Invariants of Transformation Groups Acting on Real Hypersurfaces of Complex Spaces

by

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CHAPTER I

Introduction

This thesis studies real hypersurfaces in complex spaces and their invariants relative to certain groups of holomorphic transformations. This theory was pioneered by E. Cartan and was later extended by such individuals as S. S. Chern [8], J. Moser [8], N. Tanaka [21], and S. Webster [22]. In Chapter II, we will discuss some of the techniques that they developed, and we will subsequently use these tools to address several distinct problems.

In Chapter III, we consider real hypersurfaces in $\mathbb{C}^2$ under the action of the group of volume-preserving holomorphisms of $\mathbb{C}^2$. The main object of interest for us will be a scaled surface-area measure which is invariant under such transformations. It was first introduced by C. Fefferman [9] and is referred to as Fefferman hypersurface measure. We study the extremal and isoperimetric hypersurface problems associated to Fefferman hypersurface measure. We derive the associated Euler equation for both problems. We then use the work of Webster [22] to characterize the volume-preserving images of spheres, and we show that these are the only "torsion-free" global solutions of the Euler equation for the isoperimetric problem.

In Chapter IV we turn our attention to real hypersurfaces in $\mathbb{CP}^2$ under the action of the Lie group of projective transformations. We begin by developing a theory of normal forms relative to such transformations. The normal form is chosen so that the defining function is invariant through weight 3 when passing to the dual hypersurface (in an appropriate affine chart). We also show that in general, a hypersurface is not projectively equivalent to its dual beyond weight 3.

In Section IV.2, we develop an alternate theory of projective normal forms in which we try to maximize the order of contact between a hypersurface and an osculating projectively homogeneous hypersurface. We use this normal form to investigate projectively homogeneous hypersurfaces. We successfully classify all homogeneous hypersurfaces whose symmetry algebra has dimension greater than 3, in essence, we classify the "overhomogeneous" hypersurfaces. We also use the normal form to show that L^p spheres are not projectively homogeneous and that they are not always projectively self-dual.

Finally, in Section IV.3, we examine a tensor that plays a prominent role in the projective geometry of real hypersurfaces. M. Bolt refers to this tensor as Q. In [5], he shows that Q is a projective invariant, and that if it is identically 0 on a hypersurface, then the hypersurface is projectively equivalent to the hyperquadric. We show that the only holomorphic transformations preserving Q are projective transformations, and in the process, we find an alternate proof that projective transformations are the only transformations with vanishing generalized Schwarzian.

CHAPTER II

Background Material

2.1 Real Hypersurfaces of Complex Manifolds and CR Mappings

The primary objects of study in this paper will be real hypersurfaces of two-dimensional complex manifolds. Let $\mathbb{C}^n$ represent n-dimensional complex affine space with coordinates $z = (z_1, \ldots, z_n)$. A subset $Z \subset \mathbb{C}^n$ is a real hypersurface if there exists a real-valued function $\rho : \mathbb{C}^n \to \mathbb{R}$ such that $Z = \{z : \rho(z, \overline{z}) = 0\}$ and such that for each $z \in Z, |\nabla \rho(z)| \neq 0$. ρ will be called a defining function for Z. We will use ρ_{z_j} to denote $\frac{\partial \rho}{\partial z_j}$ and $\rho_{\overline{z}_j}$ to denote $\frac{\partial \rho}{\partial \overline{z}_j}$, where $z_j = x_j + iy_j$ and

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Definition II.1. Given a real hypersurface Z, we define the Monge-Ampère operator $M(\rho)$ by

$$M(\rho) = (-1)^{n+1} \begin{vmatrix} \rho & \rho_{z_j} \\ \rho_{\overline{z}_k} & \rho_{z_j,\overline{z}_k} \end{vmatrix}.$$

Definition II.2. We say that a real hypersurface Z is pseudoconvex if $M(\rho) \geq 0$ on Z. We say that Z is strongly pseudoconvex if $M(\rho) > 0$ on Z.

Note that a real hypersurface does not have a unique defining function. However, any other defining function $\widetilde{\rho}$ for Z (with the same orientation) is given by $\widetilde{\rho} = h\rho$ where $h > 0$ on Z. If $h\rho$ is another defining function for Z, then $M(h\rho) = h^{n+1}M(\rho)$ on Z. Since $h > 0$, the notion of strong pseudoconvexity is well-defined.

We will also be interested in real hypersurfaces in complex projective space. Let $\mathbb{CP}^n$ be complex projective space, i.e., the set of lines in $\mathbb{C}^{n+1}$. Then $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\})/\sim$, where $(Z_0, \ldots, Z_{n+1}) \sim \zeta(Z_0, \ldots, Z_{n+1})$ for $\zeta \in \mathbb{C}^*$. The equivalence class of $(Z_0, \ldots, Z_{n+1})$ is denoted by $[Z_0 : \cdots : Z_{n+1}]$. $\mathbb{CP}^n$ is covered by affine charts, $\mathcal{U}_j = \{[Z_0 : \cdots : Z_{n+1}] : Z_j \neq 0\}$. We have a well-defined map $\mathcal{U}_j \to \mathbb{C}^n$ given by

$$[Z_0 : \cdots : Z_{n+1}] \mapsto \left(\frac{Z_0}{Z_j}, \ldots, \hat{Z}_j, \ldots, \frac{Z_n}{Z_j}\right),$$

where the hat means that we omit an entry. A real hypersurface in $\mathbb{CP}^n$ is a subset Z of $\mathbb{CP}^n$ which is a real hypersurface of $\mathbb{C}^n$ in any affine chart.

Remark II.3. Since we are working locally, we can restrict to just one affine chart, so we are essentially just looking at a real hypersurface in $\mathbb{C}^n$ again. The distinction is that we will consider these hypersurfaces modulo equivalence under automorphisms of projective space.

We now discuss the CR structure of real hypersurfaces of $\mathbb{C}^2$. Our exposition follows that in Burns and Shnider [7]. On $T\mathbb{C}^2$ there is the complex structure transformation J. Define a complex vector subbundle HZ over Z by

$$HZ = TZ \cap J(TZ) \subset T\mathbb{C}^2. \tag{2.1}$$

Define

$$E = (HZ)^{\perp} = \{\theta \in T^*Z : \theta(X) = 0 \text{ for all } X \in HZ\}.$$

If $\mathcal{D}$ is the sheaf of germs of $\mathbb{R}$-valued smooth functions on Z, then the sheaf of germs of smooth sections of E, denoted by $\mathcal{I}$, is a sheaf of $\mathcal{D}$-modules. Let $\mathcal{D}_{\mathbb{C}} = \mathcal{D} \otimes \mathbb{C}$, and let $\mathcal{C}$ denote the sheaf of germs of $\mathbb{C}$-valued 1-forms on Z, which are $\mathbb{C}$-linear when restricted to HZ. Then $\mathcal{C}$ is a sheaf of $\mathcal{D}_{\mathbb{C}}$-modules.

Define the CR structure on Z to be the pair $(\mathcal{I}, \mathcal{C})$. A smooth map $f : (Z, \mathcal{I}, \mathcal{C}) \to (Z', \mathcal{I}', \mathcal{C}')$ is called a CR map if

$$f^*\mathcal{I}' \subset \mathcal{I},\ f^*\mathcal{C}' \subset \mathcal{C}.$$

Equivalently,

$$df(HZ) \subset HZ'$$

and df is $\mathbb{C}$-linear on HZ. Note that the restriction of a holomorphic mapping $f : \mathbb{C}^2 \to \mathbb{C}^2$ restricted to the real hypersurface $Z \subset \mathbb{C}^2$ is a CR mapping.

We define a real hypersurface $Z \subset \mathbb{C}^2$ to be a *real-analytic* if it has a defining function which is real analytic. The following theorem is a special case of the corollary on p.29 of [4].

Theorem II.4. *Let $Z \subset \mathbb{C}^2$ be a real-analytic hypersurface and let f be function on Z, then f has a holomorphic extension in a neighborhood of $P \in \mathbb{C}^2$ if and only if f CR and real-analytic.*

Hence, if we restrict to real-analytic functions on real-analytic hypersurfaces, then CR maps are just the restrictions of holomorphic maps. Since the machinery that we shall apply throughout this paper requires the assumption of real-analyticity, we shall use "CR map" and "restriction of a holomorphic map" interchangeably.

2.2 The Weighting of Moser

In [8], Moser developed a "normal" form for the defining function of a general strongly pseudoconvex real hypersurface in $\mathbb{C}^n$, i.e., he chose special coordinates for the hypersurface. In doing this, he found it useful to introduce the notion of weight. Let $\mathbb{C}^2$ have coordinates (z, w), and let $f(z, \overline{z}, w, \overline{w})$ be a function of (z, w). We say f has weight k if

$$f(\lambda z, \lambda \overline{z}, \lambda^2 w, \lambda^2 \overline{w}) = \lambda^k f(z, \overline{z}, w, \overline{w}),$$

for $\lambda > 0$. This arises naturally, because he normalizes the defining function of the hypersurface so that it is of the form

$$\rho(z, \overline{z}, w, \overline{w}) = \frac{-(w - \overline{w})}{2i} + F\left(z, \overline{z}, \frac{w + \overline{w}}{2}\right),$$

where F has no linear terms in $\frac{w+\overline{w}}{2}$. So $\mathrm{Im}(w)$ is second order in z. Moreover, with this definition, the non-isotropic dilation

$$(z, w) \mapsto (\lambda z, \lambda^2 w)$$

is weighted homogeneous.

2.3 The Calculus of Variations

Since we address several variational problems, we will need to use some very basic notions from the calculus of variations. Let X be a space of functions. Let $\mathcal{J} : X \to \mathbb{R}$ be a functional. In order to define continuous and differentiable functionals, we will assume that X is a normed vector space with norm $\|\cdot\|$. Define $\Delta\mathcal{J}[\rho^{(0)}] : X \to \mathbb{R}$, by

$$\Delta\mathcal{J}[\rho^{(0)}](\rho) = \mathcal{J}(\rho + \rho^{(0)}) - \mathcal{J}(\rho^{(0)}).$$

Suppose that

$$\Delta\mathcal{J}[\rho^{(0)}](\rho) = \phi(\rho) + \epsilon\|\rho\|,$$

where ϕ is a linear functional and $\epsilon \to 0$ as $\|\rho\| \to 0$. Then the functional $\mathcal{J}$ is said to be differentiable, and the principal linear part of $\Delta\mathcal{J}$ is called the first variation of $\mathcal{J}(\rho^{(0)})$ and is denoted by $\delta\mathcal{J}[\rho^{(0)}]$.

Definition II.5. Let $\rho^{(0)} \in X$, then $\rho^{(0)}$ is a maximum (resp. minimum) for $\mathcal{J}$ if for all $\rho^{(1)} \in X$ one has $\mathcal{J}(\rho^{(0)}) \geq \rho^{(1)}$ (resp. $\mathcal{J}(\rho^{(0)}) \leq \rho^{(1)}$).

We cite a theorem of fundamental importance from Fomin and Gelfand [11] p.13:

Theorem II.6. *A necessary condition for the differentiable functional $\mathcal{J}$ to have an extremum at $\rho^{(0)}$ is that $\rho^{(0)}$ have vanishing first variation. This condition is known as Euler's equation.*

The following is a well-known and simple theorem from the Calculus of Variations. We refer the reader to [20], p. 223.

Theorem II.7. *If a functional is given by*

$$\mathcal{J}(F) = \int \mathcal{F}(x, F(x), F_j(x), F_{j,k}(x))dV,$$

where $\mathcal{F}(y, y_j, y_{j,k})$ is a function of $1 + n + n^2$ variables, and $F_{j,k}$ denotes $\frac{\partial^2}{\partial x_j \partial x_k}F(x)$ then for smooth functions, the vanishing of the first variation is equivalent to

$$\frac{\partial}{\partial y}\mathcal{F} \circ F^{(2)} - \sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial y_j}\mathcal{F} \circ F^{(2)}\right) + \sum \frac{\partial^2}{\partial x_j \partial x_k}\left(\frac{\partial}{\partial y_{j,k}}\mathcal{F} \circ F^{(2)}\right) = 0$$

where $F^{(2)}(x) = (x, F(x), F_j(x), F_{j,k}(x))$.

2.4 Cartan's Theory of Moving Frames

Let M be an n-dimensional manifold. Following Olver in [19], we define $\mathbb{K}^{(s)}(n)$, the sth order classifying space, to be Euclidean space of dimension $q_s(n) = \frac{1}{2}n^2(n-1)\binom{n+s}{n}$, where the coordinates are indexed by z_σ where $\sigma = (i, j, k, l_1, \ldots, l_s)$ is a multi-index of order s. Given a coframe $\mathbf{\Theta} = (\theta^1, \ldots, \theta^n)$ and a function f on M, we have that $df = \sum a_j\theta^j$. We define $\frac{\partial f}{\partial \theta^j} = a_j$. Let $T_\sigma = \frac{\partial^s T^i_{j,k}}{\partial\theta^{l_s}\cdots\partial\theta^{l_1}}$, where $\mathbf{\Theta} = (\theta^1, \ldots, \theta^n)$ is a coframe and $d\theta^i = \sum T^i_{j,k}\theta^j \wedge \theta^k$. Define the sth order structure map $\mathbf{T}^{(s)} : M \to \mathbb{K}^{(s)}$ by $z_\sigma = T_\sigma(x)$. The sth classifying space $\mathcal{C}^{(s)}(\theta, M)$ is the image of $\mathbf{T}^{(s)}$. We say that a coframe $\mathbf{\Theta}$ is *fully regular* if for all s, the sth order structure map is regular, i.e., has constant rank r_s. Note that r_s is the number of functionally independent structure invariants up to order s. Finally, we define the *order* of the coframe Θ to be the smallest s for which $r_s = r_{s+1}$.

One has

Theorem II.8. *Let $\mathbf{\Theta}$ and $\overline{\mathbf{\Theta}}$ be smooth, fully regular coframes on M and $\overline{M}$ respectively. Then there exists a local diffeomorphism $\Phi : M \to \overline{M}$ with $\Phi^*\overline{\mathbf{\Theta}} = \mathbf{\Theta}$ if and only if they have the same order $\overline{s} - 1 = s - 1$ and $\mathcal{C}^{(s)}(\mathbf{\Theta}, U) \cap \mathcal{C}^{(s)}(\overline{\mathbf{\Theta}}, \overline{U})$ is a non-empty submanifold of $\mathbb{K}^{(s)}(m)$ with the same dimension as $\mathcal{C}^{(s)}(\mathbf{\Theta}, U)$, where $U \subset M$ and $\overline{U} \subset \overline{M}$ are open neighborhoods.*

The basic idea is to construct the graph of Φ in $M \times \overline{M}$. The graph of Φ will arise as the integral manifold of a distribution, and the condition on the classifying manifolds allows us to use Frobenius' Theorem to guarantee that this integral manifold exists. As an instructive example, we shall prove this theorem in the special case where the order is 0.

Proposition II.9. *Suppose $\mathbf{\Theta} = (\theta^1, \ldots, \theta^n)$ and $\overline{\mathbf{\Theta}} = (\overline{\theta}^1, \ldots, \overline{\theta}^n)$ are coframes on M and $\overline{M}$ respectively. Suppose additionally that*

$$d\theta^i = \sum T^i_{j,k}\theta^j \wedge \theta^k$$
$$d\overline{\theta}^i = \sum \overline{T}^i_{j,k}\overline{\theta}^j \wedge \overline{\theta}^k,$$

for constants $T^i_{j,k}, \overline{T}^i_{j,k}$, for $i = 1, \ldots, n$ and $1 \leq j < k \leq n$. Then there exists a diffeomorphism $\Phi : M \to \overline{M}$ with

$$\Phi^* \overline{\mathbf{\Theta}} = \mathbf{\Theta}$$

if and only if

$$\overline{T}^i_{j,k} = T^i_{j,k}$$

for all i, j, k.

Proof. The necessity of this condition is obvious, for if $\Phi : M \to \overline{M}$ with $\Phi^* \overline{\mathbf{\Theta}} = \mathbf{\Theta}$, then for all i, j, k,

$$\begin{aligned}\sum \overline{T}^i_{j,k} \theta^j \wedge \theta^k =& \Phi^* \left(\sum \overline{T}^i_{j,k} \overline{\theta}^j \wedge \overline{\theta}^k \right) \\ =& \Phi^* d\overline{\theta}^i = d\theta^i \\ =& \sum T^i_{j,k} \theta^j \wedge \theta^k.\end{aligned}$$

Therefore

$$\overline{T}^i_{j,k} = T^i_{j,k}.$$

Conversely, suppose

$$\overline{T}^i_{j,k} = T^i_{j,k}.$$

Then on $M \times \overline{M}$, we have the natural projection maps, $\pi_1 : M \times \overline{M} \to M$ and $\pi_2 : M \times \overline{M} \to \overline{M}$. Therefore, we have 1-forms $\pi_1^* \theta^i$ and $\pi_2^* \overline{\theta}^i$ on $M \times \overline{M}$. We shall suppress the pull-back notation and refer to these forms as θ^i and $\overline{\theta}^i$. Define the 1-forms ϕ^i on $M \times \overline{M}$ by

$$\phi^i = \overline{\theta}^i - \theta^i.$$

Let $\mathcal{J}$ be the ideal that the ϕ^i generate. Note that

$$\begin{aligned}d\phi^i =& d\overline{\theta}^i - d\theta^i = \sum \left(T^i_{j,k} \overline{\theta}^j \wedge \overline{\theta}^k - T^i_{j,k} \theta^j \wedge \theta^k \right) \\ =& \sum T^i_{j,k} \left(\left(\overline{\theta}^j - \theta^j \right) \wedge \overline{\theta}^k + \theta^j \wedge \left(\overline{\theta}^k - \theta^k \right) \right) \\ =& \sum T^i_{j,k} \left(\phi^j \wedge \overline{\theta}^k + \theta^j \wedge \phi^k \right).\end{aligned}$$

Hence

$$d\mathcal{J} \subset \mathcal{J}.$$

Therefore, by Frobenius' Theorem, there exists a unique n-dimensional integral submanifold $N \subset M \times \overline{M}$ of $\mathcal{J}$ passing through each point of $M \times \overline{M}$. By definition, all of the ϕ^i restrict to 0 on N. Now, we claim that N is transverse to M and $\overline{M}$. Let X be a horizontal tangent vector. If X is tangent to N, then it must be annihilated by all of the ϕ^i. But since the $\overline{\theta}^i$ restrict to 0 on M, we have

$$\phi^i(X) = -\theta^i(X) =: C_i.$$

Since the θ^i constitute a coframe on M, all of the C_i will vanish precisely when $X = 0$. Therefore, N is transverse to M. A similar argument shows that N is transverse to $\overline{M}$. By the Implicit Function Theorem, N is locally a graph over M, so there exists

$$\Phi : M \to \overline{M},$$

such that

$$N = \{(x, \Phi(x)) : x \in M\}.$$

Since the ϕ^i restrict to 0 on N, we must have that

$$\Phi^* \overline{\theta}^i = \theta^i.$$

This completes the proof. $\square$

CHAPTER III

Volume-Preserving Invariants

3.1 Volume-Preserving Normal Form

Let $G(z,w) = (f(z,w), g(z,w))$ be a holomorphic transformation of $\mathbb{C}^2$ with Jacobian $JG := f_z g_w - f_w g_z$. We restrict to transformations where $|JG| = 1$. The set of such transformations is a group under composition, called the group of volume-preserving transformations of $\mathbb{C}^2$. Let $Z = \{\rho(z, \overline{z}, w, \overline{w}) = 0\} \subset \mathbb{C}^2$ be a strongly pseudoconvex real hypersurface. We wish to construct a normal form for the hypersurface with respect to the group of volume-preserving automorphisms of $\mathbb{C}^2$. However, we will not determine a normal form for the hypersurface, rather, using the weighting of Moser, we will construct a normal form for the terms of weight at most 4 in the defining function for our hypersurface. Since this will involve some rather lengthy computations, we used Mathematica 7.0 to verify them. The calculations done with Mathematica that are used here can be found in Appendices A.1 and A.2. The Mathematica files themselves can be found online at http://hdl.handle.net/2027.42/62185. They include: VPNormForm1.nb, VPNormForm2.nb, and VPNormForm3.nb.

At a general point (z_0, w_0) of Z, use a translation to center the hypersurface at the origin. Without loss of generality, we assume $\rho_w(0,0) \neq 0$. Let

$$f(z,w) = -2i\rho_w(0,0)z,\ g(z,w) = 2i\rho_z(0,0)z + \frac{i}{2\rho_w(0,0)}w.$$

Then our new defining function is $\widetilde{\rho} = \rho \circ G$, where $\widetilde{\rho}_z(0,0) = 0, \widetilde{\rho}_w(0,0) = \frac{i}{2}$, so by the Implicit Function Theorem, we can write

$$h\widetilde{\rho}(z, \overline{z}, w, \overline{w}) = -v + F(z, \overline{z}, u),$$

where $h > 0, w = u + iv$, and F has no linear terms. If

$$F(z, \overline{z}, u) = \alpha|z|^2 + \beta z^2 + \overline{\beta}\overline{z}^2 + \cdots,$$

then $\alpha \neq 0$ because Z is strongly pseudoconvex. Use the volume-preserving transformation

$$(z,w) \to (\alpha^{-\frac{1}{3}}z, \alpha^{\frac{1}{3}}w)$$

and our hypersurface becomes

$$-\alpha^{\frac{1}{3}}v + \alpha(\alpha^{-\frac{1}{3}})^2|z|^2 + \beta(\alpha^{-\frac{1}{3}})^2z^2 + \overline{\beta}(\alpha^{-\frac{1}{3}})^2\overline{z}^2 + \cdots = 0$$

or

$$-v + |z|^2 + \frac{\beta}{\alpha}z^2 + \frac{\overline{\beta}}{\alpha}\overline{z}^2 + \cdots = 0.$$

Using a rotation, we may assume $\overline{\beta} = \beta \geq 0$. Henceforth, we may assume that our transformations take the form

$$(z,w) \to (z + bw + \cdots, w + \cdots),$$

where b is an arbitrary constant.

Before going any further, we describe how such transformations change the equation defining Z, following Moser's approach. Suppose one begins with a hypersurface given by

$$v^* = F^*(z^*, \overline{z^*}, u^*) = F_2^* + F_3^* + \cdots ,$$

where F_i^* is a real-valued function of weight i as described in Section 2.2, i.e., $F^*(\lambda z^*, \lambda\overline{z^*}, \lambda^2 u^*) = \lambda^i F^*(z^*, \overline{z^*}, u^*)$, and $w^* = u^* + iv^*$. One wishes to change coordinate via the map

$$(z, w) \mapsto (z^*, w^*) = (f(z, w), g(z, w)),$$

so that the new equation is

$$v = F(z, \overline{z}, u) = F_2 + F_3 + \cdots .$$

We also decompose f and g into weighted homogeneous parts, homogeneous of weight 1 in z and of weight 2 in w. In order for the map to preserve the normalizations we have already made, it must be the case that

$$f(z, w) = z + f_2(z, w) + \cdots \text{ and } g(z, w) = w + g_2(z) + g_3(z, w) + \cdots .$$

Then

$$\mathrm{Im}(w^*) = F^*(f, \overline{f}, u + \mathrm{Re}(g_2(z) + g_3(z, w) + \cdots))$$

or

$$v + \mathrm{Im}(g_2(z)) + \mathrm{Im}(g_3(z, w)) + \cdots = F^*(f, \overline{f}, u + \mathrm{Re}(g_3(z, w)) + \cdots),$$

and finally

$$F = F^*(f, \overline{f}, u + \mathrm{Re}(g_2(z)) + \mathrm{Re}(g_3(z, w)) + \cdots) - \mathrm{Im}(g_2(z)) - \mathrm{Im}(g_3(z, w)) - \cdots .$$

We now use the well-known fact that any composition of shears is a volume-preserving automorphism. Consider the shear $(z, w) \to (z, w + az^2)$. If our defining function is

$$\rho(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + \beta z^2 + \beta\overline{z}^2 + \cdots ,$$

then our new defining function is

$$\widetilde{\rho}(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + \beta z^2 + \beta\overline{z}^2 - \mathrm{Im}(az^2) + \cdots .$$

If $a = 2i\beta$, then $\mathrm{Im}(az^2) = 2\beta\ \mathrm{Re}(z^2)$, and

$$\widetilde{\rho}(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + \cdots .$$

Now, let us assume that our defining function is

$$\rho(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + Az^3 + Bz^2\overline{z} + Czu + \overline{A}\overline{z}^3 + \overline{B}z\overline{z}^2 + \overline{C}\overline{z}u + \cdots ,$$

then if $G(z, w) = (z + bw, w)$ our new defining function is

$$\begin{aligned}\widetilde{\rho}(z, \overline{z}, w, \overline{w}) &= -v + |z|^2 + \beta z^2 + \beta\overline{z}^2 + (A + 2i\beta^2 b - i\beta\overline{b})z^3 + (B + 3ib - i\overline{b} - 2i\beta^2\overline{b})z^2\overline{z}\\ &+ (C + 2\beta b + \overline{b})zu + (\overline{A} - 2i\beta^2\overline{b} + i\beta b)\overline{z}^3 + (\overline{B} - 3i\overline{b} + ib + 2i\beta^2 b)z\overline{z}^2 + (\overline{C} + 2\beta\overline{b} + b)\overline{z}u + \cdots .\end{aligned}$$

We can find b to solve

$$C + 2\beta b + \overline{b} = 0.$$

Consider the shear $(z, w) \to (z, w + az^n)$. This can be used to eliminate any term of the form $Az^n + \overline{A}\overline{z}^n$. Likewise, the shears $(z, w) \to (z + bw^n, w)$ can be used to eliminate terms of the form $Bzu^n + \overline{B}\overline{z}u^n$. This is as much as can be done with shears one at a time.

Now, we assume our defining function is of the form

$$\rho(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + Bz^2\overline{z} + \overline{B}z\overline{z}^2 + Cz^3\overline{z} + D|z|^4 + Ez^2u + K|z|^2u + Lu^2 + \overline{C}z\overline{z}^3 + \overline{E}\overline{z}^2u + \cdots$$

One might expect that using the composition of two shears might be helpful here. However, this quickly becomes computationally intractable. Instead, we use a result of Forstneric [12].

Theorem III.1. *Any volume-preserving k-jet extends to a volume preserving automorphism of* $\mathbb{C}^n$.

Let
$$f(z,w) = \sum a_{i,j} z^i w^j, g(z,w) = \sum b_{i,j} z^i w^j.$$
Then
$$JG = \sum_{n,m} \left(\sum_{i=0}^{n} \sum_{l=0}^{m} (i+1)(l+1)\left(a_{i+1,m-l} b_{n-i,l+1} - a_{n-i,l+1} b_{i+1,m-l}\right) \right) z^n w^m = \sum c_{n,m} z^n w^m.$$

A k-jet is volume preserving if $c_{0,0} = 1, c_{n,m} = 0$ for $0 < n+m \leq k-1$. We would like to find a volume-preserving k-jet that will transform our defining function into one of the form
$$\widetilde{\rho}(z,\overline{z},w,\overline{w}) = -v + |z|^2 + \kappa|z|^4 + \gamma z^3 \overline{z} + \overline{\gamma} z \overline{z}^3 + \cdots .$$
Since only the 4-jet of our transformation will effect these terms, we look for a volume-preserving 4-jet that accomplishes this. We do this in stages. All of the computations involved can be done by hand, but for convenience, they can be found in Appendix A. Let
$$f(z,w) = \sum a_{i,j} z^i w^j, g(z,w) = \sum b_{i,j} z^i w^j,$$
where
$$a_{0,0} = b_{0,0} = 0, a_{1,0} = 1, b_{1,0} = 0, \text{ and } b_{0,1} = 1.$$
Suppose that initially
$$F(z,\overline{z},u) = \sum B_{j,k,l} z^j \overline{z}^k u^l,$$
then our new defining function is the graph of
$$\widetilde{F}(z,\overline{z},u) = \sum \widetilde{B}_{j,k,l} z^j \overline{z}^k u^l.$$

Our new F will be $\widetilde{F}(z,\overline{z},u) = F(f(z,w), \overline{f}(z,w), \mathrm{Re}(g(z,w))) - \mathrm{Im}(g) + v$, on Z. We solve the system of equations arising from $\widetilde{F}(z,\overline{z},u) = |z|^2 + \widetilde{F}_4(z,\overline{z},u)$ where the coefficients of $\widetilde{F}_4$ are all 0 except for possibly the coefficients of $|z|^4, z^3\overline{z}$, and $z\overline{z}^3$. We also require $c_{0,0} = 1, c_{n,m} = 0$ for $0 < n+m \leq 3$. Note that
$$\widetilde{B}_{2,0,0} = B_{2,0,0} + \frac{i}{2} b_{2,0}.$$
Since $B_{2,0,0} = 0$, we must have $b_{2,0} = 0$. Likewise
$$\widetilde{B}_{3,0,0} = B_{3,0,0} + \frac{i}{2} b_{3,0},$$
and since $B_{3,0,0} = 0$, we require that $b_{3,0} = 0$. With these restrictions, we have that
$$\begin{aligned} c_{0,0} &= 1 \\ c_{1,0} &= b_{1,1} + 2a_{2,0} \\ c_{0,1} &= a_{1,1} + 2b_{0,2} - a_{0,1} b_{1,1}. \end{aligned}$$
Since $c_{1,0} = 0$,
$$a_{2,0} = \frac{-1}{2} b_{1,1}.$$
We can choose $a_{1,1}$ so that $c_{0,1} = 0$. We have
$$\widetilde{B}_{2,1,0} = B_{2,1,0} + a_{2,0} - i\overline{a_{0,1}} - \frac{1}{2} b_{1,1} \text{ and } \widetilde{B}_{1,0,1} = B_{1,0,1} + \overline{a}_{0,1} + \frac{i}{2} b_{1,1}.$$

Since $B_{1,0,1} = 0$, $\widetilde{B}_{1,0,1} = 0$ implies that

$$a_{0,1} = \frac{i}{2}\overline{b}_{1,1}.$$

Then $\widetilde{B}_{2,1,0} = 0$ if and only if

$$b_{1,1} = \frac{2}{3}B_{2,1,0}.$$

This means that we can find the 2-jet of a volume-preserving biholomorphism such that our earlier normalizations are preserved and $\widetilde{B}_{2,1,0} = 0$. Once we have $\widetilde{B}_{2,1,0} = 0$, we impose the restrictions that $b_{1,1} = 0, a_{0,1} = 0$, and $a_{2,0} = 0$. Now

$$\widetilde{B}_{4,0,0} = B_{4,0,0} + \frac{i}{2}b_{4,0}.$$

Since $B_{4,0,0} = 0, b_{4,0} = 0$. Now we have that

$$\begin{aligned} c_{0,1} &= a_{1,1} + 2b_{0,2} \\ c_{2,0} &= 3a_{3,0} + b_{2,1} \\ c_{1,1} &= 2a_{2,1} + 2b_{1,2} \end{aligned}$$

and

$$c_{0,2} = a_{1,2} + 2a_{1,1}b_{0,2} + 3b_{0,3}.$$

We wish to obtain the normalizations $\widetilde{B}_{2,0,1} = 0, \widetilde{B}_{1,1,1} = 0, \widetilde{B}_{0,0,2} = 0$. We have

$$\begin{aligned} \widetilde{B}_{2,0,1} &= B_{2,0,1} + \frac{i}{2}b_{2,1} \\ \widetilde{B}_{1,1,1} &= B_{1,1,1} + 2\mathrm{Re}(a_{1,1}) - 2\mathrm{Re}(b_{0,2}) \end{aligned}$$

and

$$\widetilde{B}_{0,0,2} = B_{0,0,2} - \mathrm{Im}(b_{0,2}).$$

Then $\widetilde{B}_{2,0,1} = 0$ implies that

$$b_{2,1} = 2iB_{2,0,1}.$$

Also $c_{2,0} = 0$ if and only if

$$a_{3,0} = \frac{-1}{3}b_{2,1}$$

and $c_{0,1} = 0$ if and only if

$$a_{1,1} = -2b_{0,2}.$$

Then $\widetilde{B}_{1,1,1} = 0$ if and only if

$$\mathrm{Re}(b_{0,2}) = \frac{-1}{6}B_{1,1,1}.$$

Finally, $\widetilde{B}_{0,0,2} = 0$ if and only if

$$\mathrm{Im}(b_{0,2}) = B_{0,0,2}.$$

We have found the 3-jet of a volume-preserving biholomorphism that eliminates all of these terms.

We now wish to show that if

$$\rho(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + \kappa|z|^4 + \gamma z^3\overline{z} + \overline{\gamma}z\overline{z}^3 + \cdots$$

then no 4-jet of a volume-preserving transformation can alter ρ without reintroducing terms of lower weight. Again, the computations can be found in Appendix A. Suppose $G(z, w) = (f(z, w), g(z, w))$, where $f(z,w) = z + f_2(z,w) + f_3(z,w) + \cdots$ and $g(z,w) = w + g_3(z,w) + g_4(z,w) + \cdots$, where f_i and g_i have weight i. Our new F is

$$\widetilde{F}(z,\overline{z},u) = F(f(z,w), \overline{f}(z,w), \mathrm{Re}(g)(z,w)) - \mathrm{Im}(g) + v$$

on Z. The weight 3 part of $\widetilde{F}$ is

$$2\mathrm{Re}(\overline{z} f_2(z,w)) - \mathrm{Im}(g_3(z,w)) = 2\mathrm{Re}\left((a_{2,0} - i\overline{a_{0,1}} - \frac{1}{2} b_{1,1}) z^2\overline{z} - \frac{b_{3,0}}{2i} z^3 + (\overline{a_{0,1}} - \frac{b_{1,1}}{2i}) zu \right).$$

In order to preserve our earlier normalizations, we set the above coefficients equal to 0 and solve. It must be the case that

$$b_{3,0} = 0, a_{0,1} = -\frac{\overline{b_{1,1}}}{2i}, a_{2,0} = b_{1,1}.$$

We introduce these assumptions. Now, the coefficient of $z^2 u$ is

$$\frac{-i}{2} b_{1,1}^2 + \frac{i}{2} b_{2,1},$$

so $b_{2,1} = b_{1,1}^2$. Making this assumption, the coefficient of u^2 is

$$\frac{i}{2} b_{0,2} - \frac{i}{2} \overline{b_{0,2}} + \frac{1}{4} |b_{1,1}|^2,$$

so $\mathrm{Im}(b_{0,2}) = \frac{1}{4}|b_{1,1}|^2$. But $c_{1,0} = 3b_{1,1}$ and $c_{1,0} = 0$, so $b_{1,1} = 0$. Moreover,

$$c_{0,1} = a_{1,1} + 2b_{0,2}$$

so $a_{1,1} = -2b_{0,2}$ and $a_{1,1}$ is real. The new coefficient of $|z|^4$ is

$$\kappa - 2\mathrm{Im}(a_{1,1}) = \kappa.$$

Finally, $0 = c_{2,0} = 3a_{3,0}$ so $a_{3,0} = 0$. The new coefficient of $z^3\overline{z}$ is

$$\gamma + a_{3,0} = \gamma$$

and we see that ρ is unchanged. Hence, we have a normal form for ρ up through weight 4, and the coefficients κ and $\gamma, \overline{\gamma}$ represent volume-preserving invariants of our hypersurface. Actually, we have been requiring that $JG = 1$, but really, we need only impose the restriction that $|JG| = 1$. This additional freedom allows us to use a rotation in z so that γ is real and $\gamma \geq 0$. We state this result as:

Theorem III.2. *Given a strongly pseudoconvex real hypersurface $Z \subset \mathbb{C}^2$ and a point $(z_0, w_0) \in Z$, there exists the germ of a volume-preserving biholomorphic transformation $G : \mathbb{C}^2 \to \mathbb{C}^2$ at (z_0, w_0) such that $G(Z)$ has defining function $\rho(z,w) = -v + |z|^2 + \kappa|z|^4 + \gamma z^3\overline{z} + \gamma z\overline{z}^3 + \cdots$, where all other terms have weight greater than 4 and γ is real. Moreover, if there exists any other such germ of a biholomorphic volume-preserving transformation $\widetilde{G} : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\widetilde{G}(Z)$ has a defining function $\widetilde{\rho}(z,w) = -v + |z|^2 + \widetilde{\kappa}|z|^4 + \widetilde{\gamma} z^3\overline{z} + \widetilde{\gamma} z\overline{z}^3 + \cdots$ where all other terms have weight greater than 4 and $\widetilde{\gamma}$ is real, then $\widetilde{\kappa} = \kappa$ and $\widetilde{\gamma} = \gamma$.*

3.2 Classification of Volume-Preserving Images of Spheres

In this section, we will use a theorem of S. Webster [22] to determine when it is possible to find a volume-preserving CR isomorphism between a general strongly pseudoconvex real hypersurface Z in $\mathbb{C}^2$ and the sphere of radius R. Webster's work is inspired by S. S. Chern who uses techniques invented by E. Cartan. Before proceeding, we want to indicate how Cartan's approach works in general. This was succinctly and lucidly explained by Burns and Shnider [7]. The following explanation is taken from there almost verbatim, changing only notation. Associate to any manifold Z with the given structure a second manifold P, fibered over Z, such that any structure-preserving map f lifts to an $\widetilde{f}$ for which the following diagram commutes:

$$\begin{array}{ccc} P_1 & \xrightarrow{\widetilde{f}} & P_2 \\ {\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\ Z_1 & \xrightarrow{f} & Z_2 \end{array}$$

Then define on P a parallelism,i.e., a trivialization of the tangent bundle (equivalently, define a coframe Φ), such that the maps $g : P_1 \to P_2$ of the form $\widetilde{f}$ are precisely those for which dg carries the parallelism of P_1 onto that of P_2 (equivalently, such that $g^*\Phi_2 = \Phi_1$). Having reduced the problem to maps preserving a given parallelism (equivalently, coframe), one can use Cartan's method to solve the problem, as explained in Section 2.4.

