DEFORMATION SPACES OF KLEINIAN SURFACE GROUPS ARE NOT LOCALLY CONNECTED

by

Aaron D. Magid

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Doctoral Committee:

Professor Richard D. Canary, Chair
Professor Mario Bonk
Professor G. Peter Scott
Professor Ralf J. Spatzier
Assistant Professor Vilma M. Mesa
Assistant Professor Juan Souto

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ABSTRACT

For any closed surface S of genus $g \geq 2$, we show that the deformation space of marked hyperbolic 3-manifolds homotopy equivalent to S, $AH(S \times I)$, is not locally connected. This proves a conjecture of Bromberg who recently proved that the space of Kleinian punctured torus groups is not locally connected. Playing an essential role in our proof is a new version of the filling theorem that is based on the theory of cone-manifold deformations developed by Hodgson, Kerckhoff, and Bromberg.

CHAPTER 1

Introduction

Understanding and classifying 3-manifolds has been a major focus of topology during the past century. After decomposing an arbitrary compact 3-manifold using results of Kneser [43], Milnor [59], Jaco and Shalen [35], and Johannson [36], the Thurston geometrization conjecture [72] states that each of the pieces admits one of eight geometric structures (*i.e.*, locally homogeneous Riemannian metrics). Thurston [73] proved this conjecture for a large class of manifolds in the 1980s, and the recent work of Perelman [65, 66, 67] completes this geometrization program. Hyperbolic manifolds form a particularly large family of geometric manifolds, and the work of Thurston provides sufficient topological conditions for a 3-manifold to admit a hyperbolic structure.

Given a compact, orientable 3-manifold N, the existence of a hyperbolic metric on its interior is generally not enough to answer geometric questions about N. Indeed, when ∂N contains a non-toroidal boundary component, the hyperbolic metric is not uniquely determined by the topology of N. Thus it is natural to consider the set of marked hyperbolic 3-manifolds homotopy equivalent to N. We equip this set with the algebraic topology and denote it by $AH(N)$.

The work of Ahlfors [3], Bers [8], Kra [45], Marden [51], Maskit [55], Sullivan

[70], and Thurston [72] shows that the components of the interior of $AH(N)$ are in one-to-one correspondence with the marked homeomorphism types of compact 3-manifolds homotopy equivalent to N. Using the theory of quasiconformal deformations and the measurable Riemann mapping theorem, each of these components can be parameterized by analytic information.

Unfortunately, our understanding of the interior of $AH(N)$ does not extend to the entire space. When the boundary of N is incompressible, Anderson, Canary, and McCullough [5] characterized when two components of the interior of $AH(N)$ have intersecting closures. They called this phenomenon bumping. For any genus $g \geq 2$ surface S, McMullen [58] showed that the interior of $AH(S \times I)$ self-bumps. This means that there is a point $\rho \in AH(S \times I)$ such that whenever U is a sufficiently small neighborhood of ρ, the intersection of U and the interior of $AH(S \times I)$ is disconnected. Bromberg and Holt [22] generalized this result by showing that whenever N contains a primitive, essential annulus that is not homotopic into a torus boundary component of N then the interior of $AH(N)$ self-bumps.

Recent work by Agol [2], Calegari and Gabai [25], Brock, Canary, and Minsky [16], and many others has led to a classification of hyperbolic manifolds up to isometry. The existence of bumping and self-bumping points shows that the invariants used in this classification do not vary continuously at certain points on the boundary of the deformation space (see also [11]). Thus, further study of the local topology of $AH(N)$ near these points is necessary in order to fully understand these spaces of hyperbolic manifolds.

Bromberg [21] recently showed that the space of Kleinian punctured torus groups is not locally connected. The points where this deformation space fails to be locally connected are self-bumping points, but he also showed that the space is locally connected at other self-bumping points. This indicates that bumping may be considerably more complicated than we previously thought. He also conjectured that $AH(S \times I)$ would fail to be locally connected for any surface S, although his arguments in the punctured torus case made essential use of Minsky's [60] classification of punctured torus groups. The results in [60] that Bromberg uses do not generalize to higher genus surfaces.

The following theorem proves Bromberg's conjecture.

Theorem 1.1. *Let S be a closed surface of genus $g \geq 2$. Then $AH(S \times I)$ is not locally connected.*

The key technical result that we use to prove Theorem 1.1 is an improved version of the filling theorem. Given a geometrically finite hyperbolic manifold $\hat{M}$ with a rank-2 cusp, the filling theorem provides sufficient conditions for one to "Dehn-fill" the cusp. That is, if $\hat{M}$ is homeomorphic to the interior of a compact manifold $\hat{N}$ with a torus boundary component corresponding to the cusp of $\hat{M}$, and N is a Dehn-filling of $\hat{N}$, then the filling theorem provides conditions for one to construct a hyperbolic manifold M homeomorphic to the interior of N with the same conformal boundary as $\hat{M}$. Assuming the hypotheses of the theorem are satisfied, one obtains a relationship between the metrics on $\hat{M}$ and M.

We now describe some of the notation we will use in the statement of the theorem. Suppose T is a rank-2 cusp in $\hat{M}$ and β is the slope in T along which we are filling. Let L be the normalized length of β in T, and let A^2 be the reciprocal of the normalized twist of the cusp. Although we relegate the actual definitions of the normalized length and the normalized twist to Chapter 4, we now describe these quantities with respect to a particular normalization of the cusp (*i.e.*, the normalization that we will use throughout Chapters 5 and 6). Suppose the rank-2 cusp T of $\hat{M}$ is generated by parabolics $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, and that β corresponds to $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. If $Im(w) > 0$ and $\frac{|w|^2}{2|Re(w)|} > 2$, then L^2 and A^2 are given by:

$$L^2 = \frac{|w|^2}{2Im(w)} \quad \text{and} \quad A^2 = \frac{|w|^2}{2Re(w)}.$$

For any curve $\gamma \subset M$, let $B \in PSL(2, \mathbb{C})$ denote the corresponding isometry in $\pi_1(M)$. The complex length of γ is the value of $\mathcal{L} = l + i\theta$ such that $tr^2(B) = 4\cosh^2\left(\frac{\mathcal{L}}{2}\right)$, $l \geq 0$, and $\theta \in (-\pi, \pi]$. For a geodesic γ, the real part l gives the length of γ in M which is the distance that B translates along its axis. The imaginary part θ is the amount B rotates about its axis.

Let ϵ_3 denote the Margulis constant for $\mathbb{H}^3$. If γ is the core curve of the solid filling torus in M. Then for any $\epsilon_3 \geq \epsilon > 0$, let $\mathbb{T}_\epsilon(T)$ (resp. $\mathbb{T}_\epsilon(\gamma)$) denote the ϵ-Margulis tube about T (resp. γ).

Theorem 1.2. *Let $J > 1$ and $\epsilon_3 \geq \epsilon > 0$. There is some $K \geq 8(2\pi)^2$ such that the following holds: suppose $\hat{M}$ is a geometrically finite hyperbolic 3-manifold with no rank-1 cusps, T is a rank-2 cusp in $\hat{M}$, and β is a slope on T such that the normalized length of β is at least K (i.e., $L^2 \geq K^2$), then*

(i) the β-filling of $\hat{M}$, which we call M, exists;

(ii) the real part of the complex length $\mathcal{L} = l + i\theta$ of the core curve of the filling torus γ in M is approximately $\frac{2\pi}{L^2}$ with error bounded by

$$\left| l - \frac{2\pi}{L^2} \right| \leq \frac{8(2\pi)^3}{L^4 - 16(2\pi)^4};$$

(iii) in particular, the length of γ is bounded above by $\frac{2\pi}{L^2 - 4(2\pi)^2}$;

(iv) there exists a J-biLipschitz diffeomorphism

$$\phi : \hat{M} - \mathbb{T}_\epsilon(T) \to M - \mathbb{T}_\epsilon(\gamma).$$

(v) If, in addition to $L^2 \geq K^2$, we have $|A^2| \geq 3$, then the imaginary part of the complex length $\mathcal{L} = l + i\theta$ of the core curve of the filling torus γ in M (chosen so $\theta \in (-\pi, \pi]$) is approximately $\frac{2\pi}{A^2}$ with error bounded by

$$\left|\theta - \frac{2\pi}{A^2}\right| \leq \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2}.$$

The proof of Theorem 1.2 is contained in Chapter 4. Although our version may be stated differently, parts $(i)-(iii)$ can be found in the work of Hodgson and Kerckhoff [32, 33] on cone-manifold deformations which was generalized to geometrically finite manifolds by Bromberg [18, 19]. Part (iv) follows from the drilling theorem of Brock and Bromberg [13]. The most original part of this version of the filling theorem is the estimate in part (v), although its proof also relies on the Hodgson-Kerckhoff cone-manifold technology. Some of the background cone-manifold deformation theory and a summary of the work of Bromberg, Hodgson, and Kerckhoff on cone-manifolds can be found in Chapter 3.

We now outline the proof of Theorem 1.1. We begin by parameterizing a subset of $AH(S \times I)$. If $P \subset S \times \{1\}$ is a pants decomposition, then $MP(S \times I, P)$ denotes the subset of the boundary of $AH(S \times I)$ consisting of the marked hyperbolic 3-manifolds that are homeomorphic to the interior of $S \times I$, are geometrically finite, have a rank-1 cusp associated to each component of P, and contain no other cusps (see Chapter 2 for this notation).

We define a subset $\mathcal{A} \subset MP(S \times I, P) \times \hat{\mathbb{C}}^{3g-3}$ and a map

$$\Phi : \mathcal{A} \to AH(S \times I)$$

such that Φ is a local homeomorphism onto its image. That is, there is some $\sigma^0 \in MP(S \times I, P)$ and some neighborhood U of $(\sigma^0, \infty, \ldots, \infty)$ in $\mathcal{A}$ such that $\Phi(\sigma^0, \infty, \ldots, \infty) = \sigma^0$ and $\Phi|_U : U \to \Phi(U)$ is a homeomorphism. We now roughly describe the map Φ. Let $d = 3g - 3$. If $(\sigma, \infty, \ldots, \infty) \in \mathcal{A}$ then we define $\Phi(\sigma, \infty, \ldots, \infty) = \sigma$. If $(\sigma, w_1, \ldots, w_d) \in \mathcal{A}$ for some $(w_1, \ldots, w_d) \in \mathbb{C}^d$, we use the w-coordinates to define a marked hyperbolic 3-manifold with d rank-2 cusps. To each rank-2 cusp, one can associate a conformal structure on a torus, and w_i acts as a Teichmüller parameter for the ith cusp. We then use the filling theorem (Theorem 1.2) to fill in these cusps and obtain a marked hyperbolic 3-manifold in the interior of $AH(S \times I)$. We define $\mathcal{A}$ to exclude points in $MP(S \times I, P) \times \hat{\mathbb{C}}^{3g-3}$ where some, but not all, of the w-coordinates are ∞.

This parameterization of the subset $\Phi(U) \subset AH(S \times I)$ is a straightforward generalization of the results in Section 3 of Bromberg [21]. We set up the necessary background in Chapter 2 and describe the parameterization in Chapter 5. This parameterization is an application of parts $(i) - (iv)$ of Theorem 1.2 and Corollary 4.13, which is a generalization of the filling theorem for multiple cusps.

In Section 5.4, we use results of Section 4 of Bromberg [21] to show that $\mathcal{A}$ is not locally connected at $(\sigma^0, \infty, \ldots, \infty)$. Moreover, we find that in some neighborhood U of $(\sigma^0, \infty, \ldots, \infty)$ in $\mathcal{A}$, there exists $\delta > 0$ and subsets $C_n \subset U$ accumulating at $(\sigma^0, \infty, \ldots, \infty)$ such that for any $(\sigma, w_1, \ldots, w_d) \in \overline{C_n}$ and any $(\sigma', w_1', \ldots, w_d') \in \overline{U - C_n}$, we have $|w_1 - w_1'| > \delta$ for all n (see Lemma 5.7). Heuristically, we think of the sets C_n as being components of U that are bounded apart from the rest of U by a lower bound that is independent of n. In actuality, these sets are likely collections of components.

Finally, in Chapter 6, we show that $AH(S \times I)$ is not locally connected at σ^0. By

Lemma 5.7, there is a lower bound on the distance between the first w-coordinate ($i.e.$, the first coordinate of the $\hat{\mathbb{C}}^{3g-d}$ factor of $\mathcal{A}$) of a point in $\overline{C_n}$ and the first w-coordinate of a point in $\overline{U - C_n}$. We then use the filling theorem to estimate the complex length of a curve in $\Phi(\sigma, w_1, \ldots, w_d) \in AH(S \times I)$ based on $(w_1, w_2, \ldots, w_d)$. The control on the w_1-coordinate from Lemma 5.7 and the quality of the estimates in the filling theorem show that for all but finitely many n, $\overline{\Phi(C_n)}$ and $\overline{\Phi(U - C_n)}$ must be disjoint. Hence, $\overline{\Phi(U)}$ has infinitely many components that accumulate at σ^0. It follows from the Density Theorem (Theorem 2.2) that $\overline{\Phi(U)}$ contains a neighborhood of σ^0 in $AH(S \times I)$; hence, $AH(S \times I)$ is not locally connected at σ^0.

CHAPTER 2

Background Deformation Space Theory

In this chapter, we recall the definition of a pared 3-manifold (N, P) and define the relative deformation space $AH(N, P)$. This is a space of hyperbolic 3-manifolds homotopy equivalent to N with cusps associated to annuli and tori in P. We will review the Ahlfors-Bers parameterization that describes the interior of $AH(N, P)$ and set up some of the notation that will be used later. We then survey a selection of more recent results that illustrate the complexities of the topology of $AH(N, P)$ near its boundary. For more information about pared manifolds and deformation spaces, see Chapters 5 and 7 of [28] respectively. For a survey of the Density Theorem and bumponomics, see [26].

Before turning to deformation spaces, we begin with some hyperbolic geometry and Kleinian group theory.

2.1 Kleinian Groups

A Kleinian group is a discrete subgroup of $PSL(2, \mathbb{C}) \cong Isom^+(\mathbb{H}^3)$. We will assume that all of our Kleinian groups are finitely generated, torsion-free, and not virtually abelian. If Γ is a Kleinian group, then it acts properly discontinuously on $\mathbb{H}^3$ and the quotient $M_\Gamma = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold. The action of

$\Gamma \subset PSL(2, \mathbb{C})$ on $\mathbb{H}^3$ by isometries extends to an action on $\partial\mathbb{H}^3 \cong \hat{\mathbb{C}}$ by Möbius transformations. The domain of discontinuity $\Omega(\Gamma)$ is the largest open Γ-invariant subset of $\hat{\mathbb{C}}$ on which the action of Γ is properly discontinuous. The quotient $\Omega(\Gamma)/\Gamma$ is called the conformal boundary of M_Γ.

The limit set Λ_Γ of Γ is the smallest, nonempty, closed Γ-invariant subset of $\hat{\mathbb{C}}$. Equivalently, the limit set is the complement of the domain of discontinuity. Let $CH(\Lambda_\Gamma) \subset \mathbb{H}^3$ be the convex hull of the limit set. This is also Γ-invariant, and the quotient $CH(\Lambda_\Gamma)/\Gamma$ is the convex core of the manifold M_Γ.

2.1.1 Pared Manifolds

A pared 3-manifold is a pair (N, P) where N is a compact, oriented, hyperbolizable 3-manifold that is not a 3-ball, and $P \subset \partial N$ is a disjoint collection of incompressible annuli and tori satisfying the following properties:

1. P contains all of the tori in ∂N, and
2. every π_1-injective map $(S^1 \times I, S^1 \times \partial I) \to (N, P)$ is homotopic, as a map of pairs, into P.

To avoid some degenerate cases in the statements that follow, we will assume throughout this paper that $\pi_1(N)$ is not virtually abelian. This will ensure that any Kleinian group isomorphic to $\pi_1(N)$ is non-elementary.

2.1.2 Geometrically Finite Kleinian Groups

A hyperbolic 3-manifold $M_\Gamma = \mathbb{H}^3/\Gamma$ is geometrically finite if and only if the union of M_Γ with its conformal boundary, $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$, is homeomorphic to $N - P$ for some pared 3-manifold (N, P). A Kleinian group Γ is geometrically finite if and only if the corresponding 3-manifold M_Γ is geometrically finite. There are many other equivalent notions of geometric finiteness. See Bowditch [10] for a more complete discussion.

2.1.3 Thick-Thin Decomposition

A hyperbolic 3-manifold M can have two types of cusps. A cusp is an end of the manifold that is modeled on the quotient of $\mathbb{H}^3$ by a parabolic subgroup. Using the upper-half space model of $\mathbb{H}^3 = \{(x,t) \ : \ t > 0\}$, let $P_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $P_w = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ be parabolic isometries (any discrete subgroup of parabolics isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ is conjugate to $\langle P_2 \rangle \oplus \langle P_w \rangle$ for some w). A rank-1 cusp of M is an end of M, homeomorphic to $S^1 \times (-\infty, \infty) \times [0, \infty)$, that is isometric to $\{(x,t) \ : \ t \geq k\} / \langle P_2 \rangle$ for some k. A rank-2 cusp of M is a subset of M, homeomorphic to $T^2 \times [0, \infty)$, that is isometric to $\{(x,t) \ : \ t \geq k\}/(\langle P_2 \rangle \oplus \langle P_w \rangle)$ for some k and some w.

Let M be a hyperbolic manifold. For any $\epsilon > 0$, we define the ϵ-thick part of M to be the set of points $x \in M$ where the injectivity radius is at least ϵ:

$$M^{\geq \epsilon} = \{x \in M \ : \ \mathrm{inj}(x) \geq \epsilon\}.$$

The ϵ-thin part of M, denoted by $M^{\leq \epsilon}$, is the complement of $M^{> \epsilon}$. By the Margulis lemma, there is some constant ϵ_3 (depending only on the dimension) such that for any $\epsilon_3 \geq \epsilon > 0$, the ϵ-thin part of M consists of a disjoint union of metric collar neighborhoods of short geodesics and cusps.

We use the notation $\mathbb{T}_\epsilon(\gamma)$ to denote the Margulis ϵ-thin region associated to a geodesic γ and $\mathbb{T}_\epsilon(T)$ to denote the Margulis ϵ-thin region associated to a rank-2 cusp T. We let $\mathbb{T}_\epsilon^{par}$ denote the union of the Margulis ϵ-thin regions associated to parabolics (*i.e.*, the rank-1 and rank-2 cusps).

2.1.4 Klein-Maskit Combination

Let H be a subgroup of Γ. A subset $B \subset \hat{\mathbb{C}}$ is precisely invariant under H in Γ if (1) for all $h \in H$, $h(B) = B$, and (2) for all $\gamma \in \Gamma - H$, $\gamma(B) \cap B = \emptyset$.

For example, if H is the infinite cyclic group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and Γ is a geometrically finite group containing H with a rank-1 cusp corresponding to H (*i.e.*, the largest abelian subgroup of G containing H is H), then there is some R such that the two sets

$$B_R^+ = \{z \in \mathbb{C} \ : \ Im(z) > R\} \text{ and } B_R^- = \{z \in \mathbb{C} \ : \ Im(z) < -R\}$$

are precisely invariant under H in Γ (*e.g.*, see p. 125 of [51]).

Precisely invariant sets are useful for constructing Kleinian groups via a process known as Klein-Maskit combination. We will use statements similar to those in [1], but one should also refer to [52, 53, 54].

Suppose G_1, G_2 are two geometrically finite Kleinian groups with $G_1 \cap G_2 = H$. Here, H could be any subgroup, but we will only be interested in the case that H is the infinite cyclic parabolic subgroup of the previous example. If there is a Jordan curve c bounding two open discs B_1, B_2 in $\hat{\mathbb{C}}$ such that B_i is precisely invariant under H in G_i, then the group G generated by G_1 and G_2 is geometrically finite and isomorphic to the amalgamated free product $G_1 *_H G_2$. In this case, we say that the group G is obtained from G_1 and G_2 by type I Klein-Maskit combination along the subgroup H.

We now describe type II Klein-Maskit combination. Let G be a geometrically finite Kleinian group containing H. Let $f \in PSL(2, \mathbb{C})$ such that $fHf^{-1} \subset G$. Suppose there is a Jordan curve c bounding a disc $B \subset \hat{\mathbb{C}}$ such that

(1) B is precisely invariant for H in G,

(2) $\hat{\mathbb{C}} - f(\overline{B})$ is precisely invariant for $fHf^{-1} \subset G$, and

(3) $gB \cap \left(\hat{\mathbb{C}} - f(\overline{B})\right) = \emptyset$ for all $g \in G$.

Then the group Γ generated by G and f is geometrically finite and isomorphic to the HNN extension $G*_{\langle f \rangle}$.

Again, while type II Klein-Maskit combination can be applied in a more general setting, consider a geometrically finite group G containing H as above, and consider $f = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. Note that $fHf^{-1} = H$. There is some R such that B_R^- and B_R^+ are precisely invariant under H in G. Moreover, we can assume that for all $g \in G$, $gB_R^- \cap B_R^+ = \emptyset$. Then if $Im(w) = 2R$, the group $G*_{\langle f \rangle}$ is geometrically finite. In fact, one can easily see that the condition $Im(w) = 2R$ may be replaced by $Im(w) \geq 2R$, and type II Klein-Maskit combination may still be applied.

2.2 Deformation Spaces

We define the relative representation variety

$$\mathcal{R}(N, P) = Hom_P(\pi_1(N), PSL(2, \mathbb{C}))$$

to be the set of representations $\rho : \pi_1(N) \to PSL(2, \mathbb{C})$ such that $\rho(g)$ is parabolic or the identity whenever $g \in \pi_1(P)$. We then define the relative character variety $R(N, P)$ to be the Mumford quotient of the relative representation variety

$$R(N, P) = \mathcal{R}(N, P)//PSL(2, \mathbb{C}).$$

Although the Mumford quotient is defined algebraically, non-radical points in the character variety can be identified with conjugacy classes of representations (*i.e.*, points in the topological quotient $Hom_P(\pi_1(N), PSL(2, \mathbb{C}))/PSL(2, \mathbb{C})$). A representation is radical if $\rho(\pi_1(N))$ contains an infinite normal nilpotent subgroup (see

p. 62 of [38]). Since we have assumed $\pi_1(N)$ is not virtually abelian, $\rho(\pi_1(N))$ is non-elementary for any discrete, faithful representation ρ. Thus discrete, faithful representations are non-radical, and so for these representations, we will make no distinction between conjugacy classes of representations and points in $R(N, P)$. See also Section 1 of [31].

Let $AH(N, P)$ denote the subset of $R(N, P)$ consisting of the conjugacy classes of representations that are discrete and faithful. Thus $AH(N, P)$ inherits a topology from the character variety known as the algebraic topology. Results of Chuckrow [29] and Jørgensen [37] show that $AH(N, P)$ is a closed subset of $R(N, P)$ with respect to this topology. Since $\pi_1(N)$ is not virtually abelian, a neighborhood of $AH(N, P)$ is a smooth complex manifold, and the topology on $AH(N, P)$ is the same as the topology when considered as a subset of the topological quotient of $Hom_P(\pi_1(N), PSL(2, \mathbb{C}))$ by $PSL(2, \mathbb{C})$ acting by conjugation (Chapter 4 of [38]).

The space $AH(N, P)$ is a deformation space of hyperbolic 3-manifolds in the following sense. Given $\rho \in AH(N, P)$, the image group $\rho(\pi_1(M))$ defines a hyperbolic manifold $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$. Moreover (since N is aspherical) the representation determines a homotopy equivalence $f_\rho : N \to M_\rho$, defined up to homotopy. So points in $AH(N, P)$ can be identified with equivalence classes of marked hyperbolic 3-manifolds (M, f) where $f : N \to M$ is a homotopy equivalence such that $f(P)$ is homotopic into the cusps of M. Two pairs (M_1, f_1) and (M_2, f_2) correspond to the same point of $AH(N, P)$ if there is an orientation preserving isometry $g : M_1 \to M_2$ such that $f_2 \simeq g \circ f_1$.

The interior of $AH(N, P)$ is well-understood. We say that $\rho \in AH(N, P)$ is minimally parabolic if $\rho(g)$ is parabolic if and only if $g \in \pi_1(P)$. A representation $\rho \in AH(N, P)$ is geometrically finite if $\rho(\pi_1(N))$ is a geometrically finite subgroup of $PSL(2, \mathbb{C})$. Results of Marden [50] and Sullivan [70] show that when $\partial N - P \neq \emptyset$, the interior of $AH(N, P)$ consists of precisely the conjugacy classes of representations that are both geometrically finite and minimally parabolic, and we denote this set by $MP(N, P)$.

The work of Ahlfors [3], Bers [8], Kra [45], Marden [50], Maskit [55], Sullivan [70], and Thurston [72] shows that the components of the interior of $AH(N, P)$ are in one-to-one correspondence with the marked pared homeomorphism types of compact 3-manifolds pared homotopy equivalent to (N, P) and that each component of the interior can be parameterized by analytic data. We now describe this parameterization, known as the Ahlfors-Bers parameterization, in the case that $\partial(N, P) = \partial N - P$ is incompressible. See Chapter 7 of [28] for a more complete description of this parameterization including when $\partial(N, P)$ is compressible.

To enumerate the components of $MP(N, P)$, we first define $A(N, P)$ to be the set of marked pared homeomorphism types. More precisely, $A(N, P)$ is the following set of equivalence classes:

$$\begin{aligned} A(N, P) = \{[(N', P'), h] \ : \ & (N', P') \text{ is a compact, oriented, pared 3-manifold,} \\ & h : (N, P) \to (N', P') \text{ is a pared homotopy equivalence}\} / \sim \end{aligned}$$

where $[(N_1, P_1), h_1] \sim [(N_2, P_2), h_2]$ if there exists an orientation preserving pared homeomorphism $j : (N_1, P_1) \to (N_2, P_2)$ such that $j \circ h_1$ is pared homotopic to h_2.

Recall that we can identify $\rho \in AH(N, P)$ with a marked hyperbolic 3-manifold (M_ρ, f_ρ). Any 3-manifold with finitely generated fundamental group admits a compact core. This is a compact submanifold whose inclusion into M_ρ is a homotopy equivalence [69]. A relative compact core C of M_ρ is a compact core for $M_\rho - \mathbb{T}_\epsilon^{par}$ such that ∂C meets every non-compact component of the boundary of $M_\rho - \mathbb{T}_\epsilon^{par}$

in an incompressible annulus and contains every toroidal boundary component of $M_\rho - \mathbb{T}_\epsilon^{par}$. The existence of a such a core is given in [47, 56]. This definition naturally imparts a pared structure on any relative compact core whose paring locus consists of the tori and annuli that intersect $\partial\mathbb{T}_\epsilon^{par}$. When ρ is geometrically finite, we can construct a relative compact core C by intersecting the convex core of M_ρ with $M_\rho - \mathbb{T}_\epsilon^{par}$. We will refer to this as *the* relative compact core of M_ρ. If $\rho \in MP(N, P)$ then the marking f_ρ is homotopic to a pared homotopy equivalence from (N, P) to the relative compact core of M_ρ. So we can define a map $F : MP(N, P) \to A(N, P)$ by sending (M_ρ, f_ρ) to the relative compact core of M_ρ (still marked by f_ρ). The map F establishes a bijection between the components of $MP(N, P)$ and the elements of the set $A(N, P)$. That is, $F(\rho_1) = F(\rho_2)$ if and only if ρ_1 and ρ_2 are in the same component of $MP(N, P)$.

Let B be the component of $MP(N, P)$ determined by $F^{-1}([(N', P'), h])$. For $\rho \in B$, we have that M_ρ is geometrically finite and minimally parabolic, and $f_\rho \circ h^{-1}$ is homotopic to a pared homeomorphism from (N', P') to the relative compact core of M_ρ. Using $f_\rho \circ h^{-1}$, we can mark each component of the conformal boundary of M_ρ with a component of $\partial N' - P'$. This gives us a map

$$\mathcal{AB} : B \to \mathcal{T}(\partial N' - P')$$

where $\mathcal{T}(\partial N' - P')$ denotes the Teichmüller space of $\partial N' - P'$. Recall that the Teichmüller space of a disconnected surface is the product of the Teichmüller spaces of its components.

Theorem 2.1 (Ahlfors [3], Bers [8], Kra [45], Marden [50], Maskit [55], Sullivan [70], and Thurston [72])**.** *When $\partial(N, P)$ is incompressible, the map $\mathcal{AB}$ is a homeomorphism on each component of $MP(N, P)$.*

Throughout the rest of this paper, we will be primarily concerned with the case $N = S \times I$ where S is a closed surface of genus at least two. In this case, the previous theorem is known as Bers' simultaneous uniformization [7]. The interior of $AH(N)$ (in this case $P = \emptyset$) is $MP(N)$ and is connected. The Ahlfors-Bers map defines a homeomorphism

$$\mathcal{AB} : MP(N) \to \mathcal{T}(S) \times \mathcal{T}(S).$$

Although we will continue to use the term minimally parabolic when $N = S \times I$, representations in $MP(N)$ contain no parabolics. These representations are also called quasifuchsian representations because they are quasiconformally conjugate to fuchsian representations (*i.e.*, representations whose $\mathcal{AB}$-image lies in the diagonal of $\mathcal{T}(S) \times \mathcal{T}(S)$).

2.3 Density

The Bers-Sullivan-Thurston Density Conjecture states that $AH(N, P)$ is the closure of $MP(N, P)$. This has recently been proven, and we refer to it as the Density Theorem. In the case that $(N, P) = (S \times I, \emptyset)$, Brock, Canary, and Minsky obtained this result as Corollary 10.1 of the Ending Lamination Theorem [15], using results of Ohshika [63] and Thurston [74].

Theorem 2.2 (Brock-Bromberg [13], Brock-Canary-Minsky [16], Bromberg [20], Bromberg-Souto [23], Kim-Lecuire-Ohshika [41], Kleinedam-Souto [42], Lecuire [48], Namazi-Souto [62], Ohshika [64], Thurston [71]). *If (N, P) is a pared 3-manifold then*

$$AH(N, P) = \overline{MP(N, P)}.$$

There are two sets of results that can be used to prove this theorem, both of which rely on the Tameness Theorem, proven by Agol [2] and Calegari-Gabai [25]. When

$\pi_1(N)$ is freely indecomposable, as in the case $(N, P) = (S \times I, \emptyset)$, the Tameness Theorem is due to Bonahon [9].

In the drilling theorem approach, Bromberg [20] and Brock-Bromberg [13] use the drilling theorem in [13] to prove many cases of the Density Theorem. Bromberg and Souto extended these results to provide a complete proof. For an expository account of these methods, see [12]. In the subsequent sections, we will describe and use the Brock-Bromberg drilling theorem, and the cone-manifold deformation theory developed by Hodgson and Kerckhoff [32, 33], generalized by Bromberg [18, 19], on which it is based.

The other approach uses the Ending Lamination Theorem [16] and results of Kim-Lecuire-Ohshika [41], Kleinedam-Souto [42], Lecuire [48], Namazi-Souto [62], Ohshika [64], and Thurston [71] to show that the interior of $AH(N, P)$ is dense.

The significance of the Density Theorem is that any (marked) hyperbolic 3-manifold in $AH(N, P)$ can be represented as the algebraic limit of geometrically finite, minimally parabolic representations. For our purposes of understanding the topology of a neighborhood of a point ρ in the boundary of $AH(N, P)$, we will just want the topological fact that

$$AH(N, P) = \overline{MP(N, P)}.$$

See [26] for a more complete survey of the history and background to the Density Theorem.

2.4 Bumponomics

Except for some deformation spaces that are only one (complex) dimensional, the Ahlfors-Bers map cannot be naturally extended to a provide a parameterization of $AH(N, P)$. The first indication that our understanding of the topology of the interior of $AH(N, P)$ would not extend to the entire space came in the mid 1990s when Anderson and Canary showed that there exist manifolds N for which distinct components of $MP(N)$ have intersecting closures [4]. This phenomenon has become known as bumping. Later, Anderson, Canary, and McCullough characterized which components of $MP(N)$ bump for any manifold N with incompressible boundary [5]. See [34] for examples of multiple components of $MP(N)$ bumping simultaneously.

Remark. Prior to the Anderson-Canary examples, Kerckhoff and Thurston [40] had shown that the natural map from one Bers slice to another did not extend continuously to their closures. A Bers slice is a cross-section of $MP(S \times I)$ given by $\mathcal{AB}^{-1}(\{*\} \times \mathcal{T}(S))$. Later, Brock found other discontinuities in the ending invariants associated to manifolds in the boundary of a Bers slice [11].

McMullen showed that $MP(S \times I)$ self-bumps [58]. This means there is a point $\rho \in AH(S \times I)$ and a neighborhood U of ρ such that for any neighborhood $\rho \in V \subset U$, the intersection $V \cap MP(S \times I)$ is disconnected. Bromberg and Holt generalized this result by providing sufficient conditions on N for components of $MP(N)$ to self-bump.

Definition 2.3. An annulus $(A, \partial A) \subset (N, \partial N)$ is essential if the inclusion $\pi_1(A) \hookrightarrow \pi_1(N)$ is injective and $(A, \partial A)$ is not properly homotopic into ∂N. The annulus $A \subset N$ is also primitive if whenever α generates $\pi_1(A)$ (*i.e.*, $\langle \alpha \rangle \cong \pi_1(A)$), then $\alpha \neq \gamma^n$ for any $\gamma \in \pi_1(N)$ and $|n| > 1$.

Bromberg and Holt showed that whenever N contains a primitive, essential annulus that is not homotopic into a torus component of ∂N then each component of $MP(N)$ self-bumps [22].

Bumping and self-bumping illustrate that the invariants used in the classification of hyperbolic 3-manifolds in the Ending Lamination Theorem vary discontinuously in $AH(N, P)$. This is most evident in the case of bumping since the homeomorphism type varies discontinuously.

Recently, Bromberg proved the following [21].

Theorem 2.4 (Bromberg [21]). *Let $S_{1,1}$ be a punctured torus. Then $AH(S_{1,1} \times I, \partial S_{1,1} \times I)$ is not locally connected.*

The points where this deformation space fails to be locally connected are self-bumping points. On the other hand, the deformation space is locally connected at other self-bumping points so more study of the local topology near these self-bumping points is necessary in order to fully understand these pathologies. Bromberg's results also show that if $S_{0,4}$ is a four-punctured sphere then $AH(S_{0,4} \times I, \partial S_{0,4} \times I)$ is not locally connected. In our proof that $AH(S \times I)$ fails to be locally connected for any surface S with a higher dimensional Teichmüller space, we frequently refer to many of the arguments in [21]. In particular, the results in Chapter 5 rely on and/or generalize the results of Section 3 of [21]. Chapter 6 represents the most significant departure from Bromberg's methods.

Using Bromberg's description of $AH(S_{1,1} \times I, \partial S_{1,1} \times I)$, one can show that many other deformation spaces fail to be locally connected [49].

Theorem 2.5 (Magid [49]). *Let N be a hyperbolizable 3-manifold containing a primitive essential annulus A, and suppose $(S_{1,1} \times I, \partial S_{1,1} \times I)$ is pared homeomorphic to (N', A), where N' is the closure of one of the components of $N - A$. If $P \subset \partial N$ is a paring locus that contains exactly one of the components of ∂A and is otherwise disjoint from N', then $AH(N, P)$ is not locally connected.*

One can apply this theorem to $N = S \times I$ to find infinitely many deformation spaces that fail to be locally connected.

