modulus greater than 1, we say that the fixed point p is *repelling*.
- If $DF(p) = \mathrm{Id}$, we say that the fixed point p is *neutral*. In this case we say that F is *tangent to the identity at* p. Sometimes we will just say F is *tangent to the identity* when the choice of the fixed point p is obvious.

As we mentioned in the introduction we are interested in the basin of attraction of the fixed point p for F.

Definition II.2. Let F be an automorphism of $\mathbb{C}^k$ with a fixed point p i.e. $F(p) = p$.

Then we define the basin of attraction of F at p as follows:

$$\Omega_{F,p} = \{z \in \mathbb{C}^k \mid \lim_{n\to\infty} F^n(z) = p\} \tag{2.1}$$

where $F^n = F \circ F \circ \dots F$, n times.

Later we will see that this definition is appropriate in the case of an automorphim with an attracting fixed point. For an automorphism tangent to the identity we will change the definition slightly so the basin will be connected.

Definition II.3. A *Fatou-Bieberbach domain* in $\mathbb{C}^k$ is a proper subset of $\mathbb{C}^k$ that is biholomorphically equivalent to $\mathbb{C}^k$.

We would like to give some piece of history on this definition, and we reproduce the following quotation from the paper by Dixon and Esterle [Di-Es]:

> It is well known that there exists a one-to-one entire function $F : \mathbb{C}^p \to \mathbb{C}^p$, whose jacobian identically equals 1 but whose range is not dense in $\mathbb{C}^p$. Such a function was constructed by Bieberbach in [Bie] for $p = 2$.
>
> Another example of a nondegenerate entire function from $\mathbb{C}^2$ into itself whose range is not dense was given previously by Fatou in [Fa2] (but, despite some claims of the contrary, Fatou's function is not one-to-one, see ch. 6). In fact this phenomenon was explicitly pointed out by Poincaré in [Po, p.333] forty years before Bieberbach.

A very clear exposition of the construction as well as examples of these domains has been given in the paper by Rosay and Rudin [Ro-Ru]. Here we give the main theorem about Fatou-Bieberbach domains in this paper [Ro-Ru, Thm 1.1]:

Theorem II.4. *Suppose that $F \in Aut(\mathbb{C}^k)$ fixes a point p and that the eigenvalues of $DF(p)$ are all less than 1 in modulus. Then $\Omega_{F,p}$ as defined in (2.1) is biholomorphic* *to $\mathbb{C}^k$. In particular if F has another fixed point q then $\Omega_{F,p}$ is a Fatou Bieberbach Domain.*

Using this theorem, Rosay and Rudin construct different kind of examples of Fatou-Bieberbach domains with different properties. One of these examples is the a Fatou-Bieberbach domain Ω in $\mathbb{C}^2$ which does not intersect the line $\{(0, w) : w \in \mathbb{C}\}$. In [Gr] Green proves that a non-degenerate holomorphic map from $\mathbb{C}^2$ cannot miss three or more complex lines.

Rosay and Rudin then pose a list of several open questions, several of which have been answered in the last years. One of the remaining open questions is the following:

Question II.5. *Is there a biholomorphic map from $\mathbb{C}^2$ into the set $\{zw \neq 0\}$, i.e. into the complement of the union of two intersecting complex lines?*

This question is the initial point of inspiration for the work in this thesis.

2.2 Automorphisms tangent to the Identity

In one complex variable, the Leau-Fatou flower theorem says that if f is a germ of a holomorphic function at zero of the form

$$f(z) = \lambda z + O(|z|^2)$$

with λ a root of unity, then f is either linear or has a basin of attraction to the origin; that is, an open subset U, with $0 \in \partial U$, such that $f^n(z) \to 0$ locally uniformly as $n \to \infty$ for $z \in U$ (see Milnor [Mi] for a complete proof).

In $\mathbb{C}^n$, $n \geq 2$ the situation is more complicated. We consider maps tangent to the identity at the origin. That is:

$$F(z) = z + O(|z|^2)$$

Assume that $F \neq \mathrm{Id}$. Then F may or may not have an open domain of attraction to the origin. An easy example of the former situation (F has a domain of attraction) is a product map $F(z,w) = (f(z), f(w))$, with f as above; and an example of the latter situation is a shear map $F(z,w) = (z, w+g(z))$, with $g(0) = 0$. An interesting question, however, is whether an automorphism of $\mathbb{C}^n$ tangent to the identity can have such a domain, and if so, whether that domain is biholomorphic to $\mathbb{C}^n$. These questions are motivated by the classical construction of Fatou-Bieberbach domains. As we have seen in the past section Fatou-Bieberbach domains are constructed for $n \geq 2$ as basins of attraction to attracting fixed points. The fixed points we are considering are clearly not attracting but neutral. Weickert [We1] and Hakim [Hak1] provided some positive answers to these questions.

From now on, all the maps we consider have the origin as the fixed point, and when we say tangent to identity we mean it is tangent to the identity at the origin. Then Weickert proved [We1, Thm. 1]:

Theorem II.6. *There exist automorphisms of $\mathbb{C}^2$ tangent to the identity with an invariant domain of attraction to the origin, biholomorphic to $\mathbb{C}^2$, on which the automorphism is biholomorphically conjugate to the map*

$$(x, y) \to (x - 1, y)$$

Later Hakim [Hak1] proved a more general result. In order to state her result we need to introduce some definitions.

Definition II.7. A *parabolic curve* or an *invariant piece of curve* for F at the origin is an injective holomorphic map $\phi : \Delta \to \mathbb{C}^n$ satisfying the following properties:

- Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;
- ϕ is continuous at the origin, and $\phi(0) = O$;

- $\phi(\Delta)$ is invariant under F, and $\left(F^n|_{\phi(\Delta)}\right) \to O$ as $n \to \infty$.

Furthermore, if $[\phi(\zeta)] \to [v] \in \mathbb{P}^{n-1}$ as $\zeta \to 0$ (where $[.]$ is the projection of $\mathbb{C}^n \backslash \{O\}$ onto $\mathbb{P}^{n-1}$), we say that ϕ *is tangent to* $[v]$ *at the origin.* Writing

$$F(z) = z + P_k(z) + P_{k+1}(z) + ...$$

where for each $h \in \mathbb{N}, h \geq k$, P_h is an homogeneous polynomial function of degree h from $\mathbb{C}^n$ to $\mathbb{C}^n$, and $P_k \not\equiv 0$. Then $k = \nu(F)$ is the *order* of F.

A *characteristic direction* for F is a vector $[v] \in \mathbb{P}^{n-1}$ such that there is $\lambda \in \mathbb{C}$, so that $P_k(v) = \lambda v$. If $\lambda \neq 0$, we say that v is *non-degenerate*; otherwise, it is *degenerate.* Note that $[v]$ is a non-degenerate characteristic direction if the induced map of P_k on $\mathbb{P}^{n-1}$ has $[v]$ as a fixed point.

Then Hakim proved the following result [Hak1, Thm. 1.3]:

Theorem II.8. *Let F be a germ of an analytic transformation from $\mathbb{C}^k$ fixing the origin and tangent to the identity. For every nondegenerate characteristic direction $[v]$ of F, there exists a parabolic curve, tangent to $[v]$ at zero, attracted to the origin.*

This theorem is in some sense the analogue of the Fatou-Leau flower theorem in several dimensions. Later, Abate [Ab1] generalized these result for the case of dimension 2. He proved the following [Ab1, Thm.0.4]:

Theorem II.9. *Let F be a germ of an analytic transformation in $\mathbb{C}^2$ fixing the origin and tangent to the identity, with an isolated fixed point. Then F admits at least one parabolic curve tangent to some direction.*

In order to have *open* regions attracted to the origin though, we need to look at one more invariant associated to the non-degenerate characteristic direction. We follow Abate's exposition [Ab2] in order to define this invariant, because it is more illustrative than the original definition:

Definition II.10. Given F a germ of an analytic transformation of $\mathbb{C}^k$ fixing the origin and tangent to the identity, and a non-degenerate characteristic direction $[v] \in \mathbb{P}^{k-1}$, the eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \in \mathbb{C}$ of the linear operator $D(P_k)_{[v]} - \mathrm{Id} : T_{[v]}\mathbb{P}^{k-1} \to T_{[v]}\mathbb{P}^{k-1}$ are the *directors* of $[v]$.

Hakim also proves [Hak1]:

Theorem II.11. *Let F be an automorphism of $\mathbb{C}^2$ tangent to the identity. Let $[v]$ be a non-degenerate characteristic direction at 0. Assume that the real parts of all the directors of $[v]$ are strictly positive. Let Ω be defined by*

$$\Omega = \{z \in \mathbb{C}^k \backslash \{0\} : \lim_{n\to\infty} z_n = 0, \lim_{n\to\infty} [z_n] = [v]\}$$

Then Ω is biholomorphic to $\mathbb{C}^2$. In this domain the automorphism is biholomorphically conjugate to the map

$$(x, y) \to (x - 1, y)$$

It turns out that the domain of Weickert's example is the set $\Omega_{(0,V)}$ for a non-degenerate characteristic direction V associated to a positive director.

Although these are very interesting results, they do not say anything for the case of degenerate characteristic directions. In this thesis we prove:

Theorem II.12. *There exist automorphisms of $\mathbb{C}^2$ tangent to the identity with an invariant domain of attraction to the origin along a degenerate characteristic direction, biholomorphic to $\mathbb{C}^2$, on which the automorphism is biholomorphically conjugate to the map*

$$(x, y) \to (x - 1, y)$$

This will be an immediate consequence of Theorem II.15.

2.3 Automorphisms of $\mathbb{C}^* \times \mathbb{C}^*$

Related to the question of the existence of a biholomorphic map from $\mathbb{C}^2$ into the set $\{zw \neq 0\}$ is the following conjecture about $\mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$:

Conjecture II.13. *If $F \in \mathrm{Aut}(\mathbb{C}^* \times \mathbb{C}^*)$, then F preserves the form:*

$$\frac{dz \wedge dw}{zw}$$

One natural approach, in order to answer the question posed by Rosay and Rudin positively, is the following:

Find an automorphism F of $\mathbb{C}^2$ such that:

- *F is tangent to the identity i.e. $F(0) = 0$ and $DF(0) = \mathrm{Id}$.*
- *F fixes the coordinate axes i.e. $F(z, 0) = (f(z), 0)$ and $F(0, w) = (0, g(w))$. We will see later that this in fact implies $F(z, 0) = (z, 0)$ and $F(0, w) = (0, w)$.*

And, moreover:

(a) There exists an attracting fixed point for F.

or

(b) There exists a non-degenerate characteristic direction v of F at the origin such that $ReA(v) > 0$, where $A(v)$ is the director associated to the direction v.

Then, in the case of (a), using Theorem II.4 we would have a basin of attraction associated to this fixed point, or in the case of (b) we would a basin associated to v (as in Hakim's notation) by Theorem II.11 above. In either case, this basin Ω would be a Fatou Bieberbach domain and $\Omega \subset \{zw \neq 0\}$.

In the next chapter we prove the following:

Theorem II.14. *If $F : \mathbb{C}^2 \to \mathbb{C}^2$ is an automorphism such that:*

(i) F fixes both axes.

(ii) F is tangent to the identity.

(iii) F preserves the form $\dfrac{dz \wedge dw}{zw}$.

Then neither (a) nor (b) can occur.

In chapter 4, we will use a particular map tangent to the identity to prove the following:

Theorem II.15. *There exists an automorphism F of $\mathbb{C}^2$ tangent to the identity such that there is an invariant domain Ω in which every point is attracted to the origin along a trajectory tangent to v, where v is a degenerate characteristic direction of F on which the automorphism is biholomorphically conjugate to the map*

$$(x, y) \to (x - 1, y).$$

Moreover, Ω is a Fatou-Bieberbach Domain and $\Omega \cap \{(0, w) : w \in \mathbb{C}\} = \emptyset$. Furthermore, there is a biholomorphic copy of $\mathbb{C}$ injected in $\mathbb{C}^2$, locally tangent to the z-axis, that is entirely contained in the boundary of Ω.

CHAPTER III

Basins along non-degenerate characteristic directions

In this section we prove Theorem II.14.

If F is an automorphism of $\mathbb{C}^2$ that fixes the coordinate axes, then $F|_{\mathbb{C}^*\times\mathbb{C}^*}$ is an automorphism of $\mathbb{C}^* \times \mathbb{C}^*$. The automorphism group of $\mathbb{C}^* \times \mathbb{C}^*$ has been studied before by Nishimura [Ni], Varolin [Var], Andersen [An], but remains mysterious.