Let $Z = \{\rho(z, \overline{z}, w, \overline{w}) = 0\} \subset \mathbb{C}^2$ be a strongly pseudoconvex real hypersurface. We have that the real gradient of ρ is $\nabla\rho(z, \overline{z}, w, \overline{w}) = 2\left(\rho_z \frac{\partial}{\partial \overline{z}} + \rho_{\overline{z}} \frac{\partial}{\partial z} + \rho_w \frac{\partial}{\partial \overline{w}} + \rho_{\overline{w}} \frac{\partial}{\partial w}\right)$, and $|\nabla\rho(z, \overline{z}, w, \overline{w})| = 2\left(|\rho_z|^2 + |\rho_w|^2\right)^{\frac{1}{2}}$. The volume form on $\mathbb{C}^2$ is $\omega_{\mathbb{C}^2} = \frac{-1}{4} dz \wedge d\overline{z} \wedge dw \wedge d\overline{w}$. Surface area measure on Z is

$$s_Z = \frac{\nabla\rho}{|\nabla\rho|} \rfloor \omega_{\mathbb{C}^2} = \frac{1}{2|\nabla\rho|}(-\rho_{\overline{z}} d\overline{z} \wedge dw \wedge d\overline{w} + \rho_z dz \wedge dw \wedge d\overline{w} - \rho_{\overline{w}} dz \wedge d\overline{z} \wedge d\overline{w} + \rho_w dz \wedge d\overline{z} \wedge dw).$$

Fefferman hypersurface measure on Z, call it σ_Z, is given by $d\rho \wedge \sigma_Z = 2^{\frac{4}{3}} M(\rho)^{\frac{1}{3}} \omega_{\mathbb{C}^2}$, where $M(\rho)$ is the Monge-Ampère operator defined in section 2.1. Equivalently, $\sigma_Z = \frac{2^{\frac{4}{3}} M(\rho)^{\frac{1}{3}}}{|\nabla\rho|} s_Z$. See Barrett [2] for a more detailed discussion of Fefferman hypersurface measure. A holomorphic map $G : \mathbb{C}^2 \to \mathbb{C}^2$ has the property $G^*\sigma_{G(Z)} = |JG|^{\frac{4}{3}} \sigma_Z$. We see that G is a volume-preserving holomorphic map if and only if it preserves Fefferman hypersurface measure. Let $\theta_0 = i\partial\rho$. Note that on our hypersurface,

$$\begin{aligned} d\overline{z} \wedge dw \wedge d\overline{w} &= \frac{i\rho_z}{|\rho_w|^2} \theta_0 \wedge dz \wedge d\overline{z} \\ dz \wedge dw \wedge d\overline{w} &= \frac{-i\rho_{\overline{z}}}{|\rho_w|^2} \theta_0 \wedge dz \wedge d\overline{z} \\ dz \wedge d\overline{z} \wedge dw &= \frac{-i\rho_{\overline{w}}}{|\rho_w|^2} \theta_0 \wedge dz \wedge d\overline{z} \\ dz \wedge d\overline{z} \wedge d\overline{w} &= \frac{i\rho_w}{|\rho_w|^2} \theta_0 \wedge dz \wedge d\overline{z}. \end{aligned}$$

Then

$$s_Z = \frac{-i|\nabla\rho|}{4|\rho_w|^2} \theta_0 \wedge dz \wedge d\overline{z},$$

and

$$\sigma_Z = \frac{-i2^{\frac{-2}{3}} M(\rho)^{\frac{1}{3}}}{|\rho_w|^2} \theta_0 \wedge dz \wedge d\overline{z}.$$

Now, we define $\theta = 2^{-\frac{1}{3}} M(\rho)^{-\frac{1}{3}} \theta_0$. Then

$$d\theta = 2^{-\frac{1}{3}} d(M(\rho)^{-\frac{1}{3}}) \wedge \theta_0 + 2^{-\frac{1}{3}} M(\rho)^{-\frac{1}{3}} d\theta_0.$$

We compute

$$d\theta_0 = \frac{-iM(\rho)}{|\rho_w|^2} dz \wedge d\overline{z} + \theta_0 \wedge \phi_0, \tag{3.1}$$

where

$$\phi_0 = A_0 dz + \overline{A}_0 d\overline{z}, \tag{3.2}$$

with

$$A_0 = \frac{\rho_{w\overline{w}}\rho_z}{|\rho_w|^2} - \frac{\rho_{z\overline{w}}}{\rho_{\overline{w}}}. \tag{3.3}$$

So now

$$\theta \wedge d\theta = \frac{-i2^{-\frac{2}{3}}M(\rho)^{\frac{1}{3}}}{|\rho_w|^2}\theta_0 \wedge dz \wedge d\overline{z} = \sigma_Z.$$

Let $G : Z \to Z^{'}$, where the corresponding quantities on $Z^{'}$ are denoted by a prime. Then following Chern and Moser, [8] Section 4, G is the restriction of a biholomorphism if and only if the following conditions are satisfied:

$$\begin{aligned} G^*(\theta') &= u\theta, \text{ for some } u > 0; \\ G^*(dz') &= \alpha\theta + \beta dz; \\ G^*(d\overline{z}') &= \overline{\alpha}\theta + \overline{\beta} d\overline{z}. \end{aligned}$$

And such a G preserves Fefferman hypersurface measure if additionally $G^*(\sigma_{Z'}) = \sigma_Z$. But $\theta^{'} \wedge d\theta^{'} = \sigma_{Z'}$, so $G^*(\sigma_{Z'}) = \sigma_Z$ if and only if $u\theta \wedge (du \wedge \theta + ud\theta) = \theta \wedge d\theta$ if and only if $u^2\theta \wedge d\theta = \theta \wedge d\theta$ if and only if $u = 1$. So the biholomorphisms preserving Fefferman measure are precisely those preserving the 1-form θ, and Webster [22] provides a complete set of invariants for such maps.

On our hypersurface Z,

$$\overline{\theta_0} = \theta_0,\ dw = \frac{-i}{\rho_w}\theta_0 - \frac{\rho_z}{\rho_w}dz,\ d\overline{w} = \frac{i}{\rho_{\overline{w}}}\theta_0 - \frac{\rho_{\overline{z}}}{\rho_{\overline{w}}}d\overline{z}.$$

For a function f on a neighborhood of Z, the restriction of df to Z satisfies

$$\begin{aligned} df &= f_z dz + f_{\overline{z}} d\overline{z} + f_w dw + f_{\overline{w}} d\overline{w} \\ &= f_z dz + f_{\overline{z}} d\overline{z} + \left(\frac{-if_w}{\rho_w}\theta_0 - \frac{\rho_z}{\rho_w}f_w dz\right) + \left(\frac{if_{\overline{w}}}{\rho_{\overline{w}}}\theta_0 - \frac{\rho_{\overline{z}}}{\rho_{\overline{w}}}f_{\overline{w}} d\overline{z}\right) \\ &= \left(\frac{-i}{\rho_w}f_w + \frac{i}{\rho_{\overline{w}}}f_{\overline{w}}\right)\theta_0 + \left(f_z - \frac{\rho_z}{\rho_w}f_w\right)dz + \left(f_{\overline{z}} - \frac{\rho_{\overline{z}}}{\rho_{\overline{w}}}f_{\overline{w}}\right)d\overline{z}. \end{aligned}$$

So if $\widetilde{X}, \widetilde{X}_1, \widetilde{X}_{\overline{1}}$ is the dual frame to $\theta_0, dz, d\overline{z}$, then $\widetilde{X} = \frac{i}{\rho_{\overline{w}}}\frac{\partial}{\partial \overline{w}} - \frac{i}{\rho_w}\frac{\partial}{\partial w}$, $\widetilde{X}_1 = \frac{\partial}{\partial z} - \frac{\rho_z}{\rho_w}\frac{\partial}{\partial w}$, and $\widetilde{X}_{\overline{1}} = \frac{\partial}{\partial \overline{z}} - \frac{\rho_{\overline{z}}}{\rho_{\overline{w}}}\frac{\partial}{\partial \overline{w}}$.

Then

$$\begin{aligned} d\theta_0 &= id\rho_z \wedge dz + id\rho_w \wedge dw \\ &= i\left(\widetilde{X}(\rho_z)\theta_0 + \widetilde{X}_1(\rho_z)dz + \widetilde{X}_{\overline{1}}(\rho_z)d\overline{z}\right) \wedge dz \\ &+ i\left(\widetilde{X}(\rho_w)\theta_0 + \widetilde{X}_1(\rho_w)dz + \widetilde{X}_{\overline{1}}(\rho_w)d\overline{z}\right) \wedge \left(\frac{-i}{\rho_w}\theta_0 - \frac{\rho_z}{\rho_w}dz\right) \\ &= \frac{-iM(\rho)}{|\rho_w|^2}dz \wedge d\overline{z} + \theta_0 \wedge \phi_0, \end{aligned}$$

where ϕ_0 and A_0 are as in (3.1) - (3.3). Note that we have repeated some earlier work because we wish to relate these quantities to $\widetilde{X}, \widetilde{X}_1$, and $\widetilde{X}_{\overline{1}}$.

We have

$$
\begin{aligned}
d\theta &= 2^{-\frac{1}{3}} d(M^{-\frac{1}{3}}) \wedge \theta_0 + 2^{-\frac{1}{3}} M^{-\frac{1}{3}} d\theta_0 \\
&= \frac{-2^{-\frac{1}{3}}}{3} M^{-\frac{4}{3}} dM \wedge \theta_0 + 2^{-\frac{1}{3}} M^{-\frac{1}{3}} d\theta_0 \\
&= \theta \wedge (\frac{1}{3M} dM) + 2^{-\frac{1}{3}} M^{-\frac{1}{3}} d\theta_0 \\
&= \frac{1}{3M}\theta \wedge \left(\widetilde{X}(M)\theta_0 + \widetilde{X}_1(M)dz + \widetilde{X}_{\overline{1}}(M)d\overline{z}\right) \\
&+ 2^{-\frac{1}{3}} M^{-\frac{1}{3}} \left(\frac{-iM}{|\rho_w|^2}\right) dz \wedge d\overline{z} + 2^{-\frac{1}{3}} M^{-\frac{1}{3}} \theta_0 \wedge \phi_0 \\
&= \frac{-i2^{-\frac{1}{3}} M^{\frac{2}{3}}}{|\rho_w|^2} dz \wedge d\overline{z} + \theta \wedge \left(\frac{1}{3M}\widetilde{X}_1(M)dz + \frac{1}{3M}\widetilde{X}_{\overline{1}}(M)d\overline{z} + \phi_0\right).
\end{aligned}
$$

So we let

$$
A = \frac{\rho_{w,\overline{w}}\rho_z - \rho_{z,\overline{w}}\rho_w}{|\rho_w|^2} + \frac{1}{3M}\left(\frac{\partial M}{\partial z} - \frac{\rho_z}{\rho_w}\frac{\partial M}{\partial w}\right) \tag{3.4}
$$

and

$$
\phi = Adz + \overline{A}d\overline{z}. \tag{3.5}
$$

We also let

$$
g_{1,\overline{1}} = \frac{-2^{\frac{-1}{3}} M^{\frac{2}{3}}}{|\rho_w|^2}. \tag{3.6}
$$

Then

$$
d\theta = ig_{1,\overline{1}} dz \wedge d\overline{z} + \theta \wedge \phi. \tag{3.7}
$$

We are going to follow Webster in [22]. We are seeking 1-forms that satisfy certain relations. This corresponds to finding an "adapted" coframe on a bundle P over Z that depends only on the pseudo-hermitian structure of Z. We are going through Webster's construction because we will need explicit formulas for the differential invariants that he obtains.

Now we substitute

$$
\begin{aligned}
\theta^1 &= dz + i\eta^1\theta \\
\theta^{\overline{1}} &= d\overline{z} - i\eta^{\overline{1}}\theta.
\end{aligned} \tag{3.8}
$$

Note that θ^1 depends on our choice of η^1. Then

$$
\begin{aligned}
ig_{1,\overline{1}}\theta^1 \wedge \theta^{\overline{1}} &= ig_{1,\overline{1}}\left(dz + i\eta^1\theta\right) \wedge \left(d\overline{z} - i\eta^{\overline{1}}\theta\right) \\
&= ig_{1,\overline{1}} dz \wedge d\overline{z} + \theta \wedge \left(-g_{1,\overline{1}}\eta^{\overline{1}} dz - g_{1,\overline{1}}\eta^1 d\overline{z}\right) \\
&= ig_{1,\overline{1}} dz \wedge d\overline{z} + \theta \wedge \phi.
\end{aligned}
$$

Together with (3.7), this implies that if we take $\eta^1 = \frac{-1}{g_{1,\overline{1}}}\overline{A}$, we will have

$$
d\theta = ig_{1,\overline{1}}\theta^1 \wedge \theta^{\overline{1}}. \tag{3.9}
$$

At this stage, we must introduce the bundle P which was alluded to earlier. Let $(\theta, \theta^1, \theta^{\overline{1}})$ be any coframe where θ^1 is a $(1,0)$-form, i.e., $\theta^1 = dz + i\eta^1\theta$ and θ is the 1-form defining the pseudo-hermitian structure on Z. We call this coframe *admissible* if it satisfies condition (3.9). We define

P to be the bundle of admissible coframes on Z. If $(\theta, \theta^1, \theta^{\bar{1}})$ is an admissible coframe, then any other admissible coframe is of the form

$$(\theta, \widetilde{\theta}^1, \widetilde{\theta^{\bar{1}}}) = (\theta, u_1^1\theta^1, u_{\bar{1}}^{\bar{1}}\theta^{\bar{1}}),$$

where $u_1^1 : Z \to \mathbb{C}^*$. Hence the fiber of P over $z \in Z$ is

$$(\theta, u_1^1\theta^1, u_{\bar{1}}^{\bar{1}}\theta^{\bar{1}}),$$

where $u_1^1 \in \mathbb{C}^*$. Taking u_1^1 as a fiber coordinate, this defines a local trivialization of P,

$$\phi : P \to Z \times \mathbb{C}^*,$$

which is given by

$$\phi(z, (\theta, u_1^1\theta^1, u_{\bar{1}}^{\bar{1}}\theta^{\bar{1}})) = (z, u_1^1).$$

If $(\theta, \widetilde{\theta}^1, \widetilde{\theta^{\bar{1}}})$ is another admissible coframe with corresponding trivialization $\widetilde{\phi} : P \to Z \times \mathbb{C}^*$, then recall that

$$(\theta, \theta^1, \theta^{\bar{1}}) = (\theta, v_1^1\widetilde{\theta}^1, v_{\bar{1}}^{\bar{1}}\widetilde{\theta^{\bar{1}}}),$$

for some $v_1^1 : Z \to \mathbb{C}^*$. We then have the corresponding transition map $T : Z \times \mathbb{C}^* \to Z \times \mathbb{C}^*$, given by

$$T = \widetilde{\phi} \circ \phi^{-1}.$$

Note that

$$T(z, u_1^1) = (z, v_1^1(z)u_1^1) = (z, \widetilde{u}_1^1).$$

We define 1-forms Θ, Θ^1, and $\Theta^{\bar{1}}$ on the local trivialization of P by

$$\begin{aligned} \Theta &= \theta \\ \Theta^1 &= u_1^1\theta^1 \\ \Theta^{\bar{1}} &= u_{\bar{1}}^{\bar{1}}\theta^{\bar{1}}. \end{aligned}$$

We have the corresponding 1-forms $\Theta, \widetilde{\Theta}^1, \widetilde{\Theta^{\bar{1}}}$ on the trivialization $\widetilde{\phi}$. Note that $T^*\widetilde{\Theta} = \Theta$ and

$$T^*\Theta^1 = T^*(u_1^1\theta^1) = \left((v_1^1(z))^{-1}\widetilde{u}_1^1\right)\left(v_1^1(z)\widetilde{\theta}^1\right) = \widetilde{u}_1^1\widetilde{\theta}^1 = \widetilde{\Theta}^1.$$

Hence $\Theta, \Theta^1, \Theta^{\bar{1}}$ correspond to 1-forms on P. We shall sometimes abuse language and refer to these 1-forms as 1-forms on P. We have that

$$d\Theta = iG_{1,\bar{1}}\Theta^1 \wedge \Theta^{\bar{1}}, \tag{3.10}$$

where $G_{1,\bar{1}}$ is a function on P. If $(\theta, \theta^1, \theta^{\bar{1}})$ satisfy (3.9), then in the corresponding trivialization,

$$G_{1,\bar{1}} = \frac{g_{1,\bar{1}}}{|u_1^1|^2}. \tag{3.11}$$

Webster shows that there exist uniquely determined, intrinsically-defined 1-forms $\omega_1^1, \omega_{\bar{1}}^{\bar{1}}, \tau^1$, and $\tau^{\bar{1}}$ on P satisfying:

$$d\Theta^1 = \Theta^1 \wedge \omega_1^1 + \Theta \wedge \tau^1 \tag{3.12}$$

$$\tau^1 \equiv 0 \bmod \Theta^{\bar{1}} \tag{3.13}$$

$$\omega_1^1 \equiv -d(u_1^1)(u_1^1)^{-1} \bmod (\Theta, \Theta^1, \Theta^{\bar{1}}) \tag{3.14}$$

$$dG_{1,\bar{1}} = G_{1,\bar{1}}\left(\omega_1^1 + \omega_{\bar{1}}^{\bar{1}}\right). \tag{3.15}$$

Now we state the main theorem of [22]:

Theorem III.3 (Webster)**.** *Let (Z,θ) and $(\widetilde{Z},\widetilde{\theta})$ be nondegenerate, integrable pseudo-hermitian manifolds, then there exists a coframe $\Phi = \{\Theta, \Theta^1, \Theta^{\overline{1}}, \omega_1^1, \omega_{\overline{1}}^{\overline{1}}\}$ as well as a 1-form τ^1 and a function $G_{1,\overline{1}}$ on the bundle P over Z described above, satisfying the relations* (3.10) -(3.15)*, and there exists a coframe $\widetilde{\Phi} = \{\widetilde{\Theta}, \widetilde{\Theta}^1, \widetilde{\Theta}^{\overline{1}}, \widetilde{\omega}_1^1, \widetilde{\omega}_{\overline{1}}^{\overline{1}}\}$ as well as a 1-form $\widetilde{\tau}^1$ and a function $\widetilde{G}_{1,\overline{1}}$ on the corresponding bundle $\widetilde{P}$ over $\widetilde{Z}$ satisfying the corresponding relations, such that Z and $\widetilde{Z}$ are equivalent as pseudo-hermitian manifolds (i.e., there is a CR equivalence between Z and $\widetilde{Z}$ taking θ to $\widetilde{\theta}$) if and only if there exists a diffeomorphism $F : P \to \widetilde{P}$ with $F^*\widetilde{\Phi} = \Phi$.*

We shall briefly describe how one uses the uniqueness of the 1-forms in the theorem to establish that Z and $\widetilde{Z}$ are equivalent as pseudo-hermitian manifolds precisely when there exists a diffeomorphism $F : P \to \widetilde{P}$ with $F^*\widetilde{\Phi} = \Phi$.

Suppose that (Z,θ) and $(\widetilde{Z},\widetilde{\theta})$ are pseudo-hermitian manifolds with corresponding bundles of admissible coframes P and $\widetilde{P}$ respectively. Let Φ and $\widetilde{\Phi}$ be the coframes on P and $\widetilde{P}$ specified in Theorem III.3. Let $(\theta, \theta^1, \theta^{\overline{1}})$ be an admissible coframe on Z and $(\widetilde{\theta}, \widetilde{\theta}^1, \widetilde{\theta}^{\overline{1}})$ be an admissible coframe on $\widetilde{Z}$, determining local trivializations ϕ and $\widetilde{\phi}$ of P and $\widetilde{P}$, respectively. Let $\psi = \phi^{-1}$, and define the inclusion

$$j : Z \hookrightarrow Z \times \mathbb{C}^*,$$

by

$$j(z) = (z, 1).$$

We also have the natural projection $\widetilde{\pi} : \widetilde{Z} \times \mathbb{C}^* \to \widetilde{Z}$. Now we have:

$$Z \xrightarrow{\ j\ } Z \times \mathbb{C}^* \xrightarrow{\ \psi\ } P \xrightarrow{\ F\ } \widetilde{P} \xrightarrow{\ \widetilde{\phi}\ } \widetilde{Z} \times \mathbb{C}^* \xrightarrow{\ \widetilde{\pi}\ } \widetilde{Z}.$$

We define $f : Z \to \widetilde{Z}$ by

$$f = \widetilde{\pi} \circ \widetilde{\phi} \circ F \circ \psi \circ j.$$

Note that

$$\begin{aligned} \widetilde{\pi}^*\widetilde{\theta}^1 &= (\widetilde{u}_1^1)^{-1}\left(\widetilde{u}_1^1\widetilde{\theta}^1\right) \\ \widetilde{\phi}^*\left(\widetilde{u}_1^1\widetilde{\theta}^1\right) &= \widetilde{\Theta}^1 \\ F^*\widetilde{\Theta}^1 &= \Theta^1 \\ \psi^*\Theta^1 &= u_1^1\theta^1 \\ j^*\left(u_1^1\theta^1\right) &= \theta^1. \end{aligned}$$

If

$$v_1^1 = j^*\psi^*F^*\widetilde{\phi}^*(\widetilde{u}_1^1)^{-1},$$

then v_1^1 is $\mathbb{C}^*$-valued and

$$f^*\widetilde{\theta}^1 = v_1^1\theta^1.$$

Similarly,

$$f^*\widetilde{\theta}^{\overline{1}} = v_{\overline{1}}^{\overline{1}}\theta^1.$$

It is immediate that $f^*\widetilde{\theta} = \theta$. Hence, f is the desired equivalence of pseudo-hermitian manifolds.

Conversely, suppose that there exists an equivalence $f : Z \to \widetilde{Z}$ of pseudo-hermitian manifolds. Since $f^*(\widetilde{\theta}, \widetilde{\theta}^1, \widetilde{\theta}^{\overline{1}})$ is an admissible coframe on Z, we shall assume that this is the chosen coframe, i.e.,

$$(\theta, \theta^1, \theta^{\overline{1}}) = f^*(\widetilde{\theta}, \widetilde{\theta}^1, \widetilde{\theta}^{\overline{1}}).$$

Since we can identify P and $\widetilde{P}$ with the local trivializations determined by the respective coframes, we can define F by defining a mapping $Z \times \mathbb{C}^* \to \widetilde{Z} \times \mathbb{C}^*$. We shall abuse notation and refer to this map as F. Define $F : Z \times \mathbb{C}^* \to \widetilde{Z} \times \mathbb{C}^*$ by

$$F(z, u_1^1) = (f(z), u_1^1) = (\widetilde{z}, \widetilde{u}_1^1).$$

Now

$$F^*(\widetilde{u}_1^1\widetilde{\theta}^1) = u_1^1\theta^1,$$

so $F^*\widetilde{\Theta}^1 = \Theta^1$. Similarly, $F^*\widetilde{\Theta}^{\overline{1}} = \Theta^{\overline{1}}$ and $F^*\widetilde{\Theta} = \Theta$. Also note that $F^*(d\widetilde{u}_1^1(\widetilde{u}_1^1)^{-1}) = du_1^1(u_1^1)^{-1}$. Moreover,

$$iF^*(\widetilde{G}_{1,\overline{1}})\Theta^1 \wedge \Theta^{\overline{1}} = F^*d\widetilde{\Theta} = d\Theta = iG_{1,\overline{1}}\Theta^1 \wedge \Theta^{\overline{1}},$$

so we have

$$F^*(\widetilde{G}_{1,\overline{1}}) = G_{1,\overline{1}}.$$

We also have

$$F^*\widetilde{\omega}_1^1 \equiv -du_1^1(u_1^1)^{-1} \bmod (\Theta, \Theta^1, \Theta^{\overline{1}})$$

and

$$F^*\widetilde{\tau}^1 \equiv 0 \bmod \Theta^{\overline{1}}.$$

Moreover,

$$\begin{aligned} d\Theta^1 =& F^*d\widetilde{\Theta^1} = F^*\left(\widetilde{\Theta}^1 \wedge \widetilde{\omega}_1^1 + \widetilde{\Theta} \wedge \widetilde{\tau}^1\right) \\ =& \Theta^1 \wedge F^*\widetilde{\omega}_1^1 + \Theta \wedge F^*\widetilde{\tau}^1 \end{aligned}$$

and

$$dG_{1,\overline{1}} = F^*d\widetilde{G}_{1,\overline{1}} = G_{1,\overline{1}}\left(F^*\widetilde{\omega}_1^1 + F^*\widetilde{\omega}_{\overline{1}}^{\overline{1}}\right).$$

Therefore, by the uniqueness of ω_1^1 and τ^1, we must have that

$$F^*\widetilde{\omega}_1^1 = \omega_1^1 \text{ and } F^*\widetilde{\tau}^1 = \tau^1.$$

Hence, we have found a diffeomorphism $F : P \to \widetilde{P}$ such that $F^*\widetilde{\Phi} = \Phi$.

Webster then defines

$$\Omega_1^1 = d\omega_1^1. \tag{3.16}$$

He also defines

$$\Omega^1 = d\tau^1 - \tau^1 \wedge \omega_1^1. \tag{3.17}$$

He shows that (c.f., [22] p. 31)

$$\Omega_1^1 = R^1_{1,1,\overline{1}}\Theta^1 \wedge \Theta^{\overline{1}} + W^1_{1,1}\Theta^1 \wedge \Theta - W^1_{1,\overline{1}}\Theta^{\overline{1}} \wedge \Theta, \tag{3.18}$$

and

$$\Omega^1 = W^1_{1,\overline{1}}\Theta^1 \wedge \Theta^{\overline{1}} - A^1_{\overline{1}}\tau^{\overline{1}} \wedge \Theta + B^1_{\overline{1}}\Theta^{\overline{1}} \wedge \Theta. \tag{3.19}$$

Webster notes ([22] p.33) that if we compute Ω_1^1 in the local trivialization of P determined by the admissible coframe $(\theta, \theta^1, \theta^{\overline{1}})$ and if we compute $\widetilde{\Omega}_1^1$ in the local trivialization determined by another admissible coframe $(\theta, \widetilde{\theta}^1, \widetilde{\theta}^{\overline{1}})$, then (at least in $\mathbb{C}^2$)

$$\widetilde{\Omega}_1^1 = \Omega_1^1. \tag{3.20}$$

At this stage, we note that in order to explicitly compute the associated invariants, it will be convenient to work with 1-forms on the hypersurface Z. Again, we work on the local trivialization of P determined by an admissible coframe $(\theta, \theta^1, \theta^{\bar{1}})$, i.e., $P \cong Z \times \mathbb{C}^*$. We recall that we have an inclusion $j : Z \hookrightarrow Z \times \mathbb{C}^*$ given by

$$j(z) = (z, 1).$$

Then we can pull back our coframe on P to obtain 1-forms on Z. We already have that $j^*\Theta = \theta$ and $j^*\Theta^1 = \theta^1$. We define

$$\mu_1^1 = j^*\omega_1^1 \text{ and } \gamma^1 = j^*\tau^1.$$

Then we obtain analogues of relations (3.10)-(3.15) for these 1-forms.

$$d\theta = i g_{1,\bar{1}} \theta^1 \wedge \theta^{\bar{1}} \tag{3.21}$$

$$d\theta^1 = \theta^1 \wedge \mu_1^1 + \theta \wedge \gamma^1 \tag{3.22}$$

$$\gamma^1 \equiv 0 \bmod \theta^1 \tag{3.23}$$

$$\mu_1^1 \equiv 0 \bmod (\theta, \theta^1, \theta^{\bar{1}}) \tag{3.24}$$

$$dg_{1,\bar{1}} = g_{1,\bar{1}} \left(\mu_1^1 + \mu_{\bar{1}}^{\bar{1}} \right). \tag{3.25}$$

Of course, (3.24) is a trivial consequence of the fact that $(\theta, \theta^1, \theta^{\bar{1}})$ is a coframe on Z, and so is no real constraint. Now, we can use the relations (3.21)-(3.25) to obtain “invariants” that depend on the fiber coordinates, and then then find the invariants on P in terms of these. For instance, we have already seen in (3.11) that

$$G_{1,\bar{1}} = \frac{g_{1,\bar{1}}}{|u_1^1|^2}.$$

In fact, we shall only need explicit formulas for two of the invariants.

Let

$$\begin{aligned} d\mu_1^1 &\equiv r_{1,1,\bar{1}}^1 \theta^1 \wedge \theta^{\bar{1}} \bmod \theta \\ \gamma^1 &= a_{\bar{1}}^1 \theta^{\bar{1}}. \end{aligned}$$

Now, from

$$\begin{aligned} d\omega_1^1 &\equiv R_{1,1,\bar{1}}^1 \Theta^1 \wedge \Theta^{\bar{1}} \bmod \Theta \\ \tau^1 &= A_{\bar{1}}^1 \Theta^{\bar{1}}, \end{aligned}$$

we deduce

$$R_{1,1,\bar{1}}^1 = \frac{r_{1,1,\bar{1}}^1}{|u_1^1|^2} \tag{3.26}$$

$$A_{\bar{1}}^1 = \frac{u_1^1}{u_{\bar{1}}^{\bar{1}}} a_{\bar{1}}^1. \tag{3.27}$$

It follows from (3.11) and (3.26) that

$$R_{1,1}^{1,1} \stackrel{\text{def}}{=} \frac{R_{1,1,\bar{1}}^1}{G_{1,\bar{1}}}$$

is constant on fibers and

$$R_{1,1}^{1,1} = \frac{r_{1,1,\bar{1}}^1}{g_{1,\bar{1}}}.$$

Likewise, from (3.27), $|A_{\bar{1}}^1|$ is constant on fibers and

$$|A_{\bar{1}}^1| = |a_{\bar{1}}^1|.$$

Therefore, in order to compute these invariants on P, it suffices to choose an admissible coframe on Z and find 1-forms μ_1^1 and γ^1 satisfying the relations (3.21)-(3.25) and compute the corresponding quantities for these 1-forms. While computing these invariants, we shall refer to these 1-forms as ω_1^1 and τ^1, and unless we explicitly state that we are working on the bundle P, everything is occuring on the hypersurface Z. So what we shall be calling ω_1^1, τ^1 and $A_{\overline{1}}^1$ are really the quantities that we have been referring to as μ_1^1, γ^1 and $a_{\overline{1}}^1$ respectively. This is what Webster does in [22], and we shall use several of his computations from that paper.

Now, we work towards finding such 1-forms on Z. Let $\theta, \theta^1, \theta^{\overline{1}}$ have dual frame $X, X_1, X_{\overline{1}}$. Using the relation that for any function f on Z, we must have

$$\widetilde{X}(f)\theta_0 + \widetilde{X}_1(f)dz + \widetilde{X}_{\overline{1}}(f)d\overline{z} = X(f)\theta + X_1(f)\theta^1 + X_{\overline{1}}(f)\theta^{\overline{1}},$$

we obtain $X_1 = \widetilde{X}_1, X_{\overline{1}} = \widetilde{X}_{\overline{1}}$, and

$$X = -i\eta^1\frac{\partial}{\partial z} + i\eta^{\overline{1}}\frac{\partial}{\partial \overline{z}} + \left(\frac{-i2^{\frac{1}{3}}M^{\frac{1}{3}}}{\rho_w} + \frac{i\rho_z}{\rho_w}\eta^1\right)\frac{\partial}{\partial w} + \left(\frac{i2^{\frac{1}{3}}M^{\frac{1}{3}}}{\rho_{\overline{w}}} + \frac{-i\rho_{\overline{z}}}{\rho_{\overline{w}}}\eta^{\overline{1}}\right)\frac{\partial}{\partial \overline{w}}.$$

Now we want to find $A_{\overline{1}}^1$ such that if we define $\tau^1 = A_{\overline{1}}^1\theta^{\overline{1}}$, there exists $\widetilde{\omega}_1^1$ such that

$$d\theta^1 = \theta^1 \wedge \widetilde{\omega}_1^1 + \theta \wedge \tau^1.$$

Using (3.8) and (3.9), we have

$$\begin{aligned} d\theta^1 &= id\eta^1 \wedge \theta + i\eta^1 d\theta \\ &= i\left(X(\eta^1)\theta + X_1(\eta^1)\theta^1 + X_{\overline{1}}(\eta^1)\theta^{\overline{1}}\right) \wedge \theta + i\eta^1\left(ig_{1,\overline{1}}\theta^1 \wedge \theta^{\overline{1}}\right) \\ &= \theta^1 \wedge \left(-g_{1,\overline{1}}\eta^1\theta^{\overline{1}} + iX_1(\eta^1)\theta\right) + \theta \wedge (-iX_{\overline{1}}(\eta^1)\theta^{\overline{1}}). \end{aligned}$$

So $A_{\overline{1}}^1 = -iX_{\overline{1}}(\eta^1)$, and $\widetilde{\omega}_1^1 = -g_{1,\overline{1}}\eta^1\theta^{\overline{1}} + iX_1(\eta^1)\theta$. Now we wish to adjust $\widetilde{\omega}_1^1$ by a multiple of θ^1 to obtain a unique ω_1^1 with $dg_{1,\overline{1}} = g_{1,\overline{1}}(\omega_1^1 + \omega_{\overline{1}}^{\overline{1}})$. We have that

$$\begin{aligned} &dg_{1,\overline{1}} - g_{1,\overline{1}}\widetilde{\omega}_1^1 - g_{1,\overline{1}}\widetilde{\omega}_{\overline{1}}^{\overline{1}} \\ &= (X(g_{1,\overline{1}}) - ig_{1,\overline{1}}X_1(\eta^1) + ig_{1,\overline{1}}X_{\overline{1}}(\eta^{\overline{1}}))\theta + (X_1(g_{1,\overline{1}}) + g_{1,\overline{1}}^2\eta^{\overline{1}})\theta^1 + (X_{\overline{1}}(g_{1,\overline{1}}) + g_{1,\overline{1}}^2\eta^1)\theta^{\overline{1}} \\ &= (X_1(g_{1,\overline{1}}) + g_{1,\overline{1}}^2\eta^{\overline{1}})\theta^1 + (X_{\overline{1}}(g_{1,\overline{1}}) + g_{1,\overline{1}}^2\eta^1)\theta^{\overline{1}}, \end{aligned}$$

since the coefficient of θ must vanish (c.f., Webster [22] p.28). Then, again following Webster, we can define ω_1^1 by

$$\omega_{1,\overline{1}} = \omega_1^1 g_{1,\overline{1}} = B_{1,\overline{1},1}\theta^1 + C_{1,\overline{1},\overline{1}}\theta^{\overline{1}} + E_{1,\overline{1}}\theta,$$

where $B_{1,\overline{1},1} = X_1(g_{1,\overline{1}}) + g_{1,\overline{1}}^2\eta^{\overline{1}}$, $C_{1,\overline{1},\overline{1}} = -g_{1,\overline{1}}^2\eta^1$, and $E_{1,\overline{1}} = ig_{1,\overline{1}}X_1(\eta^1)$. Note that we are using $g_{1,\overline{1}}$ to raise and lower indices, i.e., by definition, $\omega_{1,\overline{1}} = \omega_1^1 g_{1,\overline{1}}$. Likewise, replacing all indices with a bar denotes the conjugate of a quantity, so $\omega_{\overline{1}}^{\overline{1}} = \overline{\omega_1^1}$. To simplify notation, we shall write $\omega_1^1 = B\theta^1 + C\theta^{\overline{1}} + E\theta$. To summarize, we have:

$$d\theta^1 = \theta^1 \wedge \omega_1^1 + \theta \wedge \tau^1 \tag{3.28}$$

$$\tau^1 = A_{\overline{1}}^1\theta^{\overline{1}} \tag{3.29}$$

$$dg_{1,\overline{1}} = g_{1,\overline{1}}(\omega_1^1 + \omega_{\overline{1}}^{\overline{1}}). \tag{3.30}$$

Now we define $\Omega_1^1 = d\omega_1^1 = R_{1,1,\overline{1}}^1\theta^1 \wedge \theta^{\overline{1}} + W_{1,1}^1\theta^1 \wedge \theta - W_{1,\overline{1}}^1\theta^{\overline{1}} \wedge \theta$. We compute

$$\begin{aligned} d\omega_1^1 &= dB \wedge \theta^1 + Bd\theta^1 + dC \wedge \theta^{\overline{1}} + Cd\theta^{\overline{1}} + dE \wedge \theta + Ed\theta \\ &= (X(B)\theta + X_1(B)\theta^1 + X_{\overline{1}}(B)\theta^{\overline{1}}) \wedge \theta^1 + B(\theta^1 \wedge \omega_1^1 + \theta \wedge \tau^1) \\ &+ (X(C)\theta + X_1(C)\theta^1 + X_{\overline{1}}(C)\theta^{\overline{1}}) \wedge \theta^{\overline{1}} + C(\theta^{\overline{1}} \wedge \omega_{\overline{1}}^{\overline{1}} + \theta \wedge \tau^{\overline{1}}) + dE \wedge \theta + E(ig_{1,\overline{1}}\theta^1 \wedge \theta^{\overline{1}}) \\ &= (X_1(C) - X_{\overline{1}}(B) + ig_{1,\overline{1}}E + BC - |C|^2)\theta^1 \wedge \theta^{\overline{1}} + \cdots \end{aligned}$$

Hence $R^1_{1,1,\overline{1}} = X_1(C) - X_{\overline{1}}(B) + ig_{1,\overline{1}}E + BC - |C|^2$. And finally, we define $R^{1,1}_{1,1} = \frac{1}{g_{1,\overline{1}}} R^1_{1,1,\overline{1}}$.