Corollary 2.6. *Let S be a closed surface of genus $g \geq 2$. Let P be a single annulus on $S \times \{1\}$ such that P separates $S \times \{1\}$ into a punctured torus and a once-punctured genus $(g-1)$ surface. Then $AH(S \times I, P)$ is not locally connected.*

These are all relative deformation spaces (*i.e.*, $P \neq \emptyset$). For any surface S and paring locus P, $AH(S \times I, P)$ naturally sits inside the boundary of $AH(S \times I)$. Theorem 1.1 provides the first examples of non-relative deformation spaces that fail to be locally connected.

CHAPTER 3

Background Cone-Manifold Deformation Theory

3.1 Geometrically Finite Cone-Manifolds

Let N be a compact 3-manifold. A hyperbolic cone-metric on the interior of N with singular locus consisting of a link $\Sigma \subset int(N)$ is an incomplete hyperbolic metric (constant sectional curvature equal to -1) on the interior of $N - \Sigma$ whose metric completion determines a singular metric on $int(N)$ with singularities along Σ. The link is totally geodesic, and in cylindrical coordinates around a component of Σ, the metric has the form

$$dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)dz^2$$

where θ is measured modulo $\alpha > 0$. We require α to be constant on each connected component of Σ, and we say α is the cone angle about that component of the singular locus. See Section 1 of [32] or Section 4 of [18] for more details. When the cone angle on each component of Σ is $\alpha = 2\pi$, this is equivalent to having a complete hyperbolic metric on the interior of N (*i.e.*, in the above definition, we require the metric on $int(N - \Sigma)$ to be complete in every end of $int(N - \Sigma)$ not associated to a component of Σ). From now on, we will only consider cone-manifolds whose singular locus is connected.

Let M_α be a hyperbolic cone-manifold homeomorphic to the interior of N with cone angle α about Σ. We now define what it means for M_α to be a geometrically finite hyperbolic cone-manifold (compare to Section 3 of [18]). To do so, we first define a geometrically finite end. Let S be a closed surface of genus at least two, and let $Y = S \times [0, \infty)$ be a hyperbolic manifold with boundary $S \times \{0\}$. That is, there is a smooth immersion $D : \tilde{Y} \to \mathbb{H}^3$ and representation $\rho : \pi_1(S) \to PSL(2, \mathbb{C})$ such that for all $x \in \tilde{Y}$ and $\gamma \in \pi_1(S)$, $D(\gamma x) = \rho(\gamma) D(x)$. We say D is the developing map for Y and ρ is the holonomy map. We say Y is a geometrically finite end if D can be extended to a local homeomorphism $\tilde{S} \times [0, \infty] \to \mathbb{H}^3 \cup \hat{\mathbb{C}}$ such that $D(\tilde{S} \times \{\infty\}) \subset \hat{\mathbb{C}}$. In this case, $S \times \{\infty\}$ inherits a conformal structure from the charts defined into $\hat{\mathbb{C}}$. In fact, Y has a projective structure at infinity since $PSL(2, \mathbb{C})$ acts by Möbius transformations, although we will only use the fact that the transition maps are conformal.

Given a hyperbolic cone-manifold M_α with cone singularity Σ, we note that $M_\alpha - \Sigma$ has a (possibly incomplete) hyperbolic metric with no cone singularities. Although one could consider hyperbolic cone-manifolds in greater generality, we have defined our cone-manifolds M_α to be homeomorphic to the interior of N and hence topologically tame. The ends of $M_\alpha - \Sigma$ (*i.e.*, the complement of a compact core) are of three types (see p. 160 of [18]). There will be one end homeomorphic to $T^2 \times [0, \infty)$ associated to Σ, some number of ends associated to the rank-2 cusps of M_α, also homeomorphic to $T^2 \times [0, \infty)$ and some number of ends homeomorphic to $S_i \times [0, \infty)$ associated to the higher genus surfaces S_i in the boundary of the compact core. We say M_α is geometrically finite if each of the ends not associated to a rank-2 cusp or to Σ is geometrically finite. We will not be considering hyperbolic cone-manifolds with rank-1 cusps.

We want to provide a meaningful way of interpreting a hyperbolic manifold with a rank-2 cusp as a hyperbolic cone-manifold with cone angle $\alpha = 0$ about the cone-singularity. The convergence results below will allow us to do this more formally. See also Section 3 of [33] and Section 6 of [19].

Definition 3.1. A sequence of metric spaces with basepoints $\{(X_i, x_i)\}$ converges to (X_∞, x_∞) geometrically if, for each $R > 0$, $K > 1$, there exists an open neighborhood U_∞ of the radius R neighborhood of x_∞ in X_∞ and some i_0 such that for all $i > i_0$, there is a map $f_i : (U_\infty, x_\infty) \to (X_i, x_i)$ that is a K-biLipschitz diffeomorphism onto its image.

We say that a sequence $X_i \to X_\infty$ geometrically if there exist basepoints such that $(X_i, x_i) \to (X_\infty, x_\infty)$ geometrically. For a more detailed discussion of geometric convergence in Kleinian group theory, see Chapter E of [6], Chapter I of [27], or Chapter 8 of [38].

The following is Theorem 6.11 of [19], although a finite volume analogue was proven in Section 3 of [33].

Theorem 3.2 (Bromberg [19]). *Let $\{M_\alpha\}$ be a family of geometrically finite hyperbolic cone-manifolds defined for $\alpha \in (0, \alpha_0)$, with fixed conformal boundary, $\alpha_0 \leq 2\pi$, and suppose there is an embedded tubular neighborhood about the cone-singularity of radius $\geq \sinh^{-1}(\sqrt{2})$ in M_α for all $\alpha \in (0, \alpha_0)$. Then*

1. *as $\alpha \to 0$, the manifolds M_α converge geometrically to a complete hyperbolic manifold M_0 homeomorphic to the interior of $N - \Sigma$ with a rank-2 cusp in the end associated to Σ and the same conformal boundary as M_α.*

2. *as $\alpha \to \alpha_0$, the manifolds M_α converge geometrically to a hyperbolic cone-manifold M_{α_0} with cone angle α_0 along Σ and the same conformal boundary*

components as M_α.

This theorem serves two purposes. First, we can interpret a manifold with a rank-2 cusp as a limit of a family of cone-manifolds. Second, a 1-parameter family of cone-manifolds M_α with fixed conformal boundary, defined for some interval $[0, \alpha_0)$, can be extended to a 1-parameter family defined over $[0, \alpha_0]$.

3.2 Cone-Manifold Deformations and Bundles of Killing Fields

Let X be the interior of $N - \Sigma$ and M_α $(0 \leq \alpha \leq 2\pi)$ a geometrically finite cone-manifold homeomorphic to the interior of N with singular locus Σ. Then the (possibly incomplete) hyperbolic metric on X is completely determined by the cone angle α and the conformal boundary components associated to each of the geometrically finite ends of M_α. See Theorem 5.8 of [18] which is restated as Theorem 1.1 of [19]. We will use the following consequence of Bromberg's result.

Theorem 3.3 (Bromberg [18])**.** *Let M_{α_0} be a geometrically finite hyperbolic cone-manifold with cone angle $\alpha_0 \in [0, 2\pi]$ about the cone singularity Σ. Suppose there is an embedded tubular neighborhood about Σ in M_{α_0} of radius $\geq \sinh^{-1}(\sqrt{2})$. Then there exists an open neighborhood W of α_0 in $[0, 2\pi]$ such that the 1-parameter family M_α, defined by varying the cone angle and keeping the conformal boundary of M_{α_0} fixed, is defined for all $\alpha \in W$.*

Suppose M_t is a 1-parameter family of hyperbolic cone-manifolds defined for some interval. By restricting this family of metrics, one obtains a 1-parameter family of hyperbolic metrics on X. Up to precomposition by isotopies of X and postcomposition by isometries of $\mathbb{H}^3$, this determines a 1-parameter family of developing maps

$$D_t : \tilde{X} \to \mathbb{H}^3.$$

We will assume our family is smooth in the sense that D_t is a smooth 1-parameter family of diffeomorphisms such that for any $x \in \tilde{X}$ and $\beta \in \pi_1(X)$, $D_t(\beta x) = \rho_t(\beta) D_t(x)$ for some holonomy representation $\rho_t : \pi_1(X) \to PSL(2, \mathbb{C})$.

If $x \in \tilde{X}$, then $D_t(x)$ determines a path in $\mathbb{H}^3$, and the pullback of the tangent vector to the path $t \mapsto D_t(x)$ at some fixed time t determines a vector $v \in T_x\tilde{X}$. Doing this for each point $x \in \tilde{X}$ determines a vector field, which we also denote v, on $\tilde{X}$.

In general, unless the deformation is trivial, v will not be a Killing field. A Killing field is a vector field whose associated flow $\phi_t : \tilde{X} \to \tilde{X}$ is an isometry for all sufficiently small t. Killing fields on $\mathbb{H}^3$ are parameterized by $sl_2(\mathbb{C})$ by taking the derivative $\frac{d}{dt}\big|_{t=0} \phi_t$. Let $\tilde{E} = \tilde{X} \times sl_2(\mathbb{C})$. We associate to v the Killing field, or equivalently, the section $s_v : \tilde{X} \to \tilde{E}$ of the bundle $\tilde{E}$ that best approximates v at x (more on how s is defined later).

The bundle $\tilde{E}$ has a natural complex structure since each of the fibers can be identified with $sl_2(\mathbb{C})$. Suppose $(x, w) \in \tilde{E}$. Then w is a Killing field on $\tilde{X}$. If $curl(w)(x) = 0$ then we say w is an infinitesimal translation at x and if $w(x) = 0$ then w is an infinitesimal rotation at x. Purely real Killing fields are infinitesimal translations and purely imaginary Killing fields are infinitesimal rotations. So given a Killing field, one can decompose it into its purely real and imaginary parts using the complex structure of $sl_2(\mathbb{C})$ and obtain its infinitesimal translational and rotational parts. One can naturally identify the infinitesimal translations with $T\tilde{X}$ and, using the curl operator on vector fields, identify infinitesimal rotations with $T\tilde{X}$ as well. So we get a decomposition $\tilde{E} \cong T\tilde{X} + iT\tilde{X}$. See p. 14-16 of [32] for more about this decomposition of $\tilde{E}$.

3.3 The Canonical Lift

Given the complex structure on $\tilde{E}$, we are ready to define the section $s : \tilde{X} \to \tilde{E}$ that best approximates the vector field v. Define $s_v : \tilde{X} \to \tilde{E}$ by

$$s_v(x) = v(x) - icurl(v)(x).$$

We say s_v is the canonical lift of the vector field v. See p. 13-14, 17-19 of [32] for more details. Note that under the identification of Killing fields with $sl_2(\mathbb{C})$, the curl operator on Killing fields acts like multiplication by i on sections of $\tilde{E}$. Here we are using twice the usual curl, which is normally defined by $curl(v) = \frac{1}{2} * \hat{d}\hat{v}$ where $\hat{d}$ is exterior differentiation and $\hat{v}$ is the 1-form corresponding to v under the identification of $T\tilde{X}$ with $T\tilde{X}^*$. So we can interpret the curl of a section s as is. Now we see the motivation for this definition of the canonical lift. The real part of s_v at x agrees with the vector field v at x and the real part of $curl(s_v) = is_v$ at x agrees with $curl(v)(x)$.

3.4 E-valued Differential Forms

Now we want to view the canonical lift s_v as an $\tilde{E}$-valued 0-form and obtain a 1-form via exterior differentiation. We briefly recall some facts about $\tilde{E}$-valued k-forms. An $\tilde{E}$-valued k-form (on $\tilde{X}$) is a section of the bundle $\wedge^k T\tilde{X}^* \otimes \tilde{E} \to \tilde{X}$. Thus an $\tilde{E}$-valued 0-form is a section $s : \tilde{X} \to \tilde{E}$. Using the identification $Hom(T\tilde{X}, \tilde{E}) \cong T\tilde{X}^* \otimes \tilde{E}$, we see an $\tilde{E}$-valued 1-form is just a map from $T_x\tilde{X}$ to $\tilde{E}$ at each point $x \in \tilde{X}$. An $\tilde{E}$-valued k-form ω can be expressed as $\omega = \alpha \otimes s$ where α is a real-valued k-form and s is an $\tilde{E}$-valued 0-form. The exterior derivative on $\tilde{E}$-valued k-forms

$$d : \wedge^k T\tilde{X}^* \otimes \tilde{E} \to \wedge^{k+1} T\tilde{X}^* \otimes \tilde{E}$$

is defined using the flat connection on $\tilde{E}$-valued 0-forms, and extended to $\tilde{E}$-valued k-forms $\omega = \alpha \otimes s$ by

$$d(\omega) = d(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge ds. \tag{3.1}$$

In other words, once we define the $\tilde{E}$-valued 1-form ds, we can use the exterior derivative on real forms and (3.1) to define the exterior derivative on $\tilde{E}$-valued forms. Since $\tilde{E}$ is a flat bundle, there is a flat connection $\nabla : \Gamma(\tilde{E}) \times T\tilde{X} \to \Gamma(\tilde{E})$, $(s, V) \mapsto \nabla_V s$, where $\Gamma(\tilde{E})$ denotes the space of smooth sections on $\tilde{E}$. Then we can define ds to be the $\tilde{E}$-valued 1-form $\nabla s : T\tilde{X} \to \Gamma(\tilde{E})$ given by $V \mapsto \nabla_V s$. Note that $\tilde{E}$ is, in fact, a trivial bundle, but we use the theory of flat bundles since we will need this for the quotient E defined below.

Since a form $\omega \in \wedge^k T\tilde{X}^* \otimes \tilde{E}$ can be thought of as having values in $\tilde{E}$ using the identification $Hom(V, W) \cong V^* \otimes W$ for vector spaces V and W, the complex structure of $\tilde{E}$ gives us real and imaginary parts D and T of the exterior derivative d. We write $d = D + T$ corresponding to the orthogonal decomposition of $\tilde{E}$ so if $d\omega = \eta_1 + i\eta_2$ then $D\omega = \eta_1$ and $T\omega = \eta_2$.

We now want to pass to the quotient X. Define E to be the quotient of $\tilde{E}$ by $\pi_1(X)$ where $\pi_1(X)$ acts on $\tilde{X}$ by covering transformations and on $sl_2(\mathbb{C})$ by the adjoint representation. This gives $E \to X$ the structure of a flat bundle. The action of $\pi_1(X)$ also preserves the complex structure of $\tilde{E}$ giving E a complex structure. We define E-valued k-forms similarly to $\tilde{E}$-valued forms, and using the exterior derivative, we can define the cohomology groups $H^k(X; E)$ to be the closed forms modulo the exact forms. We will use the notation $\Omega^k(X; E)$ to denote the set of closed forms.

Given the canonical lift s_v, we claim that ds_v is an equivariant closed 1-form and thus descends to an element $\omega \in \Omega^1(X; E)$. Moreover, we claim that the cohomology class in $H^1(X; E)$ defined by ω is independent of the choice of developing maps D_t.

By definition, $v - \gamma_* v$ is a Killing field for any $\gamma \in \pi_1(X)$. A vector field v satisfying this property is said to be automorphic. Proposition 2.3 of [32] then implies $s_v - \gamma_* s_v$ is constant, as a section of $\tilde{E}$. Thus the associated 1-form ω, defined locally as ds_v satisfies:

$$\omega - \gamma_* \omega = ds_v - \gamma_* ds_v = d(s_v - \gamma_* s_v) = 0.$$

Thus ω is a closed equivariant 1-form and therefore descends to a closed E-valued 1-form on X.

Recall that for a fixed time t, the developing map D_t is only well-defined up to precomposition with the lift (to $\tilde{X}$) of an isotopy of X and postcomposition by an isometry of $\mathbb{H}^3$. Fix $D = D_t$ and consider the effect of the following "trivial" deformations. Define a new family of developing maps (indexed by r) by $D_r = D \circ \tilde{f}_r$ where $\tilde{f}_r : \tilde{X} \to \tilde{X}$ is a lift of an isotopy $f_r : X \to X$. Then the vector field v determined by D_r is equivariant. This implies the canonical lift s_v is equivariant and thus descends from an $\tilde{E}$-valued 0-form on $\tilde{X}$ to an E-valued 0-form on X. Since ω was defined locally by $\omega = ds_v$, and s_v descends to X we have that ω is an exact form. Hence ω is trivial as a cohomology class in $H^1(X; E)$.

Now suppose that $D_r = k_r \circ D$ is a "trivial" deformation obtained by fixing $D = D_t$ and letting $k_r : \mathbb{H}^3 \to \mathbb{H}^3$ be a 1-parameter family of isometries. Without loss of generality, we can assume that k_0 is the identity. Then the vector field v is a Killing field and the canonical lift of a Killing field is a constant section. Hence $\omega = ds_v = 0$. It follows that the 1-forms associated to any two 1-parameter families of developing maps D_t representing the same deformation differ by an exact form. Hence the cohomology class of ω in $H^1(X; E)$ is well-defined by the 1-parameter family of metrics on X determined by M_t.

So to each time t in the deformation, we can associate a cohomology class $\omega_t \in H^1(X; E)$. Conversely, given a cohomology class ω_{t_0}, we can describe the infinitesimal change in the metric on X in the following sense. Given any $\gamma \in \pi_1(X)$, we can compute

$$\int_\gamma \omega_{t_0} = \frac{d}{dt}\rho_t(\gamma)\rho_0(\gamma)^{-1}\bigg|_{t=t_0}. \tag{3.2}$$

We first choose a closed 1-form representing the cohomology class ω_{t_0}, also denoted ω_{t_0}, and integrate the form along γ. The integral only depends on the homotopy class of γ and gives us an element of $sl_2(\mathbb{C})$. See p. 12-13 of [32].

3.5 Harmonic Forms

We now define an L^2-norm on the set of closed E-valued k-forms, which will allow us to pick a nice closed form to represent each cohomology class. This will be the Hodge representative we define below.

Recall that with real-valued k-forms, there is an inner product defined as

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\beta.$$

If $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta = \beta_1 \wedge \cdots \wedge \beta_k$ then one can also define the pointwise inner product at $x \in X$ by

$$\langle \alpha, \beta \rangle_x = \langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k \rangle_x = \det \langle \alpha_i, \beta_j \rangle_x$$

where the inner product on the right is the inner product on TX from the metric on X and the 1-forms α_i, β_j have been identified with their corresponding duals. This defines an inner product on arbitrary k-forms by linear extension. Then the inner product defined using the Hodge star operator agrees with the inner product defined pointwise (integrated against the volume form).

$$\langle \alpha, \beta \rangle = \int_X \langle \alpha, \beta \rangle_x \, d\text{vol}$$

We can define an inner product on $\tilde{E}$ (and E) valued k-forms similarly. Given $x \in \tilde{X}$, define an inner product on the fiber of $\tilde{E}$ over x, identified with $sl_2(\mathbb{C})$ by

$$\langle v, w \rangle_x = \langle v(x), w(x) \rangle_x + \langle iv(x), iw(x) \rangle_x$$

where the inner products on the right are both the inner product on $T_x\tilde{X}$. Recall $v, w, iv, iw \in sl_2(\mathbb{C})$ correspond to Killing fields in $T\tilde{X}$ and $v(x), w(x), iv(x), iw(x)$ are the vectors in $T_x\tilde{X}$ determined by these Killings fields. One can check that this inner product descends to an inner product on the fibers of E.

Given inner products on $\wedge^k TX^*$ and E, one can define an inner product on their tensor product $\wedge^k TX^* \otimes E$ by taking a product. We will use $(,)$ to denote this inner product. Explicitly, if α, β are E-valued k-forms, then

$$(\alpha, \beta) = \int_X \langle \alpha, \beta \rangle_x \, d\text{vol}.$$

Next we want to define an adjoint δ to the exterior derivative d with respect to this inner product. That is, for a $k+1$-form α, define $\delta\alpha$ to be the k-form such that $(\delta\alpha, \beta) = (\alpha, d\beta)$ for any k-form β. Using the inner product on E given above, we have an isomorphism $\sharp : E \to E^*$. Recall that if ω is an E-valued k-form, then $\omega = \alpha \otimes s$ for some real k-form α. Define $*\omega = *\alpha \otimes s$ and define $\sharp(\omega) = \alpha \otimes \sharp(s)$. Then the inner product on E-valued k-forms defined above is actually equal to

$$(\omega_1, \omega_2) = \int_X \omega_1 \wedge (\sharp * \omega_2).$$

The dual bundle E^* also has an exterior derivative d_* and so we define the operator

$$\delta = (-1)^{n(k+1)+1} * \sharp^{-1} d_* \sharp * .$$

One can check that if ω_1 is a $k+1$-form and ω_2 is a k-form then $(\delta\omega_1, \omega_2) = (\omega_1, d\omega_2)$, whenever the inner product is defined (the inner product could be infinite). The square of the norm of ω is (ω, ω), and we say ω is in L^2 if (ω, ω) is finite.

Also, we can define adjoints D^* and T^* for D and T respectively by

$$D^* = (-1)^{n(k+1)+1} * D * \quad \text{and} \quad T^* = (-1)^{n(k+1)+1} * T * .$$

Bromberg (p. 13-14 of [17]) calculates that while $d = D + T$,

$$\delta = (-1)^{n(k+1)+1} * (D - T) * .$$

A k-form is closed if $d\omega = 0$ and co-closed when $\delta\omega = 0$. A form ω is harmonic of $\Delta\omega = (d\delta + \delta d)\omega = 0$. When X is a closed manifold, the Hodge theorem says that any cohomology class in $H^k(X; E)$ can be represented by a closed, co-closed (and hence harmonic) form. While we are not dealing with closed manifolds, we will be able to use similar results of Hodgson, Kerckhoff, and Bromberg to find nice representatives for the cohomology classes in which we are interested. See Section 3.7 below.

3.6 Standard Form

Hodgson and Kerckhoff calculated the effects of two particular E-valued 1-forms ω_m and ω_l in a neighborhood of Σ (p. 31-33, [32]). We now review the definitions of ω_m and ω_l. Using these forms, some of the results in Section 3 of [32] will allow us to put an arbitrary E-valued 1-form into a standard form within a cohomology class of $H^1(X; E)$.

Let M_α be a cone-manifold, and let U be a metric collar neighborhood of Σ in M_α. We give U the cylindrical coordinates (r, θ, z) where r is the distance from Σ. One needs to make some normalizations to properly define θ and z; however, we will only use the r-coordinate in this section. (For instance, θ is only well-defined up to multiples of α.) Recall $X = M_\alpha - \Sigma$, so $U \cap X$ is the set of points in U with $r > 0$. Using the complex structure of E, the real and imaginary parts of an E-valued 1-form can be identified with TX-valued 1-forms, or in other words, sections of $Hom(TX, TX)$. Thus we will define ω_m and ω_l as complex valued sections of $Hom(TX, TX)$. With respect to the basis $\left(\frac{\partial}{\partial r}, \frac{1}{\sinh(r)}\frac{\partial}{\partial \theta}, \frac{1}{\cosh(r)}\frac{\partial}{\partial z}\right)$ on TX, we can define the forms ω_m and ω_l at any point (r, θ, z) of $U \cap X$ by the following matrices

$$\omega_m = \begin{pmatrix} \frac{-1}{\cosh^2(r)\sinh^2(r)} & 0 & 0 \\ 0 & \frac{1}{\sinh^2(r)} & \frac{-i}{\cosh(r)\sinh(r)} \\ 0 & \frac{-i}{\cosh(r)\sinh(r)} & \frac{-1}{\cosh^2(r)} \end{pmatrix}$$

$$\omega_l = \begin{pmatrix} \frac{-1}{\cosh^2(r)} & 0 & 0 \\ 0 & -1 & \frac{-i\sinh(r)}{\cosh(r)} \\ 0 & \frac{-i\sinh(r)}{\cosh(r)} & \frac{\cosh^2(r)+1}{\cosh^2(r)} \end{pmatrix}.$$

If $\alpha \to 0$, the neighborhoods U limit to a rank-2 cusp. We will define ω_m and ω_l in a rank-2 cusp as limits of the 1-forms defined above as $r \to \infty$, but to make this precise, we need a new coordinate system. If U is a rank-2 cusp in X, then U is isometric to the quotient of $\{(x_1, x_2, t) \in \mathbb{H}^3 \,:\, t \geq k\}$ by a discrete $\mathbb{Z}^2$-subgroup of parabolics. Without loss of generality, assume that $\frac{\partial}{\partial x_1}$ is tangent to the meridian. Let $\left(\frac{\partial}{\partial t}, t\frac{\partial}{\partial x_1}, t\frac{\partial}{\partial x_2}\right)$ be a basis for TX in U.

Define $\omega_m = 0$ and

$$\omega_l = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -i \\ 0 & -i & 1 \end{pmatrix}.$$

Even if a rank-2 cusp does not arise as the limit of neighborhoods of a cone singularity, we can still define ω_m and ω_l as above. The only difference is that if U is an arbitrary rank-2 cusp not arising as such a limit, we may not have a well-defined meridian. In this case, one may choose any isometric identification of U with the coordinates above. That is, we say that a 1-form ω_0 restricts to ω_l in U if, for some choice of coordinates in U, we have $\omega_0|_U = \omega_l$.

Integrating ω_m and ω_l along paths in U, one can compute (as on p. 31-33 of [32], see also equation (3.2)) the effect of ω_m and ω_l on (*i.e.*, infinitesimal change in) the holonomy of any element of $\pi_1(\partial U)$. More precisely, suppose M_t is a one-parameter family of cone-manifolds, and $\omega \in H^1(X; E)$ is the cohomology class representing this deformation at some time t_0. Suppose the cohomology class ω can be represented by a 1-form $\omega_0 \in \Omega^1(X; E)$ that is defined locally by $\omega_0|_U = \omega_m$ (resp. ω_l). Using the formula in (3.2), we can integrate ω_m (resp. ω_l) along a path γ in ∂U to compute $\frac{d}{dt}\rho_t(\gamma)\big|_{t=t_0}$, where $\rho_t : \pi_1(X) \to PSL(2, \mathbb{C})$ is the 1-parameter family of holonomy maps associated to M_t.

Rather than performing this computation of $\frac{d}{dt}\rho_t(\gamma)\big|_{t=t_0}$ for $\gamma \in \pi_1(\partial U)$, we will record the effect of ω_m and ω_l on $\rho_t(\gamma)$ by computing the derivative of the complex length $\frac{d}{dt}\mathcal{L}((\rho_t(\gamma)))$ in Section 3.8. See Lemma 2.1 of [33].

Definition 3.4. A closed E-valued 1-form ω is *in standard form* if there is a neighborhood U_1 of the singular locus and neighborhoods $U_2, \ldots, U_n$ of each rank-2 cusps such that in U_i, ω equals a complex linear combination of ω_m and ω_l.

Note that the complex coefficients of ω_m and ω_l will generally be different for each U_i. The following lemma (Lemma 3.3 of [32]) shows that every cohomology class can put into standard form.

Lemma 3.5 (Hodgson-Kerckhoff [32]). *Given any closed E-valued 1-form ϕ, there is a cohomologous form ω_0 that is in standard form.*

Note that the form ω_0 is not unique since there is no control over ω_0 outside the union of the neighborhoods U_i. Nevertheless, we will use a form ω_0 that is in standard form to represent our cohomology class when we want to understand the infinitesimal deformation near the cone singularities and near the cusps.

3.7 Hodge Forms

In the previous section, we showed that for any cohomology class in $H^1(X; E)$, there is a closed 1-form representing that class that is in standard form in some neighborhood of Σ and some neighborhood of each rank-2 cusp. The following Hodge theorem for cone-manifolds shows that there is a harmonic representative as well. There is no reason to believe that this harmonic representative we are about to describe should be in standard form, although the Hodge theorem bounds the difference between the harmonic form and a standard one. First we define a Hodge form.

Definition 3.6. A 1-form $\omega \in \Omega^1(X; E)$ is a Hodge form if ω is closed, co-closed, and locally ω can be expressed as ds where s is the canonical lift of a divergence-free, harmonic vector field.

Before stating Theorem 4.3 of [18] which generalizes Theorem 2.7 of [32], we need to define what it means for a 1-form to be conformal at infinity. By Lemma 3.2 of [18], there is an isomorphism $\Pi_* : H^1(X; E) \to H^1(\partial_c X; E_\infty)$ where $\partial_c X$ is the conformal boundary of X and E_∞ is the bundle of germs of projective vector fields on $\partial_c X$. A cohomology class $\omega_\infty \in H^1(\partial_c X; E_\infty)$ is conformal if it can be expressed as ds_∞ where s_∞ is the canonical lift of an automorphic, conformal vector field on $\partial_c X$. A cohomology class $\omega \in H^1(X; E)$ is conformal at infinity if $\Pi_*(\omega)$ is conformal.

We will only be concerned with 1-forms on X that arise from one-parameter deformations of hyperbolic cone manifolds M_t whose conformal boundary is fixed throughout the deformation. These 1-forms will be conformal at infinity with respect to the definition given above.

Theorem 3.7 (Bromberg [18], Hodgson-Kerckhoff [32]). *Let M be a geometrically finite hyperbolic cone-manifold, and let ω_0 be an E-valued 1-form on $X = M - \Sigma$ that is conformal at infinity and in standard form in a neighborhood U of Σ. Then there exists a unique Hodge form ω such that the following holds:*
(1) ω is cohomologous to ω_0,
(2) there exists an L^2 section s of E such that $ds = \omega_0 - \omega$
(3) $\omega_0 - \omega$ has finite L^2 norm on the complement of U.

By Proposition 2.6 of [32], a Hodge form ω can be expressed as

$$\omega = \eta + i * D\eta$$

where η and $*D\eta$ are symmetric and traceless TX-valued 1-forms. Furthermore, $D^*\eta = 0$ and $D^*D\eta = -\eta$.

In the next section, we will define a one-parameter family of hyperbolic cone-manifolds M_t, and to each time t, we will associate an element of $H^1(X; E)$. The previous subsection provides a nice way of representing this cohomology class in a neighborhood of a cusp or in a neighborhood of the cone-singularity, and the Hodge theorem provides a nice way of representing this cohomology class on the rest of the manifold.

3.8 Complex Length

Let X be the interior of $N - \Sigma$ and $\rho : \pi_1(X) \to PSL(2, \mathbb{C})$ a representation with no elliptics. If $\gamma \in \pi_1(X)$, then the complex length of γ, denoted $\mathcal{L} = l + i\theta$ or

$\mathcal{L}(\rho(\gamma))$, is defined by the formula

$$tr^2(\rho(\gamma)) = 4\cosh^2\left(\frac{\mathcal{L}}{2}\right)$$

and the normalizations $l \geq 0$ and $\theta \in (-\pi, \pi]$. If $\rho(\gamma)$ is a loxodromic element, then l is the length of the geodesic representative of γ in X (equivalently, the translation length of $\rho(\gamma)$ along its axis in $\mathbb{H}^3$), and θ gives the amount $\rho(\gamma)$ twists along its axis. If $\rho(\gamma)$ is parabolic, then the complex length is zero.

Now if M_α is a cone-manifold homeomorphic to the interior of N with cone singularity Σ, and $U' \subset M_\alpha$ is a tubular neighborhood of the cone-singularity, set $U = U' \cap X$. The boundary of U has a well-defined meridian β. This is the homotopy class of a curve on ∂U that bounds a disk (with a cone-point) in M_α. When the cone angle is $\alpha \in (0, 2\pi)$, then the meridian will be sent to an elliptic element that rotates by α about its axis. In this case, we say the meridian has (purely imaginary) complex length $i\alpha$. In our situation, when $\alpha = 0$ we will have $\rho(\beta)$ be parabolic, but when $\alpha = 2\pi$, $\rho(\beta)$ will be the identity.

If $\alpha \in (0, 2\pi]$ and U is a metric collar neighborhood, then the torus ∂U inherits a Euclidean metric as a subset of M_α, so we can pick the shortest longitude λ on ∂U (by a longitude, we mean any curve that intersects the meridian once) and normalize the complex length of λ to be $l + i\theta$ for some $\theta \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right]$. Then any other longitude will have complex length $l + i\theta + im\alpha$ for some $m \in \mathbb{Z}$. The only time the choice of λ is not well-defined is when there are two shortest longitudes on ∂U in which case we pick one and assign it the complex length $l + i\frac{\alpha}{2}$ by convention. In this case, the other will have complex length $l - i\frac{\alpha}{2}$. When $\alpha = 0$, ∂U still inherits a Euclidean metric, but every curve on ∂U will be parabolic and the complex lengths will all vanish.

We are really interested in the complex length of Σ, but since every longitude of ∂U is homotopic to Σ in M_α, the complex length of Σ is only well-defined up to the addition of multiples of $i\alpha$. So in order to work with a well-defined complex number, we pick a shortest longitude λ and keep track of its complex length $l + i\theta$, $\theta \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right]$.

Now suppose we have a one-parameter family of cone-manifolds M_t defined for $t \in (t_1, t_2)$, parameterized by $t = \alpha^2$. Let $t_0 \in (t_1, t_2)$ and let λ be the shortest longitude with complex length $l(t_0) + i\theta(t_0)$ such that $\theta(t_0) \in \left(-\frac{\alpha_0}{2}, \frac{\alpha_0}{2}\right]$ (again, if there are two such longitudes, pick one). Using the meridian β and this longitude λ as a basis, we can define the complex length of any curve $p\beta + q\lambda$ on ∂U in M_{t_0} to be $p(i\alpha) + q(l(t_0) + i\theta(t_0))$. Then there is a unique continuous extension $\mathcal{L} : \pi_1(\partial U) \times (t_1, t_2) \to \mathbb{C}$, denoted $\mathcal{L}(p\beta + q\lambda, t) = p(i\alpha) + q(l(t) + i\theta(t))$, such that $\mathcal{L}(p\beta + q\lambda, t_0) = p(i\alpha) + q(l(t_0) + i\theta(t_0))$.

Remark. The subtlety here is that the shortest longitude at $t = t_0$ may not be shortest for all $t \in (t_1, t_2)$. If at some point $t_3 \in (t_1, t_2)$ we had $\mathcal{L}(\lambda, t_3) = l(t_3) - i\frac{\alpha}{2}$, then at t_3 there are two shortest longitudes, λ and $\beta + \lambda$. If we had defined the complex length to be the continuous extension of the complex length at $t = t_3$ then we might have obtained a different function.

Notation. We will use $l(p\beta + q\lambda, t)$ and $\theta(p\beta + q\lambda, t)$ to denote the real and imaginary parts of $\mathcal{L}(p\beta + q\lambda, t)$; when the curve is specified by context or we are making a statement about any curve on ∂U, we will use $\mathcal{L}(t) = l(t) + i\theta(t)$. We will also suppress the dependence of $\mathcal{L}(t)$ on the point t_0 because this choice will not affect the derivative $\frac{d\mathcal{L}}{dt}$. Moreover, in our applications, the imaginary part of $\mathcal{L}(\lambda, t)$ will remain bounded away $\pm\frac{\alpha}{2}$ (see Lemmas 4.10 and 4.11).

3.9 Complex Length and Standard Forms

Recall that in Section 3.6 we defined the forms ω_m and ω_l and what it means for a form ω_0 to be in standard form. Let M_t be a 1-parameter family of hyperbolic cone-manifolds defined for $t \in (t_1, t_2)$. Let $t_0 \in (t_1, t_2)$ and suppose that the cohomology class determined by the deformation at $t = t_0$ is $\omega \in H^1(X; E)$. This cohomology class determines $\frac{d}{dt}\rho_t(\gamma)\big|_{t=t_0}$ for any $\gamma \in \pi_1(X)$, but we only want to consider particular peripheral elements of $\pi_1(X)$.