Proposition III.1. *Assume Conjecture II.13 holds and let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be an automorphism of $\mathbb{C}^2$ such that:*

1. $F(0,0) = (0,0)$*,* $F'(0,0) = \mathrm{Id}$
2. *F fixes the coordinate axes $\{zw = 0\}$.*

Then we can write F as follows:

$$F(z,w) = (ze^{wg(z,w)}, we^{zh(z,w)}) \tag{3.1}$$

where the power series of g and h are on the form:

$$g(z,w) = \sum_{\alpha+\beta\geq k} c_{\alpha,\beta} z^\alpha w^\beta$$
$$h(z,w) = \sum_{\alpha+\beta\geq k} d_{\alpha,\beta} z^\alpha w^\beta$$

and

$$c_{\alpha-1,\beta} = -\frac{\alpha}{\beta} d_{\alpha,\beta-1} \tag{3.2}$$

for $k \leq \alpha + \beta \leq 2k$.

Proof. An easy computation shows that $F(z,0) = (z,0)$ because $F|_{(z,0)}$ is an automorphism of $\mathbb{C}$ and F is tangent to the identity. The same argument implies $F(0,w) = (0,w)$. If we write $F(z,w) = (f_1(z,w), f_2(z,w))$ then we have: $f_1(z,0) = z$ and $f_2(z,0) = 0$. In the same way $f_1(0,w) = 0$ and $f_2(0,w) = w$. So we get $f_1(z,w) = z + zwr(z,w)$ and $f_2(z,w) = w + zws(z,w)$. So, we can write:

$$g(z,w) := \frac{\log(1 + wr(z,w))}{w}$$

and

$$h(z,w) := \frac{\log(1 + zs(z,w))}{z}$$

where g and h will be well defined function in $\mathbb{C}^2$ and therefore (3.1) follows. If $F(z,w) = (ze^{wg(z,w)}, we^{zh(z,w)})$ preserves the form $\frac{dz}{z} \wedge \frac{dw}{w}$ then we have:

$$\frac{d(ze^{wg(z,w)})}{ze^{wg(z,w)}} \wedge \frac{d(we^{zh(z,w)})}{we^{zh(z,w)}} = \frac{dz}{z} \wedge \frac{dw}{w}$$

And from here we obtain

$$d(ze^{wg(z,w)}) \wedge d(we^{zh(z,w)}) = e^{zh+wg} dz \wedge dw$$

Differentiating and comparing terms:

$$e^{wg}[(1 + zwg_z)dz + z(g + wg_w)dw] \wedge e^{zh}[w(h + zh_z)dz + (1 + zwh_w)dw] = e^{zh+wg} dz \wedge dw$$

$$(1 + zwg_z)(1 + zwh_w) - zw(g + wg_w)(h + zh_z) = 1$$

The following relationship between g, h and their derivatives follows:

$$g_z + h_w - gh - zgh_z - whg_w - zwg_wh_z + zwg_zh_w = 0 \tag{3.3}$$

Suppose that the lowest homogeneous degree in the power series of g is k, i.e. there exists terms of the form $c_{\alpha,\beta} z^\alpha w^\beta$ where $\alpha + \beta = k$ where $c_{\alpha,\beta} \neq 0$, and the lowest homogeneous degree term in the power series of h is l, i.e. there exists a term of the form $d_{\alpha,\beta} z^\alpha w^\beta$ where $\alpha + \beta = l$ and $d_{\alpha,\beta} \neq 0$. Without loss of generality suppose $k \leq l$. Looking at the terms of degree $k-1$ in the equation (3.3) we see that $k = l$. Notice that every term of the equation (3.3) has degree greater or equal than $2k$ then, except the first two terms. Fix α and β such that $k \leq \alpha + \beta \leq 2k$. The only terms of the form $z^{\alpha-1} w^{\beta-1}$ in the equation (3.3) are:

$$\beta c_{\alpha-1,\beta} + \alpha d_{\alpha,\beta-1} = 0$$

and we obtain (3.2). □

In order to compute the director of $[v]$, we will use the original definition of the director as in Hakim's definition. (This definition and the one we gave before are the same).

Let us assume $v = (1, u_0)$ is a non-degenerate characteristic direction, where $F = \mathrm{Id} + (P_k, Q_k) + h.o.t.$ If we define:

$$r(u) := Q_k(1,u) - uP_k(1,u)$$

then u_0 is a zero of the polynomial function r.

To each non-degenerate characteristic direction we associate the number:

$$A(v) := \frac{r'(u_0)}{P_k(1,u_0)} \tag{3.4}$$

This number is the director of $[v]$, and now we can use Theorem II.11.

With this in mind we prove Theorem II.14.

Proof of Thm. II.14. We will first prove that there are no attracting fixed points. If we assume F preserves the form $\dfrac{dz \wedge dw}{zw}$ and equation (3.1), an easy computation shows that the Jacobian determinant of F will be:

$$\det JF(z,w) = e^{wg(z,w) + zf(z,w)} \tag{3.5}$$

The fixed points for F are:

- $(0,0)$ where $DF(0,0) = \mathrm{Id}$, therefore the origin is not attracting.
- $(z_0, 0)$ and $(0, w_0)$. For these points an easy computation shows that they are semi-attracting (or semi-repelling) fixed points (i.e. the eigenvalues are 1 and λ); therefore not attracting either.
- (z_0, w_0) not necessarily on the axes, where $e^{w_0 g(z_0,w_0)} = e^{z_0 h(z_0,w_0)} = 1$. Using equation (3.4) we have: $JF(z_0, w_0) = 1$, therefore (z_0, w_0) will not be an attracting fixed point either (in case $DF(z_0, w_0) = \mathrm{Id}$ see Remark III.2).

Now we prove that (b) is not possible. We assume that the lowest degree in g (and therefore in h) is k. Then we have:

$$g(z,w) = \sum_{\alpha+\beta=k} c_{\alpha,\beta} z^\alpha w^\beta + h.o.t.$$

and

$$h(z,w) = \sum_{\alpha+\beta=k} d_{\alpha,\beta} z^\alpha w^\beta + h.o.t.$$

where

$$d_{\alpha-1,\beta} = -\frac{\alpha}{\beta} c_{\alpha,\beta-1}$$

for $k \le \alpha + \beta \le 2k$. Therefore the order of F is $k+2$:

$$\begin{aligned}
F(z,w) &= (ze^{wg(z,w)}, we^{zh(z,w)}) \\
&= \big(z(1 + wg(z,w) + O(w^2 g^2)), w(1 + zh(z,w) + O(z^2 h^2))\big) \\
&= \left(z + zw \sum_{\alpha+\beta=k} c_{\alpha,\beta} z^\alpha w^\beta + O(|(z,w)|^{k+3}), w + zw \sum_{\alpha+\beta=k} d_{\alpha,\beta} z^\alpha w^\beta + O(|(z,w)|^{k+3})\right)
\end{aligned}$$

We call the lowest degree homogeneous terms P_{k+2} and Q_{k+2} as in Hakim's notation:

$$P_{k+2}(z,w) = \sum_{\alpha+\beta=k} c_{\alpha,\beta} z^{\alpha+1} w^{\beta+1} \tag{3.6}$$

$$Q_{k+2}(z,w) = \sum_{\alpha+\beta=k} d_{\alpha,\beta} z^{\alpha+1} w^{\beta+1} \tag{3.7}$$

Now we want to compute characteristic directions and the directors associated to the non-degenerate ones. (Note that we have: $(P_{k+2}(1,0), Q_{k+2}(1,0)) = (0,0)$ and $(P_{k+2}(0,1), Q_{k+2}(0,1)) = (0,0)$; therefore $(1,0)$ and $(0,1)$ are degenerate characteristic directions).

From now on we assume that $v = (1,\theta)$ is a non-degenerate characteristic direction, with $\theta \neq 0$. This is equivalent to solve:

$$r(u) = Q_{k+2}(1,u) - uP_{k+2}(1,u) = 0$$

where the solutions will be $r(\theta) = 0$. Putting this back in (3.5) and (3.6) we get:

$$\begin{aligned} r(u) &= u\left(\sum_{\beta=0}^{k} d_{k-\beta,\beta} u^{\beta} - \sum_{\beta=0}^{k} c_{k-\beta,\beta} u^{\beta+1}\right) \\ &= u\left(d_{k,0} + \sum_{\beta=1}^{k} (d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1}) u^{\beta} - c_{0,k} u^{k+1}\right) \end{aligned}$$

Therefore the characteristic directions will be $(1,\theta)$ where $s(\theta) = 0$ for $r(u) = us(u)$ i.e.

$$s(u) = d_{k,0} + \sum_{\beta=1}^{k} (d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1}) u^{\beta} - c_{0,k} u^{k+1}$$

and so we have:

$$s(\theta) = d_{k,0} + \sum_{\beta=1}^{k} (d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^{\beta} - c_{0,k} \theta^{k+1} = 0 \tag{3.8}$$

We can compute the directors associated to each direction with the following formula:

$$A(\theta) = \frac{r'(\theta)}{P_{k+2}(1,\theta)} \tag{3.9}$$

Since we know:

$$r(u) = d_{k,0}u + \sum_{\beta=1}^{k}(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})u^{\beta+1} - c_{0,k}u^{k+2}$$

then we can easily get:

$$r'(u) = d_{k,0} + \sum_{\beta=1}^{k}(\beta+1)(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})u^{\beta} - (k+2)c_{0,k}u^{k+1}$$

Putting this back in (3.9) together with (3.6), we have:

$$A(\theta) = \frac{d_{k,0} + \sum_{\beta=1}^{k}(\beta+1)(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})\theta^{\beta} - (k+2)c_{0,k}\theta^{k+1}}{\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta+1}} \tag{3.10}$$

(When $k = 0$ the sum from $\beta = 1$ to k is the empty sum.)

By (3.8) we have:

$$d_{k,0} = -\sum_{\beta=1}^{k}(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})\theta^{\beta} + c_{0,k}\theta^{k+1} \tag{3.11}$$

and putting this in (3.10) we can have some cancelations:

$$\begin{aligned} A(\theta) = \frac{1}{\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta+1}}[-\sum_{\beta=1}^{k}(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})\theta^{\beta} + c_{0,k}\theta^{k+1} + \\ + \sum_{\beta=1}^{k}(\beta+1)(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})\theta^{\beta} - (k+2)c_{0,k}\theta^{k+1}] \end{aligned}$$

After simplifying and canceling θ (which is not equal to 0), we obtain:

$$\begin{aligned} A(\theta) &= \frac{\sum_{\beta=1}^{k} \beta(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})\theta^{\beta} - (k+1)c_{0,k}\theta^{k+1}}{\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta+1}} \\ &= \frac{\theta\left(\sum_{\beta=1}^{k} \beta(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})\theta^{\beta-1} - (k+1)c_{0,k}\theta^{k}\right)}{\theta\left(\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta}\right)} \\ &= \frac{\sum_{\beta=1}^{k} \beta(d_{k-\beta,\beta} - c_{k-\beta+1,\beta-1})\theta^{\beta-1} - (k+1)c_{0,k}\theta^{k}}{\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta}} \qquad (3.12) \end{aligned}$$

For all the automorphism for which the conjecture II.13 holds true, we have:

$$d_{\alpha-1,\beta} = -\frac{\alpha}{\beta} c_{\alpha,\beta-1}$$

for $k \leq \alpha + \beta - 1 \leq 2k$. So, using $\alpha = k + 1 - \beta$ we have:

$$d_{k-\beta,\beta} = -\frac{k - \beta + 1}{\beta} c_{k-\beta+1,\beta-1}$$

(For $k = 0$ the condition is empty, but the result still holds). Back in our equation (3.12):

$$\begin{aligned} A(\theta) &= \frac{\sum_{\beta=1}^{k} \beta(-\frac{k-\beta+1}{\beta} c_{k-\beta+1,\beta-1} - c_{k-\beta+1,\beta-1})\theta^{\beta-1} - (k+1)c_{0,k}\theta^{k}}{\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta}} \\ &= \frac{-(k+1)\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta}}{\sum_{\beta=0}^{k} c_{k-\beta,\beta}\theta^{\beta}} \\ &= -(k+1) \end{aligned}$$

This finishes the proof of Theorem II.14. $\square$

Remark III.2. If (z_0, w_0) is a fixed point different from the origin, where $DF(z_0, w_0) = \mathrm{Id}$, then we can use the same technique to show that the result above still holds i.e. non-degenerate characteristic directions at (z_0, w_0) will have a negative number associated to them.