We shall compute these invariants for the sphere of radius R. Let $\rho(z, \overline{z}, w, \overline{w}) = |z|^2 + |w|^2 - R^2$. Then $M(\rho) = |z|^2 + |w|^2$. $A = \frac{\overline{z}}{|w|^2}, g_{1,\overline{1}} = \frac{-2^{\frac{-1}{3}}}{|w|^2}(|z|^2 + |w|^2)^{\frac{2}{3}}$, and $\eta^1 = 2^{\frac{1}{3}} z(|z|^2 + |w|^2)^{\frac{-2}{3}}$. We compute $A^1_1 = -iX_{\overline{1}}(\eta^1) = -i\left(\frac{\partial \eta^1}{\partial \overline{z}} - \frac{\rho_{\overline{z}}}{\rho_{\overline{w}}} \frac{\partial \eta^1}{\partial \overline{w}}\right) = 0$ and hence $\tau^1 = 0$. We also have $\omega^1_1 = B\theta^1 + C\theta^{\overline{1}} + E\theta$, where $B = \frac{1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}}) + g_{1,\overline{1}}\eta^{\overline{1}} = 0, C = -g_{1,\overline{1}}\eta^1$, and $E = iX_1(\eta^1)$. We compute $E = i2^{\frac{1}{3}}(|z|^2 + |w|^2)^{\frac{-2}{3}}$ and $C = \frac{z}{|w|^2}$, so $\omega^1_1 = i2^{\frac{1}{3}}(|z|^2 + |w|^2)^{\frac{-2}{3}}\theta + \frac{z}{|w|^2}\theta^{\overline{1}}$. Before proceeding, note that since $(|z|^2 + |w|^2) = \rho(z, \overline{z}, w, \overline{w}) + R^2, X_1((|z|^2 + |w|^2)) = X_{\overline{1}}((|z|^2 + |w|^2)) = 0$. Now,

$$d\omega^1_1 = 2^{\frac{1}{3}} i(|z|^2 + |w|^2)^{\frac{-2}{3}} d\theta + \left(X(\frac{z}{|w|^2})\theta + X_1(\frac{z}{|w|^2})\theta^1 + X_{\overline{1}}(\frac{z}{|w|^2})\theta^{\overline{1}}\right) \wedge \theta^{\overline{1}} + \frac{z}{|w|^2} d\theta^{\overline{1}}$$
$$= 2^{\frac{1}{3}} i(|z|^2 + |w|^2)^{\frac{-2}{3}}(ig_{1,\overline{1}})\theta^1 \wedge \theta^{\overline{1}} + X(\frac{z}{|w|^2})\theta \wedge \theta^{\overline{1}} + X_1(\frac{z}{|w|^2})\theta^1 \wedge \theta^{\overline{1}} + \frac{z}{|w|^2}\theta^{\overline{1}} \wedge \omega^{\overline{1}}_{\overline{1}}.$$

The last equality is because $\tau^1 = 0$. We now substitute $\omega^{\overline{1}}_{\overline{1}} = -i2^{\frac{1}{3}}(|z|^2 + |w|^2)^{\frac{-2}{3}}\theta + \frac{\overline{z}}{|w|^2}\theta^1$ and compute. We find that the above expression reduces to $d\omega^1_1 = \frac{2}{|w|^2}\theta^1 \wedge \theta^{\overline{1}}$. We then have

Proposition III.4. *For the sphere of radius R in $\mathbb{C}^2$,*

$$R^{1,1}_{1,1} = \frac{1}{g_{1,\overline{1}}} R^1_{1,1,\overline{1}}$$
$$= -2^{\frac{4}{3}}(|z|^2 + |w|^2)^{\frac{-2}{3}} = -2^{\frac{4}{3}} R^{\frac{-4}{3}},$$

and $\tau^1 = 0$.

Since it will prove to be quite useful, we will now find a simplified expression for $R^{1,1}_{1,1}$ assuming that we have chosen a defining function ρ such that $M(\rho) = 1 + \mathcal{O}(\rho^3)$ (according to Fefferman [9], this is always possible). Then $A = \frac{1}{|\rho_w|^2}(\rho_{w,\overline{w}}\rho_z - \rho_{z,\overline{w}}\rho_w) + \mathcal{O}(\rho^2)$ and $g_{1,\overline{1}} = \frac{-2^{\frac{-1}{3}}}{|\rho_w|^2} + \mathcal{O}(\rho^3)$. So $\eta^1 = \frac{-1}{g_{1,\overline{1}}}\overline{A} = 2^{\frac{1}{3}}(\rho_{w,\overline{w}}\rho_{\overline{z}} - \rho_{\overline{z},w}\rho_{\overline{w}}) + \mathcal{O}(\rho^2)$. Moreover, $B = \frac{1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}}) + g_{1,\overline{1}}\eta^{\overline{1}}, C = -g_{1,\overline{1}}\eta^1$, and $E = iX_1(\eta^1)$. Then we have $R^1_{1,1,\overline{1}} = X_1(C) - X_{\overline{1}}(B) + ig_{1,\overline{1}}E + BC - |C|^2$ and so

$$R^{1,1}_{1,1} = \frac{1}{g_{1,\overline{1}}} X_1(C) - \frac{1}{g_{1,\overline{1}}} X_{\overline{1}}(B) + iE + \frac{1}{g_{1,\overline{1}}} BC - \frac{1}{g_{1,\overline{1}}}|C|^2$$
$$= \frac{1}{g_{1,\overline{1}}} X_1(-g_{1,\overline{1}}\eta^1) - \frac{1}{g_{1,\overline{1}}} X_{\overline{1}}(\frac{1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}})) - \frac{1}{g_{1,\overline{1}}} X_{\overline{1}}(g_{1,\overline{1}}\eta^{\overline{1}})$$
$$- X_1(\eta^1) + \frac{1}{g_{1,\overline{1}}}(\frac{1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}}) + g_{1,\overline{1}}\eta^{\overline{1}})(-g_{1,\overline{1}}\eta^1) - g_{1,\overline{1}}|\eta^1|^2.$$

This simplifies to

$$R^{1,1}_{1,1} = -\frac{1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}}\eta^1) - \frac{1}{g_{1,\overline{1}}} X_{\overline{1}}(g_{1,\overline{1}}\eta^{\overline{1}})$$
$$- \frac{1}{g_{1,\overline{1}}} X_{\overline{1}}(\frac{1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}})) - X_1(\eta^1) - \frac{\eta^1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}}) - 2g_{1,\overline{1}}|\eta^1|^2.$$

Now, substituting our formula for $g_{1,\overline{1}}$, we find that

$$\frac{1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}}) = -g_{1,\overline{1}}\eta^{\overline{1}} + \frac{1}{\rho_w^2}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w) + \mathcal{O}(\rho^2).$$

Substituting this back into our expression for $R^{1,1}_{1,1}$, we obtain

$$R^{1,1}_{1,1} = -2X_1(\eta^1) - \frac{2\eta^1}{g_{1,\overline{1}}} X_1(g_{1,\overline{1}}) - \frac{1}{g_{1,\overline{1}}} X_{\overline{1}}(\frac{1}{\rho_w^2}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w)) - 2g_{1,\overline{1}}|\eta^1|^2 + \mathcal{O}(\rho).$$

Again using our formula for $\frac{1}{g_{1,\bar{1}}}X_1(g_{1,\bar{1}})$, we find that

$$R^{1,1}_{1,1} = -2X_1(\eta^1) - \frac{2\eta^1}{\rho_w^2}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w) - \frac{1}{g_{1,\bar{1}}}X_{\bar{1}}(\frac{1}{\rho_w^2}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w)) + \mathcal{O}(\rho).$$

More direct computation shows us that $X_{\bar{1}}(\frac{1}{\rho_w^2}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w)) = -2(\frac{1}{\rho_w^2}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w))g_{1,\bar{1}}\eta^1 + \frac{1}{\rho_w^2}X_{\bar{1}}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w) + \mathcal{O}(\rho^2)$. Then we have $R^{1,1}_{1,1} = -2X_1(\eta^1) + \frac{2^{\frac{1}{3}}|\rho_w|^2}{\rho_w^2}X_{\bar{1}}(\rho_{w,w}\rho_z - \rho_{z,w}\rho_w) + \mathcal{O}(\rho)$. At this stage, computing both quantities as well as $\frac{\partial M(\rho)}{\partial w}$, we find that

$$\begin{aligned}
R^{1,1}_{1,1} &= 2^{\frac{1}{3}}(2(\rho_{\bar{z},w}\rho_{z,\overline{w}} - \rho_{z,\bar{z}}\rho_{w,\overline{w}}) + \frac{1}{\rho_w}(M_w + (\rho_{\bar{z},w}\rho_{z,\overline{w}} - \rho_{z,\bar{z}}\rho_{w,\overline{w}})\rho_w)) + \mathcal{O}(\rho) \\
&= 2^{\frac{1}{3}}3(\rho_{\bar{z},w}\rho_{z,\overline{w}} - \rho_{z,\bar{z}}\rho_{w,\overline{w}}) + \mathcal{O}(\rho),
\end{aligned}$$

where the last equality is because $M_w = 0 + \mathcal{O}(\rho^2)$. Therefore, on our hypersurface,

$$R^{1,1}_{1,1} = 2^{\frac{1}{3}}3(\rho_{\bar{z},w}\rho_{z,\overline{w}} - \rho_{z,\bar{z}}\rho_{w,\overline{w}}). \tag{3.31}$$

If $G(z,w) = (f(z,w), g(z,w))$ is a volume-preserving local biholomorphism, then if we set $\widetilde{\rho} = \rho \circ G$, direct computation yields that

$$M(\widetilde{\rho}) = 1 + \mathcal{O}(\widetilde{\rho}^3) \tag{3.32}$$

and

$$(\widetilde{\rho}_{z,\overline{w}}\widetilde{\rho}_{\bar{z},w} - \widetilde{\rho}_{z,\bar{z}}\widetilde{\rho}_{w,\overline{w}}) = (\rho_{z,\overline{w}}\rho_{\bar{z},w} - \rho_{z,\bar{z}}\rho_{w,\overline{w}}). \tag{3.33}$$

So we see again that $R^{1,1}_{1,1}$ is an absolute invariant with respect to volume-preserving local biholomorphisms.

Now we wish to find the relationship between $R^{1,1}_{1,1}$ and our invariants κ and γ. For this purpose, we adopt a ghastly notation. We shall write

$$\mathcal{R}(z^{j_1}\bar{z}^{k_1}u^{l_1}, \ldots, z^{j_q}\bar{z}^{k_q}u^{l_q})$$

to denote a sum of the form

$$K_{j_1,k_1,l_1}z^{j_1}\bar{z}^{k_1}u^{l_1} + \cdots + K_{j_q,k_q,l_q}z^{j_q}\bar{z}^{k_q}u^{l_q}$$

for some arbitrary constants K_{j_m,k_m,l_m} with $m = 1, \ldots, q$ together with terms of order greater than or equal to 5. Suppose that our defining function is in normal form, so

$$\begin{aligned}
\rho(z,\bar{z},w,\overline{w}) &= -\frac{w-\overline{w}}{2i} + |z|^2 + \kappa|z|^4 + \gamma z^3\bar{z} + \gamma z\bar{z}^3 \\
&+ \mathcal{R}(u^3, zu^2, \bar{z}u^2, u^4, zu^3, \bar{z}u^3, z^2u^2, \bar{z}^2u^2, |z|^2u^2, z^3u, \bar{z}^3u, z^2\bar{z}u, z\bar{z}^2u).
\end{aligned}$$

Then

$$\begin{aligned}
M(\rho) &= \frac{1}{4}(1 + 4\kappa|z|^2 + 3\gamma z^2 + 3\gamma\bar{z}^2) \\
&+ \mathcal{R}(u^2, zu, \bar{z}u, |z|^2u, \bar{z}u^2, zu^2, z^2u, \bar{z}^2u, z^3, \bar{z}^3, z^2\bar{z}, z\bar{z}^2, |z|^2u^2, z^2\bar{z}u, z\bar{z}^2u, z^3\bar{z}, z\bar{z}^3, |z|^4).
\end{aligned}$$

Also note that $M_z = \kappa\bar{z} + \frac{3}{2}\gamma z + \mathcal{R}(u, \bar{z}u, zu, u^2, z^2, \bar{z}^2, |z|^2)$ and
$M_w = \mathcal{R}(u, z, \bar{z}, |z|^2, zu, \bar{z}u, |z|^2, z^2, \bar{z}^2)$. We then have that

$$\begin{aligned}
A &= \frac{4}{3}(\kappa\bar{z} + \frac{3}{2}\gamma z) \\
&+ \mathcal{R}(u, zu, \bar{z}u, u^2, |z|^2u, \bar{z}u^2, zu^2, z^2u, \bar{z}^2u, z^3, \bar{z}^3, z^2\bar{z}, z\bar{z}^2, |z|^2u^2, z^2\bar{z}u, z\bar{z}^2u, z^3\bar{z}, z\bar{z}^3, |z|^4).
\end{aligned}$$

We also find that

$$\begin{aligned} g_{1,\bar{1}} &= -2^{\frac{1}{3}}(1 + \frac{8}{3}\kappa|z|^2 + 2\gamma z^2 + 2\gamma\bar{z}^2) \\ &+ \mathcal{R}(u^2, zu, \bar{z}u, |z|^2u, \bar{z}u^2, zu^2, z^2u, \bar{z}^2u, z^3, \bar{z}^3, z^2\bar{z}, z\bar{z}^2, |z|^2u^2, z^2\bar{z}u, z\bar{z}^2u, z^3\bar{z}, z\bar{z}^3, |z|^4) \end{aligned}$$

and

$$\begin{aligned} \eta^1 &= 2^{\frac{-1}{3}}\frac{4}{3}(\kappa z + \frac{3}{2}\gamma\bar{z}) \\ &+ \mathcal{R}(u, zu, \bar{z}u, , u^2, |z|^2u, \bar{z}u^2, zu^2, z^2u, \bar{z}^2u, z^3, \bar{z}^3, z^2\bar{z}, z\bar{z}^2, |z|^2u^2, z^2\bar{z}u, z\bar{z}^2u, z^3\bar{z}, z\bar{z}^3, |z|^4). \end{aligned}$$

Then

$$R_{1,1}^{1,1} = \frac{-1}{g_{1,\bar{1}}}X_1(g_{1,\bar{1}}\eta^1) - \frac{1}{g_{1,\bar{1}}}X_{\bar{1}}(g_{1,\bar{1}}\eta^{\bar{1}}) - \frac{1}{g_{1,\bar{1}}}X_{\bar{1}}(\frac{1}{g_{1,\bar{1}}}X_1(g_{1,\bar{1}})) - X_1(\eta^1) - \frac{\eta^1}{g_{1,\bar{1}}}X_1(g_{1,\bar{1}}) - 2g_{1,\bar{1}}|\eta^1|^2.$$

We need to compute

$$\begin{aligned} X_1(\eta^1) &= 2^{\frac{-1}{3}}\frac{4}{3}\kappa + \mathcal{R}(u, \bar{z}, |z|^2, \bar{z}^2, z^2, u^2, zu, \bar{z}u, \dots), \\ X_{\bar{1}}(\eta^{\bar{1}}) &= 2^{\frac{-1}{3}}\frac{4}{3}\kappa + \mathcal{R}(u, z, |z|^2, \bar{z}^2, z^2, u^2, zu, \bar{z}u, \dots), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{g_{1,\bar{1}}}X_1(g_{1,\bar{1}}) &= \frac{8}{3}\kappa\bar{z} + 4\gamma z \\ &+ \mathcal{R}(u, zu, \bar{z}u, u^2, |z|^2u, \bar{z}u^2, zu^2, z^2u, \bar{z}^2u, z^3, \bar{z}^3, z^2\bar{z}, z\bar{z}^2, |z|^2u^2, z^2\bar{z}u, z\bar{z}^2u, z^3\bar{z}, z\bar{z}^3, |z|^4), \end{aligned}$$

so $\frac{1}{g_{1,\bar{1}}}X_{\bar{1}}(\frac{1}{g_{1,\bar{1}}}X_1(g_{1,\bar{1}})) = -2^{\frac{-1}{3}}\frac{8}{3}\kappa + \mathcal{R}(z, u, \dots)$. So we find that, at the origin,

$$R_{1,1}^{1,1} = \frac{-2^{\frac{5}{3}}}{3}\kappa. \tag{3.34}$$

Our expression for η^1 also allows us to compute $A_{\bar{1}}^1$. Recall that $A_{\bar{1}}^1 = -iX_{\bar{1}}(\eta^1)$. Using the above expression, we see that at the origin,

$$A_{\bar{1}}^1 = -i2^{\frac{2}{3}}\gamma. \tag{3.35}$$

Therefore $A_{\bar{1}}^1 = 0$ at a point precisely when $\gamma = 0$.

We are now in a position to prove the following:

Theorem III.5. *Let $Z = \{\rho(z, \bar{z}, w, \overline{w} = 0\} \subset \mathbb{C}^2$ be a strongly pseudoconvex, real-analytic real hypersurface. Then there exists a locally volume-preserving CR equivalence between Z and the sphere of radius R if and only if $R_{1,1}^{1,1} \equiv 2^{\frac{4}{3}}R^{\frac{-4}{3}}$, and $A_{\bar{1}}^1 \equiv 0$.*

Proof By Theorem III.3, this reduces to showing an equivalence of the coframes defined on the corresponding coframe bundles P and P' for Z and the sphere, respectively. Cartan provides us with the means of solving this problem. Recall the notation from Section 2.4. Suppose

$$R_{1,1}^{1,1} = -2^{\frac{4}{3}}R^{\frac{-4}{3}},$$

and

$$A_{\bar{1}}^1 = 0.$$

Since

$$\tau^1 = A_{\bar{1}}^1\Theta^{\bar{1}},$$

this implies that $\tau^1 = 0$. So

$$\Omega^1 = d\tau^1 - \tau^1 \wedge \omega_1^1 = 0,$$

but

$$\Omega^1 = W^1_{1,\bar{1}}\Theta^1 \wedge \Theta^{\bar{1}} - A^1_{\bar{1}}\tau^{\bar{1}} \wedge \Theta + B^1_{\bar{1}}\Theta^{\bar{1}} \wedge \Theta.$$

Hence

$$W^1_{1,\bar{1}}\Theta^1 \wedge \Theta^{\bar{1}} + B^1_{\bar{1}}\Theta^{\bar{1}} \wedge \Theta = 0.$$

Since $\Theta, \Theta^1, \Theta^{\bar{1}}$ are linearly independent, so are $\Theta^1 \wedge \Theta^{\bar{1}}$ and $\Theta^{\bar{1}} \wedge \Theta$. It follows that $W^1_{1,\bar{1}} = 0$ and $B^1_{\bar{1}} = 0$. So all of the invariants except $R^1_{1,1,\bar{1}}$ are 0. For convenience, we shall work with a real coframe and use the notation of Chapter 2.4. Let

$$\begin{aligned}\Phi &= (\phi^1, \phi^2, \phi^3, \phi^4, \phi^5, \phi^6, \phi^7)\\ &= (\frac{1}{2}(\Theta^1 + \Theta^{\bar{1}}), \frac{1}{2}(\omega_1^1 + \omega_{\bar{1}}^{\bar{1}}), \frac{1}{2}(\tau^1 + \tau^{\bar{1}}), \Theta, \frac{1}{2i}(\Theta^1 - \Theta^{\bar{1}}), \frac{1}{2i}(\omega_1^1 - \omega_{\bar{1}}^{\bar{1}}), \frac{1}{2}(\tau^1 - \tau^{\bar{1}}))).\end{aligned}$$

Then as in Olver, we will denote

$$d\phi^j = \sum T^j_{k,l}\phi^k \wedge \phi^l.$$

Using the relations

$$\begin{aligned}d\Theta &= iG_{1,\bar{1}}\Theta^1 \wedge \Theta^{\bar{1}}\\ d\Theta^1 &= \Theta^1 \wedge \omega_1^1\\ d\omega_1^1 &= G_{1,\bar{1}}R^{1,1}_{1,1}\Theta^1 \wedge \Theta^{\bar{1}},\end{aligned}$$

and

$$d\tau^1 = 0,$$

as well as the fact that

$$\Theta^1 = \phi^1 + i\phi^5,$$

and

$$\omega_1^1 = \phi^2 + i\phi^6,$$

we obtain

$$\begin{aligned}d\phi^1 &= \phi^1 \wedge \phi^2 - \phi^5 \wedge \phi^6,\\ d\phi^2 &= 0,\\ d\phi^3 &= 0,\\ d\phi^4 &= 2G_{1,\bar{1}}\phi^1 \wedge \phi^5,\\ d\phi^5 &= \phi^1 \wedge \phi^6 - \phi^2 \wedge \phi^5,\\ d\phi^6 &= -2G_{1,\bar{1}}R^{1,1}_{1,1}\phi^1 \wedge \phi^5,\end{aligned}$$

and

$$d\phi^7 = 0,$$

so

$$\begin{aligned}T^4_{1,5} &= 2G_{1,\bar{1}},\\ T^6_{1,5} &= -2G_{1,\bar{1}}R^{1,1}_{1,1},\end{aligned}$$

and all other coefficients are constant. We recall that by assumption, $R^{1,1}_{1,1}$ is a real constant. We see that all of the nonconstant coefficients $T^j_{k,l}$ are constant multiples of $G_{1,\bar{1}}$, so $r_0 = 1$. In the process of constructing our coframe, we imposed the condition (3.15), that

$$dG_{1,\bar{1}} = G_{1,\bar{1}}\omega^1_1 + G_{1,\bar{1}}\omega^{\bar{1}}_{\bar{1}},$$

so

$$dG_{1,\bar{1}} = 2G_{1,\bar{1}}\phi^2,$$

and so

$$\frac{\partial G_{1,\bar{1}}}{\partial \phi^2} = 2G_{1,\bar{1}},$$

and all other derivatives are zero. Hence,

$$\frac{\partial T^j_{k,l}}{\partial \phi^m} = C_{j,k,l,m}G_{1,\bar{1}},$$

where $C_{j,k,l,m}$ is a constant with

$$C_{4,1,5,2} = 4, C_{6,1,5,2} = -4R^{1,1}_{1,1},$$

and all other constants are 0. We see then that $r_1 = 1 = r_0$. Therefore the order of P is equal to the order of P' (the coframe bundle of $\mathbb{S}^3$) is 1. We see that the classifying manifold of P will coincide with that of the P' on a set of full dimension if and only if $\mathrm{Range}(G'_{1,\bar{1}}) \cap \mathrm{Range}(G_{1,\bar{1}})$ contains an interval, where $G'_{1,\bar{1}}$ is the corresponding quantity for the sphere. We shall see that this is always possible. If u^1_1 is the fiber coordinate on the trivialization of P determined by an admissible coframe $(\theta, \theta^1, \theta^{\bar{1}})$ and if u'^1_1 is the corresponding fiber coordinate on the trivialization of P' determined by $(\theta', \theta'^1, \theta'^{\bar{1}})$, let

$$d\theta = g_{1,\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$$
$$d\theta' = g'_{1,\bar{1}}\theta'^1 \wedge \theta'^{\bar{1}}.$$

We found in (3.11) that

$$G_{1,\bar{1}} = \frac{g_{1,\bar{1}}}{|u^1_1|^2}$$
$$G'_{1,\bar{1}} = \frac{g'_{1,\bar{1}}}{|u'^1_1|^2}.$$

Pick any points $z \in Z$ and $z' \in \mathbb{S}^3$. By assumption, Z is strongly pseudoconvex, as is $\mathbb{S}^3$, so

$$g_{1,\bar{1}}(z) < 0 \text{ and } g'_{1,\bar{1}}(z') < 0.$$

By choosing our fiber coordinates u^1_1 and u'^1_1 appropriately, we can make $G_{1,\bar{1}}(z, u^1_1)$ and $G'_{1,\bar{1}}(z', u'^1_1)$ obtain any negative value. Hence, it is always the case that $\mathrm{Range}(G'_{1,\bar{1}}) \cap \mathrm{Range}(G_{1,\bar{1}})$ contains an interval.

Hence by Theorem II.8, there exists a diffeomorphism from P to the P' preserving the given coframe. Then by Theorem III.3, there exists a CR isomorphism between Z and the sphere preserving the specified 1-forms, so Z is equivalent to the sphere. □

Definition III.6. Given a strongly pseudoconvex real hypersurface $Z \subset \mathbb{C}^2$ and a point $(z, w) \in Z$, Z is umbilic at (z, w) if the normal form of Z at (z, w) agrees with that of a sphere through weight 4.

We obtain the following:

Corollary III.7. *A strongly pseudoconvex real hypersurface $Z \subset \mathbb{C}^2$ is locally volume-preserving equivalent to the sphere of radius R if and only if it is everywhere umbilic to the sphere of radius R.*

Remark III.8. As Webster notes in [22], $R_{1,1}^{1;1}$ is the holomorphic sectional curvature for the induced connection on Z, and τ^1 is its torsion. We can restate our earlier result as follows: A hypersurface is locally volume-preserving equivalent to a sphere if and only if it has zero torsion and constant holomorphic sectional curvature.

Remark III.9. In [22] Webster notes that there is a geometric interpretation associated to a hypersurface having zero torsion. Let $X, X_1, X_{\overline{1}}$ be the frame that is dual to the coframe $\theta, \theta^1, \theta^{\overline{1}}$. Then the hypersurface has zero torsion precisely when X is an infinitesimal pseudo-conformal transformation. Webster notes that this is a prominent assumption in the earlier work of Tanaka [21].

3.3 Isoperimetric and Extremal Hypersurfaces For Fefferman Hypersurface Measure

We now wish to apply our differential invariants to the study of two related variational problems, which are both inspired by classical problems in differential geometry. The first is the minimal surface problem. Here one wishes to characterize those surfaces which minimize surface area (with a prescribed boundary). The second is the isoperimetric problem. Here one wishes to characterize the compact surfaces which minimize the ratio of (the appropriate power of) surface area to volume. Since surface area is not invariant with respect to biholomorphic transformations, these problems are not natural in our setting. However, if one replaces surface area measure with Fefferman hypersurface measure, both questions are well-posed in this context.

We now wish to study the extremal hypersurface problem for Fefferman measure. This is inspired by the minimal surface problem, but the hypersurfaces that we shall find do not minimize Fefferman measure. Let $Z \subset \mathbb{C}^2$ have defining function ρ. By assumption, $|\nabla\rho| \neq 0$ on Z. Therefore, at each point one of ρ_x, ρ_y, ρ_u, or ρ_v is nonzero, where $z = x + iy, w = u + iv$. Since we are working locally, as soon as one of these derivatives is nonzero at a point, we will assume it is everywhere nonzero (since it is on the appropriate neighborhood). Without loss of generality, we assume $\rho_v \neq 0$. We view $Z \mapsto \int_Z \sigma_Z$ as a functional on the space of defining functions, i.e.,

$$\sigma : \rho \mapsto \int_{\{\rho=0\}} \sigma_{\{\rho=0\}}.$$

In order to compute the first variation of this functional at $\rho^{(0)}$, we must consider

$$\rho^{(0)} + \epsilon\rho^{(1)} \tag{3.36}$$

for arbitrary perturbations $\rho^{(1)}$. As is standard in the calculus of variations, it suffices to consider $\rho^{(1)}$ with small support. Since, by assumption, $\rho_v^{(0)} \neq 0$, by the Implicit Function Theorem, we can write $\{\rho^{(0)} = 0\}$ locally as a graph, i.e., $\{\rho^{(0)} = 0\} = \{-v + F^{(0)}(z, \overline{z}, u) = 0\}$, for some function $F^{(0)}$. So we shall assume that $\rho^{(0)} = -v + F^{(0)}(z, \overline{z}, u)$. Then a perturbation of the form (3.36) is equivalent to a perturbation of the graph. To be more precise, let $\rho = \rho^{(0)} + \epsilon\rho^{(1)}$. For small enough perturbations, we will have that $\rho_v \neq 0$. Therefore we can write $\{\rho = 0\}$ as a graph, i.e., we can find a positive function $h(z, \overline{z}, w, \overline{w})$ and $F(z, \overline{z}, u)$ such that

$$h(z, \overline{z}, w, \overline{w})\rho(z, \overline{z}, w, \overline{w}) = -v + F(z, \overline{z}, u).$$

So

$$h \cdot (\rho^{(0)} + \epsilon\rho^{(1)}) = -v + F^{(0)} + \epsilon F^{(1)},$$

where $F^{(1)} = \frac{1}{\epsilon}(F - F^{(0)})$.

Let Z be parametrized by $G(z,\overline{z},u)=(z,\overline{z},u+iF(z,\overline{z},u),u-iF(z,\overline{z},u))$. Then the Fefferman measure of Z is

$$\int_Z \sigma_Z=\int_{\mathbb{C}\times\mathbb{R}} G^*(\sigma_Z)=2^{\frac{4}{3}}\int_{\mathbb{C}\times\mathbb{R}} G^*(M(\rho))^{\frac{1}{3}}dV,$$

where

$$G^*M(\rho)=M(F)=\frac{1}{4}(F_{z,\overline{z}}(F_u^2+1)-F_{z,u}(F_u+i)F_{\overline{z}}-F_{\overline{z},u}(F_u-i)F_z+F_{u,u}|F_z|^2).$$

Now recall how one obtains Euler's equation. If a functional is given by

$$\mathcal{J}(F)=\int \mathcal{F}(x,F(x),F_j(x),F_{j,k}(x))dV,$$

where $\mathcal{F}(y,y_j,y_{j,k})$ is a function of $1+n+n^2$ variables, and $F_{j,k}$ denotes $\frac{\partial^2}{\partial x_j\partial x_k}F(x)$ then, by Theorem II.7, for smooth functions, the vanishing of the first variation is equivalent to

$$\frac{\partial}{\partial y}\mathcal{F}\circ F^{(2)}-\sum_j\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial y_j}\mathcal{F}\circ F^{(2)}\right)+\sum\frac{\partial^2}{\partial x_j\partial x_k}\left(\frac{\partial}{\partial y_{j,k}}\mathcal{F}\circ F^{(2)}\right)=0$$

where $F^{(2)}(x)=(x,F(x),F_j(x),F_{j,k}(x))$.

So our equation becomes:

$$\begin{aligned}&-\frac{\partial}{\partial u}(2M^{\frac{-2}{3}}F_uF_{z,\overline{z}})+\frac{\partial^2}{\partial z\partial\overline{z}}(M^{\frac{-2}{3}}(F_u^2+1))+\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}F_{z,u}(F_u+i))+\frac{\partial}{\partial u}(M^{\frac{-2}{3}}F_{z,u}F_{\overline{z}})\\&-\frac{\partial^2}{\partial z\partial u}(M^{\frac{-2}{3}}(F_u+i)F_{\overline{z}})+\frac{\partial}{\partial z}(M^{\frac{-2}{3}}F_{\overline{z},u}(F_u-i))+\frac{\partial}{\partial u}(M^{\frac{-2}{3}}F_{\overline{z},u}F_z)-\frac{\partial^2}{\partial\overline{z}\partial u}(M^{\frac{-2}{3}}(F_u-i)F_z)\\&-\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}F_{u,u}F_z)-\frac{\partial}{\partial z}(M^{\frac{-2}{3}}F_{u,u}F_{\overline{z}})+\frac{\partial^2}{\partial u\partial u}(M^{\frac{-2}{3}}F_zF_{\overline{z}})=0.\end{aligned}$$

Note that

$$\begin{aligned}&\rho_z=F_z,\rho_{\overline{z}}=F_{\overline{z}},\rho_w=\frac{1}{2}(F_u+i),\rho_{\overline{w}}=\frac{1}{2}(F_u-i),\\&\rho_{w,\overline{w}}=\frac{1}{4}F_{u,u},\rho_{z,\overline{z}}=F_{z,\overline{z}},\rho_{z,\overline{w}}=\frac{1}{2}F_{z,u},\rho_{\overline{z},w}=\frac{1}{2}F_{\overline{z},u},\end{aligned}$$

so that the preceding equation is equivalent to:

$$\begin{aligned}&-\frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_{\overline{w}})-\frac{\partial}{\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_w)+\frac{\partial^2}{\partial z\partial\overline{z}}(M^{\frac{-2}{3}}\rho_w\rho_{\overline{w}})+\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_w)\\&+\frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_{\overline{z}})-\frac{\partial^2}{\partial z\partial\overline{w}}(M^{\frac{-2}{3}}\rho_w\rho_{\overline{z}})+\frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_{\overline{w}})+\frac{\partial}{\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_z)\\&-\frac{\partial^2}{\partial\overline{z}\partial w}(M^{\frac{-2}{3}}\rho_{\overline{w}}\rho_z)-\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_z)-\frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_{\overline{z}})+\frac{\partial^2}{\partial w\partial\overline{w}}(M^{\frac{-2}{3}}\rho_z\rho_{\overline{z}})=0.\end{aligned}$$

Given a function $\rho(z,\overline{z},w,\overline{w})$ on $\mathbb{C}^2$, we define the differential operatior $\mathcal{L}$, by

$$\begin{aligned}\mathcal{L}(\rho)=&-\frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_{\overline{w}})-\frac{\partial}{\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_w)+\frac{\partial^2}{\partial z\partial\overline{z}}(M^{\frac{-2}{3}}\rho_w\rho_{\overline{w}})+\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_w)\\&+\frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_{\overline{z}})-\frac{\partial^2}{\partial z\partial\overline{w}}(M^{\frac{-2}{3}}\rho_w\rho_{\overline{z}})+\frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_{\overline{w}})+\frac{\partial}{\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_z)\\&-\frac{\partial^2}{\partial\overline{z}\partial w}(M^{\frac{-2}{3}}\rho_{\overline{w}}\rho_z)-\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_z)-\frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_{\overline{z}})+\frac{\partial^2}{\partial w\partial\overline{w}}(M^{\frac{-2}{3}}\rho_z\rho_{\overline{z}}).\end{aligned}$$

Then we have shown:

Lemma III.10. *Let $Z \subset \mathbb{C}^2$ be a strongly pseudoconvex, four times continuously differentiable real hypersurface with defining function ρ such that ρ is a graph. Then Z satisfies the Euler equation for the extremal hypersurface problem if and only if $\mathcal{L}(\rho) = 0$ on Z.*

Through a difficult computation, which we have done with the help of the computer, we find that if $h > 0$ is a positive function, then on the hypersurface

$$\mathcal{L}(h\rho) = \mathcal{L}(\rho). \tag{3.37}$$

This computation can be found online in the Mathematica file VPInvarianceCheck2.nb at http://hdl.handle.net/2027.42/62185. In order to perform this computation, we expand ρ through its 4-jet at a generic point, noting that $\mathcal{L}(\rho)$ is a fourth order differential operator, and we compute $\mathcal{L}(\rho)$ at this point. We do the same for $h\rho$, and we find that the two expressions coincide. This is unsatisfying, and we are convinced there is a "cleaner" proof of this fact. Now we prove:

Theorem III.11. *Let $Z \subset \mathbb{C}^2$ be a strongly pseudoconvex, four times continuously differentiable real hypersurface. Then Z satisfies the Euler equation for the extremal hypersurface problem if and only if $\mathcal{L}(\rho) = 0$ on Z.*

Proof. Suppose ρ satisfies $\mathcal{L}(\rho) = 0$. Then for some $h > 0$, $h\rho$ is a graph, and by 3.37, $\mathcal{L}(h\rho) = 0$, and so by Lemma III.10, $h\rho$ satisfies Euler's equation. Since satisfying Euler's equation does not depend on the choice of defining function, ρ satisfies Euler's equation. Conversely, suppose that ρ satisfies Euler's equation. Then again there exists $h > 0$ such that $h\rho$ is a graph which satisfies Euler's equation. Then by Lemma III.10, $\mathcal{L}(h\rho) = 0$, so by 3.37, $\mathcal{L}(\rho) = 0$. $\square$

Remark III.12. Given our explicit formula for $\mathcal{L}(\rho)$, it is easy to check that the Heisenberg group (the hyperquadric by another name), i.e., the hypersurface defined by $\rho(z, \overline{z}, w, \overline{w}) = -\mathrm{Im}(w) + |z|^2$, satisfies $\mathcal{L}(\rho) = 0$.