There is a cohomologous form ω_0 that is in standard form in U_i. Recall U_1 is a neighborhood of the singular locus and $U_2, \ldots, U_n$ are neighborhoods of the rank-2 cusps of X. Let β be the meridian of ∂U_1. After choosing a longitude λ and defining the complex length $\mathcal{L}(p\beta + q\lambda, t)$ for any curve on ∂U_1 as above, one can obtain $\frac{d}{dt}\mathcal{L}(p\beta + q\lambda, t)\big|_{t=t_0}$ from $\frac{d}{dt}\rho_t(p\beta + q\lambda)\big|_{t=t_0}$. So for the particular forms ω_m and ω_l, we measure their effects on the complex length of curves on ∂U_1 instead of measuring their affects on the holonomy representation. This is done in Lemma 2.1 of [33], which we now state.

If $\omega_0|_{U_1} = \omega_m$ then for any curve on ∂U_1, the derivative of $\mathcal{L}$ with respect to t is given by

$$\frac{d\mathcal{L}}{dt}\bigg|_{t=t_0} = -2\mathcal{L}(t_0). \tag{3.3}$$

If $\omega_0|_{U_1} = \omega_l$ then for any curve on ∂U_1,

$$\frac{d\mathcal{L}}{dt}\bigg|_{t=t_0} = 2Re(\mathcal{L}(t_0)) = 2l(t_0). \tag{3.4}$$

Note that since this holds for *any* curve on ∂U_1 we ignore the dependence on the curve $p\beta + q\lambda$. Also, the same holds for the other neighborhoods U_i, but we will not use this. As a consequence, if $\omega_0|_{U_1} = z_m\omega_m + z_l\omega_l$ for $z_m, z_l \in \mathbb{C}$ then for any curve on ∂U_1,

$$\left.\frac{d\mathcal{L}}{dt}\right|_{t=t_0} = z_m\left(-2\mathcal{L}(t_0)\right) + z_l\left(2l(t_0)\right).$$

3.10 Complex Length and the Teichmüller Space of a Torus

Finally, we relate the Euclidean metric on ∂U_1 to the complex length $\mathcal{L}$ of Σ in $M_{2\pi}$. Here we are assuming U_1 is a radius R collar neighborhood of Σ so that ∂U_1 inherits a Euclidean metric, and that the cone angle about Σ is 2π (*i.e.*, the subscript on $M_{2\pi}$ refers to the cone angle, not the parameter $t = \alpha^2 = 2\pi$). Consider the quantity $\frac{2\pi i}{\mathcal{L}}$. This is the point in $\mathcal{T}(T^2)$ determined by $(\hat{\mathbb{C}} - Fix(\Gamma))/\Gamma$, where Γ is the infinite cyclic group of isometries generated by the isometry corresponding to Σ. Since the imaginary part of the complex length is only defined up to multiples of 2π, the point we get in $\mathcal{T}(T^2)$ (identified with the upper-half plane model of $\mathbb{H}^2$) is only defined up to the transformation $w \mapsto \frac{w}{w+1}$. Note that this Euclidean torus does not embed in $M_{2\pi}$, but there is a $\coth(R)$-quasiconformal map between ∂U_1 and $(\hat{\mathbb{C}} - Fix(\Gamma))/\Gamma$. Thus the distance between $\frac{2\pi i}{\mathcal{L}}$ and ∂U_1 in $\mathcal{T}(T^2)$ (again, with the appropriate choice of marking of ∂U_1) is bounded by a quantity that only depends on R. In particular, this bound goes to 0 as $R \to \infty$. Note that for any R, there is some l_0 such that if the length of Σ is less than l_0 then there exists an embedded neighborhood U_1 of radius R about Σ. See Section 6 of [60].

CHAPTER 4

The Drilling and Filling Theorems

We are now ready to apply the cone-manifold machinery developed by Hodgson-Kerckhoff [32, 33] and generalized by Bromberg [18, 19] that we outlined in the previous section to prove our version of the filling theorem (Theorem 1.2).

If M is a geometrically finite hyperbolic manifold and γ is a disjoint collection of simple closed geodesics in M, then Kojima showed, using an argument he attributes to Kerckhoff, that $M - \gamma$ admits a unique complete hyperbolic metric such that the natural inclusion of $(M - \gamma) \subset M$ extends to a conformal map between the conformal boundaries of $M - \gamma$ and M [44]. We call this process drilling (*i.e.*, finding a new metric on $M - \gamma$). If the curves in γ are sufficiently short, the Brock-Bromberg drilling theorem bounds the difference between the original metric on the complement of γ in M and the new complete metric on $M - \gamma$ [13]. That is, they show the metrics are close (in a biLipschitz sense) on the complement of a neighborhood of the drilled curves.

The filling theorem provides an inverse construction. A geometrically finite hyperbolic manifold with a rank-2 cusp is homeomorphic to the interior of a compact manifold with a torus boundary component. One can Dehn-fill this compactification along any boundary slope and attempt to hyperbolize the interior of the filled manifold. Under certain conditions on the boundary slope, the filling theorem provides a way of doing this hyperbolic Dehn-filling while preserving the conformal boundary components. Moreover, we obtain estimates on the complex length of the core curve of the filling torus in the new metric.

4.1 Main Results: Setup and Outline

To describe the drilling process more precisely, let M be a geometrically finite hyperbolic 3-manifold without rank-1 cusps and suppose $\gamma_1, \ldots, \gamma_n \subset M$ is a disjoint collection of simple closed geodesics. Define the $\cup\gamma_i$-drilling of M to be the unique complete hyperbolic metric on $M - \cup\gamma_i$ with the same conformal boundary as M (for existence, see Theorem 1.2.1 of [44]). We denote this new geometrically finite hyperbolic manifold by $\hat{M}$.

Although we can drill out these n curves without changing the conformal boundary components of each of the geometrically finite ends, we want more control over the change in the geometry of M. The drilling theorem of Brock and Bromberg (Theorem 6.2 of [13]) provides such control. Let ϵ_3 denote the Margulis constant in dimension 3, and for any $\epsilon_3 \geq \epsilon > 0$, let $\mathbb{T}_\epsilon(\gamma)$ (resp. $\mathbb{T}_\epsilon(T)$) denote the ϵ-Margulis tube about some short geodesic γ (resp. rank-2 cusp T).

Theorem 4.1 (Brock-Bromberg [13])**.** *Given any $J > 1$, $\epsilon_3 \geq \epsilon > 0$, there exists some $l_0 > 0$ such that the following holds: if M is a geometrically finite manifold with no rank-1 cusps, and $\gamma_1, \ldots, \gamma_n$ is a collection of geodesics in M with*

$$\sum_{i=1}^{n} l(\gamma_i) < l_0,$$

then there exists a J-biLipschitz diffeomorphism

$$\phi : \hat{M} - \cup_{i=1}^{n} \mathbb{T}_\epsilon(T_i) \to M - \cup_{i=1}^{n} \mathbb{T}_\epsilon(\gamma_i)$$

where $\hat{M}$ is the $\cup\gamma_i$-drilling of M, and T_i is the cusp corresponding to the drilling of γ_i.

Remark. Suppose T' is a rank-2 cusp in M. Then the drilling map ϕ sends T' to a cusp in $\hat{M}$. Let $T'(M)$ denote the cusp in M and let $T'(\hat{M})$ denote the same cusp in $\hat{M}$ so that we have a way of distinguishing the geometry of the cusp before and after filling. In their proof of the drilling theorem, Brock and Bromberg actually conclude that there exists a J-biLipschitz diffeomorphism ϕ such that ϕ is level-preserving on cusps in the following sense. For all $0 < \epsilon' \leq \epsilon$, $\phi(\partial\mathbb{T}_{\epsilon'}(T'(M))) = \partial\mathbb{T}_{\epsilon'}(T'(\hat{M}))$. See Theorem 6.12 of [13] (in particular, see Lemma 6.17 of [13] which was used to prove 6.12).

Now let $\hat{M}$ be a geometrically finite hyperbolic manifold with n rank-2 cusps. We want to describe a way of filling in these cusps to obtain a hyperbolic manifold M with the same conformal boundary but no rank-2 cusps. Although methods developed by Hodgson, Kerckhoff, and Purcell [68] can be used to fill multiple cusps simultaneously, this introduces some unnecessary complications. We will proceed by describing the filling theorem for one cusp, which up to renumbering we can assume is the first cusp. Then we derive the multiple cusp case by filling one cusp at a time (see Corollary 4.13).

Let $\hat{N}$ be a compact 3-manifold with interior homeomorphic to $\hat{M}$. On the first torus boundary component of $\hat{N}$ fix a slope β. Let N be the manifold obtained by Dehn-filling $\hat{N}$ along β. If possible, we hyperbolize the interior of N to obtain a hyperbolic manifold M with the same conformal boundary as $\hat{M}$ and one fewer cusp. If it exists, we call M the β-filling of $\hat{M}$. Let γ be the geodesic representative in M of the core curve of the solid torus used to Dehn-fill $\hat{N}$.

The following theorem gives sufficient conditions for the β-filling of $\hat{M}$ to exist, and when these conditions are satisfied, gives information about the complex length of the geodesic γ in M. However, before stating the theorem, we set up some notation.

Let $T \cong T^2 \times [0, \infty)$ be the rank-2 cusp of $\hat{M}$ we are trying to fill, and let β be the filling slope, henceforth called the meridian. That is, let β be any nontrivial simple closed curve on $T^2 \times \{0\}$. The torus $T^2 \times \{0\}$ inherits a flat metric, and after choosing β, this torus has a well-defined meridian.

Let μ be a geodesic on $T^2 \times \{0\}$ in the homotopy class of β and let m be the length of μ. Fix a point $x \in \mu$, and let ν be a geodesic ray perpendicular to μ at x. Let y denote the next point on ν where ν meets μ (after x). Orient μ so that if $\vec{\mu}$ and $\vec{\nu}$ denote the tangent vectors to μ and ν at x then $\vec{\mu} \times \vec{\nu}$ points into the cusp T. Define b to be the value in $\left(-\frac{m}{2}, \frac{m}{2}\right]$ such that $|b|$ is the distance between x and y, and after orienting μ as described, the sign of b gives the orientation of the shortest path beginning at x and ending at y realizing this distance. When y is exactly half-way around μ from x then there are two shortest paths. In this case, we choose the positively oriented one so $b = \frac{m}{2}$ instead of $-\frac{m}{2}$; consequently, b lies in the interval $\left(-\frac{m}{2}, \frac{m}{2}\right]$. See Figure 4.1. The value b is the twist associated to the flat structure on $T^2 \times \{0\}$ with meridian β.

As Hodgson and Kerckhoff explain on p. 386-388 of [33], the flat structure on $T^2 \times \{0\}$ can be reconstructed by taking a flat cylinder with circumference m, height $h := \text{length}(\nu)$ and identifying the boundary circles with a twist of b. In [33], they use the notation tw to denote the twist. Although we have defined b and m for the flat structure on $T^2 \times \{0\}$, the same could be done for any flat torus with a well-defined meridian.

Definition 4.2. The normalized length L of β is the length of μ divided by the square root of the area of $T^2 \times \{0\}$:

$$L = \frac{m}{\sqrt{Area(T^2 \times \{0\})}}.$$

Note that one could define L to be the length of any geodesic representative of β on any cross-section $T^2 \times \{*\}$ of the cusp T divided by the square root of the area of $T^2 \times \{*\}$. Thus L is a well-defined invariant associated to the meridian on the cusp T.

Definition 4.3. The normalized twist of the flat metric on $T^2 \times \{0\}$ with meridian β is the ratio $\frac{b}{m}$. We let $A^2 = \frac{m}{b}$ denote the reciprocal of the normalized twist.

Although this is an invariant associated to the flat structure on $T^2 \times \{0\}$ with meridian β, the normalized twist is independent of the cross section $T^2 \times \{0\}$. Thus, A^2 is really an invariant associated to slope β on the cusp T. Because the twist was defined to lie in the interval $b \in \left(-\frac{m}{2}, \frac{m}{2}\right]$, the normalized twist lies in the interval $\frac{b}{m} = \frac{1}{A^2} \in (-2, 2]$. Despite the "square" notation, the quantity A^2 could be negative, and we make no use of any complex number A in this paper. We use this notation to emphasize that A^2 is a counterpart to L^2 in the following sense.

Assume $A^2 \neq 2$ so there is a unique shortest longitude λ on $T^2 \times \{0\}$. (If $A^2 = 2$, then one could pick a shortest longitude, but we will be assuming $|A^2| \geq 3$ in the filling theorem so we will disregard this case.) If we conjugate $\pi_1(\hat{M})$ in $PSL(2, \mathbb{C})$ so that the isometry corresponding to λ is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and the isometry corresponding to the meridian β is $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ for some $w \in \mathbb{C}$ with $Im(w) > 0$, then

$$L = \frac{|w|}{\sqrt{2Im(w)}}.$$

The point w is really a point in the Teichmüller space of the torus, $\mathcal{T}(T^2)$. Had we chosen a different longitude to mark the torus, then the point w would transform by an iterate of the map $w \mapsto \frac{2w}{w+2}$. Since the quantity $\frac{|w|}{\sqrt{2Im(w)}}$ is invariant under the transformation $w \mapsto \frac{2w}{w+2}$, one could define L by $L = \frac{|w|}{\sqrt{2Im(w)}}$ independently of the choice of longitude. With this normalization, the square of the normalized length is given by

$$L^2 = \frac{|w|^2}{2Im(w)}.$$

We now claim that

$$A^2 = \frac{|w|^2}{2Re(w)}.$$

Note that $\frac{|w|^2}{2Re(w)} \neq \frac{\left|\frac{2w}{w+2}\right|^2}{2Re\left(\frac{2w}{w+2}\right)}$, so this claim depends on λ being the unique shortest longitude.

To prove this claim, let φ be the angle formed by $\vec{\lambda}$ and $-\vec{\mu}$. That is, $\varphi \leq \frac{\pi}{2}$ if $b \in \left[0, \frac{m}{2}\right]$ and $\varphi > \frac{\pi}{2}$ if $b \in \left(-\frac{m}{2}, 0\right)$. Then $\cos(\varphi) = \frac{b}{\text{length}(\lambda)}$ since λ is the shortest longitude (see Figure 4.1).

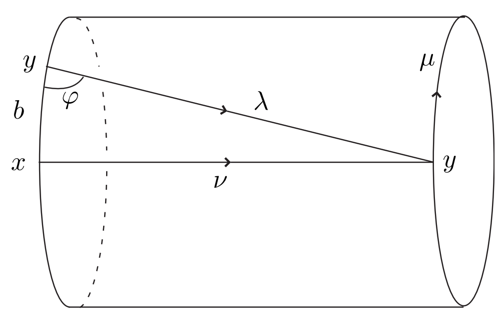

Figure 4.1: The torus $T^2 \times \{0\}$ is obtained by taking a cylinder $\mu \times \nu$ and identifying the boundary circles with a twist of b. In this picture, b is positive.

After conjugating so that $\lambda \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\beta \mapsto \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$, the Euclidean length of the longitude is 2 so $\cos(\varphi) = \frac{b}{2}$. Also, φ is the angle formed by the segment joining 0 to w with the real line at the origin, so $\cos(\varphi) = \frac{Re(w)}{|w|}$. Hence

$$A^2 = \frac{m}{b} = \frac{|w|}{b} = \frac{|w|}{2\cos(\varphi)} = \frac{|w|^2}{2Re(w)}.$$

See Figure 4.2.

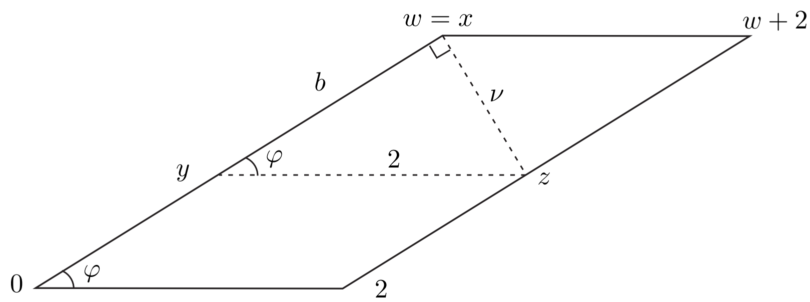

Figure 4.2: A fundamental domain for the torus that is the quotient of $\mathbb{C}$ by the $\mathbb{Z}^2$ subgroup generated by $z \mapsto z + 2$ and $z \mapsto z + w$. The meridian, μ, is the path joining w to 0, oriented so that the path joining w to 0 is positively oriented. If we identify $\mathbb{C}$ with a horosphere in $\mathbb{H}^3$ tangent to the point at ∞, then this is the orientation such that $\vec{\mu} \times \vec{\nu}$ points upward.

Remark. Unlike L^2, if we choose a different longitude that is not shortest on $T^2 \times \{0\}$ to correspond to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then the quantity $\frac{|w|^2}{2Re(w)}$ will no longer be equal to A^2. One could remedy this by saying that A^2 is the value of $\frac{|w|^2}{2Re(w)}$ such that $\frac{|w|^2}{2|Re(w)|}$ is largest among all possible choices of λ (*i.e.* iterates of $w \mapsto \frac{2w}{w+2}$). When $\frac{|w|^2}{2|Re(w)|} = 2$, there are two shortest longitudes, one of which will give $\frac{|w|^2}{2Re(w)} = 2$ and the other will give $\frac{|w|^2}{2Re(w)} = -2$. We take $A^2 = 2$ in this case, although this situation will not arise in this paper.

Conversely, suppose we are given a hyperbolic manifold with a rank-2 cusp T that has already been normalized so that $\pi_1(T)$ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ for some $w \in \mathcal{T}(T^2)$. Moreover, suppose one specifies the meridian corresponds to $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. Then $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ corresponds to some longitude, but not necessarily the shortest one. However, if $\frac{|w|^2}{2|Re(w)|} > 2$, then we claim that the longitude $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is the unique shortest longitude. Any other longitude will have length $|2 + nw|$ for some $n \neq 0$, and whenever $|w|^2 > 4|Re(w)|$, one can check that when $n \neq 0$

$$|2 + nw|^2 = 4 + 4nRe(w) + n^2|w|^2 > 4 + 4nRe(w) + 4n^2|Re(w)| \geq 4.$$

Thus, if we are given a hyperbolic manifold with a cusp defined this way, and $\frac{|w|^2}{2|Re(w)|} > 2$, then $\frac{|w|^2}{2Re(w)} = \frac{b}{m} = A^2$. In other words, even if we do not choose the normalization of the cusp so that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ corresponds to the unique shortest longitude, it will automatically be thus if $\frac{|w|^2}{2|Re(w)|} > 2$. Note that $\frac{|w|^2}{2|Re(w)|} \geq \frac{|w|}{2}$. So whenever we have that $|w| > 4$, $\frac{|w|^2}{2|Re(w)|} > 2$. We make these comments since we will be applying the filling theorem in situations where the rank-2 cusp has been normalized this way and $|w|$ is large.

Now that we have defined the quantities L^2 and A^2, we are ready to state the filling theorem.

Theorem 1.2. *Let $J > 1$ and $\epsilon_3 \geq \epsilon > 0$. There is some $K \geq 8(2\pi)^2$ such that the following holds: suppose $\hat{M}$ is a geometrically finite hyperbolic 3-manifold with no rank-1 cusps, T is a rank-2 cusp in $\hat{M}$, and β is a slope on T such that the normalized length of β is at least K (i.e., $L^2 \geq K^2$), then*

(i) the β-filling of $\hat{M}$, which we call M, exists;

(ii) the real part of the complex length $\mathcal{L} = l + i\theta$ of the core curve of the filling torus γ in M is approximately $\frac{2\pi}{L^2}$ with error bounded by

$$\left| l - \frac{2\pi}{L^2} \right| \leq \frac{8(2\pi)^3}{L^4 - 16(2\pi)^4};$$

(iii) in particular, the length of γ is bounded above by $\frac{2\pi}{L^2-4(2\pi)^2}$;

(iv) there exists a J-biLipschitz diffeomorphism

$$\phi : \hat{M} - \mathbb{T}_\epsilon(T) \to M - \mathbb{T}_\epsilon(\gamma).$$

(v) If, in addition to $L^2 \geq K^2$, we have $|A^2| \geq 3$, then the imaginary part of the complex length $\mathcal{L} = l + i\theta$ of the core curve of the filling torus γ in M (chosen so $\theta \in (-\pi, \pi]$) is approximately $\frac{2\pi}{A^2}$ with error bounded by

$$\left|\theta - \frac{2\pi}{A^2}\right| \leq \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2}.$$

Outline of the Proof and Prior Results

As the proof is rather technical and spans multiple sections, we begin by outlining the strategy. Many of the results of the filling theorem follow directly from results of Brock, Bromberg, Hodgson, and Kerckhoff. Before beginning our proof, we clearly delineate these contributions.

Part (i) was shown by Bromberg in [19]. This was a generalization of the work of Hodgson and Kerckhoff in [33] to geometrically finite manifolds. We intend to revisit these arguments in order to reprove (i) and draw the conclusions stated in (ii) and (v). Part (iii) follows from (ii), although both are proven in Lemma 4.7. Part (iv) follows from parts (i), (iii) and the drilling theorem. That is, once we show that the filling exists and γ is sufficiently short in M, we can then apply the drilling theorem to M to recover $\hat{M}$ and get the J-biLipschitz map from the drilling theorem.

We view the geometrically finite manifold $\hat{M}$ with n cusps as a geometrically finite hyperbolic cone-manifold with $n - 1$ rank-2 cusps and cone singularity $\Sigma = \gamma$

with cone angle $\alpha = 0$ about Σ. Part (i) amounts to showing that we can increase the cone angle from 0 to 2π. We will use the parameterization $t = \alpha^2$. While we increase the cone angle, we will derive estimates on the derivative of the complex length of γ that will allow us to obtain part (ii). We start by using Theorem 3.3 to show that the 1-parameter family is defined in some interval $[0, t')$. Then at any t for which M_t is defined, we prove Proposition 4.4 which estimates the derivative at t of the complex length of any peripheral curve, where the estimates depend on the radius of an embedded tube about the cone-singularity. Lemma 4.5 shows that for all $t \in [0, t')$, the radius remains bounded below; therefore, we can extend the 1-parameter family to $M_{t'}$ using Theorem 3.2 from the previous section. Provided L^2 is sufficiently large, we can then continue the deformation to $t = (2\pi)^2$. This is shown in Lemma 4.6. Then we return to the estimates in Proposition 4.4 which can be applied for all $t \leq (2\pi)^2$. Lemma 4.7 gives the estimate on the real part of the length of γ we claimed in part (ii). Technically, we work with the complex length of any longitude on the boundary of a neighborhood U_1 of γ. Any longitude on ∂U_1 is homotopic to γ in M_t; choosing a longitude enables us to define the complex length as in Section 3.8.

To get the estimate on the imaginary part of the complex length of γ in M that we claimed in part (v), we again work with a longitude on ∂U_1 homotopic to γ. Actually, it is more convenient to work with a metric collar neighborhood V_t of γ in M_t. The torus ∂V_t inherits a flat metric and we can define the twist $b(t)$ and the normalized twist $\frac{b(t)}{m(t)}$ associated to ∂V_t in the same way that we defined the twist b and normalized twist $\frac{b}{m}$ for ∂V_0 (the boundary of the cusp we are filling in $\hat{M}$). Since $|A^2| > 2$, there is a unique shortest longitude λ on ∂V_0 and some δ such that for all $t \in [0, \delta)$, the longitude λ continues to be the unique shortest longitude on

∂V_t. Recall from Section 3.8 that for any $t \in [0, \delta)$, we can define the imaginary part of the complex length of λ, denoted $\theta(\lambda, t)$. Since λ is the shortest longitude, we will have $\theta(\lambda, t) \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right)$ for all $t \in (0, \delta)$. Then for this longitude, we will have $\frac{\theta(\lambda,t)}{\alpha} = \frac{b(t)}{m(t)}$ for all $t \in (0, \delta)$. We show in Lemmas 4.9 and 4.10 that this implies $\lim_{t\to 0} \frac{\theta(\lambda,t)}{\alpha} = \lim_{t\to 0} \frac{b(t)}{m(t)} = \frac{b}{m} = \frac{1}{A^2}$.

In Lemma 4.8, we define the quantity $v(\lambda', t) = \frac{\theta(\lambda',t)}{\alpha}$ for any longitude λ' and bound $\frac{dv}{d\alpha}$ at any fixed time t. This bound follows from the bounds in Proposition 4.4. Using $\lim_{t\to 0} v(\lambda, t) = \frac{1}{A^2}$ and the bound on the derivative of v from Lemma 4.8, we estimate $\theta(\lambda, (2\pi)^2)$ in Lemma 4.11. From this, we deduce part (v) of Theorem 1.2 when $|A^2| \geq 3$.

4.2 Existence of the Deformation and Derivative Estimates

We now begin the proof of Theorem 1.2. In this section, our goal is to show that the 1-parameter family M_t, defined by setting $M_0 = \hat{M}$ and increasing the cone angle with with the parameterization $t = \alpha^2$, can be defined for all $t \in [0, (2\pi)^2]$. In Lemma 4.6, we show that if $L^2 \geq 8(2\pi)^2$, then we can do this.

Proof. Let $M_0 = \hat{M}$, and define a 1-parameter family of hyperbolic cone-manifolds M_t by increasing the cone angle about γ according to the parameterization $t = \alpha^2$ and keeping the conformal boundary fixed. This is well defined for some interval $[0, t')$ by the local rigidity results of Bromberg (see Theorem 5.8 of [18]) which generalize the work of Hodgson and Kerckhoff [32] on finite volume manifolds. See also Theorem 3.3 in the previous chapter.

At each time t for which the 1-parameter deformation has been defined we can let $X = M_t - \Sigma$ and represent the infinitesimal deformation (the infinitesimal change in the hyperbolic metric on X) by a 1-form in $H^1(X; E)$ that is conformal at infinity.

We can represent this cohomology class in two ways. First, by Lemma 3.5, we can choose ω_0 to be in standard form in a neighborhood U_1 of the singular locus and in each of the rank-2 cusps U_i $(i = 2, \ldots, n)$ of X. By our choice of parameterization $t = \alpha^2$, in a neighborhood U_1 we must have

$$\omega_0 = \frac{-1}{4\alpha^2}\omega_m + (x + iy)\omega_l \tag{4.1}$$

for some constants x and y since equations (3.3) and (3.4) tell us that the derivative of the complex length of the meridian is determined by the coefficient of ω_m. Note that X and ω depend on t, and thus x and y also depend on t. For now, we will be using these forms at a particular time t in order to compute a derivative. Therefore, we will suppress this dependence on t until after the proof of Proposition 4.4.

Theorem 4.3 of [18] which generalizes Theorem 2.7 of [32] says that we can find a Hodge form ω in the same cohomology class as ω_0 such that $\omega_c := \omega - \omega_0$ has finite L^2-norm outside U_1 (see Theorem 3.7 in the previous chapter). Lemma 3.4 of [32] shows that ω_c does not effect the holonomy of any of the peripheral elements. By Proposition 2.6 of [32], we can write ω as

$$\omega = \eta + i * D\eta$$

such that $D^*\eta = D * D\eta + \eta = 0$, and both η and $D\eta$ are symmetric and traceless.

Since $\omega = \omega_0 + \omega_c$, we can decompose the real part of ω as $\eta = \eta_0 + \eta_c$ where η_c does not effect the holonomy of the peripheral elements.

Proposition 4.4. *Suppose the* 1*-parameter family of cone manifolds* M_t *has been defined for* $t \in [0, t')$*, and suppose there is an embedded tube* U_1 *of radius* R *about* γ *in* M_t*. If* $\mathcal{L} = l + i\theta$ *denotes the complex length of any curve on* ∂U_1*, then the* *derivative of $\mathcal{L}$ is given by*

$$\frac{d\mathcal{L}}{dt} = \frac{-1}{4\alpha^2}(-2\mathcal{L}) + (x+iy)(2l)$$

and therefore

$$\frac{dl}{d\alpha} = \frac{l}{\alpha}(1+4\alpha^2 x) \quad \textit{and} \quad \frac{d\theta}{d\alpha} = \frac{\theta}{\alpha} + 4\alpha y l \tag{4.2}$$

where x and y satisfy

$$\frac{-1}{\sinh^2(R)}\left(\frac{2\sinh^2(R)+1}{2\sinh^2(R)+3}\right) \leq 4\alpha^2 x \leq \frac{1}{\sinh^2(R)} \tag{4.3}$$

and

$$|4\alpha^2 y| \leq \frac{2}{\sinh^2(R)} \frac{\cosh^2(R)}{(2\cosh^2(R)+1)}. \tag{4.4}$$

Proof. Recall $\omega = \omega_0 + \omega_c$ and ω_0, being in standard form, was equal to a complex linear combination of ω_m and ω_l in U_1 which we determined in (4.1). It follows immediately from the effects of ω_m and ω_l (see eq.(3.3) and (3.4)) that if $\mathcal{L}$ is the complex length of any curve in U_1

$$\frac{d\mathcal{L}}{dt} = \frac{-1}{4\alpha^2}(-2\mathcal{L}) + (x+iy)(2l).$$

From this, the real part gives us $\frac{dl}{dt}$ and the imaginary part gives us $\frac{d\theta}{dt}$. We now use the parameterization $t = \alpha^2$ and the chain rule to obtain the formulas for $\frac{dl}{d\alpha}$ and $\frac{d\theta}{d\alpha}$ given in (4.2).

Now we want to derive the bounds on x and y. We can find neighborhoods U_i of the ith cusp in X and a smoothly embedded convex surface S cutting off the geometrically finite ends of X such that the U_i and S are all pairwise disjoint. Note that such a surface exists by Theorem 4.3 of [18] and may have multiple components (one for each end). Recall that we are using U_1 to denote a neighborhood of the singular locus and U_i, $i \geq 2$, to denote the rank-2 cusps of X.

By Lemma 2.3 of [33], we have that for any compact submanifold $N \subset X$ with ∂N oriented with an inward pointing normal

$$\int_N \|\omega\|^2 = \int_{\partial N} *D\eta \wedge \eta.$$

Letting N be the complement of the union $\cup_{i=1}^n U_i$ and the ends cut off by S, we can decompose the boundary integral:

$$\int_N \|\omega\|^2 - \int_S *D\eta \wedge \eta = \sum_{i=1}^n \int_{\partial U_i} *D\eta \wedge \eta$$

Let $b_i(\alpha, \beta) = \int_{\partial U_i} *D\alpha \wedge \beta$.

Lemma 2.5 of [33] says that $b_i(\eta, \eta) = b_i(\eta_0, \eta_0) + b_i(\eta_c, \eta_c)$ and Lemma 2.6 of [33] says that $b_i(\eta_c, \eta_c)$ is non-positive. So we have

$$\int_N \|\omega\|^2 - \int_S *D\eta \wedge \eta \leq \sum_{i=1}^n b_i(\eta_0, \eta_0).$$

Since this holds for any compact submanifold N, we can apply this to a family of compact submanifolds N_T obtained by shrinking the neighborhood of the geometrically finite ends (but keeping the torus boundary components of ∂N_T the same as before). That is, let S_T be the surface obtained by taking a parallel copy of S a distance T further out the ends. When we do this, $\int_{S_T} *D\eta \wedge \eta \to 0$ as $T \to \infty$, while the other term on the left, $\int_{N_T} \|\omega\|^2$, can only get larger (Theorem 4.6 of [18]). So we must have

$$\sum_{i=1}^n b_i(\eta_0, \eta_0) \geq 0$$

with equality if and only if the deformation is trivial (see eq. (16) of [68]).

We also note that Lemmas 3.1 and 3.2 in [68] imply that for $i \geq 2$,

$$-b_i(\eta_0, \eta_0) = 2|\zeta_i'(t)|^2 Area(\partial U_i)$$

where $\zeta_i(t)$ is the path in the Teichmüller space (using the Teichmüller metric) of ∂U_i throughout the deformation. Note that the marking of ∂U_i is irrelevant and we can essentially work in moduli space since we only care about $|\zeta_i'(t)|^2$ and $Area(\partial U_i)$. In particular $b_i(\eta_0, \eta_0) \leq 0$ for all $i \geq 2$. It follows that $b_1(\eta_0, \eta_0) \geq 0$. We now are in a position to follow the calculations on p. 382-384 of [33] identically.

Remark. The only difference between our work and Section 2 of [33] is that we have to deal with geometrically finite ends and rank-2 cusps. Using [33] directly, we only know that $\sum_{i=1}^n b_j(\eta_0, \eta_0) \geq 0$. If the singular locus was disconnected and we were varying cone angles around multiple curves we would have to be careful about the rates at which the cone angles increased (see Section 4 of Purcell [68]). However, because we are only varying a single cone angle, we can conclude that the $b_1(\eta_0, \eta_0) \geq 0 \geq \sum_{i=2}^n b_i(\eta_0, \eta_0)$.

In order to bound x and y, we begin by computing $b_1(\eta_0, \eta_0)$ in terms of x and y and some constants that only depend on R and α (within this proposition, R and α are fixed so we refer to a_R, b_R, c_R as constants). This is done on p. 382-383 of [33].

$$\frac{b_1(\eta_0, \eta_0)}{area(\partial V)} = a_R(x^2 + y^2) + b_R x + c_R$$

where

$$\begin{aligned}
a_R &= -\tanh(R)\frac{2\cosh^2(R) + 1}{\cosh^2(R)} \\
b_R &= \frac{\tanh(R)}{2\alpha^2\cosh^2(R)\sinh^2(R)} \\
c_R &= \frac{\tanh(R) + \tanh^3(R)}{16\alpha^4\sinh^4(R)}.
\end{aligned}$$

Using the fact that $b_1(\eta_0, \eta_0) \geq 0$, we get

$$a_R(x^2 + y^2) + b_R x + c_R = \frac{b_1(\eta_0, \eta_0)}{area(\partial V)} \geq 0$$
$$a_R \left[\left(x + \frac{b_R}{2a_R} \right)^2 + y^2 \right] + \frac{4a_R c_R - b_R^2}{4a_R} \geq 0$$
$$\left(x + \frac{b_R}{2a_R} \right)^2 + y^2 \leq \frac{b_R^2 - 4a_R c_R}{4a_R^2}.$$

Note that the last inequality is reversed since a_R is negative. Thus plugging in a_R, b_R, c_R we get

$$\left(x + \frac{b_R}{2a_R} \right)^2 + y^2 \leq \frac{1}{4\alpha^4 \sinh^4(R)} \frac{\cosh^4(R)}{(2\cosh^2(R) + 1)^2}.$$

Remark. When substituting a_R, b_R, c_R to evaluate $\frac{b_R^2 - 4a_R c_R}{4a_R^2}$, we recommend first finding $b_R^2 - 4a_R c_R = \frac{\tanh^2(R)}{\alpha^4 \sinh^4(R)}$ which makes use of the identity $\cosh^2(R) - \sinh^2(R) = 1$.