CHAPTER IV

Basins along degenerate characteristic directions

In this section we prove Theorem II.15. The proof of the theorem is similar in spirit to the proof of Weickert's Theorem (Theorem II.6). We first explain the general structure of both proofs. In the remarks that follow we point out the major differences between the case of non-degenerate characteristic directions and that of degenerate directions.

Given an automorphism F of $\mathbb{C}^2$ tangent to the identity at the origin. The steps we follow to show that the region of attraction is a Fatou-Bieberbach domain are:

1. First we find a domain D such that $F(D) \subset D$ and $0 \in \partial D$. Also we prove that for any $z \in D, F^n(z) \to 0$. Moreover, $[F^n(z)] \to [v]$ where $[v] \in \mathbb{P}^1$ is a *degenerate* characteristic direction.
2. We work locally in D and find an Abel-Fatou coordinate. That is, $\phi : D \to \mathbb{C}$ such that $\phi(F(p)) = \phi(p) - 1$.
3. A global basin of attraction Ω is obtained as $\Omega = \cup_{n=0}^{\infty} F^{-n}(D)$. We extend ϕ to all of Ω. We prove that ϕ is surjective onto $\mathbb{C}$ (this normally comes for free with the use of the Abel-Fatou coordinates).
4. After this we prove that $V_t = \phi^{-1}(t)$ is biholomorphic to $\mathbb{C}$ for each t. We call the biholomorphism $\psi_t : V_t \to \mathbb{C}$.

5. Using ϕ and ψ_t, we construct a global map $G = (\phi, \psi) : \Omega \to \mathbb{C}^2$. We prove G is a biholomorphism, i.e. G is injective and surjective.

Note that many of these steps could also be carried out for a germ of a biholomorphism. Therefore we find explicit examples of germs for which there exists an open basin along a degenerate characteristic direction. This phenomenon has already been explored by Abate in [Ab2], where he gives an example of a germ of $\mathbb{C}^2$ with $(0, 1)$ a degenerate characteristic direction and an open basin along $(0, 1)$. See case $1_{(10)}$ Page 8.

The major difference between the case of degenerate and non-degenerate characteristic direction is in Step 2. The condition of being a non-degenerate characteristic direction allows us to make a simple change coordinates in order to prove Step 2. In the general case, this change of coordinates does not always exist. In our case we prove the existence of ϕ by solving a differential equation. We remark that given Step 1 through 4, Step 5 is exactly the same as in Weickert's case.

We should point out that for most of these steps to be completed we need a very good estimate of the size of the change of variables. The difficulty of proving surjectivity depends in general on how the shape of our domain is changing with the change of coordinate.

The presentation of the proof is as follows. We start with an automorphism of $\mathbb{C}^2$ tangent to the identity. Then each section, from section 4.1 through section 4.5 will be used to prove steps 1 through 5 of the outline of the proof. We also prove the existence of an F-invariant curve Γ biholomorphic to $\mathbb{C}$ and contained in $\partial\Omega$, where Ω is the Fatou-Bieberbach domain. In Section 4.1 (Proposition IV.4) we obtain estimates for the size of this curve in a local neighborhood of the origin. In Section 4.3 (Proposition IV.16) we prove that the curve Γ is indeed in the boundary of Ω.

4.1 Invariant Domain

We first describe a class of automorphism where we are able to find a basin.

Theorem IV.1. *Let F be an automorphism tangent to the identity at the origin of the form:*

$$F(z,w) = (z + z^2 + O(z^3, z^2w), w - zw^2 + O(z^4, z^3w, z^2w^2, zw^3)). \tag{4.1}$$

Then the vector $(0,1)$ *is a degenerate characteristic direction for F. Moreover, there exists a basin of attraction D, for which every point $p \in D$ is attracted to the origin along* $(0,1)$.

It is not obvious that there exist automorphisms of $\mathbb{C}^2$ that have a germ as in (4.1). Note that Buzzard-Forstneric [Bu-Fo] and Weickert [We2] have proven that given a finite jet germ of a biholomorphic map, it is possible to construct a global automorphism with the prescribed jet. However, this theorem is not enough for our purposes, since we want to prescribe *which* higher order terms can appear. Specifically we want $F(0,w) = (0,w)$.

Below we construct automorphisms of the form (4.1) using a composition of shears and overshears.

Lemma IV.2. *There exists $F \in Aut(\mathbb{C}^2)$ such that:*

$$F(0) = 0$$

$$F'(0) = \mathrm{Id}$$

and

$$F(z,w) = (z + z^2 + O(z^3, z^2w), w - w^2z - \frac{z^4}{3} + \frac{8z^3w}{3} + O(z^5, z^4w, z^2w^2, zw^3)) \tag{4.2}$$

is the germ of F at (0,0).

Proof. We will define F as the compositions of shears and overshears in $\mathbb{C}^2$. Let us start with:

$$\begin{aligned}
(z,w) &\xrightarrow{f_1} (z, w+z)\\
(z,w) &\xrightarrow{f_2} (ze^{w}, w)\\
(z,w) &\xrightarrow{f_3} (z, w-z)\\
(z,w) &\xrightarrow{f_4} (ze^{-w}, w)
\end{aligned}$$

If we compute the power series of $f = f_4 \circ f_3 \circ f_2 \circ f_1$ around the origin we obtain:

$$f(z,w) = (z + z^2 + O(z^3, z^2w), w - zw - z^2 - \frac{z^3}{2} - z^2w - \frac{z^3}{2} + O(z^4, z^3w, z^2w^2, zw^3))$$

We want to get rid of the terms $-zw, -z^2, -z^2w$ and $-\frac{z^3}{2}$ in the second coordinate. So, we apply a shear and an overshear to cancel these terms.

The terms z^α can be canceled by using shears of the kind $(z,w) \to (z, w + f(z))$ and the terms $z^\beta w$ can be canceled using overshears of the kind $(z,w) \to (z, we^{g(z)})$.

We call $O(4) = O(|z|^4, |z|^3|w|, |z|^2|w|^2, |z||w|^3)$.

- To cancel $-z^2$; we use the shear $s_1(z,w) = (z, w + z^2)$. Now we compute the power series expansion of $s_1 \circ f$ around $(0,0)$ and we get:

$$s_1 \circ f(z,w) = (z + z^2 + O(z^3, z^2w), w + \frac{3z^3}{2} - zw - z^2w - \frac{zw^2}{2} + O(4))$$

- To cancel $\frac{3z^3}{2}$; we use the shear $s_2(z,w) = (z, w - \frac{3z^3}{2})$ and now we get:

$$g(z,w) = s_2 \circ s_1 \circ f(z,w) = (z + z^2 + O(z^3, z^2w), w - zw - z^2w - \frac{zw^2}{2} + O(4))$$

- To cancel $-zw$; we use the overshear $o_1(z,w) = (z, we^{z})$ and now we get:

$$o_1 \circ g(z,w) = (z + z^2 + O(z^3, z^2w), w - \frac{z^2w}{2} - \frac{zw^2}{2} + O(4))$$

- To cancel $\frac{-z^2w}{2}$; we use the overshear $o_2(z,w) = (z, we^{z^2/2})$ and now we get:

$$h(z,w) = o_2 \circ o_1 \circ g(z,w) = (z + z^2 + O(z^3, z^2w), w - \frac{zw^2}{2} + O(4))$$

which is the desired power series around 0.

We can also conjugate by $\phi(z,w) = (z, 2w)$ and finally we have:

$$F(z,w) = \phi^{-1} \circ h \circ \phi(z,w) = (z + z^2 + O(z^3, z^2w), w - zw^2 + O(z^4, z^3w, z^2w^2, zw^3))$$

To summarize we are composing the following shears and overshears:

$$\begin{array}{ccc}
(z,w) & \xrightarrow{\phi} & (z, 2w) \\
(z,w) & \xrightarrow{f_1} & (z, w+z) \\
(z,w) & \xrightarrow{f_2} & (ze^w, w) \\
(z,w) & \xrightarrow{f_3} & (z, w-z) \\
(z,w) & \xrightarrow{f_4} & (ze^{-w}, w) \\
(z,w) & \xrightarrow{s_1} & (z, w+z^2) \\
(z,w) & \xrightarrow{s_2} & (z, w - \dfrac{3z^3}{2}) \\
(z,w) & \xrightarrow{o_1} & (z, we^z) \\
(z,w) & \xrightarrow{o_2} & (z, we^{z^2/2}) \\
(z,w) & \xrightarrow{\phi^{-1}} & (z, w/2)
\end{array}$$

and

$$F = \phi^{-1} \circ o_2 \circ o_1 \circ s_2 \circ s_1 \circ f_4 \circ f_3 \circ f_2 \circ f_1 \circ \phi$$

so

$$F(z,w) = (F_1(z,w), F_2(z,w))$$

where

$$F_1(z,w) = z\exp(ze^{2w+z}) \tag{4.3}$$

and

$$F_2(z,w) = \left[w + \frac{z}{2} - \frac{z}{2}e^{2w+z} + \frac{z^2}{2}\exp(2ze^{2w+z}) - \frac{3z^3}{4}\exp(3ze^{2w+z})\right] \times \exp\left(z\exp(ze^{2w+z}) + \frac{z^2}{2}\exp(2ze^{2w+z})\right) \tag{4.4}$$

which gives the expansion we were looking for. □

Notice that by using more shears and overshears we can get rid of the pure terms z^α and $z^\beta w$ for α and β as large as we want meaning we can get an automorphism of $\mathbb{C}^2$ with power series around $(0,0)$:

$$F(z,w) = (z + z^2 + O(z^3, z^2w), w - w^2z + O(z^\alpha, z^\beta w, z^2w^2, zw^3))$$

Since $F(0,w) = (0,w)$ it follows that F restricted to $\mathbb{C}^* \times \mathbb{C}$ is an automorphism of $\mathbb{C}^* \times \mathbb{C}$.

As we said before, it is easy to see that F has two characteristic directions: $(1,0)$, which is nondegenerate and $(0,1)$, which is degenerate.

Hakim's Theorem II.8 says that for any nondegenerate characteristic direction $[v]$, there exists an invariant piece of curve attracted to the origin tangent to $[v]$. Applying this theorem in our setup, it follows there exists an invariant piece of curve attracted to $(1,0)$ for the map F in Lemma IV.2.

The assumption of the existence of a non-degenerate characteristic direction allows us to blow-up the origin to get a simpler expression for F. In these new coordinates Hakim proves that the invariant piece of curve is locally a holomorphic graph over the z-axis, and she gives precise estimates of the size of the graph function and its derivative. We will also need estimates in our case therefore we present her results and apply them to our case. The estimates of her paper that we use are scattered in different parts of the paper. Therefore for the reader's convenience we summarize them in the following proposition (see Main Theorem 1.3, Lemma 4.5, Lemma 4.6 and Proposition 4.8 in [Hak1]).

Proposition IV.3. *Assume that a transformation tangent to the identity is written as follows:*

$$\begin{aligned} x_1 &= f(x,u) = x - x^2 + O(ux^2, x^3), \\ u_1 &= \Psi(x,u) = u - axu + O(u^2x, ux^2) + x^{k+1}\psi_k(x). \end{aligned} \tag{4.5}$$

where $a \notin \mathbb{N}$. Then there exists an invariant piece of curve $(x, u(x))$, where u is defined in some $D_r = \{x \in \mathbb{C}; |x-r| < r\}$ and where $\lim_{x\in D_r, x\to 0} u(x) = 0$. Moreover,

$$|u(x)| \leq C_1|x|^k \quad \text{and} \quad |u'(x)| \leq C_2|x|^{k-1} \tag{4.6}$$

for $x \in D_r$, where C_1 and C_2 are positive constants.

We refer to Hakim's proof of Main Theorem 1.3 for this result, but let us point out that we should compare (4.5) with equation (4.1) on section 4 of [Hak1]. Note that, as stated, Proposition IV.3 does not involve log terms as opposed to equation (4.1) in [Hak1]. This is because we are assuming $a \notin \mathbb{N}$. The estimates we are quoting for the size of u should be compared to Lemmas 4.5 (p.420) and 4.6 (p.421) of [Hak1]. In our case we do not need the log terms since $a \notin \mathbb{N}$, and we will use $k = 4$ in both lemmas.

Let us define, for any $\epsilon > 0$:

$$V_\epsilon = \{z \in \mathbb{C} \mid 0 < |z| < \epsilon, |\mathrm{Arg}(z) - \pi| < \pi/8\} \subset \mathbb{C} \tag{4.7}$$

It is clear that for ϵ small enough, $z \in V_\epsilon$ implies $-z \in D_r$. Note that V_ϵ is a local basin of attraction for the map $z \mapsto z + z^2$.