Now we turn our attention to the Isoperimetric Functional. Let $Z = \{\rho(z, \overline{z}, w, \overline{w}) = 0\} \subset \mathbb{C}^2$ be a strongly pseudoconvex real hypersurface. Suppose that $Z = \partial\Omega$, where Ω is a bounded open set in $\mathbb{C}^2$, then the volume of Ω is

$$\int_\Omega \omega_{\mathbb{C}^2} = \int_Z \tau,$$

where

$$\tau = \frac{-1}{16}(z d\overline{z} \wedge dw \wedge d\overline{w} - \overline{z} dz \wedge dw \wedge d\overline{w} + w dz \wedge d\overline{z} \wedge d\overline{w} - \overline{w} dz \wedge d\overline{z} \wedge dw).$$

Now we define the isoperimetric functional by

$$\mathcal{I}(Z) = \frac{(\int_Z \sigma_Z)^{\frac{3}{2}}}{\int_Z \tau},$$

where σ_Z is the Fefferman measure on Z and recalling from section 3.2, it is given by $\sigma_Z \wedge d\rho = 2^{\frac{4}{3}} M(\rho)^{\frac{1}{3}} \omega_{\mathbb{C}^2}$, i.e., $\sigma_Z = \frac{2^{\frac{4}{3}} M(\rho)^{\frac{1}{3}}}{|\nabla\rho|} s_Z$, where

$$s_Z = \frac{\nabla\rho}{|\nabla\rho|} \rfloor \omega_{\mathbb{C}^2} = \frac{1}{2|\nabla\rho|}(-\rho_{\overline{z}} d\overline{z} \wedge dw \wedge d\overline{w} + \rho_z dz \wedge dw \wedge d\overline{w} - \rho_{\overline{w}} dz \wedge d\overline{z} \wedge d\overline{w} + \rho_w dz \wedge d\overline{z} \wedge dw)$$

is surface area measure on Z. This functional was introduced by D. Barrett. The motivation for the power of Fefferman hypersurface measure in the numerator is that with this particular choice, this functional is invariant under constant-Jacobian maps. Since the isoperimetric functional is invariant under constant-Jacobian maps, we shall assume that the volume of Ω is 1.

Since we are ultimately interested in bounded smooth domains that are extrema for the isoperimetric functional, we shall look for hypersurfaces that satisfy Euler's equation everywhere. In order to derive Euler's equation for the isoperimetric functional, we work locally. More precisely, we let U be an open subset of the hypersurface Z, where $Z = \partial\Omega$ and Ω is a smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^2$. In a sufficiently small open subset U of $\partial\Omega$, we can assume that our hypersurface is a graph. Suppose $\rho = -v + F(z,\overline{z},u)$, then $Z \cap U$ is parametrized by $G : W \to \mathbb{C}^2$, where W is an open subset of $\mathbb{C} \times \mathbb{R}$ and

$$G(z,\overline{z},u) = (z,\overline{z},u + iF(z,\overline{z},u), u - iF(z,\overline{z},u))$$

and

$$\int_{Z\cap U} \tau = \int_W G^*(\tau).$$

Now,

$$G^*(dz \wedge dw \wedge d\overline{w}) = 2iF_{\overline{z}} dz \wedge d\overline{z} \wedge du, G^*(dz \wedge d\overline{z} \wedge dw) = (1 + iF_u) dz \wedge d\overline{z} \wedge du,$$
$$G^*(d\overline{z} \wedge dw \wedge d\overline{w}) = -2iF_z dz \wedge d\overline{z} \wedge du, \text{ and } G^*(dz \wedge d\overline{z} \wedge d\overline{w}) = (1 - iF_u) dz \wedge d\overline{z} \wedge du,$$

so

$$G^*(\tau) = \frac{i}{8}(zF_z + \overline{z}F_{\overline{z}} + uF_u - F) dz \wedge d\overline{z} \wedge du = \frac{1}{4}(zF_z + \overline{z}F_{\overline{z}} + uF_u - F) dV,$$

where dV represents the volume form on $\mathbb{C} \times \mathbb{R}$. Note that

$$\int_{Z\cap U} \sigma_Z = \int_W G^*(\sigma_Z) = 2^{\frac{4}{3}} \int_W G^*(M(\rho))^{\frac{1}{3}} dV$$

and

$$G^* M(\rho) = M(F) = \frac{1}{4}(F_{z,\overline{z}}(F_u^2 + 1) - F_{z,u}(F_u + i)F_{\overline{z}} - F_{\overline{z},u}(F_u - i)F_z + F_{u,u}|F_z|^2).$$

In order to derive Euler's equation, we consider a local perturbation $\widetilde{Z}$ of Z over U. More precisely, $\widetilde{Z} = (Z \cap U^c) \cup Y$, where Y is a graph over $Z \cap U$, i.e., $Y = \{(z,w) + g(z,w)\hat{n}(z,w) : (z,w) \in Z \cap U\}$, where $g : Z \cap U \to \mathbb{R}$ has compact support in U and $\hat{n}(z,w)$ is the unit normal to Z at (z,w). Such a perturbation will also be a graph over W on Y. If $Z \cap U$ is the graph $v = F^0(z,\overline{z},u)$ then Y is the graph $v = F(z,\overline{z},u)$, where $F = F^0 + \epsilon F^1$ and F^1 has compact support in W. Now $M(F) = M(F^0) + \epsilon M_1(F^0,F^1) + \mathcal{O}(\epsilon^2)$, where

$$M_1(F^0,F^1) = \frac{1}{4}(2F_u^0 F_{z,\overline{z}}^0 F_u^1 + ((F_u^0)^2 + 1)F_{z,\overline{z}}^1 - F_{z,u}^0(F_u^0 + i)F_{\overline{z}}^1 - F_{z,u}^0 F_{\overline{z}}^0 F_u^1$$
$$- (F_u^0 + i)F_{\overline{z}}^0 F_{z,u}^1 - (F_u^0 - i)F_{\overline{z},u}^0 F_z^1 - F_{\overline{z},u}^0 F_z^0 F_u^1 - (F_u^0 - i)F_z^0 F_{\overline{z},u}^1 + F_{u,u}^0 F_z^0 F_{\overline{z}}^1$$
$$+ F_{u,u}^0 F_{\overline{z}}^0 F_z^1 + F_z^0 F_{\overline{z}}^0 F_{u,u}^1).$$

We see that if $V(\widetilde{Z})$ denotes the volume of $\widetilde{Z}$, then $V(\widetilde{Z}) = V(Z) + \epsilon V(F^1) = 1 + \epsilon V(F^1)$, abusing notation. We also note that

$$\begin{aligned}\int_{\widetilde{Z}} \sigma_{\widetilde{Z}} =& \int_{Z\cap U^c} \sigma_Z + \int_Y \sigma_{\widetilde{Z}} = \sigma(Z \cap U^c) + 2^{\frac{4}{3}} \int_W G^*(M(\rho))^{\frac{1}{3}} dV \\ =& \sigma(Z) + \frac{2^{\frac{4}{3}}}{3}\epsilon \int_W M(F^0)^{-\frac{2}{3}} M_1(F^0,F^1) dV.\end{aligned}$$

Taking power series expansions, we have

$$\mathcal{I}(\widetilde{Z}) = \mathcal{I}(Z) + \epsilon \mathcal{I}_1(Z,F^1) + \mathcal{O}(\epsilon^2),$$

where

$$\mathcal{I}_1(Z,F^1) = \frac{3}{2}\sigma(Z)^{\frac{1}{2}} \frac{2^{-\frac{4}{3}}}{3} \int_W M(F^0)^{-\frac{2}{3}} M_1(F^0,F^1) dV - V(F^1)\sigma(Z)^{\frac{3}{2}}.$$

Abbreviating $M := M(F^0)$, the equation $\mathcal{I}_1(Z,F^1) = 0$ becomes

$$\int_W M^{\frac{-2}{3}} M_1 dV - CV(F^1) = 0,$$

where
$$C = 2^{\frac{-1}{3}}\sigma(Z).$$
Integrating by parts, we have that
$$\begin{aligned}
&-\frac{\partial}{\partial u}(2M^{\frac{-2}{3}}F_u^0F_{z,\overline{z}}^0)+\frac{\partial^2}{\partial z\partial\overline{z}}(M^{\frac{-2}{3}}((F_u^0)^2+1))+\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}F_{z,u}^0(F_u^0+i))+\frac{\partial}{\partial u}(M^{\frac{-2}{3}}F_{z,u}^0F_{\overline{z}}^0)\\
&-\frac{\partial^2}{\partial z\partial u}(M^{\frac{-2}{3}}(F_u^0+i)F_{\overline{z}}^0)+\frac{\partial}{\partial z}(M^{\frac{-2}{3}}F_{\overline{z},u}^0(F_u^0-i))+\frac{\partial}{\partial u}(M^{\frac{-2}{3}}F_{\overline{z},u}^0F_z^0)-\frac{\partial^2}{\partial\overline{z}\partial u}(M^{\frac{-2}{3}}(F_u^0-i)F_z^0)\\
&-\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}F_{u,u}^0F_z^0)-\frac{\partial}{\partial z}(M^{\frac{-2}{3}}F_{u,u}^0F_{\overline{z}}^0)+\frac{\partial^2}{\partial u\partial u}(M^{\frac{-2}{3}}F_z^0F_{\overline{z}}^0)+4C=0.
\end{aligned}$$
Note that
$$\begin{aligned}
&\rho_z=F_z,\rho_{\overline{z}}=F_{\overline{z}},\rho_w=\frac{1}{2}(F_u+i),\rho_{\overline{w}}=\frac{1}{2}(F_u-i),\\
&\rho_{w,\overline{w}}=\frac{1}{4}F_{u,u},\rho_{z,\overline{z}}=F_{z,\overline{z}},\rho_{z,\overline{w}}=\frac{1}{2}F_{z,u},\rho_{\overline{z},w}=\frac{1}{2}F_{\overline{z},u},
\end{aligned}$$
so that the preceding equation is equivalent to:
$$\begin{aligned}
&-\frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_{\overline{w}})-\frac{\partial}{\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_w)+\frac{\partial^2}{\partial z\partial\overline{z}}(M^{\frac{-2}{3}}\rho_w\rho_{\overline{w}})+\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_w)\\
&+\frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_{\overline{z}})-\frac{\partial^2}{\partial z\partial\overline{w}}(M^{\frac{-2}{3}}\rho_w\rho_{\overline{z}})+\frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_{\overline{w}})+\frac{\partial}{\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_z)\\
&-\frac{\partial^2}{\partial\overline{z}\partial w}(M^{\frac{-2}{3}}\rho_{\overline{w}}\rho_z)-\frac{\partial}{\partial\overline{z}}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_z)-\frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_{\overline{z}})+\frac{\partial^2}{\partial w\partial\overline{w}}(M^{\frac{-2}{3}}\rho_z\rho_{\overline{z}})+C=0,
\end{aligned}$$
where $M = M(\rho)$ and $C = 2^{\frac{-1}{3}}\sigma(Z), V(Z) = 1$. Recalling the differential operator $\mathcal{L}$ from the previous section, we have that

Lemma III.13. *Where ρ is a graph, Euler's equation for the isoperimetric functional is given by*
$$\mathcal{L}(\rho) = -C.$$

Recalling the invariance property of $\mathcal{L}(\rho)$ stated in 3.37 and the proof of Theorem III.11, it is immediate that:

Theorem III.14. *If Z is the boundary of a smoothly bounded strongly pseudoconvex domain with unit volume in $\mathbb{C}^2$, then Z is a global solution of Euler's equation for the isoperimetric functional if and only if*
$$\mathcal{L}(\rho) = -C,$$
where $C = 2^{\frac{-1}{3}}\sigma(Z)$.

Remark III.15. Note that this is the inhomogeneous version of Euler's Equation for the Fefferman Hypersurface Measure functional σ.

We wish to study this operator by first simplifying it. In [9], Fefferman shows that we can choose a defining function ρ for our hypersurface such that $M(\rho) = 1 + h\rho^3$, where h is a real-valued function. Let us compute $\mathcal{L}(\rho)$ for this particular (essentially unique) choice of defining function.

Our expression reduces to
$$\begin{aligned}
\mathcal{L}(\rho)=&-\frac{\partial}{\partial w}(\rho_{z,\overline{z}}\rho_{\overline{w}})-\frac{\partial}{\partial\overline{w}}(\rho_{z,\overline{z}}\rho_w)+\frac{\partial^2}{\partial z\partial\overline{z}}(\rho_w\rho_{\overline{w}})+\frac{\partial}{\partial\overline{z}}(\rho_{z,\overline{w}}\rho_w)\\
&+\frac{\partial}{\partial w}(\rho_{z,\overline{w}}\rho_{\overline{z}})-\frac{\partial^2}{\partial z\partial\overline{w}}(\rho_w\rho_{\overline{z}})+\frac{\partial}{\partial z}(\rho_{\overline{z},w}\rho_{\overline{w}})+\frac{\partial}{\partial\overline{w}}(\rho_{\overline{z},w}\rho_z)\\
&-\frac{\partial^2}{\partial\overline{z}\partial w}(\rho_{\overline{w}}\rho_z)-\frac{\partial}{\partial\overline{z}}(\rho_{w,\overline{w}}\rho_z)-\frac{\partial}{\partial z}(\rho_{w,\overline{w}}\rho_{\overline{z}})+\frac{\partial^2}{\partial w\partial\overline{w}}(\rho_z\rho_{\overline{z}})+\mathcal{O}(\rho)\\
&=6(\rho_{z,\overline{w}}\rho_{\overline{z},w}-\rho_{z,\overline{z}}\rho_{w,\overline{w}})+\mathcal{O}(\rho).
\end{aligned}$$

The last equality above is merely computation. Therefore, we can find a defining function for our hypersurface so that $\mathcal{L}(\rho) = 6(\rho_{z,\overline{w}}\rho_{\overline{z},w} - \rho_{z,\overline{z}}\rho_{w,\overline{w}})$ on our hypersurface. By our earlier computations (3.32) and (3.33), $\mathcal{L}(\rho)$ is an absolute invariant with respect to volume-preserving biholomorphisms. Actually, this is a special case a more general result. Namely, the Euler-Langrange equation associated to a variational problem is invariant under tranformations under which the variational problem is invariant. This is a nontrivial result that can be found in Olver [20]. Note that for the same choice of defining function, our expression for $R^{1,1}_{1,1}$ reduces to

$$R^{1,1}_{1,1} = 2^{\frac{1}{3}}3(\rho_{z,\overline{w}}\rho_{\overline{z},w} - \rho_{z,\overline{z}}\rho_{w,\overline{w}}).$$

Since neither $R^{1,1}_{1,1}$ nor $\mathcal{L}(\rho)$ depends on the choice of defining function, we have that

$$\mathcal{L}(\rho) = 2^{\frac{2}{3}}R^{1,1}_{1,1}.$$

The above relationship tells us the $\mathcal{L}(\rho)$ is also a multiple of the invariant κ, however, as we could have used κ to relate $\mathcal{L}(\rho)$ and $R^{1,1}_{1,1}$, let us now also derive the relationship between $\mathcal{L}(\rho)$ and invariant κ.

Suppose that our defining function is in normal form, so

$$\begin{aligned}\rho(z,\overline{z},w,\overline{w}) = &-\frac{w-\overline{w}}{2i} + |z|^2 + \kappa|z|^4 + \gamma z^3\overline{z} + \gamma z\overline{z}^3\\ &+ \mathcal{R}(u^3, zu^2, \overline{z}u^2, u^4, zu^3, \overline{z}u^3, z^2u^2, \overline{z}^2u^2, |z|^2u^2, z^3u, \overline{z}^3u, z^2\overline{z}u, z\overline{z}^2u).\end{aligned}$$

Here we have ignored any term that is not in the 4-jet, since this will not appear in our expression evaluated at the origin. Then

$$\begin{aligned}M(\rho) = &\frac{1}{4}(1 + 4\kappa|z|^2 + 3\gamma z^2 + 3\gamma\overline{z}^2)\\ &+ \mathcal{R}(u^2, zu, \overline{z}u, |z|^2u, \overline{z}u^2, zu^2, z^2u, \overline{z}^2u, z^3, \overline{z}^3, z^2\overline{z}, z\overline{z}^2, |z|^2u^2, z^2\overline{z}u, z\overline{z}^2u, z^3\overline{z}, z\overline{z}^3, |z|^4).\end{aligned}$$

Also note that

$$M_z = \kappa\overline{z} + \frac{3}{2}\gamma z + \mathcal{R}(u, \overline{z}u, zu, u^2, z^2, \overline{z}^2, |z|^2)$$

and

$$M_w = \mathcal{R}(u, z, \overline{z}, |z|^2, zu, \overline{z}u, |z|^2, z^2, \overline{z}^2).$$

Now, recall that

$$\begin{aligned}\mathcal{L}(\rho) = &-\frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_{\overline{w}}) - \frac{\partial}{\partial \overline{w}}(M^{\frac{-2}{3}}\rho_{z,\overline{z}}\rho_{w}) + \frac{\partial^2}{\partial z\partial\overline{z}}(M^{\frac{-2}{3}}\rho_{w}\rho_{\overline{w}}) + \frac{\partial}{\partial \overline{z}}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_{w})\\ &+ \frac{\partial}{\partial w}(M^{\frac{-2}{3}}\rho_{z,\overline{w}}\rho_{\overline{z}}) - \frac{\partial^2}{\partial z\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{w}\rho_{\overline{z}}) + \frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_{\overline{w}}) + \frac{\partial}{\partial \overline{w}}(M^{\frac{-2}{3}}\rho_{\overline{z},w}\rho_{z})\\ &- \frac{\partial^2}{\partial\overline{z}\partial w}(M^{\frac{-2}{3}}\rho_{\overline{w}}\rho_{z}) - \frac{\partial}{\partial \overline{z}}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_{z}) - \frac{\partial}{\partial z}(M^{\frac{-2}{3}}\rho_{w,\overline{w}}\rho_{\overline{z}}) + \frac{\partial^2}{\partial w\partial\overline{w}}(M^{\frac{-2}{3}}\rho_{z}\rho_{\overline{z}}).\end{aligned}$$

By our earlier observation, any term that is multiplied by a first-order derivative of $M^{\frac{-2}{3}}$ may be discarded. Likewise, any term multiplying ρ_z or $\rho_{\overline{z}}$. Moreover, since

$$\rho_w = \frac{-1}{2i} + \mathcal{R}(u^2, zu, \overline{z}u, u^3, zu^2, \overline{z}u^2, z^2u, \overline{z}^2u, |z|^2u, z^3, \overline{z}^3, z^2\overline{z}, z\overline{z}^2),$$

we see that any term multiplying $\rho_{z,w,z}, \rho_{z,w,\overline{z}}, \rho_{\overline{z},w,\overline{z}}, \rho_{\overline{z},\overline{w},\overline{z}}, \rho_{\overline{z},\overline{w},z}, \rho_{z,\overline{w},z}$ may be discarded. Likewise, any term multiplying $\rho_{w,\overline{w}}, \rho_{w,w}, \rho_{\overline{w},\overline{w}}$ may be discarded. Implementing all of these simplifications, we find $M^{\frac{2}{3}}\mathcal{L}(\rho) = \frac{-2}{3}M^{-1}|\rho_w|^2M_{z,\overline{z}} + \mathcal{R}(z,\overline{z},u)$. Since $M_{z,\overline{z}} = \kappa + \mathcal{R}(z,\overline{z},u)$, we find that at the origin,

$$\mathcal{L}(\rho) = \frac{-2^{\frac{7}{3}}}{3}\kappa. \tag{3.38}$$

For a sphere of radius R, $\rho = z\overline{z} + w\overline{w} - R^2$. We have $V(Z) = \frac{\pi^2}{2}R^4$ and $s(Z) = 2\pi^2 R^3$ are the volume and surface area, respectively. Now, $|\nabla\rho| = 2R$ and $M(\rho) = R^2$, so $\sigma_Z = 2^{\frac{4}{3}}R^{-\frac{1}{3}}s_Z$, which implies that the Fefferman measure of the sphere is merely $\sigma(Z) = 2^{\frac{4}{3}}R^{-\frac{1}{3}}s(Z) = 2^{\frac{4}{3}}2\pi^2 R^{\frac{8}{3}}$. For the sphere of radius R,

$$\mathcal{L}(\rho) = -4R^{\frac{-4}{3}}.$$

If we assume the sphere has unit volume, $V = 1$, then $R^4 = \frac{2}{\pi^2}$. We merely check that the left and right-hand sides are equal. Since $\mathcal{L}(\rho) = 2^{\frac{2}{3}}R^{1,1}_{1,1}$, we immediately obtain the following:

Theorem III.16. *Let $Z \subset \mathbb{C}^2$ be a strongly pseudoconvex, four times continuously differentiable real hypersurface bounding a domain with unit volume. Then Z is a global solution of Euler's equation for the isoperimetric functional if and only if $R^{1,1}_{1,1} = -\frac{1}{2}\sigma(Z)$ on Z.*

Which, by Theorem III.5 gives:

Corollary III.17. *A torsion-free strongly pseudoconvex real-analytic, real hypersurface $Z \subset \mathbb{C}^2$ bounding a domain with unit volume is a global solution of Euler's equation for the isopermetric functional if and only if Z is locally the image of a sphere under a volume-preserving holomorphic map.*

We also have the following:

Theorem III.18. *Let $Z \subset \mathbb{C}^2$ be a strongly pseudoconvex, four times continuously differentiable real hypersurface. Then Z satisfies the Euler equation for the extremal hypersurface problem if and only if $R^{1,1}_{1,1} = 0$ on Z.*

Remark III.19. There are several interesting questions that remain here. We would like to analyze the second variation of the Isoperimetric Functional. It is already clear that spheres are not minima for the Isoperimetric Functional but rather saddle points. However, Barrett has shown that if one restricts to a special class of hypersurfaces, that changes. In particular, he has obtained an Isoperimetric Inequality for complete circular domains.

We would also like to determine whether or not the spheres are the only compact, simply connected hypersurfaces having constant holomorphic sectional curvature.

CHAPTER IV

Projective Invariants

4.1 Projective Normal Form and Dual Hypersurfaces

Let $S \subset \mathbb{C}^2$ be a real analytic, strongly pseudoconvex hypersurface with real analytic defining function $\rho(z, \overline{z}, w, \overline{w})$. S is said to be strongly $\mathbb{C}$-linearly convex if all complex tangent hyperplanes to S lie to one side of S with minimal order of contact (c.f., [1] for a thorough discussion). However, since this is not essential for what we are doing, we shall use an equivalent, non-standard definition (c.f., the equivalence of this definition and much more can be found in Barrett's paper [3]).

Definition IV.1. For a point $P \in S$ and for vectors $s = (s_1, s_2), t = (t_1, t_2) \in \mathbb{C}^2$, let

$$L_{\rho,P}(s, \bar{t}) = \rho_{z,\overline{z}} s_1 \overline{t_1} + \rho_{z,\overline{w}} s_1 \overline{t_2} + \rho_{w,\overline{z}} s_2 \overline{t_1} + \rho_{w,\overline{w}} s_2 \overline{t_2}$$
$$Q_{\rho,P}(s, t) = \rho_{z,z} s_1 t_1 + \rho_{z,w} s_1 t_2 + \rho_{w,z} s_2 t_1 + \rho_{w,w} s_2 t_2.$$

We shall often abuse notation by writing $L(s, \bar{t})$ and $Q(s, t)$. Let $X = (\frac{\partial \rho}{\partial w}, -\frac{\partial \rho}{\partial z})$, then X spans the complex tangent space of S, and we see that L is the Levi form, or the Hermitian part of the second fundamental form with:

$$L(X, \overline{X}) = \frac{\partial^2 \rho}{\partial z \partial \overline{z}} \frac{\partial \rho}{\partial w} \frac{\partial \rho}{\partial \overline{w}} - \frac{\partial^2 \rho}{\partial z \partial \overline{w}} \frac{\partial \rho}{\partial w} \frac{\partial \rho}{\partial \overline{z}} - \frac{\partial^2 \rho}{\partial w \partial \overline{z}} \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial \overline{w}} + \frac{\partial^2 \rho}{\partial w \partial \overline{w}} \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial \overline{z}}.$$

Similarly, Q is the anti-Hermitian part of the second fundamental form with:

$$Q(X, X) = \frac{\partial^2 \rho}{\partial z \partial z} \left(\frac{\partial \rho}{\partial w}\right)^2 - 2 \frac{\partial^2 \rho}{\partial w \partial z} \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial w} + \frac{\partial^2 \rho}{\partial w \partial w} \left(\frac{\partial \rho}{\partial z}\right)^2.$$

Finally, we define:

Definition IV.2. The real hypersurface Z is called strongly $\mathbb{C}$-linearly convex if

$$\left| \frac{Q(X, X)}{L(X, \overline{X})} \right| < 1.$$

Now, given a strongly $\mathbb{C}$-linearly convex hypersurface S in $\mathbb{C}^2$, we can define a map from S into $(\mathbb{CP}^2)^*$. If $P \in S$ and $H_P S$ is the complex tangent hyperplane, $H_P S$ is given by $H_P S = \{(z, w) : \phi(z, w) = 0\}$, where $\phi(z, w) = Az + Bw + C$ for some constants $A, B, C \in \mathbb{C}$. In fact, $\lambda\phi$ will define the same complex hyperplane for any $\lambda \in \mathbb{C}^*$. We define a map from S to $\mathbb{CP}^2$ via $P \mapsto [A : B : C]$. One can show (c.f., [1]) that the image of S under this mapping is a $\mathbb{C}$-linearly convex hypersurface in $\mathbb{CP}^2$. We call this hypersurface, S^*, the dual hypersurface of S. Note that if $P = (z_0, w_0)$, then

$$H_P S = \{(z, w) : \rho_z(z_0, \overline{z_0}, w_0, \overline{w_0}) \cdot (z - z_0) + \rho_w(z_0, \overline{z_0}, w_0, \overline{w_0}) \cdot (w - w_0) = 0\}.$$

We wish to introduce a normal form for a strongly $\mathbb{C}$-linearly convex real analytic hypersurface S in $\mathbb{CP}^2$ with respect to the group of automorphisms of $\mathbb{CP}^2$ in such a way that the normal form behaves well when one passes to the corresponding dual hypersurface. If we are in an affine chart where the coordinates are z, w, where $w = u + iv$, then locally, our hypersurface can be written as $v = F(z, \overline{z}, u)$, where F is real analytic. The defining function for S is $\rho(z, \overline{z}, w, \overline{w}) = -v + F(z, \overline{z}, u)$. In an appropriate affine chart, the map taking S to its dual hypersurface S^* is

$$G(z, w) = (f(z, w), g(z, w)) = \left(-\frac{\rho_z}{\rho_w}, -w - z\frac{\rho_z}{\rho_w}\right).$$

This was suggested by D. Barrett so that the transformation would mirror the Legendre transform in $\mathbb{R}^2$, and it is almost immediate that $G \circ G = \mathrm{Id}$, where Id is the identity (more on this in [3]). The computations that follow are rather lengthy, so we refer the reader to Appendix B, where all computations are done using Mathematica 7.0. The computations needed to find the defining function of the dual hypersurface are found in Appendix B.1. In Appendix B.2 we have included the computations necessary to construct a normal form, and in Appendix B.3, we have computed the isotropy group of a hypersurface in normal form.

Let S^* have defining function $\widetilde{\rho}(\zeta, \eta)$, then

$$H(\zeta, \eta) := G^{-1}(\zeta, \eta) = (s(\zeta, \eta), t(\zeta, \eta)) = \left(-\frac{\widetilde{\rho_\zeta}}{\widetilde{\rho_\eta}}, -\eta - \zeta\frac{\widetilde{\rho_\zeta}}{\widetilde{\rho_\eta}}\right).$$

Now,

$$F(z, \overline{z}, u) = \alpha|z|^2 + \beta z^2 + \beta\overline{z}^2 + Az^3 + Bz^2\overline{z} + Czu + \overline{A}\overline{z}^3 + \overline{B}\overline{z}^2 z + \overline{C}\overline{z}u + \cdots,$$

where α, β are real and nonnegative. By direct computation, one has that

$$\frac{\rho_z}{\rho_w} = -4i\beta z - 2i\alpha\overline{z} + (2\alpha C + 4\beta\overline{C} - 4iB)|z|^2 + (4\beta C - 6iA)z^2 + (2\alpha\overline{C} - 2i\overline{B})\overline{z}^2 - 2iCu + \cdots,$$

so of course

$$f(z, w) = 4i\beta z + 2i\alpha\overline{z} - (2\alpha C + 4\beta\overline{C} - 4iB)|z|^2 - (4\beta C - 6iA)z^2 - (2\alpha\overline{C} - 2i\overline{B})\overline{z}^2 + 2iCu + \cdots.$$

Let $\eta = \mu + i\nu$. Then for some unknown coefficients

$$s(\zeta, \eta) := \widetilde{\gamma}\zeta + \widetilde{\delta}\overline{\zeta} + \widetilde{\mathcal{A}}|\zeta|^2 + \widetilde{\mathcal{B}}\zeta^2 + \widetilde{\mathcal{C}}\overline{\zeta}^2 + \widetilde{\mathcal{E}}\mu + \cdots.$$

Now we impose the condition that $s(f(z, w), g(z, w)) = z$, so working only with the linear terms at first, we have $(4i\beta\widetilde{\gamma} - 2i\alpha\widetilde{\delta})z + (2i\alpha\widetilde{\gamma} - 4i\beta\widetilde{\delta})\overline{z} = z$, which implies that

$$\begin{aligned} 4i\beta\widetilde{\gamma} - 2i\alpha\widetilde{\delta} &= 1 \\ 2i\alpha\widetilde{\gamma} - 4i\beta\widetilde{\delta} &= 0. \end{aligned}$$

Solving this system yields

$$\begin{aligned} \widetilde{\gamma} &= \frac{i\beta}{\alpha^2 - 4\beta^2} \\ \widetilde{\delta} &= \frac{i\alpha}{2(\alpha^2 - 4\beta^2)}. \end{aligned}$$

Now, we define $\Delta = \alpha^2 - 4\beta^2$. If $\Delta \equiv \frac{1}{4}$ and if $\widetilde{\rho}(\zeta, \overline{\zeta}, \mu, \nu) = -\nu + \widetilde{\alpha}|\zeta|^2 + \widetilde{\beta}\zeta^2 + \widetilde{\beta}\overline{\zeta}^2 + \cdots$, then $\widetilde{\alpha} = \frac{\widetilde{\delta}}{2i}$ and $\widetilde{\beta} = \frac{\widetilde{\gamma}}{4i}$, so $\widetilde{\alpha} = \frac{\alpha}{4\Delta} = \alpha$ and $\widetilde{\beta} = \frac{\beta}{4\Delta} = \beta$. So if α, β satisfy the relation $\Delta = \frac{1}{4}$, then the equation defining the hypersurface is invariant up through weight 2. We shall assume that $\alpha^2 - 4\beta^2 \neq 0$ and we shall justify our assumption when we construct the normal form. We shall see that this assumption corresponds to strong $\mathbb{C}$-linear convexity.