Since both of the terms $\left(x^2 + \frac{b_R}{2a_R} \right)^2$ and y^2 are positive, we get the inequalities

$$\left(x - \frac{1}{(4\alpha^2 \sinh^2(R))(2\cosh^2(R) + 1)} \right)^2 \leq \left(\frac{\cosh^2(R)}{(2\alpha^2 \sinh^2(R))(2\cosh^2(R) + 1)} \right)^2$$

and

$$y^2 \leq \left(\frac{\cosh^2(R)}{(2\alpha^2 \sinh^2(R))(2\cosh^2(R) + 1)} \right)^2.$$

The inequalities (4.3) and (4.4) follow immediately. This completes the proof which gives us a generalized version of Theorem 2.7 in [33] for geometrically finite manifolds with rank-2 cusps. Moreover, we obtain an additional bound on y which will be useful later. $\square$

Because the hypotheses in Proposition 4.4 require the existence of an embedded tube U_1 of radius R about γ, we would like to show there is some interval on which there is a lower bound to the size of such a tube. Then we can integrate the differential inequalities over that interval. From now on we will use R_t to denote the maximum radius of an embedded tubular neighborhood about γ in M_t. We want to show that R_t is bounded below by some positive constant for all $t \in [0, t')$. Although it is not optimal, it will be convenient to show $R_t \geq \sinh^{-1}(\sqrt{2})$. In particular, this will allow us to invoke Theorem 1.2 of [19] in the proof of Lemma 4.6. (See the comments preceding Theorem 3.5 on p. 796 of [19].) Also, we will let V_t denote the R_t-neighborhood of Σ in M_t. Note that this replaces the neighborhood U_1 we were using earlier because we are now interested in the parameter t and no longer care about the other neighborhoods U_i, $i \geq 2$.

Lemma 4.5. *Suppose M_t is defined for some interval $[0, t') \subset [0, (2\pi)^2]$, and let R_t denote the maximal radius such that if V_t is an R_t-neighborhood of Σ in M_t then V_t is embedded. Suppose $L^2 \geq 8(2\pi)^2$ where L is the normalized length of the meridian of ∂V_0. Then $R_t > \sinh^{-1}(\sqrt{2})$ for all $t \in [0, t')$.*

Proof. When $t = 0$, M_0 has a rank-2 cusp. We can interpret V_0 as this rank-2 cusp and $R_0 = \infty$. As we vary the metric, R_t varies continuously. Suppose there was some first time $t'' < t'$ such that $R_{t''} = \sinh^{-1}(\sqrt{2})$. Let $l(t)$ denote the length of γ in M_t. This is the same as the length of any curve on ∂V_t homotopic to γ so we can apply the bounds on $\frac{dl}{dt}$ in Proposition 4.4. We will find a contradiction by showing that $R_{t''}$ is bounded below by a function of $l(t'')$ and estimate $l(t'')$ using L^2.

The area, A_t, of ∂V_t satisfies

$$A_t \geq 1.6978 \frac{\sinh^2(R_t)}{\cosh(2R_t)}$$

by Theorem 4.4 of [33] (see also Proposition 3.4 of [19] in the geometrically finite setting). The value 1.6978 is an approximation for $2\sqrt{6}\sinh^{-1}\left(\frac{1}{2\sqrt{2}}\right)$ but we won't need this precision. Define

$$h(r) = 1.6978\frac{\tanh(r)}{\cosh(2r)}.$$

Remark. In Section 5 of [33], Hodgson and Kerckhoff define $h(r) = 3.3957\frac{\tanh(r)}{\cosh(2r)}$. Our definition differs by a factor of 2 since we are allowing our manifold to have multiple cusps and geometrically finite ends (see Theorem 4.4 of [33]). This also allows us to directly apply Propositions 3.2 and 3.4 in [19].

Since $A_t = \alpha l(t)\sinh(R_t)\cosh(R_t)$ (see p. 403 of [33]), the maximal radius R_t satisfies

$$\alpha l(t) \geq h(R_t).$$

In Lemma 5.2 of [33], Hodgson and Kerckhoff show that h has a unique maximum, $h_{max} \approx 0.5098$, when $r \approx 0.531$ and is decreasing for all $r \geq 0.531$. For any $0 \leq a \leq h_{max}$ we can define an inverse function $h^{-1}(a)$ to be the value of r such that $r \geq 0.531$ and $h(r) = a$. One can easily see from the definition of $h(r)$ that $\lim_{r\to\infty} h(r) = 0$, so we interpret $h^{-1}(0) = \infty$.

Then we have

$$R_t \geq h^{-1}(\alpha l(t))$$

whenever $\alpha l(t) \leq h_{max}$ and $R_t \geq 0.531$. We are assuming $R_t \geq \sinh^{-1}(\sqrt{2}) \approx 1.4622$ for all $0 \leq t \leq t''$ so the condition that $R_t \geq 0.531$ is immediately satisfied for all t in this interval.

If $\alpha l(t) \leq h_{max}$, set $\rho(t) = h^{-1}(\alpha l(t))$ which is clearly bounded above by R_t. Note that α and $l(t)$ both start at zero when $t = 0$ so the condition that $\alpha l(t) \leq h_{max}$ holds in some interval around $t = 0$.

Substituting ρ for R_t in the inequality (4.3) gives us

$$\frac{-1}{\sinh^2(\rho)}\left(\frac{2\sinh^2(\rho)+1}{2\sinh^2(\rho)+3}\right) \leq 4\alpha^2 x \leq \frac{1}{\sinh^2(\rho)}.$$

One needs to check that the function on the left is increasing for $R > 0$ and the function on the right is decreasing, but this is done in Proposition 5.5 of [33]. Now set

$$u(t) = \frac{\alpha}{l(t)}.$$

Differentiating with respect to t, we find that

$$\begin{aligned}
\frac{du}{dt} &= \frac{l\frac{d\alpha}{dt} - \alpha\frac{dl}{dt}}{l^2} \\
&= \frac{\frac{l}{2\alpha} - \alpha\frac{dl}{dt}}{l^2} \\
&= \frac{1}{2\alpha l} - \frac{\alpha}{l^2}\frac{dl}{dt} \\
&= \frac{1}{2\alpha l} - \frac{\alpha}{l^2}\frac{l}{2\alpha^2}(1+4\alpha^2 x) \\
&= \frac{-1}{2\alpha l}(4\alpha^2 x).
\end{aligned}$$

Now substituting $z = \tanh(\rho)$, we claim that

$$-\frac{1+z^2}{3.3956(z^3)} \leq \frac{du}{dt} \leq \frac{(1+z^2)^2}{3.3956(z^3)(3-z^2)}. \tag{4.5}$$

First observe that $\sinh^2(\rho) = \frac{z^2}{1-z^2}$, and thus

$$\alpha l = h(\rho) = 1.6978\frac{\tanh(\rho)}{\cosh(2\rho)} = 1.6978\tanh(\rho)\frac{1}{\cosh^2(\rho)+\sinh^2(\rho)} = 1.6978z\frac{1-z^2}{1+z^2}.$$

Again using $\sinh^2(\rho) = \frac{z^2}{1-z^2}$, we bound $4\alpha^2 x$ by

$$-\left(\frac{(1-z^2)(1+z^2)}{z^2(3-z^2)}\right) \leq 4\alpha^2 x \leq \frac{1-z^2}{z^2}.$$

The inequality in (4.5) follows from the identity $\frac{du}{dt} = \frac{1}{2\alpha l}(-4\alpha^2 x)$ by rewriting $\frac{1}{2\alpha l} = \frac{1+z^2}{3.3957(z)(1-z^2)}$ and from the bounds on $-4\alpha^2 x$.

As long as $\alpha l \leq h_{max}$ we have $0.531 \leq \rho = h^{-1}(\alpha l)$ and therefore $0.48 \leq z \leq 1$. Since $\frac{1+z^2}{3.3957(z^3)}$ and $\frac{(1+z^2)^2}{3.3957(z^3)(3-z^2)}$ are both decreasing functions of z over this interval, we can replace z with 0.48 to obtain the somewhat liberal bound

$$\left|\frac{du}{dt}\right| \leq 4. \tag{4.6}$$

Eq. (37) of [33] shows that $\lim_{\alpha \to 0} u = L^2$ which implies that as long as $\alpha l(t) \leq h_{max}$ and $0 \leq t \leq t''$ we have

$$\left|\frac{\alpha}{l(t)} - L^2\right| \leq 4t.$$

Since $L^2 \geq 8(2\pi)^2$, we have that $L^2 \pm 4t$ is positive for any $t \leq (2\pi)^2$, so

$$\frac{\alpha}{L^2 + 4t} \leq l(t) \leq \frac{\alpha}{L^2 - 4t}. \tag{4.7}$$

Multiplying by α and substituting $t = \alpha^2$ we get

$$\alpha l(t) \leq \frac{t}{L^2 - 4t}. \tag{4.8}$$

Since $L^2 \geq 8(2\pi)^2$ we have $L^2 - 4(2\pi)^2 \geq 4(2\pi)^2$. Thus

$$\frac{(2\pi)^2}{L^2 - 4(2\pi)^2} \leq \frac{1}{4} < h_{max}.$$

This implies that for any $0 \leq t \leq (2\pi)^2$,

$$\frac{t}{L^2 - 4t} \leq \frac{(2\pi)^2}{L^2 - 4(2\pi)^2} < h_{max}$$

which in particular implies that $\alpha l(t) < h_{max}$ for all $t \in [0, t'']$. It also follows that

$$R_{t''} \geq h^{-1}(\alpha l(t'')) \geq h^{-1}\left(\frac{t''}{L^2 - 4t''}\right) \geq h^{-1}\left(\frac{(2\pi)^2}{L^2 - 4(2\pi)^2}\right) \geq h^{-1}\left(\frac{1}{4}\right).$$

Direct calculation shows that $\frac{1}{4} < h(\sinh^{-1}(\sqrt{2})) \approx 0.27725$. Thus we have

$$R_{t''} > \sinh^{-1}(\sqrt{2}).$$

This contradicts that $R_{t''} = \sinh^{-1}(\sqrt{2})$ for any time $t'' < t'$, and so we have $R_t > \sinh^{-1}(\sqrt{2})$ for all $0 \leq t < t'$. $\square$

We now extend the 1-parameter family M_t to be defined for all $t \in [0,(2\pi)^2]$. The following lemma proves part (i) of the filling theorem. See also Theorem 1.2 of [19].

Lemma 4.6. *If $L^2 \geq 8(2\pi)^2$, then the 1-parameter family is defined for all $t \in [0,(2\pi)^2]$.*

Proof. The previous lemma shows that for any $t' \in (0,(2\pi)^2]$, if M_t is defined for $t \in [0,t')$, we have $R_t > \sinh^{-1}(\sqrt{2})$ for all $t \in [0,t')$. Then by Theorem 1.2 of [19] (see also Theorem 3.12 and 5.4 in [33]) the 1-parameter family can be extended to $M_{t'}$ using a geometric limit argument (see also Theorem 3.2 in the previous section). So the maximal subinterval of $[0,(2\pi)^2]$ containing 0 for which M_t is defined is closed.

By Theorem 3.3 (see Theorem 1.1 of [19]), this maximal subinterval for which M_t is defined is open. This implies M_t is defined for all $t \in [0,(2\pi)^2]$. $\square$

4.3 Complex Length Estimates

Now that we have defined the 1-parameter family for all $t \in [0,(2\pi)^2]$, we can integrate the estimates we found for $\frac{dl}{d\alpha}$ and $\frac{d\theta}{d\alpha}$ in Proposition 4.4. This will allow us to estimate the complex length of any longitude on ∂V_t at any t. When $t = (2\pi)^2$, this produces estimates on the complex length of γ in M.

First we consider the real part of the complex length of γ. As in the proof of Lemma 4.5, we consider $u(t) = \frac{\alpha}{l(t)}$, which approaches L^2 as $t \to 0$. We can integrate the bounds on $\frac{du}{dt}$ in (4.6) to estimate the length of γ in M. In other words, the inequalities in (4.7) hold for all $t \in [0,(2\pi)^2]$. Thus we have shown

Lemma 4.7. *If $L^2 \geq 8(2\pi)^2$, the length of γ in M is given by $l((2\pi)^2)$ which satisfies*

$$\frac{2\pi}{L^2 + 4(2\pi)^2} \leq l((2\pi)^2) \leq \frac{2\pi}{L^2 - 4(2\pi)^2}.$$

This immediately implies part (iii) of the filling theorem. It also implies part (ii) since

$$\left| l((2\pi)^2) - \frac{2\pi}{L^2} \right| \leq \frac{2\pi}{L^2 - 4(2\pi)^2} - \frac{2\pi}{L^2 + 4(2\pi)^2} = \frac{8(2\pi)^3}{L^4 - (16)(2\pi)^4}.$$

Next we consider the imaginary part of the complex length of γ. Again we consider any longitude on ∂V_t. Recall from Section 3.8 that for a fixed t we can choose a shortest longitude on ∂V_t and thus identify the imaginary part of the complex length of any longitude on ∂V_t with a real number as opposed to modulo α. We begin by using the bounds from Proposition 4.4 to bound the change in $\frac{\theta(t)}{\alpha}$ for any longitude on ∂V_t.

Lemma 4.8. *Suppose $L^2 \geq 8(2\pi)^2$. For any fixed t such that $0 < t \leq (2\pi)^2$ and any longitude on ∂V_t, let $\mathcal{L} = l + i\theta$ denote the complex length of that longitude. Define*

$$v = \frac{\theta}{\alpha}.$$

Then

$$\left| \frac{dv}{d\alpha} \right| \leq \frac{5(2\pi)}{(L^2 - 4(2\pi)^2)^2}.$$

Remark. Because we are considering the derivative at a fixed time t, we use $l = l(t)$ to denote the length at time t and $\theta = \theta(t)$. Also note that the role of v is similar to that of the reciprocal of u, rather than u itself.

Proof. Because $L^2 \geq 8(2\pi)^2$, the 1-parameter family M_t can be defined for all $t \in [0, (2\pi)^2]$ by Lemma 4.6. Lemma 4.5 implies that $R_t \geq \sinh^{-1}(\sqrt{2})$ for all t, and so we can apply the results of Proposition 4.4 with a lower bound on R.

Recall from (4.2) in the statement of Proposition 4.4 that $\frac{d\theta}{d\alpha} = \frac{\theta}{\alpha} + 4\alpha y l$, so differentiating v with respect to α gives us

$$\frac{dv}{d\alpha} = \frac{\left(\frac{d\theta}{d\alpha}\right)}{\alpha} - \frac{\theta}{\alpha^2} = 4yl.$$

In order to obtain a bound on this quantity, we rewrite $4yl$ in the following way since we can bound $|4\alpha^2 y|$ using (4.4).

$$\left|\frac{dv}{d\alpha}\right| = |4yl| = \left|\frac{l^2}{\alpha^2 l}4\alpha^2 y\right| = (l)\left(\frac{1}{u}\right)\left(\frac{1}{\alpha l}\right)\left|4\alpha^2 y\right|. \tag{4.9}$$

We will bound each of these four quantities separately. First, an upper bound for l at any time t is given in (4.7). For any $t \leq (2\pi)^2$ this upper bound satisfies the uniform bound

$$l \leq \frac{\alpha}{L^2 - 4t} \leq \frac{2\pi}{L^2 - 4(2\pi)^2}. \tag{4.10}$$

Recall that $u = \frac{\alpha}{l}$ and so the quantity $\frac{1}{u} = \frac{l}{\alpha}$ can bounded using (4.6). Since u approaches L^2 as $\alpha \to 0$ and $\left|\frac{du}{dt}\right| \leq 4$, we have that $|u - L^2| \leq 4\alpha^2$. Therefore at any $t \leq (2\pi)^2$, a lower bound for u is given by $u \geq L^2 - 4(2\pi)^2$ which implies

$$\frac{1}{u} \leq \frac{1}{L^2 - 4(2\pi)^2}. \tag{4.11}$$

Making the same changes of variables $\rho = h^{-1}(\alpha l)$ and $z = \tanh(\rho)$ that we made in the proof of the inequality (4.5) in Proposition 4.4, we see that (as in eq. (38) of [33])

$$\frac{1}{\alpha l} = \frac{1 + z^2}{1.6978(z)(1 - z^2)}. \tag{4.12}$$

Finally we must bound $4\alpha^2 y$ in terms of z. As in the proof of Proposition 4.4, we can replace R by ρ in the inequality (4.4) to get

$$|4\alpha^2 y| \leq \frac{2}{\sinh^2(\rho)}\frac{\cosh^2(\rho)}{(2\cosh^2(\rho) + 1)}.$$

Using $\sinh^2(\rho) = \frac{z^2}{1-z^2}$ and $\cosh^2(\rho) = \frac{1}{1-z^2}$ gives us

$$|4\alpha^2 y| \leq \frac{2(1 - z^2)}{3z^2 - z^4}. \tag{4.13}$$

Now we combine the bounds on l, $\frac{1}{u}$, $\frac{1}{\alpha l}$, and $|4\alpha^2 y|$ in (4.10), (4.11), (4.12), and (4.13) to get an estimate replacing eq. (4.9):

$$\left|\frac{dv}{d\alpha}\right| \leq \left(\frac{2\pi}{L^2 - 4(2\pi)^2}\right)\left(\frac{1}{L^2 - 4(2\pi)^2}\right)\left(\frac{1+z^2}{1.6978(z)(1-z^2)}\right)\left(\frac{2(1-z^2)}{3z^2 - z^4}\right).$$

Since $L^2 \geq 8(2\pi)^2$ we know (as in the proof of Lemma 4.5) that αl remains bounded above by h_{max} for all t and therefore $\rho \geq 0.531$. This implies $1 \geq z \geq 0.4862$ throughout the deformation so we can bound the following function of z by its value when $z = 0.48$ since it is decreasing on $[0.48, 1]$.

$$\frac{2(1+z^2)(1-z^2)}{1.6978(z)(1-z^2)(3z^2-z^4)} \leq \frac{2(1+(0.48)^2)(1-(0.48)^2)}{1.6978(0.48)(1-(0.48)^2)(3(0.48)^2-(0.48)^4)}.$$

This upper bound is approximately 4.73191, so for any $z \in [0.48, 1]$,

$$\frac{2(1+z^2)(1-z^2)}{1.6978(z)(1-z^2)(3z^2-z^4)} \leq 5.$$

Thus

$$\left|\frac{dv}{d\alpha}\right| \leq \frac{5(2\pi)}{(L^2 - 4(2\pi)^2)^2}.$$

□

The quantity $v = v(\lambda', t)$ depends on both a longitude λ' and the parameter t. The previous Lemma shows that if L^2 is sufficiently large, then v remains roughly constant. Thus for any longitude λ', $v(\lambda', (2\pi)^2)$ can be approximated by $\lim_{t\to 0} v(\lambda', t)$. We claim that since $|A^2| > 2$, there is a longitude λ such that this limit exists and is equal to $\frac{1}{A^2}$, the normalized twist of the cusp we are filling $\hat{M}$.

Define R_t and V_t as in Lemma 4.5. The torus ∂V_t inherits a flat metric as a subset of M_t, and ∂V_t has a well-defined meridian for all t. Therefore we can define the twist $b(t)$ and the length of a geodesic representative of the meridian $m(t)$ in the same way that we defined b and m for the flat torus ∂V_0 in the cusp of $\hat{M} = M_0$. Since

$M_t \to M_0$ geometrically, the tori ∂V_t converge to ∂V_0 in $\mathcal{T}(T^2)/\mathbb{Z}$ after normalizing with respect to area. Here, the group $\mathbb{Z}$ acts on $\mathcal{T}(T^2)$ by Dehn twists about the meridian. Since the normalized twist $\frac{b(t)}{m(t)}$ is independent of the area of ∂V_t, this ratio converges to $\frac{b}{m}$ unless $\frac{b}{m} = \frac{1}{2}$ (in which case $\lim \frac{b(t)}{m(t)}$ could be $\frac{1}{2}$, $-\frac{1}{2}$, or not exist). Thus we have

Lemma 4.9. *Suppose that $|A^2| > 2$. Then*

$$\lim_{t\to 0} \frac{b(t)}{m(t)} \to \frac{1}{A^2}.$$

When $\frac{b(t)}{m(t)} \neq \frac{1}{2}$, then there is a unique shortest longitude λ on ∂V_t. In particular, if $|A^2| > 2$, there is a unique shortest longitude λ on ∂V_0.

Lemma 4.10. *If $|A^2| > 2$, then there is some $\delta > 0$ such that the imaginary part of the complex length of λ lies in the interval $\theta(\lambda, t) \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right)$ for all $0 < t < \delta$. Moreover,*

$$\lim_{t\to 0} v(\lambda, t) = \lim_{t\to 0} \frac{\theta(\lambda, t)}{\alpha} = \frac{1}{A^2}.$$

Proof. Since $\left|\frac{1}{A^2}\right| < \frac{1}{2}$, there is some $\delta > 0$ such that for all $t \in [0, \delta)$, $\frac{b(t)}{m(t)} \neq \frac{1}{2}$ by the previous lemma. Hence, the longitude λ that is shortest on ∂V_0 is the unique shortest longitude on ∂V_t for all $t \in [0, \delta)$. Thus, the imaginary part of the complex length satisfies $\theta(\lambda, t) \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right)$ for all $t \in (0, \delta)$.

For any $t \in (0, \delta)$, we have $\alpha > 0$, so it is clear from the definitions that

$$\frac{\theta(\lambda, t)}{\alpha} = \frac{b(t)}{m(t)}.$$

Thus

$$\lim_{t\to 0} v(\lambda, t) = \lim_{t\to 0} \frac{\theta(\lambda, t)}{\alpha} = \lim_{t\to 0} \frac{b(t)}{m(t)} = \frac{1}{A^2}.$$

$\square$

We now use the bounds on $\frac{dv}{d\alpha}$ and the fact that $\lim_{t\to 0} v(\lambda, t) = \frac{1}{A^2}$ to estimate $v(\lambda, (2\pi)^2)$.

Lemma 4.11. *If $|A^2| > 2$ and $L^2 \geq 8(2\pi)^2$, then the complex length $l((2\pi)^2) + i\theta((2\pi)^2)$ of the longitude λ on $\partial V_{(2\pi)^2}$ satisfies:*

$$\left|\theta(\lambda, (2\pi)^2) - \frac{2\pi}{A^2}\right| \leq \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2}.$$

Proof. Recall that for any longitude, we obtained the bound

$$\left|\frac{dv}{d\alpha}\right| \leq \frac{5(2\pi)}{(L^2 - 4(2\pi)^2)^2}$$

in Lemma 4.8. For the longitude λ, we have $\lim_{t\to 0} v(\lambda, t) = \frac{1}{A^2}$, so we can integrate to find that, for any $t \leq (2\pi)^2$, we have

$$\left|\frac{\theta(\lambda, t)}{\alpha} - \frac{1}{A^2}\right| \leq \frac{5(2\pi)(t)}{(L^2 - 4(2\pi)^2)^2}.$$

Since $L^2 \geq 8(2\pi)^2$ we can define the deformation for all $t \in [0, (2\pi)^2]$, and therefore setting $t = (2\pi)^2$ gives us

$$\left|\theta(\lambda, (2\pi)^2) - \frac{2\pi}{A^2}\right| \leq \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2}. \tag{4.14}$$

□

We can now prove part (v) of Theorem 1.2.

Lemma 4.12. *Let γ be the core curve of the filling torus in $M = M_{(2\pi)^2}$. If $L^2 \geq 8(2\pi)^2$ and $|A^2| \geq 3$, then the imaginary part of the complex length $\mathcal{L}(\gamma) = l(\gamma) + i\theta(\gamma)$, normalized so that $\theta(\gamma) \in (-\pi, \pi]$, satisfies:*

$$\left|\theta(\gamma) - \frac{2\pi}{A^2}\right| \leq \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2}. \tag{4.15}$$

Proof. Since any longitude on $\partial V_{(2\pi)^2}$ is homotopic to γ, the complex length $l(\gamma) + i\theta(\gamma)$ of γ in $M_{(2\pi)^2}$ is given by the complex length of λ on $V_{(2\pi)^2}$; although, the imaginary part of the complex length of λ may not lie in the interval $(-\pi, \pi]$. The conditions $|A^2| \geq 3$ and $L^2 \geq 8(2\pi)^2$, together with the inequality in (4.14) imply

$$\left|\theta(\lambda, (2\pi)^2)\right| \leq \frac{5}{16(2\pi)} + \frac{2\pi}{3} < \pi.$$

Hence $\theta(\gamma) = \theta(\lambda, (2\pi)^2) \in (-\pi, \pi)$, and inequality (4.15) holds. $\square$

We now complete the proof of the filling theorem by summarizing what we have done to prove parts $(i), (ii), (iii)$, and deriving part (iv). Part (i) was proven in Lemma 4.6 when we showed that one can increase the cone angle from 0 to 2π. Parts (ii) and (iii) were completed in Lemma 4.7. Part (iv) follows from parts (i), (iii), and the drilling theorem (Theorem 4.1) in the following way. Part (i) provides the existence of M (*i.e.*, the β-filling of $\hat{M}$), and by the drilling theorem, there exists some l_0 such that if $l(\gamma) < l_0$ then there is a J-biLipschitz diffeomorphism

$$\phi : \hat{M} - \mathbb{T}_\epsilon(T) \to M - \mathbb{T}_\epsilon(\gamma).$$

By part (iii) of the filling theorem, there exists some K such that if the normalized length, L, of β is at least K then $l(\gamma) \leq \frac{2\pi}{L^2 - 4(2\pi)^2} < l_0$. Thus we can apply the drilling theorem to reverse the filling and obtain the desired biLipschitz map. $\square$

Remark. Note that in parts $(i), (ii), (iii), (v)$ of the filling theorem, we only used the uniform bounds $L^2 \geq 8(2\pi)^2$ and $|A^2| \geq 3$. These four parts do not depend on the condition that $L \geq K$. The constant K depends on J and ϵ and is therefore only necessary to conclude that if $L \geq K$, then the map ϕ is J-biLipschitz outside a Margulis ϵ-thin region about the cusp T. We also remark that since the filling map ϕ

is obtained by applying the drilling theorem, we can assume that ϕ is level-preserving on cusps (see the remark following Theorem 4.1).

Before moving on to the next section, we remark that one could remove the assumption that $|A^2| \geq 3$ by not requiring the normalization $\theta(\gamma) \in (-\pi, \pi]$. If we only assume $|A^2| > 2$, then (4.14) still holds. Thus, we could claim that the imaginary part of the complex length of γ, normalized so $\theta(\gamma) \in \left(\frac{2\pi}{A^2} - \pi, \frac{2\pi}{A^2} + \pi\right]$, satisfies (4.15).

If $A^2 = 2$, one could adapt our methods to show that with the normalization $\theta(\gamma) \in (0, 2\pi]$, we have

$$|\theta(\gamma) - \pi| \leq \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2}. \tag{4.16}$$

The problem with applying our methods directly is that there are two shortest longitudes on ∂V_0 when $b = \frac{m}{2}$. To circumvent this problem, one could define the normalized twist $\frac{b(t)}{m(t)}$ as a value on the circle $S^1 \cong \left[-\frac{1}{2}, \frac{1}{2}\right] / \{-\frac{1}{2}\} \sim \{\frac{1}{2}\}$. Since Lemma 4.8 applies to all longitudes on ∂V_t, this lemma can be used to show that if $L^2 \geq 8(2\pi)^2$, then for any $t \in [0, (2\pi)^2]$,

$$\left|\frac{b(0)}{m(0)} - \frac{b(t)}{m(t)}\right| \leq \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2} \tag{4.17}$$

where the absolute value signs on the left-hand side of (4.17) denote the distance on the circle. One can then derive (4.16) from (4.17) with the appropriate normalization of $\theta(\gamma)$.

4.4 Consequences and Generalizations

In this section, we explain some of the consequences of the drilling and filling theorems. First, we draw the following corollary of Theorems 4.1 and 1.2 that will allow us to fill a manifold with multiple cusps. If $\hat{M}$ has multiple cusps, we can fill them one at a time, using the filling theorem each time to obtain bounds on the lengths of the core curves of the filling tori. In the statement of the following corollary, we will suppose our manifold has d cusps, but only $n \leq d$ are being filled. We will assume they have been ordered so that the first n are filled. We label the cusps in $\hat{M}$ by T_i and we label the core curves of the solid tori by γ_i.

Corollary 4.13. *Let $J > 1$, $l_0 > 0$, $\epsilon_3 \geq \epsilon > 0$, and n be given. There exists some K such that the following holds: suppose $\hat{M}$ is a geometrically finite manifold with $d \geq n$ cusps. Suppose β_i is a slope on the ith cusp of $\hat{M}$, $1 \leq i \leq n \leq d$. If the normalized length of β_i is at least K for each i, then*
(i) we can fill in the n cusps with labeled meridians, obtaining a manifold M such that each β_i bounds a disk in M (in other words, the $\cup\beta_i$-filling of $\hat{M}$ exists);
(ii) if γ_i is the core curve of the torus used to fill the ith cusp, then $\sum_{i=1}^{n} l(\gamma_i) < l_0$;
(iii) there exists a J-biLipschitz diffeomorphism

$$\phi : \hat{M} - \cup_{i=1}^{n} \mathbb{T}_\epsilon(T_i) \to M - \cup_{i=1}^{n} \mathbb{T}_\epsilon(\gamma_i).$$

Proof. If the filled manifold M exists, the drilling theorem says that there exists some l_0', depending only on J and ϵ, such that if $\sum_{i=1}^{n} l(\gamma_i) < l_0'$, then there is a J-biLipschitz diffeomorphism

$$\phi : \hat{M} - \cup_{i=1}^{n} \mathbb{T}_\epsilon(T_i) \to M - \cup_{i=1}^{n} \mathbb{T}_\epsilon(\gamma_i).$$

Let $\ell = \min\{l_0, l_0'\}$ where l_0 is the constant given in the statement of the Corollary. We will show there exists a K such that if the normalized length of β_i is at least K for $1 \leq i \leq n$, then the filled manifold M exists, and the length of γ_i in M is less than $\frac{\ell}{n}$.

We want to fill the cusps one at a time. Let $M^0 = \hat{M}$, and if it exists let M^i be the β_i filling of M^{i-1}. We will use the notation $l_{M^i}(\gamma_j)$ to denote the length of γ_j in M^i. In this notation, we want to show that $M^n = M$ exists and that $l_{M^n}(\gamma_i) < \frac{\ell}{n}$ for all $1 \leq i \leq n$.

By the filling theorem, there exists some K' (independent of i) such that if the normalized length of β_i in M^{i-1} is at least K' then we have the following:

- the β_i-filling of M^{i-1}, which we will call M^i, exists;
- the length of γ_i in M^i satisfies $l_{M^i}(\gamma_i) < \frac{\ell}{n2^n}$;
- there is a 2-biLipschitz map $\phi_i : M^{i-1} - \mathbb{T}_{\epsilon'}(T_i) \rightarrow M^i - \mathbb{T}_{\epsilon'}(\gamma_i)$ for some $\epsilon_3 \geq \epsilon' > 0$.

Now let $K > 4^n K'$. If the normalized length of β_1 is at least K, we can do the first filling to obtain M^1. To apply the filling theorem inductively, we must show that in the ith filling, the length of γ_j, $1 \leq j < i$, does not increase by more than a factor of 2, and the normalized length of β_j, $i < j \leq n$, does not decrease by more than a factor of 4. The fact that the length of γ_j does not double follows immediately from the fact that ϕ_i is 2-biLipschitz. The normalized length of β_j takes more consideration.

To prove that the normalized length of β_j, $i < j \leq n$, does not decrease by a factor of 4 during the β_i-filling of M^{i-1}, fix a torus cross-section $T = T^2 \times \{1\}$ of the cusp $T_j \cong T^2 \times [0, \infty)$ contained in $M^{i-1} - \mathbb{T}_\epsilon(T_i)$. Let μ be a curve on T homotopic to the meridian β_j. The normalized length of β_j in M^{i-1} is $\frac{l(\mu)}{\sqrt{area(T)}}$, where $l(\mu)$ is the length of a geodesic representative of μ on T with respect to the induced Euclidean metric on T. Since ϕ_i is 2-biLipschitz, the length of a geodesic representative of $\phi_i(\mu)$

on $\phi_i(T)$ is bounded by $l(\phi_i(\mu)) > \frac{l(\mu)}{2}$, and also $area(\phi_i(T)) < 4(area(T))$. Thus

$$\frac{l(\phi_i(\mu))}{\sqrt{area(\phi_i(T))}} > \frac{l(\mu)/2}{\sqrt{4(area(T))}}.$$

By Theorem 6.12 of [13] (see the remark following Theorem 4.1), we can assume that $\phi_i(T)$ is a flat cross-section of the jth cusp in M^i. Thus the ratio $\frac{l(\phi_i(\mu))}{\sqrt{area(\phi_i(T))}}$ in the left-hand side of the inequality above is the normalized length of β_j in M^i. This completes the proof that the normalized length of β_j does not shrink by more than a factor of 4 during the ith filling.

Thus, for any $1 \leq i \leq n$, the normalized length of β_i in M^{i-1} is at least $4^{n-i}K'$. So we can apply the filling theorem n times to get M, the $\cup\beta_i$-filling of $\hat{M}$. This completes part (i). Since the length of γ_i in M^i is less than $\frac{\ell}{n2^n}$, the length of γ_i in M is less than $\frac{\ell}{n}\frac{2^{n-i}}{2^n} \leq \frac{\ell}{n}$. Hence $\sum_{i=1}^n l(\gamma_i) < \ell$. Since $\ell = \min\{l_0, l_0'\}$ this completes parts (ii) and (iii). $\square$

Now suppose that $\hat{M}$ is the γ-drilling of M, and let T denote the new cusp of $\hat{M}$. Recall that if γ is sufficiently short, the drilling theorem provides a biLipschitz diffeomorphism $\phi : \hat{M} - \mathbb{T}_{\epsilon_3}(T) \to M - \mathbb{T}_{\epsilon_3}(\gamma)$. There is a unique slope β on $\partial\mathbb{T}_{\epsilon_3}(T)$ such that $\phi(\beta)$ bounds a disk in $\mathbb{T}_{\epsilon_3}(\gamma) \subset M$, but β does not bound a disk in $\hat{M}$. Equivalently, the β-filling of $\hat{M}$ (if it exists) is M. We say that β is the meridian of $\hat{M}$. If γ is sufficiently short, then one can bound the normalized length of β in $\hat{M}$ from below. This is stated without proof in part (2) of Theorem 2.4 of [21].

Proposition 4.14. *Let $K > 0$ be given. Let $\hat{M}$ be the γ-drilling of M. There exists some l_0 such that if the length of γ is less than l_0, then the normalized length L of the meridian β in $\hat{M}$ is at least K.*

Proof. By the drilling theorem, there exists l_1 such that if $l(\gamma) < l_1$ then there exists a 2-biLipschitz map

$$\phi : \hat{M} - \mathbb{T}_{\epsilon_3}(T) \to M - \mathbb{T}_{\epsilon_3}(\gamma).$$

Suppose that R is the distance between γ and $\partial\mathbb{T}_{\epsilon_3}(\gamma)$. The area of the boundary of this Margulis tube in M is equal to

$$A(\partial\mathbb{T}_{\epsilon_3}(\gamma)) = A = 2\pi l(\gamma)\sinh(R)\cosh(R).$$

(For example, see p. 403 of [33].) Here $2\pi\sinh(R)$ gives the length of the meridian on $\partial\mathbb{T}_{\epsilon_3}(\gamma)$ in M.