Now we can give our result:

Proposition IV.4. *Let F be as in Lemma IV.2. Then there exists a parabolic curve attracted to $(0,0)$ along $(1,0)$ i.e. there exists γ defined in the closure of V_ϵ with values in $\mathbb{C}$ such that γ is analytic in V_ϵ, continuous up to the closure of V_ϵ, $\gamma(0) = 0$ and such that if $p \in \mathcal{C} = \{(z, \gamma(z)), z \in V_\epsilon\}$ then $F(p) \in \mathcal{C}$ and $F^n(p) \to 0$ when $n \to \infty$. Also we have the following estimates on the size of γ and its derivative, i.e. there exist constants C_1 and C_2 such that:*

$$|\gamma(z)| \leq C_1 |z|^3 \tag{4.8}$$

and

$$|\gamma'(z)| \leq C_2 |z|^2 \tag{4.9}$$

for $z \in V_\epsilon$.

Proof. This result follows almost directly from Proposition IV.3. The existence of an attracting piece of curve is immediate, since $(1,0)$ is a non-degenerate characteristic direction. For the estimates on the size of the graph, we want to change coordinates so that

$$F(z,w) = (z + z^2 + O(z^3, z^2 w), w - w^2 z + O(z^4, z^3 w, z^2 w^2, z w^3))$$

is locally of the form (4.5).

Let our invariant curve for our original map be $\mathcal{C} = \{(z, \gamma(z)), z \in V_\epsilon\}$. We know this curve exists by Hakim's theorem, since $(1,0)$ is a non-degenerate characteristic direction.

We use the change of coordinates:

$$(x,u) := \varphi(z,w) = (-z,w)$$

$$(z,w) = \varphi^{-1}(x,y) = (-x,u)$$

In these coordinates we have:

$$\begin{aligned}
\tilde{F}(x,y) &= \varphi \circ F \circ \varphi^{-1}(x,y) = \varphi \circ F(-x,y) \\
&= \varphi(-x+x^2+O(x^3,x^2u), u+xu^2+cx^4+O(x^5,x^3u,x^2u^2,xu^3)) \\
&= (x-x^2+O(x^3,x^2u), u+O(x^3u,xu^2)+x^4\psi(x))
\end{aligned}$$

Comparing with (4.5), we can see our map is in the desired form, for $a=0$ and $k=4$. Under the change of coordinate φ the invariant curve $(z,\gamma(z))$ will be $(x,u(x))$ where:

$$(x,u(x)) = \varphi(z,\gamma(z))$$

and $(x,u(x))$ is the attractive invariant curve for $\tilde{F}$. We use the estimates we have in (4.6) and the relationship:

$$\begin{aligned}
x &= -z \\
u(x) &= \gamma(z)
\end{aligned}$$

And we obtain:

$$|\gamma(z)| \leq C_1|z|^3$$

$$|\gamma'(z)| \leq C_2|z|^2$$

for $z \in V_\epsilon$. $\square$

We can compute the director associated to $v = (1, 0)$. We use the definition we give in (3.4):

$$A(v) := \frac{r'(u_0)}{P_k(1, u_0)}$$

in our case we have $P_2(z, w) = z^2$, $Q_2(z, w) = 0$. We defined $r(u) = Q_2(1, u) - uP_2(1, u)$, then $r(u) = 0 - u = -u$, and $r'(u_0) = -1$. We have then $A((1, 0)) = -1$. Therefore we can expect there is not an open invariant region around $\mathcal{C}$.

We can extend the curve $\mathcal{C}$ and prove:

Proposition IV.5. *If we define*

$$\Gamma = \bigcup_{n \geq 0} F^{-n}(\mathcal{C}) \tag{4.10}$$

then we have that

(i) $F(\Gamma) = \Gamma$

(ii) Γ *is biholomorphic to* $\mathbb{C}$.

Proof. The first part is clear. For the second part we need to understand the action of F in $\mathcal{C}$. We have $\mathcal{C} = \{(z, \gamma(z)); z \in V_\epsilon\}$, with $F(\mathcal{C}) \subset \mathcal{C}$. Clearly this action is conjugated to the following action:

$$z_1 = z + z^2 + O(z^3, z^2 w) = z + z^2 + O(z^3)$$

on V_ϵ. It is a consequence of Fatou's work [Fa2] that the transformation is conjugated to $\zeta_1 = \zeta + 1$ in a domain of type $U = \{\Re e\zeta > R\}$, for R big enough. Define the following holomorphic map from Γ to $\mathbb{C}$:

$$\Phi : \Gamma \to \mathbb{C}$$

$$\Phi(p) = \zeta(\pi_1 \circ F^N(p)) - N$$

where π_1 is the projection in the first coordinate and N is large enough so $F^N(p) \in \mathcal{C}$. We check now that Φ is well-defined, injective and surjective.

- Φ is well defined. We want to prove that Φ is independent of N. If N and M are both integers such that $F^N(p)$ and $F^M(p)$ are both in $\mathcal{C}$, then without loss of generalization we can assume $N < M$. Then $F^{M-N}(F^N(p)) = F^M(p)$ implies $\zeta(\pi_1(F^M(p))) = \zeta(\pi_1(F^N(p))) + M - N$. Therefore $\zeta(\pi_1(F^M(p))) - M = \zeta(\pi_1(F^N(p))) - N$, and we see $\Phi(p)$ is well-defined.
- Φ is injective. Let p and q in Γ such that $\Phi(p) = \Phi(q)$. We know there exists N large enough such that $F^N(p)$ and $F^N(q)$ are both in $\mathcal{C}$. Therefore we will have $\zeta(\pi_1 \circ F^N(p)) - N = \zeta(\pi_1 \circ F^N(q)) - N$. Since ζ is injective in V_ϵ we have $F^N(p) = F^N(q)$ and therefore $p = q$.
- Φ is surjective. By the definition of Φ, we clearly have:
$$\Phi(\Gamma) = \bigcup_{n=0}^{\infty} U - n = \bigcup_{n=0}^{\infty} \{\Re e\zeta > R - n\} = \mathbb{C}$$
and we get Φ is surjective.

Therefore Γ is biholomorphic to $\mathbb{C}$. $\square$

Now we describe the open region U attracted to the origin along the *degenerate* characteristic direction $(0,1)$ as in Step 1 in the outline of the proof we presented at the beginning of the chapter.

We will show later that the attracting curve $\mathcal{C}$, as in (4.10) and this open region U are disjoint. In fact, we will prove that the attracting curve is contained in the boundary of the basin of the open region.

The main proposition is the following:

Proposition IV.6. *Define* $D_{(z,w)} := \{(z,w) \in \mathbb{C}^2 : z \in V_\epsilon, w - \gamma(z) \in V_\epsilon, |z| < |w - \gamma(z)|\}$. *Then* $F(D_{(z,w)}) \subset D_{(z,w)}$.

Moreover, denote $F^n(z,w) = (z_n, w_n)$ *for* $(z,w) \in D_{(z,w)}$. *Then:*

$$|z_n| \sim \frac{1}{n} \tag{4.11}$$

and

$$|w_n| \sim \frac{1}{\log n} \tag{4.12}$$

Proof. Let $(z,w) \in D_{(z,w)}$, i.e. $z \in V_\epsilon, u = w - \gamma(z) \in V_\epsilon$ and $|z| < |u|$ where we introduce the new variable $u := w - \gamma(z)$. We want to prove:

(i) $\underline{z_1 \in V_\epsilon}$

$$\begin{aligned} z_1 &= z + z^2 + O(z^3, z^2 w) \\ &= z + z^2 + O(z^3, z^2(u + \gamma(z))) \\ &= z + z^2 + O(z^3, z^2 u) \end{aligned}$$

Proving that $z_1 \in V_\epsilon$ is equivalent to proving that $1/z_1 \in U_R$ where

$$U_R := \{\zeta \in \mathbb{C} : |\zeta| > R, |\mathrm{Arg}(\zeta) - \pi| < \pi/8\} \tag{4.13}$$

and $R = 1/\epsilon$. Rewriting the equation for z_1 we see:

$$\begin{aligned} \frac{1}{z_1} &= \frac{1}{z + z^2 + O(z^3, z^2 u)} \\ &= \frac{1}{z}\left(\frac{1}{1 + z + O(z^2, zu)}\right) \\ &= \frac{1}{z}(1 - z + O(z^2, zu)) \end{aligned}$$

$$\frac{1}{z_1} = \frac{1}{z} - 1 + O(z, u) \tag{4.14}$$

By decreasing the value of ϵ as necessary (or equivalently, increasing the value of R) it is clear than $|O(z,u)| < 1/8$. So $1/z \in U_R$ implies $1/z_1 \in U_R$ and therefore $z_1 \in V_\epsilon$. Later we will need a more refined estimate for $1/z_1$.

(ii) $\underline{u_1 = w_1 - \gamma(z_1) \in V_\epsilon}$. This is the most delicate part of the proof. We recall the second coordinate of F and rewrite it in a convenient way for our purposes:

$$w_1 = w - w^2 z - \frac{z^4}{3} + O(z^5, z^3 w, z^2 w^2, z w^3)$$

$$w_1 = w - w^2 z - l(z) + z w \theta(z, w) \tag{4.15}$$

where $\theta(z, w) = \sum_{i+j \geq 2} a_{ij} z^i w^j$. We would like to express $u_1 = w_1 - \gamma(z_1)$ in terms of u and z. We need therefore to estimate $\gamma(z_1)$. Substituting $u + \gamma(z)$ for w in (4.15) we obtain:

(4.16)

$$u_1 + \gamma(z_1) = (u + \gamma(z)) - (u + \gamma(z))^2 z - l(z) + z(u + \gamma(z))\theta(z, u + \gamma(z))$$

Recall that the curve $\mathcal{C}$ is invariant. This implies that $F(z, \gamma(z)) \in \mathcal{C}$, since $z \in V_\epsilon$. Let $F(z, \gamma(z)) = (z', \gamma(z')) \in \mathcal{C}$. Then $z' \in V_\epsilon$ and using equation (4.15):

$$\gamma(z') = \gamma(z) - \gamma(z)^2 z - l(z) + z\gamma(z)\theta(z, \gamma(z)) \tag{4.17}$$

Noting that Equation (4.17) and Equation (4.16) are very similar and subtracting one from the other, we get:

$$\begin{aligned} u_1 + \gamma(z_1) - \gamma(z') =& u - (u^2 + 2u\gamma(z))z + zu\theta(z, u + \gamma(z)) + \\ & z\gamma(z)[\theta(z, u + \gamma(z)) - \theta(z, \gamma(z))] \end{aligned}$$

Then solving for u_1:

$$\begin{aligned} u_1 =& u - u^2 z - 2uz\gamma(z) + zu\theta(z, u + \gamma(z)) + \\ & z\gamma(z)(\theta(z, u + \gamma(z)) - \theta(z, \gamma(z)) + \gamma(z') - \gamma(z_1) \end{aligned} \tag{4.18}$$

Claim: The following holds for $z \in V_\epsilon$ and $u \in V_\epsilon$:

* $\gamma(z') - \gamma(z_1) = O(z^4 u)$

* $\theta\left(z,u+\gamma(z)\right)-\theta(z,\gamma(z))=O(uz)$
* $zu\theta\left(z,u+\gamma(z)\right)=O(z^3u,z^2u^2,zu^3)$

Assume the claim is proved. Equation (4.18) yields:

$$u_1=u-u^2z+O(uz^3,z^2u^2,zu^3) \tag{4.19}$$

We use again the same idea as in (i). Proving that $u\in V_\epsilon$ is equivalent to proving that $1/u\in U_R$, where U_R is defined as in equation (4.13). Rewriting the Equation (4.19) we get:

$$\begin{aligned}\frac{1}{u_1}&=\frac{1}{u}\left(\frac{1}{1-uz+O(z^3,z^2u,zu^2)}\right)\\&=\frac{1}{u}(1+uz+O(z^3,z^2u,zu^2))\\&=\frac{1}{u}+z+O(z^3/u,z^2,zu)\end{aligned}$$

Since we are assuming $|z|<|u|$ we have $|z^3/u|<|z^2|$. Shrinking ϵ if necessary, we get $|O(z^3/u,z^2,zu)|<\frac{1}{8}|z|$. Since $z\in V_\epsilon$, we see that $1/u\in U_R$ implies $1/u_1\in U_R$, and hence $u_1\in V_\epsilon$.

