T H E   U N I V E R S I T Y   O F   M I C H I G A N

COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

Technical Report

ON SOLUTIONS TO n-PERSON GAMES IN PARTITION FUNCTION FORM

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ORA Project 04637

under contract with:

NATIONAL SCIENCE FOUNDATION
GRANT NO. G-18914
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION   ANN ARBOR

April 1963

This report was also a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The University of Michigan, 1963.

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ABSTRACT

John von Neumann and Oskar Morgenstern (Theory of Games and Economic Behavior, Princeton University Press, 1943) formulated a theory of n-person games in terms of a characteristic function which is defined on the set of all subsets of the set of players. Since then several reformulations of the theory have appeared. Among the more recent developments in this direction is a presentation by R. M. Thrall ("Generalized characteristic functions for n-person games," Proceedings of the Princeton University Conference on game theory, October 4-6, 1961, pp. 157-160) of a theory of n-person cooperative games with side payments in terms of a partition function which is defined on the set of all partitions of the set of players. This formulation assigns a real numbered outcome to each coalition in each partition of the set of players. For each such partition, the sum of the outcomes of its coalitions then determines an imputation simplex. The concepts of dominance and solution are similar to those in the von Neumann-Morgenstern theory, except that an imputation can dominate via a certain coalition only if it is on an imputation simplex realized by a partition containing that coalition. This approach reduces to the von Neumann-Morgenstern approach when all the imputation simplices are the same.

The object of this thesis is to present solutions (stable sets) for games in partition function form as defined by Thrall. Chapter 1 summarizes the formulation of the theory of these games, and gives the solutions for the 2-person games and the n-person games in which the largest payoff goes to the partition made up of the single coalition containing all of the players. All solutions for all 3-person games are discussed in Chapter 2. Chapter 3 gives a polyhedral solution for each 4-person game in which distinct partitions give rise to distinct imputation simplices. Partial results for n-person games in which only partitions of type (n), (n-1,1) and (1,...,1) have large outcomes are presented in Chapter 4. The final chapter lists some of the unsolved problems in the theory.

In comparison with the von Neumann-Morgenstern games, the Thrall theory gives more cases to consider and usually more imputations in a solution. But there are fewer solutions in the latter theory as the number of imputation simplices increases. The discriminatory solutions and the bargaining curves in the von Neumann-Morgenstern theory do not seem to appear in the Thrall games unless some of the imputation simplices coincide.

CHAPTER 1

n-PERSON GAMES IN PARTITION FUNCTION FORM

1. INTRODUCTION

In 1944, von Neumann and Morgenstern [8] formulated a theory of n-person games in terms of a characteristic function which is defined on the set of all subsets of the set of players. This formulation was followed by much criticism of it, and several reformulations of the theory have appeared. An excellent survey of this work, through 1957, is given in the book by Luce and Raiffa [2]. A complete bibliography of all of game theory up to 1958 is contained in [7]. Among the more recent developments in this direction is a formulation by R. M. Thrall [6] of a theory of n-person cooperative games with side payments in terms of a partition function which is defined on the set of all partitions of the set of players. Thrall's formulation assigns a real numbered outcome to each coalition (coset) in each partition of the set of players. For each partition, the sum of the outcomes of its coalitions then determines an imputation simplex. The concepts of dominance and solution are similar to those in the von Neumann-Morgenstern theory, except that an imputation can dominate via a certain coalition only if it is on an imputation simplex realized by a partition containing that coalition. This approach reduces to the von Neumann-Morgenstern approach when all the imputation simplices are the same. R. J. Aumann, Morton Davis, Michael Maschler and Bezalel Peleg (see references in [3]) are also working on a more local theory of bargaining sets for n-person games which makes use of partitions of the set of players. It is safe to say that none of the models for n-person games proposed to date is completely satisfactory and no such single model is likely to appear. But the various models do give some insight into the problems of bargaining and conflict resolution.

The object of this thesis is to present solutions (stable sets) for games in partition function form as defined by Thrall. Chapter 1 summarizes the formulation of the theory of these games, and gives the solutions for the 2-person games and the n-person games in which the largest payoff goes to the partition made up of the single coalition containing all of the players. Much of this material was presented by Thrall in [6]. All solutions for all 3-person games are discussed in Chapter 2. Chapter 3 gives a polyhedral solution for all 4-person games in which distinct partitions give rise to distinct imputation simplices whenever their payoffs are greater or equal to the payoff to the partition containing the single coalition of all players. Partial results for n-person games in which only partitions of type (n), (n-1,1) and (1,...,1) are significant are presented in Chapter 4. The final chapter lists some of the outstanding problems in the theory.

2. A GAME AND THE VALUE OF A COALITION

We prodeed to define an n-person game in partition function form and the value of a coalition in such a game.

Let

$$N = \{1,\ldots,n\}$$

be a set of n players who are represented by $1,\ldots,n$. Let

$$P = \{P_1,\ldots,P_r\}$$

be an arbitrary partition of N into coalitions $P_1,\ldots,P_r$. The set of all partitions of N is denoted by

$$\Pi = \{P\}$$

Denote the real numbers by R^1. Then for each partition P assume there is an <u>outcome function</u>

$$F_P \;:\; P \to R^1$$

which assigns the real numbered outcome $F_P(P_i)$ to the coalition P_i when the partition P forms. The function

$$F \;:\; \Pi \to \{F_P\}$$

which assigns to each partition its outcome function is called the <u>pay-off function</u> or <u>partition function</u> for the game. Finally the ordered pair

$$\Gamma = (N,F)$$

is called an <u>n-person game in partition function form</u>.

For each non-empty subset M of N define the <u>value</u> of M as

$$(1) \qquad v(M) = \min_{\{P | M \in P\}} F_P(M)$$

and define $v(\emptyset) = 0$. Let $v(\{i\}) = v_i$ for each $i \in N$. This minimum is over all partitions P which contain M as a coset and not the partitions in which M is a union of more than one coset. That is, secret coalitions are prohibited. This function $v: 2^N \to R^1$ need not satisfy the superadditivity condition

$$v(M_1 \cup M_2) \geq v(M_1) + v(M_2)$$

whenever $M_1 \cap M_2 = \emptyset$. In fact, there exist games for any choice of the values v(M). If we were to take the above minimum over all P that have M as a union of cosets of P, we would obtain a superadditive function as in the von Neumann-Morgenstern theory. This was proved by D. B. Gillies on p. 68 in [7].

3. IMPUTATION, DOMINATION, AND SOLUTION

For a given game, the concepts of imputation, domination, and solution can now be introduced in a manner similar to that in the von Neumann-Morgenstern theory.

A vector $\underline{a} = (a_1,\ldots,a_n)$ is called an imputation if

(2) $\quad a_i \geq v_i \qquad\qquad i = 1,\ldots,n$

and

(3) $\quad \sum_{i\in N} a_i = \sum_{P_j\in P} F_P(P_j) \qquad\qquad$ for some $P\in\Pi$.

Let $\quad R$ = the set of all imputations

of a game. Conditions (2) and (3) are called individual rationality and realizability respectively. An imputation $\underline{a}$ is a possible set of payoffs to the individual players, amount a_i to player i, at the end of a game. (2) states that no player need accept a payoff less than what he is assured of if he forms a coalition by himself. And (3) states that the total payoff to all of the players equals the sum of the coalition outcomes for some partition. This definition of an imputation allows for side payments between coalitions. Replacing the equality in (3) by "less than or equals" would allow for a disposal of wealth, and does not appear to change the following theory significantly.

If $\underline{a}$ and $\underline{b}$ are imputations and M is a non empty subset of N, then $\underline{a}$ dominates $\underline{b}$ via M, denoted $\underline{a} \operatorname{dom}_M \underline{b}$, means that

(4) $\quad a_i > b_i \qquad\qquad$ for all $i\in M$,

(5) $$\sum_{i\in M} a_i \leq v(M)$$

and

(6) $$\sum_{i\in N} a_i = \sum_{P_j\in P} F_P(P_j) \qquad \text{for some } P\in\Pi \text{ with } M\in P.$$

These conditions are called, respectively, M-preferable, M-effective, and M-realizable. Condition (4) says that each player in M prefers his payoff in $\underline{a}$ to that in $\underline{b}$. (5) states that M can be assured of getting at least what they get in $\underline{a}$ no matter what N - M does, and (6) states that $\underline{a}$ could arise when M is actually acting as a coalition. This last restriction does not appear in the von Neumann-Morgenstern theory. $\underline{a}$ is called exactly M-effective if the equality holds in (5), and strictly M-effective if the inequality holds. If (5) fails, then we call $\underline{a}$ M-ineffective. We say $\underline{a}$ dominates $\underline{b}$, denoted $\underline{a}$ dom $\underline{b}$, if there exists such an M such that $\underline{a} \operatorname{dom}_M \underline{b}$. The relation "dom" is neither transitive nor antisymmetric. Also, if $A \subseteq R$, let $\operatorname{dom}_M A = \{\underline{b}\in R \mid \underline{a} \operatorname{dom}_M \underline{b} \text{ for some } \underline{a}\in A\}$, and $\operatorname{dom} A = \{\underline{b}\in R \mid \underline{a} \operatorname{dom} \underline{b} \text{ for some } \underline{a}\in A\}$. Clearly, $\operatorname{dom}(A\cup B) = \operatorname{dom} A \cup \operatorname{dom} B$ and $\operatorname{dom}(A\cap B) \subseteq \operatorname{dom} A \cap \operatorname{dom} B$.

A set of imputations K is a solution if

(7) $$K \cap \operatorname{dom} K = \emptyset$$

and

(8) $K \cup \text{dom } K = R$.

These two conditions are equivalent to the one condition

$$R - \text{dom } K = K.$$

In words, these two equations say that

(7') if $\underline{a}$ and $\underline{b}$ are in K, then neither dominates the other,

and

(8') if $\underline{c}$ is not in K, then there exists an $\underline{a}$ in K which dominates $\underline{c}$.

If B is some subset of R, we will also say that a subset K of B is a <u>solution for</u> B if $K \cap \text{dom } K = \emptyset$ and $K \cup \text{dom } K \supseteq B$.

A set that is more stable than a solution set is the <u>core</u> defined by

$$C = R - \text{dom } R,$$

that is, the undominated imputations in R. For games in partition function form the core is similar to this concept in the classical theory as discussed by Gillies on p. 71 of [7]. Since $C = \emptyset$ for many games, we will proceed to find solutions.

4. S-EQUIVALENCE

Let Γ and Γ' be two n-person games (in partition function form). Γ and Γ' are called <u>S-equivalent</u> if there exists constants

$c > 0$, $a_1, \ldots, a_n$ and a permutation σ of N such that

$$(9) \qquad F_{P^\sigma}'(P_i^\sigma) = cF_P(P_i) + \sum_{j \in P_i} a_j$$

for all $P_i \in P$ and all $P \in \mathbb{P}$. Intuitively, σ relabels the players, c changes the unit of wealth, and a_j is an ante or subsidy that player j makes before the play of the game. Γ is in <u>normal</u> <u>form</u> if $v_i = 0$ for all $i \in N$. Γ is in <u>strict</u> <u>normal</u> <u>form</u> if Γ is in normal form and

$$\max_{P \in \mathbb{P}} \sum_{P_i \in P} F_P(P_i) = 1.$$

It is easy to prove the following facts about S-equivalence. S-equivalence is an equivalence relation. Each equivalence class contains a game in normal form. If Γ has some $P \in \mathbb{P}$ for which

$\sum_{P_i \in P} F_P(P_i) > 0$, then there exists Γ' in strict normal form such that

Γ' is S-equivalent to Γ. Two games in strict normal form are S-equivalent if and only if they are identical up to some permutation σ of N. Two S-equivalent games Γ and Γ' are isomorphic, that is, there is a one to one correspondence between R and R' that preserves the relation "$\underset{M}{\text{dom}}$" for all $M \subseteq N$, and thus preserves solutions.