Now, taking the weight 2 terms of $s \circ G(z,w)$ and setting them equal to 0, we obtain

$$\left(\frac{-i\alpha^2\overline{C}}{\Delta} - \frac{4\beta B}{\Delta} - \frac{4i\beta^2\overline{C}}{\Delta} + 4\alpha^2\widetilde{\mathcal{A}} + \frac{2\alpha\overline{B}}{\Delta} - 16\alpha\beta\widetilde{\mathcal{B}} + 16\beta^2\widetilde{\mathcal{A}} - \frac{4i\alpha\beta C}{\Delta} - 16\alpha\beta\widetilde{\mathcal{C}}\right)|z|^2$$
$$+\left(\frac{-6\beta A}{\Delta} + \frac{\alpha B}{\Delta} - \frac{4i\beta^2 C}{\Delta} - 4\alpha^2\widetilde{\mathcal{C}} - 16\beta^2\widetilde{\mathcal{B}} + 8\alpha\beta\widetilde{\mathcal{A}} + 2i\beta\widetilde{\mathcal{E}} - \frac{i\alpha^2 C}{\Delta}\right)z^2$$
$$+\left(-2i\beta\widetilde{\mathcal{E}} - 16\beta^2\widetilde{\mathcal{C}} + \frac{3\alpha\overline{A}}{\Delta} - \frac{2\beta\overline{B}}{\Delta} + 8\alpha\beta\widetilde{\mathcal{A}} - \frac{4i\alpha\beta\overline{C}}{\Delta} - 4\alpha^2\widetilde{\mathcal{B}}\right)\overline{z}^2 + \left(\frac{\alpha\overline{C}}{\Delta} - \frac{2\beta C}{\Delta} - \widetilde{\mathcal{E}}\right)u = 0.$$

This yields a linear system of equations in the unknowns $\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{E}}$ by setting the coefficient of each term equal to 0. Solving this system gives us:

$$\widetilde{\mathcal{A}} =$$
$$\frac{i}{4\Delta^3}\left(\overline{C}\alpha^4 + 2i\overline{B}\alpha^3 - 12i\overline{A}\beta\alpha^2 - 8iB\beta\alpha^2 - 8\beta^2\overline{C}\alpha^2 + 24iA\beta^2\alpha + 16i\overline{B}\beta^2\alpha - 16iB\beta^3 + 16\beta^4\overline{C}\right)$$
$$\widetilde{\mathcal{B}} = -\frac{1}{4\Delta^3}\left(32iC\beta^4 + 24A\beta^3 - 16i\alpha\overline{C}\beta^3 - 12\alpha B\beta^2 - 8i\alpha^2 C\beta^2 + 6\alpha^2\overline{B}\beta + 4i\alpha^3\overline{C}\beta - 3\overline{A}\alpha^3\right)$$
$$\widetilde{\mathcal{C}} = -\frac{1}{4\Delta^3}\cdot$$
$$\left(iC\alpha^4 - B\alpha^3 - 4i\beta\overline{C}\alpha^3 + 6A\beta\alpha^2 + 4\overline{B}\beta\alpha^2 - 12\overline{A}\beta^2\alpha - 8B\beta^2\alpha + 16i\beta^3\overline{C}\alpha + 8\overline{B}\beta^3 - 16i\beta^4 C\right)$$
$$\widetilde{\mathcal{E}} = -\frac{1}{\Delta}\left(2\beta C - \alpha\overline{C}\right).$$

If $\widetilde{\rho}(\zeta,\overline{\zeta},\mu,\nu) = -\nu + \widetilde{\alpha}|\zeta|^2 + \widetilde{\beta}\zeta^2 + \widetilde{\beta}\overline{\zeta}^2 + \widetilde{A}\zeta^3 + \widetilde{B}\zeta^2\overline{\zeta} + \widetilde{C}\zeta\mu + \overline{\widetilde{A}}\overline{\zeta}^3 + \overline{\widetilde{B}}\overline{\zeta}^2\zeta + \overline{\widetilde{C}}\overline{\zeta}\mu + \cdots$, then

$$\widetilde{\mathcal{A}} = 4i\widetilde{B} - \frac{\alpha\widetilde{C}}{2\left(\alpha^2 - 4\beta^2\right)} - \frac{\beta\overline{\widetilde{C}}}{\alpha^2 - 4\beta^2}$$
$$\widetilde{\mathcal{B}} = 6i\widetilde{A} - \frac{\beta\widetilde{C}}{\alpha^2 - 4\beta^2}$$
$$\widetilde{\mathcal{C}} = 2i\overline{\widetilde{B}} - \frac{\alpha\overline{\widetilde{C}}}{2\left(\alpha^2 - 4\beta^2\right)}$$
$$\widetilde{\mathcal{E}} = 2i\widetilde{C}.$$

We can solve this system of equations for $\widetilde{A}, \widetilde{B}, \widetilde{C}$ to obtain

$$\widetilde{A} = \frac{1}{48}\left(-8i\mathcal{B} - \frac{\mathcal{E}}{\alpha - 2\beta} + \frac{\mathcal{E}}{\alpha + 2\beta}\right)$$
$$\widetilde{B} = \frac{-4i\mathcal{A}\alpha^2 - \mathcal{E}\alpha + 16i\mathcal{A}\beta^2 + 2\beta\overline{\mathcal{E}}}{16(\alpha - 2\beta)(\alpha + 2\beta)}$$
$$\widetilde{C} = \frac{-i}{2}\widetilde{\mathcal{E}}.$$

So in terms of the coefficients of the original hypersurface, the coefficients for the dual hypersurface are as follows:

$$\begin{aligned}
\widetilde{A} &= \frac{2\beta\left(-\overline{C}\alpha^3 + (i\overline{B} + 2\beta C)\alpha^2 + 2\beta(2\beta\overline{C} - iB)\alpha + 4i\beta^2(A + 2i\beta C)\right) - i\overline{A}\alpha^3}{8\left(\alpha^2 - 4\beta^2\right)^3} \\
\widetilde{B} &= \frac{i\left((\overline{B} - 2i\beta C)\alpha^3 + 2\beta(-3\overline{A} - 2B + 2i\beta\overline{C})\alpha^2 + 4\beta^2(3A + 2\overline{B} + 2i\beta C)\alpha - 8\beta^3(B + 2i\beta\overline{C})\right)}{8\left(\alpha^2 - 4\beta^2\right)^3} \\
\widetilde{C} &= -\frac{i(\alpha\overline{C} - 2\beta C)}{2\left(\alpha^2 - 4\beta^2\right)}.
\end{aligned}$$

We note that if $C = 0$ then $\widetilde{C} = 0$. Assuming $C = 0$, the equations for the dual hypersurface are as follows:

$$\begin{aligned}
\widetilde{A} &= -\frac{i\left(\overline{A}\alpha^3 - 2\beta\left(\overline{B}\alpha^2 - 2B\beta\alpha + 4A\beta^2\right)\right)}{8\left(\alpha^2 - 4\beta^2\right)^3} \\
\widetilde{B} &= \frac{i\left(\overline{B}\alpha^3 - 2(3\overline{A} + 2B)\beta\alpha^2 + 4(3A + 2\overline{B})\beta^2\alpha - 8B\beta^3\right)}{8\left(\alpha^2 - 4\beta^2\right)^3} \\
\widetilde{C} &= 0.
\end{aligned}$$

Suppose $\Delta = \frac{1}{4}$. Define the matrix M as folllows:

$$M = 4^3 i \begin{pmatrix} \beta^3 & -\frac{\alpha\beta^2}{2} & -\frac{\alpha^3}{8} & \frac{\alpha^2\beta}{4} \\ \frac{3\alpha\beta^2}{2} & \frac{-\alpha^2\beta}{2} - \beta^3 & -\frac{3\alpha^2\beta}{4} & \frac{\alpha^3}{8} + \alpha\beta^2 \\ \frac{\alpha^3}{8} & \frac{-\alpha^2\beta}{4} & -\beta^3 & \frac{\alpha\beta^2}{2} \\ \frac{3\alpha^2\beta}{4} & -\frac{\alpha^3}{8} - \alpha\beta^2 & -\frac{3\alpha\beta^2}{2} & \frac{\alpha^2\beta}{2} + \beta^3 \end{pmatrix}.$$

Then if $v = \begin{pmatrix} A \\ B \\ \overline{A} \\ \overline{B} \end{pmatrix}$ and $\widetilde{v} = \begin{pmatrix} \widetilde{A} \\ \widetilde{B} \\ \overline{\widetilde{A}} \\ \overline{\widetilde{B}} \end{pmatrix}$ one has that $\widetilde{v} = Mv$. And one can check that $M^2 = \mathrm{Id}$. If $A = x + iy, B = r + is$, then if

$$D = \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}$$

and if $u = \begin{pmatrix} x \\ y \\ r \\ s \end{pmatrix}$, then $v = Du$, and we look for eigenvectors of the matrix $D^{-1}MD$ with eigenvalue 1. We find that if $\lambda = \frac{\alpha^2+4\beta^2}{6\alpha\beta} + \frac{i}{6\alpha}$ and $\xi = -\frac{1}{3} - \frac{i}{12\beta}$, then if

$$A = \lambda B + \xi\overline{B}$$

or if

$$B = \frac{\overline{\lambda}}{|\lambda|^2 - |\xi|^2}A - \frac{\xi}{|\lambda|^2 - |\xi|^2}\overline{A}$$

v is an eigenvector for M. If this were the case, then the equations defining the dual hypersurface would be invariant up through weight 3. Note, that in order for this to make sense, we must assume that $\alpha\beta \neq 0$. Again, we shall make this assumption now, and we shall justify it later when we construct the normal form. We shall see that $\alpha \neq 0$ corresponds to strong pseudoconvexity, and $\beta = 0$ characterizes "spherical" points, which must be treated separately.

Now, we set about showing that one can in fact use projective transformations to obtain the desired normalizations for the hypersurface S. In an affine chart of $\mathbb{CP}^2$ projective transformations take the form

$$z \mapsto \frac{az + bw + c}{dz + ew + h}, w \mapsto \frac{kz + lw + m}{dz + ew + f}.$$

In particular, affine maps are projective transformations.

Suppose $S = \{(z,w) | \rho(z, \overline{z}, w, \overline{w}) = 0\}$. We wish to normalize S at the point (z_0, w_0). Using a translation, we may assume that the point in question is the origin. Also, we assume that $\rho_w(0) \neq 0$. Then after using a linear change of coordinates, one may invoke the Implicit Function Theorem to find a defining function for our hypersurface of the form:

$$\rho(z, \overline{z}, w, \overline{w}) = -v + F(z, \overline{z}, u) = -v + \alpha|z|^2 + \beta z^2 + \overline{\beta}\overline{z}^2 + \cdots .$$

To see how to do this, suppose we apply a transformation of the form

$$G(z,w) = (f(z,w), g(z,w)) = (z, aw + bz).$$

If the original defining function is $\rho^*(\zeta, \eta)$, then our new defining function is

$$\rho(z,w) = \rho^* \circ G(z,w).$$

We would like

$$\rho_z(0,0) = 0 \tag{4.1}$$

and

$$\rho_w(0,0) = \frac{1}{2i}. \tag{4.2}$$

But

$$\rho_z(0,0) = \rho^*_\zeta(0,0) + b\rho^*_\eta(0,0),$$

and

$$\rho_w(0,0) = a\rho^*_\eta(0,0).$$

By assumption, $\rho^*_\eta(0,0) \neq 0$, so taking

$$b = \frac{-\rho^*_\zeta(0,0)}{\rho^*_\eta(0,0)} \text{ and } a = \frac{1}{2i\rho^*_\eta(0,0)},$$

we obtain (4.1) and (4.2). Then by the Implicit Function Theorem, we can find a function $h > 0$ such that $h \cdot \rho$ has the desired form.

Now, having performed this first normalization, our defining function is of the form

$$\rho(z, \overline{z}, w, \overline{w}) = -v + \alpha|z|^2 + \beta z^2 + \overline{\beta}\overline{z}^2 + \cdots .$$

If, as before,

$$X = \rho_w \frac{\partial}{\partial z} - \rho_z \frac{\partial}{\partial w},$$

then one can compute that at the origin,

$$L(X, \overline{X}) = \alpha$$

and

$$Q(X, X) = 2\beta.$$

Therefore our assumption that S is strongly $\mathbb{C}$-linearly convex implies that

$$\alpha^2 - 4\beta^2 > 0. \tag{4.3}$$

It follows that $\alpha > 0$. Of course, it is entirely possible that $\beta = 0$, but this case exhibits special behavior. We shall now assume that $\beta \neq 0$, and we shall address the case $\beta = 0$ later.

Note that we have already normalized four of the group's complex parameters. Now, as we want our transformations to preserve this normalization, we will be considering projective transformations of the form $(z, w) \mapsto (z^*, w^*) = (f(z, w), g(z, w))$, where

$$f(z, w) = \frac{az + bw}{cz + dw + 1}, g(z, w) = \frac{w}{cz + dw + 1}.$$

Before going any further, we describe how such transformations change the equation defining S, following Moser's approach. Suppose one begins with a hypersurface given by

$$v^* = F^*(z^*, \overline{z^*}, u^*) = F_2^* + F_3^* + \cdots,$$

where F_i^* is a real-valued function of weight i, and $w^* = u^* + iv^*$. One wishes to change coordinates via the map $(z, w) \mapsto (z^*, w^*) = (f(z, w), g(z, w))$, so that the new equation is $v = F(z, \overline{z}, u) = F_2 + F_3 + \cdots$. We also decompose f and g into weighted homogeneous parts. In order for the map to preserve the normalizations we have already made, it must be the case that

$$f(z, w) = f_2(z, w) + \cdots, \text{ and } g(z, w) = w + g_3(z, w) + \cdots.$$

Then

$$\mathrm{Im}(w^*) = F^*(f, \overline{f}, u + \mathrm{Re}(g_3 + \cdots))$$

or

$$v + \mathrm{Im}(g_3) + \cdots = F^*(f, \overline{f}, u + \mathrm{Re}(g_3) + \cdots) \tag{4.4}$$

and finally

$$F = F^*(f, \overline{f}, u + \mathrm{Re}(g_3) + \cdots) - \mathrm{Im}(g_3) - \cdots. \tag{4.5}$$

However, we note that if our original defining function is a graph over $\mathrm{Im}(w^*)$, i.e.,

$$\rho^*(z^*, \overline{z}^*, w^*, \overline{w}^*) = -\mathrm{Im}(w^*) + F^*(z^*, \overline{z}^*, \mathrm{Re}(w^*)),$$

then it is not necessarily the case that $\rho(z, \overline{z}, w, \overline{w}) = \rho^*(f(z, w), g(z, w))$ is a graph,i.e., it may not be the case that

$$\rho(z, \overline{z}, w, \overline{w}) = -\mathrm{Im}(w) + F(z, \overline{z}, \mathrm{Re}(w)). \tag{4.6}$$

In fact, there is an essentially unique defining function for which this is the case. We know that any two defining functions for the same hypersurface S differ by a multiple of h, where $h : S \to \mathbb{R}$ is a positive function on S (assuming that we have fixed an orientation). Suppose we have two defining function for S of the form (4.6), $\rho = -\mathrm{Im}(w) + F(z, \overline{z}, \mathrm{Re}(w))$ and $r = -\mathrm{Im}(w) + H(z, \overline{z}, \mathrm{Re}(w))$, with $\rho = h \cdot r$. We have, letting $w = u + iv$,

$$-v + F(z, \overline{z}, u) - h(z, \overline{z}, w, \overline{w}) \cdot (-v + H(z, \overline{z}, u)) = 0. \tag{4.7}$$

Applying $\frac{\partial}{\partial v}$ to both sides of (4.7), we obtain:

$$-1 - \frac{\partial h}{\partial v}(z, \overline{z}, w, \overline{w}) \cdot (-v + H(z, \overline{z}, u) + h(z, \overline{z}, w, \overline{w})) = 0. \tag{4.8}$$

Since $-v + H(z,\overline{z},u) = 0$ on S, we have that (4.8) reduces to:

$$-1 + h(z,\overline{z},w,\overline{w}) = 0 \text{ on } S.$$

So $h \equiv 1$ on S, and

$$H(z,\overline{z},u) = F(z,\overline{z},u) \text{ on } S.$$

In order to find F after we apply a transformation of the form $(z,w) \mapsto (z^*,w^*) = (f(z,w),g(z,w))$, we use equation (4.4). Since, by assumption, there are no linear terms in u or v, we can use (4.4) to find the weight 2 terms of F, which we denote by F_2. Then we substitute

$$w = u + iF_2(z,\overline{z})$$

back into (4.4). We can then find the weight 3 terms of F, denote them by F_3. Then again we substitute

$$w = u + i\left(F_2(z,\overline{z}) + F_3(z,\overline{z},u)\right)$$

into (4.4) to obtain the weight 4 terms of F. This provides us with an algorithm for computing F. We obtain

$$F(z,\overline{z},u) = F_2(z,\overline{z}) + F_3(z,\overline{z},u) + F_4(z,\overline{z},u) + \cdots .$$

We now set out to find how projective transformations will modify our defining function. We begin by taking the power series expansion of our transformations:

$$\begin{aligned}
f(z,w) =& \frac{az+bw}{cz+dw+1}\\
=& az + (-acz^2 + bw) + (ac^2z^3 + (-ad-bc)zw)\\
& + (-ac^3z^4 + (2acd + bc^2)z^2w - bdw^2) + \cdots ,
\end{aligned}$$

and

$$\begin{aligned}
g(z,w) =& \frac{w}{cz+dw+1}\\
=& w + (-czw) + (c^2z^2w - dw^2) + (-c^3z^3w + 2cdzw^2) + \cdots .
\end{aligned}$$

If

$$F^*(z^*,\overline{z^*},u^*) = \alpha|z^*|^2 + \beta(z^*)^2 + \overline{\beta(z^*)^2} + \cdots ,$$

then when we change coordinates,

$$F(z,\overline{z},u) = \alpha|a|^2|z|^2 + a^2\beta z^2 + \overline{a^2\beta z^2} + \cdots .$$

If $\beta = \beta_0 e^{i\theta}$, where $\beta_0 \geq 0$, we let

$$a = (4(\alpha^2 - 4\beta_0^2))^{-\frac{1}{4}} e^{-i\frac{\theta}{2}},$$

then $a^2\beta$ is real and by (4.3) , $a \neq 0$. Then

$$(|a|^2\alpha)^2 - 4(a^2\beta)^2 = \frac{1}{4}.$$

Now, we may assume $\beta \geq 0$ and $\Delta = \alpha^2 - 4\beta^2 = \frac{1}{4}$, and in order to preserve this normalization, we must take $a = 1$.

Now we deal with the terms of weight 3. Suppose

$$F^*(z^*,\overline{z^*},u^*) = \alpha|z^*|^2 + \beta(z^*)^2 + \beta(\overline{z^*})^2 + A(z^*)^3 + B(z^*)^2\overline{z^*} + Cz^*u^* + \overline{A(z^*)^3 + B(z^*)^2\overline{z^*} + Cz^*u^*}$$

where $\Delta = \frac{1}{4}$. Then after we apply our transformation,

$$\begin{aligned} F(z, \overline{z}, u) = &\alpha|z|^2 + \beta z^2 + \beta\overline{z}^2 + (A - \frac{3}{2}\beta c - i\alpha\beta\overline{b} + 2i\beta^2 b)z^3 \\ &+ (B + 3i\alpha\beta b + \frac{1}{2}\beta\overline{c} - 2i\beta^2\overline{b} - \frac{1}{2}c\alpha - i\alpha^2\overline{b})z^2\overline{z} \\ &+ (C + 2\beta b - \frac{i}{2}c + \alpha\overline{b})zu + \text{Conjugate Terms} + \cdots . \end{aligned}$$

We wish to solve

$$C + 2\beta b - \frac{i}{2}c + \alpha\overline{b} = 0,$$

which we can do by choosing

$$c = -2iC - 4i\beta b - 2i\alpha\overline{b}.$$

So we may assume $C = 0$ and $c = -4i\beta b - 2i\alpha\overline{b}$. Also, let

$$\lambda = \frac{\alpha^2 + 4\beta^2}{6\alpha\beta} + \frac{i}{6\alpha},$$

and let

$$\xi = \frac{-1}{3} - \frac{i}{12\beta},$$

and finally, let

$$\eta = |\lambda|^2 - |\xi|^2.$$

Note that if $\alpha^2 - 4\beta^2 = \frac{1}{4}$, then provided $\beta \neq 0$, one has that $\eta \neq 0$. As noted above, if we have the relation

$$B = \frac{\overline{\lambda}}{\eta}A - \frac{\xi}{\eta}\overline{A}, \tag{4.9}$$

then A, B will be invariant when passing to the dual hypersurface. Let

$$p = 6i\alpha\beta\eta - 4i\alpha\beta\xi - 8i\beta^2\overline{\lambda},$$

and

$$q = -2i\alpha\beta\overline{\lambda} - 4i\beta^2\xi,$$

and finally let

$$r = -\eta B + \overline{\lambda}A.$$

By direct computation, if $\alpha^2 - 4\beta^2 = \frac{1}{4}$, then $|p|^2 - |q|^2 = \frac{5}{3}\beta^2 + \frac{1}{48} > 0$. Recall that after applying our transformation,

$$\begin{aligned} A &\mapsto A - \frac{3}{2}\beta c - i\alpha\beta\overline{b} + 2i\beta^2 b \\ B &\mapsto B + 3i\alpha\beta b + \frac{1}{2}\beta\overline{c} - 2i\beta^2\overline{b} - \frac{1}{2}c\alpha - i\alpha^2\overline{b}, \end{aligned}$$

where, by assumption

$$c = -4i\beta b - 2i\alpha\overline{b}.$$

Then finding a transformation so that our new A and B satisfy (4.9) reduces to solving $Mv = w$, where $v = \begin{pmatrix} b \\ \overline{b} \end{pmatrix}$, $w = \begin{pmatrix} r \\ \overline{r} \end{pmatrix}$, and $M = \begin{pmatrix} p & q \\ \overline{q} & \overline{p} \end{pmatrix}$. This is always possible since $|p|^2 - |q|^2 \neq 0$. So for all $\beta \neq 0$, this is possible. Therefore, we may assume that the equation defining S is invariant up through weight 3 under taking the dual hypersurface, and the only parameter in the projective transformation that is not determined by our normalization is d. So we return to our earlier computation of the coefficients in the defining function of the dual hypersurface.

We set the weight 3 terms of $s \circ G(z, w)$ equal to zero. If

$$F(z, \overline{z}, u) = F_2(z, \overline{z}, u) + F_3(z, \overline{z}, u) + F_4(z, \overline{z}, u) + \cdots,$$

where

$$F_4(z, \overline{z}, u) = Ez^4 + Nz^3\overline{z} + J|z|^4 + Kz^2u + L|z|^2u + Mu^2 + \overline{E}\overline{z}^4 + \overline{N}z\overline{z}^3 + \overline{K}\overline{z}^2u,$$

and if the weight 4 part of the dual hypersurface takes the form

$$\widetilde{E}z^4 + \widetilde{N}z^3\overline{z} + \widetilde{J}|z|^4 + \widetilde{K}z^2u + \widetilde{L}|z|^2u + \widetilde{M}u^2 + \overline{\widetilde{E}}\overline{z}^4 + \overline{\widetilde{N}}z\overline{z}^3 + \overline{\widetilde{K}}\overline{z}^2u,$$

then this gives us a system of equations which allows us to solve for $\widetilde{E}, \widetilde{N}, \widetilde{J}, \widetilde{K}, \widetilde{L}, \widetilde{M}$ in terms of $\alpha, \beta, A, B, E, N, J, K, L, M$. We have done this using Mathematica 7.0. However, since the output is rather lengthy, we have not included it. The computations are contained in the files DualWeight4.nb and DualHypersurfaceWeight42.nb, which are available at the website http://hdl.handle.net/2027.42/62185. It is done following the same method used to obtain the weight 3 coefficients, but it is lengthier. We find that if $w = \begin{pmatrix} \widetilde{E} \\ \widetilde{N} \\ \widetilde{J} \\ \widetilde{K} \\ \widetilde{L} \\ \widetilde{M} \\ \overline{\widetilde{E}} \\ \overline{\widetilde{N}} \\ \overline{\widetilde{K}} \end{pmatrix}$ and if $v = \begin{pmatrix} E \\ N \\ J \\ K \\ L \\ M \\ \overline{E} \\ \overline{N} \\ \overline{K} \end{pmatrix}$, then $w = \mathcal{M}(\alpha, \beta, A, B)v$, where $\mathcal{M}$ is a matrix whose entries are functions of α, β, A, B. Since $\mathcal{M}^2 = \mathrm{Id}$, the eigenvalues of $\mathcal{M}$ are $1, -1$. The dimension of the -1-eigenspace is 5 and the dimension of the $+1$-eigenspace is 4. With only 2 real degrees of freedom, we cannot in general guarantee that the the defining equation for our hypersurface is invariant up through weight 4. To be more precise, the transformation $\mathcal{M}$ is determined by the weight 3 terms of the defining function for our hypersurface, and so its $+1$- and -1-eigenspaces depend only on the weight 3 terms. The set of all weight 4 terms is unrestricted and is independent of $\mathcal{M}$, so it is in one-to-one correspondence with $\mathbb{R}^9$. The remaining freedom to normalize the weight 4 coefficients is provided by the two real parameters $\mathrm{Re}(d)$ and $\mathrm{Im}(d)$. These parameters act on the weight 4 coefficients by translation. Treating $\mathbb{R}^9$ as a vector space, and letting $s = \mathrm{Re}(d)$ and $t = \mathrm{Im}(d)$, we have that s and t act on $\mathbb{R}^9$ by $u \mapsto u + sv + tw$ for $u \in \mathbb{R}^9$ and $v, w \in \mathbb{R}^9$ whose components depend on the weight 2 and 3 coefficients. Two vectors are equivalent if they are in the same orbit of this action. Let V be the vector space generated by v and w. Then the space of weight 4 coefficients modulo equivalence can be identified with $\mathbb{R}^9/V$. The linear transformation $\mathcal{M}$ induces a linear transformation $\overline{\mathcal{M}}$ on $\mathbb{R}^9/V$ with the same eigenvalues. The dimension of the $+1$-eigenspace does not increase, so it is at most 4, but $\mathbb{R}^9/V$ is at least a 7-dimensional vector space, so $\overline{\mathcal{M}}$ is not the identity on $\mathbb{R}^9/V$. Therefore, it is not possible to choose a normal form so that $\mathcal{M}$ is the identity. However, one does have that $\widetilde{M} = -M$, so if one can normalize the original hypersurface so that $M = 0$, then this normalization will remain when one passes to the dual hypersurface. However, when one uses the projective transformation above to change coordinates, if the coefficient of u^2 is initially M, it becomes

$$M + \alpha|b|^2 + 2\mathrm{Re}(\beta b^2) + \mathrm{Im}(d).$$

So choosing

$$\mathrm{Im}(d) = -M - \alpha|b|^2 - 2\beta\mathrm{Re}(b^2),$$

we can guarantee that the coefficient of u^2 is 0.

At this point, our hypersurface is almost in normal form. The only additional freedom we have is $\mathrm{Re}(d)$. We can use this freedom to perform an additional normalization, but there does not seem to be a normalization that provides additional symmetry between the hypersurface and its dual.

As a result of the preceding, we have the following proposition.

Theorem IV.3. *Given a strongly $\mathbb{C}$-linearly convex real hypersurface $Z \subset \mathbb{CP}^2$ and a point $P \in Z$, there exist a projective transformation G, such that if $Y = G(Z)$, then Y and its dual Y^* have the same expansion through weight 3 at P. This will still be the case if we adjust Y by a transformation of the form*

$$(z, w) \mapsto \left(\frac{z}{d\,w+1}, \frac{w}{d\,w+1}\right)$$

for $d \in \mathbb{R}$. Generically, these are the only transformations with this property. For a generic hypersurface Z and $P \in Z$, it is not possible to find a projective transformation G, such that $G(Z)$ and its dual have the same expansion through weight 4.

4.2 Projective Normal Form and Homogeneity

In the previous section, we constructed a normal form for a real analytic, strongly $\mathbb{C}$-linearly convex hypersurface in $\mathbb{CP}^2$ with respect to the group of projective transformations, and the motivation for this normal form was to make a hypersurface Z agree with its dual to maximal order. In this section, we will construct an alternate normal form, but now our motivation will be to achieve the maximal order of contact between a hypersurface Z and a special osculating, projectively homogeneous hypersurface. Among other things, this allows us to characterize the strongly $\mathbb{C}$-linearly convex, real analytic projectively homogeneous hypersurfaces whose automorphism groups have dimension at least 4.

As our model hypersurfaces, we shall use the following family:

Definition IV.4. Given $\beta \geq 0$, define the hypersurface $M_\beta \subset \mathbb{C}^2$ to be the hypersurface with the defining function $\rho_\beta(z, \overline{z}, w, \overline{w}) = -\frac{w-\overline{w}}{2i} + |z|^2 + \beta z^2 + \beta \overline{z}^2$. We shall refer to these hypersurfaces as the Bolt hypersurfaces.

Michael Bolt introduces these hypersurfaces in [5]. Note that if $\beta = 0$, this is nothing but the hyperquadric (i.e., the Heisenberg group). In this paper, Bolt shows that for strongly $\mathbb{C}$-linearly convex hypersurface Z in $\mathbb{CP}^n$, the vanishing of the anti-Hermitian part of the second fundamental form implies that Z is projectively equivalent to the hyperquadric.

Since we intend to use these as our model projectively homogenous hypersurfaces, we must show that they are in fact homogeneous. For $0 \leq \beta < \frac{1}{2}$, we will characterize the automorphism group of M_β, which we shall denote by $\mathrm{Aut}(M_\beta)$. We shall also denote by $\mathrm{Aut}_0(M_\beta)$ the isotropy group of M_β at the origin.

One can directly compute that M_β is invariant under the transformations

$$\begin{aligned}
&(z, w) \mapsto \left(z + z_0, w + 2i\left(\overline{z}_0 + 2\beta z_0\right) z + i\left(|z_0|^2 + \beta z_0^2 + \beta \overline{z_0}^2\right)\right), \text{ for } z_0 \in \mathbb{C} \\
&(z, w) \mapsto (z, w + u_0), \text{ for } u_0 \in \mathbb{R} \\
&(z, w) \mapsto (\lambda z, \lambda^2 w), \text{ for } \lambda \in \mathbb{R}
\end{aligned} \tag{4.10}$$

and if $\beta = 0$, one has many additional transformations including:

$$(z, w) \mapsto (e^{i\theta} z, w), \text{ for } \theta \in \mathbb{R}.$$

We shall see later that for $\beta \neq 0$, these are all of the projective automorphisms of M_β, and so these hypersurfaces correspond to subgroups of the affine group, $A(2, \mathbb{C})$. The group $A(2, \mathbb{C})$ of affine transformations of $\mathbb{C}^2$ can be represented as a matrix group where

$$(z, w) \mapsto (az + bw + c, dz + ew + f)$$

corresponds to

$$\begin{pmatrix} z \\ w \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ w \\ 1 \end{pmatrix}.$$

Then if we let $G = \mathrm{Aut}(M_\beta)$ and $H = \mathrm{Aut}_0(M_\beta)$, for $\beta \neq 0$, we shall see that

$$G = \left\{ \begin{pmatrix} \lambda & 0 & z_0 \\ 2i(\overline{z}_0 + 2\beta z_0) & \lambda^2 & u_0 + i(|z_0|^2 + \beta z_0^2 + \beta \overline{z}_0^2) \\ 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{R}, z_0 \in \mathbb{C}, u_0 \in \mathbb{R} \right\} \tag{4.11}$$

and

$$H = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

with the corresponding Lie algebras

$$\mathfrak{g} = \left\{ \begin{pmatrix} t & 0 & \zeta \\ 4i\beta\zeta + 2i\overline{\zeta} & 2t & s \\ 0 & 0 & 0 \end{pmatrix} : s, t \in \mathbb{R}, \zeta \in \mathbb{C} \right\}$$

and

$$\mathfrak{h} = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & 2t & 0 \\ 0 & 0 & 0 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

So later we shall prove the following:

Proposition IV.5. *For each β, M_β is homogeneous. For $0 < \beta < \frac{1}{2}$, Aut(M_β) has dimension 4 and Aut$_0(M_\beta)$ consists of the non-isotropic dilations $(z, w) \mapsto (\lambda z, \lambda^2 w)$ for $\lambda \in \mathbb{R}$.*

We shall now try to construct a normal form for a real hypersurface Z that will make the defining function of Z agree with ρ_β for some β. As in the previous section, given $P \in Z$ we begin by translating so that P is the origin. We assume that Z is given by a defining function ρ, and without loss of generality $\rho_w(0,0) \neq 0$. Then we can apply a linear change of coordinates so that, after invoking the Implicit Function Theorem, we can assume that

$$\rho(z, \overline{z}, w, \overline{w}) = -v + \alpha|z|^2 + \beta z^2 + \overline{\beta}\overline{z}^2 + \cdots,$$

and now in order to preserve this form, we must restrict to transformations of the form $(z, w) \mapsto (z^*, w^*) = (f(z,w), g(z,w))$, where

$$f(z, w) = \frac{az + bw}{cz + dw + 1}, g(z, w) = \frac{w}{cz + dw + 1}. \tag{4.12}$$

We can take $a = \alpha^{-\frac{1}{2}} e^{-\frac{\arg(\beta)}{2}}$. Then $(z, w) \mapsto (az, w)$ will transform our hypersurface into one of the form

$$\rho(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + \beta z^2 + \beta \overline{z}^2 + Az^3 + Bz^2\overline{z} + Czu + \overline{A}\overline{z}^3 + \overline{B}z\overline{z}^2 + \overline{C}\overline{z}u + \cdots,$$

where

$$\beta \in \mathbb{R}. \tag{4.13}$$

Then in what follows, we shall assume that $a = 1$.

As in the previous section, a map of the form (4.12) will lead to a new defining function

$$\begin{aligned} \widetilde{\rho}(z, \overline{z}, w, \overline{w}) =& \rho(f(z,w), \overline{f(z,w)}, g(z,w), \overline{g(z,w)}) = -v + |z|^2 + \beta z^2 + \beta\overline{z}^2 + \widetilde{A}z^3 + \widetilde{B}z^2\overline{z} \\ &+ \widetilde{C}zu + \overline{\widetilde{A}}\overline{z}^3 + \overline{\widetilde{B}}z\overline{z}^2 + \overline{\widetilde{C}}\overline{z}u + \cdots, \end{aligned}$$

where

$$\widetilde{A} = A + 2ib\beta^2 - i\bar{b}\beta - \frac{3c\beta}{2}$$
$$\widetilde{B} = B - i\bar{b} + 3ib\beta - \frac{c}{2} - 2i\bar{b}\beta^2 + \frac{\beta\overline{c}}{2}, \text{ and}$$
$$\widetilde{C} = C + \bar{b} + 2b\beta - \frac{ic}{2}.$$

Assume now for the remainder of this section that $\beta \neq 0$. We solve the system $\widetilde{B} = 0, \widetilde{C} = 0$ for b and c, and we obtain:

$$b = -\frac{-iB + C + \beta\overline{C}}{6\beta},$$

and

$$c = -\frac{-2i\overline{C}\beta^2 - 2B\beta + 3iC\beta + \overline{B} - i\overline{C}}{3\beta}.$$

Now, we will assume that we have imposed these normalizations, so we have

$$B = 0 \tag{4.14}$$

and

$$C = 0, \tag{4.15}$$

and we require that we restrict to transformations of the form (4.12) for which $b = c = 0$. So our defining function is of the form:

$$\rho(z, \overline{z}, w, \overline{w}) = -v + |z|^2 + \beta z^2 + \beta\overline{z}^2 + Az^3 + \overline{A}\overline{z}^3 + Hz^4 + Jz^3\overline{z} + Kz^2\overline{z}^2 + Lz^2u + Mz\overline{z}u + Pu^2 + \overline{H}\overline{z}^4 + \overline{J}z\overline{z}^3 + \overline{L}\overline{z}^2u + \cdots,$$

Then after applying our transformation of the form (4.12) with $b = c = 0$, our new defining function will be

$$\widetilde{\rho}(z, \overline{z}, w, \overline{w}) = \rho(f(z,w), \overline{f(z,w)}, g(z,w), \overline{g(z,w)}) = -v + |z|^2 + \beta z^2 + \beta\overline{z}^2 + \widetilde{A}z^3 + \overline{\widetilde{A}}\overline{z}^3 + \widetilde{H}z^4 + \widetilde{J}z^3\overline{z} + \widetilde{K}z^2\overline{z}^2 + \widetilde{L}z^2u + \widetilde{M}z\overline{z}u + \widetilde{P}u^2 + \overline{\widetilde{H}}\overline{z}^4 + \overline{\widetilde{J}}z\overline{z}^3 + \overline{\widetilde{L}}\overline{z}^2u + \cdots,$$

where

$$\widetilde{H} = H - \frac{3}{2}id\beta^2 - \frac{1}{2}i\overline{d}\beta^2 \tag{4.16}$$
$$\widetilde{J} = J - 2i\beta d \tag{4.17}$$
$$\widetilde{K} = K - id\beta^2 + i\overline{d}\beta^2 - \frac{id}{2} + \frac{i\overline{d}}{2} \tag{4.18}$$
$$\widetilde{L} = L - \beta d + \beta\overline{d} \tag{4.19}$$
$$\widetilde{M} = M \tag{4.20}$$
$$\widetilde{P} = P - \frac{id}{2} + \frac{i\overline{d}}{2}. \tag{4.21}$$

(4.22)

From (4.16) above, we see that if we take

$$\text{Re}(d) = \frac{\text{Im}H}{2\beta^2} \tag{4.23}$$

$$\text{Im}(d) = -P, \tag{4.24}$$

then we can achieve

$$\text{Im}(\widetilde{H}) = 0$$
$$\widetilde{P} = 0.$$

So we can assume that we have imposed the normalizations:

$$\text{Im}(H) = 0 \tag{4.25}$$

and

$$P = 0. \tag{4.26}$$

Therefore, at points where $\beta \neq 0$, we can assume that our hypersurface is of the form

$$\begin{aligned}\rho(z, \overline{z}, w, \overline{w}) = &-v + |z|^2 + \beta z^2 + \beta \overline{z}^2 + Az^3 + \overline{A}\overline{z}^3 + Hz^4 \\ &+ Jz^3\overline{z} + Kz^2\overline{z}^2 + Lz^2u + Mz\overline{z}u + H\overline{z}^4 + \overline{J}z\overline{z}^3 + \overline{L}\overline{z}^2u + \cdots .\end{aligned} \tag{4.27}$$

Now, let us suppose that we start with a hypersurface of the form (4.27) for which $\beta \neq 0$. Let us examine the isotropy subgroup of such a hypersurface at the origin. We must restrict to transformations of the form $(z, w) \mapsto (f(z, w), g(z, w)$ where

$$f(z, w) = \frac{az + bw}{cz + dw + 1}, g(z, w) = \frac{ew}{cz + dw + 1}, \tag{4.28}$$

and e is real in order to preserve the first-order normalizations. Then after applying a transformation of the form (4.28), our defining function will take the form

$$\rho(f(z, w), \overline{f(z, w)}, g(z, w), \overline{g(z, w)}).$$

After we apply the procedure explained in the previous section, using (4.4) to write our new hypersurface as a graph, our defining function takes the form:

$$\begin{aligned}\widetilde{\rho}(z, \overline{z}, w, \overline{w}) = \rho(f(z, w), \overline{f(z, w)}, g(z, w), \overline{g(z, w)}) = -v + \widetilde{\alpha}|z|^2 + \widetilde{\beta}z^2 + \overline{\widetilde{\beta}}\overline{z}^2 + \widetilde{A}z^3 + \widetilde{B}z^2\overline{z} \\ + \widetilde{C}zu + \overline{\widetilde{A}}\overline{z}^3 + \overline{\widetilde{B}}z\overline{z}^2 + \overline{\widetilde{C}}\overline{z}u + \widetilde{H}z^4 + \widetilde{J}z^3\overline{z} \\ + \widetilde{K}z^2\overline{z}^2 + \widetilde{L}z^2u + \widetilde{M}z\overline{z}u + \widetilde{P}u^2 + \overline{\widetilde{H}}\overline{z}^4 + \overline{\widetilde{J}}z\overline{z}^3 + \overline{\widetilde{L}}\overline{z}^2u + \cdots .\end{aligned}$$

As before, we can compute the new coefficients. We find that

$$\widetilde{\alpha} = a\overline{a} - e + 1$$
$$\widetilde{\beta} = \beta a^2 + \beta - \beta e.$$

In order to preserve our hypersurface, we must have that $\widetilde{\alpha} = 1$ and $\widetilde{\beta} = \beta$. Solving these equations we find that $e = |a|^2$ and $a^2 = |a|^2$, so a must be real.