Define the normalized length of β in M to be $L_M(\beta) = \frac{2\pi\sinh(R)}{\sqrt{A}}$. This is the length of (a geodesic representative of) $\phi(\beta)$ on $\partial\mathbb{T}_{\epsilon_3}(\gamma)$ divided by the square root of the area of $\partial\mathbb{T}_{\epsilon_3}(\gamma)$. Unlike the normalized length of β in $\hat{M}$, which is the length of β on $\partial\mathbb{T}_{\epsilon_3}(T)$ divided by the square root of the area of $\partial\mathbb{T}_{\epsilon_3}(T)$, the normalized length of β in M depends on the size of the Margulis tube (in this case ϵ_3).

Now $L_M(\beta) = \frac{2\pi\sinh(R)}{\sqrt{A}} = \frac{\sqrt{A}}{l(\gamma)\cosh(R)}$. Thus

$$L_M(\beta) = \sqrt{\frac{2\pi\tanh(R)}{l(\gamma)}}.$$

The estimates in Brooks-Matelski [24] imply that given any R_0, there is some l_2' such that if $l(\gamma) < l_2'$ then $R > R_0$. Hence, there is some l_2 such that if $l(\gamma) < l_2$, then $L_M(\beta) > 4K$.

Now let $l_0 = \min\{l_1, l_2\}$. This implies the filling map restricts to a 2-biLipschitz diffeomorphism on the boundary tori: $\phi^{-1} : \partial\mathbb{T}_{\epsilon_3}(\gamma) \to \partial\mathbb{T}_{\epsilon_3}(T)$. Hence, as we saw in the proof of the previous corollary, the normalized length of β in $\hat{M}$ is no less than $\frac{1}{4}$ times the normalized length of β on $\partial\mathbb{T}_{\epsilon_3}(\gamma)$. Thus, the normalized length of the meridian in $\hat{M}$ (which we are simply denoting by L) is at least

$$L \geq \frac{1}{4}L_M(\beta) > K.$$

$\square$

Remark. One can also prove Proposition 4.14 using the tools developed in the proofs of Proposition 4.4, Lemma 4.5. One defines a 1-parameter family of cone-manifolds by decreasing the cone-angle about γ from 2π to 0, showing that the maximal radius of a neighborhood of γ does not become too small. There is some l_1 such that if the length of γ is less than l_1 then the one-parameter family of cone-manifolds can be defined for all $t \in [0, (2\pi)^2]$, and the estimate

$$\left|\frac{du}{dt}\right| \leq 4$$

can be applied for all t. Recall that $u(t) \to L^2$ as $t \to 0$ where L^2 is the square of the normalized length of β in $\hat{M}$, and $u((2\pi)^2) = \frac{2\pi}{l(\gamma)}$ when $t = (2\pi)^2$. From this, one can see that given any K, there exists some l_0 such that if $l(\gamma) < l_0$ then $L^2 > K^2$.

CHAPTER 5

A Local Homeomorphism

Let S be a closed surface of genus at least two, set $N = S \times I$, and define the paring locus $P \subset \partial N$ to be a collection of annuli forming a pants decomposition of $S \times \{1\}$. In this chapter, we parameterize a neighborhood in $MP(N) \cup MP(N, P)$ of a point $\sigma \in MP(N, P)$. To do this, we define a model space $\mathcal{A}$ and a map $\Phi : \mathcal{A} \to MP(N) \cup MP(N, P)$. We show that there is some neighborhood $U \subset \mathcal{A}$ and some neighborhood V of σ in $MP(N) \cup MP(N, P)$ such that $\Phi|_U : U \to V$ is a homeomorphism. We show that $\mathcal{A}$ is not locally connected at one of the points where Φ is a local homeomorphism; hence, there is a point where $MP(N) \cup MP(N, P)$ is not locally connected. In fact, in Section 5.4, we find a point of $\mathcal{A}$ with the property that any sufficiently small neighborhood of this point contains infinitely many components that are bounded apart from each other. In Chapter 6, we will use this description of the components of a neighborhood $U \subset \mathcal{A}$ and the filling theorem, which is used in the definition of Φ, to show that there is a point $\sigma^0 \in MP(N, P)$ such that for any sufficiently small neighborhood $\sigma^0 \in V \subset MP(N) \cup MP(N, P)$, the closure of V has infinitely many components. Along with the Density Theorem, this will be used to show that $AH(N)$ is not locally connected.

We now outline this chapter. Let $S_{1,1}$ and $S_{0,4}$ denote the punctured torus and four-punctured sphere respectively. In Section 5.1, we define spaces $\mathcal{A}_{1,1}$ and $\mathcal{A}_{0,4}$ which, by Bromberg's results [21], locally model the deformation spaces $AH(S_{1,1} \times I, \partial S_{1,1} \times I)$ and $AH(S_{0,4} \times I, \partial S_{0,4} \times I)$ respectively. We then define $\mathcal{A}$ in Section 5.2. The construction of $\mathcal{A}$ is analogous to the constructions of $\mathcal{A}_{1,1}$ and $\mathcal{A}_{0,4}$, although more technical. In addition to being defined similarly, we relate $\mathcal{A}$ to $\mathcal{A}_{1,1}$ by showing there is a continuous surjection $\Pi : \mathcal{A} \to \mathcal{A}_{1,1}$ in Section 5.3. We use this in Section 5.4, along with the fact that $\mathcal{A}_{1,1}$ is not locally connected [21], to show that $\mathcal{A}$ is not locally connected. As we alluded to in the introductory paragraph, we use Bromberg's description of $\mathcal{A}_{1,1}$ (Section 4, [21]) to show that there is a point $(\sigma^0, \infty, \ldots, \infty) \in \mathcal{A}$, a neighborhood U of $(\sigma^0, \infty, \ldots, \infty)$, and collections of components $C_n \subset U$ such that the distance between $\overline{C_n}$ and $\overline{U - C_n}$ is uniformly bounded from below. We will be more precise about the notion of distance in Section 5.4 since, as subsets of $\mathcal{A}$, the collections C_n accumulate at $(\sigma^0, \infty, \ldots, \infty)$.

In Section 5.5, we define the map Φ on a subset of $\mathcal{A}$ and prove that it is continuous. In Sections 5.6 and 5.7, we show that there a subset $V \subset MP(N) \cup MP(N, P)$ and a map $\Psi : V \to \mathcal{A}$ that is an inverse to Φ. We use this to show that Φ is a local homeomorphism. Sections 5.5, 5.6, and 5.7 closely parallel Section 3 of [21]. Finally, in Section 5.8, we combine the results of the chapter in order to show that $MP(N) \cup MP(N, P)$ is not locally connected.

5.1 The Punctured Torus and Four-Punctured Sphere

Define $N_{1,1} = S_{1,1} \times I$ and $P_{1,1} = \partial S_{1,1} \times I$. Let $P'_{1,1}$ be the union of $P_{1,1}$ with a non-peripheral annulus in $S_{1,1} \times \{1\}$ about a curve $b \times \{1\}$ (see Figure 5.1). Let $\hat{N}_{1,1}$ be the manifold obtained by removing an open tubular neighborhood of $b \times \{\frac{1}{2}\}$ from $N_{1,1}$, and let $\hat{P}_{1,1}$ be the union of $P_{1,1}$ with the toroidal boundary component of $\hat{N}_{1,1}$.

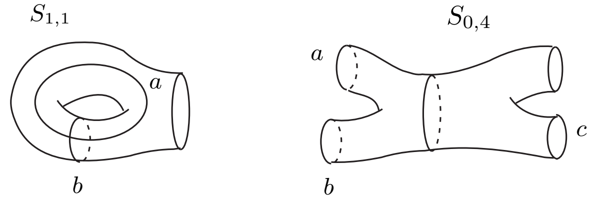

Figure 5.1: Orient the curves a, b on $S_{1,1}$ and identify a presentation $\pi_1(S_{1,1}) = \langle a, b\rangle$. Similarly, orient a, b, c on $S_{0,4}$ and identify a presentation $\pi_1(S_{0,4}) = \langle a, b, c\rangle$.

Recall from Chapter 2 that the components of $MP(N_{1,1}, P'_{1,1})$ are in one-to-one correspondence with the marked pared homeomorphism types of manifolds pared homotopy equivalent to $(N_{1,1}, P'_{1,1})$. Fix an orientation on $N_{1,1}$ and let $\overline{N}_{1,1}$ denote $N_{1,1}$ with the opposite orientation. Then there are two components of $MP(N_{1,1}, P'_{1,1})$ corresponding to $F^{-1}([(N_{1,1}, P'_{1,1}), id])$ and $F^{-1}([(\overline{N}_{1,1}, P'_{1,1}), id])$. We denote the former by $MP_0(N_{1,1}, P'_{1,1})$.

Given any $z \in \mathbb{C}$, one can define a representation $\sigma_z : \pi_1(N_{1,1}) \to PSL(2, \mathbb{C})$ by

$$\sigma_z(a) = \begin{pmatrix} iz & i \\ i & 0 \end{pmatrix} \quad \sigma_z(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

For any $\sigma \in MP(N_{1,1}, P'_{1,1})$ there is a unique z such that σ_z is in the conjugacy class of σ (see Section 6.3 of [46] or [39]). This defines an embedding of $MP(N_{1,1}, P'_{1,1})$ into $\mathbb{C}$. The Maskit slice, $\mathcal{M}^+$, denotes the set of all $z \in \mathbb{C}$ such that $\sigma_z \in MP_0(N_{1,1}, P'_{1,1})$, and its mirror image in the lower half plane will be denoted by $\mathcal{M}^-$ (this corresponds to the other component of $MP(N_{1,1}, P'_{1,1})$). Let $\mathcal{M} = \mathcal{M}^+ \cup \mathcal{M}^-$.

Remark. We are following the Keen-Series convention by defining $\mathcal{M}^+$ and $\mathcal{M}^-$ to be open sets, whereas Bromberg uses $\mathcal{M}^\pm$ to denote the closures of these sets in Section 4 of [21]. See Proposition 4.4 of [21] or [39] for more on the Maskit slice and its closure. The set $\mathcal{M}^+$ is also known as the Maskit embedding of the Teichmüller space of punctured tori.

Given $w \in \mathbb{C}$ and a conjugacy class of representations $\sigma \in MP_0(N_{1,1}, P'_{1,1})$, one can define a representation $\sigma_{z,w}$ of $\pi_1(\hat{N}_{1,1}) = \langle a, b, c \mid [b, c] = 1\rangle$ in the following way. There is a unique $z \in \mathcal{M}^+$ such that the representation σ_z represents the conjugacy class σ. Define $\sigma_{z,w}(a) = \sigma_z(a)$, $\sigma_{z,w}(b) = \sigma_z(b)$ and $\sigma_{z,w}(c) = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. Note that $\sigma_{z,w}$ is an actual representation of $\pi_1(\hat{N}_{1,1})$, but we will also use $\sigma_{z,w}$ to refer to the conjugacy class.

Define $\mathcal{A}_{1,1}$ to be

$$\mathcal{A}_{1,1} = \{(\sigma, w) \in MP_0(N_{1,1}, P'_{1,1}) \times \hat{\mathbb{C}} \ : w = \infty, \text{ or } \sigma_{z,w} \in MP(\hat{N}_{1,1}, \hat{P}_{1,1}) \text{ and } Im(w) > 0\}.$$

Note that this is what Bromberg calls $\mathring{\mathcal{A}}$ in [21].

One defines $\mathcal{A}_{0,4}$ similarly. Let $N_{0,4} = S_{0,4} \times I$, let $P_{0,4} = \partial S_{0,4} \times I$, and define $P'_{0,4}$ to be the union of $P_{0,4}$ with a non-peripheral annulus in $S_{0,4} \times \{1\}$ whose core curve is $ab \times \{1\}$ (see Figure 5.1). Let $\hat{N}_{0,4}$ be the manifold obtained by removing an open tubular neighborhood of $ab \times \{\frac{1}{2}\}$ from $N_{0,4}$, and define $\hat{P}_{0,4}$ to be the union of $P_{0,4}$ with the toroidal boundary component of $\hat{N}_{0,4}$.

Like the space $MP(N_{1,1}, P'_{1,1})$, the space $MP(N_{0,4}, P'_{0,4})$ has two components corresponding to $F^{-1}([(N_{0,4}, P'_{0,4}), id])$ and $F^{-1}([(\overline{N}_{0,4}, P'_{0,4}), id])$. We denote the former by $MP_0(N_{0,4}, P'_{0,4})$. As we discussed for $(N_{1,1}, P'_{1,1})$, $MP(N_{0,4}, P'_{0,4})$ admits a natural embedding into $\mathbb{C}$. Given any $\sigma \in MP(N_{0,4}, P'_{0,4})$, there is a unique $z \in \mathbb{C}$ such that the representation $\sigma_z : \pi_1(N_{0,4}) \to PSL(2, \mathbb{C})$ defined by

$$\sigma_z(a) = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}, \quad \sigma_z(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \sigma_z(c) = \begin{pmatrix} -1+2z & -2z^2 \\ 2 & -1-2z \end{pmatrix}$$

represents the conjugacy class of σ (Section 6.1 of [46]). Note that for any z, one can check that $\sigma_z(ab) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

The Maskit slice for $S_{0,4}$, denoted $\mathcal{M}^+_{0,4}$, is the set of all $z \in \mathbb{C}$ such that $\sigma_z \in MP_0(N_{0,4}, P'_{0,4})$, and its mirror image in the lower half plane will be denoted by $\mathcal{M}^-_{0,4}$. Again, we let $\mathcal{M}_{0,4} = \mathcal{M}^+_{0,4} \cup \mathcal{M}^-_{0,4}$. Kra shows that $z \in \mathcal{M}_{0,4}$ if and only if $2z \in \mathcal{M}^+$ (p. 558 of [46]).

Given $w \in \mathbb{C}$ and a conjugacy class of representations $\sigma \in MP_0(N_{0,4}, P'_{0,4})$, one can define a representation $\sigma_{z,w}$ of $\pi_1(\hat{N}_{0,4}) = \langle a, b, c, d \mid [ab, d] = 1\rangle$ in the following way. There is a unique $z \in \mathcal{M}^+_{0,4}$ such that the representation σ_z represents the conjugacy class σ. Define $\sigma_{z,w}(a) = \sigma_z(a)$, $\sigma_{z,w}(b) = \sigma_z(b)$, $\sigma_{z,w}(c) = \sigma_z(c)$, and $\sigma_{z,w}(d) = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$.

Define $\mathcal{A}_{0,4}$ to be

$$\begin{aligned}\mathcal{A}_{0,4} = \{(\sigma, w) \in MP_0(N_{0,4}, P'_{0,4}) \times \hat{\mathbb{C}}\ : w = \infty, \text{ or} \\ \sigma_{z,w} \in MP(\hat{N}_{0,4}, \hat{P}_{0,4}) \text{ and } Im(w) > 0\}.\end{aligned}$$

Given $\sigma \in MP_0(N_{1,1}, P'_{1,1})$ and $w \in \mathbb{C}$, Bromberg characterizes when $(\sigma, w) \in \mathcal{A}_{1,1}$. For any $\sigma \in MP_0(N_{1,1}, P'_{1,1})$, there is a unique z for which the conjugacy class of σ_z is σ. Bromberg shows in Proposition 4.7 of [21] that $(\sigma, w) \in \mathcal{A}_{1,1}$ if and only if there exists an integer n such that $z - nw \in \mathcal{M}^+$ and $z - (n+1)w \in \mathcal{M}^-$.

This is essential in Bromberg's proof that $\mathcal{A}_{1,1}$ is not locally connected. A similar statement holds for $\mathcal{A}_{0,4}$, although in the rest of this paper we will only need the necessity of the existence of n.

Lemma 5.1. *(i)* (Bromberg [21]) *Let $\sigma_z \in MP_0(N_{1,1}, P'_{1,1})$ and $w \in \mathbb{C}$ with $Im(w) > 0$. Then $\sigma_{z,w} \in MP(\hat{N}_{1,1}, \hat{P}_{1,1})$ if and only if there exists an integer n such that*

$z - nw \in \mathcal{M}^+$ and $z - (n+1)w \in \mathcal{M}^-$.

(ii) Let $\sigma_z \in MP_0(N_{0,4}, P'_{0,4})$ and $w \in \mathbb{C}$ with $Im(w) > 0$. If $\sigma_{z,w} \in MP(\hat{N}_{0,4}, \hat{P}_{0,4})$, then there exists an integer n such that $z - nw \in \mathcal{M}^+_{0,4}$ and $z - (n+1)w \in \mathcal{M}^-_{0,4}$.

Remark. As in part (i), one could show that the existence of such an n in part (ii) is sufficient to ensure $\sigma_{z,w} \in MP(\hat{N}_{0,4}, \hat{P}_{0,4})$, but we will not need this fact so we omit the proof.

Proof. The first statement is Proposition 4.7 of [21]. The proof of (ii) is nearly identical to the proof of (i) so we only sketch the argument. First we claim that for all $n \in \mathbb{Z}$, $z - nw \in \mathcal{M}_{0,4}$ (*i.e.*, the union of the two components of the Maskit slice). This is true since the restriction of $\sigma_{z,w}$ to the subgroup generated by $\langle a, b, d^n c d^{-n} \rangle$ is the representation σ_{z-nw}. Clearly σ_{z-nw} must be discrete and faithful. Also, σ_{z-nw} is geometrically finite since $\sigma_{z-nw}(\pi_1(S_{0,4}))$ is a finitely generated subgroup of a geometrically finite Kleinian group with nonempty domain of discontinuity (Proposition 7.1 of [61]). Finally, one can check that since $\sigma_{z,w} \in MP(\hat{N}_{0,4}, \hat{P}_{0,4})$, $\sigma_{z-nw} \in MP(N_{0,4}, P'_{0,4})$ for all n.

Since $Im(w) > 0$, there is a unique n such that $Im(z - nw) > 0$ and $Im(z - (n+1)w) \leq 0$. Since $z - nw, z - (n+1)w \in \mathcal{M}_{0,4}$ and $\mathcal{M}^+_{0,4}$ lies entirely in the upper-half plane (and $\mathcal{M}^-_{0,4}$ lies entirely in the lower-half plane) we must have that $z - nw \in \mathcal{M}^+_{0,4}$ and $z - (n+1)w \in \mathcal{M}^-_{0,4}$. $\square$

Although in the definitions of $\mathcal{A}_{1,1}$ and $\mathcal{A}_{0,4}$ we only require $Im(w) > 0$, the following lemma allows us to give a positive lower bound. This will be used to obtain Corollary 5.5 which is used in the proof of Lemma 5.14.

Lemma 5.2. *If $(\sigma, w) \in \mathcal{A}_{1,1}$, then $w = \infty$ or $Im(w) > 2$. If $(\sigma, w) \in \mathcal{A}_{0,4}$ then $w = \infty$ or $Im(w) > 1$.*

Proof. If $(\sigma, w) \in \mathcal{A}_{1,1}$, then there is some z such that the conjugacy class of σ_z is σ. If $w \neq \infty$, then by Proposition 4.7 of [21], there is some integer n such that $z - nw \in \mathcal{M}^+$ and $z - (n+1)w \in \mathcal{M}^-$. Thus $Im(w)$ is at least as large as the vertical distance between the components of $\mathcal{M}$. Wright shows that if $Im(z) \leq 1$ then $z \notin \mathcal{M}^+$ [76] (see also p. 534, 558 of [46] and the comments after Proposition 2.6 of [39]). Since $\zeta \in \mathcal{M}^-$ if and only if $-\zeta \in \mathcal{M}^+$, the distance between these two components of the Maskit slice is at least 2. It follows that if $(\sigma, w) \in \mathcal{A}_{1,1}$ and $w \neq \infty$ then $Im(w) > 2$.

By the previous lemma, one can do the same thing for the four-punctured sphere. Suppose $(\sigma, w) \in \mathcal{A}_{0,4}$ and $w \neq \infty$. Let z be such that the conjugacy class of σ_z is σ. There is an integer n such that $z - nw \in \mathcal{M}^+_{0,4}$ and $z - (n+1)w \in \mathcal{M}^-_{0,4}$. It follows that $Im(w)$ is at least the vertical distance between the two components of $\mathcal{M}_{0,4}$. Since $\zeta \in \mathcal{M}_{0,4}$ if and only if $2\zeta \in \mathcal{M}$ (p. 558 [46]), we must have $Im(w) > 1$. □

5.2 The Model Space $\mathcal{A}$

For surfaces with higher dimensional Teichmüller spaces, the construction of $\mathcal{A}$ takes more bookkeeping but is otherwise similar to $\mathcal{A}_{1,1}$ and $\mathcal{A}_{0,4}$.

Recall $N = S \times I$. Let $\{\gamma_i\}_{i=1}^d$ be a pants decomposition of S (recall from Chapter 1 that we will abbreviate $d = 3g - 3$). Although fixing any pants decomposition would be acceptable, we will make some choices that make it more convenient to apply Bromberg's work. We will choose γ_2 to be a curve that separates S into a punctured torus and a punctured genus $g-1$ surface, and let γ_1 be a curve in the punctured torus component of $S - \gamma_2$. Also, we fix an orientation on each γ_i to distinguish γ_i

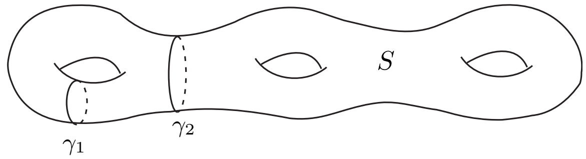

Figure 5.2: Part of the pants decomposition that we will fix throughout the rest of the argument.

from its inverse in $\pi_1(S)$.

For each i, define a homeomorphism G_i from either $S_{1,1}$ or $S_{0,4}$ to the component of $(S - \cup_{j \neq i} \gamma_j)$ containing γ_i. Moreover, using the markings of $S_{1,1}$ and $S_{0,4}$ from Figure 5.1, define G_i on $S_{1,1}$ so that $(G_i)_*(b) = \gamma_i$ and define G_i on $S_{0,4}$ so that $(G_i)_*(ab) = \gamma_i$.

Let P be a collection of d disjoint annuli in $S \times \{1\}$ such that $\gamma_i \times \{1\}$ is core curve of the ith annulus of P (see Figure 5.2). Then $MP(N, P)$ has $\binom{d}{2}$ components corresponding to whether γ_i is parabolic to one side or the other. For two of these components, all of the parabolics will be on the same side of N. In other words, for any ρ in one of these components, M_ρ will have exactly one closed component of its conformal boundary homeomorphic to S. Using the notation from Chapter 2, these two components can be identified with $F^{-1}[(N, P), id]$ and $F^{-1}[(\overline{N}, P), id]$ where $\overline{N}$ denotes N with the opposite orientation. Label the former component by $MP_0(N, P)$. These will be the marked hyperbolic manifolds with a rank-1 cusp associated to each γ_i such that the cusps all occur to the "top" of the manifold.

We now elaborate on what we mean by the "top" of a hyperbolic manifold. This discussion will be useful in distinguishing whether or not a point $\rho \in MP(N, P)$ lies in $MP_0(N, P)$. Given $\rho \in MP(N, P)$, let $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$ be the corresponding hyperbolic 3-manifold. There exists an embedding $f : S \to M$ such that $f_* = \rho$. The orientation on S induces an orientation on $f(S)$. This orientation, together with a normal direction to $f(S)$, defines an orientation on M_ρ, and for only one of the two normal directions will this orientation be compatible with the orientation induced on M_ρ as a quotient of $\mathbb{H}^3$. This distinguishes a top side of $f(S)$, and we say the top of the manifold M_ρ with respect to $f(S)$ is the component of $M_\rho - f(S)$ that lies to the top side of $f(S)$. If $\rho \in MP(N,P)$, then there are d rank-1 cusps associated to each of the components of P. If each of these cusps lies in the top of $M_\rho - f(S)$, then we say $\rho \in MP_0(N,P)$. Since any two embeddings $f : S \to M_\rho$ such that $f_* = \rho$ are isotopic [75], this notion is independent of the map f. Likewise, if $\rho \in MP(N,P)$ and X is a conformal boundary component of M_ρ, then we can distinguish whether X lies on the top or bottom side of M_ρ.

Let $\hat{N}$ be obtained by drilling a set of d curves out of N. Specifically, let $\gamma_i \times \{1/2\}$ be a collection of d curves in $S \times \{1/2\}$. Let U_i be an open collar neighborhood of $\gamma_i \times \{1/2\}$ such that the elements of the collection $\cup U_i$ are pairwise disjoint. Let

$$\hat{N} = N - \bigcup_{i=1}^{d} U_i.$$

Note that $\pi_1(\hat{N})$ is generated by $\pi_1(N)$ and d new elements β_i corresponding to meridians of ∂U_i. Since the meridian of ∂U_i commutes with any longitude of ∂U_i, there are new relations. We will use γ_i to denote both a curve and the element of $\pi_1(N)$ corresponding to that curve. Then $[\beta_i, \gamma_i] = 1$. We write

$$\pi_1(\hat{N}) = \langle \pi_1(N), \beta_1, \ldots, \beta_d \mid [\beta_i, \gamma_i] = 1 \rangle$$

with the understanding that $\pi_1(N)$ has generators besides γ_i and some of its own relations.

Given $\sigma \in MP_0(N,P)$ and $\underline{w} = (w_1, \ldots, w_d) \in \mathbb{C}^d$, we describe a process for constructing a representation $\sigma_{\underline{w}} : \pi_1(\hat{N}) \to PSL(2,\mathbb{C})$. We can find a representative in the conjugacy class determined by σ (which by an abuse of notation, we still refer to as σ) such that

$$\sigma(\gamma_1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$\sigma(\gamma_2) = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.$$

We choose these matrices to parallel the construction of $\mathcal{A}_{1,1}$ in the previous section. Recall that we fixed some homeomorphism G_1 from $S_{1,1}$ to the subsurface of S bounded by γ_2 that contains γ_1. There is a unique $z \in \mathcal{M}^+$ such that $\sigma \circ (G_1)_*$ is conjugate to σ_z. Note that for any z, $\sigma_z(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \sigma \circ (G_1)_*(b)$ and $\sigma_z(b^{-1}a^{-1}ba) = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} = \sigma \circ (G_1)_*(b^{-1}a^{-1}ba)$. Thus, specifying σ on γ_1 and γ_2 determines a well-defined representation in the conjugacy class of σ that restricts to σ_z for some z on $G_1(S_{1,1})$.

We define $\sigma_{w_1} : \pi_1(N - U_1) \to PSL(2, \mathbb{C})$ by $\sigma_{w_1}(\alpha) = \sigma(\alpha)$ for all $\alpha \in \pi_1(N)$ and

$$\sigma_{w_1}(\beta_1) = \begin{pmatrix} 1 & w_1 \\ 0 & 1 \end{pmatrix}.$$

We then inductively define $\sigma_{w_1,\ldots,w_i} : \pi_1(N - \cup_{j=1}^{i} U_j) \to PSL(2, \mathbb{C})$ by conjugating $\sigma_{w_1,\ldots,w_{i-1}}$ so that

$$\sigma_{w_1,\ldots,w_{i-1}}(\gamma_i) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

(there is some ambiguity here that will be clarified below). Then define $\sigma_{w_1,\ldots,w_i}$ by

$\sigma_{w_1,\dots,w_i}(\alpha) = \sigma_{w_1,\dots,w_{i-1}}(\alpha)$ for all $\alpha \in \pi_1(N - \cup_{j=1}^{i-1} U_j)$, and

$$\sigma_{w_1,\dots,w_i}(\beta_i) = \begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix}.$$

As we indicated above, specifying that we should conjugate $\sigma_{w_1,\dots,w_{i-1}}$ so that γ_i is sent to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ does not determine a unique representation, but we now show how a unique representation can be specified. Each curve γ_i lies in either a four-punctured sphere or punctured torus component of

$$S - \bigcup_{j \neq i} \gamma_j$$

that we have marked by a homeomorphism G_i from $S_{1,1}$ or $S_{0,4}$. If γ_i lies in a punctured-torus component bounded by some curve γ_j then we conjugate $\sigma_{w_1,\dots,w_{i-1}}$ such that

$$\sigma_{w_1,\dots,w_{i-1}} \circ (G_i)_*(b) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_{w_1,\dots,w_{i-1}} \circ (G_i)_*(b^{-1}a^{-1}ba) = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}.$$

Since G_i was chosen so that $(G_i)_*(b) = \gamma_i$, this ensures $\sigma_{w_1,\dots,w_{i-1}}(\gamma_i) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and the condition that $\sigma_{w_1,\dots,w_{i-1}}(\gamma_j) = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$ specifies $\sigma_{w_1,\dots,w_{i-1}}$ uniquely.

If γ_i lies in a four-punctured sphere component, then we conjugate $\sigma_{w_1,\dots,w_{i-1}}$ so that

$$\sigma_{w_1,\dots,w_{i-1}} \circ (G_i)_*(a) = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_{w_1,\dots,w_{i-1}} \circ (G_i)_*(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

It follows that

$$\sigma_{w_1,\dots,w_{i-1}} \circ (G_i)_*(ab) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

After d steps, we get a well-defined representation $\sigma_{w_1,\ldots,w_d}$ which we also denote by $\sigma_{\underline{w}} : \pi_1(\hat{N}) \to PSL(2,\mathbb{C})$. By construction, for each i there exists some representation in the conjugacy class of $\sigma_{\underline{w}}$ such that the generators γ_i, β_i of $\pi_1(\partial U_i)$ are sent to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix}$ respectively.

Given $\sigma \in MP_0(N,P)$, not every choice of $(w_1,\ldots,w_d) \in \mathbb{C}^d$ will result in $\sigma_{\underline{w}} \in MP(\hat{N},\hat{P})$. So we will only consider the following set

$$\begin{aligned}\mathcal{A} = \{(\sigma,\underline{w}) \in MP_0(N,P) \times \hat{\mathbb{C}}^d \ : \underline{w} = (\infty,\ldots,\infty), \text{ or} \\ Im(w_i) > 0 \text{ and } \sigma_{\underline{w}} \in MP(\hat{N},\hat{P})\}.\end{aligned}$$

5.3 Projections of $\mathcal{A}$ to $\mathcal{A}_{1,1}$ and $\mathcal{A}_{0,4}$

Now that we have defined the model space $\mathcal{A}$, we want to use the fact that $\mathcal{A}_{1,1}$ is not locally connected [21] to show that $\mathcal{A}$ is not locally connected. In this section, we show there is a continuous surjection $\Pi : \mathcal{A} \to \mathcal{A}_{1,1}$, and in the sequel we explain how this can be used to relate the local connectivity of $\mathcal{A}$ and $\mathcal{A}_{1,1}$.

By our choice of pants decomposition (see Fig. 5.2), the annulus $\gamma_2 \times [0,1]$ cuts N into two pieces, one of which is homeomorphic to $N_{1,1}$ (and so we will refer to this component as $N_{1,1}$). Given $\sigma : \pi_1(N) \to PSL(2,\mathbb{C})$, the restriction of σ to this punctured torus defines a representation $\sigma|_{\pi_1(N_{1,1})} : \pi_1(N_{1,1}) \to PSL(2,\mathbb{C})$.

Lemma 5.3. *If $\sigma \in MP_0(N,P)$, then $\sigma|_{\pi_1(N_{1,1})} \in MP_0(N_{1,1},P'_{1,1})$.*

Proof. If this restriction was not discrete, faithful, geometrically finite, and minimally parabolic with respect to $P'_{1,1} = P \cap N_{1,1}$ then σ would not be in $MP(N,P)$. Note that we are using the fact (attributed to Thurston) that finitely generated subgroups of geometrically finite Kleinian groups with nonempty domain of discontinuity are geometrically finite (Proposition 7.1 of [61]). Hence, $\sigma|_{\pi_1(N_{1,1})} \in MP(N_{1,1},P'_{1,1})$.

Recall in the previous section, we defined the notion of top and bottom. If $\sigma \in MP_0(N, P)$ then any embedding of $f : S \to \mathbb{H}^3/\sigma(\pi_1(N))$ such that $f_* = \sigma$ will divide $\mathbb{H}^3/\sigma(\pi_1(N))$ into a top and bottom piece such that the top piece contains all of the cusps.

If $\sigma|_{\pi_1(N_{1,1})}$ is in $MP(N_{1,1}, P'_{1,1}) \backslash MP_0(N_{1,1}, P'_{1,1})$, then for any proper embedding of $f_{1,1} : int(S_{1,1}) \to \mathbb{H}^3/\sigma|_{\pi_1(N_{1,1})}(\pi_1(N_{1,1}))$ inducing $\sigma|_{\pi_1(N_{1,1})}$, the cusp corresponding to γ_1 will lie to the bottom side of $f_{1,1}(int(S_{1,1}))$. Since $\sigma|_{\pi_1(N_{1,1})}$ is defined as the restriction of σ, this implies that the cusp corresponding to γ_1 in $\mathbb{H}^3/\sigma(\pi_1(N))$ lies on the bottom with respect to any embedding $f : S \to \mathbb{H}^3/\sigma(\pi_1(N))$. This contradicts that $\sigma \in MP_0(N, P)$. Thus $\sigma|_{\pi_1(N_{1,1})} \in MP_0(N_{1,1}, P'_{1,1})$. $\square$

Lemma 5.3 allows us to define the projection map in the following lemma.

Lemma 5.4. *The map* $\Pi : \mathcal{A} \to MP_0(N_{1,1}, P'_{1,1}) \times \hat{\mathbb{C}}$ *defined by*

$$\Pi(\sigma, w_1, \ldots, w_d) = (\sigma|_{\pi_1(N_{1,1})}, w_1).$$

is a continuous map such that $\Pi(\mathcal{A}) = \mathcal{A}_{1,1}$.

Proof. We first claim $\Pi(\mathcal{A}) \subset \mathcal{A}_{1,1}$. Recall the definitions of $\mathcal{A}$ and $\mathcal{A}_{1,1}$. If a point $(\sigma, w_1, \ldots, w_d) \in \mathcal{A}$ satisfies $(w_1, \ldots w_d) \neq (\infty, \ldots, \infty)$ then the extension $\sigma_{w_1,\ldots,w_d} \in MP(\hat{N}, \hat{P})$. There is a π_1-injective pared embedding $\iota : \hat{N}_{1,1} \to \hat{N}$, such that the representation $\sigma_{w_1,\ldots,w_d} \circ \iota_* : \pi_1(\hat{N}_{1,1}) \to PSL(2, \mathbb{C})$ is conjugate to the extension of $\sigma|_{\pi_1(N_{1,1})}$ by w_1. So this extended representation is discrete, faithful, geometrically finite, and minimally parabolic with respect to $\hat{P}_{1,1}$. Note that we are again using that finitely generated subgroups of geometrically finite groups are geometrically finite, provided the domain of discontinuity is nonempty. Thus the extension of $\sigma|_{\pi_1(N_{1,1})}$ by w_1 lies in $MP(\hat{N}_{1,1}, \hat{P}_{1,1})$, and so by the definition of $\mathcal{A}_{1,1}$, we have $\Pi(\sigma, w_1, \ldots, w_d) = (\sigma|_{\pi_1(N_{1,1})}, w_1) \in \mathcal{A}_{1,1}$. If $(w_1, \ldots, w_d) = (\infty, \ldots, \infty)$ then it follows immediately from Lemma 5.3 and the definition of Π that $\Pi(\sigma, \infty, \ldots, \infty) = (\sigma|_{\pi_1(N_{1,1})}, \infty) \in \mathcal{A}_{1,1}$.