In what follows, we assume that all games are in normal form but not necessarily strict normal form. Thus equation (2) now becomes

$$(2') \qquad a_i \geq 0.$$

For a partition P in a game Γ let

$$|P| = \sum_{P_i \in P} F_P(P_i).$$

Also, for any constant b let

$$A(b) = \{\underline{a} | \sum_{i \in N} a_i = b \text{ and } a_i \geq 0\}.$$

Then for each $P \in \Pi$ conditions (2') and (3) give an imputation simplex $A(|P|)$ which we will also denote by $A(P)$.

5. 2-PERSON GAMES

For a 2-person game, $N = \{1,2\}$ and $\Pi = \{P^0, P^1\}$ where

$$P^0 = \{N\} \text{ and } P^1 = \{\{1\}, \{2\}\}.$$

Assume $F_{P^0}(N) = c$ and $F_{P^1}(\{i\}) = 0$ for $i = 1,2$. Then

$$v(N) = c \text{ and } v_i = 0 \qquad i = 1,2.$$

And

$$R = A(c) \cup \{(0,0)\}.$$

Clearly, if $c > 0$ then the unique solution is $K = A(c)$, and if $c \leq 0$ then the unique solution is $K = R = \{(0,0)\}$. That is, the two players must agree on some way to split the amount $c > 0$ or else both get nothing.

6. THE PARTITION $\{N\}$ AND GAMES WITH LARGE $v(N)$.

Define $\underline{a} \underset{M}{\geq} \underline{b}$ to mean $a_i \geq b_i$ for all $i \in M$. Similarly define $\underset{M}{>}$ and $\underset{M}{=}$.

Lemma 1. If $\underline{a} \underset{M}{\text{dom}} \underline{b}$ and $\underline{b} \underset{S}{\geq} \underline{c}$ where $M \subseteq S \subseteq N$, then $\underline{a} \underset{M}{\text{dom}} \underline{c}$.

Proof. Since $\underline{a} \underset{M}{\text{dom}} \underline{b}$, $\underline{a}$ is M-effective and M-realizable. And clearly $a_i > c_i$ for all $i \in M$.

Corollary. If $\underline{b} \in \underset{N}{\text{dom}}\ A$ and $A \subseteq K \cup \text{dom}\ K$, then $\underline{b} \in \text{dom}\ K$.

The following theorem states that no part of a solution can be below the imputation simplex $A(c)$ realized by the partition $\{N\}$. That is, no imputation in a solution can be on an imputation simplex $A(a)$ with $a < c$. And Theorem 2 then shows that this partition $\{N\}$ causes no trouble in finding solutions.

Theorem 1. If K is any solution and $a < c = v(N)$, then $K \cap A(a) = \emptyset$.

Proof. Let $\underline{b} \in A(a)$ and define $\underline{a}$ by $a_i = b_i + d$ where $nd = c - \sum_{i \in N} b_i > 0$. Then $\underline{a} \in A(c)$ and $\underline{a} \underset{N}{\text{dom}} \underline{b}$. If $\underline{a} \in K$ then $\underline{b} \in \underset{N}{\text{dom}}\ K$. If $\underline{a} \notin K$, then $\underline{a} \in \underset{M}{\text{dom}}\ K$ for some $M \subseteq N$ and so $\underline{b} \in \underset{M}{\text{dom}}\ K$ by Lemma 1. In either case $\underline{b} \notin K$.

Theorem 2. If K is a solution for $T = \cup A(P)$ where this union is over all $P \in \Pi$ with $|P| \geq c = v(N)$ and $P \neq \{N\}$, then $K' = K \cup (A(c)\text{-dom } K)$ is a solution for all of R.

Proof. If $A(c) \subseteq T$ then $A(c) \subseteq K \cup \text{dom } K$ by hypothesis and $K' = K$. If $A(c) \cap T = \emptyset$ then clearly $A(c) \subseteq K' \cup \text{dom } K \subseteq K' \cup \text{dom } K'$. In either case, if $\underline{b} \in R - A(c) - T$ then $\underline{b} \in A(a)$ for some $a < c$ and so $\underline{b} \in \text{dom}_N A(c)$ as was shown in the proof of Theorem 1. So $\underline{b} \in \text{dom}_N A(c) \subseteq K' \cup \text{dom } K'$ and then the above Corollary gives $\underline{b} \in \text{dom } K'$. Thus $K' \cup \text{dom } K' = R$. If $A(c) \subseteq T$ then $K' \cap \text{dom } K' = K \cap \text{dom } K = \emptyset$. If $A(c) \cap T = \emptyset$ then $K' \cap \text{dom } K' = [K \cup (A(c) - \text{dom } K)] \cap \text{dom}[K \cup (A(c) - \text{dom } K)] = [K \cap \text{dom } K] \cup [(A(c) - \text{dom } K) \cap \text{dom } K] \cup [K \cap \text{dom } (A(c) - \text{dom } K)] \cup [(A(c) - \text{dom } K) \cap \text{dom } (A(c) - \text{dom } K)] \subseteq \emptyset \cup \emptyset \cup [K \cap \text{dom } A(c)] \cup [A(c) \cap \text{dom } A(c)]$. But $\underline{b} \in \text{dom } A(c)$ implies $\sum_{i \in N} b_i < c$, and $\underline{a} \in K$ or $\underline{a} \in A(c)$ implies $\sum_{i \in N} a_i \geq c$. So $(K \cup A(c)) \cap \text{dom } A(c) = \emptyset$. Thus $K' \cap \text{dom } K' = \emptyset$. Therefore, K' is a solution for R.

We will now give the solution for n-person games in which the outcome to the partition $\{N\}$ is greater than the sum of the outcomes for any other partition.

Theorem 3. For an n-person game with $F_{\{N\}}(N) = v(N) = c > \sum_{P_j \in P} F_P(P_j)$ for all partitions P different from $\{N\}$, the unique solution is $K = A(c)$.

Proof. Note that $c > 0$ since $v(N) > \sum_{i\in N} F_P(\{i\}) \geq \sum_{i\in N} v_i = 0$ where $P = \{\{1\},\ldots,\{n\}\}$. Let $K = \emptyset$ in Theorem 2 and then $K' = A(c)$. So $A(c)$ is a solution. It is the unique solution by Theorem 1.

For many games of the type in Theorem 3, the solution $A(c)$ seems to contain too many imputations, but if all the players in N are actually acting as a coalition, then any imputation in $A(c)$ seems reasonable, at least as reasonable as the discriminatory solutions in the classical theory. In general, a solution for a game in partition function form has at least as many imputations as a similar solution in the classical theory. However, Thrall games usually have fewer solutions than the corresponding von Neumann-Morgenstern games as the number of distinct imputation simplices $A(P)$ ($P \neq \{N\}$ nor $\{\{1\},\ldots,\{n\}\}$) increases. It is also clear from Theorem 3 that the intersection of all solutions need not be the core for Thrall games, because $A(c)$ is the unique solution but an imputation below $A(c)$ may dominate some imputations on $A(c)$.

7. THE PARTITION $\{\{1\},\ldots,\{n\}\}$.

We will now show that the partition $\{\{1\},\ldots,\{n\}\}$ causes no trouble in finding solutions.

Lemma 2. Domination via one element subsets $\{i\}$ is impossible.

Proof. $\underline{a} \underset{\{i\}}{\text{dom}} \underline{b}$ implies $0 = v_i \geq a_i > b_i \geq 0$ which is impossible.

Theorem 4. If K is a solution for $V = \cup A(P)$ where the union is over all $P \in \mathbb{P}$ except $P = P' = \{\{1\},\ldots,\{n\}\}$, then $K' = K \cup (A(g) - \text{dom } K)$ is a solution for all of R where $g = |P'| = \sum_{i \in N} F_{P'}(\{i\})$.

Proof. If $A(g) \subseteq V$ then $R = V$ and $K' = K$ and K' is a solution for R. So assume $A(g) \cap V = \emptyset$. Then $K' \cup \text{dom } K' = [K \cup (A(g)\text{-dom } K)] \cup \text{dom}[K \cup (A(g)\text{-dom } K)] = [K \cup \text{dom } K] \cup [K \cup \text{dom } (A(g)\text{-dom } K)] \cup [(A(g)\text{-dom } K) \cup \text{dom } K] \cup [(A(g)\text{-dom } K) \cup \text{dom } (A(g)\text{-dom } K)] \supseteq (K \cup \text{dom } K) \cup A(g) = R$. And $K' \cap \text{dom } K' = [K \cup (A(g)\text{-dom } K)] \cap \text{dom } [K \cup (A(g)\text{-dom } K)] \subseteq [K \cap \text{dom } K] \cup [(A(g)\text{-dom } K) \cap \text{dom } K] \cup ([K \cup (A(g)\text{-dom } K)] \cap \text{dom } A(g)) = \emptyset$, because $\text{dom } A(g) = \emptyset$ since the only coalitions realizable for $\underline{a} \in A(g)$ are one element coalitions and domination by these is impossible by Lemma 2. Therefore, K' is a solution for R.

As a result of Theorems 2 and 4, only the imputation simplices that are at least as "high" as $A(c)$ where $c = v(N)$ and that are realized by the non trivial partitions will be considered in finding solutions. So the appropriate points on $A(c)$ and $A(g)$ must be appended to the solutions that are described in what follows in order to get a complete solution for all of R.

CHAPTER 2

THE 3-PERSON GAMES

1. INTRODUCTION

In this chapter, we will discuss the 3-person games in partition function form. The set of players is

$$N = \{1,2,3\}$$

and the set $\Pi = \{P^p\}$ of partitions of N consists of

$$P^0 = \{N\}$$

$$P^i = \{\{i\},\{j,k\}\} \qquad i = 1,2,3$$

$$P^4 = \{\{1\},\{2\},\{3\}\}$$

where in this chapter i,j,k always stand for distinct elements of N. We denote the values of the outcome functions by

$$F_{P^0}(N) = c$$

$$F_{P^i}(\{i\}) = d_i,\ F_{P^i}(\{j,k\}) = e_i \qquad i = 1,2,3$$

$$F_{P^4}(\{i\}) = g_i \qquad i = 1,2,3.$$

Then the values of the coalitions of N are

$$v(\emptyset) = 0$$

$$v_i \;=\; \min\{d_i, g_i\} \qquad i = 1,2,3$$

$$v(\{j,k\}) \;=\; e_i \qquad i = 1,2,3$$

$$v(N) \;=\; c.$$

Further reducing to normal form under S-equivalence gives

$$v_i \;=\; 0$$

and so one of d_i or $g_i = 0$ and the other is non negative, and also $e_i \leq c_i$ where we define

$$c_i \;=\; d_i + e_i.$$

The set R of all imputations consists of the vectors $\underline{a} = (a_1, a_2, a_3)$ with $a_i \geq 0$ which satisfy one of the equations

$$a_1 + a_2 + a_3 \;=\; c$$

$$a_1 + a_2 + a_3 \;=\; d_i + e_i \;=\; c_i \qquad i = 1,2,3$$

$$a_1 + a_2 + a_3 \;=\; g_1 + g_2 + g_3 \;=\; g.$$

These five imputation simplices are realized by the partitions P^0, P^i, P^4 and are written as $A(c)$, $A(c_i)$, $A(g)$, respectively.

If $\underline{a}$ and $\underline{b}$ are imputations and $N \supseteq M \neq \emptyset$, then $\underline{a}\ \underset{M}{\mathrm{dom}}\ \underline{b}$ means

$$a_i > b_i \text{ for } i \in M,$$

$$\sum_{i \in M} a_i \leq v(M), \text{ and}$$

$\underline{a}$ is on an imputation simplex realized by a partition that contains M.