We now prove the claim.

* $\gamma(z')-\gamma(z_1)$. For this term we have to use the previous estimates for γ and its derivative, as well as an estimate for $z'-z_1$:

$$z'-z_1=F_1(z,\gamma(z))-F_1(z,\gamma(z)+u)=O(z^2u)$$

where F_1 is the first coordinate of F. Note that both z_1 and z' are in V_ϵ. Thus (4.8) and (4.9) imply:

$$\begin{aligned}\gamma(z')-\gamma(z_1)&=\gamma'(z_1)(z'-z_1)+O((z'-z_1)^2)\\&=O(z^2)O(z^2u)+O(z^4u^2)=O(z^4u)\end{aligned}$$

∗ $\theta\left(z, u+\gamma(z)\right)-\theta(z,\gamma(z)) = O(uz)$. For this term we use $\theta(z,w) = O(z^2, zw, w^2)$ and therefore

$$\theta\left(z, u+\gamma(z)\right)-\theta(z,\gamma(z)) = O(zu, u^2)$$

∗ $zu\theta\left(z, u+\gamma(z)\right) = O(z^3u, z^2u^2, zu^3)$: This follows from $\theta(z, u+\gamma(z)) = O(z^2, zu, u^2)$.

This completes the proof of the claim and of (ii).

(iii) $\underline{|z_1| < |u_1|}$. This is equivalent to proving that $|1/z_1| > |1/u_1|$. From part (i) and (ii) we obtain $|1/z_1| > |1/z| + 1/2$ and $|1/u_1| < |1/u| + |2z|$. Since $|z| < |u|$ we got $|1/z| > |1/u|$. By shrinking ϵ we have $|1/z_1| > |1/z| + 1/2 > |1/u| + 2|z| > 1/|u_1|$, and we get $|z_1| < |u_1|$.

Therefore, we proved $F(D_{(z,w)}) \subset D_{(z,w)}$. If $(z,w) \in D_{(z,w)}$, then $(z_k, w_k) := F^k(z,w) \in D_{(z,w)}$ for all $k \geq 0$. Set $u_k = w_k - \gamma(z_k)$. Therefore we have:

$$\frac{1}{z_{k+1}} = \frac{1}{z_k} - 1 + O(z_k, u_k)$$

summing from $k = 0$ to N, and dividing by N, we get:

$$\frac{1}{Nz_N} = \frac{1}{Nz_0} - 1 + O(\sum_k z_k, u_k)/N.$$

Letting N tend to infinity we get

$$\lim_{N\to\infty} \frac{1}{Nz_N} = -1$$

In the same way for u:

$$\frac{1}{u_{k+1}} = \frac{1}{u_k} + z_k + O(z_k^3/u_k, z_k^2, z_k u_k),$$

summing from $k = 0$ to N, and dividing by $\log N$:

$$\frac{1}{\log N u_N} = -\frac{\sum_k z_k}{\log N} + O(\sum_k z_k^3/u_k, z_k^2, z_k u_k)/\log N$$

and, letting N tend to infinity,

$$\lim_{N\to\infty} \frac{1}{\log N u_N} = -1$$

which implies:

$$|z_N| \sim 1/N$$

$$|w_N| \sim 1/\log N.$$

□

Note that this does not prove that our curve $\mathcal{C}$ is in the boundary of the region $D_{(z,w)}$. We prove a weaker statement now:

Proposition IV.7. *There exists $\delta > 0$ and $N > 0$, such that if $(z, w) \in \mathbb{C}^2$, $z \in V_\delta$ and $w - \gamma(z) \in V_\delta$, then $F^N(z, w) \in D_{(z,w)}$.*

Proof. Let $u = w - \gamma(z)$. We are considering any $z, u \in V_\delta$, and we want to prove that there exists N large such that $|z_N| < |u_N|$ and $z_N, u_N \in V_\epsilon$. If $|z| < |u|$, then there is nothing to do. If $|z| > |u|$, going back to the proof of the last proposition we will still have $z_1 \in V_\epsilon$, since we did not use $|z| < |u|$ in the proof of $z_1 \in V_\epsilon$. The only part of the proof in which we use $|z| < |u|$ was to prove $u_1 \in V_\epsilon$. Recall from equation (4.19):

$$u_1 = u - u^2 z + O(z^3 u, z^2 u^2, z u^3).$$

Since we do not have $|z| < |u|$, we can not replace $|z^3 u| < |zu|^2$. Therefore we do not get $u_1 \in V_\epsilon$. Nonetheless, we have the following estimate:

$$\frac{1}{u_1} = \frac{1}{u}(1 + O(z^3)) + z + O\left(z^2, zu\right)$$

and therefore

$$\frac{1}{u_N} = \frac{1}{u}\prod_i (1 + O(z_i^3)) + \sum_i z_i + \sum_i O\left({z_i}^2, z_i u_i\right).$$

We choose δ small enough, so we have $|u_i| < \epsilon$, for all $1 \leq 1i \leq N$. Consequently we have $z_i \in V_e$ for all $1 \leq i \leq N$, since $z_{i+1} \in V_\epsilon$ depends only on the size of u_i. We will have:

$$\frac{1}{z_N} \sim \frac{1}{z} - N$$

and

$$\frac{1}{u_N} = \frac{1}{u}\prod_i (1 + O(z_i^3)) + \sum_i z_i + \sum_i O\left({z_i}^2, z_i u_i\right).$$

Since $|z_k| = O(1/k)$, for every $k < N$, the term $\prod_i (1 + O(z_i^3)) \sim \exp(\sum_i (O(z_i^3)) < C|z_N| < 2C/N$, where C is a finite number. Therefore, the dominant term in the expression for u is $\sum_i z_i \sim \log N$. This means that choosing N large enough, or equivalently choosing δ small enough, we have $|z_N| \sim 1/N < 1/\log N \sim |u_N|$. $\square$

Corollary IV.8. *There exists $\delta > 0$ and $N > 0$; such that*

$$F^N(\mathcal{C} \cap B_\delta(O)) \subset \partial D_{(z,w)} \tag{4.20}$$

Proof. We have $\mathcal{C} \cap B_\delta(O) \subset \partial(\{(z,w), z \in V_\delta, w - \gamma(z) \in V_\delta\})$. Applying F^N and using Proposition IV.7 we get (4.20). $\square$

Remark IV.9. Note that the curve $\mathcal{C}$ is tangent to the z-axis to order 3.

It will be more useful for our purposes to change coordinates in $D_{(z,w)}$. We use the following change of coordinates (we have already used it implicitly):

$$(x,y) := \phi(z,w) = \left(\frac{1}{z}, \frac{1}{w - \gamma(z)}\right) \tag{4.21}$$

Let $D_{(x,y)} = \phi(D_{(z,w)})$. Clearly:

$$D_{(x,y)} = \{(x,y) \in \mathbb{C}^2 : x, y \in U_R, |y| < |x|\} \tag{4.22}$$

where U_R is defined as in (4.13). We will work from now on with the map $\tilde{F} = \phi^{-1} \circ F \circ \phi$. As we said before, we need more precise equations for $\tilde{F}$. We give them in the following proposition:

Proposition IV.10. *Let $\tilde{F}(x, y) = (x_1, y_1)$, where $\tilde{F}$ is defined above and $(x, y) \in D_{(x,y)}$. In these coordinates we have:*

$$x_1 = x - 1 + g\left(\frac{1}{y}\right) + \frac{c}{x} + O\left(\frac{1}{x^2}, \frac{1}{xy}\right) \tag{4.23}$$

$$y_1 = y + \frac{1}{x} + \frac{1}{x} h\left(\frac{1}{y}\right) + O\left(\frac{1}{x^2}\right) \tag{4.24}$$

for all $(x, y) \in D_{(x,y)}$, where g and h analytic functions in V_ϵ, $g(1/y) = O(1/y)$ and $h(1/y) = O(1/y)$.

Proof. It is clear from the Proposition IV.7 that the map $\tilde{F}$ is well-defined. Indeed $F(D_{(z,w)}) \subset D_{(z,w)}$ and because $\mathcal{C} = \{(z, \gamma(z)), z \in V_\epsilon\} \cap D_{(z,w)} = \emptyset$ we can invert $w - \gamma(z)$ in $D_{(z,w)}$. In the proof of the last proposition we saw equation (4.14):

$$\frac{1}{z_1} = \frac{1}{z} - 1 + O(z, u)$$

We put together all the terms in $O(z, u)$ that contain only u terms and we call this function $g(u)$. This function is analytic in a neighborhood of 0 and in particular in V_ϵ. We also separate the linear term on z:

$$\frac{1}{z_1} = \frac{1}{z} - 1 + g(u) + cz + O(z^2, zu)$$

where $c \in \mathbb{C}$ is a constant. In the new coordinates (x, y) we have:

$$x_1 = x - 1 + g\left(\frac{1}{y}\right) + \frac{c}{x} + O\left(\frac{1}{x^2}, \frac{1}{xy}\right)$$

Similarly for $y = 1/(w - \gamma(z)) = 1/u$ we have from equation (4.19)

$$\frac{1}{u_1} = \frac{1}{u} + z + O\left(\frac{z^3}{u}, z^2, zu\right)$$

and rewriting it in terms of x and y, we obtain:

$$\begin{aligned} y_1 &= y + \frac{1}{x} + O\left(\frac{y}{x^3}, \frac{1}{x^2}, \frac{1}{xy}\right) \\ &= y + \frac{1}{x} + \frac{1}{x}h\left(\frac{1}{x}\right) + O\left(\frac{y}{x^3}, \frac{1}{x^2}\right) \end{aligned}$$

Since we have $|y| < |x|$, $O(y/x^3)$ is bounded by $O(1/x^2)$. Also, we separated the terms of the form zu^k, or equivalently, the terms $1/xy^k$. So we have:

$$y_1 = y + \frac{1}{x} + \frac{1}{x}h\left(\frac{1}{y}\right) + O\left(\frac{1}{x^2}\right)$$

where h is an analytic function defined for $y \in U_R$. The proposition is proved. $\square$

Notice that $D_{(z,w)}$ is the invariant region for F and $D_{(x,y)}$ is the invariant region for $\tilde{F}$. The following diagram commutes:

$$\begin{array}{ccc} D_{(z,w)} & \xrightarrow{F} & D_{(z,w)} \\ \phi\big\downarrow & & \phi\big\downarrow \\ D_{(x,y)} & \xrightarrow{\tilde{F}} & D_{(x,y)} \end{array}$$

In the following section we introduce new coordinates on $D_{(z,w)}$ in which our map takes a simpler form.

4.2 Semiconjugacy to translation

In this section we change coordinates $(z,w) \to (\psi(z,w), w)$ in $D_{(z,w)}$ so we have (as in Weickert, or Hakim) that ψ is an Abel-Fatou coordinate. More precisely, we want to find a map $\psi : D_{(z,w)} \to \mathbb{C}$ such that

$$\psi \circ F(z,w) = \psi(z,w) - 1 \tag{4.25}$$

for all $(z,w) \in D_{(z,w)}$.

This coordinate is known in the literature as the Abel-Fatou coordinate. It always exists for maps tangent to the identity in one dimension; see Milnor [Mi].

So far we have introduced one biholomorphic change of coordinates as in equation (4.21) that transform $D_{(z,w)}$ as follows:

$$(x, y) := \phi(z, w) = \left(\frac{1}{z}, \frac{1}{w - \gamma(z)}\right)$$

Notice that this can be translated to our new coordinates (x, y) in the following way: if μ is defined as $\mu := \psi \circ \phi^{-1}$ then we have:

$$\mu : D_{(x,y)} \to \mathbb{C}$$

and

$$\mu \circ \tilde{F}(x, y) = \mu(x, y) - 1 \tag{4.26}$$

for all $(x, y) \in D_{(x,y)}$.

So, finding a solution for (4.26) and for (4.25) are equivalent problems. We will first find μ that solves (4.26) and then translate it to the original coordinates.

Let us recall how our map looks in the (x, y) coordinates.

$$\begin{aligned} x_1 &= x - 1 + g\left(\frac{1}{y}\right) + \frac{c}{x} + O\left(\frac{1}{x^2}, \frac{1}{xy}\right) \\ y_1 &= y + \frac{1}{x} + \frac{1}{x}h\left(\frac{1}{y}\right) + O\left(\frac{1}{x^2}\right) \end{aligned}$$

So, our coordinate $x \to x_1$ is close to being a translation. Nonetheless we have several terms to deal with, such as $g(1/y)$ and c/x. We prove the following lemma, which will help us clean up some of these inconvenient terms.