We then find that

$$\widetilde{B} = -2ia\overline{b}\beta^2 + 2iab\beta - ia\overline{b} + a(ib\beta - ac) + \frac{1}{2}i\left(-ica^2 - i\beta\overline{c}a^2\right)$$

$$\widetilde{C} = -\frac{1}{2}ica^2 + \overline{b}a + 2b\beta a.$$

We must have that $\widetilde{B} = 0$ and $\widetilde{C} = 0$. This gives us a homogeneous linear system of equations in $b, \bar{b}, c$, and $\bar{c}$. The corresponding matrix has determinant $9a^6\beta^2$. Since $a \neq 0$ and $\beta > 0$, the unique solution of this system is $b = 0$ and $c = 0$. We impose these conditions, and then we have that since $\mathrm{Im}(H) = 0$,

$$\widetilde{P} = a^2 \mathrm{Im}(d)$$
$$\mathrm{Im}(\widetilde{H}) = -2a^2\beta^2 \mathrm{Re}(d).$$

Therefore in order for $\widetilde{P} = 0$ and $\mathrm{Im}(\widetilde{H}) = 0$, we must have that $d = 0$. Therefore, the isotropy group of our hypersurface is a subgroup of the set of non-isotropic dilations

$$(z, w) \mapsto (\lambda z, \lambda^2 w),$$

where λ is real. Moreover, we note that unless all coefficients of weight greater than 2 vanish, there exists an integer m such that $\lambda^m = \lambda$. Since λ is a nonzero real number, λ is $+1$ or -1. If λ is $+1$, the transformation is the identity. Therefore, the isotropy group is trivial or consists of precisely two transformations unless our hypersurface is M_β. We have proved the following:

Theorem IV.6. *Given a strongly pseudo-convex real hypersurface $Z \subset \mathbb{C}^2$ and a point $(z_0, w_0) \in Z$, at which $Q(X, X) \neq 0$, there exists a projective transformation G taking (z_0, w_0) to the origin and, letting $Y = G(Z)$, such that Y is of the form (4.27). Then the isotropy group of Y, i.e., the group of projective transformations taking Y to itself and fixing the origin, is a subgroup of the non-isotropic dilations. Moreover, unless $Y = M_\beta$, for some β, the isotropy group is either trivial or it consists of the identity and the transformation $(z, w) \mapsto (-z, w)$.*

Remark IV.7. Our normal form may be thought of as a way of reducing the isotropy subgroup at each point on our hypersurface to a subgroup of the isotropy group of one of our "model" hypersurface M_β.

We are now in a position to prove Proposition IV.5.

Proof of Proposition IV.5. It is easy to check that the list of transformations (4.10) are indeed automorphisms of M_β, so that M_β is homogeneous. Since M_β is in normal form, by Theorem IV.6, the isotropy group of M_β at the origin is comprised solely of the non-istropic dilations. We know that the three-dimensional homogeneous manifold M_β is diffeomorphic to $\mathrm{Aut}(M_\beta)/\mathrm{Aut}_0(M_\beta)$, where $\mathrm{Aut}(M_\beta)$ is the group of projective transformations of M_β and $\mathrm{Aut}_0(M_\beta)$ is the isotropy subgroup at the origin. Since we know that $\mathrm{Aut}_0(M_\beta)$ is one dimensional, $\mathrm{Aut}(M_\beta)$ must be four dimensional, since M_β is three dimensional. Therefore, the four dimensional subgroup G of $\mathrm{Aut}(M_\beta)$ from (4.11) must be all of $\mathrm{Aut}(M_\beta)$. $\square$

We recall Theorem I. from [5], which we mentioned earlier.

Theorem IV.8 (Bolt)**.** *Given a three-times differentiable, strongly pseudoconvex real hypersurface $Z \subset \mathbb{C}^n$ such that $Q(X, X)$ is identically equal to 0, Z is the projective image of a hyperquadric.*

Using Theorem IV.8, we can prove the following:

Corollary IV.9. *The only strongly pseudoconvex real hypersurfaces in $\mathbb{C}^2$ which are projectively homogeneous and whose automorphism groups have dimension strictly larger than 3 are the hypersurfaces M_β.*

Proof. Suppose Z is a strongly pseudoconvex hypersurface in $\mathbb{C}^2$. Let $(z_0, w_0) \in Z$. Since $\beta = \frac{|Q|}{L}$ is constant on homogeneous hypersurfaces, if $\beta = 0$, then by Theorem IV.8, Z is the projective image of M_0. Suppose that $\beta \neq 0$. Then by Theorem IV.6, there exists a projective transformation G taking (z_0, w_0) to the origin such that the isotropy group of $G(Z)$ at the origin is a subset of the non-isotropic dilations. The dimension of this group will be larger than 3 precisely when $G(Z)$ is M_β. $\square$

There is one additional application of Theorem IV.6 which we would like to point out. It can be used to address questions of homogeneity and self-duality. For example consider the L^p-spheres.

Definition IV.10. We call the hypersurface

$$\mathbb{S}_p = \{(z,w) : |z|^p + |w|^p = 1\}$$

the L^p-sphere.

It is a standard fact that the dual of the L^p-sphere $\mathbb{S}_p$ is projectively equivalent to $\mathbb{S}_q$, where $\frac{1}{p} + \frac{1}{q} = 1$. We let $\rho(z,\overline{z},w,\overline{w}) = |z|^p + |w|^p - 1$ be the defining function for $\mathbb{S}_p$. At each point $(z_0,w_0) \in \mathbb{S}_p$, we can use projective transformations to put ρ in normal form, so that at (z_0,w_0), $\mathbb{S}_p$ is projectively equivalent to the hypersurface with defining function

$$\begin{aligned} &- v + |z|^2 + \beta(z_0,w_0)z^2 + \beta(z_0,w_0)\overline{z}^2 + A(z_0,w_0)z^3 + \overline{A}(z_0,w_0)\overline{z}^3 + H(z_0,w_0)z^4 \\ &+ J(z_0,w_0)z^3\overline{z} + K(z_0,w_0)z^2\overline{z}^2 + L(z_0,w_0)z^2u + M(z_0,w_0)z\overline{z}u \\ &+ H(z_0,w_0)\overline{z}^4 + \overline{J}(z_0,w_0)z\overline{z}^3 + \overline{L}(z_0,w_0)\overline{z}^2u + \cdots . \end{aligned} \tag{4.29}$$

The coefficients in (4.29) can be interpreted as numerical invariants at the point (z_0,w_0), or they may be thought of as invariant functions on the L^p-sphere. We can keep track of how the defining function changes in each step required to put it in normal form (as in Theorem IV.6), and we can explicitly compute the invariants as functions of (z_0,w_0). This requires some rather lengthy computation, but we have done this using Mathematica 7.0. These computations may be found in the files ExplicitMobInv1.nb, ExplicitMobInv2.nb, and ExplicitMobInv3.nb available online at http://hdl.handle.net/2027.42/62185. For $\mathbb{S}_p$, one finds that β is a constant

$$\beta \equiv \left|\frac{1}{2} - \frac{1}{p}\right|. \tag{4.30}$$

Likewise, A is a constant

$$A \equiv 0. \tag{4.31}$$

When computing the other invariants, it is convenient to note that through rotations, every point on $\mathbb{S}_p$ is projectively equivalent to a point of the form

$$\left(x, (1-x^p)^{\frac{1}{p}}\right) \text{ for } x \in [0,1],$$

so it suffices to compute the invariants at these points. Then we find that

$$H\left(x, (1-x^p)^{\frac{1}{p}}\right) = \frac{(p+2)(p-2)(p-3)}{18p^3} \cdot \frac{x^{2p} - x^p + 1}{x^p(1-x^p)}, \text{ for } 0 < x < 1. \tag{4.32}$$

Since

$$\frac{x^{2p} - x^p + 1}{x^p(1-x^p)} > 0, \text{ for } 0 < x < 1,$$

the sign of H is an invariant and is equal to the sign of

$$\frac{(p+2)(p-2)(p-3)}{18p^3}.$$

With only this information, we can draw some interesting conclusions about the L^p-spheres. Since all of the invariant functions in (4.29) must be constant on a homogeneous surface, and since H is not constant on $\mathbb{S}_p$, the L^p-spheres are not projectively homogeneous.

We also know that the dual of $\mathbb{S}_p$ is $\mathbb{B}_q$, where $q = \frac{p}{p-1}$. It immediately follows that the second order invariant β is the same for $\mathbb{S}_p$ and $\mathbb{S}_q$. Likewise, both have vanishing weight 3 invariant A. By Theorem IV.3, a hypersurface can always be made to agree with its dual up through weight 3, so we could have deduced this without actually computing these invariants. However, we notice that for $p \in (0, 3/5]$, the sign of H is positive, whereas the sign of H on the dual hypersurface is negative. Therefore $\mathbb{S}_p$ is not projectively equivalent to its dual hypersurface for $p \in (0, 3/5]$.

In particular, we can see that the L^p-spheres are not projectively homogeneous, and there are L^p-spheres which are not projectively equivalent to their duals.

Remark IV.11. The author has expanded the work done in this section. He has used Cartan's moving frame construction for the affine group to classify all of the affinely homogeneous, strongly $\mathbb{C}$-linearly convex hypersurfaces in $\mathbb{C}^2$ in terms of third-order differential invariants. He wishes to find more concrete representations of these hypersurfaces and to relate the invariants arising from this construction to the affine normal form. He has also used the moving frame construction for the special affine group to give an alternate proof of (a slightly weaker version of) Bolt's result, which roughly, states that if $\dfrac{Q(X,X)}{L(X,\overline{X})}$ is a nonzero constant on a strongly $\mathbb{C}$-linearly convex hypersurface, then the hypersurface is affinely equivalent to one of the Bolt hypersurfaces (c.f., Bolt [6]). In theory, this construction should allow one to determine generically the order of contact between two real hypersurfaces needed to guarantee that they are affinely equivalent. This work should appear in the forthcoming paper "The Affine Geometry of Hypersurfaces."

4.3 Transformations Preserving Q and the Generalized Schwarzian

In the previous section, we noted that the quantity

$$\left|\frac{Q(X,X)}{L(X,\overline{X})}\right|$$

is independent of the choice of defining function. In fact, Bolt observed that $Q(X,X)$ is itself a projective invariant. More specifically, in [5], he proved

Proposition IV.12 (Bolt)**.** *Let $Z \subset \mathbb{C}^n$ be a real hypersurface that is twice differentiable near $P \in Z$ and $w = F(z)$ is a biholomorphism in a neighborhood U of P. Let $\rho \in \mathcal{C}^2(U)$ be a defining function for Z near P. Then $Z' = F(Z \cap U)$ is twice differentiable and has defining function $\rho \circ F^{-1}$. If F is a projective transformation and s and t are in the complex tangent space of Z at P, then*

$$Q_{\rho,P}(s,t) = Q_{\rho\circ F^{-1},F(P)}(F'(P)s, F'(P)t). \tag{4.33}$$

We shall show that the projective transformations are in general the only transformations of $\mathbb{C}^2$ for which $Q(X,X)$ satisfies the transformation law (4.33).

Let $s = t = (-\rho_w, \rho_z), F = (f,g) = (f_1,f_2), (z_1,z_2) = (z,w)$, and let $\widetilde{\rho} = \rho \circ F^{-1}$. Also let $F(z_1,z_2) = (w_1,w_2) = (\zeta,\eta)$. Then

$$Q_{\rho,P}(s,t) = \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P) s_j t_k \tag{4.34}$$

$$= \left(\sum_{j,k,l,m} \frac{\partial^2 \widetilde{\rho}}{\partial w_l \partial w_k}(F(P)) \frac{\partial f_l}{\partial z_j}\frac{\partial f_m}{\partial z_k} s_j t_k\right) + \sum_{j,k,l} \frac{\partial \widetilde{\rho}}{\partial w_l}\frac{\partial^2 f_l}{\partial z_j \partial z_k} s_j t_k. \tag{4.35}$$

So from 4.34, we see that a transformation F preserves Q in he sense of (4.33) if and only if

$$\sum_{j,k,l} \frac{\partial \widetilde{\rho}}{\partial w_l}\frac{\partial^2 f_l}{\partial z_j \partial z_k} s_j t_k = 0. \tag{4.36}$$

This is equivalent to

$$\frac{\partial \widetilde{\rho}}{\partial \zeta} \circ F \left(\frac{\partial^2 f}{\partial z^2}\rho_w^2 - 2\frac{\partial^2 f}{\partial z \partial w}\rho_z \rho_w + \frac{\partial^2 f}{\partial w^2}\rho_z^2\right) + \frac{\partial \widetilde{\rho}}{\partial \eta} \circ F \left(\frac{\partial^2 g}{\partial z^2}\rho_w^2 - 2\frac{\partial^2 g}{\partial z \partial w}\rho_z \rho_w + \frac{\partial^2 g}{\partial w^2}\rho_z^2\right) = 0.$$

If we let $F^{-1}(\zeta,\eta)=(h,k)$, $J=f_z g_w - f_w g_z$, then

$$\begin{pmatrix} h_\zeta & h_\eta \\ k_\zeta & k_\eta \end{pmatrix} = \frac{1}{J}\begin{pmatrix} g_w & -f_w \\ -g_z & f_z \end{pmatrix}.$$

So

$$\begin{aligned}\widetilde{\rho}_\zeta &= \rho_z h_\zeta + \rho_w k_\zeta = \frac{1}{J}\left(\rho_z g_w - \rho_w g_z\right)\\ \widetilde{\rho}_\eta &= \rho_z h_\eta + \rho_w k_\eta = \frac{1}{J}\left(\rho_w f_z - \rho_z f_w\right).\end{aligned}$$

Hence F preserves Q if and only if

$$\begin{aligned}(4.37)\quad (f_{ww}g_w - g_{ww}f_w)\,\rho_z^3 &+ (g_{ww}f_z + 2g_{zw}f_w - 2f_{zw}g_w - f_{ww}g_z)\,\rho_z^2\rho_w \\ &+ (f_{zz}g_w + 2f_{zw}g_z - 2g_{zw}f_z - g_{zz}f_w)\,\rho_z\rho_w^2 + (g_{zz}f_z - f_{zz}g_z)\,\rho_w^3 = 0.\end{aligned}$$

This equation will hold for all defining functions ρ when the above coefficients vanish, i.e., when:

$$(4.38)\qquad \begin{aligned} f_{ww}g_w - g_{ww}f_w &= 0\\ g_{ww}f_z + 2g_{zw}f_w - 2f_{zw}g_w - f_{ww}g_z &= 0\\ f_{zz}g_w + 2f_{zw}g_z - 2g_{zw}f_z - g_{zz}f_w &= 0\\ g_{zz}f_z - f_{zz}g_z &= 0.\end{aligned}$$

We shall now provide a new proof of the following:

Lemma IV.13. *The locally invertible holomorphic mapping from $\mathbb{C}^2$ to $\mathbb{C}^2$ given by $(z,w)\mapsto (f(z,w),g(z,w))$ is a projective transformation if and only if it satisfies the system of equations* (4.38).

Remark IV.14. Lemma IV.13 is proved in Gunning [13], where he introduces the Generalized Schwarzian tensor. The system of PDE's (4.38) is easily seen to be equivalent to the vanishing of the components of the Generalized Schwarzian in $\mathbb{C}^2$.

The following theorem is an immediate consequence of (4.37) and Lemma IV.13.

Theorem IV.15. *The only transformations $F=(f,g)$ for which the transformation law* (4.33) *is satisfied for a generic hypersurface are projective transformations.*

Before proceeding, let us review the familiar Schwarzian from one complex variable. In $\mathbb{C}$, a holomorphic function f is a projective transformation when it is the quotient of two affine functions. A holomorphic function is affine when its second derivative vanishes identically. So f is a projective transformation when there exists a holomorphic function $h\neq 0$ with $h''=0$ and $(hf)''=0$. Hence

$$2h'f' + hf'' = 0.$$

Differentiating the above equation again,

$$3h'f'' + hf''' = 0,$$

so in matrix form

$$\begin{pmatrix} f''' & 3f'' \\ f'' & 2f' \end{pmatrix}\begin{pmatrix} h \\ h' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since h is non-zero, the matrix

$$(4.39)\qquad \begin{pmatrix} f''' & 3f'' \\ f'' & 2f' \end{pmatrix}$$

has a nontrivial null space, therefore its determinant is zero. In other words,

$$2f'''f' - 3(f'')^2 = 0.$$

However, the Schwarzian derivative of f is

$$S(f) = f''' - \frac{3}{2}\frac{(f'')^2}{f'},$$

so the above condition is equivalent to $S(f) = 0$. We see that having a vanishing Schwarzian is a necessary condition for a holomorphic function to be a projective transformation. Suppose, conversely, that $S(f) = 0$. Then given a point z_0, there is a non-zero element $\begin{pmatrix} s_0 \\ t_0 \end{pmatrix}$ of the null space of the matrix (4.39) at z_0. Unless f is an affine transformation, f'' vanishes on a discrete set, so we can work away from this set, and when we find f, by analytic continuation, our formula for f will extend to this set as well. Assume $f'' \neq 0$, then the null space of our matrix is one-dimensional and $t_0 \neq 0$. Near z_0 we can choose a unique element of the null space $\begin{pmatrix} s \\ t \end{pmatrix}$ by requiring that $t = t_0$. This is possible near z_0 since $t_0 \neq 0$ and the matrices are continuous in z. Then s is completely determined by $f''s + 2f't_0 = 0$ which implies that $s = -\frac{2f't_0}{f''}$. Note that

$$s' = \frac{(-2f''t_0)f'' + 2f'''f't_0}{(f'')^2} = \frac{2t_0}{(f'')^2}\left(f'''f' - (f'')^2\right) = \frac{2t_0(\frac{1}{2}(f'')^2}{(f'')^2} = t_0.$$

So $s' = t_0$ and $s'' = 0$. If we take $h = s$, then we see that f is a projective transformation.

We now generalize these ideas to prove Lemma IV.13. A holomorphic mapping (f, g) is a projective transformation if and only if there exists a holomorphic function $h \neq 0$ with $h_{zz} = h_{zw} = h_{ww} = 0$ such that

$$\begin{aligned} (hf)_{zz} = (hf)_{zw} = (hf)_{ww} = 0, \text{ and} \\ (hg)_{zz} = (hg)_{zw} = (hg)_{ww} = 0. \end{aligned}$$

Note that

$$\begin{aligned} (hf)_{zz} &= 2h_z f_z + hf_{zz} \\ (hg)_{zz} &= 2h_z g_z + hg_{zz} \\ (hf)_{ww} &= 2h_w f_w + hf_{ww} \\ (hg)_{ww} &= 2h_w g_w + hg_{ww} \\ (hf)_{zw} &= h_z f_w + h_w f_z + hf_{zw} \\ (hg)_{zw} &= h_z g_w + h_w g_z + hg_{zw}. \end{aligned}$$

So we have

$$\begin{pmatrix} f_{zz} & 2f_z \\ g_{zz} & 2g_z \end{pmatrix} \begin{pmatrix} h \\ h_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies that the determinant of the above matrix is zero, so

$$f_{zz}g_z - g_{zz}f_z = 0.$$

Likewise we see that

$$f_{ww}g_w - g_{ww}f_w = 0.$$

Similarly we find that

$$\begin{pmatrix} f_{zw} & f_w & f_z \\ g_{zw} & g_w & g_z \\ f_{ww} & 0 & 2f_w \end{pmatrix} \begin{pmatrix} h \\ h_z \\ h_w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore

$$\begin{vmatrix} f_{zw} & f_w & f_z \\ g_{zw} & g_w & g_z \\ f_{ww} & 0 & 2f_w \end{vmatrix} = 0.$$

In other words,

$$2f_{zw}f_wg_w - 2g_{zw}f_w^2 + f_{ww}g_zf_w - f_{ww}g_wf_z = 0.$$

Since

$$f_{ww}g_w = g_{ww}f_w,$$

the above equation reduces to

$$f_w\left(2f_{zw}g_w - 2g_{zw}f_w + f_{ww}g_z - g_{ww}f_z\right) = 0.$$

Assuming $f_w \neq 0$, we have

$$2f_{zw}g_w - 2g_{zw}f_w + f_{ww}g_z - g_{ww}f_z = 0.$$

If $f_w = 0$, then since our transformation is locally invertible, $g_w \neq 0$, and we can get the same equation replacing f_w with g_w. By the same reasoning, we arrive at

$$-2f_{zw}g_z + 2g_{zw}f_z + g_{zz}f_w - f_{zz}g_w = 0.$$

So we see that the equations (4.38) must be satisfied by a projective transformation.

Suppose, conversely, that $(z, w) \mapsto (f(z, w), g(z, w))$ is a local biholomorphism and that the equations (4.38) are satisfied. Consider the matrix $m : \mathbb{C}^3 \to \mathbb{C}^6$

$$m = \begin{pmatrix} f_{zw} & f_w & f_z \\ g_{zw} & g_w & g_z \\ f_{zz} & 2f_z & 0 \\ g_{zz} & 2g_z & 0 \\ f_{ww} & 0 & 2f_w \\ g_{ww} & 0 & 2g_w \end{pmatrix}.$$

Since the determinant $\begin{vmatrix} f_z & f_w \\ g_z & g_w \end{vmatrix} \neq 0$ the rank of m is at least 2. The equations (4.38) imply that the rank of m is exactly 2. So at each point, m has a one-dimensional null space. At the point (z_0, w_0), suppose that we have a nonzero element of the null space $\begin{pmatrix} r_0 \\ s_0 \\ t_0 \end{pmatrix}$, and assume that $t_0 \neq 0$. Then near (z_0, w_0), m will have a unique element of the null space $\begin{pmatrix} r \\ s \\ t \end{pmatrix}$ with $t = t_0$. Then r and s will be uniquely determined. So we have the relations

$$rf_{ww} + 2f_wt_0 = 0 \tag{4.40}$$

$$rg_{ww} + 2g_wt_0 = 0 \tag{4.41}$$

$$rf_{zz} + 2f_zs = 0 \tag{4.42}$$

$$rg_{zz} + 2g_zs = 0 \tag{4.43}$$

$$rf_{zw} + sf_w + t_0f_z = 0 \tag{4.44}$$

$$rg_{zw} + sg_w + t_0g_z = 0. \tag{4.45}$$

Taking $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ of equations (4.40)-(4.43), we can solve for all third order derivatives of f and g in terms of lower order derivatives. We find that

$$rf_{zww} = -r_z f_{ww} - 2f_{zw}t_0 \tag{4.46}$$
$$rf_{www} = -r_w f_{ww} - 2f_{ww}t_0 \tag{4.47}$$
$$rg_{zww} = -r_z g_{ww} - 2g_{zw}t_0 \tag{4.48}$$
$$rg_{www} = -r_w g_{ww} - 2g_{ww}t_0 \tag{4.49}$$
$$rf_{zzz} = -r_z f_{zz} - 2f_{zz}s - 2f_z s_z \tag{4.50}$$
$$rf_{zzw} = -r_w f_{zz} - 2f_{zw}s - 2f_z s_w \tag{4.51}$$
$$rg_{zzz} = -r_z g_{zz} - 2g_{zz}s - 2g_z s_z \tag{4.52}$$
$$rg_{zzw} = -r_w g_{zz} - 2g_{zw}s - 2g_z s_w. \tag{4.53}$$

Differentiating equations (4.44)-(4.45), we obtain

$$r_z f_{zw} + rf_{zzw} + s_z f_w + sf_{zw} + t_0 f_{zz} = 0 \tag{4.54}$$
$$r_w f_{zw} + rf_{zww} + s_w f_w + sf_{ww} + t_0 f_{zw} = 0 \tag{4.55}$$
$$r_z g_{zw} + rg_{zzw} + s_z g_w + sg_{zw} + t_0 g_{zz} = 0 \tag{4.56}$$
$$r_w g_{zw} + rg_{zww} + s_w g_w + sg_{ww} + t_0 g_{zw} = 0. \tag{4.57}$$

Eliminating all third order derivatives from equations (4.54) - (4.57), one obtains

$$r_z f_{zw} - r_w f_{zz} - 2f_{zw}s - 2f_z s_w + s_z f_w + sf_{zw} + t_0 f_{zz} = 0 \tag{4.58}$$
$$r_w f_{zw} - r_z f_{ww} - 2f_{zw}t_0 + s_w f_w + sf_{ww} + t_0 f_{zw} = 0 \tag{4.59}$$
$$r_z g_{zw} - r_w g_{zz} - 2g_{zw}s - 2g_z s_w + s_z g_w + sg_{zw} + t_0 g_{zz} = 0 \tag{4.60}$$
$$r_w g_{zw} - r_z g_{ww} - 2g_{zw}t_0 + s_w g_w + sg_{ww} + t_0 g_{zw} = 0. \tag{4.61}$$

If one eliminates s_z, s_w, r_w from the equations (4.58)-(4.61) and solves for r_z, one obtains $r_z = s$. Likewise, if one eliminates s_z, s_w, r_z and solves for r_w, one obtains that $r_w = t_0$. Therefore $r_{zw} = 0$ and so $s_w = r_{zw} = 0$. However, if one imposes the conditions that $s_w = 0, r_z = s, r_w = t_0$ on the equation in (4.60), one obtains that $s_z g_w = 0$. Provided that $g_w \neq 0$, this implies that $s_z = 0$. The equation (4.58) then reduces to $s_z f_w = 0$. If $g_w = 0$, then $f_w \neq 0$, so either way, $s_z = 0$. Letting $h = s$, we see that (f, g) is a projective transformation. This proves Lemma IV.13.

APPENDICES

APPENDIX A

Volume-Preserving Normal Form Computations

We wish to compute a normal form for our hypersurface with respect to volume-preserving biholomorphisms up through weight 4. Here we have included the Mathematica notebooks "VP-NormalForm1.nb" and "VPNormalForm2.nb." In the first notebook, our goal is to show that it is possible to make the desired normalizations, and in the second notebook, our aim is to show that no additional normalizations are possible. There is an additional notebook, "VPNormalForm3.nb" in which we obtain an explicit formula for the invariants. It is available for download on my website, but it is not included here.

In the following, note that an expression followed by a "b" denotes its complex conjugate. The "coefficient" function in Mathematica gives us the coefficient of a particular term. For instance "Coefficient[Coefficient[Coefficient[Fnew,z,1],zb,2],u,3]" yields the coefficient of $z\overline{z}^2u^3$ in the expression Fnew.

A.1 VPNormalForm1

$\mathbf{F2[z\_, zb\_, u\_]{:=}}z*\mathbf{zb}+B[2,0,0]*z\hat{}2+B[0,2,0]*\mathbf{zb}\hat{}2;$
$\mathbf{F3[z\_, zb\_, u\_]{:=}}B[3,0,0]*z\hat{}3+B[2,1,0]*z\hat{}2*\mathbf{zb}+B[1,0,1]*z*u+B[0,3,0]*\mathbf{zb}\hat{}3$
$+B[1,2,0]*z*\mathbf{zb}\hat{}2+B[0,1,1]*\mathbf{zb}*u;$
$\mathbf{F4[z\_, zb\_, u\_]{:=}}B[4,0,0]*z\hat{}4+B[3,1,0]*z\hat{}3*\mathbf{zb}+B[2,2,0]*z\hat{}2*\mathbf{zb}\hat{}2+B[2,0,1]*z\hat{}2*u$
$+B[1,1,1]*z*\mathbf{zb}*u+$
$B[0,0,2]*u\hat{}2+B[0,4,0]*\mathbf{zb}\hat{}4+B[1,3,0]*z*\mathbf{zb}\hat{}3+B[0,2,1]*\mathbf{zb}\hat{}2*u;$
$F[\mathbf{z\_, zb\_, u\_}]{:=}\mathbf{F2}[z,\mathbf{zb},u]+\mathbf{F3}[z,\mathbf{zb},u]+\mathbf{F4}[z,\mathbf{zb},u];$
$f[\mathbf{z\_, w\_}]{:=}\sum_{i=0}^{4}\sum_{j=0}^{4-i}a[i,j]*z\hat{}i*w\hat{}j;$
$\mathbf{fb[zb\_, wb\_]}{:=}\sum_{i=0}^{4}\sum_{j=0}^{4-i}\mathbf{ab}[i,j]*\mathbf{zb}\hat{}i*\mathbf{wb}\hat{}j;$
$g[\mathbf{z\_, w\_}]{:=}\sum_{i=0}^{4}\sum_{j=0}^{4-i}b[i,j]*z\hat{}i*w\hat{}j;$
$\mathbf{gb[zb\_, wb\_]}{:=}\sum_{i=0}^{4}\sum_{j=0}^{4-i}\mathbf{bb}[i,j]*\mathbf{zb}\hat{}i*\mathbf{wb}\hat{}j;$
$a[0,0]{:=}0;\mathbf{ab}[0,0]{:=}0;b[0,0]{:=}0;\mathbf{bb}[0,0]{:=}0;a[1,0]{:=}1;\mathbf{ab}[1,0]{:=}1;b[1,0]{:=}0;\mathbf{bb}[1,0]{:=}0;b[0,1]{:=}1;$
$\mathbf{bb}[0,1]{:=}1$
$\mathbf{Img[z\_, zb\_, w\_, wb\_]}{:=}(-I/2)*(g[z,w]-\mathbf{gb}[\mathbf{zb},\mathbf{wb}]);$
$\mathbf{Reg[z\_, zb\_, w\_, wb\_]}{:=}(1/2)*(g[z,w]+\mathbf{gb}[\mathbf{zb},\mathbf{wb}]);$
$\mathbf{Jac[n\_, m\_]}{:=}\sum_{i=0}^{n}\sum_{l=0}^{m}(i+1)*(l+1)*((a[i+1,m-l])*(b[n-i,l+1])$
$-(a[n-i,l+1])*(b[i+1,m-l]))$
$\mathbf{Jacb[n\_, m\_]}{:=}\sum_{i=0}^{n}\sum_{l=0}^{m}(i+1)*(l+1)*((\mathbf{ab}[i+1,m-l])*(\mathbf{bb}[n-i,l+1])$
$-(\mathbf{ab}[n-i,l+1])*(\mathbf{bb}[i+1,m-l]))$

$w{:=}u+I*z*\mathbf{zb};\mathbf{wb}{:=}u-I*z*\mathbf{zb};$

$\mathbf{Fnew}{:=}F[f[z,w],\mathbf{fb}[\mathbf{zb},\mathbf{wb}],\mathbf{Reg}[z,\mathbf{zb},w,\mathbf{wb}]]-\mathbf{Img}[z,\mathbf{zb},w,\mathbf{wb}]+(w-\mathbf{wb})/(2*I)$

$G[2,0,0]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 2], zb, 0], u, 0]
$G[1,1,0]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 1], zb, 1], u, 0]
$G[0,2,0]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 0], zb, 2], u, 0]
$G[3,0,0]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 3], zb, 0], u, 0]
$G[2,0,0]$
$\frac{1}{2}ib[2,0]+B[2,0,0]$
$B[2,0,0]$:=0; $B[0,2,0]$:=0;
$b[2,0]$:=0; bb[2,0]:=0;
$G[1,1,0]$
1
$G[3,0,0]$
$\frac{1}{2}ib[3,0]+B[3,0,0]$
$B[3,0,0]$:=0; $b[0,3,0]$:=0;
$b[3,0]$:=0; bb[3,0]:=0;
Jac[0, 0]
1
Jac[1, 0]
$2a[2,0]+b[1,1]$
Jac[0, 1]
$a[1,1]+2b[0,2]-a[0,1]b[1,1]$
$G[2,1,0]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 2], zb, 1], u, 0]
$G[1,0,1]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 1], zb, 0], u, 1]
Simplify[$G[2,1,0]$]
$a[2,0]-i\mathrm{ab}[0,1]-\frac{1}{2}b[1,1]+B[2,1,0]$
$G[1,0,1]$
$\mathrm{ab}[0,1]+\frac{1}{2}ib[1,1]+B[1,0,1]$
$B[1,0,1]$:=0
$a[2,0]$:=(−1/2) ∗ $b[1,1]$; ab[2,0]:=(−1/2) ∗ bb[1,1];
$a[0,1]$:=$\frac{1}{2}i$bb[1,1]; ab[0,1]:= $-\frac{1}{2}ib[1,1]$;
Simplify[$G[2,1,0]$]
$-\frac{3}{2}b[1,1]+B[2,1,0]$
$B[2,1,0]$:=0; $B[1,2,0]$:=0;
$b[1,1]$:=0; bb[1,1]:=0;
$G[4,0,0]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 4], zb, 0], u, 0]
$G[4,0,0]$
$\frac{1}{2}ib[4,0]+B[4,0,0]$
$B[4,0,0]$:=0; $B[0,4,0]$:=0;
$b[4,0]$:=0; $b[0,4]$:=0;
Simplify[Jac[0, 1]]
$a[1,1]+2b[0,2]$
Jac[2, 0]
$3a[3,0]+b[2,1]$
Jac[1, 1]
$2a[2,1]+2b[1,2]$
Jac[0, 2]
$a[1,2]+2a[1,1]b[0,2]+3b[0,3]$
$G[2,0,1]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 2], zb, 0], u, 1]
$G[1,1,1]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 1], zb, 1], u, 1]
$G[0,0,2]$:=Coefficient[Coefficient[Coefficient[Fnew, z, 0], zb, 0], u, 2]
$G[2,0,1]$
$\frac{1}{2}ib[2,1]+B[2,0,1]$
$G[1,1,1]$
$a[1,1]+\mathrm{ab}[1,1]-b[0,2]+B[1,1,1]-\mathrm{bb}[0,2]$
$G[0,0,2]$

$\frac{1}{2}ib[0,2] + B[0,0,2] - \frac{1}{2}i\mathrm{bb}[0,2]$
$\boldsymbol{b[2,1] \to 2iB[2,0,1]; \mathrm{bb}[2,1] \to -2iB[0,2,1];}$
$\boldsymbol{a[3,0]{:=}(-1/3) * b[2,1]; \mathrm{ab}[3,0]{:=}(-1/3) * \mathrm{bb}[2,1];}$
$\boldsymbol{a[1,1]{:=} -2b[0,2]; \mathrm{ab}[1,1]{:=} -2\mathrm{bb}[0,2];}$
Simplify[G[1, 1, 1]]
$-3b[0,2] + B[1,1,1] - 3\mathrm{bb}[0,2]$
Simplify[G[0, 0, 2]]
$\frac{1}{2}ib[0,2] + B[0,0,2] - \frac{1}{2}i\mathrm{bb}[0,2]$
Jac[1, 0]
0
Jac[0, 1]
0
Jac[2, 0]
0
Jac[1, 1]
$2a[2,1] + 2b[1,2]$
Jac[0, 2]
$a[1,2] - 4b[0,2]^2 + 3b[0,3]$