But domination via $M = \{i\}$ is impossible by Lemma 2. And $\underline{a} \operatorname{dom}_N \underline{b}$ means

$\underline{a} \in A(c)$ and $\underline{b}$ is an imputation in the open octant below $\underline{a}$. By Theorem 1, such $\underline{b}$ are never in a solution, so we do not have to worry about domination via N. Finally, $\underline{a} \operatorname{dom}_{\{j,k\}} \underline{b}$ means

$$a_j > b_j, a_k > b_k$$

$$a_j + a_k \leq v(\{j,k\}) = e_i \text{ (or } a_i \geq d_i\text{), and}$$

$$\underline{a} \in A(c_i).$$

So $\underline{a}$ dominates all imputations $\underline{b}$ that are in the open wedge $\{\underline{x} | x_j < a_j, x_k < a_k\}$. This wedge meets the imputation simplices A(P) with $|P| \geq a_j + a_k$ in congruent parallelogram shaped regions similar to the regions in the von Neumann-Morgenstern theory, for example, see Figure 72 on p. 408 of [8]. On the A(P) with $|P| < a_j + a_k$ these parallelograms are truncated by $x_j + x_k = |P|$ (or $x_i = 0$).

2. SOLUTIONS

The nature of the solutions for the 3-person games depends upon the number of the simplices $A(c_i)$ that have $c_i \geq c$. So for the purpose of discussing solutions, four genera will be considered.

Genus 0. $c > c_1 \geq c_2 \geq c_3$

Genus 1. $c_1 \geq c > c_2 \geq c_3$

Genus 2. $c_1 \geq c_2 \geq c > c_3$

Genus 3. $c_1 \geq c_2 \geq c_3 \geq c$.

We assume that $c_1 \geq c_2 \geq c_3$ since the other possibilities can be obtained from these by symmetry. The solutions in some genera also depend upon the distribution of equalities in the above relations. Genera 2 and 3 are thus subdivided into species.

Genus 2. Species A. $c_1 > c_2$

Species B. $c_1 = c_2$

Genus 3. Species A. $c_1 > c_2 > c_3$

Species B. $c_1 = c_2 > c_3$

Species C. $c_1 > c_2 = c_3$

Species D. $c_1 = c_2 = c_3$

From Theorem 4, the simplex $A(g)$ need not be considered in determining solutions. Except for describing genera, the simplex $A(c)$ also need not be considered in determining solutions (Theorem 2). And by Theorem 1, no part of a solution is below $A(c)$. Therefore, only the parts of the solutions on the simplices $A(c_i)$ with $c_i \geq c$ will be described in the following. For complete solutions, we must add the appropriate parts on $A(g)$ and $A(c)$. So we now proceed to discuss the solutions for each of the above cases. The verification that these are solutions is fairly evident from the geometry of Figures 1-6. The technical proofs are long but easy and will be omitted. The verification of the uniqueness of these solutions (except for Genus 0) will be discussed in the following section.

Our discussion of solutions is followed by corresponding figures in which the solutions K consist of the closed regions bounded by the heavy lines. In these figures, those imputation simplices $A(c_i)$ that are at least as high as $A(c)$ appear as if they were viewed from a distance in the $(1,1,1)$ direction, that is, in barycentric coordinates. Points on the same 120°, 60°, 0° lines have the same first, second, third coordinates respectively. The figures show the cases where $c_i \geq 0$ for there are no imputations in $A(c_i)$ to consider if $c_i < 0$.

Genus 0. The unique solution is $A(c)$. This was proved in Theorem 3 for the case of arbitrary n.

Genus 1. The unique solution is

$$A(c_1) - \{\underline{x} \mid x_2 + x_3 < e_1\}$$

where recall that $e_1 = v(\{2,3\}) \leq c_1$.

Genus 2, Species A. The unique soluton is

$$\bigcup_{p=1}^{2} A(c_p) - \{\underline{x} \mid x_2 + x_3 < e_1\} - \{\underline{x} \mid x_1 < d_1 - \Delta_{12}, x_1 + x_3 < e_2\}$$

where $\Delta_{pq} = c_p - c_q$. The last term could also be written as

$$\bigcup_{p=1}^{2} \{\underline{x} \in A(c_p) \mid e_1 + \Delta_{p2} < x_2 + x_3,\ x_1 + x_3 < e_2\}.$$

Genus 2, Species B. The solution is

$$A(c_1) - \{\underline{x} | x_2 + x_3 < e_1\} - \{\underline{x} | x_1 + x_3 < e_2\} \cup K_1$$

where K_1 is a continuous curve on $A(c_1)$ from the point $(d_1, d_2, c_1 - d_1 - d_2)$ to the edge $\{x_3 = 0\}$ whose x_1 and x_2 coordinates are non decreasing as x_3 decreases. K_1 is the same as the curves in the von Neumann-Morgenstern 3-person games, for example, see Figure 82 on p. 412 of [8]. This solution is unique up to the choice of K_1.

Genus 3, Species A. The unique solution is

$$\bigcup_{p=1}^{3} A(c_p) - \{\underline{x} | x_2 + x_3 < e_1\} - \{\underline{x} | x_1 < d_1 - \Delta_{12}, x_1 + x_3 < e_2\}$$

$$-\{\underline{x} | x_1 < d_1 - \Delta_{13}, x_2 < d_2 - \Delta_{23}, x_1 + x_2 < e_3\}.$$

Genus 3, Species B. The solution is

$$\bigcup_{p=1}^{3} A(c_p) - \{\underline{x} | x_2 + x_3 < e_1\} - \{\underline{x} | x_1 + x_3 < e_2\}$$

$$-\{\underline{x} | x_1 < d_1 - \Delta_{13}, x_2 < d_2 - \Delta_{23}, x_1 + x_2 < e_3\} \cup K_1$$

where K_1 is the same curve as in Genus 2, Species B. This solution is also unique up to the choice of K_1.

Genus 3, Species C. The solution is

$$\bigcup_{p=1}^{3} A(c_p) - \{\underline{x} | x_2 + x_3 < e_1\} - \{\underline{x} | x_1 < d_1 - \Delta_{12}, x_1 + x_3 < e_2\}$$

$$-\{\underline{x} | x_1 < d_1 - \Delta_{13}, x_1 + x_2 < e_3\} \cup K_2 \cup K_3$$

where two cases need to be considered.

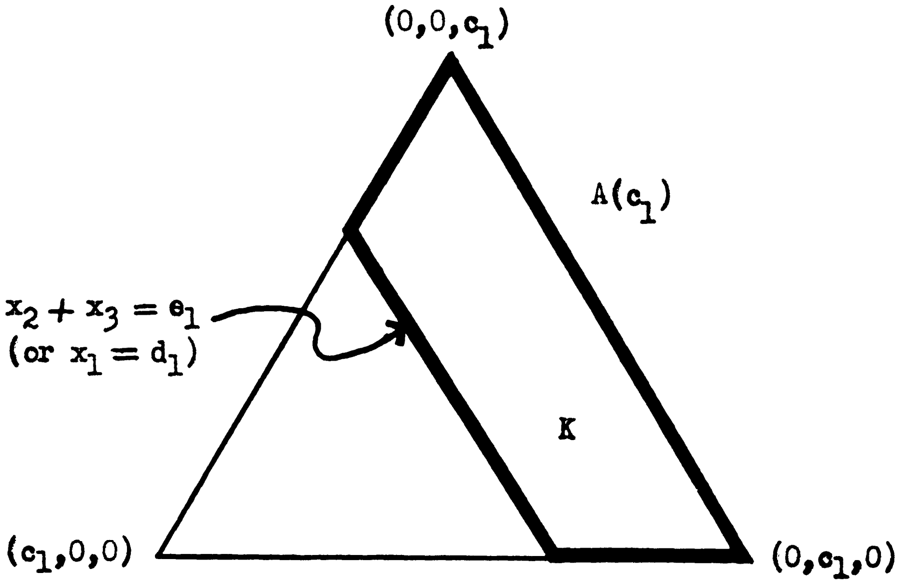

Figure 1. Genus 1.

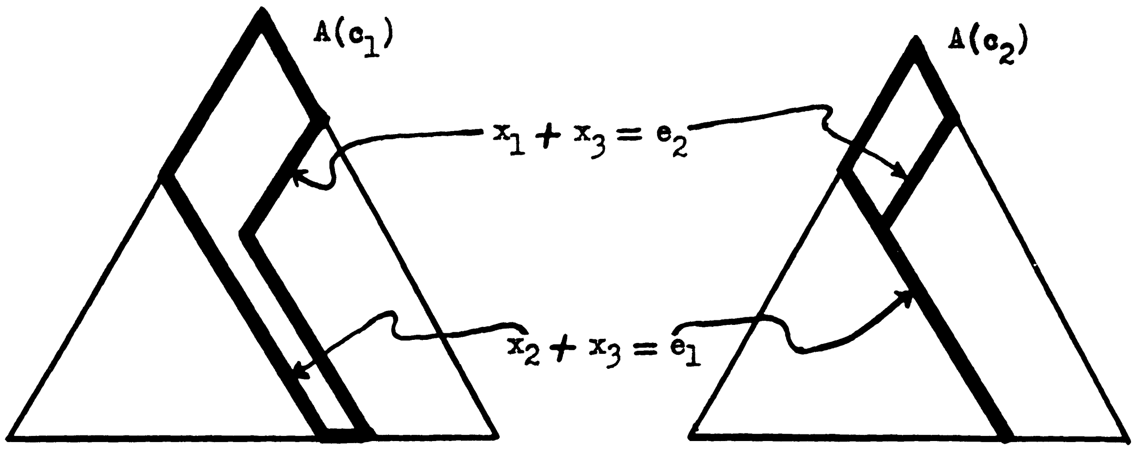

Figure 2. Genus 2, Species A.

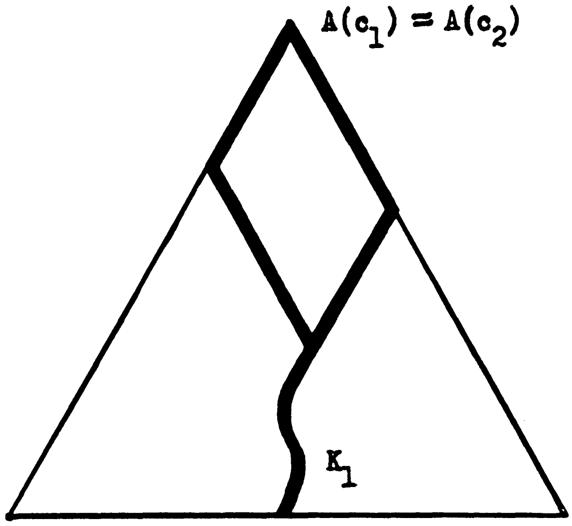

Figure 3. Genus 2, Species B.

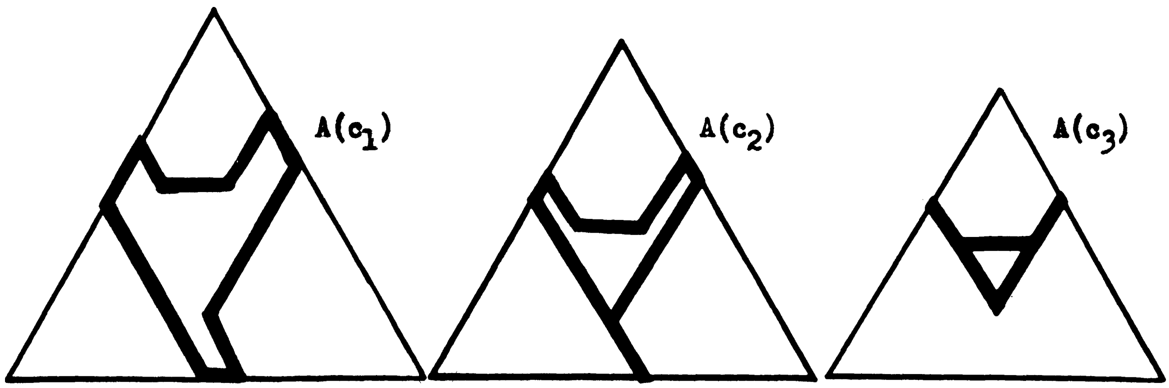

Figure 4. Genus 3, Species A.