We now use Klein-Maskit combination to show that $\mathcal{A}_{1,1} \subset \Pi(\mathcal{A})$. We begin by defining some new pared manifolds that arise as pieces of N and $\hat{N}$ (see Figure 5.3). Recall that $N = S \times I$ and $\hat{N} = N - \cup U_i$ where $\{U_i\}$ is a collection of disjoint collar neighborhoods of the curves $\gamma_1, \ldots, \gamma_d$. Recall $P \subset \partial N$ is a collection of d annuli in $S \times \{1\}$ whose core curves are homotopic (in N) to $\gamma_1, \ldots, \gamma_d$, and $\hat{P} \subset \partial \hat{N}$ consists of the d torus boundary components of $\hat{N}$. The annulus $\gamma_2 \times [0,1]$ divides N into two pieces. Let $N_{1,1}$ denote the closure of the piece containing γ_1 and let N_0 denote the closure of the remaining piece containing $\gamma_3, \ldots, \gamma_d$. Let $\hat{N}_{1,1} = N_{1,1} - U_1$ and $\hat{N}_0 = N_0 - \cup_{i=3}^d U_i$. Define $\hat{P}_{1,1} = \partial U_1 \cup (\gamma_2 \times [0,1])$ and $\hat{P}_0 = \left(\cup_{i=3}^d \partial U_i\right) \cup (\gamma_2 \times [0,1])$.

Next define $N_2 = N - \cup_{i \neq 2} U_i$ and set P_2 to be the union of the $d-1$ tori $\cup_{i\neq 2} \partial U_i$ with the annulus in $P \subset S \times \{1\}$ whose core curve is homotopic to γ_2. Roughly speaking, (N_2, P_2) is the pared manifold we will get by gluing $\hat{N}_{1,1}$ to $\hat{N}_0$ along $\gamma_2 \times [0,1]$. We then obtain $(\hat{N}, \hat{P})$ from (N_2, P_2) by drilling out γ_2. Again, refer to Figure 5.3.

For any $(\rho, w) \in \mathcal{A}_{1,1}$ ($w \neq \infty$), we have $\rho_w \in MP(\hat{N}_{1,1}, \hat{P}_{1,1})$. We can find a representation $\hat{\rho}_1 : \pi_1(\hat{N}_{1,1}) \to PSL(2,\mathbb{C})$ conjugate to ρ_w such that $\hat{\rho}_1(\gamma_2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B^+ = \{z \in \mathbb{C} \ : \ Im(z) > C_1\}$ and $B^- = \{z \in \mathbb{C} \ : \ Im(z) < -C_1\}$ are precisely invariant for the subgroup $\langle \hat{\rho}_1(\gamma_2) \rangle$ in $\hat{\rho}_1(\pi_1(\hat{N}_{1,1}))$. Moreover, we can find $\hat{\rho}_1$ such that the component of the domain of discontinuity $\Omega(\hat{\rho}_1)$ containing B^+ projects to the top surface in the conformal boundary $\Omega(\hat{\rho}_1)/\hat{\rho}_1(\pi_1(\hat{N}_{1,1}))$, and the component containing B^- projects to the bottom. Unlike $N_{1,1}$, the manifold

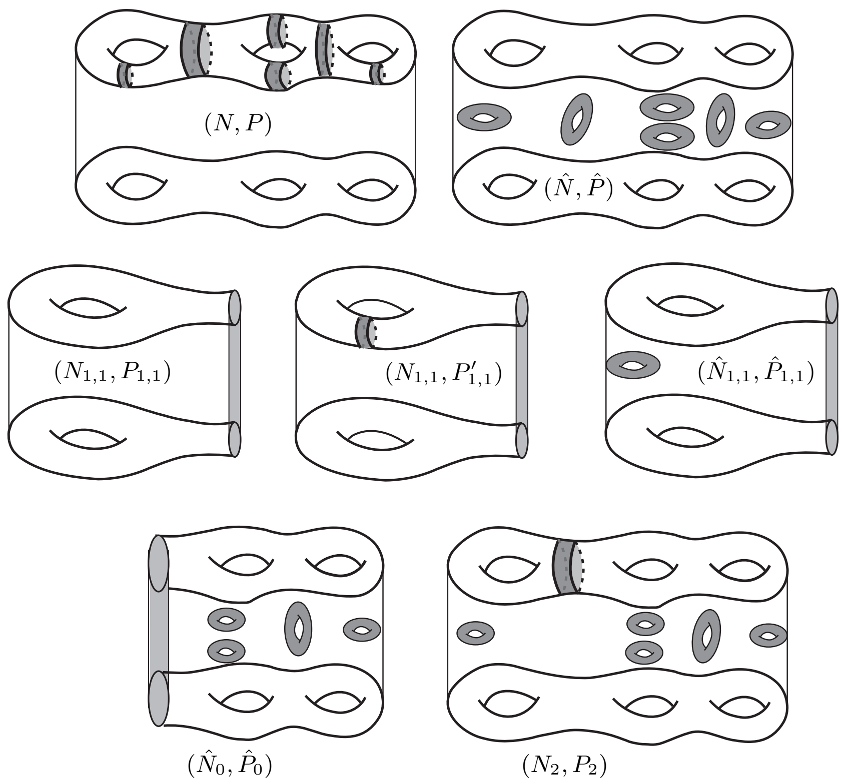

Figure 5.3: Some of the pared manifolds we are using (illustrated in genus 3). The shaded regions indicate the paring locus.

$\hat{N}_{1,1}$ is not an I-bundle so we make the distinction between the top and bottom components of $\Omega(\hat{\rho}_1)/\hat{\rho}_1(\pi_1(\hat{N}_{1,1}))$ in the following way. There is a discrete, faithful representation $\sigma_1 : \pi_1(N_{1,1}) \to \hat{\rho}_1(\pi_1(\hat{N}_{1,1}))$ such that the conjugacy class of σ_1 is $\rho \in MP_0(N_{1,1}, P'_{1,1})$. This follows from the fact that ρ_w is an extension of ρ (and that ρ_w is conjugate to $\hat{\rho}_1$). Since $\sigma_1(\pi_1(N_{1,1})) \subset \hat{\rho}_1(\pi_1(\hat{N}_{1,1}))$, we have $\Omega(\hat{\rho}_1) \subset \Omega(\sigma_1)$. Since σ_1 is a representation of a surface group, there is a well-defined top component of $\Omega(\sigma_1)/\sigma_1(\pi_1(N_{1,1}))$. So we say $B^+ \subset \Omega(\hat{\rho}_1) \subset \Omega(\sigma_1)$ projects to the top surface in the conformal boundary $\Omega(\hat{\rho}_1)/\hat{\rho}_1(\pi_1(\hat{N}_{1,1}))$ if it projects to the top surface in $\Omega(\sigma_1)/\sigma_1(\pi_1(N_{1,1}))$.

Next, let $\hat{M}_0$ be a geometrically finite hyperbolic 3-manifold homeomorphic to the interior of $\hat{N}_0$ whose only cusps are those associated to $\hat{P}_0$. Let h_0 be an orientation preserving pared homeomorphism from $(\hat{N}_0, \hat{P}_0)$ to the relative compact core of $\hat{M}_0$. The boundary $\partial\hat{N}_0 - \hat{P}_0$ has a top and bottom component, both of which are homeomorphic to a punctured genus $g-1$ surface $S_{g-1,1}$. We will call these $S_{g-1,1,top}$ and $S_{g-1,1,bot}$. The homeomorphism h_0 distinguishes a top and bottom component of the relative compact core of $\hat{M}_0$ and thus distinguishes a top and bottom of the conformal boundary of $\hat{M}_0$. Define a representation $\hat{\rho}_0 : \pi_1(\hat{N}_0) \to PSL(2,\mathbb{C})$ conjugate to $(h_0)_*$ such that

(a) $\hat{\rho}_0(\gamma_2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

(b) $B_0 = \{z \in \mathbb{C} \ : \ Im(z) > -C_1 - 1\}$ is a precisely invariant set for the subgroup $\langle\hat{\rho}_0(\gamma_2)\rangle$ in $\hat{\rho}_0(\pi_1(\hat{N}_0))$

(c) the component of $\Omega(\hat{\rho}_0)$ containing B_0 projects to the bottom conformal boundary component of $\hat{M}_0$ (note that $\hat{M}_0$ is isometric to $\mathbb{H}^3/\hat{\rho}_0(\pi_1(\hat{N}_0))$ since $\hat{\rho}_0$ is conjugate to $(h_0)_*$).

Let $\sigma_0 = \hat{\rho}_0|_{\pi_1(S_{g-1,1,bot})}$. That is, σ_0 is the restriction of $\hat{\rho}_0$ to the natural inclusion of the fundamental group of the bottom surface into $\pi_1(\hat{N}_0)$. Thus, $\mathbb{H}^3/\sigma_0(\pi_1(S_{g-1,1}))$ has a rank-1 cusp associated to each of the curves $\gamma_2, \ldots, \gamma_d$, and all of these cusps are on the top since the representation σ_0 was constructed from the inclusion of the bottom surface into $\hat{M}_0$. Hence, $\sigma_0 \in MP_0(S_{g-1,1} \times I, (S_{g-1,1} \times \{1\}) \cap P)$. Here, the intersection $(S_{g-1,1} \times \{1\}) \cap P$ is defined by naturally identifying $S_{g-1,1} \times I$ with the component of $(S \times I) - (\gamma_2 \times I)$ not containing γ_1. In other words, the paring locus is a collection of $d-1$ annuli in $S_{g-1,1} \times \{1\}$ whose core curves are $\gamma_i \times \{1\}$ for $i = 2, \ldots, d$.

Now we can apply type I Klein-Maskit combination along the subgroup $\langle \hat{\rho}_0(\gamma_2) \rangle = \langle \hat{\rho}_1(\gamma_2) \rangle$. See Section 2.1.4 for references and notation. Note that $\pi_1(N_2)$ is the amalgamated free product of $\pi_1(\hat{N}_{1,1})$ and $\pi_1(\hat{N}_0)$ along the infinite cyclic subgroup corresponding to γ_2. Define a representation $\rho_2 : \pi_1(N_2) \to PSL(2, \mathbb{C})$ by setting $\rho_2(x) = \hat{\rho}_0(x)$ for all $x \in \pi_1(\hat{N}_0)$ and $\rho_2(x) = \hat{\rho}_1(x)$ for all $x \in \pi_1(\hat{N}_{1,1})$. By construction, the representation ρ_2 is discrete, faithful, and geometrically finite. Moreover, it is minimally parabolic with respect to the pared manifold (N_2, P_2).

We can also apply Klein-Maskit combination to the subgroups $\sigma_1(\pi_1(S_{1,1}))$ and $\sigma_0(\pi_1(S_{g-1,1}))$. Note that $\pi_1(S)$ is the amalgamated free product of $\pi_1(S_{1,1})$ and $\pi_1(S_{g-1,1})$ along the subgroup $\mathbb{Z}$ generated by γ_2. We can define a representation $\sigma : \pi_1(S) \to PSL(2, \mathbb{C})$ by $\sigma(x) = \sigma_1(x)$ for all $x \in \pi_1(S_{1,1})$ and $\sigma(x) = \sigma_0(x)$ for $x \in \pi_1(S_{g-1,1})$. This defines a discrete, faithful, geometrically finite representation whose parabolics consist precisely of the curves γ_i in S. Hence $\sigma \in MP(N, P)$. Recall $\sigma_1 \in MP_0(N_{1,1}, P'_{1,1})$ and $\sigma_0 \in MP_0(S_{g-1,1} \times I, (S_{g-1,1} \times \{1\}) \cap P)$. It follows immediately from this construction that $\gamma_1, \gamma_3, \ldots, \gamma_d$ are cusped to the top in $\mathbb{H}^3/\sigma(\pi_1(S))$. It also follows, since the precisely invariant sets B^- and B_0 corresponded to the bottoms of their respective manifolds, that γ_2 is cusped to the top. So $\sigma \in MP_0(N, P)$.

To summarize, we now have a representation $\sigma \in MP_0(N, P)$ such that $\sigma|_{\pi_1(N_{1,1})} = \sigma_1$ is conjugate to ρ. The next step is to find $w_1, \ldots, w_d$ such that $\Pi(\sigma, w_1, \ldots, w_d) = (\rho, w)$.

Since $\pi_1(\hat{N})$ is generated by $\pi_1(N_2)$ and the meridian of the second torus boundary component ∂U_2, we can extend the representation ρ_2 to a representation $\hat{\rho} : \pi_1(\hat{N}) \to PSL(2, \mathbb{C})$ by defining $\hat{\rho}$ to equal ρ_2 on $\pi_1(N_2)$ and sending the additional generator to $\begin{pmatrix} 1 & w_2 \\ 0 & 1 \end{pmatrix}$ for $w_2 \in \mathbb{C}$. There is some constant C_2 such that if $Im(w_2) > C_2$, then we can apply type II Klein-Maskit combination (see Section 2.1.4). In this case, $\hat{\rho}$ defines a geometrically finite representation of $\pi_1(\hat{N})$ which is minimally parabolic with respect to $(\hat{N}, \hat{P})$.

On each torus boundary component of $\hat{N}$ there is a well-defined meridian. Up to conjugation (that depends on i), we can assume $\hat{\rho}$ sends γ_i to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. In this case, there will be some w_i such that the meridian of ∂U_i is sent to $\begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix}$. With this definition of w_i we now have a point $(\sigma, w_1, \ldots, w_d) \in \mathcal{A}$.

Next, we check that $w_1 = w$ and therefore $\Pi(\sigma, w_1, \ldots, w_d) = (\rho, w)$. This follows by construction. The extension of σ by $(w_1, \ldots, w_d)$ is conjugate to $\hat{\rho}$. The restriction of $\hat{\rho}$ to $\pi_1(\hat{N}_{1,1})$ is conjugate to ρ_w (which is conjugate to $\hat{\rho}_1$). Thus $w_1 = w$.

Finally, if $(\rho, \infty) \in \mathcal{A}_{1,1}$, then we can pick any $w' \neq \infty$ such that $(\rho, w') \in \mathcal{A}_{1,1}$. Following the same construction, we can find a point $(\sigma, w', w_2, \ldots, w_d) \in \mathcal{A}$ such that $\sigma|_{\pi_1(N_{1,1})}$ is conjugate to ρ. Then the point $(\sigma, \infty, \ldots, \infty)$ is also in $\mathcal{A}$ and satisfies $\Pi(\sigma, \infty, \ldots, \infty) = (\rho, \infty)$. $\square$

Recall that each γ_i was contained in a four-punctured sphere or punctured torus component of $S - \cup_{j \neq i} \gamma_j$. Given $\sigma \in MP_0(N, P)$, one can define Π_i similarly to $\Pi = \Pi_1$. The first coordinate is obtained by restricting σ to the ith such subsurface and the second coordinate is defined by projecting $(w_1, \ldots, w_d) \mapsto w_i$. Lemma 5.4 generalizes to show that $\Pi_i(\mathcal{A}) = \mathcal{A}_{1,1}$ if γ_i lives in a punctured torus or $\Pi_i(\mathcal{A}) = \mathcal{A}_{0,4}$ if γ_i lives in a four-punctured sphere. In fact, for $i > 1$, we will only need the first paragraph of the proof of this Lemma which shows that $\Pi_i(\mathcal{A}) \subset \mathcal{A}_{1,1}$ or $\Pi_i(\mathcal{A}) \subset \mathcal{A}_{0,4}$.

We now get the following corollary to Lemmas 5.2 and 5.4.

Corollary 5.5. *For all $(\sigma, w_1, \ldots, w_d) \in \mathcal{A}$ with $(w_1, \ldots, w_d) \neq (\infty, \ldots, \infty)$, the imaginary part of w_i is bounded below by*

$$Im(w_i) > 1$$

for all i.

5.4 $\mathcal{A}$ is Not Locally Connected

In Lemma 4.14 of [21], Bromberg shows that there exists a point $(\sigma_z, \infty) \in \mathcal{A}_{1,1}$ at which $\mathcal{A}_{1,1}$ fails to be locally connected. We will use Lemma 5.4 to extend this failure of local connectivity to $\mathcal{A}$. In our notation, Lemma 4.14 of [21] can be restated as follows:

Lemma 5.6 (Bromberg [21])**.** *There exists a point $z^0 \in \mathcal{M}^+$, a closed rectangle R, and some $\delta > 0$ such that if O is the δ-neighborhood of z^0 in $\mathbb{C}$ then $O \subset \mathcal{M}^+$ and for all $z \in O$, the set*

$$\mathcal{A}_z = \{w \in \hat{\mathbb{C}} \ : \ (\sigma_z, w) \in \mathcal{A}_{1,1}\}$$

satisfies

(i) $\mathcal{A}_z \cap int(R) \neq \emptyset$*, and*

(ii) The distance between $\overline{\mathcal{A}_z}$ *and* ∂R *is at least* δ*.*

Moreover, we can choose R *such that its sides are parallel to the axes and its width is* < 2*.*

As our statement of Lemma 5.6 and the notation we use differs somewhat from Lemma 4.14 of [21], we include a proof. Bromberg states a weaker version of (*ii*) that says $\overline{\mathcal{A}_z} \cap \partial R = \emptyset$; however, his proof really shows the version we have stated here.

Proof. We claim there exists a $\delta > 0$, a rectangle

$$Q = \{x + iy \in \mathbb{C} \,:\, 0 < 4\delta < y_0 \leq y \leq y_1,\ 0 \leq x_0 \leq x \leq x_1 < 2\}$$

and a point $q \in \mathbb{C}$ with $Im(q) = \frac{y_0+2y_1}{3}$ such that

(1) the δ-ball centered at q is contained in $\mathcal{M}^+$,

(2) the 4δ-neighborhood of $\mathcal{M}^+$ is disjoint from the vertical sides of ∂Q and the horizontal line with imaginary part y_0 (*i.e.*, the horizontal line containing the lower horizontal side of ∂Q). The existence of δ, Q, and q follows from several properties of the Maskit slice. The Maskit slice is invariant under the translation $z \mapsto z + 2$, and $\partial\mathcal{M}^+$ is a Jordan curve contained in the upper half plane that is not a horizontal line [21, 39, 46, 60, 76]. One can find $0 < y_0 < y_1$ and $0 \leq x_0 < x_1 < 2$ such that $\overline{\mathcal{M}^+}$ is disjoint from the horizontal line with imaginary part y_0, the vertical sides of ∂Q are disjoint from $\overline{\mathcal{M}^+}$, and there is a point $q \in \mathcal{M}^+$ with $Im(q) = \frac{y_0+2y_1}{3}$. One can then choose $\delta > 0$ smaller than the following four quantities: $\frac{y_0}{4}$, the distance between q and $\partial\mathcal{M}^+$, one fourth of the distance between $\partial\mathcal{M}^+$ and the union of the

vertical sides of Q, and one fourth of the distance between $\partial\mathcal{M}^+$ and the horizontal line with imaginary part y_0.

Let $z^0 = 3q$, and define

$$\begin{aligned} R &= \{w \in \mathbb{C} \, : \, z^0 - w \in Q\} \\ &= \{w \, : \, y_0 + y_1 \leq Im(w) \leq 2y_1, \; Re(3q) - x_1 \leq Re(w) \leq Re(3q) - x_0\}. \end{aligned}$$

Note that since $q \in \mathcal{M}^+$, we must have $Im(q) > 1$ (see [76], p. 558 of [46]). Hence $Im(z^0) > 3$. Since any point with imaginary part > 2 is contained in $\mathcal{M}^+$ (see Proposition 2.6 of [39]), a 1-neighborhood of z^0 is contained in $\mathcal{M}^+$. Since the δ satisfying the properties above must be less than 1, a δ-neighborhood, O, of z^0 is contained in $\mathcal{M}^+$.

Recall from part (i) of Lemma 5.1, a point w lies in $\mathcal{A}_z - \{\infty\}$ if and only if there exists some n such that $z - nw \in \mathcal{M}^+$ and $z - (n+1)w \in \mathcal{M}^-$. We claim $2q \in \mathcal{A}_z$ for all $z \in O$. Let $n = 1$ in the criterion above. Since $z - 2q$ is within δ of $z^0 - 2q = 3q - 2q = q$, and a δ-ball about q is contained in $\mathcal{M}^+$, we must have $z - 2q \in \mathcal{M}^+$ for all $z \in O$. Likewise $z - 2(2q)$ is in a δ-ball about $-q$, which is in $\mathcal{M}^-$ by symmetry. It also follows directly from the definition of R that $2q \in int(R)$. Thus, for all $z \in O$, $\mathcal{A}_z \cap int(R)$ contains $2q$; hence $\mathcal{A}_z \cap int(R) \neq \emptyset$.

Next we show that if w is within δ of ∂R then $w \notin \overline{\mathcal{A}_z}$ for any $z \in O$. First consider any w in a δ-neighborhood of R. Clearly

$$y_0 + y_1 - \delta < Im(w) < 2y_1 + \delta.$$

Hence $Im(z^0 - w) = Im(3q) - Im(w) = (y_0 + 2y_1) - Im(w) > y_0 - \delta$ and $Im(z^0 - 2w) < -y_0 + 2\delta$. Thus, if $z \in O$, $Im(z - w) > y_0 - 2\delta > 0$ and $Im(z - 2w) < -y_0 + 3\delta < 0$.

Thus for any w in a δ-neighborhood of R and any $z \in O$, if there was an n such that $z - nw \in \overline{\mathcal{M}^+}$ and $z - (n+1)w \in \overline{\mathcal{M}^-}$ then it must be the case that $n = 1$

since $Im(z-w) > y_0 - 2\delta > 0$, $Im(z-2w) < -y_0 + 3\delta < 0$. Thus, if w is within δ of R and $z \in O$ then either of the conditions $z - w \notin \overline{\mathcal{M}^+}$ or $z - 2w \notin \overline{\mathcal{M}^-}$ would imply $w \notin \overline{\mathcal{A}_z}$.

Let w be in the δ-neighborhood of the union of the two vertical sides of ∂R and the top horizontal side of ∂R. For any $z \in O$, we have $z - w$ is in the 2δ-neighborhood of the union of the two vertical sides of ∂Q and the lower horizontal side of ∂Q. Since these three sides of Q were at least a distance of 4δ away from $\mathcal{M}^+$, we must have $w \notin \overline{\mathcal{A}_z}$ for any $z \in O$.

Now let w be in a δ-neighborhood of the lower horizontal side of ∂R.

$$y_0 + y_1 - \delta < Im(w) < y_0 + y_1 + \delta.$$

For any $z \in O$,

$$-y_0 - 3\delta < Im(z-2w) < -y_0 + 3\delta.$$

Since the horizontal line with imaginary part y_0 is disjoint from a 4δ-neighborhood of $\mathcal{M}^+$, we must have $z - 2w \notin \overline{\mathcal{M}^-}$. Hence, a δ-neighborhood of ∂R is disjoint from $\overline{\mathcal{A}_z}$ for all $z \in O$. $\square$

Let W be an open neighborhood of σ_{z^0} in $MP_0(N_{1,1}, P'_{1,1})$ such that for all $\sigma_z \in \overline{W}$, $z \in O$. In other words, if $\tau : \mathcal{M}^+ \to MP_0(N_{1,1}, P'_{1,1})$ is the homeomorphism $z \mapsto \sigma_z$, then W is a neighborhood of σ_{z^0} such that $\tau^{-1}(\overline{W}) \subset O$.

By Lemma 5.6, $\mathcal{A}_{z^0} \cap int(R) \neq \emptyset$, so let $w_1^0 \in int(R)$ such that $(\sigma_{z^0}, w_1^0) \in \mathcal{A}_{1,1}$. Lemma 5.4 shows that $\Pi : \mathcal{A} \to \mathcal{A}_{1,1}$ is a surjection, thus there is some $(\sigma^0, w_1^0, \ldots, w_d^0) \in \Pi^{-1}(\sigma_{z^0}, w_1^0)$. Note that since $(\sigma^0, w_1^0, \ldots, w_d^0) \in \mathcal{A}$, we have $\sigma^0 \in MP_0(N, P)$. Thus, by the definition of $\mathcal{A}$, we also have $(\sigma^0, \infty, \ldots, \infty) \in \mathcal{A}$.

Let U be a neighborhood of $(\sigma^0, \infty, \ldots, \infty)$ in $\mathcal{A}$ such that for all $(\sigma, w_1, \ldots, w_d)$ in U, the first coordinate of $\Pi(\sigma, w_1, \ldots, w_d)$ lies in W. That is, for all $(\sigma, w_1, \ldots, w_d) \in$

U, $\sigma|_{\pi_1(N_{1,1})} \in W$.

For each n, let C_n be the collection of components of U defined by

$$C_n = U \cap \Pi^{-1}(W \times (R + 2n))).$$

Equivalently,

$$C_n = \{(\sigma, w_1, \ldots, w_d) \in U \ : \ w_1 \in (R + 2n)\}.$$

Here, $R + 2n$ denotes the box R translated by $2n$.

Observe that $(\sigma^0, w_1^0, \ldots, w_d^0) \in \Pi^{-1}(W \times R)$, since W was defined to be a neighborhood of σ_{z^0} and w_1^0 was defined to be in $int(R)$. Next we claim that $(\sigma^0, w_1^0 + 2n, \ldots, w_d^0 + 2n) \in \mathcal{A}$ for all n. The definition of $(\sigma^0, w_1^0, \ldots, w_d^0)$ belonging to the set $\mathcal{A}$ is that $\sigma^0_{(w_1^0, \ldots, w_d^0)} \in MP(\hat{N}, \hat{P})$. The representation defined to be the extension of σ^0 by $(w_1^0 + 2n, \ldots, w_d^0 + 2n)$ (as in Section 5.2) has the same image as the extension of σ^0 by $(w_1^0, \ldots, w_d^0)$. Hence, $(\sigma^0, w_1^0 + 2n, \ldots, w_d^0 + 2n) \in \mathcal{A}$, and therefore $(\sigma^0, w_1^0 + 2n, \ldots, w_d^0 + 2n) \in \Pi^{-1}(W \times (R + 2n))$.

Since these points converge to $(\sigma^0, \infty, \ldots, \infty)$ as $n \to \infty$, we have $(\sigma^0, w_1^0 + 2n, \ldots, w_d^0 + 2n) \in U$ for all but finitely many n. Hence, C_n is nonempty for all but finitely many n.

For any $(\sigma, w_1, \ldots, w_d) \in \overline{U}$, we have $\sigma|_{\pi_1(N_{1,1})} \in \overline{W}$. Since $\overline{W} \subset \tau^{-1}(O)$, if we let $z = \tau(\sigma|_{\pi_1(N_{1,1})})$, then Lemma 5.6 implies that $\overline{\mathcal{A}_z}$ and ∂R are at least a distance δ apart. Since the set $\mathcal{A}_z$ is invariant under the translation $w \mapsto w + 2$, points in $\overline{\mathcal{A}_z} \cap (R + 2n)$ are bounded away from points in $\overline{\mathcal{A}_z} \cap (\mathbb{C} - (R + 2n))$ by a distance of at least 2δ. This gives us a lower bound on the distance between points in $\overline{C_n}$ and $\overline{U - C_n}$ that is independent of n. By using δ instead of 2δ, we can make the inequality strict.

Lemma 5.7. *There exists some $\delta > 0$ such that for any $(\sigma, w_1, \ldots, w_d) \in \overline{C_n}$ and any $(\sigma', w'_1, \ldots, w'_d) \in \overline{U - C_n}$ we have*

$$|w_1 - w'_1| > \delta.$$

for all n.

Since C_n is nonempty for all but finitely many n, Lemma 5.7 shows that U has infinitely many components. Note that we do not need that $\overline{C_n}$ and $\overline{U - C_n}$ are disjoint to conclude this, but we will need the full strength of Lemma 5.7 in the following chapter.

Moreover, any neighborhood $(\sigma^0, \infty, \ldots, \infty) \in U' \subset U$ will have infinitely many components. Hence, we have shown

Proposition 5.8. *There is a point $\sigma^0 \in MP_0(N, P)$ such that $\mathcal{A}$ is not locally connected at $(\sigma^0, \infty, \ldots, \infty)$.*

5.5 Definition of Φ

Now that we have defined $\mathcal{A}$ and shown that it fails to be locally connected at some point $(\sigma^0, \infty, \ldots, \infty)$, we want to construct a map Φ from a subset of $\mathcal{A}$ containing this point into $AH(S \times I)$. In this section, we show that Φ is well-defined on some subset of $\mathcal{A}$ and in the subsequent sections, we will show that Φ is a local homeomorphism at $(\sigma^0, \infty, \ldots, \infty)$.

As in Section 3 of [21], we construct the map Φ in two steps. Heuristically, points in $\mathcal{A}$ with $(w_1, \ldots, w_d) = (\infty, \ldots, \infty)$ parameterize $MP_0(N, P)$, and points $(\sigma, w_1, \ldots, w_d) \in \mathcal{A}$ parameterize a subset of $MP(\hat{N}, \hat{P})$. For points $(\sigma, \infty, \ldots, \infty) \in \mathcal{A}$, the representation $\sigma \in MP_0(N, P) \subset AH(N)$, so we will define $\Phi(\sigma, \infty, \ldots, \infty) = \sigma$. For all other points $(\sigma, w_1, \ldots, w_d) \in \mathcal{A}$, the representation $\sigma_{(w_1, \ldots, w_d)} \in MP(\hat{N}, \hat{P})$

and so $\mathbb{H}^3/\sigma_{(w_1,\ldots,w_d)}(\pi_1(\hat{N}))$ is a marked hyperbolic manifold with d rank-2 cusps. For these points, we will define $\Phi(\sigma, w_1, \ldots, w_d)$ to be the marked hyperbolic manifold in $MP(N)$ obtained by filling in these cusps. We use the filling theorem to show that Φ is well-defined on some subset of $\mathcal{A}$ and that Φ is continuous.

Let $(\sigma, w) \in \mathcal{A}$ such that $w \neq (\infty, \ldots, \infty)$ (we will drop the underline on w when there can be no ambiguity whether w refers to a coordinate in $\hat{\mathbb{C}}$ or $\hat{\mathbb{C}}^d$). By the definition of $\mathcal{A}$ we have that $\sigma_w \in MP(\hat{N}, \hat{P})$. Let $\hat{M}_{\sigma,w} = \mathbb{H}^3/\sigma_w(\pi_1(\hat{N}))$ be the corresponding geometrically finite manifold with d cusps.

Recall that ϵ_3 denotes the Margulis constant in dimension 3. By Corollary 4.13, there is a constant K such that if

$$\frac{|w_i|}{\sqrt{2Im(w_i)}} > K$$

for all i, then we can β_i-fill the ith cusp ($i = 1, \ldots, d$) to get a hyperbolic manifold $M_{\sigma,w}$ with the same conformal boundary as $\hat{M}_{\sigma,w}$, and there exists a biLipschitz diffeomorphism

$$\phi_{\sigma,w} : \hat{M}_{\sigma,w} - \mathbb{T}_{\epsilon_3}(T) \to M_{\sigma,w} - \mathbb{T}_{\epsilon_3}(\gamma).$$

Here T denotes the union of the cusps T_i and γ denotes the union of the curves γ_i.

Define

$$\mathcal{A}_K = \{(\sigma, w) \in \mathcal{A} \ : \ w = (\infty, \ldots, \infty), \text{ or } \frac{|w_i|}{\sqrt{2Im(w_i)}} > K \text{ for all } i\}.$$

Recall that $\sigma \in MP_0(N, P)$ can be identified with a marked hyperbolic manifold (M_σ, f_σ). Without loss of generality, we can assume that f_σ is a smooth immersion and that $f_\sigma(N)$ does not intersect the ϵ_3-parabolic thin part of M_σ since f_σ is only defined up to homotopy. As $\sigma(\pi_1(N))$ is a subgroup of $\sigma_w(\pi_1(\hat{N}))$ we have a covering map

$$\pi_{\sigma,w} : M_\sigma \to \hat{M}_{\sigma,w}$$

Now define

$$f_{\sigma,w} = \phi_{\sigma,w} \circ \pi_{\sigma,w} \circ f_\sigma.$$

Since we have assumed $f_\sigma(N)$ avoids the ϵ_3-parabolic thin part of M_σ, $\pi_{\sigma,w} \circ f_\sigma(N)$ is contained in $\hat{M}_{\sigma,w} - \mathbb{T}_{\epsilon_3}(T)$. Thus, post-composition by the filling map $\phi_{\sigma,w}$ makes sense here.

We next claim (as in Lemma 3.3 of [21]) that $(f_{\sigma,w})_*$ is an isomorphism from $\pi_1(N)$ to $\pi_1(M_{\sigma,w})$ and therefore $(M_{\sigma,w}, f_{\sigma,w}) \in AH(N)$. First observe that f_σ is a homotopy equivalence so we only need to show that

$$(\phi_{\sigma,w})_* \circ (\pi_{\sigma,w})_* : \pi_1(M_\sigma) \to \pi_1(M_{\sigma,w})$$

is an isomorphism. Recall $\pi_1(\hat{N}) = \langle \pi_1(N), \beta_1, \ldots, \beta_d \mid [\beta_i, \gamma_i] = 1 \rangle$. By the definition of the covering map $\pi_{\sigma,w}$ and the definition of the representation σ_w,

$$\pi_1(\hat{M}_{\sigma,w}) = \langle (\pi_{\sigma,w})_*(\pi_1(M_\sigma)), \sigma_w(\beta_1), \ldots, \sigma_w(\beta_d) \mid [\sigma_w(\beta_i), \sigma_w(\gamma_i)] = 1 \rangle \,.$$

Now the filling map $(\phi_{\sigma,w})_*$ kills the meridians in $\hat{M}_{\sigma,w}$ which were precisely the group elements $\sigma_w(\beta_i)$. Thus

$$(\pi_{\sigma,w})_*(\pi_1(M_\sigma)) \cap Ker((\phi_{\sigma,w})_*) = \{1\}$$

and therefore $(\phi_{\sigma,w})_* \circ (\pi_{\sigma,w})_*$ is an isomorphism from $\pi_1(M_\sigma)$ onto its image, which is $\pi_1(M_{\sigma,w})$.

Moreover, as the filling preserves the conformal boundary components of $\hat{M}_{\sigma,w}$ and the filled manifold $M_{\sigma,w}$ has no cusps, $(f_{\sigma,w})_*$ is a minimally parabolic, geometrically finite representation in $AH(N)$.

When $w = (\infty, \ldots, \infty)$, we define $\Phi(\sigma, \infty, \ldots, \infty) = \sigma \in MP_0(N, P)$.

So we have

$$\Phi(\sigma, w) = \begin{cases} (f_{\sigma,w})_* & \text{if } w \neq (\infty, \ldots, \infty) \\ \\ \sigma & \text{if } w = (\infty, \ldots, \infty). \end{cases}$$

Thus we have defined Φ on some subset $\mathcal{A}_K \subset \mathcal{A}$ such that $\Phi(\mathcal{A}_K) \subset MP(N) \cup MP_0(N, P)$.