Lemma IV.11. *Choose and fix a branch of the logarithm in U_R. For (x, y) in $D_{(x,y)} = U_R \times U_R$, with R big enough we have:*

a)

$$\log(x_1) - \log(x) = -\frac{1}{x} + O\left(\frac{1}{x^2}, \frac{1}{xy}\right) \tag{4.27}$$

b)

$$\log(y_1) - \log(y) = \frac{1}{xy} + O\left(\frac{1}{x^2y}, \frac{1}{xy^2}\right) \tag{4.28}$$

c) There exists a holomorphic solution $\beta : U_R \to \mathbb{C}$ to the following first-order linear differential equation:

$$\left(1 + h\left(\frac{1}{y}\right)\right)\beta'(y) - \left(1 - g\left(\frac{1}{y}\right)\right)\beta(y) = -g\left(\frac{1}{y}\right) \tag{4.29}$$

such that $\beta(y) = O(1/y)$ and $\beta'(y) = O(1/y^2)$, where $x, y \in U_R$. If $\alpha(x,y) = x\beta(y)$ then we have the following estimate:

$$\alpha(x_1, y_1) = \alpha(x,y) - g\left(\frac{1}{y}\right) + O\left(\frac{1}{x^2}, \frac{1}{xy}\right). \tag{4.30}$$

Proof. We start by proving (4.27):

$$\begin{aligned}
\log\left(\frac{x_1}{x}\right) &= \log\left(\frac{x - 1 + O(1/x, 1/y)}{x}\right) \\
&= \log\left(1 - \frac{1}{x} + O\left(\frac{1}{x^2}, \frac{1}{xy}\right)\right) \\
&= -\frac{1}{x} + O\left(\frac{1}{x^2}, \frac{1}{xy}\right)
\end{aligned}$$

Now, we prove Equation (4.28):

$$\begin{aligned}
\log\left(\frac{y_1}{y}\right) &= \log\left(\frac{y + 1/x + O(1/x^2, 1/xy)}{y}\right) \\
&= \log\left(1 + \frac{1}{xy} + O\left(\frac{1}{x^2y}, \frac{1}{xy^2}\right)\right) \\
&= \frac{1}{xy} + O\left(\frac{1}{x^2y}, \frac{1}{xy^2}\right)
\end{aligned}$$

Finally we prove (4.30). This is probably the hardest part of the proof, so we give each step very carefully.

First we want to compute $\alpha(x_1, y_1) - \alpha(x, y)$, where α is defined as in the proposition.

$$\begin{aligned}\alpha(x_1, y_1) - \alpha(x, y) &= x_1\beta(y_1) - x\beta(y)\\ &= x_1\beta(y_1) - x_1\beta(y) + x_1\beta(y) - x\beta(y)\\ &= x_1[\beta(y_1) - \beta(y)] + \beta(y)[x_1 - x]\end{aligned}$$

Recall that $y_1 - y = \frac{1}{x}[1 + h\left(\frac{1}{y}\right)] + O(1/x^2)$ and we will prove later that $\beta'(y) = O(1/y^2)$. We first compute $\beta(y_1) - \beta(y)$:

$$\begin{aligned}\beta(y_1) - \beta(y) =&(y_1 - y)\beta'(y) + O\left(\beta''(y)|y_1 - y|^2\right)\\ =&\beta'(y)\left[\frac{1}{x}\left(1 + h\left(\frac{1}{y}\right)\right) + O\left(\frac{1}{x^2}\right)\right] + O\left(\frac{1}{x^2}\right)\\ =&\frac{1}{x}\left(1 + h\left(\frac{1}{y}\right)\right)\beta'(y) + O\left(\frac{1}{x^2}\right)\end{aligned}$$

so, putting it back we get:

$$\begin{aligned}\alpha(x_1, y_1) - \alpha(x, y) =&x_1\left[\frac{1}{x}\left(1 + h\left(\frac{1}{y}\right)\right)\beta'(y) + O\left(\frac{1}{x^2y^2}\right)\right]\\ &+ \beta(y)\left[-1 + g\left(\frac{1}{y}\right) + O\left(\frac{1}{x}\right)\right]\\ =&(x + O(1))\left[\frac{1}{x}\left(1 + h\left(\frac{1}{y}\right)\right)\beta'(y)\right]\\ &- \left(1 - g\left(\frac{1}{y}\right)\right)\beta(y) + O\left(\frac{1}{xy}\right),\end{aligned}$$

since $\beta(y) = O\left(1/y\right)$. So, we see:

$$\alpha(x_1, y_1) - \alpha(x, y) = \left(1 + h\left(\frac{1}{y}\right)\right)\beta'(y) - \left(1 - g\left(\frac{1}{y}\right)\right)\beta(y) + O\left(\frac{1}{x^2}, \frac{1}{xy}\right)$$

From (4.29) we then get:

$$\alpha(x_1, y_1) - \alpha(x, y) = -g\left(\frac{1}{y}\right) + O\left(\frac{1}{x^2}, \frac{1}{xy}\right)$$

and Equation (4.30) is proved. It remains to prove is the estimates on the solution of the differential equation (4.29).

Recall that g and h are holomorphic equations defined in a neighborhood of the origin and $g(z) = O(z)$ and also $h(z) = O(z)$. Choosing R large enough we can guarantee that $|h(1/y)| < 1/4$ for all $y \in U_{\mathcal{R}}$, and therefore we can divide by $1 + h(1/y)$ to obtain:

$$\beta'(y) - \frac{1 - g(1/y)}{1 + h(1/y)}\beta(y) = -\frac{g(1/y)}{1 + h(1/y)},$$

renaming for convenience we have the following differential equation

$$\beta'(y) - (1 + s(y))\,\beta(y) = t(y),$$

where $s(z) = O(1/z)$ and similarly $t(z) = O(1/z)$.

If we call $\sigma(y) = \beta(y) + t(y)$, the differential equation will become

$$\sigma'(y) - (1 + s(y))\,\sigma(y) = \tau(y)$$

where $s(z) = O(1/z)$ and similarly $\tau(z) = O(1/z^2)$. We will prove that the solution of the equation $\sigma(y) = O(1/y)$, and therefore we will have $\beta(y) = O(1/y)$.

By increasing R if necessary we have all the estimates for s, t and τ in an open neighborhood of $\overline{U_R}$. Using differential equations theory, we see that one solution of this differential equation is

$$\sigma(y) = \frac{\int_{-R}^{y} H(\zeta)\tau(\zeta)d\zeta}{H(y)},$$

where the integral is in any path joining the points $-R$ and y in U_R and:

$$\frac{H'(y)}{H(y)} = -1 - s(y).$$

Notice that all the integrals are independent of the path chosen because we are working in U_R, which is a simply connected region. We prove the following

$$\max_{\zeta\in\gamma} |H(\zeta)\tau(\zeta)| = |H(y)\tau(y)|$$

If $\tau \equiv 0$ there is nothing to prove. Assume then $\tau(y) = \sum_{n\geq k} a_n/y^n$, where k is the lowest index such that $a_k \neq 0$. Then $\tau(y) = \frac{a_k}{y^k}(1 + O(1/y))$ by increasing R we can make $|O(1/y)| < 1/2$. So we have $\tau(y) \neq 0$ for all $y \in U_R$. We can choose a branch of logarithm for $\tau(y)$. Let $M(y) = \log(\tau(y))$ or equivalently $\tau(y) = \exp(M(y))$. We have

$$\begin{aligned} M(y) &= \log\left(\frac{a_k}{y^k}\left(1 + O\left(\frac{1}{y}\right)\right)\right) = \log\left(\frac{a_k}{y^k}\right) + \log\left(1 + O\left(\frac{1}{y}\right)\right) \\ &= \log a_k - k\log y + O\left(\frac{1}{y}\right). \end{aligned}$$

From the definition of H, we have:

$$\log(H(y)) = L(y) = -y - b_1 \ln y - \sum_{i\geq 2} \frac{b_i}{(i-1)y^{i-1}},$$

where $L'(y) = -1 - s(y)$ and $s(y) = \sum_{i\geq 1} \frac{b_i}{y^i}$.

We call $u(t) = |\exp(M(\zeta(t)) + L(\zeta(t))|$, where $\zeta(t) = (1-t)(-R) + ty$ is a parametrization of the line that joins the points $-R$ and y. We rewrite what we want to prove as $\max\limits_{0\leq t\leq 1} u(t) = u(1)$. We are gonna prove $u'(t) \geq 0$ for $0 \leq t \leq 1$, if we choose y large enough.

We have

$$u(t) = |\exp(M(\zeta(t)) + L(\zeta(t))| = \exp \Re\left[M(\zeta(t)) + L(\zeta(t))\right]$$

Therefore $u'(t) = u(t)\Re[(M(\zeta) + L(\zeta))'|_{\zeta=\zeta(t)}(y + R)]$ and

$$\begin{aligned} [M(\zeta) + L(\zeta)]' &= M'(\zeta) + L'(\zeta) \\ &= [-\frac{k}{\zeta} + O\left(\frac{1}{\zeta^2}\right)] + [-1 + O(1/\zeta)] \\ &= -1 + O(1/\zeta) \end{aligned}$$

Then $u'(t) = u(t)\Re[(y + R)(-1 + O(1/\zeta(t))]$ By considering y large enough we can guarantee $\Re[(y + R)(-1 + O(\frac{1}{R+tR+ty}))]$ is positive. So, we have $u'(t) \geq 0$, then $\max_{\zeta\in\gamma} |H(\zeta)\tau(\zeta)| = |H(y)\tau(y)|$. Putting back on the solution σ we will have $|\sigma(y)| \leq |\tau(y)|O(y) = O(1/y)$. Therefore $\beta(y) = O(1/y)$.

□

We use these functions to define the Abel-Fatou coordinate. In the (x, y) coordinates, the first coordinate map looks as follows:

$$x_1 \quad = \quad x - 1 + g\left(\frac{1}{y}\right) + \frac{c}{x} + O\left(\frac{1}{x^2}, \frac{1}{xy}\right).$$

By using the last lemma and the auxiliary functions we found we will be able to prove the existence of μ such that $\mu(x_1, y_1) = \mu(x, y) - 1$.

Theorem IV.12. *Choose and fix a branch of the logarithm on U_R. Define:*

$$\mu_n(x, y) = x_n + n + r\log(x_n) + s\log(y_n) + \alpha(x_n, y_n),$$

with r and s constants well chosen. Then $\mu_n(x, y)$ is Cauchy with the uniform norm topology and therefore we define the limit function as $\mu(x, y)$.

Proof. We want to show that $\mu_n(x, y)$ is Cauchy and therefore converges uniformly in $U_R \times U_R$. Using the lemma above, we have:

$$\begin{aligned}
\mu_{n+1} - \mu_n =& (x_{n+1} + n + 1 + r \log x_{n+1} + s \log y_{n+1} + \alpha(x_{n+1}, y_{n+1})) - \\
& - (x_n + n + r \log x_n + s \log y_n + \alpha(x_n, y_n)) \\
=& (x_{n+1} - x_n) + 1 + r(\log x_{n+1} - \log x_n) + \\
& + s(\log y_{n+1} - \log y_n) + (\alpha(x_{n+1}, y_{n+1}) - \alpha(x_n, y_n)) \\
=& \left(-1 + g\left(\frac{1}{y_n}\right) + 1 + \frac{c}{x_n}\right) - \frac{r}{x_n} + \\
& + s\left(\frac{1}{x_n y_n}\right) - g\left(\frac{1}{y_n}\right) + \frac{k}{x_n y_n} + O\left(\frac{1}{x_n^2}, \frac{1}{x_n y_n^2}\right)
\end{aligned}$$

We choose $r = c$ and $s = -k$, where the term $k/x_n y_n$ is the sum of all the terms of this form in the other factors. Therefore:

$$|\mu_{n+1} - \mu_n| = \left| O\left(\frac{1}{x_n^2}, \frac{1}{x_n y_n^2}\right) \right|$$

Using the fact that $|x_n| \sim n$ and $|y_n| \sim \log n$ we can see that these terms add up and converge. Therefore:

$$\mu_n - \mu_0 = \sum_{i=0}^{n} (\mu_{i+1} - \mu_i) \tag{4.31}$$

converges absolutely uniformly on $D_{(x,y)}$ to a holomorphic limit μ. Let $\mu = \lim \mu_n = \mu_0 + \eta$. We have:

$$|\eta| = |\sum_n (\mu_{n+1} - \mu_n)| \leq \sum_n \left| O\left(\frac{1}{n^2}, \frac{1}{n \log^2 n}\right) \right| = O\left(\frac{1}{n}, \frac{1}{\log n}\right)$$