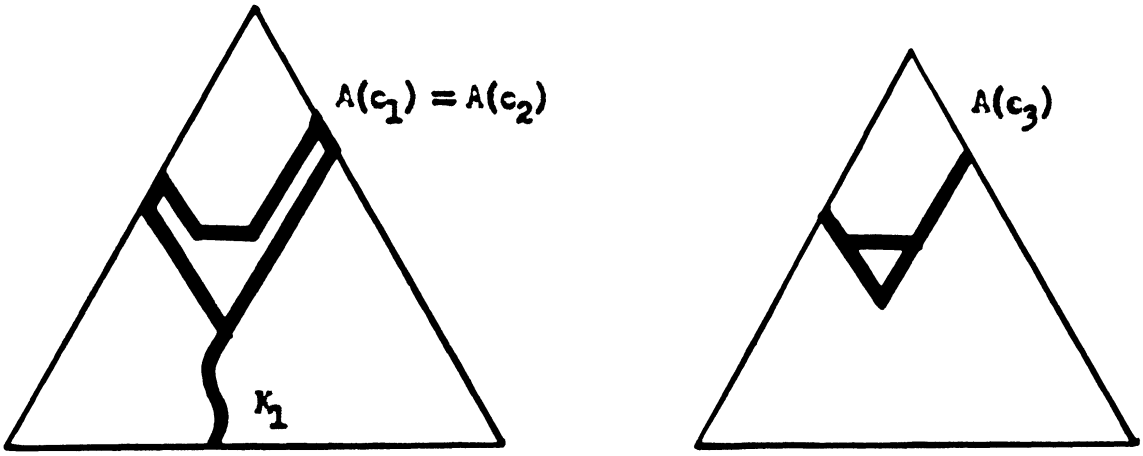

Figure 5. Genus 3, Species B.

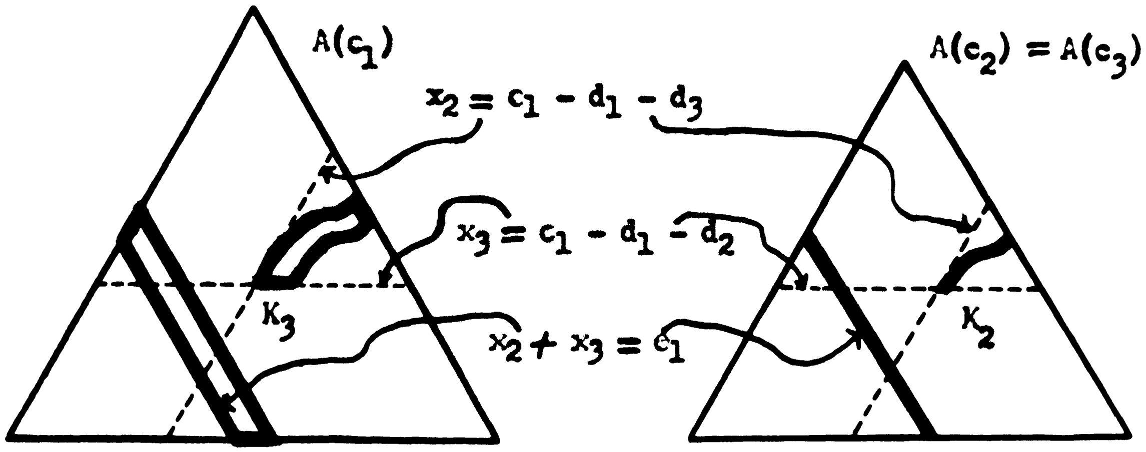

Figure 6. Genus 3, Species C, Case $c_1 \geq d_1 + d_2 + d_3$.

(i) If $c_1 \leq d_1+d_2+d_3$, then K_2 is a curve on $A(c_2)$ from the point $(c_2-d_2-d_3, d_2, d_3)$ to the edge $\{x_1 = 0\}$ and is analogous to K_1 above, and K_3 is a curved bar on $A(c_1)$ of width $\sqrt{2}\Delta_{12}$ whose sides are the same shape as K_2 and whose ends are determined by $x_1 \geq 0$, and $x_2 \geq d_2$, $x_3 \geq d_3$.

(ii) If $c_1 \geq d_1+d_2+d_3$, then K_2 is a similar curve on $A(c_2)$ from the point $(c_2+d_2+d_3-2e_1,\ c_1-d_1-d_3,\ c_1-d_1-d_2)$ to the edge $\{x_1 = 0\}$, and K_3 is a similar bar on $A(c_1)$ whose ends are determined by $x_1 \geq 0$ and $x_2 \geq c_1-d_1-d_3$, $x_3 \geq c_1-d_1-d_2$. Case (ii) is the one shown in Figure 6.

Genus 3, Species D. The solutions (still not counting $A(g)$ and $A(c)$) in this case are the same as those in the von Neumann-Morgenstern 3-person games. Again, two cases need to be considered.

(i) If $c_1 \leq d_1+d_2+d_3$, then the solutions are the type pictured in Figure 86 on p. 415 of [8].

(ii) If $c_1 \geq d_1+d_2+d_3$, then the solutions are the type pictured in Figure 82 or Figure 83 on p. 412 of [8].

In comparison with the von Neumann-Morgenstern 3-person games, the Thrall games give more cases to consider and usually more points in a solution, but fewer solutions in most cases. In the cases where the $A(c_i)$ are distinct, a unique solution exists and it is polyhedral with boundaries parallel to the edges of the imputation simplices $A(c_i)$.

When some c_i are equal and as large as c, then an infinity of solutions (the K_i parts) exist; and often one of these solutions is the limit of the distinct simplex case. However, this is not true when in Genus 3, Species C is considered as the limiting case of Species A. Note that the union of all solutions for some games of the type in Genus 3, Species C need not meet the $A(c_i)$ in a connected set. Also, if $c > d_1+d_2+d_3$, then the core of the games in Genus 0 is empty, and since A(c) is the unique solution and need not be empty, we have an example where the intersection of all solutions is not the core.

3. UNIQUENESS

The following theorems will be used to prove the uniqueness asserted in the above discussion of solutions.

Theorem 5. If $c_1 > c_2 \geq c_3$, $c_1 \geq c$ and K is any solution, then

$$K \cap \{\underline{x} \mid x_2 + x_3 < e_1\} = \emptyset.$$

Proof. If $e_1 = c_1 - d_1 \leq 0$, then $x_2+x_3 < e_1 \leq 0$ implies $\underline{x} \notin R \supseteq K$ and the theorem is true. But always $d_1 \geq 0$ or $e_1 \leq c_1$. So we can assume $0 < e_1 \leq c_1$ Let $B^n = \{\underline{x} \in A(c_1) \mid x_2 + x_3 \leq d^n\}$ where $d^n = \min\{e_1, n\Delta_{12}\}$ and again $\Delta_{pq} = c_p - c_q$.

The idea of our proof is to show that there exists a neighborhood B^1 of the vertex $(c_1,0,0)$ on $A(c_1)$ that is contained in $K \cup \operatorname{dom}_{\{2,3\}} K$ and is $\{2,3\}$ effective. Then the neighborhood $\{\underline{x} \in A(a) \mid x_2+x_3 < d^1\}$ of the vertex

$(a,0,0)$ on each $A(a)$ is in $\underset{\{2,3\}}{\text{dom}}\, K$ and thus does not meet K. Using these facts, we can continue to enlarge our neighborhood B^1 to B^2, B^3,...until it includes all of $\{\underline{x}\epsilon A(c_1)\,|\,x_2+x_3 < e_1 = v(\{2,3\})\}$. Then each region $\{\underline{x}\epsilon A(a)\,|\,x_2+x_3 < e_1\}$ is in $\underset{\{2,3\}}{\text{dom}}\, K$ and thus does not meet K.

First, we will show that $B^1 \subseteq K \cup \underset{\{2,3\}}{\text{dom}}\, K$. Since K is a solution, $B^1 \subseteq K \cup \text{dom}\, K$. But $B^1 \cap \underset{M}{\text{dom}}\, K = \emptyset$ for all $M \neq \{2,3\}$. Because domination via one player coalitions is impossible; and since $c_1 > c$, $A(c_1) \cap \underset{N}{\text{dom}}\, R = \emptyset$. And because $B^1 \cap \underset{\{1,i\}}{\text{dom}}\, K = \emptyset$, for if not then there exists $\underline{a}\epsilon A(c_j)$ ($j \neq 1$ nor i) and $\underline{b}\epsilon B^1$ such that $\underline{a} \underset{\{1,i\}}{\text{dom}} \underline{b}$ which implies $b_1 < a_1 \leq a_1+a_2+a_3 = c_j$ and $b_2+b_3 = c_1-b_1 > c_1-c_j = \Delta_{1j} \geq \Delta_{12} \geq \min\{e_1,\Delta_{12}\} = d^1$, which contradicts $\underline{b}\epsilon B^1$. Therefore, $B^1 \subseteq K \cup \underset{\{2,3\}}{\text{dom}}\, K$. Next, it is then clear that

$$\{\underline{x}\,|\,x_2+x_3 < d^1\} \subseteq \underset{\{2,3\}}{\text{dom}}\, \{\underline{x}\epsilon A(c_1)\,|\,x_2+x_3 = d^1\} \subseteq \underset{\{2,3\}}{\text{dom}}\, B^1 \subseteq \underset{\{2,3\}}{\text{dom}}\, (K \cup \underset{\{2,3\}}{\text{dom}}\, K) = \underset{\{2,3\}}{\text{dom}}\, K.$$

So $\{\underline{x}\,|\,x_2+x_3 < d^1\} \cap K = \emptyset$. If $d^1 = e_1$, then the proof is complete.

If $d^1 < e_1$, then $d^1 = \Delta_{12}$ and $\{\underline{x}\,|\,x_2+x_3 < \Delta_{12}\} \cap K = \emptyset$. So if $\underline{a}\epsilon K \cap A(c_i)$ ($i = 2$ or 3), then $a_2+a_3 \geq \Delta_{12}$ and $a_1 \leq c_i - \Delta_{12}$. Now, we can show that $B^2 \subseteq K \cup \underset{\{2,3\}}{\text{dom}}\, K$. Because $B^2 \cap \underset{\{1,i\}}{\text{dom}}\, K = \emptyset$, for if not, then there exists $\underline{a}\epsilon A(c_j) - \{\underline{x}\,|\,x_2+x_3 < \Delta_{12}\}$ and $\underline{b}\epsilon B^2$ such that $\underline{a} \underset{\{1,i\}}{\text{dom}} \underline{b}$, which implies $a_2+a_3 \geq \Delta_{12}$ and $b_1 < a_1 = c_j - a_2 - a_3 \leq c_j - \Delta_{12}$, and so $b_2+b_3 = c_1-b_1 > c_1-c_j +\Delta_{12} = \Delta_{1j}+\Delta_{12} \geq 2\Delta_{12}$, which contradicts $\underline{b}\epsilon B^2$. Therefore, $B^2 \subseteq K \cup \underset{\{2,3\}}{\text{dom}}\, K$.

Next, it is then clear that $\{\underline{x} | x_2+x_3 < d^2\} \subseteq \underset{\{2,3\}}{\text{dom}} \{\underline{x}\epsilon A(c_1) | x_2+x_3 = d^2\} \subseteq$

$\underset{\{2,3\}}{\text{dom}} B^2 \subseteq \underset{\{2,3\}}{\text{dom}} (K \cup \underset{\{2,3\}}{\text{dom}} K) = \underset{\{2,3\}}{\text{dom}} K$. So $\{\underline{x} | x_2+x_3 < d^2\} \cap K = \emptyset$.

If $d^2 = e_1$, then the proof is complete.