A.2 VPNormalForm2

$\boldsymbol{\mathrm{F2}[\mathrm{z\_}, \mathrm{zb\_}, \mathrm{u\_}]{:=}z * \mathrm{zb};}$
$\boldsymbol{\mathrm{F3}[\mathrm{z\_}, \mathrm{zb\_}, \mathrm{u\_}]{:=}0;}$
$\boldsymbol{\mathrm{F4}[\mathrm{z\_}, \mathrm{zb\_}, \mathrm{u\_}]{:=}\gamma * z\hat{}3 * \mathrm{zb} + \kappa * z\hat{}2 * \mathrm{zb}\hat{}2 + \gamma * z * \mathrm{zb}\hat{}3}$
$\boldsymbol{F[\mathrm{z\_}, \mathrm{zb\_}, \mathrm{u\_}]{:=}\mathrm{F2}[z, \mathrm{zb}, u] + \mathrm{F3}[z, \mathrm{zb}, u] + \mathrm{F4}[z, \mathrm{zb}, u];}$
$\boldsymbol{f[\mathrm{z\_}, \mathrm{w\_}]{:=} \sum_{i=0}^{4} \sum_{j=0}^{4-i} a[i,j] * z\hat{}i * w\hat{}j;}$
$\boldsymbol{\mathrm{fb}[\mathrm{zb\_}, \mathrm{wb\_}]{:=} \sum_{i=0}^{4} \sum_{j=0}^{4-i} \mathrm{ab}[i,j] * \mathrm{zb}\hat{}i * \mathrm{wb}\hat{}j;}$
$\boldsymbol{g[\mathrm{z\_}, \mathrm{w\_}]{:=} \sum_{i=0}^{4} \sum_{j=0}^{4-i} b[i,j] * z\hat{}i * w\hat{}j;}$
$\boldsymbol{\mathrm{gb}[\mathrm{zb\_}, \mathrm{wb\_}]{:=} \sum_{i=0}^{4} \sum_{j=0}^{4-i} \mathrm{bb}[i,j] * \mathrm{zb}\hat{}i * \mathrm{wb}\hat{}j;}$
$\boldsymbol{a[0,0]{:=}0; \mathrm{ab}[0,0]{:=}0; b[0,0]{:=}0; \mathrm{bb}[0,0]{:=}0; a[1,0]{:=}1; \mathrm{ab}[1,0]{:=}1; b[1,0]{:=}0;}$
$\boldsymbol{\mathrm{bb}[1,0]{:=}0; b[0,1]{:=}1;}$
$\boldsymbol{\mathrm{bb}[0,1]{:=}1}$
$\boldsymbol{\mathrm{Img}[\mathrm{z\_}, \mathrm{zb\_}, \mathrm{w\_}, \mathrm{wb\_}]{:=}(-I/2) * (g[z,w] - \mathrm{gb}[\mathrm{zb}, \mathrm{wb}]);}$
$\boldsymbol{\mathrm{Reg}[\mathrm{z\_}, \mathrm{zb\_}, \mathrm{w\_}, \mathrm{wb\_}]{:=}(1/2) * (g[z,w] + \mathrm{gb}[\mathrm{zb}, \mathrm{wb}]);}$
$\boldsymbol{\mathrm{Jac}[\mathrm{n\_}, \mathrm{m\_}]{:=} \sum_{i=0}^{n} \sum_{l=0}^{m} (i+1) * (l+1) * ((a[i+1, m-l]) * (b[n-i, l+1])}$
$\boldsymbol{-(a[n-i, l+1]) * (b[i+1, m-l]))}$
$\boldsymbol{\mathrm{Jacb}[\mathrm{n\_}, \mathrm{m\_}]{:=} \sum_{i=0}^{n} \sum_{l=0}^{m} (i+1) * (l+1) * ((\mathrm{ab}[i+1, m-l]) * (\mathrm{bb}[n-i, l+1])}$
$\boldsymbol{-(\mathrm{ab}[n-i, l+1]) * (\mathrm{bb}[i+1, m-l]))}$

$\boldsymbol{w{:=}u + I * z * \mathrm{zb}; \mathrm{wb}{:=}u - I * z * \mathrm{zb};}$

$\boldsymbol{\mathrm{Fnew}{:=}F[f[z,w], \mathrm{fb}[\mathrm{zb}, \mathrm{wb}], \mathrm{Reg}[z, \mathrm{zb}, w, \mathrm{wb}]] - \mathrm{Img}[z, \mathrm{zb}, w, \mathrm{wb}] + (w - \mathrm{wb})/(2 * I)}$
$\boldsymbol{G[2,0,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew}, z, 2], \mathrm{zb}, 0], u, 0]}$
$\boldsymbol{G[1,1,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew}, z, 1], \mathrm{zb}, 1], u, 0]}$
$\boldsymbol{G[0,2,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew}, z, 0], \mathrm{zb}, 2], u, 0]}$
$\boldsymbol{G[3,0,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew}, z, 3], \mathrm{zb}, 0], u, 0]}$
$\boldsymbol{G[2,0,0]}$
$\frac{1}{2}ib[2,0]$
$\boldsymbol{b[2,0]{:=}0; \mathrm{bb}[2,0]{:=}0;}$
$\boldsymbol{G[1,1,0]}$
0

$\boldsymbol{G[3,0,0]}$
$\frac{1}{2}ib[3,0]$
$\boldsymbol{b[3,0]{:=}0;\mathrm{bb}[3,0]{:=}0;}$
$\mathbf{Jac[0,0]}$
1
$\mathbf{Jac[1,0]}$
$2a[2,0]+b[1,1]$
$\mathbf{Jac[0,1]}$
$a[1,1]+2b[0,2]-a[0,1]b[1,1]$
$\boldsymbol{a[2,0]{:=}-\frac{1}{2}b[1,1];\mathrm{ab}[2,0]{:=}-\frac{1}{2}\mathrm{bb}[1,1];}$
$\boldsymbol{G[2,1,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,2],\mathrm{zb},1],u,0]}$
$\boldsymbol{G[1,0,1]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,1],\mathrm{zb},0],u,1]}$
$\boldsymbol{\mathrm{Simplify}[G[2,1,0]]}$
$-\frac{1}{2}b[1,1]$
$\boldsymbol{G[1,0,1]}$
$\frac{1}{2}ib[1,1]$
$\boldsymbol{b[1,1]{:=}0;\mathrm{bb}[1,1]{:=}0;}$
$\mathbf{Jac[0,1]}$
$a[1,1]+2b[0,2]$
$\boldsymbol{a[1,1]{:=}-2b[0,2];a[1,1]{:=}-2b[0,2];}$
$\boldsymbol{G[4,0,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,4],\mathrm{zb},0],u,0]}$
$\boldsymbol{G[4,0,0]}$
$\frac{1}{2}ib[4,0]$
$\boldsymbol{b[4,0]{:=}0;b[0,4]{:=}0;}$
$\mathbf{Simplify[Jac[1,0]]}$
0
$\mathbf{Simplify[Jac[0,1]]}$
0
$\mathbf{Jac[2,0]}$
$3a[3,0]+b[2,1]$
$\mathbf{Jac[1,1]}$
$2b[1,2]+2(a[2,1]-a[0,1]b[2,1])$
$\mathbf{Jac[0,2]}$
$a[1,2]-4b[0,2]^2+3b[0,3]-a[0,1]b[1,2]$
$\boldsymbol{G[2,0,1]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,2],\mathrm{zb},0],u,1]}$
$\boldsymbol{G[1,1,1]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,1],\mathrm{zb},1],u,1]}$
$\boldsymbol{G[0,0,2]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,0],\mathrm{zb},0],u,2]}$
$\boldsymbol{G[2,0,1]}$
$\frac{1}{2}ib[2,1]$
$\boldsymbol{b[2,1]{:=}0;\mathrm{bb}[2,1]{:=}0;}$
$\boldsymbol{G[1,1,1]}$
$-b[0,2]-\mathrm{bb}[0,2]$
$\boldsymbol{G[0,0,2]}$
$\frac{1}{2}ib[0,2]-\frac{1}{2}i\mathrm{bb}[0,2]$
$\boldsymbol{b[0,2]{:=}0;\mathrm{bb}[0,2]{:=}0}$
$\mathbf{Jac[2,0]}$
$3a[3,0]$
$\boldsymbol{a[3,0]{:=}0;\mathrm{ab}[3,0]{:=}0;}$
$\mathbf{Jac[1,1]}$
$2a[2,1]+2b[1,2]$
$\mathbf{Jac[0,2]}$
$a[1,2]+3b[0,3]-a[0,1]b[1,2]$
$\boldsymbol{G[2,2,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,2],\mathrm{zb},2],u,0]}$
$\boldsymbol{G[3,1,0]{:=}\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Coefficient}[\mathrm{Fnew},z,3],\mathrm{zb},1],u,0]}$

$\boldsymbol{G[2,2,0]}$
κ
$\boldsymbol{G[3,1,0]}$
γ

APPENDIX B

Projective Normal Form Computations

B.1 ProjNormForm1

Unprotect[C]

$\{C\}$

$\mathbf{rho[z\_,zb\_,w\_,wb\_]} := -(w-\mathrm{wb})/(2*I) + \mathrm{alpha}*z*\mathrm{zb} + \mathrm{beta}*z\textasciicircum 2 + \mathrm{beta}*\mathrm{zb}\textasciicircum 2 + A*z\textasciicircum 3$
$+B*z\textasciicircum 2*\mathrm{zb}$
$+C*z*((w+\mathrm{wb})/2) + \mathrm{Ab}*\mathrm{zb}\textasciicircum 3 + \mathrm{Bb}*\mathrm{zb}\textasciicircum 2*z + \mathrm{Cb}*\mathrm{zb}*((w+\mathrm{wb})/2);$

$h[\mathrm{z\_,zb\_,w\_,wb\_}] := \mathrm{rho}^{(1,0,0,0)}[z,\mathrm{zb},w,\mathrm{wb}]/\mathrm{rho}^{(0,0,1,0)}[z,\mathrm{zb},w,\mathrm{wb}]$

$\mathrm{hb}[\mathrm{z\_,zb\_,w\_,wb\_}] := \mathrm{rho}^{(0,1,0,0)}[z,\mathrm{zb},w,\mathrm{wb}]/\mathrm{rho}^{(0,0,0,1)}[z,\mathrm{zb},w,\mathrm{wb}]$

$\sum_{j=0}^{3}\sum_{k=0}^{3-j}\sum_{l=0}^{3-j-k}\sum_{m=0}^{3-j-k-l}\left(\left(h^{(j,k,l,m)}[0,0,0,0]/(j!*k!*l!*m!)\right)*z\textasciicircum j*\mathrm{zb}\textasciicircum k*w\textasciicircum l*\mathrm{wb}\textasciicircum m\right)$

$-iCw - iC\mathrm{wb} - 4i\mathrm{beta}z + C^2wz + C^2\mathrm{wb}z + \frac{1}{2}(-12iA+8\mathrm{beta}C)z^2 + iC^3wz^2 + iC^3\mathrm{wb}z^2 + \frac{1}{6}\left(36AC+24i\mathrm{beta}C^2\right)z^3 - 2i\mathrm{alphazb} + C\mathrm{Cb}w\mathrm{zb} + C\mathrm{Cbwbzb} + (-4iB+2\mathrm{alpha}C+4\mathrm{betaCb})z\mathrm{zb} + 2iC^2\mathrm{Cb}wz\mathrm{zb} + 2iC^2\mathrm{Cbwb}z\mathrm{zb} + \frac{1}{2}\left(8BC+4i\mathrm{alpha}C^2+12A\mathrm{Cb}+16i\mathrm{beta}C\mathrm{Cb}\right)z^2\mathrm{zb} + \frac{1}{2}(-4i\mathrm{Bb}+4\mathrm{alphaCb})\mathrm{zb}^2 + iC\mathrm{Cb}^2w\mathrm{zb}^2 + iC\mathrm{Cb}^2\mathrm{wbzb}^2 + \frac{1}{2}\left(4\mathrm{Bb}C+8B\mathrm{Cb}+8i\mathrm{alpha}C\mathrm{Cb}+8i\mathrm{betaCb}^2\right)z\mathrm{zb}^2 + \frac{1}{6}\left(12\mathrm{BbCb}+12i\mathrm{alphaCb}^2\right)\mathrm{zb}^3$

$h[\mathrm{z\_,zb\_,w\_,wb\_}] := -iCw - iC\mathrm{wb} - 4i\mathrm{beta}z + C^2wz + C^2\mathrm{wb}z + \frac{1}{2}(-12iA+8\mathrm{beta}C)z^2 + iC^3wz^2$
$+iC^3\mathrm{wb}z^2$
$+\frac{1}{6}\left(36AC+24i\mathrm{beta}C^2\right)z^3 - 2i\mathrm{alphazb}+$
$C\mathrm{Cb}w\mathrm{zb} + C\mathrm{Cbwbzb} + (-4iB+2\mathrm{alpha}C+4\mathrm{betaCb})z\mathrm{zb} + 2iC^2\mathrm{Cb}wz\mathrm{zb} + 2iC^2\mathrm{Cbwb}z\mathrm{zb}$
$+\frac{1}{2}\left(8BC+4i\mathrm{alpha}C^2+12A\mathrm{Cb}+16i\mathrm{beta}C\mathrm{Cb}\right)z^2\mathrm{zb}+$
$\frac{1}{2}(-4i\mathrm{Bb}+4\mathrm{alphaCb})\mathrm{zb}^2 + iC\mathrm{Cb}^2w\mathrm{zb}^2 + iC\mathrm{Cb}^2\mathrm{wbzb}^2$
$+\frac{1}{2}\left(8B\mathrm{Cb}+8i\mathrm{betaCb}^2-\frac{1}{2}C(-8\mathrm{Bb}-16i\mathrm{alphaCb})\right)z\mathrm{zb}^2$
$+\frac{1}{6}\left(12\mathrm{BbCb}+12i\mathrm{alphaCb}^2\right)\mathrm{zb}^3$

$\sum_{j=0}^{3}\sum_{k=0}^{3-j}\sum_{l=0}^{3-j-k}\sum_{m=0}^{3-j-k-l}$
$\left(\left(\mathrm{hb}^{(j,k,l,m)}[0,0,0,0]\middle/(j!*k!*l!*m!)\right)*z\textasciicircum j*\mathrm{zb}\textasciicircum k*w\textasciicircum l*\mathrm{wb}\textasciicircum m\right)$

$i\mathrm{Cb}w + i\mathrm{Cbwb} + 2i\mathrm{alpha}z + C\mathrm{Cb}wz + C\mathrm{Cbwb}z + \frac{1}{2}(4iB+4\mathrm{alpha}C)z^2 - iC^2\mathrm{Cb}wz^2 - iC^2\mathrm{Cbwb}z^2 + \frac{1}{6}\left(12BC-12i\mathrm{alpha}C^2\right)z^3 + 4i\mathrm{betazb} + \mathrm{Cb}^2w\mathrm{zb} + \mathrm{Cb}^2\mathrm{wbzb} + (4i\mathrm{Bb}+4\mathrm{beta}C+2\mathrm{alphaCb})z\mathrm{zb} - 2iC\mathrm{Cb}^2wz\mathrm{zb} - 2iC\mathrm{Cb}^2\mathrm{wb}z\mathrm{zb} + \frac{1}{2}\left(8\mathrm{Bb}C-8i\mathrm{beta}C^2+4B\mathrm{Cb}-8i\mathrm{alpha}C\mathrm{Cb}\right)z^2\mathrm{zb} + \frac{1}{2}(12i\mathrm{Ab}+8\mathrm{betaCb})\mathrm{zb}^2 - i\mathrm{Cb}^3w\mathrm{zb}^2 - i\mathrm{Cb}^3\mathrm{wbzb}^2 + \frac{1}{2}\left(12\mathrm{Ab}C+8\mathrm{BbCb}-16i\mathrm{beta}C\mathrm{Cb}-4i\mathrm{alphaCb}^2\right)z\mathrm{zb}^2 + \frac{1}{6}\left(36\mathrm{AbCb}-24i\mathrm{betaCb}^2\right)\mathrm{zb}^3$

$\mathrm{hb}[\mathrm{z\_,zb\_,w\_,wb\_}] := i\mathrm{Cb}w + i\mathrm{Cbwb} + 2i\mathrm{alpha}z + C\mathrm{Cb}wz + C\mathrm{Cbwb}z + \frac{1}{2}(4iB+4\mathrm{alpha}C)z^2$
$-iC^2\mathrm{Cb}wz^2$
$-iC^2\mathrm{Cbwb}z^2 + \frac{1}{6}\left(12BC-12i\mathrm{alpha}C^2\right)z^3 + 4i\mathrm{betazb}+$
$\mathrm{Cb}^2w\mathrm{zb} + \mathrm{Cb}^2\mathrm{wbzb} + (4i\mathrm{Bb}+4\mathrm{beta}C+2\mathrm{alphaCb})z\mathrm{zb} - 2iC\mathrm{Cb}^2wz\mathrm{zb} - 2iC\mathrm{Cb}^2\mathrm{wb}z\mathrm{zb}$
$+\frac{1}{2}\left(8\mathrm{Bb}C-8i\mathrm{beta}C^2+4B\mathrm{Cb}-8i\mathrm{alpha}C\mathrm{Cb}\right)z^2\mathrm{zb} + \frac{1}{2}(12i\mathrm{Ab}+8\mathrm{betaCb})\mathrm{zb}^2-$

$i\mathrm{Cb}^3 w\mathrm{zb}^2 - i\mathrm{Cb}^3\mathrm{wbzb}^2 + \frac{1}{2}\left(8\mathrm{BbCb} - 4i\mathrm{alphaCb}^2 - \frac{1}{2}C(-24\mathrm{Ab} + 32i\mathrm{betaCb})\right) z\mathrm{zb}^2$
$+\frac{1}{6}\left(36\mathrm{AbCb} - 24i\mathrm{betaCb}^2\right)\mathrm{zb}^3$

Coefficient[Coefficient[Coefficient[h[z, zb, $u + I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2), $u - I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2)], z, 1], zb, 0],u, 0]

$-4i\mathrm{beta}$

Coefficient[Coefficient[Coefficient[h[z, zb, $u + I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2), $u - I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2)], z, 0], zb, 1],u, 0]

$-2i\mathrm{alpha}$

Coefficient[Coefficient[Coefficient[h[z, zb, $u + I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2), $u - I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2)], z, 1], zb, 1],u, 0]

$-4iB + 2\mathrm{alpha}C + 4\mathrm{betaCb}$

Coefficient[Coefficient[Coefficient[h[z, zb, $u + I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2), $u - I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2)], z, 2], zb, 0],u, 0]

$-6iA + 4\mathrm{beta}C$

Coefficient[Coefficient[Coefficient[h[z, zb, $u + I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2), $u - I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2)], z, 0], zb, 2],u, 0]

$-2i\mathrm{Bb} + 2\mathrm{alphaCb}$

Coefficient[Coefficient[Coefficient[h[z, zb, $u + I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2), $u - I *$ (alpha $* z *$ zb + beta $* z$^2 + beta $*$ zb^2)], z, 0], zb, 0],u, 1]

$-2iC$

f[z_, zb_, w_, wb_]:= $-$ h[z, zb, w, wb];
fb[z_, zb_, w_, wb_]:= $-$ hb[z, zb, w, wb];
g[z_, zb_, w_, wb_]:= $- w - z *$ h[z, zb, w, wb];
gb[z_, zb_, w_, wb_]:= $-$ wb $-$ zb $*$ hb[z, zb, w, wb];
s[zeta_, zetab_, eta_, etab_]:=gamma $*$ zeta + delta $*$ zetab + As $*$ zeta $*$ zetab + Bs $*$ zeta^2
+Cs $*$ zetab^2
+Es $*$ ((eta + etab)/2);
sb[zeta_, zetab_, eta_, etab_]:=gammab $*$ zetab + deltab $*$ zeta + Asb $*$ zeta $*$ zetab
+Bsb $*$ zetab^2
+Csb $*$ zeta^2 + Esb $*$ ((eta + etab)/2);
Express:=s[f[z, zb, $u + I * v$, $u - I * v$], fb[z, zb, $u + I * v$, $u - I * v$], g[z, zb, $u + I * v$, $u - I * v$], gb[z, zb, $u + I * v$, $u - I * v$]]

Coefficient[Coefficient[Coefficient[Coefficient[Express, z, 0], zb, 0], u, 0], v, 0]

0

Coefficient[Coefficient[Coefficient[Coefficient[Express, z, 1], zb, 0], u, 0], v, 0]

$-2i\mathrm{alphadelta} + 4i\mathrm{betagamma}$

Coefficient[Coefficient[Coefficient[Coefficient[Express, z, 0], zb, 1], u, 0], v, 0]

$-4i\mathrm{betadelta} + 2i\mathrm{alphagamma}$

Solve [{$-2i$alphadelta + $4i$betagamma == 1, $-4i$betadelta + $2i$alphagamma == 0}, {gamma, delta}]

$\left\{\left\{\mathrm{gamma} \to -\frac{i\mathrm{beta}}{-\mathrm{alpha}^2+4\mathrm{beta}^2}, \mathrm{delta} \to \frac{i\mathrm{alpha}}{2(\mathrm{alpha}^2-4\mathrm{beta}^2)}\right\}\right\}$

r[zeta_, zetab_, eta_, etab_]:= $-$ ((eta $-$ etab)/(2 $* I$)) + alphap $*$ zeta $*$ zetab + betap $*$ zeta^2
+betap $*$ zetab^2
+Ap $*$ zeta^3 + Bp $*$ zeta^2 $*$ zetab+
Cp $*$ zeta $*$ ((eta + etab)/2) + Apb $*$ zetab^3 + Bpb $*$ zetab^2 $*$ zeta
+Cpb $*$ zetab $*$ ((eta + etab)/2);
psi[zeta_, zetab_, eta_, etab_]:=$r^{(1,0,0,0)}$[zeta, zetab, eta, etab]/$r^{(0,0,1,0)}$[zeta, zetab, eta, etab];
psib[zeta_, zetab_, eta_, etab_]:=$r^{(0,1,0,0)}$[zeta, zetab, eta, etab]/$r^{(0,0,0,1)}$[zeta, zetab, eta, etab];
$\sum_{j=0}^{3}\sum_{k=0}^{3-j}\sum_{l=0}^{3-j-k}\sum_{m=0}^{3-j-k-l}\left(\left(\mathrm{psi}^{(j,k,l,m)}[0,0,0,0]\middle/(j! * k! * l! * m!)\right) * \mathrm{zeta}\text{^}j * \mathrm{zetab}\text{^}k *\right.$ eta^l $*$ etab^m)

$-i\mathrm{Cpeta} - i\mathrm{Cpetab} - 4i\mathrm{betapzeta} + \mathrm{Cp}^2\mathrm{etazeta} + \mathrm{Cp}^2\mathrm{etabzeta} + \frac{1}{2}(-12i\mathrm{Ap} + 8\mathrm{betapCp})\mathrm{zeta}^2 + i\mathrm{Cp}^3\mathrm{etazeta}^2 + i\mathrm{Cp}^3\mathrm{etabzeta}^2 + \frac{1}{6}\left(36\mathrm{ApCp} + 24i\mathrm{betapCp}^2\right)\mathrm{zeta}^3 - 2i\mathrm{alphapzetab} + \mathrm{CpCpbetazetab} +$

$\text{CpCpbetabzetab} + (-4i\text{Bp} + 2\text{alphapCp} + 4\text{betapCpb})\text{zetazetab}$
$+ 2i\text{Cp}^2\text{Cpbetazetazetab} + 2i\text{Cp}^2\text{Cpbetabzetazetab}$
$+ \frac{1}{2}\left(8\text{BpCp} + 4i\text{alphapCp}^2 + 12\text{ApCpb} + 16i\text{betapCpCpb}\right)\text{zeta}^2\text{zetab}$
$+ \frac{1}{2}(-4i\text{Bpb} + 4\text{alphapCpb})\text{zetab}^2 + i\text{CpCpb}^2\text{etazetab}^2 + i\text{CpCpb}^2\text{etabzetab}^2$
$+ \frac{1}{2}\left(4\text{BpbCp} + 8\text{BpCpb} + 8i\text{alphapCpCpb} + 8i\text{betapCpb}^2\right)\text{zetazetab}^2$
$+ \frac{1}{6}\left(12\text{BpbCpb} + 12i\text{alphapCpb}^2\right)\text{zetab}^3$

$\mathbf{psi[zeta\_, zetab\_, eta\_, etab\_]{:=} - \mathit{i}Cpeta - \mathit{i}Cpetab - 4\mathit{i}betapzeta + Cp^2etazeta + Cp^2etabzeta}$
$\mathbf{+\frac{1}{2}(-12\mathit{i}Ap + 8betapCp)zeta^2 + \mathit{i}Cp^3etazeta^2 +}$
$\mathbf{\mathit{i}Cp^3etabzeta^2 + \frac{1}{6}\left(36ApCp + 24\mathit{i}betapCp^2\right)zeta^3 - 2\mathit{i}alphapzetab + CpCpbetazetab}$
$\mathbf{+CpCpbetabzetab}$
$\mathbf{+(-4\mathit{i}Bp + 2alphapCp + 4betapCpb)zetazetab +}$
$\mathbf{2\mathit{i}Cp^2Cpbetazetazetab + 2\mathit{i}Cp^2Cpbetabzetazetab}$
$\mathbf{+\frac{1}{2}\left(8BpCp + 4\mathit{i}alphapCp^2 + 12ApCpb + 16\mathit{i}betapCpCpb\right)zeta^2zetab}$
$\mathbf{+\frac{1}{2}(-4\mathit{i}Bpb + 4alphapCpb)zetab^2 +}$
$\mathbf{\mathit{i}CpCpb^2etazetab^2 + \mathit{i}CpCpb^2etabzetab^2}$
$\mathbf{+\frac{1}{2}\left(8BpCpb + 8\mathit{i}betapCpb^2 - \frac{1}{2}Cp(-8Bpb - 16\mathit{i}alphapCpb)\right)zetazetab^2}$
$\mathbf{+\frac{1}{6}\left(12BpbCpb + 12\mathit{i}alphapCpb^2\right)zetab^3}$

$\mathbf{\sum_{j=0}^{3}\sum_{k=0}^{3-j}\sum_{l=0}^{3-j-k}\sum_{m=0}^{3-j-k-l}\left(\left(psib^{(j,k,l,m)}[0,0,0,0]\middle/(j!*k!*l!*m!)\right)*zeta\hat{}j*zetab\hat{}k*\right.}$
$\mathbf{\left.eta\hat{}l*etab\hat{}m\right)}$

$i\text{Cpbeta}+i\text{Cpbetab}+2i\text{alphapzeta}+\text{CpCpbetazeta}+\text{CpCpbetabzeta}+\frac{1}{2}(4i\text{Bp}+4\text{alphapCp})\text{zeta}^2-$
$i\text{Cp}^2\text{Cpbetazeta}^2 - i\text{Cp}^2\text{Cpbetabzeta}^2 + \frac{1}{6}\left(12\text{BpCp} - 12i\text{alphapCp}^2\right)\text{zeta}^3 + 4i\text{betapzetab}$
$+ \text{Cpb}^2\text{etazetab} + \text{Cpb}^2\text{etabzetab}$
$+ (4i\text{Bpb} + 4\text{betapCp} + 2\text{alphapCpb})\text{zetazetab} - 2i\text{CpCpb}^2\text{etazetazetab}$
$- 2i\text{CpCpb}^2\text{etabzetazetab} + \frac{1}{2}\left(8\text{BpbCp} - 8i\text{betapCp}^2 + 4\text{BpCpb} - 8i\text{alphapCpCpb}\right)\text{zeta}^2\text{zetab}$
$+ \frac{1}{2}(12i\text{Apb} + 8\text{betapCpb})\text{zetab}^2 - i\text{Cpb}^3\text{etazetab}^2$
$- i\text{Cpb}^3\text{etabzetab}^2 + \frac{1}{2}\left(12\text{ApbCp} + 8\text{BpbCpb} - 16i\text{betapCpCpb} - 4i\text{alphapCpb}^2\right)\text{zetazetab}^2$
$+ \frac{1}{6}\left(36\text{ApbCpb} - 24i\text{betapCpb}^2\right)\text{zetab}^3$

$\mathbf{psib[zeta\_, zetab\_, eta\_, etab\_]{:=}\mathit{i}Cpbeta + \mathit{i}Cpbetab + 2\mathit{i}alphapzeta + CpCpbetazeta}$
$\mathbf{+CpCpbetabzeta}$
$\mathbf{+\frac{1}{2}(4\mathit{i}Bp + 4alphapCp)zeta^2 - \mathit{i}Cp^2Cpbetazeta^2 -}$
$\mathbf{\mathit{i}Cp^2Cpbetabzeta^2 + \frac{1}{6}\left(12BpCp - 12\mathit{i}alphapCp^2\right)zeta^3}$
$\mathbf{+4\mathit{i}betapzetab + Cpb^2etazetab + Cpb^2etabzetab + (4\mathit{i}Bpb + 4betapCp + 2alphapCpb)zetazetab -}$
$\mathbf{2\mathit{i}CpCpb^2etazetazetab - 2\mathit{i}CpCpb^2etabzetazetab}$
$\mathbf{+\frac{1}{2}\left(8BpbCp - 8\mathit{i}betapCp^2 + 4BpCpb - 8\mathit{i}alphapCpCpb\right)zeta^2zetab}$
$\mathbf{+\frac{1}{2}(12\mathit{i}Apb + 8betapCpb)zetab^2 -}$
$\mathbf{\mathit{i}Cpb^3etazetab^2 - \mathit{i}Cpb^3etabzetab^2}$
$\mathbf{+\frac{1}{2}\left(8BpbCpb - 4\mathit{i}alphapCpb^2 - \frac{1}{2}Cp(-24Apb + 32\mathit{i}betapCpb)\right)zetazetab^2}$
$\mathbf{+\frac{1}{6}\left(36ApbCpb - 24\mathit{i}betapCpb^2\right)zetab^3}$

Express2:= − psi[zeta, zetab, mu + *I* ∗ nu, mu − *I* ∗ nu]

Coefficient[Coefficient[Coefficient[Coefficient[Express2, zeta, 1], zetab, 0], mu, 0], nu, 0]

$4i\text{betap}$

Coefficient[Coefficient[Coefficient[Coefficient[Express2, zeta, 0], zetab, 1], mu, 0], nu, 0]

$2i\text{alphap}$

$\mathbf{gamma{:=} - \frac{\mathit{i}beta}{-alpha^2+4beta^2}; delta{:=}\frac{\mathit{i}alpha}{2(alpha^2-4beta^2)};}$

Solve[{4*i*betap == gamma, 2*i*alphap == delta}, {alphap, betap}]

$\left\{\left\{\text{betap} \to \frac{\text{beta}}{4(\text{alpha}^2-4\text{beta}^2)}, \text{alphap} \to \frac{\text{alpha}}{4(\text{alpha}^2-4\text{beta}^2)}\right\}\right\}$

$\mathbf{betap{:=}\frac{beta}{4(alpha^2-4beta^2)}; alphap{:=}\frac{alpha}{4(alpha^2-4beta^2)};}$

Coefficient[Coefficient[Coefficient[Coefficient[Express2, zeta, 1], zetab, 1], mu, 0], nu, 0]

$4i\text{Bp} - \frac{\text{alphaCp}}{2(\text{alpha}^2-4\text{beta}^2)} - \frac{\text{betaCpb}}{\text{alpha}^2-4\text{beta}^2}$

Coefficient[Coefficient[Coefficient[Coefficient[Express2, zeta, 2], zetab, 0], mu, 0], nu, 0]

$$6i\text{Ap}-\frac{\text{betaCp}}{\text{alpha}^2-4\text{beta}^2}$$

Coefficient[Coefficient[Coefficient[Coefficient[Express2, zeta, 0], zetab, 2], mu, 0], nu, 0]

$$2i\text{Bpb}-\frac{\text{alphaCpb}}{2(\text{alpha}^2-4\text{beta}^2)}$$

Coefficient[Coefficient[Coefficient[Coefficient[Express2, zeta, 0], zetab, 0], mu, 1], nu, 0]

$$2i\text{Cp}$$

$$\mathbf{Solve}\left[\left\{4i\text{Bp}-\frac{\text{alphaCp}}{2(\text{alpha}^2-4\text{beta}^2)}-\frac{\text{betaCpb}}{\text{alpha}^2-4\text{beta}^2}==\text{As},6i\text{Ap}-\frac{\text{betaCp}}{\text{alpha}^2-4\text{beta}^2}==\text{Bs},2i\text{Cp}==\text{Es},\right.\right.$$
$$\left.\left.-2i\text{Cpb}==\text{Esb}\right\},\{\text{Ap},\text{Bp},\text{Cp}\}\right]$$

$$\left\{\left\{\text{Cp}\to-\frac{i\text{Es}}{2},\text{Ap}\to\frac{1}{48}\left(-8i\text{Bs}-\frac{\text{Es}}{\text{alpha}-2\text{beta}}+\frac{\text{Es}}{\text{alpha}+2\text{beta}}\right),\right.\right.$$
$$\left.\left.\text{Bp}\to\frac{-4i\text{alpha}^2\text{As}+16i\text{Asbeta}^2-\text{alphaEs}+2\text{betaEsb}}{16(\text{alpha}-2\text{beta})(\text{alpha}+2\text{beta})}\right\}\right\}$$

$$\text{Cp}:=-\frac{i\text{Es}}{2};\text{Ap}:=\frac{1}{48}\left(-8i\text{Bs}-\frac{\text{Es}}{\text{alpha}-2\text{beta}}+\frac{\text{Es}}{\text{alpha}+2\text{beta}}\right);$$
$$\text{Bp}:=\frac{-4i\text{alpha}^2\text{As}+16i\text{Asbeta}^2-\text{alphaEs}+2\text{betaEsb}}{16(\text{alpha}-2\text{beta})(\text{alpha}+2\text{beta})};\text{Cpb}:=\frac{i\text{Esb}}{2};$$

Coefficient[Coefficient[Coefficient[Coefficient[Express, z, 1], zb, 1], u, 0], v, 0]

$$4\text{alpha}^2\text{As}+16\text{Asbeta}^2+\frac{2\text{alphaBb}}{\text{alpha}^2-4\text{beta}^2}+\frac{4B\text{beta}}{-\text{alpha}^2+4\text{beta}^2}-16\text{alphabetaBs}-\frac{2i\text{alphabeta}C}{\text{alpha}^2-4\text{beta}^2}$$
$$+\frac{2i\text{alphabeta}C}{-\text{alpha}^2+4\text{beta}^2}-\frac{i\text{alpha}^2\text{Cb}}{\text{alpha}^2-4\text{beta}^2}+\frac{4i\text{beta}^2\text{Cb}}{-\text{alpha}^2+4\text{beta}^2}-16\text{alphabetaCs}$$

Coefficient[Coefficient[Coefficient[Coefficient[Express, z, 2], zb, 0], u, 0], v, 0]

$$8\text{alphaAsbeta}+\frac{\text{alpha}B}{\text{alpha}^2-4\text{beta}^2}+\frac{6A\text{beta}}{-\text{alpha}^2+4\text{beta}^2}-16\text{beta}^2\text{Bs}-\frac{i\text{alpha}^2C}{\text{alpha}^2-4\text{beta}^2}+\frac{4i\text{beta}^2C}{-\text{alpha}^2+4\text{beta}^2}$$
$$-4\text{alpha}^2\text{Cs}+2i\text{betaEs}$$

Coefficient[Coefficient[Coefficient[Coefficient[Express, z, 0], zb, 2], u, 0], v, 0]

$$8\text{alphaAsbeta}+\frac{3\text{Abalpha}}{\text{alpha}^2-4\text{beta}^2}+\frac{2\text{Bbbeta}}{-\text{alpha}^2+4\text{beta}^2}-4\text{alpha}^2\text{Bs}-\frac{2i\text{alphabetaCb}}{\text{alpha}^2-4\text{beta}^2}+\frac{2i\text{alphabetaCb}}{-\text{alpha}^2+4\text{beta}^2}$$
$$-16\text{beta}^2\text{Cs}-2i\text{betaEs}$$