Our estimates above show that $\eta \to 0$ uniformly in $D_{(x,y)}$ as $(x, y) \to \infty$. $\square$

We summarize what we have found in the following proposition:

Proposition IV.13. *There exists a map $\psi : D_{(z,w)} \to \mathbb{C}$ such that:*

$$\psi(F(z, w)) = \psi(z, w) - 1$$

for all $(z,w) \in D_{(z,w)}$. *We have:*

$$\psi(z,w) = \mu(x,y) = \mu_0(x,y) + \eta(x,y)$$

$$= x + r\log(x) + s\log(y) + \alpha(x,y) + \eta(x,y) \tag{4.32}$$

Proof. Let $\mu = \lim \mu_n = \mu_0 + \eta = x + r\log(x) + s\log(y) + \alpha(x,y) + \eta$; and $\psi(z,w) = \mu(x,y) = \mu(\phi(z,w))$. Then:

$$\begin{aligned} \psi \circ F(z,w) &= \lim_{n\to\infty} [x_{n+1} + n + r\log x_{n+1} + s\log y_{n+1} + \alpha(x_{n+1}, y_{n+1}) + \eta(x_{n+1}, y_{n+1})] \\ &= \lim_{n\to\infty} [x_{n+1} + n + 1 + r\log x_{n+1} + s\log y_{n+1} + \alpha(x_{n+1}, y_{n+1}) + \eta(x_{n+1}, y_{n+1})] - 1 \\ &= \psi(z,w) - 1, \end{aligned}$$

and so (4.25) is satisfied. □

Consider the mapping from $D_{(z,w)}$ to $\mathbb{C}^2$ given by:

$$\Theta(z,w) = \left(\psi(z,w), \frac{1}{w-\gamma(z)}\right) =: (t,y)$$

In the next proposition we prove Θ is a change of coordinates.

Proposition IV.14. $\Theta = (t,y)$ *is a biholomorphism from* $D_{(z,w)}$ *onto its image* $D_{(t,y)}$.

Proof. We have to show that (t,y) is injective in $D_{(z,w)}$, or equivalently in $D_{(x,y)}$. If

$$(\mu(x_1,y_1), y_1) = (\mu(x_2,y_2), y_2)$$

for (x_1,y_1) and $(x_2,y_2) \in D_{(x,y)}$ then we have:

$$y_1 = y_2$$

and

$$\mu(x_1,y) = \mu(x_2,y).$$

Recalling how we define $\mu(x,y)$ then we should have:

$$x_1 + r\log x_1 + s\log y + \alpha(x_1,y) + \eta(x_1,y) = x_2 + r\log x_2 + s\log y + \alpha(x_2,y) + \eta(x_2,y)$$

$$x_1 - x_2 + r\left(\log x_1 - \log x_2\right) + \alpha(x_1,y) - \alpha(x_2,y) + \eta(x_1,y) - \eta(x_2,y) = 0$$

In case $x_1 \neq x_2$ we can divide by $x_1 - x_2$:

$$1 + r\frac{\log x_1 - \log x_2}{x_1 - x_2} + \frac{\alpha(x_1,y) - \alpha(x_2,y)}{x_1 - x_2} + \frac{\eta(x_1,y) - \eta(x_2,y)}{x_1 - x_2} = 0 \tag{4.33}$$

We have that $\eta(x,y)$ is bounded, so by increasing R we can assume $\dfrac{\eta(x_1,y) - \eta(x_2,y)}{x_1 - x_2}$ is less than ϵ. Also:

$$\left|\frac{\log(x_1) - \log(x_2)}{x_1 - x_2}\right| = \left|\frac{1}{\tilde{x}}\right|$$

for some $\tilde{x} \in V_R$ therefore: $\left|r\dfrac{\log(x_1) - \log(x_2)}{x_1 - x_2}\right| < \dfrac{|r|}{R}$.

And for the third term:

$$\frac{\alpha(x_1,y) - \alpha(x_2,y)}{x_1 - x_2} = \beta(y) = O\left(\frac{1}{y}\right),$$

we have $|\beta(y)| < C/|y| < C/R$.

Therefore we can choose R large enough such that:

$$1 > \frac{|r|}{R} + \frac{C}{R} + \epsilon$$

So our map is injective and we have a valid change of coordinates. □

In the following section we modify the open region $D_{(t,y)}$ for convenience, so it will be easier to visualize each fiber of Ψ.

4.3 Modification of $D_{(z,w)}$

In this section we investigate the geometry of the image $D_{(t,y)}$ of $D_{(z,w)}$ in the new coordinates $\Theta = (t,y)$.

We want to prove:

Lemma IV.15. *$D_{(t,y)} \supset D'_{(t,y)}$ where*

$$D'_{(t,y)} = \{(t,y) \in U_{2R} \times U_R : |y| < \frac{|t|}{2}\}.$$

Proof. Fix $y_o \in U_R$, we call $\mu_{y_o}(x) := \mu(x, y_o)$. Define:

$$D_{y_o} = \{x \in \mathbb{C} : (x, y_o) \in D_{(x,y)}\}$$

By the definition of $D_{(x,y)}$ we have $D_{y_o}(x) = \{x \in U_R, |y_o| < |x|\}$. We want to prove $\mu_{y_o}(D_{y_o}) \supset U_{2|y_o|}$. We have $\mu_{y_o}(x) = x + r\log(x) + s\log(y_o) + \alpha(x, y_o) + \eta(x, y_o)$. Notice that $\log(y) = \log(|y_o|e^{i\theta}) = \log|y_o| + i\theta$ where $\theta \in [\pi - \pi/8, \pi + \pi/8]$, which is basically just a translation along the real axis, since $\log|y_o| \gg \pi$ and since $x \in D_{y_o}$ we have $\log|y_o| < \log|x|$. Also $\eta(x, y_o)$ is bounded, therefore we can assume $|\eta(x, y_o)| < 1$. The only term we need to estimate is $\alpha(x, y_o)$. We can choose $|\alpha(x, y)| < |x|/10$, since $\alpha(x, y) = x\beta(y)$ and $\beta(y) = O(1/y^2)$. Putting all together we have:

$$|\mu_{y_o}(x) - (x + (r+s)\log(x))| < \frac{|x|}{10} + 1$$

We will have that the image of $D_{y_o} = \{x \in U_R : |y_o| < |x|\}$ by μ will not change the asymptotic behavior of U_R and $\mu_{y_o}(D_{y_o}) \supset U_{2|y_o|}$. This proves that $D_{(t,y)}$ contains $D'_{(t,y)} = \{(t,y) \in U_{2R} \times U_R : |t| > 2|y|\}$. □

It follows from the asymptotic behavior of F in the coordinates (t, y) that the region $D_{(t,y)}$ will be mapped into the region $D'_{(t,y)}$ under a finite number of iterates of F. So, we have $F^k(D_{(z,w)}) \subset D'_{(z,w)}$ for some k big enough. Therefore, if we define Ω as follows:

$$\Omega = \bigcup_{n \geq 0} F^{-n}(D'_{(z,w)}) \tag{4.34}$$

then we also have:

$$\Omega = \bigcup_{n \geq 0} F^{-n}(D_{(z,w)})$$

From now on we work in $D'_{(z,w)}$, $D'_{(x,y)}$ and $D'_{(t,y)}$, where:

$$\begin{aligned} D'_{(t,y)} &= (t,y)(D'_{(z,w)}) = \Theta(D'_{(z,w)}) \\ D'_{(x,y)} &= (x,y)(D'_{(z,w)}) = \phi(D'_{(z,w)}) \\ D'_{(t,y)} &= \{(t,y) \in U_{2R} \times U_R : |y| < |t|/2\} \end{aligned}$$

All the open regions are displayed in the following diagram, where each vertical arrow is a biholomorphic change of coordinates and $\tilde{F}$ and $\tilde{\tilde{F}}$ is defined so the diagram commutes.

$$\begin{array}{ccccc} D'_{(z,w)} & \xrightarrow{\subset} & D_{(z,w)} & \xrightarrow{F} & D_{(z,w)} \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi}\downarrow \\ D'_{(x,y)} & \xrightarrow{\subset} & D_{(x,y)} & \xrightarrow{\tilde{F}} & D_{(x,y)} \\ {\scriptstyle(\mu,id)}\downarrow & & {\scriptstyle(\mu,id)}\downarrow & & {\scriptstyle(\mu,id)}\downarrow \\ D'_{(t,y)} & \xrightarrow{\subset} & D_{(t,y)} & \xrightarrow{\tilde{\tilde{F}}} & D_{(t,y)} \end{array}$$

We are now ready to prove that the invariant curve attracted to the origin along the nondegenerate characteristic direction is contained in the boundary of the invariant region attracted to the origin along the degenerate characteristic direction.

Proposition IV.16. *For Γ defined as $\Gamma = \cup F^{-n}(\mathcal{C})$, we have:*

$$\Gamma \subset \partial\Omega \tag{4.35}$$

Proof. First we claim that for K large, $F^K(\mathcal{C}) \subset \partial D_{(z,w)}$. Assume the claim is proved. By definition $\Gamma = \bigcup_{n\geq 0} F^{-n}(\mathcal{C})$ and $\Omega = \bigcup_{n\geq 0} F^{-n}(D_{(z,w)})$. Let $p \in \Gamma$, then $p \notin \Omega$, since iterating forward a finite number of times we should be in $\mathcal{C} \cap D_{(z,w)}$ which is empty. If $p \notin \partial\Omega$, then there exits a small ball around p, which is entirely outside Ω. By iterating forward with F a finite number of times, we should have a ball around $F^k(p) \in F^K(\mathcal{C}) \in \partial D_{(z,w)}$, but by shrinking enough, the whole ball should be outside

$D_{(z,w)}$, which is a contradiction. Therefore $p \in \partial\Omega$. So $\Gamma \subset \partial\Omega$. Now, we prove the claim. We have seen that $F(\mathcal{C}) \subset \mathcal{C}$. Let $\delta > 0$ and $N > 0$ be as in corollary IV.8. We proved $F^N(\mathcal{C} \cap B_\delta(0)) \subset \partial D_{(z,w)}$. Since $\mathcal{C}$ is invariant, attracted to the origin by F, there exists M large enough so $F^M(\mathcal{C}) \subset \mathcal{C} \cap B_\delta(0)$. So, $F^{M+N}(\mathcal{C}) \subset \partial D_{(z,w)}$ and the claim is proved. $\square$

For Ω as in (4.34):

$$\Omega = \bigcup_{n \geq 1} F^{-n}(D'_{(z,w)})$$

we can extend our coordinate $\psi = \mu \circ \phi$ to all of Ω in the usual way: Given $(z,w) \in \Omega$, there exists some N such that $(z_N, w_N) := F^N(z,w) \in D'_{(z,w)}$. We define:

$$\psi(z,w) := \psi(z_N, w_N) + N$$

It is straightforward to show that ψ is well defined. From the equation it follows that $\psi(\Omega)$ covers $\mu(D'_{(x,y)}) + N$ for all N. Since $\psi(D'_{(z,w)}) = \mu(D'_{(x,y)}) = U_{2R}$, for R fixed, then we have:

$$\psi(\Omega) = \mathbb{C} \tag{4.36}$$

Now we want to choose new coordinates in each fiber of ψ and prove that Ω is biholomorphic to $\mathbb{C}^2$.

4.4 New coordinates on the fibers of ψ

We start by proving that each fiber is connected and simply connected.

Proposition IV.17. *For each $t \in \mathbb{C}$, $\psi^{-1}(t)$ is connected and simply connected.*

Proof. For each $t \in \mathbb{C}$ we can exhaust $\psi^{-1}(t)$ by the sequence:

$$\psi^{-1}(t) = \bigcup_{n \geq 0} W_n$$

where

$$W_n := \psi^{-1}(t) \cap F^{-n}(D'_{(z,w)}),$$

We have the following biholomorphism:

$$F^n : \psi^{-1}(t) \cap F^{-n}(D'_{(z,w)}) \to \psi^{-1}(t-n) \cap D'_{(z,w)}$$

From the last section we have that Θ biholomorphically maps $(z, w) \to (\psi(z, w), y)$. Therefore for n large enough, $\Theta = (t, y)$ maps $\psi^{-1}(t-n) \cap D'_{(z,w)}$ biholomorphically to $\{t-n\} \times T_{R,2|t-n|}$ where we introduce the new notation

$$T_{b,a} := \{\zeta \in \mathbb{C} : \zeta \in U_b, |\zeta| < a\} \tag{4.37}$$

for $b < a$. Clearly, for R large enough, $T_{R,2|t-n|}$ is connected and simply connected.