If $d^2 < e_1$, then we continue as above to get $B^m \subseteq K \cup \underset{\{2,3\}}{\text{dom}} K$ for $m = 3,4,\ldots,m_0$ where finally $d^{m_0} = e_1$. Then $\{\underline{x} | x_2+x_3 < d^{m_0} = e_1\} \subseteq$ $\underset{\{2,3\}}{\text{dom}} \{\underline{x}\epsilon A(c_1) | x_2+x_3 = e_1\} \subseteq \underset{\{2,3\}}{\text{dom}} B^{m_0} \subseteq \underset{\{2,3\}}{\text{dom}} (K \cup \underset{\{2,3\}}{\text{dom}} K) = \underset{\{2,3\}}{\text{dom}} K$. So $\{\underline{x} | x_2+x_3 < e_1\} \cap K = \emptyset$, and this completes the proof of the theorem.

The following two theorems can be proved in a manner similar to Theorem 5 and so their proofs will be omitted. By starting with a neighborhood of the vertex $(0,c_2,0)$ on $A(c_2)$ and staying where $\{2,3\}$ is ineffective, we can enlarge this neighborhood to get Theorem 6. And by starting with a neighborhood of the vertex $(0,0,c_3)$ on $A(c_3)$ and staying where $\{2,3\}$ and $\{1,3\}$ are ineffective, we can enlarge this neighborhood to get Theorem 7.

Theorem 6. If $c_2 > c_3$, $c_2 \geq c$ and K is any solution, then

$$K \cap \{\underline{x} | x_1+x_3 < e_2,\ x_1 < d_1 - \Delta_{12}\} = \emptyset$$

where $\Delta_{pq} = c_p - c_q$.

Theorem 7. If $c_3 \geq c$ and K is any solution, then

$$K \cap \{\underline{x} | x_1+x_2 < e_3,\ x_1 < d_1 - \Delta_{13},\ x_2 < d_2 - \Delta_{23}\} = \emptyset.$$

Theorem 8. If $c_1 \geq c_2 > c_3$ and K is any solution, then

$$\{\underline{x} \mid x_1 \geq d_1 - \Delta_{13} \text{ or } x_2 \geq d_2 - \Delta_{23}\} \cap \underset{\{1,2\}}{\text{dom}}\, K = \emptyset.$$

Proof. Let $D = \{\underline{x} \epsilon A(c_3) \mid x_1 > d_1 - \Delta_{13} \text{ or } x_2 > d_2 - \Delta_{23}\}$. We will first show that $D \cap K = \emptyset$. For let $\underline{a} \epsilon D$, so $a_i > d_i - \Delta_{i3}$ and $a_j + a_k < e_i$ for i = 1 or 2. Then define $\underline{b}$ by $b_j = a_j + \epsilon$, $b_k = a_k + \epsilon$, $b_i = a_i + \Delta_{i3} - 2\epsilon$ where $3\epsilon = \min\{e_i - a_j - a_k, \Delta_{i3}\} > 0$. And then $\underline{b} \underset{N}{>} \underline{a}$ and $\underline{b} \underset{\{j,k\}}{\text{dom}}\, \underline{a}$, because ϵ and $\Delta_{i3} - 2\epsilon > 0$, $b_j + b_k = a_j + a_k + 2\epsilon < e_i = v(\{j,k\})$, and $b_1 + b_2 + b_3 = a_1 + a_2 + a_3 + \Delta_{i3} = c_i$ so $\underline{b} \epsilon A(c_i)$. But $\underline{b} \epsilon K \cup \text{dom}\, K$. If $\underline{b} \epsilon K$, then $\underline{a} \epsilon \underset{\{j,k\}}{\text{dom}}\, \underline{b} \subseteq \text{dom}\, K$. If $\underline{b} \epsilon$ dom K, then clearly $\underline{a} \epsilon$ dom K (Lemma 1). In either case $\underline{a} \notin K$. Therefore, $D \cap K = \emptyset$. That is, if $\underline{a} \epsilon A(c_3) \cap K$, then $a_1 \leq d_1 - \Delta_{13}$ and $a_2 \leq d_2 - \Delta_{23}$. So if $\underline{b} \epsilon \underset{\{1,2\}}{\text{dom}}\, K$ then $b_1 < d_1 - \Delta_{13}$ and $b_2 < d_2 - \Delta_{23}$. This completes the proof.

The uniqueness of the solutions in Genera 1, 2A and 3A now follows immediately, because Theorems 5, 6 and 7 show that exactly those points on an $A(c_p)$ that are not in the described solutions cannot be in any solution. But extending a solution from $\bigcup_{p=1}^{3} A(c_p)$ to all of R, that is, adding appropriate parts on A(c) and A(g), does not destroy the uniqueness. Theorem 8 is used in Genera 2B and 3B to show that although some imputations in $\{\underline{x} \epsilon A(c_1) \mid x_2 + x_3 < e_1 \text{ and } x_1 + x_3 < e_2\}$ may have $x_1 + x_2 < e_3 = v(\{1,2\})$, there is still only domination via $\{2,3\}$ and $\{1,3\}$ to be considered in this region. Then the uniqueness of the solutions in Genera 2B, 3B and 3C, up to the K_i parts, also follows from the above theorems and the characterization of the K_i curves in [8]. It is also shown in [8] that the solutions discussed in Genus 3D are all possible ones. Consequently, Section 2 lists all possible solutions for all 3-person games.

CHAPTER 3

THE 4-PERSON GAMES WITH DISTINCT IMPUTATION SIMPLICES

1. INTRODUCTION

For the 4-person games, the set of players is $N = \{1,2,3,4\}$. In this chapter h,k,p,q will represent a permutation of $1,2,3,4$. There are 15 partitions $P = \{P_1,\ldots,P_r\}$ of N: $\{N\}$, $\{\{1\},\{2\},\{3\},\{4\}\}$, four of type $\{\{h\},\{k,p,q\}\}$, three of type $\{\{h,k\},\{p,q\}\}$ and six of type $\{\{h,k\}, \{p\},\{q\}\}$. The 15 outcome functions F_P are specified when the 37 real numbers $F_P(P_i)$ are given. For $M \subseteq N$ equation (1) then gives the value of the coalition M as $v(M) = \min\limits_{M \in P} F_P(M)$. Assuming the games in normal form gives $v(\{h\}) = v_h = 0$, and hence $F_P(\{h\}) \geq 0$ for all P containing $\{h\}$. The set R of all imputations then consists of up to 15 imputation simplices in the first orthant of four dimensional space.

Domination via one player coalitions $\{h\}$ is impossible by Lemma 2, and domination via N is discussed in Theorems 1, 2 and 3. If $\underline{a} \operatorname{dom}_M \underline{b}$ and $M = \{h,k\}$, then equations (4), (5), and (6) gives $a_h > b_h$ and $a_k > b_k$, $\sum\limits_{j \in M} a_j \leq v(M)$, and $\underline{a}$ is on an imputation simplex realized by the partition $\{\{h,k\}, \{p,q\}\}$ or $\{\{h,k\}, \{p\}, \{q\}\}$. For such an M, $\operatorname{dom}_M \underline{a}$ meets the imputation simplex containing $\underline{a}$ in an open region shown in broken lines in Figure 7. (The tetrahedra in the figures of this chapter picture the imputation simplices A(P) in barycentric coordinates.) The domination cone $\operatorname{dom}_M \underline{a}$ meets the other imputation simplices in a similar manner. If $\underline{a} \operatorname{dom}_M \underline{b}$ and $M = \{h,k,p\}$, then $a_h > b_h$, $a_k > b_k$, and $a_p > b_p$, $\sum_{j \in M} a_j \leq v(M)$, and $\underline{a}$ is on the imputation simplex $A(\{\{h,k,p\},\{q\}\})$. For such an M, Figure 8 shows $\operatorname{dom}_M \underline{a}$ intersected with the imputation simplex containing $\underline{a}$.

In this chapter, we will consider the 4-person games that have distinct simplices. This hypothesis means that for the different partitions $P \in \Pi$ the numbers

$$(10) \qquad |P| = \sum_{j=1}^{r} F_P(P_j)$$

are distinct. We will give a solution for each 4-person game that satisfies (10). The nature of this solution depends upon the manner in which the 15 numbers $|P|$ are ordered for the given P, the relative magnitudes of these numbers, and values v(M) of the 10 non trivial coalitions M of N. Thus, the general method of this chapter applies to a great number of 4-person games (over ten billion). The many solutions so obtained are a valuable source of examples to test additonal conjectures on solution theory. It will further be clear that our solution is

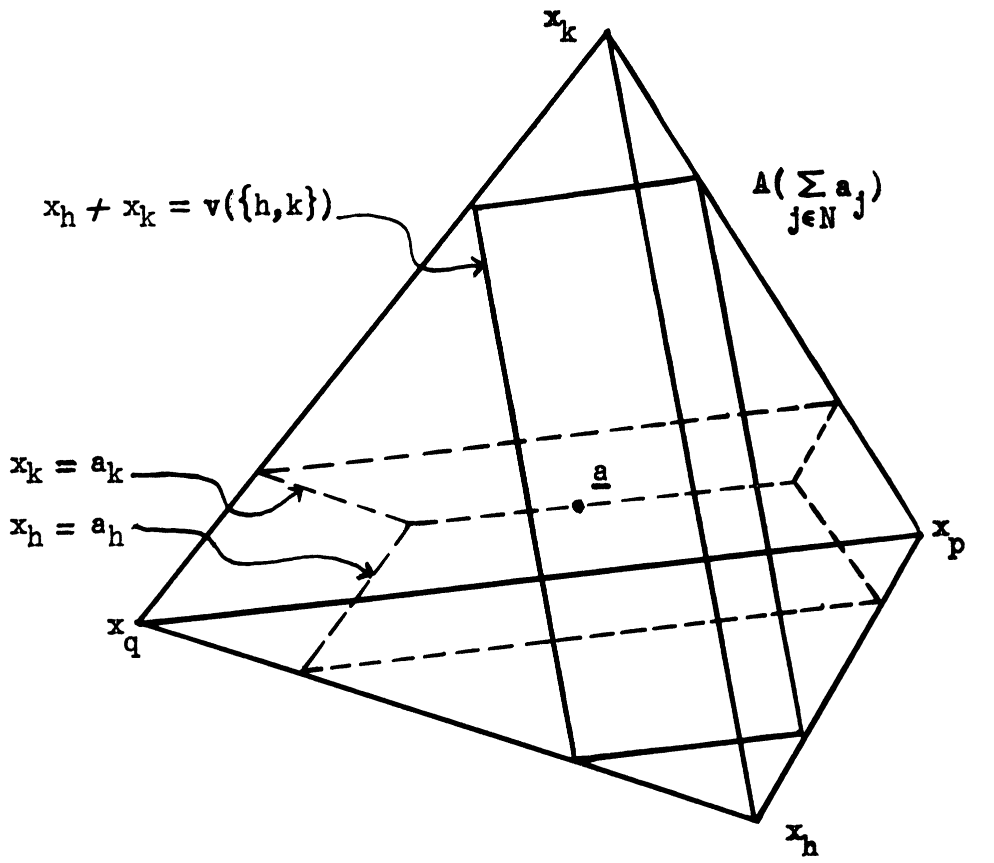

Figure 7. dom $\underline{a}$ via $\{h,k\}$.