Lemma 5.9. *The map Φ is continuous on $\mathcal{A}_K$.*

Proof. Let (σ_0, w_0) be a point in $\mathcal{A}_K$ with $w_0 \neq (\infty, \ldots, \infty)$. Let B be the component of $(\mathcal{A}_K - \{(\sigma, w) : w = (\infty, \ldots, \infty)\})$ containing (σ_0, w_0). Clearly the correspondence $(\sigma, w) \mapsto \sigma_w$ is a continuous map from $(\mathcal{A}_K - \{(\sigma, w) : w = (\infty, \ldots, \infty)\})$ to $MP(\hat{N}, \hat{P})$ and thus takes the component B into one of the components C of $MP(\hat{N}, \hat{P})$. Recall from Chapter 2 that $C = F^{-1}([(\hat{N}_C, \hat{P}_C), h_C])$ for some $[(\hat{N}_C, \hat{P}_C), h_C] \in A(\hat{N}, \hat{P})$. For any point $(\hat{M}_{\hat{\rho}}, f_{\hat{\rho}}) \in C$, the map $f_{\hat{\rho}} \circ h_C^{-1}$ is homotopic to a pared homeomorphism from $(\hat{N}_C, \hat{P}_C)$ to the relative compact core of $\hat{M}_{\hat{\rho}}$, and thus we can use $f_{\hat{\rho}} \circ h_C^{-1}$ to define a marking from $\partial \hat{N}_C - \hat{P}_C$ to the conformal boundary of $\hat{M}_{\hat{\rho}}$. The Ahlfors-Bers parameterization $\widehat{\mathcal{AB}}_C : C \to \mathcal{T}(\partial \hat{N}_C - \hat{P}_C)$ is defined by sending $(\hat{M}_{\hat{\rho}}, f_{\hat{\rho}})$ to the conformal boundary of $\hat{M}_{\hat{\rho}}$ marked by $f_{\hat{\rho}} \circ h_C^{-1}$. Similarly, let $\mathcal{AB} : MP(N) \to \mathcal{T}(\partial N)$ be the Ahlfors-Bers parameterization of $MP(N)$.

For any $(\sigma, w) \in B$, we showed in the definition of Φ that $(f_{\sigma,w})_*$ is an isomorphism, which implies $f_{\sigma,w}$ is homotopic to a homeomorphism [75]. Thus the $\cup h_C(\beta_i)$-Dehn filling of $(\hat{N}_C, \hat{P}_C)$ is homeomorphic to N, where $\cup h_C(\beta_i)$ refers to the collection of filling slopes corresponding to $\beta_1, \ldots, \beta_d$ under the homotopy equivalence $h_C : (\hat{N}, \hat{P}) \to (\hat{N}_C, \hat{P}_C)$. This filling gives us an inclusion $i_C : (\hat{N}_C, \hat{P}_C) \to N$ which defines a homeomorphism $i_C : (\partial \hat{N}_C - \hat{P}_C) \to \partial N$. Using this homeomorphism, we can identify $\mathcal{T}(\partial \hat{N}_C - \hat{P}_C)$ with $\mathcal{T}(\partial N) \cong \mathcal{T}(S) \times \mathcal{T}(S)$.

With this identification of the Teichmüller spaces of $(\partial\hat{N}_C - \hat{P}_C)$ and ∂N, it follows that $\Phi(\sigma, w) = \mathcal{AB}^{-1} \circ \widehat{\mathcal{AB}}_C(\sigma_w)$ for any $(\sigma, w) \in B$ since the filling map $\phi_{\sigma,w}$ extends to a conformal map from the conformal boundary of $\hat{M}_{\sigma,w}$ to the conformal boundary of $M_{\sigma,w}$. Since the Ahlfors-Bers maps are homeomorphisms, this shows that Φ is continuous on the component B of $(\mathcal{A}_K - \{(\sigma, w) : w = (\infty, \ldots, \infty)\})$ containing (σ_0, w_0). Since (σ_0, w_0) was arbitrary, we have that Φ is continuous on all of $(\mathcal{A}_K - \{(\sigma, w) : w = (\infty, \ldots, \infty)\})$.

Next, we show Φ is continuous at points where $w = (\infty, \ldots, \infty)$. Suppose

$$(\sigma_i, w_{1,i}, \ldots, w_{d,i}) \to (\sigma, \infty, \ldots, \infty).$$

We claim that $\Phi(\sigma_i, w_{1,i}, \ldots, w_{d,i}) \to \Phi(\sigma, \infty, \ldots, \infty) = \sigma$. If $(w_{1,i}, \ldots, w_{d,i}) = (\infty, \ldots, \infty)$ for all i, then clearly $\Phi(\sigma_i, w_{1,i}, \ldots, w_{d,i}) = \sigma_i \to \sigma$.

Now suppose that $(w_{1,i}, \ldots, w_{d,i}) \neq (\infty, \ldots, \infty)$ for all i. Let (M_σ, f_σ) be the marked hyperbolic 3-manifold corresponding to σ. Again, assume that f_σ is smooth. Since $\sigma_i \to \sigma$, there is a sequence $L_i \to 1$ and smooth homotopy equivalences $g_i : M_\sigma \to M_{\sigma_i}$ such that $(g_i \circ f_\sigma)_* = \sigma_i$ and g_i is an L_i-biLipschitz local diffeomorphism on a compact core of M_σ (*i.e.*, the maps g_i converge to a local isometry). If we let $f_{\sigma_i} = g_i \circ f_\sigma$, then the pullback metrics on N via f_{σ_i} converge to the pullback metrics on N via f_σ. See p. 154 of [14] for this geometric definition of algebraic convergence (see also p. 43 of [57]).

Recall that by definition, $\Phi(\sigma_i, \underline{w}_i) = (M_{\sigma,\underline{w}_i}, f_{\sigma_i,\underline{w}_i})$ where

$$f_{\sigma_i,\underline{w}_i} = \phi_{\sigma_i,\underline{w}_i} \circ \pi_{\sigma_i,\underline{w}_i} \circ f_{\sigma_i}.$$

Since each $w_{j,i} \to \infty$, we can find a sequence $J_i \to 1$ such that $\phi_{\sigma_i,\underline{w}_i}$ is J_i-biLipschitz away from the ϵ_3-neighborhood of the cusps of $\hat{M}_{\sigma_i,\underline{w}_i}$. In particular, $\phi_{\sigma_i,\underline{w}_i}$ is J_i-biLipschitz on $\pi_{\sigma_i,\underline{w}_i}(f_{\sigma_i}(N))$.

It follows that the limit of the pullback metrics on N via the maps $f_{\sigma_i,\underline{w}_i} : N \to M_{\sigma_i,\underline{w}_i}$ is the same as the limit of the pullback metrics on N via $\pi_{\sigma_i,\underline{w}_i} \circ f_{\sigma_i}$ since $f_{\sigma_i,\underline{w}_i} = \phi_{\sigma_i,\underline{w}_i} \circ \pi_{\sigma_i,\underline{w}_i} \circ f_{\sigma_i}$. The covering map is a local isometry so this limit is the limit of the pullback metrics on N under f_{σ_i}. Since $\sigma_i \to \sigma$, the limit of the pullback metrics on N via f_{σ_i} is the pullback metric on N via f_σ. To summarize, the limit of the pullback metrics on N via the maps $f_{\sigma_i,\underline{w}_i} : N \to M_{\sigma_i,\underline{w}_i}$ is the pullback metric on N via $f_\sigma : N \to M_\sigma$. This convergence of metrics implies that $(f_{\sigma_i,\underline{w}_i})_*$ converges to σ as a sequence of representations in $AH(N)$ [14]. $\square$

Remark. The space $MP(N)$ is connected since Waldhausen showed that any homotopy equivalence of N is homotopic to a homeomorphism [75]. On the contrary, the manifold $(\hat{N}, \hat{P})$ has double trouble (see [5]) and therefore $MP(\hat{N}, \hat{P})$ has infinitely many components. This is why, in the first part of the proof of Lemma 5.9, we had to work componentwise.

We also remark that Lemma 5.9 is essentially the same as Proposition 3.7 in [21], but when there are multiple cusps we need to use the multiple cusp version of the filling theorem (Corollary 4.13) which requires all of the w-coordinates to go to infinity. This is the why we have defined $\mathcal{A}$ to exclude points $(\sigma, w_1, \ldots, w_d)$ where some but not all of the w-coordinates are ∞.

5.6 An Inverse to Φ

We now construct a map Ψ from a subset of $MP(N) \cup MP_0(N, P)$ to $\mathcal{A}$. For any $\sigma \in MP_0(N, P)$, and some sufficiently small neighborhood of σ in $MP(N) \cup MP_0(N, P)$, Ψ will be an inverse to Φ.

Fix a representation $\sigma_0 \in MP_0(N, P)$. For ρ in some neighborhood of σ_0, the definition of Ψ will have two coordinates $\Psi(\rho) = (\xi(\rho), q(\rho)) \in MP_0(N,P) \times \hat{\mathbb{C}}^d$. We will actually begin by defining a neighborhood V' of σ_0 such that for $\rho \in V'$, $\xi(\rho) \in AH(N,P)$. We will then restrict to a smaller neighborhood V such that $\xi(V) \subset MP_0(N,P)$ and $(\xi(\rho), q(\rho)) \in \mathcal{A}$. Before defining this neighborhood of σ_0 on which Ψ is defined, we set up some notation and background.

Let $\mathcal{H}(N)$ denote the space of smooth, hyperbolic metrics on N with the C^∞-topology (see I.1.1 of [27] for the definition of a $(PSL(2,\mathbb{C}), \mathbb{H}^3)$-structure on a manifold with boundary, and I.1.5 for a description of the space $\mathcal{H}(N)$ which is denoted $\Omega(N)$ in [27]). If we let $\mathcal{D}(N)$ be the space of smooth developing maps $\tilde{N} \to \mathbb{H}^3$ with the compact-C^∞ topology, then $\mathcal{H}(N)$ is the quotient of $\mathcal{D}(N)$ by $PSL(2,\mathbb{C})$ acting by postcomposition. Note that $\mathcal{H}(N)$ is still infinite dimensional since we are not identifying developing maps that differ by the lift of an isotopy. Let $H : \mathcal{H}(N) \to AH(N)$ be the holonomy map. Theorem I.1.7.1 of [27] locally describes $\mathcal{H}(N)$. See Chapter I of [27] for more details.

Theorem 5.10 (Canary-Epstein-Green [27])**.** *Let N_{th} be a thickening of N (i.e., the union of N with a collar $\partial N \times I$). Let $D_0 : \tilde{N}_{th} \to \mathbb{H}^3$ be a fixed developing map. A small neighborhood of $D_0|_{\tilde{N}}$ in $\mathcal{D}(N)$ is homeomorphic to $X \times Y$ where X is a small neighborhood of the obvious inclusion $N \subset N_{th}$ in the space of locally flat embeddings, and Y is a neighborhood of the holonomy map $H(D_0)$ in $Hom(\pi_1(N), PSL(2,\mathbb{C}))$. A small neighborhood of D_0 in $\mathcal{H}(N)$ is homeomorphic to $X \times Z$ where Z is a small neighborhood of the conjugacy class of $H(D_0)$ in $R(N)$.*

We now let V' be a neighborhood of $\sigma_0 \in V' \subset MP(N) \cup MP_0(N,P)$ that satisfies the properties (1)-(4) given below. Roughly, V' is a neighborhood on which we can define a section $\varsigma : V' \subset AH(N) \to \mathcal{H}(N)$ and such that if $\rho \in V'$ then the length of $\rho(\gamma_i)$ is short in M_ρ. The existence of such a neighborhood follows from the arguments given in Section 3.2 of [21] and Theorem 5.10, although we include justification for why we can define V' with these properties after the statement of each property.

Fix a smooth embedding $s_{\sigma_0} : N \to M_{\sigma_0}$ such that $(s_{\sigma_0})_* = \sigma_0$. Let g_{σ_0} be the pullback of the hyperbolic metric on M_{σ_0}. We can choose s_{σ_0} so that the core curves of the annuli in P (*i.e.*, the curves $\gamma_i \times \{1\}$) have length less than $\epsilon_3/4$ in the g_{σ_0} metric.

(1) There exists a continuous section $\varsigma : V' \to \mathcal{H}(N)$ to the holonomy map such that $\varsigma(\sigma_0) = g_{\sigma_0}$.

The existence is given by Theorem 5.10. For any $\rho \in V'$, define $g_\rho = \varsigma(\rho)$. We emphasize that, by the definition of a section, $H(g_\rho) = \rho$.

(2) For any $\rho_1, \rho_2 \in V'$, the identity map

$$(N, g_{\rho_1}) \xrightarrow{id} (N, g_{\rho_2})$$

is 2-biLipschitz.

This follows from the continuity of ς and the topology on $\mathcal{H}(N)$.

(3) For any $\rho \in V'$, there is a locally isometric immersion $s_\rho : (N, g_\rho) \to M_\rho$ where $M_\rho = \mathbb{H}^3/\rho(\pi_1(N))$ is equipped with the hyperbolic metric, such that $(s_\rho)_* = \rho$. Moreover, there is some $\epsilon_3 > \epsilon_0 > 0$ such that $s_\rho(N)$ is contained in the ϵ_0-thick part of M_ρ.

The existence of $s_\rho : N \to M_\rho$ with $(s_\rho)_* = \rho$ is given by Theorem 5.10. We now find ϵ_0. There is some K such that for any point $x \in (N, g_{\sigma_0})$, there are loops α, β based at x of length less than K such that the group generated by α and β is not virtually abelian. For example, one can find a point x_0 and loops α_0 and β_0 based at

x_0 that generate a free group, and then let K be larger than the sum of the diameter of (N, g_{σ_0}) and the lengths of α_0 and β_0. Since for any $\rho \in V'$, $(N, g_{\sigma_0}) \xrightarrow{id} (N, g_\rho)$ is 2-biLipschitz by (2), at any point $x \in (N, g_\rho)$ there are loops based at x of length less than $2K$ generating a free group. There exists some $\epsilon_3 > \epsilon_0 > 0$ such that for any component $\mathbb{T}_{\epsilon_0}$ of the ϵ_0-thin part of any hyperbolic manifold M, the distance between $\partial\mathbb{T}_{\epsilon_0}$ and $\partial\mathbb{T}_{\epsilon_3}$ is at least K.

Suppose $s_\rho(x) \in s_\rho(N) \cap M_\rho^{\leq\epsilon_0}$ for some x. Then since s_ρ is a homotopy equivalence, there are loops based at $s_\rho(x)$ that generate a free group and therefore must leave the ϵ_3-thin part of M_ρ; however, to do so they must have length greater than $2K$ contradicting that s_ρ is a locally isometric immersion. Thus there exists some ϵ_0 such that $s_\rho(N)$ is contained in ϵ_0-thick part of M_ρ for all $\rho \in V'$.

(4) Let ϵ_0 be the constant in property (3). Let l_0 be the constant from the drilling theorem such that the drilling map is a biLipschitz diffeomorphism outside an ϵ_0-Margulis tube about the drilling. Let $l_1 = \min\{\epsilon_0/8, l_0\}$. Then for any $\rho \in V'$ we have the length of γ_i in M_ρ is less than l_1, for each $i = 1, \ldots, d$.

Notation. Here, the length of γ_i in M_ρ is really the length of the unique geodesic representative of $s_\rho(\gamma_i)$ in M_ρ. For the remainder of this section, we distinguish this geodesic representative by $s_\rho(\gamma_i)^*$. This curve is homotopic to $s_\rho(\gamma_i \times \{t\})$ for any t, but its length is less than or equal to the length of $s_\rho(\gamma_i \times \{t\})$. We make this distinction since we will also be using the length of $s_\rho(\gamma_i \times \{t\})$, which is the length of $\gamma_i \times \{t\} \subset (N, g_\rho)$.

Now we construct the map ξ which will be the first coordinate of Ψ. If $\rho \in V' \cap MP_0(N, P)$, then set $\xi(\rho) = \rho$. Otherwise $\rho \in V' \cap MP(N)$ so let (M_ρ, s_ρ) be the associated marked hyperbolic 3-manifold. Note that by properties (1) and (3) of the neighborhood V' we can use $s_\rho : N \to M_\rho$ to mark M_ρ.

By property (4), the length of each $s_\rho(\gamma_i)^*$ will be short in M_ρ so we can drill out $s_\rho(\gamma)^* = s_\rho(\gamma_1)^* \cup \cdots \cup s_\rho(\gamma_d)^*$ and get a hyperbolic manifold $\hat{M}_\rho$. Let

$$\psi_\rho : M_\rho - \mathbb{T}_{\epsilon_0}(s_\rho(\gamma)^*) \to \hat{M}_\rho - \mathbb{T}_{\epsilon_0}(T)$$

be the inverse of the map ϕ from the drilling theorem (Theorem 4.1). Let $\overline{M}_\rho$ be the cover of $\hat{M}_\rho$ associated to $(\psi_\rho \circ s_\rho)_*(\pi_1(N))$. Let $\overline{f}_\rho : N \to \overline{M}_\rho$ be the lift of $\psi_\rho \circ s_\rho : N \to \hat{M}_\rho$. Note that $s_\rho(N)$ will be contained in $M_\rho - \mathbb{T}_{\epsilon_0}(s_\rho(\gamma)^*)$ by (3), so it makes sense to compose with ψ_ρ. We show in the following lemma that $(\overline{M}_\rho, \overline{f}_\rho) \in AH(N, P)$. This is also done in Lemma 3.4 in [21], with different notation. Note this is where property (2) is used. Essentially, we have to show that for each i, $s_\rho(\gamma_i)$ is isotopic to $s_\rho(\gamma_i)^*$. These curves are clearly homotopic; however, if the homotopy was not an isotopy, we may not have $\psi_\rho \circ s_\rho(\gamma_i)$ homotopic into the cusp T_i.

Lemma 5.11. *The representation $(\overline{f}_\rho)_* : \pi_1(N) \to \pi_1(\overline{M}_\rho) \subset PSL(2, \mathbb{C})$ is in $AH(N, P)$.*

Proof. Note that $\overline{M}_\rho$ was defined as a cover of the hyperbolic manifold $\hat{M}_\rho$ and therefore $\pi_1(\overline{M}_\rho)$ is a discrete subgroup of $PSL(2, \mathbb{C})$. Since $(\psi_\rho \circ s_\rho)_*$ is injective, the lift $(\overline{f}_\rho)_* : \pi_1(N) \to \pi_1(\overline{M}_\rho)$ is an isomorphism. Hence $(\overline{f}_\rho)_*$ is discrete and faithful.

We now check that $(\overline{f}_\rho)_*$ has the appropriate parabolics, or equivalently, $\overline{f}_\rho(\gamma_i)$ is homotopic into a cusp of $\overline{M}_\rho$. Since $\overline{f}_\rho$ is the lift of $(\psi_\rho \circ s_\rho)$, it is sufficient to check that $s_\rho(P) \subset \mathbb{T}_{\epsilon_3}(\gamma) - \mathbb{T}_{\epsilon_0}(\gamma)$ in M_ρ. Recall P is a union of annuli in $N \times \{1\}$ with core curves $\gamma_i \times \{1\}$.

By (2), the identity $(N, g_{\sigma_0}) \to (N, g_\rho)$ is 2-biLipschitz, and by the choice of g_{σ_0}, the curves $\gamma_i \times \{1\}$ have length less than $\epsilon_3/4$ in the g_{σ_0} metric. Thus, the curves $\gamma_i \times \{1\}$ have length less than $\epsilon_3/2$ in the g_ρ metric. Since s_ρ is a local isometry, the images $s_\rho(\gamma_i \times \{1\})$ will be contained in the ϵ_3-thin part of M_ρ.

By (3), $s_\rho(N)$ is contained in the ϵ_0-thick part of M_ρ. It follows that $s_\rho(P) \subset \mathbb{T}_{\epsilon_3}(\gamma) - \mathbb{T}_{\epsilon_0}(\gamma)$, and therefore $(\overline{M}_\rho, \overline{f}_\rho) \in AH(N, P)$. □

Now define ξ by

$$\xi(\rho) = \begin{cases} (\overline{f}_\rho)_* & \text{if } \rho \in MP(N) \\ \\ \rho & \text{if } \rho \in MP_0(N, P). \end{cases}$$

The following Lemma and subsequent Corollary are the same as Lemma 3.5 and Corollary 3.6 of [21].

Lemma 5.12. *The map ξ is continuous at all points in $V' \cap MP_0(N, P)$.*

Proof. Let $\rho_i \to \sigma$ be a sequence in V' that converges to $\sigma \in MP_0(N, P)$. If $\{\rho_i\} \subset MP_0(N, P)$ then $\lim \xi(\rho_i) = \lim \rho_i = \sigma$, so assume $\{\rho_i\} \subset V' \cap MP(N)$ and $\xi(\rho_i) = (\overline{M}_{\rho_i}, \overline{f}_{\rho_i})$.

Let g_i be the pullback metric on N from the map $s_{\rho_i} : N \to M_{\rho_i}$ and let g be the pullback metric on N from $s_\sigma : N \to M_\sigma$. Since ς is continuous on V', we must have that that $g_i \to g$ in $\mathcal{H}(N)$. Let $\overline{g}_i$ be the pullback metric on N from $\overline{f}_{\rho_i} : N \to \overline{M}_{\rho_i}$. Since this map was defined as a lift of $(\psi_{\rho_i} \circ s_{\rho_i})$, the metric $\overline{g}_i$ is the same as the pullback via $(\psi_{\rho_i} \circ s_{\rho_i}) : N \to \hat{M}_{\rho_i}$.

As $i \to \infty$, $\rho_i(\gamma_j)$ limits to a parabolic for each $j = 1, \ldots, d$. Thus, for each j, the length of $s_\rho(\gamma_j)^*$ in M_{ρ_i} goes to zero as $i \to \infty$. By Theorem 4.1, we can find drilling maps ψ_{ρ_i} whose biLipschitz constants limit to 1 on $M_{\rho_i} - \mathbb{T}_{\epsilon_0}(s_{\rho_i}(\gamma)^*)$.

Thus the limit (as $i \to \infty$) of the pullback metrics on N from $(\psi_{\rho_i} \circ s_{\rho_i})$ is the same as the limit of the pullback metric using just s_{ρ_i}. This implies

$$\lim_{i\to\infty} \overline{g}_i = \lim_{i\to\infty} g_i = g.$$

This implies $(\overline{M}_{\rho_i}, \overline{f}_{\rho_i}) \to (M_\sigma, s_\sigma)$ which implies $\xi(\rho_i) \to \xi(\sigma) = \sigma$ proving ξ is continuous at σ. $\square$

We now have $\xi : V' \to AH(N, P)$, and by Lemma 5.12, ξ is continuous on $V' \cap MP_0(N, P)$. Since $MP_0(N, P)$ is an open subset of $AH(N, P)$, we can restrict ξ to a smaller neighborhood of σ_0 so that its image is contained in $MP_0(N, P)$. See also Corollary 3.6 of [21].

Corollary 5.13. *There is a neighborhood $\sigma_0 \in V \subset V' \subset MP(N) \cup MP_0(N, P)$ such that $\xi(V) \subset MP_0(N, P)$.*

Now that we have a neighborhood V such that $\xi(V) \subset MP_0(N, P)$, we can use $\xi(\rho)$ as the first coordinate of $\Psi(\rho)$ and begin to define the second coordinate $q(\rho)$.

If $\rho \in V \cap MP_0(N, P)$, then we set $q(\rho) = (\infty, \ldots, \infty)$. Otherwise, we consider the covering $\pi_\rho : \overline{M}_\rho \to \hat{M}_\rho$ induced by the image of the injection $(\psi_\rho \circ s_\rho)_* : \pi_1(N) \to \pi_1(\hat{M}_\rho)$. The group $\pi_1(\hat{M}_\rho)$ is obtained from $\pi_1(\overline{M}_\rho)$ by the same construction described in Section 5.2. That is, $\xi(\rho) = (\overline{M}_\rho, \overline{f}_\rho)$ corresponds to some representation $\sigma \in MP_0(N, P)$ and there is a unique $(w_1, \ldots, w_d)$ such that the extension $\sigma_{w_1,\ldots,w_d}(\pi_1(\hat{N})) = \pi_1(\hat{M}_\rho)$. We define this to be $q(\rho) = (w_1, \ldots, w_d)$.

Equivalently, w_i is defined so that if we conjugate $(\psi_\rho \circ s_\rho)_*$ so that γ_i is mapped to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then the unique nontrivial element $\beta_i \in \pi_1(\partial U_i) \subset \pi_1(\hat{M}_\rho)$ that bounds a disk in M_ρ will be $\begin{pmatrix} 1 & w_i \\ 0 & 1 \end{pmatrix}$.

Now that we have defined $q(\rho)$, we can define $\Psi : V \to MP_0(N,P) \times \hat{\mathbb{C}}^d$ by

$$\Psi(\rho) = (\xi(\rho), q(\rho))$$

for any $\rho \in V$. Note that we have defined $q(\rho)$ so that $\Psi(\rho) \in \mathcal{A}$ for all $\rho \in V$. Unlike Φ, we only show Ψ is continuous for points on the boundary of $MP(N)$.

Lemma 5.14. *The map Ψ is continuous on $V \cap MP_0(N,P)$.*

Proof. Lemma 5.12 shows that ξ is continuous on $V \cap MP_0(N,P)$. Now consider a sequence $\rho_i \to \sigma$ where $\sigma \in MP_0(N,P)$. Since $\Psi(\sigma) = (\sigma, \infty, \ldots, \infty)$ and we know $\xi(\rho_i) = \sigma$, it suffices to show that $q(\rho_i) \to (\infty, \ldots, \infty)$. If $\rho_i \in MP_0(N,P)$ then $q(\rho_i) = (\infty, \ldots, \infty)$ so assume that $\rho_i \in V \cap MP(N)$. We will use the notation $q(\rho_i) = (w_{1,i}, \ldots, w_{d,i})$ and show $w_{j,i} \to \infty$ as $i \to \infty$ for $j = 1, \ldots, d$.

Since for each $j = 1, \ldots, d$, the length of $\rho_i(\gamma_j) \to 0$ as $i \to \infty$, Proposition 4.14 shows that the normalized length of the jth meridian, β_j, goes to infinity as i goes to infinity. The normalized length is given by

$$\frac{|w_{j,i}|}{\sqrt{2Im(w_{j,i})}}$$

If $w_{j,i}$ does not go to ∞, then we must have $Im(w_{j,i}) \to 0$ as $i \to \infty$. This cannot happen by Corollary 5.5.

It follows that $q(\rho_i) \to (\infty, \ldots, \infty) = q(\sigma)$ proving q is continuous at any point $\sigma \in V \cap MP_0(N,P)$. Thus, Ψ is continuous on $V \cap MP_0(N,P)$. $\square$

5.7 Local Homeomorphism

Recall, in Section 5.5 we defined Φ on a subset $\mathcal{A}_K \subset \mathcal{A}$ and showed Φ is continuous. We now claim that there is some subset of $\mathcal{A}_K$ on which Φ is continuous and injective. See also Lemma 3.9 and Proposition 3.10 of [21].

Lemma 5.15. *Let $\sigma_0 \in MP_0(N, P)$. There is some neighborhood U of $(\sigma_0, \infty, \ldots, \infty)$ in $\mathcal{A}$ such that $\Psi \circ \Phi|_U = id$. In particular, Φ is injective on U.*

Proof. Let V be the neighborhood of σ_0 on which Ψ was defined. By the continuity of Φ, we can find a neighborhood U' so that $\Phi(U')$ is contained in V, and for any $(\sigma, w_1, \ldots, w_d) \in U'$, $\sigma \in V$. We now consider $\Psi \circ \Phi|_{U'}$.

Let $(\sigma, w_1, \ldots, w_d) \in U'$. If $(w_1, \ldots, w_d) = (\infty, \ldots, \infty)$ then $\Phi(\sigma, \infty, \ldots, \infty) = \sigma$ and $\Psi(\sigma) = (\sigma, \infty, \ldots, \infty)$. If $(w_1, \ldots, w_d) \neq (\infty, \ldots, \infty)$ then $\sigma_{w_1,\ldots,w_d} \in MP(\hat{N}, \hat{P})$. Recall that the definition of Φ in this case was

$$\Phi(\sigma, w_1, \ldots, w_d) = (M_{\sigma,w}, f_{\sigma,w})$$

where $M_{\sigma,w}$ was the filling of $\hat{M}_{\sigma,w} = \mathbb{H}^3/\sigma_{w_1,\ldots,w_d}(\pi_1(\hat{N}))$ and $f_{\sigma,w} = \phi_{\sigma,w} \circ \pi_{\sigma,w} \circ f_\sigma$. By our choice of U', we have $\sigma \in V$ and therefore f_σ is homotopic to a local isometry $s_\sigma : (N, g_\sigma) \to M_\sigma$ such that $s_\sigma(N) \subset M_\sigma^{\geq \epsilon_0}$ (see the four properties of the neighborhood V' defined in Section 5.6). Thus we can redefine the marking $f_{\sigma,w} = \phi_{\sigma,w} \circ \pi_{\sigma,w} \circ s_\sigma$ without changing the definition of $\Phi(\sigma, w)$. Also recall that $\pi_{\sigma,w}$ is a covering map and therefore a local isometry, and

$$\phi_{\sigma,w} : \hat{M}_{\sigma,w} - \mathbb{T}_{\epsilon_0}(T) \to M_{\sigma,w} - \mathbb{T}_{\epsilon_0}(\gamma)$$

is a biLipschitz diffeomorphism. (Recall that Φ was originally defined on $\mathcal{A}_K$ so that for any $(\sigma, w) \in \mathcal{A}_K$, $\phi_{\sigma,w}$ is a biLipschitz diffeomorphism on $\hat{M}_{\sigma,w} - \mathbb{T}_{\epsilon_3}(T)$. By possibly making U' smaller, we can assume that $\phi_{\sigma,w}$ is biLipschitz on the $\hat{M}_{\sigma,w} - \mathbb{T}_{\epsilon_0}(T)$.) Thus $f_{\sigma,w} = \phi_{\sigma,w} \circ \pi_{\sigma,w} \circ s_\sigma$ is smooth, and we let $g'_{\sigma,w}$ be the pullback metric on N via $f_{\sigma,w} : N \to M_{\sigma,w}$.

By the assumption that $\Phi(U') \subset V$, we can find a homotopic marking $s_{\sigma,w} \simeq f_{\sigma,w} : N \to M_{\sigma,w}$ satisfying the properties (1)-(4) listed prior to the definition of Ψ.

However, we need that $f_{\sigma,w}$ is homotopic to $s_{\sigma,w}$ in $M_{\sigma,w} - \gamma$ in order to have the drilling construction in Ψ be the inverse to the filling construction in Φ.

Let W be a neighborhood of g_{σ_0} in $H^{-1}(V) \subset \mathcal{H}(N)$ such that Theorem 5.10 applies. We first claim there is some $U \subset U' \subset \mathcal{A}$ such that if $(\sigma, w) \in U$, then $g'_{\sigma,w} \in W$. There is some J such that if $\phi_{\sigma,w}$ is a J-biLipschitz diffeomorphism and σ is sufficiently close to σ_0, then $g'_{\sigma,w} \in W$. So, let $U \subset U'$ be a neighborhood of $(\sigma_0, \infty, \ldots, \infty)$ in $\mathcal{A}$ such that for all $(\sigma, w_1, \ldots, w_d) \in U$, σ is sufficiently close to σ_0, and for all i, $|w_i|$ is large enough so that the filling map $\phi_{\sigma,w}$ is a J-biLipschitz diffeomorphism.

Next we claim that there is a metric $g_{\sigma,w} \in \varsigma(V) \subset W$ and a locally isometry $s_{\sigma,w} : (N, g_{\sigma,w}) \to M_{\sigma,w}$ such that $f_{\sigma,w}$ is homotopic to $s_{\sigma,w}$ as a map into $M_{\sigma,w} - \gamma$. This claim follows from product structure of W described in Theorem 5.10. More precisely, let N_{th} be a thickening of N. Then we can extend the local isometry $f_{\sigma,w} : (N, g'_{\sigma,w}) \to M_{\sigma,w}$ to a local isometry $f_{\sigma,w,th} : (N_{th}, g'_{\sigma,w,th}) \to M_{\sigma,w}$, where $g'_{\sigma,w,th}$ is a hyperbolic metric on N_{th} that restricts to $g_{\sigma,w}$ on $N \subset N_{th}$. Then there exists a locally flat embedding $i : N \to N_{th}$ isotopic to the identity such that $s_{\sigma,w} = f_{\sigma,w,th} \circ i$. Thus $s_{\sigma,w}$ and $f_{\sigma,w}$ are homotopic as maps inside $f_{\sigma,w,th}(N_{th}) \subset M_{\sigma}$. Since $f_{\sigma,w}(N) \subset M_{\sigma,w} - \mathbb{T}_{\epsilon_0}(\gamma)$, we can assume that the neighborhood W in $\mathcal{H}(N)$ is small enough so that $f_{\sigma,w,th}(N_{th}) \subset M_{\sigma,w} - \gamma$. Thus, $f_{\sigma,w}$ and $s_{\sigma,w}$ are homotopic in $M_{\sigma,w} - \gamma$.

Now we want to show that $\Psi(M_{\sigma,w}, s_{\sigma,w}) = (\sigma, w)$. Recall that in the definition of Ψ, we drill out the geodesic representative of $s_{\sigma,w}(\gamma)$ from $M_{\sigma,w}$ to get $\hat{M}_{\sigma,w}$. Let $\psi_{\sigma,w} : M_{\sigma,w} - \mathbb{T}_{\epsilon_0}(\gamma) \to \hat{M}_{\sigma,w} - \mathbb{T}_{\epsilon_0}(T)$ be the inverse of $\phi_{\sigma,w}$. Since $s_{\sigma,w}(N)$ is contained in the ϵ_0-thick part of $M_{\sigma,w}$, we can define

$$\psi_{\sigma,w} \circ s_{\sigma,w} : N \to \hat{M}_{\sigma,w}.$$

As $s_{\sigma,w}$ was homotopic to $f_{\sigma,w}$ as a map to $M_{\sigma,w} - \gamma$, we have that $\psi_{\sigma,w} \circ s_{\sigma,w} \simeq \psi_{\sigma,w} \circ f_{\sigma,w}$. Since $\psi_{\sigma,w}$ is the inverse of the filling map $\phi_{\sigma,w}$, this implies that

$$\psi_{\sigma,w} \circ s_{\sigma,w} \simeq \psi_{\sigma,w} \circ f_{\sigma,w} = \psi_{\sigma,w} \circ \phi_{\sigma,w} \circ \pi_{\sigma,w} \circ s_\sigma = \pi_{\sigma,w} \circ s_\sigma.$$

It follows that M_σ is the cover of $\hat{M}_{\sigma,w}$ associated to $(\psi_{\sigma,w} \circ s_{\sigma,w})_*(\pi_1(N))$ and the covering map is $\pi_{\sigma,w}$. Moreover the lift of $\psi_{\sigma,w} \circ s_{\sigma,w} : N \to \hat{M}_{\sigma,w}$ is homotopic to $s_\sigma : N \to M_\sigma$. It follows immediately that $\xi(M_{\sigma,w}, s_{\sigma,w}) = \sigma$ and $q(M_{\sigma,w}, s_{\sigma,w}) = (w_1, \ldots, w_d)$. Hence for $(\sigma, w_1, \ldots, w_d) \in U$, we have $\Psi(\Phi(\sigma, w_1, \ldots, w_d)) =$
$\Psi(M_{\sigma,w}, f_{\sigma,w}) = \Psi(M_{\sigma,w}, s_{\sigma,w}) = (\xi(M_{\sigma,w}, s_{\sigma,w}), q(M_{\sigma,w}, s_{\sigma,w})) = (\sigma, w_1, \ldots, w_d)$. □

Lemma 5.16. *Let $\rho \in MP(N) \cup MP_0(N,P)$. If $\Phi \circ \Psi$ is defined at ρ then $\Phi \circ \Psi(\rho) = \rho$.*

Proof. If $\rho \in V \cap MP_0(N,P)$ then clearly $\Psi(\rho) = (\rho, \infty, \ldots, \infty)$ and $\Phi(\Psi(\rho)) = \rho$. If $\rho \in V \cap MP(N)$, then recall that we can choose the marking $s_\rho : N \to M_\rho$ and define $\hat{M}_\rho$ to be the γ-drilling of M_ρ. Then we let $\overline{M}_\rho$ be the cover of $\hat{M}_\rho$ associated to $(\psi_\rho \circ s_\rho)_*(\pi_1(N))$. If $\Psi(\rho) = (\sigma, w_1, \ldots, w_d)$ then $\overline{M}_\rho = M_\sigma$, $\overline{f}_\rho \simeq f_\sigma$, $\hat{M}_\rho = \hat{M}_{\sigma,w}$, and $M_\rho = M_{\sigma,w}$. Thus

$$\pi_{\sigma,w} \circ f_\sigma \simeq \pi_\rho \circ \overline{f}_\rho = \psi_\rho \circ s_\rho$$

since $\overline{f}_\rho$ was the lift of $\psi_\rho \circ s_\rho$. But then

$$f_{\sigma,w} = \phi_{\sigma,w} \circ \pi_{\sigma,w} \circ f_\sigma \simeq \phi_{\sigma,w} \circ \psi_\rho \circ s_\rho = \psi_\rho^{-1} \circ \psi_\rho \circ s_\rho = s_\rho.$$

It follows that when we apply Φ to $(\sigma, w_1, \ldots, w_d)$ we get $(M_{\sigma,w}, f_{\sigma,w}) = (M_\rho, s_\rho) = \rho$. □

Theorem 5.17. *Let $\sigma_0 \in MP_0(N,P)$. The map Φ is a local homeomorphism from $\mathcal{A}_K$ to $MP(N) \cup MP_0(N,P)$ at $(\sigma_0, \infty, \ldots, \infty)$.*

Proof. It follows from Lemma 5.9 that Φ is continuous and from Lemma 5.15 that Φ is injective on some neighborhood U of $(\sigma_0, \infty, \ldots, \infty)$.