$\square$

We would like to prove now that each fiber is biholomorphic to $\mathbb{C}$. As before, defining a function in $D'_{(z,w)}$ is equivalent to defining a function on $D'_{(t,y)}$. We will define a function ξ on $D'_{(t,y)}$ and therefore, by composing with Θ we will get a function Υ in $D'_{(z,w)}$.

In the commutative diagram before we know the form for F, for $\tilde{F}$ but we have not computed the exact form for $\tilde{\tilde{F}}$. We will not need to compute it explicitly but have an estimate so we can map each fiber to $\mathbb{C}$.

We have:

$$x_1 = x - 1 + g\left(\frac{1}{y}\right) + \frac{c}{x} + O\left(\frac{1}{x^2}, \frac{1}{x^2 y}, \frac{1}{xy^2}\right)$$
$$y_1 = y + \frac{1}{x} + O\left(\frac{1}{x^2}, \frac{1}{x^2 y}\right)$$

and we have:

$$t_1 = t - 1$$
$$y_1 = G(x, y) = H(t, y)$$

We will not find H exactly, but instead we will see it in terms of t and x. For that purpose we use:

$$t = x + r\log x + s\log y + x\beta(y) + \eta(x,y).$$

Now:

$$\begin{aligned}
\frac{1}{t} - \frac{1}{x} &= \frac{1}{x + r\log x + s\log y + x\beta(y) + \eta} - \frac{1}{x} \\
&= (-r\log x - s\log y - x\beta(y) - \eta)\,\frac{1}{x^2\left[1 + r\log x/x + s\log y/x + \beta(y) + \eta/x\right]} \\
&= \left(-r\frac{\log x}{x^2} - s\frac{\log y}{x^2} - \frac{\beta(y)}{x} - \frac{\eta}{x^2}\right)\frac{1}{1 + r\log x/x + s\log y/x + \beta(y) + \eta/x} \\
&= \left(\frac{-r\log x}{x^2} - \frac{s\log y}{x^2} - \frac{\beta(y)}{x} - \frac{\eta}{x^2}\right)\left[1 + O\left(\frac{\log x}{x}, \frac{\log y}{x}, \beta(y), \frac{\eta}{x}\right)\right] \\
&= -\frac{\beta(y)}{x} + O\left(\frac{\beta(y)^s 2}{x}, \frac{\log x}{x^2}\right) \\
&= -\frac{1}{xy} + O\left(\frac{1}{xy^2}, \frac{1}{x^{4/3}}\right).
\end{aligned}$$

So, we have:

$$-\frac{1}{t} + \frac{1}{x} = \frac{1}{xy} + O\left(\frac{1}{xy^2}, \frac{1}{x^{4/3}}\right),$$

where we write these terms since they are the largest ones asymptotically. The term that we need to cancel is therefore $1/xy$. We can use the equation (4.28), which tells us:

$$\log(y_{n+1}) - \log(y_n) = \frac{1}{x_n y_n} + O\left(\frac{1}{x_n^2 y_n}, \frac{1}{x_n y_n^2}\right)$$

to get rid of $1/xy$.

Proposition IV.18. *Define:*

$$\xi_n(t,y) = y_n - \log(y_n) + \log(t_n)$$

Then ξ_n is Cauchy with the uniform norm topology.

Proof. Using the relationships worked out above, we compute:

$$\begin{aligned}
\xi_{n+1} - \xi_n &= [y_{n+1} - \log(y_{n+1}) + \log(t_{n+1})] - [y_n - \log(y_n) + \log(t_n)] \\
&= y_{n+1} - y_n - [\log(y_{n+1}) - \log(y_n)] + [\log(t_{n+1}) - \log(t_n)] \\
&= \frac{1}{x_n} + O\left(\frac{1}{x_n^2}\right) - \frac{1}{x_n y_n} + O\left(\frac{1}{x_n^2 y_n}, \frac{1}{x_n y_n^2}\right) - \frac{1}{t_n} + O\left(\frac{1}{t_n^2}\right) \\
&= O\left(\frac{1}{x_n y_n^2}, \frac{1}{x_n^{4/3}}, \frac{1}{x_n^2}, \frac{1}{t_n^2}, \frac{1}{x_n^2 y_n}\right).
\end{aligned}$$

Using $|x_n| \sim n$, $|t_n| \sim n$ and $|y_n| \sim \log n$, we have:

$$|\xi_{n+1} - \xi_n| = O\left(\frac{1}{n^2 \log n}, \frac{1}{n^{4/3}}, \frac{1}{n \log^2 n}\right) = O\left(\frac{1}{n \log^2 n}\right)$$

which is summable. $\square$

Therefore we can define:

$$\xi(t, y) := \lim_{n\to\infty} \xi_n(t, y). = \xi_0(t, y) + \eta'(t, y),$$

where $\eta'(t, y)$ is a bounded function. Summarizing we get:

$$\Upsilon(z, w) := \xi(t, y) = y - \log(y) + \log(t) + \eta'(t, y). \tag{4.38}$$

Notice here that

$$|\xi(t, y) - y| = |-\log(y) + \log(t) + \eta'(t, y)| < 2 \log |y| + \log |t|$$

which means that $\Upsilon(z, w)$, or equivalently $\xi(t, y)$, is still close to y, when t is fixed. We will be able to extend this function ξ to a larger domain in $D'_{(z,w)}$. In the next proposition we give an estimate for the image of $D'_{(t,y)}$ under the map ξ. Increase R if necessary so we have $2R - 2\log 2R > R + 2\log R$.

Lemma IV.19. *Given the function ξ defined as in equation* (4.38)*, we have:*

1. *Fix* $t_o \in U_{4R}$*, we call* $D^{t_o} = \{y \in \mathbb{C}; (t_o, y) \in D'_{(t,y)}\}$*. Then* $\xi_{t_o}(y)$ *is injective in* D^{t_o} *and* $\xi(\{t_o\} \times D^{t_o}) \supset T'_{R+2\log R, \frac{|t_o|}{2} - 2\log\frac{|t_o|}{2}} + \log(t_o)$*. Where* $T'_{a,b} = \{z \in \mathbb{C}; a < |z| < b, |Arg(z) - \pi| < \pi/20\}$ *and* $U + w = \{z + w : z \in U\}$*.*

2. *For* $(z, w) \in D_{(z,w)}$*, we have:*

$$\Upsilon(z_1, w_1) = \Upsilon(z, w)$$

Proof. It is immediate that ξ_{t_o} is injective, by using the relationship

$$|\xi_{t_o}(y) - \log(t_o) - y| < 2\log|y|.$$

Recall that $D'_{(t,y)} = \{(t, y) \in U_{2R} \times U_R; |y| < |t|/2\}$. Fix $y_o \in U_{4R}$, the fiber is $D^{t_o} = \{y \in U_R, |y| < |t_o|/2\}$. Recall the definition of $T_{a,b}$ from equation (4.37), we have then $D^{t_o} = T_{R,|t_o|/2}$. Since $|\xi_{t_o}(y) - \log(t_o) - y| < 2\log|y|$ and for $y \in D^{t_o}$ we have $R < |y| < |t_o|/2$. Choose R large enough so we have $2R\sin(\pi/8) > 4\log R$, which implies $2S\sin(\pi/8) > 4\log S$, for any $S > R$. Fix, S any number $R < S < |t_o|/2$. We have $y \in D^{t_o}$, $|y| = S$ if and only if $y = Se^{i\pi\theta}$, where $|\theta - \pi| < \pi/8$. Then $|\xi_{t_o}(y) - \log(t_o) - y| < 2\log S$ i.e. $\xi_{t_o}(y) - \log(t_o) = Se^{i\pi\theta} + 2\log Se^{i\alpha}$, where $|\theta - \pi| < \pi/8$ and α is any number. Using the relation $2S\sin(\pi/8) > 4\log S$ we get $\mathrm{Arg}(\xi_{t_o}(y)) > \pi/20$. Also, $|\xi_{t_o}(y) - \log(t_o)| < |y| + 2\log|y| = S + 2\log S$ and $|\xi_{t_o}(y)| > S - 2\log S$. Run S from R to $t_o/2$ and we will get that $\xi_{t_o}(D^{t_o}) \supset T'_{R+2\log R, \frac{|t_o|}{2} - 2\log\frac{|t_o|}{2}} + \log(t_o)$ and the first claim is proved.

Now let us look at the relationship between $\Upsilon(z_1, w_1)$ and $\Upsilon(z, w)$:

$$\begin{aligned}
\Upsilon(z_1, w_1) &= \xi(t_1, y_1) \\
&= \lim_{n\to\infty} (y_{n+1} - \log(y_{n+1}) + \log(t_{n+1})) \\
&= \lim_{n\to\infty} (y_n - \log(y_n) + \log(t_n)) \\
&= \xi(t, y) \\
&= \Upsilon(z, w)
\end{aligned}$$

where $t = \psi(z, w)$. This completes the proof of the second claim. □

We extend Υ to $\{p \in \Omega : t = \psi(p) \in U_{4R}\}$ by defining:

$$\Upsilon(p) = \Upsilon(F^n(p)) \tag{4.39}$$

where n is chosen so that $F^n(p) \in D'_{(z,w)}$. It is clear that this definition is independent of n and that, for $t \in U_{4R}$, the mapping from $\psi^{-1}(t)$ to $\mathbb{C}$ defined by $\Upsilon_t(p) = \Upsilon(t(p), y(p))$ is injective.

Lemma IV.20. *The mapping Υ_t defined above is a biholomorphism of $\psi^{-1}(t)$ onto $\mathbb{C}$, for every $t \in U_{4R}$.*

Proof. We want to show that it is surjective. Fix $t \in U_{4R}$. Consider again the sets w_n. Recall $\psi^{-1}(t) = \bigcup_{n\geq 0} W_n$, where $W_n := \psi^{-1}(t) \cap F^{-n}(D'_{(z,w)})$. We have defined Υ_t in $\psi^{-1}(t)$ by defining it on W_0 first, as in (4.38), and defining it recursively in W_n with the relationship (4.39). We will prove that $\bigcup_{n\geq 0} \Upsilon_t(W_n) = \mathbb{C}$. For this purpose, we fix n. By definition, the image of W_n by Υ_t is equal to the image of $\psi^{-1}(t-n) \cap D'_{(z,w)}$ by Υ_{t-n}. We first analyze the image of $\psi^{-1}(t-n) \cap D'_{(z,w)}$ by Υ_{t-n}. This set is the image of $\{t-n\} \times D^{t-n}$ by ξ_{t-n}. Using Lemma IV.19 we get $\xi(\{t-n\} \times D^{t-n}) \supset T'_{R+2\log R, \frac{|t-n|}{2} - 2\log\frac{|t-n|}{2}} + \log(t-n)$. We see that, for a large n,

$\Upsilon_t(W_n) \supset T'_{2R,n} + \log n$, Clearly we have $\bigcup_{n>N} T'_{2R,n} + \log n = \mathbb{C}$, for any N. This concludes the proof. $\square$

We now have that the mapping (ψ, Υ) is a biholomorphism of $\psi^{-1}(U_{4R})$ onto $U_{4R} \times \mathbb{C}$, and furthermore, for $n \in \mathbb{N}$, $(\psi, \Upsilon \circ F^n)$ is a biholomorphism of $\psi^{-1}(U_{4R}+n)$ onto $(U_{4R} + n) \times \mathbb{C}$.

So we have the function $\psi : \Omega \to \mathbb{C}$, the open cover $\{\psi^{-1}(U_{4R} + n)\}_{n\in\mathbb{N}}$ of Ω, and the coordinates $(\psi, \Upsilon \circ F^n)$ on each $\psi^{-1}(U_{4R} + n)$ define on Ω a structure of locally trivial fiber bundle with base $\mathbb{C}$ and fiber $\mathbb{C}$.

4.5 Biholomorphism to $\mathbb{C}^2$

We can define the following biholomorphic map to $\mathbb{C}^2$

$$(\psi, \Upsilon) : \Omega \to \mathbb{C}^2$$

and

$$(\psi, \Upsilon)(p) \quad = (\psi(F^n(p)) - n, \Upsilon(F^n(p)))$$

for $p \in \Omega$. We have:

$$(\psi, \Upsilon)(F(p)) = (\psi(p) - 1, \Upsilon(p))$$

for $p \in \Omega$.

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