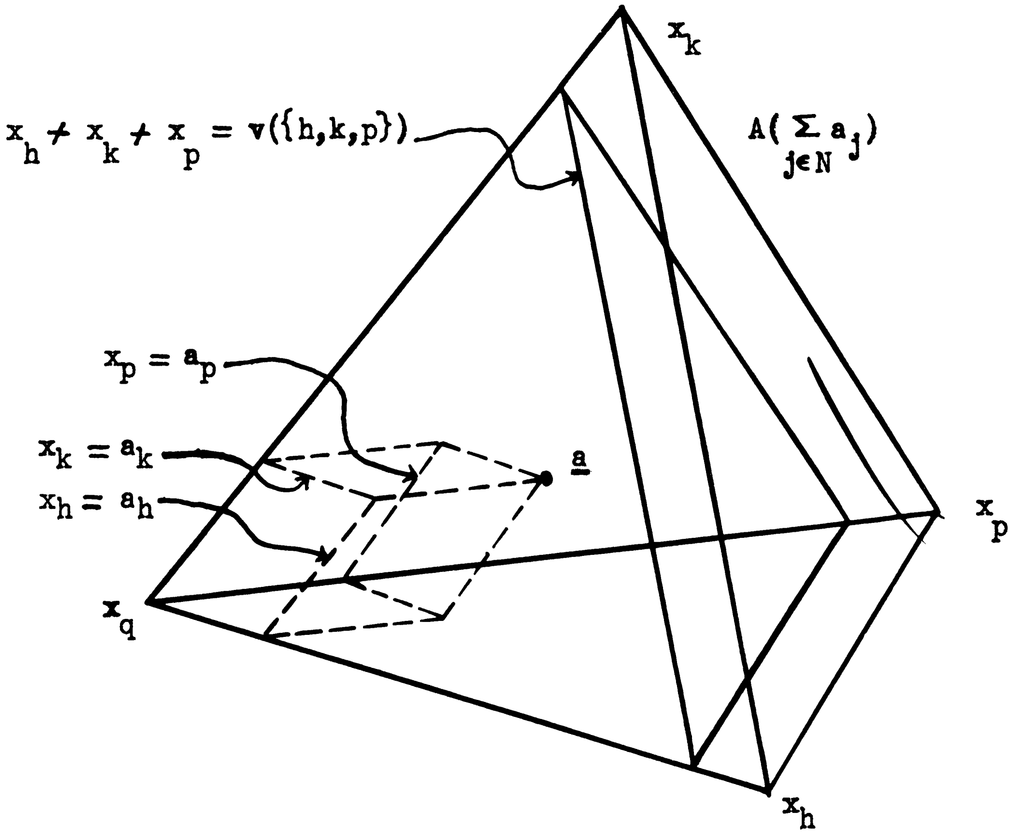

Figure 8. dom $\underline{a}$ via $\{h,k,p\}$.

also valid for many games in which some of the numbers given by (10) are not distinct. In addition, for those remaining games in which the imputation simplices are not all distinct, it appears that solutions can always be found by combining the methods of this chapter on the distinct imputation simplices with the known solutions for the von Neumann-Morgenstern 4-person games on the simplices realized by more than one partition. But no such general existence theorem has yet been proved for all 4-person Thrall games.

Let

$$P^i = (P^i_1, \ldots, P^i_{r_i})$$

be a partition in Π and we assume that the numbers $|P^i|$ are ordered by

$$|P^i| = \sum_{j=1}^{r_i} F_{P^i}(P_j) > \sum_{j=1}^{r_{i+1}} F_{P^{i+1}}(P_j) = |P^{i+1}|$$

for $i = 1,2,\ldots,14$. Then let

$$A^i = A(P^i).$$

Also define

$$E^i = \left\{ \underline{x} \;\middle|\; \sum_{j \in M} x_j \geq v(M) \text{ for all } M \in \bigcup_{j=1}^{i} P^j \right\}.$$

Clearly $E^i \cap A^j$ is a compact convex polyhedron.

To find a solution K for a 4-person game that satisfies condition

(10) we proceed as follows. First, $K^1 = E^1 \cap A^1$ is a solution for $A^1 \cup (R-E^1)$ where all domination is via coalitions $M \epsilon P^1$. Second, take the elements G^2 in $E^1 \cap A^2$ that are maxiaml with respect to dom_M for all $M \epsilon P^2$. Then $K^2 = G^2 \cup (K^1\text{-dom } G^2)$ is a solution for $A^1 \cup A^2 \cup (R-E^2)$. We continue this way. Take the elements G^i in $E^{i-1} \cap A^i$ that are maximal with respect to dom_M for all $M \epsilon P^i$. Then $G^i \cup K^{i-1}$ -dom G^i is <u>usually</u> a solution for $A^1 \cup \ldots \cup A^i \cup (R-E^i)$. There is only one case that may cause trouble in this process. If $M = \{h,q\} \epsilon P^i$, and $\{h,p\}$ and $\{h,k\}$ occurred in earlier partitions P^n and P^s where $s < n < i$, then $\operatorname{dom}_M G^i$ may dominate away too much from the previous solution K^{i-1}. Then $G^i \cup K^{i-1}$ - dom G^i may leave some imputations in $A^1 \cup \ldots \cup A^i \cup (R-E^i)$ undominated. But if we repeat the above process of taking successive maximal elements, but now only in a region $J(h)$ that contains the undominated imputations, then we obtain a solution $K_1^i(h)$ for $J(h)$ which can be added to $G^i \cup K^{i-1}$ - dom G^i to get the solution K^i for $A^1 \cup \ldots \cup A^i \cup (R-E^i)$. After this digression to obtain $K_1^i(h)$, we continue the above process until $R \subseteq A^1 \cup \ldots \cup A^f \cup (R-E^f)$ and then $K = K^f$ is the desired solution for the given game. We will discuss this solution K in more detail after we give some preliminary lemmas in the next section.

2. SOME LEMMAS

In order to start our induction process to find K, we must have the following lemma.

Lemma 3. $E^1 \cap A^1 \neq \emptyset$.

Proof. First, note that $R \neq \emptyset$, since if $P^g = \{\{1\},\{2\},\{3\},\{4\}\}$,

then $\sum_{j=1}^{4} F_{P^g}(\{j\}) \geq 0$, and thus $R \supseteq A(P^g) \neq \emptyset$. Thus $|P^1| \geq |P^g| > 0$

and $A^1 \neq \emptyset$. Next, consider the partition Q made up of the $P^1_j \in P^1$ with $v(P^1_j) \geq 0$ and all other cosets containing just one element. Using (10) and $F_Q(\{j\}) \geq 0$, we get
$$|P^1| = \sum_{P^1_j \in P^1} F_{P^1}(P^1_j) \geq \sum_{Q_j \in Q} F_Q(Q_j) \geq \sum_{P^1_j \in Q} F_Q(P^1_j) \geq$$
$$\sum_{P^1_j \in Q} \min_P F_P(P^1_j) = \sum_{P^1_j \in Q} v(P^1_j) = \sum_{\substack{P^1_j \in P^1 \\ v(P^1_j) \geq 0}} v(P^1_j).$$
So there exists $\underline{x} \in A^1$,

that is $\sum_{j \in N} x_j = |P^1|$ and $x_j \geq 0$, such that $\sum_{j \in P^1_j} x_j \geq v(P^1_j)$ for all $P^1_j \in P^1$.

That is, $E^1 \cap A^1 \neq \emptyset$.

If $\underline{a}$ and $\underline{b}$ are imputations, $N \supseteq M \neq \emptyset$, and $\underline{a}$ is M-realizable and M-effective, then $\underline{a} \operatorname{dom}_M \underline{b}$ is equivalent to $\underline{a} >_M \underline{b}$. The existence of the maxiamal elements G^i in our induction follows from Zorn's Lemma. The nature of these sets of maximal elements is given in the following lemma. For additional results of this type, see Stearns [5].

Lemma 4. If E is a compact convex polyhedron and G is the set of elements in E that are maximal with respect to $\underset{M}{>}$, then G is a union of closed faces of E.

Proof. Assume $\underline{c} \in G$. If $\underline{c}$ is a vertex of E, then $\underline{c}$ is a closed face. So let $\underline{c}$ be in a closed face H of E of dimension > 0. We can assume $\underline{c}$ is in the relative interior of H or otherwise, it is in a closed face of less dimension. Then we must show that $H \subseteq G$. Assume to the contrary that $\underline{a}$ is in H-G. Then there exists $\underline{b} \epsilon G$ such that $\underline{b} \underset{M}{>} \underline{a}$. By the convexity of H, the line segment $L(\underline{a},\underline{c}) \subseteq H$. Since $\underline{c}$ is interior to H, $L(\underline{a},\underline{c})$ can be extended slightly beyond $\underline{c}$ to $\underline{c}'$ such that $\underline{c} \epsilon L(\underline{a},\underline{c}') \subseteq H$. Then $\underline{c} = \lambda\underline{a}+(1-\lambda)\underline{c}'$ for $0 < \lambda < 1$. Since $\underline{b} \underset{M}{>} \underline{a}$ and $\underline{c}' \underset{M}{=} \underline{c}'$, $\underline{b}' = \lambda\underline{b} +(1-\lambda)\underline{c}' \underset{M}{>} \underline{c}$. But $\underline{b}' \epsilon E$ by the convexity of E. Thus $\underline{b}' \underset{M}{>} \underline{c}$ which contradicts $\underline{c} \epsilon G$. Therefore, $H \subseteq G$.

The two previous lemmas hold for n-person games with arbitrary n, whereas the following one holds only for $n \leq 4$.

Lemma 5. In a 4-person game which satisfies condition (10), any $\underline{a} \epsilon R$ can be strictly effective and realizable for at most one $M \subseteq N$.

Proof. Since the imputation simplices are distinct, $\underline{a}$ can be realized by only one partition P. But no imputation is strictly effective for a one element coalition. And the only partitions P with more than one M with more than one element are of the type $Q = \{\{h,k\}, \{p,q\}\}$. But $\underline{a} \epsilon A(Q)$ implies $\sum_{j \epsilon N} a_j = F_Q(\{h,k\})+F_Q(\{p,q\}) \geq v(\{h,k\})+v(\{p,q\})$. So if $a_h+a_k < v(\{h,k\})$, then $a_p+a_q > v(\{p,q\})$, that is, if $\underline{a}$ is $\{h,k\}$-strictly effective then $\underline{a}$ is $\{p,q\}$-ineffective.

3. SOLUTION

In this section, we give a detailed discussion of the solution K described in the first section of this chapter. First, we will prove that $K^1 = E^1 \cap A^1 = \{\underline{x} \epsilon A^1 \mid \sum_{j \epsilon M} a_j \geq v(M)$ for all $M \epsilon P^1\}$ is a solution for $A^1 \cup (R-E^1)$. If $P^1 = \{N\}$ then $E^1 \cap A^1 = A^1$ is a solution for all of R by Theorem 3. Thus assume $P^1 \neq \{N\}$, so $M \epsilon P^1$ implies $N-M \neq \emptyset$. To show that $K^1 \cup \text{dom}\, K^1 = A^1 \cup (R-E^1)$, let $\underline{a} \epsilon [A^1 \cup (R-E^1)]-K^1$. Then $\sum_{j \epsilon N} a_j \leq |P^1|$ and $\sum_{j \epsilon M} a_j < v(M)$ for some $M \epsilon P^1$. Let $\epsilon = v(M) - \sum_{j \epsilon M} a_j > 0$ and $\delta = |P^1| - \sum_{j \epsilon N} a_j \geq 0$. Also let $|M|$ = the number of elements in M. Then define $\underline{b}$ by

$$b_j = a_j + \epsilon/|M| \qquad \text{for } j \epsilon M$$

$$b_j = a_j - \delta_j \qquad \text{for } j \epsilon N-M$$

where $\sum_{j \epsilon N-M} \delta_j = \epsilon - \delta$ and $a_j \geq \delta_j$ for $j \epsilon N-M$. Such δ_j exist, because

$$\sum_{j \epsilon N-M} a_j = |P^1| - \sum_{j \epsilon M} a_j - \delta = |P^1| - v(M) + v(M) - \sum_{j \epsilon M} a_j - \delta = |P^1| - v(M) + \epsilon - \delta \geq \epsilon - \delta$$

since $|P^1| - v(M) \geq |Q| - v(M) \geq F_Q(M) - v(M) \geq 0$ where Q is the partition composed of M and one element sets. Then $\underline{b} \epsilon A^1$, $\sum_{j \epsilon M} b_j = v(M)$ and $\underline{b} \underset{M}{>} \underline{a}$, and thus $\underline{b} \underset{M}{\text{dom}} \underline{a}$. And $\underline{b} \epsilon K^1 = E^1 \cap A^1$ is clear from the proof of Lemma 5. (The separation property in Lemma 5 is only valid for $n \leq 4$, but with some extra restrictions on the δ_j above we could get $\underline{b} \epsilon E^1 \cap A^1$ and thus

K^1 would still be a solution for $A^1 \cup (R-E^1)$ for the case of arbitrary n.) This proves that $K^1 \cup \operatorname{dom} K^1 \supseteq A^1 \cup (R-E^1)$. Clearly $E^1 \cap \operatorname{dom} (E^1 \cap A^1) = \emptyset$, and so the inclusion is an equality. Finally, $K^1 \cap \operatorname{dom} K^1 = (E^1 \cap A^1) \cap \operatorname{dom}(E^1 \cap A^1) \subseteq E^1 \cap \operatorname{dom} A^1 = \emptyset$. This completes the proof that $K^1 = E^1 \cap A^1$ is a solution for $A^1 \cup (R-E^1)$. K^1 for $P^1 = \{\{h\},\{k,p,q\}\}$ and $\{\{h,k\},\{p,q\}\}$ is shown by heavy lines in Figures 9 and 10 respectively. Next, we can descend to A^2 and the part on A^2 that is not dominated by $K^1 = E^1 \cap A^1$ is $E^1 \cap A^2$, which is a compact convex polyhedron bounded by $x_j \geq 0$ and $\sum_{j \in M} x_j \geq v(M)$ for all $M \in P^1$.