Certainly $\Phi(U)$ contains σ_0. We claim that $\Phi(U)$ contains some neighborhood V of σ_0 in $MP(N) \cup MP_0(N,P)$. Suppose no such neighborhood exists. Then we can find a nested sequence of neighborhoods V_i whose intersection is σ_0 and a sequence $\rho_i \in V_i$ such that $\rho_i \notin \Phi(U)$. Since $\rho_i \to \sigma_0$, and Lemma 5.14 says that Ψ is continuous at σ_0 we have $\Psi(\rho_i) \to \Psi(\sigma_0) = (\sigma_0, \infty, \ldots, \infty)$. It follows that $\Psi(\rho_i) \in U$ for all sufficiently large i; however, this contradicts Lemma 5.16 which says that $\Phi(\Psi(\rho_i)) = \rho_i \notin \Phi(U)$ for sufficiently large i.

Hence, there is some neighborhood V of σ_0 contained in $\Phi(U)$. Since Φ is continuous, $\Phi^{-1}(V)$ is a neighborhood of $(\sigma_0, \infty, \ldots, \infty)$ in $\mathcal{A}$ such that $\Phi|_{\Phi^{-1}(V)} : \Phi^{-1}(V) \to V$ is a continuous bijection. The inverse map is given by Ψ, which is continuous on $V \cap MP_0(N,P)$ by Lemma 5.14 and on $V \cap MP(N)$ by invariance of domain. Hence Φ is a local homeomorphism at σ_0. $\square$

Remark. Since the point $\sigma_0 \in MP_0(N,P)$ that we fixed in the beginning of Section 5.6 and used throughout Sections 5.6 and 5.7 was arbitrary, we have actually shown that Φ is a local homeomorphism at any $\sigma \in MP_0(N,P)$.

5.8 $MP(N) \cup MP_0(N,P)$ is not locally connected

In Proposition 5.8, we saw that there was a point $\sigma^0 \in MP_0(N,P)$ such that $\mathcal{A}$ is not locally connected at $(\sigma^0, \infty, \ldots, \infty)$. By Theorem 5.17, Φ is a local homeomorphism from $\mathcal{A}$ to $MP(N) \cup MP_0(N,P)$ at $(\sigma^0, \infty, \ldots, \infty)$. Hence, $MP(N) \cup$

$MP_0(N,P)$ is not locally connected at $\Phi(\sigma^0, \infty, \ldots, \infty) = \sigma^0 \in MP_0(N,P)$. Thus we have shown

Theorem 5.18. *There exists $\sigma^0 \in MP_0(N,P)$ such that $MP(N) \cup MP_0(N,P)$ is not locally connected at σ^0.*

By the Density Theorem (Theorem 2.2), $AH(N)$ is the closure of $MP(N) \cup MP_0(N,P)$. Of course, it does not follow directly from this that $AH(N)$ is not locally connected at σ^0. In order to conclude anything about the closure, we need more quantitative control over the components of a neighborhood U of $(\sigma^0, \infty, \ldots, \infty)$ in $\mathcal{A}$, and what happens to these components under the map Φ. By Lemma 5.7, there is lower bound to the distance between some of the components of U. In the next Chapter, we will use the filling theorem to show that this implies $\overline{\Phi(U)}$ has infinitely many components.

CHAPTER 6

$AH(S \times I)$ is not locally connected

In this chapter, we prove Theorem 1.1 by contradiction. Specifically, if one assumes $AH(S\times I)$ is locally connected, then one may use the filling theorem (Theorem 1.2) and Lemma 5.7 to derive a contradiction.

Theorem 1.1. *Let S be a closed surface of genus $g \geq 2$. Then $AH(S \times I)$ is not locally connected.*

The filling theorem is the principle tool that we use to prove Theorem 1.1. The w_1-coordinate of a point $(\sigma, w_1, \ldots, w_d) \in \mathcal{A}$ (when $\underline{w} \neq (\infty, \ldots, \infty)$) can be used to estimate the complex length of γ_1 in $\Phi(\sigma, w_1, \ldots, w_d) \in MP(N)$. Before beginning the proof of Theorem 1.1, we prove a lemma that provides an intermediate step in this estimate. Recall that when $\underline{w} \neq (\infty, \ldots, \infty)$, $\Phi(\sigma, w_1, \ldots, w_d)$ is the marked hyperbolic manifold $M_{\sigma,w}$ in $MP(N)$ obtained by filling the d cusps of $\hat{M}_{\sigma,w} = \mathbb{H}^3/\sigma_w(\pi_1(\hat{N}))$. The manifold $M_{\sigma,w}$ obtained in this filling is independent of the order in which the filling is done, so we can assume that the cusps are filled one at a time, with the cusp corresponding to w_1 filled last. After filling $d-1$ cusps, we have a manifold $M'_{\sigma,w}$ with one rank-2 cusp. Equivalently, $M'_{\sigma,w}$ is the γ_1-drilling of $\Phi(\sigma, w_1, \ldots, w_d) \in MP(N)$. Lemma 6.1 bounds the change in the geometry of the first cusp while we perform the other $d-1$ fillings.

Let q_1 be the first coordinate of the map q in the definition of Ψ. That is,

$$q_1 : V \cap MP(N) \to \mathcal{T}(T^2)$$

is defined so that if $\Psi(\eta) = (\sigma, w_1, \ldots, w_d)$, then $q_1(\eta) = w_1$. This is a Teichmüller parameter for the first cusp in $\hat{M}_{\sigma,w}$ in the sense that σ_w is conjugate to a representation that sends γ_1 to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and the meridian β_1 of ∂U_1 to $\begin{pmatrix} 1 & q_1(\eta) \\ 0 & 1 \end{pmatrix}$.

Now define $r_1 : V \cap MP(N) \to \mathcal{T}(T^2)$ so that $r_1(\eta)$ is the Teichmüller parameter of the cusp of $M'_{\sigma,w}$. That is, after $d-1$ cusps have been filled, we can conjugate $\pi_1(M'_{\sigma,w})$ so that the remaining cusp is marked by

$$\gamma_1 \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta_1 \mapsto \begin{pmatrix} 1 & r_1(\eta) \\ 0 & 1 \end{pmatrix}.$$

The drilling theorem can be used to show that q_1 and r_1 are close in the following sense. One obtains $\hat{M}_{\sigma,w}$ from $M'_{\sigma,w}$ by drilling out $\gamma_2, \ldots, \gamma_d$. So if the sum of the lengths of $\gamma_2, \ldots, \gamma_d$ is small enough in $M'_{\sigma,w}$ then the drilling theorem can be used to bound $|q_1(\eta) - r_1(\eta)|$. The following lemma provides a quantitative bound based on the lengths of $\gamma_1, \ldots, \gamma_d$ in M_η.

Lemma 6.1. *Let $\delta > 0$, $\kappa > 0$. There is some $l_0 > 0$ such that for any $\eta \in MP(N)$ with $\min\{Im(q_1(\eta)), Im(r_1(\eta))\} < \kappa$ and*

$$\sum_{i=1}^{d} l(\eta(\gamma_i)) < l_0,$$

then

$$|q_1(\eta) - r_1(\eta)| < \frac{\delta}{4}.$$

Proof. For any $\eta \in MP(N)$, let $M_\eta = \mathbb{H}^3/\eta(\pi_1(N))$, let $M'_{\sigma,w}$ denote the γ_1-drilling of M_η, and let $\hat{M}_{\sigma,w}$ denote the $\cup_{i=2}^{d}\gamma_i$-drilling of $M'_{\sigma,w}$.

The drilling theorem says that there exists l_1 such that if the length of $l(\eta(\gamma_1)) < l_1$ then there is a 2-biLipschitz map

$$M_\eta - \mathbb{T}_{\epsilon_3}(\gamma_1) \to M'_{\sigma,w} - \mathbb{T}_{\epsilon_3}(T_1).$$

This implies the lengths of $\gamma_2, \ldots, \gamma_d$ do not double when we drill γ_1. That is, if $l(\eta(\gamma_1)) < l_1$ then for $i = 2, \ldots, d$,

$$l_{M'_{\sigma,w}}(\gamma_i) < 2l(\eta(\gamma_i)).$$

Choose some $\varepsilon > 0$ such that $\varepsilon e^{\varepsilon} < \frac{\delta}{4\kappa}$. There exists some $J > 1$ such that if X_1, X_2 are two points in $\mathcal{T}(T^2)$ and $\phi : X_1 \to X_2$ is a J-biLipschitz diffeomorphism, then $d_{\mathcal{T}(T^2)}(X_1, X_2) < \varepsilon$.

By the drilling theorem, there is some l_2 such that if $\sum_{i=2}^{d} l_{M'_{\sigma,w}}(\gamma_i) < l_2$ then there exists a J-biLipschitz diffeomorphism

$$\phi : M'_{\sigma,w} - \cup_{i=2}^{d} \mathbb{T}_{\epsilon_3}(\gamma_i) \to \hat{M}_{\sigma,w} - \cup_{i=2}^{d} \mathbb{T}_{\epsilon_3}(T_i).$$

Now choose any $0 < l_0 < \min\{l_1, \frac{l_2}{2}\}$. If $\sum_{i=1}^{d} l(\eta(\gamma_i)) < l_0$ then $l(\eta(\gamma_1)) < l_0 < l_1$. This implies the lengths of $\gamma_2, \ldots, \gamma_d$ do not double as we do the first drilling. Thus,

$$\sum_{i=2}^{d} l_{M'_{\sigma,w}}(\gamma_i) < \sum_{i=2}^{d} 2l(\eta(\gamma_i)) < 2l_0 < l_2.$$

Now since $\sum_{i=2}^{d} l_{M'_{\sigma,w}}(\gamma_i) < l_2$, there exists a J-biLipschitz diffeomorphism

$$\phi : M'_{\sigma,w} - \cup_{i=2}^{d} \mathbb{T}_{\epsilon_3}(\gamma_i) \to \hat{M}_{\sigma,w} - \cup_{i=2}^{d} \mathbb{T}_{\epsilon_3}(T_i)$$

when we drill $\gamma_2, \ldots, \gamma_d$. As in the proof of Corollary 4.13 (see also the remarks following Theorem 4.1), we can assume that ϕ restricts to a J-biLipschitz diffeomorphism on T_1 that takes torus cross-sections of the first cusp in $M'_{\sigma,w}$ to torus cross-sections of the first cusp in $\hat{M}_{\sigma,w}$ (Theorem 6.12 of [13]). Since the Teichmüller metric for $\mathcal{T}(T^2)$ agrees with the hyperbolic metric for the upper-half plane model of $\mathbb{H}^2$, this implies

$$d_{\mathcal{T}(T^2)}(q_1(\eta), r_1(\eta)) = d_{\mathbb{H}^2}(q_1(\eta), r_1(\eta)) < \varepsilon.$$

See also Theorem 7.2 of [19].

Since either $Im(q_1(\eta)) < \kappa$ or $Im(r_1(\eta)) < \kappa$,

$$|q_1(\eta) - r_1(\eta)| < \kappa e^{\varepsilon} \left(d_{\mathbb{H}^2}(q_1(\eta), r_1(\eta))\right) < \kappa \varepsilon e^{\varepsilon} < \frac{\delta}{4}.$$

□

With Lemma 6.1 providing some control on r_1 based on q_1, we are now ready to prove Theorem 1.1. If $(\sigma^0, \infty, \ldots, \infty)$ is the point where we found $\mathcal{A}$ fails to be locally connected in Proposition 5.8, then $MP(N) \cup MP_0(N, P)$ is not locally connected at σ^0 (Theorem 5.18). Essentially what we will show is that if $AH(S \times I)$ is locally connected at σ^0, then there are points $\eta, \eta' \in MP(N)$ such that $|q_1(\eta) - q_1(\eta')|$ bounded from below, $|r_1(\eta) - r_1(\eta')|$ is bounded from above, and these bounds form a contradiction to Lemma 6.1.

Proof. Let $(\sigma^0, \infty, \ldots, \infty) \in \mathcal{A}$ be the point that we described in Section 5.4. Recall this was a point such that $\sigma^0|_{\pi_1(N_{1,1})} = \sigma_{z^0}$ where z^0 was the point described in Lemma 5.6. We claim there exists a neighborhood U of $(\sigma^0, \infty, \ldots, \infty)$ with the following properties:

(1) There is a neighborhood V of σ^0 in $MP(N) \cup MP_0(N, P)$ such that $\Phi|_U : U \to V$ is a homeomorphism. Such a neighborhood exists by Theorem 5.17.

(2) For any $(\eta, w_1, \ldots, w_d) \in U$, $\eta|_{\pi_1(N_{1,1})}$ lies in the neighborhood W of $\sigma^0|_{\pi_1(N_{1,1})}$ that we defined in Section 5.4. Recall W is a neighborhood of σ_{z^0} in $MP_0(N_{1,1}, P'_{1,1})$

such that for all $\sigma_z \in \overline{W}$, the coordinate z lies in the δ-neighborhood O of z^0 (see Lemma 5.6).

(3) Recall from Section 5.4 that $C_n = \{(\eta, w_1, \ldots, w_d) \in U \ : \ w_1 \in R + 2n\}$. Then there exists $1 > \delta > 0$ such that for any $(\eta, w_1, \ldots, w_d) \in \overline{C_n}$ and any $(\eta', w_1', \ldots, w_d') \in \overline{U - C_n}$, we have $|w_1 - w_1'| > \delta$ for any n. This follows from property (2) and Lemma 5.7. Note that the quantity δ is the radius of the neighborhood O of z^0 from Lemma 5.6.

(4) Let $\delta > 0$ be the constant from (3). Let $\kappa > 80(2\pi)^2$ be some constant such that for any $(\eta, w_1, \ldots, w_d) \in \overline{C_n}$, we have $Im(w_1) < \kappa - 1$. Then for any $\eta \in \Phi(U) \cap MP(N) = V \cap MP(N)$, we have $|r_1(\eta) - q_1(\eta)| < \frac{\delta}{4}$ or $\min\{Im(r_1(\eta)), Im(q_1(\eta))\} \geq \kappa$.

We can assume U satisfies property (4) for the following reason. By Lemma 6.1, given any $\delta, \kappa > 0$ there exists some l_0 such that if $\sum_{i=1}^{d} l(\eta(\gamma_i)) < l_0$ then either $|r_1(\eta) - q_1(\eta)| < \frac{\delta}{4}$ or $\min\{Im(r_1(\eta)), Im(q_1(\eta))\} \geq \kappa$. Since $\Phi|_U : U \to V$ is a homeomorphism, and V is a neighborhood of σ^0 where $\sigma^0(\gamma_1), \ldots, \sigma^0(\gamma_d)$ are parabolic, we can make U small enough so that $\sum_{i=1}^{d} l(\eta(\gamma_i)) < l_0$ for any $\eta \in \Phi(U) \cap MP(N)$. One can check that shrinking U does not change properties (1), (2), and (3).

Again, since making U smaller does not affect the above properties, we can assume that U satisfies:

(5) For any $(\eta, w_1, \ldots, w_d) \in U$, $|w_1| > 81(2\pi)^2$.

As an easy consequence of properties (3), (4), and (5), we have the following:

(5′) For any $(\eta, w_1, \ldots, w_d) \in U$, $|r_1(\Phi(\eta, w_1, \ldots, w_d))| > 80(2\pi)^2$.

This follows directly from the other properties since $q_1(\Phi(\eta, w_1, \ldots, w_d)) = w_1$ by definition. So (4) and (5) imply that either $|r_1(\Phi(\eta, w_1, \ldots, w_d))| > 81(2\pi)^2 - \frac{\delta}{4} >$

$80(2\pi)^2$ (since we assumed that $\delta < 1$ in property (3)), or $Im(r_1(\Phi(\eta, w_1, \ldots, w_d))) \geq \kappa > 80(2\pi)^2$, which implies that $|r_1(\Phi(\eta, w_1, \ldots, w_d))| > 80(2\pi)^2$.

Now that we have set up a neighborhood U of $(\sigma^0, \infty, \ldots, \infty)$ in $\mathcal{A}$, suppose $AH(N)$ was locally connected at $\Phi(\sigma^0, \infty, \ldots, \infty) = \sigma^0$. Then we claim

$$\overline{\Phi(C_n)} \cap \overline{\Phi(U - C_n)} \neq \emptyset$$

for all but finitely many n.

To prove the claim, let V_{AH} be a neighborhood (in $AH(N)$) of σ^0 contained inside $\overline{\Phi(U)}$. Note that the closure of $\Phi(U)$ contains such a neighborhood of σ^0 in $AH(N)$ by the Density Theorem (Theorem 2.2). If $AH(N)$ is locally connected, then there exists a connected neighborhood $\sigma^0 \in V_{conn} \subset V_{AH}$. We claim that for any such V_{conn}, both $V_{conn} \cap \Phi(C_n)$ and $V_{conn} \cap \Phi(U - C_n)$ are nonempty for all sufficiently large n. This will follow from the claim that any neighborhood of $(\sigma^0, \infty, \ldots, \infty)$ in $\mathcal{A}$ contains points in C_n and $U - C_n$ for all sufficiently large n. We showed this in Section 5.4, but we reiterate the argument here. If $(\eta, w_1, \ldots, w_d) \in \mathcal{A}$ then $(\eta, w_1 + 2n, \ldots, w_d + 2n) \in \mathcal{A}$ for all n since the extension of η by $(w_1 + 2n, \ldots, w_d + 2n)$ is a representation with the same image as the extension of η by $(w_1, \ldots, w_d)$. By Lemma 5.4 and Lemma 5.6, one can find a sequence of points $(\sigma^0, w_1^0 + 2n, \ldots, w_d^0 + 2n) \to (\sigma^0, \infty, \ldots, \infty)$ such that $(\sigma^0, w_1^0 + 2n, \ldots, w_d^0 + 2n) \in C_n$. Similarly, one can find a sequence of points in $U - C_n$ approaching $(\sigma^0, \infty, \ldots, \infty)$. Since Φ is a local homeomorphism from $\mathcal{A}$ to $MP(N) \cup MP_0(N, P)$ at σ^0, we have verified the claim that $V_{conn} \cap \Phi(C_n)$ and $V_{conn} \cap \Phi(U - C_n)$ are nonempty for all sufficiently large n.

If the closures of $\Phi(C_n)$ and $\Phi(U - C_n)$ were disjoint then we could form a separation of V_{conn}. Thus we must have

$$\overline{\Phi(C_n)} \cap \overline{\Phi(U - C_n)} \neq \emptyset$$

for all but finitely many n.

Now let

$$\rho \in \overline{\Phi(C_n)} \cap \overline{\Phi(U - C_n)}.$$

for some sufficiently large n. We will determine n later, but for now notice that there are only finitely many n for which this intersection is empty.

Although ρ is not in the image of Φ, we can find sequences

$$\rho = \lim_{m\to\infty} \eta_m = \lim_{m\to\infty} \eta'_m$$

where $\eta_m \in \Phi(C_n)$ and $\eta'_m \in \Phi(U - C_n)$ are representations in $MP(N)$.

Up to subsequence, we can assume that $q_1(\eta_m)$ and $q_1(\eta'_m)$ converge, so we define w_1 and w'_1 by

$$w_1 = \lim_{m\to\infty} q_1(\eta_m)$$

and

$$w'_1 = \lim_{m\to\infty} q_1(\eta'_m).$$

Equivalently, w_1 and w'_1 are the second coordinates of $\lim_{m\to\infty} \Psi(\eta_m) \in \overline{C_n}$ and $\lim_{m\to\infty} \Psi(\eta'_m) \in \overline{U - C_n}$. Note that $w_1 \in R + 2n$ since $\eta_m \in \Phi(C_n)$ for all m. Also, by passing to further subsequences if necessary, we define ζ_1 and ζ'_1 by

$$\zeta_1 = \lim_{m\to\infty} r_1(\eta_m)$$

and

$$\zeta'_1 = \lim_{m\to\infty} r_1(\eta'_m).$$

By property (3) of U, there is some $1 > \delta > 0$ such that

$$|w_1 - w'_1| > \delta.$$

Note that κ was chosen so that $Im(w_1) < \kappa - 1$ since $w_1 = \lim q_1(\eta_m)$ and $\eta_m \in \Phi(C_n)$. Thus by property (4) of the neighborhood U, we have

$$|\zeta_1 - w_1| \leq \frac{\delta}{4}.$$

If we also have $\min\{Im(w_1'), Im(\zeta_1')\} < \kappa$, then $|\zeta_1' - w_1'| \leq \frac{\delta}{4}$ and thus

$$|\zeta_1 - \zeta_1'| \geq \delta - \frac{\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2}. \tag{6.1}$$

Otherwise, we have $\min\{Im(w_1'), Im(\zeta_1')\} \geq \kappa$. But since $Im(w_1) < \kappa - 1$ and $|w_1 - \zeta_1| \leq \frac{\delta}{4}$, we must have $Im(\zeta_1) < \kappa - 1 + \frac{\delta}{4}$. Thus

$$|\zeta_1 - \zeta_1'| \geq |Im(\zeta_1) - Im(\zeta_1')| > \kappa - \left(\kappa - 1 + \frac{\delta}{4}\right) = 1 - \frac{\delta}{4} > \frac{\delta}{2}$$

so inequality (6.1) still holds.

Next we will use the complex length estimates in the filling theorem to produce a contradiction to (6.1). Consider the complex length, $\mathcal{L}(\rho(\gamma_1))$. We can estimate the complex length of $\rho(\gamma_1)$ in two ways, corresponding to each of the two sequences η_m and η_m'.

For any $\eta \in V$, parts (ii) and (v) of the filling theorem (Theorem 1.2) can be used to estimate $\mathcal{L}(\eta(\gamma_1))$. If we let

$$L_\eta^2 = \frac{|r_1(\eta)|^2}{2Im(r_1(\eta))} \quad \text{and} \quad A_\eta^2 = \frac{|r_1(\eta)|^2}{2Re(r_1(\eta))},$$

then the filling theorem gives us the following estimates on $\mathcal{L}(\eta(\gamma_1)) = l(\eta(\gamma_1)) + i\theta(\eta(\gamma_1))$.

$$\left|l(\eta(\gamma_1)) - \frac{2\pi}{L_\eta^2}\right| \leq \frac{8(2\pi)^3}{L_\eta^4 - (16)(2\pi)^4} \quad \text{and} \quad \left|\theta(\eta(\gamma_1)) - \frac{2\pi}{A_\eta^2}\right| \leq \frac{5(2\pi)^3}{(L_\eta^2 - 4(2\pi)^2)^2}.$$

If $\rho = \lim_{m\to\infty} \eta_m = \lim_{m\to\infty} \eta_m'$, then we get the following two sets of estimates on $\mathcal{L}(\rho(\gamma_1)) = l(\rho(\gamma_1)) + i\theta(\rho(\gamma_1))$. Let

$$L^2 = \lim_{m\to\infty} L_{\eta_m}^2,\ A^2 = \lim_{m\to\infty} A_{\eta_m}^2,\ (L')^2 = \lim_{m\to\infty} L_{\eta_m'}^2,\ (A')^2 = \lim_{m\to\infty} A_{\eta_m'}^2$$

Then

$$\left|l(\rho(\gamma_1)) - \frac{2\pi}{L^2}\right| \le \frac{8(2\pi)^3}{L^4 - (16)(2\pi)^4} \quad \text{and} \quad \left|\theta(\rho(\gamma_1)) - \frac{2\pi}{A^2}\right| \le \frac{5(2\pi)^3}{(L^2 - 4(2\pi)^2)^2},$$

and

$$\left|l(\rho(\gamma_1)) - \frac{2\pi}{(L')^2}\right| \le \frac{8(2\pi)^3}{(L')^4 - (16)(2\pi)^4} \quad \text{and} \quad \left|\theta(\rho(\gamma_1)) - \frac{2\pi}{(A')^2}\right| \le \frac{5(2\pi)^3}{((L')^2 - 4(2\pi)^2)^2}.$$

Recall that by property $(5')$ of the neighborhood U, we have $|r_1(\eta_m)|, |r_1(\eta'_m)| > 80(2\pi)^2$ for all m. So after passing to the limit, $|\zeta_1|, |\zeta'_1| \ge 80(2\pi)^2$ and therefore $L^2, (L')^2 \ge 40(2\pi)^2$. In particular, $L^2, (L')^2 > 8(2\pi)^2$, which together with the triangle inequality implies

$$\left|\frac{2\pi}{L^2} - \frac{2\pi}{(L')^2}\right| \le 16(2\pi)^3 \left(\frac{1}{L^4} + \frac{1}{(L')^4}\right) \tag{6.2}$$

and

$$\left|\frac{2\pi}{A^2} - \frac{2\pi}{(A')^2}\right| \le 20(2\pi)^3 \left(\frac{1}{L^4} + \frac{1}{(L')^4}\right). \tag{6.3}$$

Next, we combine the inequalities (6.2) and (6.3) to show that ζ_1 and ζ'_1 are close. The following lemma provides a way of doing this. This lemma is a calculation in $\mathbb{C}$ whose proof we postpone until after the completion of Theorem 1.1.

Lemma 6.2. *Let $z_1, z_2 \in \mathbb{C}$, $|z_i| \ge (80)(2\pi)^2$, and set*

$$L_i^2 = \frac{|z_i|^2}{2Im(z_i)} \quad \textit{and} \quad A_i^2 = \frac{|z_i|^2}{2Re(z_i)}.$$

Suppose

$$\left|\frac{2\pi}{L_1^2} - \frac{2\pi}{L_2^2}\right| \le 16(2\pi)^3 \left(\frac{1}{L_1^4} + \frac{1}{L_2^4}\right)$$
$$\left|\frac{2\pi}{A_1^2} - \frac{2\pi}{A_2^2}\right| \le 20(2\pi)^3 \left(\frac{1}{L_1^4} + \frac{1}{L_2^4}\right).$$

Then

$$|z_1 - z_2| < 560(2\pi)^2 \frac{Im(z_1)}{|z_1|}.$$

Setting $z_1 = \zeta_1$ and $z_2 = \zeta_1'$, the inequalities (6.2) and (6.3), together with Lemma 6.2, imply

$$|\zeta_1 - \zeta_1'| < 560(2\pi)^2 \frac{Im(\zeta_1)}{|\zeta_1|}. \tag{6.4}$$

By combining the lower bound from (6.1) and the upper bound from (6.4), we find that

$$\frac{\delta}{2} < 560(2\pi)^2 \frac{Im(\zeta_1)}{|\zeta_1|}.$$

Recall that the constant κ was chosen in property (4) of U so that $Im(q_1(\eta_m)) < \kappa - 1$ for any $\eta_m \in \Phi(C_n)$. Thus $|r_1(\eta_m) - q_1(\eta_m)| < \frac{\delta}{4}$ for all m. It follows that $Im(\zeta_1)$ is bounded above by a quantity that is independent of n:

$$Im\,(\zeta_1) \leq \kappa - 1 + \frac{\delta}{4} < \kappa.$$

Recall from Section 5.4 that $w_1 = \lim_{m\to\infty}(q_1(\eta_m))$ lies in the box $R + 2n$. For any point $w \in R$, $Re(w) > -2$ (see the proof of Lemma 5.6). Since ζ_1 lies in a closed $\frac{\delta}{4}$-neighborhood of $R + 2n$, we have $|\zeta_1| \geq 2n - 2 - \frac{\delta}{4} > 2n - 3$. It follows that

$$\frac{\delta}{2} < 560(2\pi)^2 \frac{Im(\zeta_1)}{|\zeta_1|} < \frac{560(2\pi)^2\kappa}{2n-3}. \tag{6.5}$$

Since κ is independent of n, there are only finitely many n that satisfy (6.5). Hence, for any $\rho \in \overline{\Phi(C_n)} \cap \overline{\Phi(U - C_n)}$ with $n > \frac{560(2\pi)^2\kappa}{\delta} + \frac{3}{2}$, inequality (6.5) produces a contradiction. However, our assumption that $AH(N)$ was locally connected implied that $\overline{\Phi(C_n)} \cap \overline{\Phi(U - C_n)}$ is non-empty for all but finitely many n.

It follows that $AH(N)$ is not locally connected at the point $\sigma^0 \in MP_0(N, P)$. $\square$

We now prove Lemma 6.2. Again, this is just a fact about complex numbers, although we include the proof since the calculation is somewhat nonstandard and nontrivial.

Lemma 6.2. *Let $z_1, z_2 \in \mathbb{C}$, $|z_i| \geq (80)(2\pi)^2$, and set*

$$L_i^2 = \frac{|z_i|^2}{2Im(z_i)} \quad \textit{and} \quad A_i^2 = \frac{|z_i|^2}{2Re(z_i)}.$$

Suppose

$$\left|\frac{2\pi}{L_1^2} - \frac{2\pi}{L_2^2}\right| \leq 16(2\pi)^3 \left(\frac{1}{L_1^4} + \frac{1}{L_2^4}\right) \tag{6.6}$$

$$\left|\frac{2\pi}{A_1^2} - \frac{2\pi}{A_2^2}\right| \leq 20(2\pi)^3 \left(\frac{1}{L_1^4} + \frac{1}{L_2^4}\right). \tag{6.7}$$

Then

$$|z_1 - z_2| < 560(2\pi)^2 \frac{Im(z_1)}{|z_1|}.$$

Note that one could reverse the roles of z_1, z_2 to show

$$|z_1 - z_2| < 560(2\pi)^2 \frac{Im(z_2)}{|z_2|}$$

but we only need one of these inequalities.

Proof. First we claim that $\frac{1}{2}L_1^2 \leq L_2^2 \leq 2L_1^2$. Note that the lower bounds on the size of z_1 and z_2 imply lower bounds on L_1 and L_2.

$$L_i^2 = \frac{|z_i|^2}{2Im(z_i)} \geq \frac{|z_i|^2}{2|z_i|} = \frac{|z_i|}{2} \geq 40(2\pi)^2. \tag{6.8}$$

Set

$$B = \frac{L_2^2}{L_1^2}.$$

Then inequality (6.6) can be rewritten as

$$\frac{L_1^2}{16(2\pi)^2}\left|1 - \frac{1}{B}\right| \leq \left(1 + \frac{1}{B^2}\right)$$

If $B > 2$ then

$$\frac{L_1^2}{16(2\pi)^2}\left(\frac{1}{2}\right) < \frac{5}{4}$$

This implies $L_1^2 < \frac{5(16)(2\pi)^2(2)}{4}$ which contradicts (6.8). Thus $B \leq 2$ which implies that $L_2^2 \leq 2L_1^2$. Reversing the roles of L_1^2 and L_2^2 in the previous argument shows that $L_2^2 \geq \frac{1}{2}L_1^2$.

One consequence of $L_2^2 \geq \frac{1}{2}L_1^2$ is that inequality (6.6) can be replaced by

$$\left|\frac{2\pi}{L_1^2} - \frac{2\pi}{L_2^2}\right| \leq 16(2\pi)^3\left(\frac{1}{L_1^4} + \frac{4}{L_1^4}\right) \leq \frac{16(5)(2\pi)^3}{L_1^4} \tag{6.9}$$

and similarly (6.7) can be replaced by

$$\left|\frac{2\pi}{A_1^2} - \frac{2\pi}{A_2^2}\right| \leq \frac{20(5)(2\pi)^3}{L_1^4}. \tag{6.10}$$

Next we will bound $|z_2 - z_1|^2$.

$$\begin{aligned}
|z_2 - z_1|^2 &= |z_2|^2|z_1|^2\left|\frac{1}{z_1} - \frac{1}{z_2}\right|^2 \\
&= |z_2|^2|z_1|^2\left|\frac{\overline{z_1}}{|z_1|^2} - \frac{\overline{z_2}}{|z_2|^2}\right|^2 \\
&= |z_2|^2|z_1|^2\left|\frac{Re(z_1)}{|z_1|^2} - \frac{Re(z_2)}{|z_2|^2} - i\frac{Im(z_1)}{|z_1|^2} + i\frac{Im(z_2)}{|z_2|^2}\right|^2 \\
&= \frac{|z_2|^2|z_1|^2}{4}\left|\left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right) + i\left(\frac{1}{L_2^2} - \frac{1}{L_1^2}\right)\right|^2 \\
&= \frac{|z_2|^2|z_1|^2}{4}\left(\left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right)^2 + \left(\frac{1}{L_2^2} - \frac{1}{L_1^2}\right)^2\right) \\
&\leq \frac{|z_2|^2|z_1|^2}{4}\left(\frac{(20)^2(5)^2(2\pi)^4}{L_1^8} + \frac{(16)^2(5)^2(2\pi)^4}{L_1^8}\right) \\
&= \frac{|z_2|^2|z_1|^2(2)^2(5)^2(41)(2\pi)^4}{L_1^8}.
\end{aligned}$$

Now recall that $L_2^2 \leq 2L_1^2$. Thus

$$|z_2| = \frac{|z_2|^2}{|z_2|} \leq \frac{|z_2|^2}{Im(z_2)} = 2L_2^2 \leq 4L_1^2.$$

Thus $|z_2|^2 \leq 16L_1^4$. It follows from the previous inequalities that

$$|z_2 - z_1|^2 \leq \frac{|z_2|^2|z_1|^2(2)^2(5)^2(41)(2\pi)^4}{L_1^8} \leq \frac{|z_1|^2(2)^6(5)^2(41)(2\pi)^4}{L_1^4}$$

We can now take the positive square root of both sides, rounding 41 to the nearest square to make the calculation simpler and the inequality strict:

$$|z_2 - z_1| < (2)^4(5)(7)(2\pi)^2\frac{Im(z_1)}{|z_1|}.$$

$\square$

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