Coefficient[Coefficient[Coefficient[Coefficient[Express, z, 0], zb, 0], u, 1], v, 0]

$$\frac{2\text{beta}C}{-\text{alpha}^2+4\text{beta}^2}+\frac{\text{alphaCb}}{\text{alpha}^2-4\text{beta}^2}-\text{Es}$$

$$\mathbf{Solve}\left[\left\{4\text{alpha}^2\text{As}+16\text{Asbeta}^2+\frac{2\text{alphaBb}}{\text{alpha}^2-4\text{beta}^2}+\frac{4B\text{beta}}{-\text{alpha}^2+4\text{beta}^2}-16\text{alphabetaBs}-\right.\right.$$
$$\frac{2i\text{alphabeta}C}{\text{alpha}^2-4\text{beta}^2}+\frac{2i\text{alphabeta}C}{-\text{alpha}^2+4\text{beta}^2}-\frac{i\text{alpha}^2\text{Cb}}{\text{alpha}^2-4\text{beta}^2}$$
$$+\frac{4i\text{beta}^2\text{Cb}}{-\text{alpha}^2+4\text{beta}^2}-16\text{alphabetaCs}==0,$$
$$8\text{alphaAsbeta}+\frac{\text{alpha}B}{\text{alpha}^2-4\text{beta}^2}+\frac{6A\text{beta}}{-\text{alpha}^2+4\text{beta}^2}$$
$$-16\text{beta}^2\text{Bs}-\frac{i\text{alpha}^2C}{\text{alpha}^2-4\text{beta}^2}+\frac{4i\text{beta}^2C}{-\text{alpha}^2}+4\text{beta}^2-4\text{alpha}^2\text{Cs}+2i\text{betaEs}==0,$$
$$8\text{alphaAsbeta}+\frac{3\text{Abalpha}}{\text{alpha}^2-4\text{beta}^2}$$
$$+\frac{2\text{Bbbeta}}{-\text{alpha}^2+4\text{beta}^2}-4\text{alpha}^2\text{Bs}$$
$$-\frac{2i\text{alphabetaCb}}{\text{alpha}^2-4\text{beta}^2}+\frac{2i\text{alphabetaCb}}{-\text{alpha}^2+4\text{beta}^2}-16\text{beta}^2\text{Cs}-2i\text{betaEs}==0,$$
$$\left.\frac{2\text{beta}C}{-\text{alpha}^2+4\text{beta}^2}+\frac{\text{alphaCb}}{\text{alpha}^2-4\text{beta}^2}-\text{Es}==0\right\},$$
$$\left.\{\text{As},\text{Bs},\text{Cs},\text{Es}\}\right]$$

$$\left\{\left\{\text{Bs}\to-\frac{1}{4(\text{alpha}^2-4\text{beta}^2)^3}\left(-3\text{Abalpha}^3+6\text{alpha}^2\text{Bbbeta}\right.\right.\right.$$
$$-12\text{alpha}B\text{beta}^2+24A\text{beta}^3-8i\text{alpha}^2\text{beta}^2C+32i\text{beta}^4C$$
$$\left.+4i\text{alpha}^3\text{betaCb}-16i\text{alphabeta}^3\text{Cb}\right),\text{Cs}\to-\frac{1}{4(\text{alpha}^2-4\text{beta}^2)^3}\left(-\text{alpha}^3B+6A\text{alpha}^2\text{beta}\right.$$
$$+4\text{alpha}^2\text{Bbbeta}-12\text{Abalphabeta}^2-8\text{alpha}B\text{beta}^2+8\text{Bbbeta}^3+i\text{alpha}^4C$$
$$\left.-16i\text{beta}^4C-4i\text{alpha}^3\text{betaCb}+16i\text{alphabeta}^3\text{Cb}\right),\text{As}\to\frac{i}{4(\text{alpha}^2-4\text{beta}^2)^3}\left(2i\text{alpha}^3\text{Bb}\right.$$
$$-12i\text{Abalpha}^2\text{beta}-8i\text{alpha}^2B\text{beta}+24iA\text{alphabeta}^2+16i\text{alphaBbbeta}^2$$
$$\left.\left.\left.-16iB\text{beta}^3+\text{alpha}^4\text{Cb}-8\text{alpha}^2\text{beta}^2\text{Cb}+16\text{beta}^4\text{Cb}\right),\text{Es}\to-\frac{2\text{beta}C-\text{alphaCb}}{\text{alpha}^2-4\text{beta}^2}\right\}\right\}$$

$$\text{Bs}:=-\frac{1}{4(\text{alpha}^2-4\text{beta}^2)^3}\left(-3\text{Abalpha}^3+6\text{alpha}^2\text{Bbbeta}\right.$$
$$-12\text{alpha}B\text{beta}^2+24A\text{beta}^3-8i\text{alpha}^2\text{beta}^2C+32i\text{beta}^4C$$
$$\left.+4i\text{alpha}^3\text{betaCb}-16i\text{alphabeta}^3\text{Cb}\right);$$

$$\text{Cs:=}-\frac{1}{4(\text{alpha}^2-4\text{beta}^2)^3}\left(-\text{alpha}^3B+6A\text{alpha}^2\text{beta}\right.$$
$$+4\text{alpha}^2\text{Bbbeta}-12\text{Abalphabeta}^2-8\text{alpha}B\text{beta}^2$$
$$\left.+8\text{Bbbeta}^3+i\text{alpha}^4C-16i\text{beta}^4C-4i\text{alpha}^3\text{betaCb}+16i\text{alphabeta}^3\text{Cb}\right);$$
$$\text{As:=}\frac{i}{4(\text{alpha}^2-4\text{beta}^2)^3}\left(2i\text{alpha}^3\text{Bb}\right.$$
$$-12i\text{Abalpha}^2\text{beta}-8i\text{alpha}^2B\text{beta}+24iA\text{alphabeta}^2$$
$$\left.+16i\text{alphaBbbeta}^2-16iB\text{beta}^3+\text{alpha}^4\text{Cb}-8\text{alpha}^2\text{beta}^2\text{Cb}+16\text{beta}^4\text{Cb}\right);$$
$$\text{Es:=}-\frac{2\text{beta}C-\text{alphaCb}}{\text{alpha}^2-4\text{beta}^2};$$
$$\text{Esb:=}-\frac{2\text{betaCb}-\text{alpha}C}{\text{alpha}^2-4\text{beta}^2};$$
Simplify[Ap]
$$\frac{-i\text{Abalpha}^3+2\text{beta}\left(4i\text{beta}^2(A+2i\text{beta}C)+\text{alpha}^2(i\text{Bb}+2\text{beta}C)-\text{alpha}^3\text{Cb}+2\text{alphabeta}(-iB+2\text{betaCb})\right)}{8(\text{alpha}^2-4\text{beta}^2)^3}$$
Simplify[Bp]
$$\frac{i\left(\text{alpha}^3(\text{Bb}-2i\text{beta}C)+4\text{alphabeta}^2(3A+2\text{Bb}+2i\text{beta}C)+2\text{alpha}^2\text{beta}(-3\text{Ab}-2B+2i\text{betaCb})-8\text{beta}^3(B+2i\text{betaCb})\right)}{8(\text{alpha}^2-4\text{beta}^2)^3}$$
Simplify[Cp]
$$-\frac{i(-2\text{beta}C+\text{alphaCb})}{2(\text{alpha}^2-4\text{beta}^2)}$$
C:=0; Cb:=0;
Simplify[Ap, Assumptions → alpha^2 − 4 ∗ beta^2 == 1/4]
$$-8i\left(\text{Abalpha}^3-2\text{beta}\left(\text{alpha}^2\text{Bb}-2\text{alpha}B\text{beta}+4A\text{beta}^2\right)\right)$$
Simplify[Ap]
$$-\frac{i\left(\text{Abalpha}^3-2\text{beta}\left(\text{alpha}^2\text{Bb}-2\text{alpha}B\text{beta}+4A\text{beta}^2\right)\right)}{8(\text{alpha}^2-4\text{beta}^2)^3}$$
Simplify[Bp, Assumptions → alpha^2 − 4 ∗ beta^2 == 1/4]
$$2i\left(-2\text{beta}\left(3\text{Ab}+2B+48(\text{Ab}+B)\text{beta}^2\right)+\text{alpha}\left(\text{Bb}+48(A+\text{Bb})\text{beta}^2\right)\right)$$
Simplify[Bp]
$$\frac{i\left(\text{alpha}^3\text{Bb}-2\text{alpha}^2(3\text{Ab}+2B)\text{beta}+4\text{alpha}(3A+2\text{Bb})\text{beta}^2-8B\text{beta}^3\right)}{8(\text{alpha}^2-4\text{beta}^2)^3}$$
$$\text{Apb:=}8i\left(A\text{alpha}^3-2\text{beta}\left(\text{alpha}^2B-2\text{alphaBbbeta}+4\text{Abbeta}^2\right)\right);$$
$$\text{Bpb:=}-2i\left(-2\text{beta}\left(3A+2\text{Bb}+48(A+\text{Bb})\text{beta}^2\right)+\text{alpha}\left(B+48(\text{Ab}+B)\text{beta}^2\right)\right);$$
$$\text{Coefficient}\left[-\frac{i\left(\text{Abalpha}^3-2\text{beta}\left(\text{alpha}^2\text{Bb}-2\text{alpha}B\text{beta}+4A\text{beta}^2\right)\right)}{8(\text{alpha}^2-4\text{beta}^2)^3},\text{Bb}\right]$$
$$\frac{i\text{alpha}^2\text{beta}}{4(\text{alpha}^2-4\text{beta}^2)^3}$$
$$\text{Coefficient}\left[\frac{i\left(\text{alpha}^3\text{Bb}-2\text{alpha}^2(3\text{Ab}+2B)\text{beta}+4\text{alpha}(3A+2\text{Bb})\text{beta}^2-8B\text{beta}^3\right)}{8(\text{alpha}^2-4\text{beta}^2)^3},\text{Bb}\right]$$
$$\frac{i\left(\text{alpha}^3+8\text{alphabeta}^2\right)}{8(\text{alpha}^2-4\text{beta}^2)^3}$$
$$\text{Coefficient}\left[-\frac{-i\left(A\text{alpha}^3-2\text{beta}\left(\text{alpha}^2B-2\text{alphaBbbeta}+4\text{Abbeta}^2\right)\right)}{8(\text{alpha}^2-4\text{beta}^2)^3},\text{Bb}\right]$$
$$\frac{i\text{alphabeta}^2}{2(\text{alpha}^2-4\text{beta}^2)^3}$$
$$\text{Coefficient}\left[\frac{-i\left(\text{alpha}^3B-2\text{alpha}^2(3A+2\text{Bb})\text{beta}+4\text{alpha}(3\text{Ab}+2B)\text{beta}^2-8\text{Bbbeta}^3\right)}{8(\text{alpha}^2-4\text{beta}^2)^3},\text{Bb}\right]$$
$$-\frac{i\left(-4\text{alpha}^2\text{beta}-8\text{beta}^3\right)}{8(\text{alpha}^2-4\text{beta}^2)^3}$$
MyMat:=
$$\begin{pmatrix}
\frac{i\text{beta}^3}{(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{i\text{alphabeta}^2}{2(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{i\text{alpha}^3}{8(\text{alpha}^2-4\text{beta}^2)^3} & \frac{i\text{alpha}^2\text{beta}}{4(\text{alpha}^2-4\text{beta}^2)^3} \\
\frac{3i\text{alphabeta}^2}{2(\text{alpha}^2-4\text{beta}^2)^3} & \frac{i\left(-4\text{alpha}^2\text{beta}-8\text{beta}^3\right)}{8(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{3i\text{alpha}^2\text{beta}}{4(\text{alpha}^2-4\text{beta}^2)^3} & \frac{i\left(\text{alpha}^3+8\text{alphabeta}^2\right)}{8(\text{alpha}^2-4\text{beta}^2)^3} \\
\frac{i\text{alpha}^3}{8(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{i\text{alpha}^2\text{beta}}{4(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{i\text{beta}^3}{(\text{alpha}^2-4\text{beta}^2)^3} & \frac{i\text{alphabeta}^2}{2(\text{alpha}^2-4\text{beta}^2)^3} \\
\frac{3i\text{alpha}^2\text{beta}}{4(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{i\left(\text{alpha}^3+8\text{alphabeta}^2\right)}{8(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{3i\text{alphabeta}^2}{2(\text{alpha}^2-4\text{beta}^2)^3} & -\frac{i\left(-4\text{alpha}^2\text{beta}-8\text{beta}^3\right)}{8(\text{alpha}^2-4\text{beta}^2)^3}
\end{pmatrix}$$
MatrixForm[Simplify[MyMat.MyMat, Assumptions → alpha^2 − 4 ∗ beta^2 == 1/4]]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \textbf{Lambda:=} \begin{pmatrix} 1 & I & 0 & 0 \\ 0 & 0 & 1 & I \\ 1 & -I & 0 & 0 \\ 0 & 0 & 1 & -I \end{pmatrix};$$

LambdaIn:=Inverse[Lambda];
Simplify[Eigenvalues[LambdaIn.MyMat.Lambda], Assumptions → alpha^2 − 4 ∗ beta^2 == 1/4]

$\{-1, -1, 1, 1\}$

Simplify[Eigenvectors[LambdaIn.MyMat.Lambda], Assumptions->alpha^2 − 4 ∗ beta^2 == 1/4]

$$\left\{\left\{\frac{1}{12}\left(\frac{2}{\text{alpha}}+\frac{1}{\text{beta}}\right), \frac{1}{6}\left(2+\frac{\text{alpha}}{\text{beta}}+\frac{4\text{beta}}{\text{alpha}}\right), 0, 1\right\},\right.$$
$$\left\{\frac{1}{6}\left(-2+\frac{\text{alpha}}{\text{beta}}+\frac{4\text{beta}}{\text{alpha}}\right), \frac{1}{12}\left(-\frac{2}{\text{alpha}}+\frac{1}{\text{beta}}\right), 1, 0\right\},$$
$$\left.\left\{-\frac{\text{alpha}+2\text{beta}}{12\text{alphabeta}}, \frac{1}{6}\left(2+\frac{\text{alpha}}{\text{beta}}+\frac{4\text{beta}}{\text{alpha}}\right), 0, 1\right\}, \left\{\frac{1}{6}\left(-2+\frac{\text{alpha}}{\text{beta}}+\frac{4\text{beta}}{\text{alpha}}\right), \frac{1}{12}\left(\frac{2}{\text{alpha}}-\frac{1}{\text{beta}}\right), 1, 0\right\}\right\}$$

$$\textbf{v1:=}\begin{pmatrix} -\frac{\text{alpha}+2\text{beta}}{12\text{alphabeta}} \\ \frac{1}{6}\left(2+\frac{\text{alpha}}{\text{beta}}+\frac{4\text{beta}}{\text{alpha}}\right) \\ 0 \\ 1 \end{pmatrix}; \textbf{v2:=}\begin{pmatrix} \frac{1}{6}\left(-2+\frac{\text{alpha}}{\text{beta}}+\frac{4\text{beta}}{\text{alpha}}\right) \\ \frac{1}{12}\left(\frac{2}{\text{alpha}}-\frac{1}{\text{beta}}\right) \\ 1 \\ 0 \end{pmatrix};$$

w1:=Lambda.v1;
w2:=Lambda.v2;
MatrixForm[Simplify[MyMat.w1 − w1, Assumptions->alpha^2 − 4 ∗ beta^2 == 1/4]]

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

MatrixForm[Simplify[MyMat.w2 − w2, Assumptions->alpha^2 − 4 ∗ beta^2 == 1/4]]

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

MatrixForm[Simplify[w1]]

$$\begin{pmatrix} \frac{i\left(2\text{alpha}^2+\text{alpha}(i+4\text{beta})+2\text{beta}(i+4\text{beta})\right)}{12\text{alphabeta}} \\ i \\ -\frac{\text{alpha}+2i\text{alpha}^2+2\text{beta}+4i\text{alphabeta}+8i\text{beta}^2}{12\text{alphabeta}} \\ -i \end{pmatrix}$$

MatrixForm[Simplify[w2]]

$$\begin{pmatrix} \frac{2\text{alpha}^2-\text{alpha}(i+4\text{beta})+2\text{beta}(i+4\text{beta})}{12\text{alphabeta}} \\ 1 \\ \frac{i\text{alpha}+2\text{alpha}^2-2i\text{beta}-4\text{alphabeta}+8\text{beta}^2}{12\text{alphabeta}} \\ 1 \end{pmatrix}$$

B.2 ProjNormForm2

Unprotect[C]

$\{C\}$

ftemp[z_, w_]:=$(z + b * w)/(c * z + d * w + 1)$; gtemp[z_, w_]:=$w/(c * z + d * w + 1)$;
fbtemp[zb_, wb_]:=(zb + bb ∗ wb)/(cb ∗ zb + db ∗ wb + 1);
gbtemp[zb_, wb_]:=wb/(cb ∗ zb + db ∗ wb + 1);

$$f[\text{z\_}, \text{w\_}] := \sum_{k=0}^{3}\sum_{j=0}^{3-k}\left(\left(\text{ftemp}^{(j,k)}[0,0]\right)\Big/(j! * k!)\right) * z\hat{\ }j * w\hat{\ }k$$

$$\text{fb}[\text{zb\_}, \text{wb\_}] := \sum_{k=0}^{3}\sum_{j=0}^{3-k}\left(\left(\text{fbtemp}^{(j,k)}[0,0]\right)\Big/(j! * k!)\right) * \text{zb}\hat{\ }j * \text{wb}\hat{\ }k$$

$$g[\text{z\_}, \text{w\_}] := \sum_{k=0}^{3}\sum_{j=0}^{3-k}\left(\left(\text{gtemp}^{(j,k)}[0,0]\right)/(j! * k!)\right) * z\hat{\ }j * w\hat{\ }k$$

$\mathbf{gb[zb\_, wb\_]{:=}\sum_{k=0}^{3}\sum_{j=0}^{3-k}\left(\left(gbtemp^{(j,k)}[0,0]\right)\big/(j!*k!)\right)*zb\char`^j*wb\char`^k}$

$\boldsymbol{f[z,w]}$

$bw - bdw^2 + bd^2w^3 + z + (-bc-d)wz + \frac{1}{2}\left(4bcd+2d^2\right)w^2z - cz^2 + \frac{1}{2}\left(2bc^2+4cd\right)wz^2 + c^2z^3$

$\boldsymbol{g[z,w]}$

$w - dw^2 + d^2w^3 - cwz + 2cdw^2z + c^2wz^2$

F[z_, zb_, u_]:=alpha * z * zb + beta * z^2 + beta * zb^2 + A * z^3 + Ab * zb^3
+B * z^2 * zb + Bb * z * zb^2 + C * z * u + Cb * zb * u;
Express:=F[f[z, w], fb[zb, wb], (g[z, w] + gb[zb, wb])/2]
−((g[z, w] − gb[zb, wb])/(2 * I)) + ((w − wb)/(2 * I))
w:=u + I * (alpha * z * zb + beta * z^2 + beta * zb^2);
wb:=u − I * (alpha * z * zb + beta * z^2 + beta * zb^2)
Coefficient[Coefficient[Coefficient[Express, z, 3], zb, 0], u, 0]

$A - i\text{alphabbbeta} + 2ib\text{beta}^2 - \frac{3\text{betac}}{2}$

Coefficient[Coefficient[Coefficient[Express, z, 2], zb, 1], u, 0]

$B - i\text{alpha}^2\text{bb} + 3i\text{alpha}b\text{beta} - 2i\text{bbbeta}^2 - \frac{\text{alphac}}{2} + \frac{\text{betacb}}{2}$

Coefficient[Coefficient[Coefficient[Express, z, 1], zb, 0], u, 1]

$\text{alphabb} + 2b\text{beta} - \frac{ic}{2} + C$

Coefficient[Coefficient[Coefficient[Express, z, 0], zb, 3], u, 0]

$\text{Ab} + i\text{alpha}b\text{beta} - 2i\text{bbbeta}^2 - \frac{3\text{betacb}}{2}$

Coefficient[Coefficient[Coefficient[Express, z, 1], zb, 2], u, 0]

$i\text{alpha}^2b + \text{Bb} - 3i\text{alphabbbeta} + 2ib\text{beta}^2 + \frac{\text{betac}}{2} - \frac{\text{alphacb}}{2}$

Coefficient[Coefficient[Coefficient[Express, z, 0], zb, 1], u, 1]

$\text{alpha}b + 2\text{bbbeta} + \frac{i\text{cb}}{2} + \text{Cb}$

$\mathbf{Solve\left[\left\{alphabb + 2\boldsymbol{b}beta - \frac{\boldsymbol{ic}}{2} + \boldsymbol{C}{==}0, alpha\boldsymbol{b} + 2bbbeta + \frac{\boldsymbol{i}cb}{2} + Cb == 0\right\}, \{\boldsymbol{c}, cb\}\right]}$

$\{\{c \to -2i(\text{alphabb} + 2b\text{beta} + C), \text{cb} \to 2i(\text{alpha}b + 2\text{bbbeta} + \text{Cb})\}\}$

c:= − 2i(alphabb + 2bbeta + C); cb:=2i(alphab + 2bbbeta + Cb)
C:=0; Cb:=0;
Coefficient[Coefficient[Coefficient[Express, z, 3], zb, 0], u, 0]

$A + 2i\text{alphabbbeta} + 8ib\text{beta}^2$

Coefficient[Coefficient[Coefficient[Express, z, 2], zb, 1], u, 0]

$B + 6i\text{alpha}b\text{beta}$

Coefficient[Coefficient[Coefficient[Express, z, 0], zb, 3], u, 0]

$\text{Ab} - 2i\text{alpha}b\text{beta} - 8i\text{bbbeta}^2$

Coefficient[Coefficient[Coefficient[Express, z, 1], zb, 2], u, 0]

$\text{Bb} - 6i\text{alphabbbeta}$

Solve[{B + 6ialphabbeta == 0, Bb − 6ialphabbbeta == 0}, {b, bb}]

$\left\{\left\{b \to \frac{iB}{6\text{alphabeta}}, \text{bb} \to -\frac{i\text{Bb}}{6\text{alphabeta}}\right\}\right\}$

(*Restart*)
ftemp[z_, w_]:=(z + b * w)/(c * z + d * w + 1);
gtemp[z_, w_]:=w/(c * z + d * w + 1);
fbtemp[zb_, wb_]:=(zb + bb * wb)/(cb * zb + db * wb + 1);
gbtemp[zb_, wb_]:=wb/(cb * zb + db * wb + 1);

$\mathbf{\boldsymbol{f}[z\_, w\_]{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(ftemp^{(j,k)}[0,0]\right)\big/(j!*k!)\right)*z\char`^j*w\char`^k}$

$\mathbf{fb[zb\_, wb\_]{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(fbtemp^{(j,k)}[0,0]\right)\big/(j!*k!)\right)*zb\char`^j*wb\char`^k}$

$\mathbf{\boldsymbol{g}[z\_, w\_]{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(gtemp^{(j,k)}[0,0]\right)/(j!*k!)\right)*z\char`^j*w\char`^k}$

$\mathbf{gb[zb\_, wb\_]{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(gbtemp^{(j,k)}[0,0]\right)\big/(j!*k!)\right)*zb\char`^j*wb\char`^k}$

F[z_, zb_, u_]:=alpha * z * zb + beta * z^2 + beta * zb^2 + A * z^3 + Ab * zb^3
+B * z^2 * zb + Bb * z * zb^2 + C * z * u + Cb * zb * u + k400 * z^4+
k310 * z^3 * zb + k220 * z^2 * zb^2 + k130 * z * zb^3 + k040 * zb^4
+k201 * z^2 * u + k111 * z * zb * u + k021 * zb^2 * u + k002 * u^2;

$B\text{:=}0;\text{Bb:=}0;C\text{:=}0;\text{Cb:=}0;b\text{:=}0;\text{bb:=}0;c\text{:=}0;\text{cb:=}0;$

$\text{Express:=}F[f[z,w],\text{fb}[\text{zb},\text{wb}],(g[z,w]+\text{gb}[\text{zb},\text{wb}])/2]$

$-((g[z,w]-\text{gb}[\text{zb},\text{wb}])/(2*I))+((w-\text{wb})/(2*I))$

$w\text{:=}u+I*(\text{alpha}*z*\text{zb}+\text{beta}*z\text{^}2+\text{beta}*\text{zb}\text{^}2+A*z\text{^}3+\text{Ab}*\text{zb}\text{^}3);$

$\text{wb:=}u-I*(\text{alpha}*z*\text{zb}+\text{beta}*z\text{^}2+\text{beta}*\text{zb}\text{^}2+A*z\text{^}3+\text{Ab}*\text{zb}\text{^}3);$

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,3],\text{zb},0],u,0]$

A

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,4],\text{zb},0],u,0]$

$-\frac{3}{2}i\text{beta}^2d-\frac{1}{2}i\text{beta}^2\text{db}+\text{k400}$

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,3],\text{zb},1],u,0]$

$-2i\text{alphabeta}d+\text{k310}$

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,2],\text{zb},2],u,0]$

$-\frac{1}{2}i\text{alpha}^2d-i\text{beta}^2d+\frac{1}{2}i\text{alpha}^2\text{db}+i\text{beta}^2\text{db}+\text{k220}$

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,2],\text{zb},0],u,1]$

$-\text{beta}d+\text{betadb}+\text{k201}$

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,1],\text{zb},1],u,1]$

k111

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,0],\text{zb},0],u,2]$

$-\frac{id}{2}+\frac{i\text{db}}{2}+\text{k002}$

$d\text{:=dre}+I*\text{dim};\text{db:=dre}-I*\text{dim};$

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,0],\text{zb},0],u,2]$

$\text{dim}+\text{k002}$

$\text{Solve}[\{\text{dim}+\text{k002}==0\},\{\text{dim}\}]$

$\{\{\text{dim}\to-\text{k002}\}\}$

$\text{dim}:=-\text{k002}$

$\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,4],\text{zb},0],u,0]$

$-2i\text{beta}^2\text{dre}-\text{beta}^2\text{k002}+\text{k400}$

$\text{Simplify}[\text{Coefficient}[\text{Coefficient}[\text{Coefficient}[\text{Express},z,2],\text{zb},2],u,0]]$

$-\text{alpha}^2\text{k002}-2\text{beta}^2\text{k002}+\text{k220}$

$\text{Im}\left[-2i\text{beta}^2\text{dre}-\text{beta}^2\text{k002}+\text{k400}\right]$

$\text{Im}\left[-\text{beta}^2\text{k002}+\text{k400}\right]-2\text{Re}\left[\text{beta}^2\text{dre}\right]$

$\text{Solve}\left[\left\{\text{k400im}-2\text{beta}^2\text{dre}==0\right\},\{\text{dre}\}\right]$

$\left\{\left\{\text{dre}\to\frac{\text{k400im}}{2\text{beta}^2}\right\}\right\}$

B.3 ProjNormForm3

$\text{Unprotect}[C]$

$\{C\}$

$F[\text{z\_},\text{zb\_},\text{u\_}]\text{:=alpha}*z*\text{zb}+\text{beta}*z\text{^}2+\text{beta}*\text{zb}\text{^}2+A*z\text{^}3+$

$\text{Ab}*\text{zb}\text{^}3+B*z\text{^}2*\text{zb}+\text{Bb}*z*\text{zb}\text{^}2+C*z*u+\text{Cb}*\text{zb}*u+\text{k400}*z\text{^}4+$

$\text{k310}*z\text{^}3*\text{zb}+\text{k220}*z\text{^}2*\text{zb}\text{^}2+\text{k130}*z*\text{zb}\text{^}3+\text{k040}*\text{zb}\text{^}4+$

$\text{k201}*z\text{^}2*u+\text{k111}*z*\text{zb}*u+\text{k021}*\text{zb}\text{^}2*u+\text{k002}*u\text{^}2;$

$\text{alpha:=}1;B\text{:=}0;\text{Bb:=}0;C\text{:=}0;\text{Cb:=}0;\text{k002:=}0;\text{k040:=k400};$

$\text{ftemp}[\text{z\_},\text{w\_}]\text{:=}(a*z+b*w)/(c*z+d*w+1);$

$\text{gtemp}[\text{z\_},\text{w\_}]\text{:=}(e*w)/(c*z+d*w+1);$

$\text{fbtemp}[\text{zb\_},\text{wb\_}]\text{:=}(\text{ab}*\text{zb}+\text{bb}*\text{wb})/(\text{cb}*\text{zb}+\text{db}*\text{wb}+1);$

$\text{gbtemp}[\text{zb\_},\text{wb\_}]\text{:=}(e*\text{wb})/(\text{cb}*\text{zb}+\text{db}*\text{wb}+1);$

$f[\text{z\_},\text{w\_}]\text{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(\text{ftemp}^{(j,k)}[0,0]\right)\big/(j!*k!)\right)*z\text{^}j*w\text{^}k$

$\text{fb}[\text{zb\_},\text{wb\_}]\text{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(\text{fbtemp}^{(j,k)}[0,0]\right)\big/(j!*k!)\right)*\text{zb}\text{^}j*\text{wb}\text{^}k$

$g[\text{z\_},\text{w\_}]\text{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(\text{gtemp}^{(j,k)}[0,0]\right)/(j!*k!)\right)*z\text{^}j*w\text{^}k$

$\text{gb}[\text{zb\_},\text{wb\_}]\text{:=}\sum_{k=0}^{4}\sum_{j=0}^{4-k}\left(\left(\text{gbtemp}^{(j,k)}[0,0]\right)\big/(j!*k!)\right)*\text{zb}\text{^}j*\text{wb}\text{^}k$

Express:=F[f[z,w],fb[zb,wb],(g[z,w]+gb[zb,wb])/2]−
((g[z,w]−gb[zb,wb])/(2*I))+((w−wb)/(2*I))
w:=
u+
I*(alpha*z*zb+beta*z^2+beta*zb^2+A*z^3+Ab*zb^3+
B*z^2*zb+Bb*z*zb^2+C*z*u+Cb*zb*u+k400*z^4+
k310*z^3*zb+k220*z^2*zb^2+k130*z*zb^3+k040*zb^4+
k201*z^2*u+k111*z*zb*u+k021*zb^2*u+k002*u^2);
wb:=
u−
I*(alpha*z*zb+beta*z^2+beta*zb^2+A*z^3+Ab*zb^3+
B*z^2*zb+Bb*z*zb^2+C*z*u+Cb*zb*u+k400*z^4+
k310*z^3*zb+k220*z^2*zb^2+k130*z*zb^3+k040*zb^4+
k201*z^2*u+k111*z*zb*u+k021*zb^2*u+k002*u^2);
Coefficient[Coefficient[Coefficient[Express,z,1],zb,1],u,0]
$1+a\text{ab}-e$
Coefficient[Coefficient[Coefficient[Express,z,2],zb,0],u,0]
$\text{beta}+a^2\text{beta}-\text{beta}e$
Coefficient[Coefficient[Coefficient[Express,z,0],zb,2],u,0]
$\text{beta}+\text{ab}^2\text{beta}-\text{beta}e$
Solve[{1+aab−e==1},{e}]
$\{\{e\to a\text{ab}\}\}$
e:=aab;
Coefficient[Coefficient[Coefficient[Express,z,2],zb,0],u,0]
beta
Coefficient[Coefficient[Coefficient[Express,z,0],zb,2],u,0]
beta
$\text{Solve}\left[\left\{\text{beta}+a^2\text{beta}-a\text{abbeta}==\text{beta}\right\},\{\text{ab}\}\right]$
$\{\{\text{ab}\to a\}\}$
ab:=a;
Coefficient[Coefficient[Coefficient[Express,z,3],zb,0],u,0]
$A+a^3A-ia\text{bbbeta}+\text{beta}\left(2iab\text{beta}-2a^2c\right)+\frac{1}{2}i\left(2ia^2A-ia^2\text{beta}c\right)$
Coefficient[Coefficient[Coefficient[Express,z,0],zb,3],u,0]
$\text{Ab}+a^3\text{Ab}+iab\text{beta}+\text{beta}\left(-2ia\text{bbbeta}-2a^2\text{cb}\right)+\frac{1}{2}i\left(2ia^2\text{Ab}-ia^2\text{betacb}\right)$
Coefficient[Coefficient[Coefficient[Express,z,2],zb,1],u,0]
$-ia\text{bb}+2iab\text{beta}-2ia\text{bbbeta}^2+a(ib\text{beta}-ac)+\frac{1}{2}i\left(-ia^2c-ia^2\text{betacb}\right)$
Coefficient[Coefficient[Coefficient[Express,z,1],zb,2],u,0]
$iab-2ia\text{bbbeta}+2iab\text{beta}^2+a(-i\text{bbbeta}-a\text{cb})+\frac{1}{2}i\left(-ia^2\text{beta}c-ia^2\text{cb}\right)$
Coefficient[Coefficient[Coefficient[Express,z,1],zb,0],u,1]
$a\text{bb}+2ab\text{beta}-\frac{1}{2}ia^2c$
Coefficient[Coefficient[Coefficient[Express,z,0],zb,1],u,1]
$ab+2a\text{bbbeta}+\frac{1}{2}ia^2\text{cb}$
$\text{Solve}\left[\left\{a\text{bb}+2ab\text{beta}-\frac{1}{2}ia^2c==0,ab+2a\text{bbbeta}+\frac{1}{2}ia^2\text{cb}==0\right\},\right.$
$\left.\{c,\text{cb}\}\right]$
$\left\{\left\{c\to-\frac{2i(\text{bb}+2b\text{beta})}{a},\text{cb}\to\frac{2i(b+2\text{bbbeta})}{a}\right\}\right\}$
$c:=-\frac{2i(\text{bb}+2b\text{beta})}{a};\text{cb}:=\frac{2i(b+2\text{bbbeta})}{a}$
Coefficient[Coefficient[Coefficient[Express,z,2],zb,1],u,0]
$-ia\text{bb}+2iab\text{beta}-2ia\text{bbbeta}^2+a(ib\text{beta}+2i(\text{bb}+2b\text{beta}))$
$+\frac{1}{2}i(-2a(\text{bb}+2b\text{beta})+2a\text{beta}(b+2\text{bbbeta}))$
Coefficient[Coefficient[Coefficient[Express,z,1],zb,2],u,0]
$iab-2ia\text{bbbeta}+2iab\text{beta}^2+a(-i\text{bbbeta}-2i(b+2\text{bbbeta}))$
$+\frac{1}{2}i(-2a\text{beta}(\text{bb}+2b\text{beta})+2a(b+2\text{bbbeta}))$

Solve[
$\{-ia\mathrm{bb} + 2iab\mathrm{beta} - 2ia\mathrm{bbbeta}^2 + a(ib\mathrm{beta} + 2i(\mathrm{bb} + 2b\mathrm{beta}))+$
$\frac{1}{2}i(-2a(\mathrm{bb} + 2b\mathrm{beta}) + 2a\mathrm{beta}(b + 2\mathrm{bbbeta})) == 0,$
$iab - 2ia\mathrm{bbbeta} + 2iab\mathrm{beta}^2 + a(-i\mathrm{bbbeta} - 2i(b + 2\mathrm{bbbeta}))+$
$\frac{1}{2}i(-2a\mathrm{beta}(\mathrm{bb} + 2b\mathrm{beta}) + 2a(b + 2\mathrm{bbbeta})) == 0\}, \{b, \mathrm{bb}\}]$
$\{\{b \to 0, \mathrm{bb} \to 0\}\}$
b**:=0; bb:=0**
Coefficient[Coefficient[Coefficient[Express, z**, 0], zb, 0],** u**, 2]**
$\frac{1}{2}i\left(-a^2d + a^2\mathrm{db}\right)$
d**:=dre +** I *** dim; db:=dre −** I *** dim;**
Simplify[Coefficient[Coefficient[Coefficient[Express, z**, 0], zb, 0],** u**, 2]]**
$a^2\,\mathrm{dim}$
dim :=0;
Coefficient[Coefficient[Coefficient[Express, z**, 4], zb, 0],** u**, 0]**
$-2ia^2\mathrm{beta}^2\mathrm{dre} + \mathrm{k400} - a^2\mathrm{k400} + a^4\mathrm{k400}$
Im $\left[-2ia^2\mathrm{beta}^2\mathrm{dre} + \mathrm{k400} - a^2\mathrm{k400} + a^4\mathrm{k400}\right]$
$\mathrm{Im}\left[\mathrm{k400} - a^2\mathrm{k400} + a^4\mathrm{k400}\right] - 2\mathrm{Re}\left[a^2\mathrm{beta}^2\mathrm{dre}\right]$
dre:=0;

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