Now, proceed to the induction step. Assume that K^{i-1} where $i \geq 2$ is a solution for $A^1 \cup \ldots \cup A^{i-1} \cup (R-E^{i-1})$ with $K^{i-1} \subseteq A^1 \cup \ldots \cup A^{i-1}$ and $K^{i-1} \cap A^j \subseteq E^{i-1} \cap A^j$. This condition holds for $i = 2$ since from above $K^1 = E^1 \cap A^1$ is a solution for $A^1 \cup (R-E^1)$ and it trivially has $K^1 \subseteq A^1$ and $K^1 \cap A^j \subseteq E^{j-1} \cap A^j$ (where let $E^0 = R$). We then need to find a K^i that is a solution on $A^1 \cup \ldots \cup A^i \cup (R-E^i)$ with $K^i \subseteq A^1 \cup \ldots \cup A^i$ and $K^i \cap A^j \subseteq E^{j-1} \cap A^j$.

Since imputations in E^{i-1} are not strictly effective for any $M \in \bigcup_{j=1}^{i-1} P^j$ and since $K^{i-1} \subseteq A^1 \cup \ldots \cup A^{i-1}$, $E^{i-1} \cap \operatorname{dom} K^{i-1} = \emptyset$ and $E^{i-1} \cap (A^i \cup \ldots \cup A^{15})$ is the part in R below A^{i-1} that remains undominated by K^{i-1}. On A^i this part is $E^{i-1} \cap A^i$. If $E^{i-1} \cap A^i = \emptyset$, then $A^i \subseteq R-E^{i-1}$ and so $K^i = K^{i-1}$ is a solution for $A^1 \cup \ldots \cup A^i \cup (R-E^i)$ with the desired properties. In fact K^{i-1} is then a solution for all of R. So assume that $E^{i-1} \cap A^i \neq \emptyset$. Clearly, $E^{i-1} \cap A^i$ is a compact convex polyhedron.

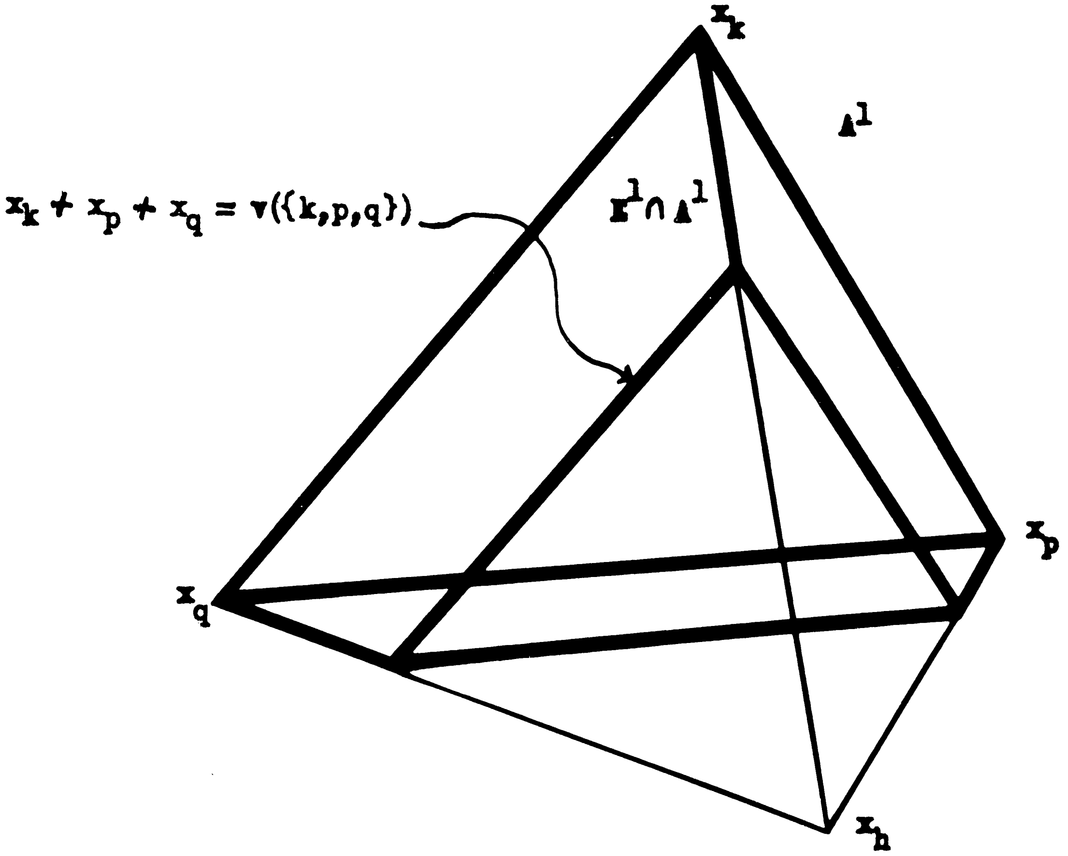

Figure 9. $E^1 \cap A^1$ for $P^1 = \{\{h\},\{k,p,q\}\}$.

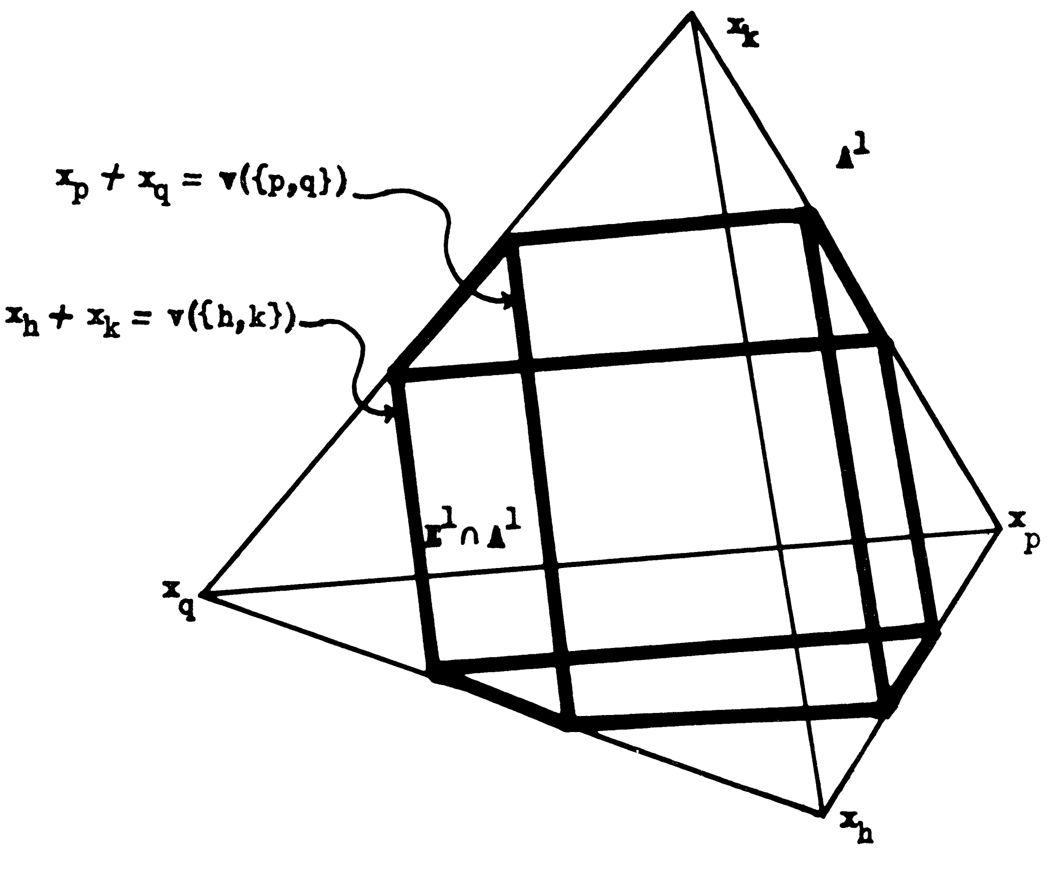

Figure 10. $E^1 \cap A^1$ for $P^1 = \{\{h,k\},\{p,q\}\}$.

For each $M\epsilon P^i$ let $G^i(M)$ be the elements in $E^{i-1}\cap A^i$ that are maximal with respect to $\underset{M}{\text{dom}}$. Then $G^i(M) \supseteq E^{i-1} \cap A^i \cap \{\underline{x} \mid \sum_{j\epsilon M} x_j \geq v(M)\}$.

But $E^{i-1} \cap A^i \cap \{\underline{x} \mid \sum_{j\epsilon M} x_j \leq v(M)\}$ is a compacet convex polyhedron of dimension ≤ 3 in which $\underset{M}{\text{dom}}$ is equivalent to $\underset{M}{>}$. So by Lemma 4, $G^i(M)$ meets this latter set in a union of closed faces that clearly are contained in $\{\underline{x}\epsilon A^i \mid \sum_{j\epsilon S} x_j = v(S)$ for some $S \epsilon \bigcup_{j=1}^{i} P^j\}$. Next, let $G^i = \bigcap_{M\epsilon P^i} G^i(M)$. Then $G^i \supseteq E^i \cap A^i$ and

$$G^i \cap \text{dom}\ G^i = \emptyset.$$

And using Lemma 5,

$$G^i \cup \text{dom}\ G^i \supseteq E^{i-1} \cap A^i.$$

Also

$$\text{dom}\ G^i \supseteq (A^{i+1} \cup \ldots \cup A^{15}) \cap (E^{i-1} - E^i).$$

Because if $\underline{a}$ is in this set, then $\sum_{j\epsilon N} a_j < |P^i|$, $\sum_{j\epsilon S} a_j \geq v(S)$ for all $S\epsilon \bigcup_{j=1}^{i-1} P^j$, and $\sum_{j\epsilon M} a_j < v(M)$ for some $M\epsilon P^i$. So pick $\underline{b}\epsilon A^i$ such that $\underline{b} \underset{N}{\geq} \underline{a}$ and $\sum_{j\epsilon M} b_j \leq v(M)$, and then $\sum_{j\epsilon S} b_j > v(S)$ for all $S\epsilon \bigcup_{j=1}^{i-1} P^j$ so $\underline{b}\epsilon E^{i-1}\cap A^i$.

Then $\underline{b}\ \underset{M}{\text{dom}}\ \underline{a}$ and $\underline{b}\epsilon G^i \cup \underset{M}{\text{dom}}\ G^i$ which implies $\underline{a}\epsilon$ dom G^i.

Finally, we define

$$(11) \quad K^i = (G^i \cup K^{i-1} - \text{dom } G^i) \cup K^i_1$$

where $K^i_1 = \emptyset$ unless we are in one of the four exceptional cases where $\{h,q\} \epsilon P^i$, and $\{h,p\}$ and $\{h,k\}$ occurred in partitions P^n and P^s respectively with $i > n > s \geq 1$, and G^i dominates away too much from K^{i-1}. (Since $G^i \cup (K^{i-1}\text{-dom } G^i) = (G^i \cup K^{i-1})\text{-dom } G^i$, no such parenthesis appear in (11).) These exceptional cases will be discussed in Section 5. When $K^i_1 = \emptyset$, it is clear that $K^i \subseteq A^1 \cup \ldots \cup A^i$ and $K^i \cap A^j \subseteq E^{j-1} \cap A^j$. So in this case when $K^i_1 = \emptyset$, it only remains to prove that K^i is a solution for $A^1 \cup \ldots \cup A^i \cup (R\text{-}E^i)$. But $[A^1 \cup \ldots \cup A^i \cup (R\text{-}E^i)] - [A^1 \cup \ldots \cup A^{i-1} \cup (R\text{-}E^{i-1})] = [E^{i-1} \cap A^i] \cup [(A^{i+1} \cup \ldots \cup A^{15}) \cap (E^{i-1}\text{-}E^i)]$, and from above $G^i \cup \text{dom } G^i$ contains this set. And dom $G^i \supseteq K^{i-1} \cap \text{dom } G^i$. So if $G^i \cup K^{i-1}$ -dom G^i dominates as much as K^{i-1} did, then K^i is a solution for $A^1 \cup \ldots \cup A^i \cup (R\text{-}E^i)$. By considering cases in the next section, we will show that this is true. So completion of the next two sections will finish the induction step of the proof.

4. CASES

We will now show, except for the exceptional case handled in the next section, that

$$(12) \quad \text{dom } K^{i-1} \subseteq \text{dom } (G^i \cup K^{i-1} - \text{dom } G^i).$$

Let $\underline{a} \epsilon K^{i-1}$ be S-effective and S-realizable where $S \epsilon \bigcup_{j=1}^{i-1} P^j$. Then

$\sum_{j\in S} a_j \leq v(S)$ and $\underline{a}\epsilon A^1 \cup \ldots \cup A^{i-1}$. Consider an arbitrary $\underline{b}\epsilon G^i$ which is M-effective for $M\epsilon P^i$. Then $\sum_{j\in M} b_j \leq v(M)$, $\left|P^i\right| = \sum_{j\in N} b_j < \sum_{j\in N} a_j$, and $\sum_{j\in T} b_j \geq v(T)$ for all $T\epsilon \bigcup_{j=1}^{i-1} P^j$. To prove (12), it is sufficient to show that either (i) $\underline{b}$ cannot dominate $\underline{a}$ via M, or (ii) if $\underline{b} \operatorname{dom}_M \underline{a}$ then $\operatorname{dom}_M \underline{b} \supseteq \operatorname{dom}_S \underline{a}$, or (iii) if $\underline{b} \operatorname{dom}_M \underline{a}$ then there exists $\underline{a}'\epsilon K^{i-1}$ such that no $\underline{b}'\epsilon G^i$ can dominate $\underline{a}'$ via M and $\operatorname{dom}_S \underline{a}' \supseteq \operatorname{dom}_S \underline{a}$. We proceed to consider all non trivial cases of M and S.

(i) $M = \{h,k,p\}$, $S = \{k,p,q\}$. Then $\sum_{j\in S} b_j \geq v(S) \geq \sum_{j\in S} a_j$, $\sum_{j\in N} b_j < \sum_{j\in N} a_j$, and so $a_h > b_h$. Thus $\underline{b} \operatorname{dom}_M \underline{a}$ is impossible.

(ii) $M = \{h,k,p\}$, $S = \{p,q\}$. Then $\sum_{j\in S} b_j \geq v(S) \geq \sum_{j\in S} a_j$ and $\sum_{j\in N} b_j < \sum_{j\in N} a_j$, which implies that $\sum_{j\in N-S} b_j < \sum_{j\in N-S} a_j$. Since $N-S\subseteq M$, $\underline{b} \operatorname{dom}_M \underline{a}$ is impossible.

(iii) $M = \{h,k\}$, $S = \{h,k,p\}$. If $\underline{b} \operatorname{dom}_M \underline{a}$ then $\operatorname{dom}_M \underline{b} \supseteq \operatorname{dom}_S \underline{a}$ by Lemma 1.

(iv) $M = \{h,k\}$, $S = \{k,p,q\}$. Then $\sum_{j\in S} b_j \geq v(S) \geq \sum_{j\in S} a_j$ and

$\sum_{j\in N} b_j < \sum_{j\in N} a_j$, which implies that $b_h < a_h$. Thus $\underline{b} \underset{M}{\text{dom}} \underline{a}$ is impossible.

(v) $M = \{h,k\}$, $S = \{p,q\}$. Then $\sum_{j\in S} b_j \geq v(S) \geq \sum_{j\in S} a_j$ and

$\sum_{j\in N} b_j < \sum_{j\in N} a_j$, which implies $\sum_{j\in M} b_j < \sum_{j\in M} a_j$ since $M = N\text{-}S$. Thus

$\underline{b} \underset{M}{\text{dom}} \underline{a}$ is impossible.

(vi) $M = \{h,k,p\}$, $S = \{h,k\}$. Assume $\underline{b} \underset{M}{\text{dom}} \underline{a}$ where $\underline{b}\epsilon G^i$ and

$\underline{a}\epsilon K^{i-1}$. Let $B = \{\underline{x}\epsilon G^i \mid \underline{x} \underset{M}{\text{dom}} \underline{a}\}$ and let $z = \text{lub}\ \{x_p \mid \underline{x}\epsilon B\}$. Define

$\underline{a}'\epsilon A(\sum_{j\in N} a_j)$ by $a_h' = a_h$, $a_k' = a_k$, $a_p' = z$, and $a_q' = a_q-(z-a_p) =$

$x_q+(x_h-a_h')+(x_k-a_k')-(z-x_p)+\Delta \geq 0$ for any $\underline{x}\epsilon B$ where $\Delta = \sum_{j\in N} a_j' - \sum_{j\in N} x_j$.

Clearly $\underset{S}{\text{dom}}\ \underline{a} = \underset{S}{\text{dom}}\ \underline{a}'$ and no $\underline{b}'\epsilon G^i$ can dominate $\underline{a}'$ via M. So if

$\underline{a}'\epsilon K^{i-1}$ then $\underline{a}'\epsilon K^i$. We then need to show that $\underline{a}'\epsilon K^{i-1}$. Pick $\underline{b}'\epsilon B$

such that $z-b_p' < \Delta$ and assume $T\epsilon \bigcup_{j=1}^{i-1} P^j$. If $q\epsilon T$, then $\sum_{j\in T} a_j' >$

$\sum_{j\in T} b_j' \geq v(T)$, and thus $\underline{a}'$ is T-ineffective. If $q\notin T$, then $\underline{a}' \underset{T}{\geq} \underline{a}$,

and so if $\underline{c} \underset{T}{\text{dom}} \underline{a}'$, then $\underline{c} \underset{T}{\text{dom}} \underline{a}$. So if $\underline{a}'\notin K^{i-1}$, then $\underline{a}\notin K^{i-1}$ which

is a contradiction. Therefore, $\underline{a}'\epsilon K^{i-1}$.

(vii a) $M = \{h,q\}$, $S = \{h,k\}$, and assume that $\{h,p\}\notin \bigcup_{j=1}^{i-1} P^j$.

Let $B = \{\underline{x}\epsilon G^i \mid \underline{x} \underset{M}{\text{dom}} \underline{a}\}$ and let $z = \text{lub}\ \{x_q \mid \underline{x}\epsilon B\}$. Define $\underline{a}'\epsilon A(\sum_{j\epsilon N} a_j)$

by $a_h' = a_h$, $a_k' = a_k$, $a_q' = z$, $a_p' = a_p-(a_q'-a_q)$. Clearly $\underset{S}{\text{dom}}\ \underline{a}' = \underset{S}{\text{dom}}\ \underline{a}$

and no $\underline{b}'\epsilon G^i$ can dominate $\underline{a}'$ via M. We need to show that $\underline{a}'\epsilon K^{i-1}$. That is, we need to show that $\underline{a}' \notin \underset{T}{\text{dom}}\ K^{i-1}$ for all $T\epsilon \bigcup_{j=1}^{i} P^j$. So we now consider all non trivial cases for various T. We assumed that $T \neq \{h,p\}$. If $T = \{p,q\}$, then $a_p'+a_q' = a_p+a_q \geq |P^s|-a_h-a_k \geq |P^s|-v(\{h,k\}) \geq v(\{p,q\})$ by Lemma 5 where $T\epsilon P^s$, and so $\underline{a}'$ is not $\{p,q\}$-strictly effective. If $T = \{k,p\}$, $\{k,p,q\}$ or $\{h,k,p\}$, then there exists $\underline{b}\epsilon B \subseteq G^i$ such that $\sum_{j\epsilon N-T} b_j+\epsilon > \sum_{j\epsilon N-T} a_j'$ where $\epsilon < \Delta = \sum_{j\epsilon N} a_j' - \sum_{j\epsilon N} b_j$, which implies $\sum_{j\epsilon T} a_j' > \sum_{j\epsilon T} b_j \geq v(T)$, and so $\underline{a}'$ is T-ineffective. If $T = \{k,q\}$ or $\{h,k,q\}$,

then $\underline{a}' \underset{T}{\geq} \underline{a}$ so $\underline{a}'\epsilon \underset{T}{\text{dom}}\ K^{i-1}$ implies $\underline{a}\epsilon \underset{T}{\text{dom}}\ K^{i-1}$, a contradiction. Finally, if $T = \{h,p,q\}$, then either $T\epsilon P^j$ for $j < s$ where $S\epsilon P^s$ and $\sum_{j\epsilon T} a_j'$

$= \sum_{j\epsilon T} a_j \geq v(T)$ so $\underline{a}'$ is not strictly effective for T, or $T\epsilon P^j$ for

$i > j > s$ and $b_h+b_k \geq v(\{h,k\}) \geq a_h+a_k$ so $b_p+b_q < a_p+a_q$ which says $\underline{b}$ cannot dominate $\underline{a}$ via T. This proves that $\underline{a}' \notin \underset{T}{\text{dom}}\ K^{i-1}$ for all T when $\{h,p\} \notin \bigcup_{j=1}^{i-1} P^j$. This completes case (vii a).

(vii b) $M = \{h,q\}$, $S = \{h,k\}$, and assume that $\{h,p\}\epsilon \bigcup_{j=1}^{i-1} P^j$. In this case, K_1^i may not be $\emptyset$. So this case is treated separately in the next section.

Therefore, except for this exceptional case (vii b) which will be treated in the next section, we have verified equation (12) and thus completed our induction step.

5. THE EXCEPTIONAL CASES

In this section, we consider the more difficult case (vii b), that is, we assume $M = \{h,q\} \epsilon P^i$, $S = \{h,k\} \epsilon P^s$ and $\{h,p\} \epsilon P^n$ where $s < n < i$. As shown in (vii a) above, $\underset{\{h,p\}}{\text{dom}}\, G^n$ cannot contain the $\underline{x} \epsilon G^s - \bigcup_{j=s+1}^{n-1} \text{dom}\, G^j$ with large values of x_p (small values of x_q), and the dominion via $\{h,k\}$ of all such $\underline{x}$ equals $\underset{\{h,k\}}{\text{dom}}\, G^s$. But $\underset{\{h,q\}}{\text{dom}}\, G^i$ may contain some such $\underline{x}$ but not the corresponding $\underset{\{h,k\}}{\text{dom}}\, \underline{x}$. So some imputations that were previously in $\underset{\{h,k\}}{\text{dom}}\, K^{i-1}$ or $\underset{\{h,p\}}{\text{dom}}\, K^{i-1}$ may now be left undominated by $G^i \cup K^{i-1}$-dom G^i. The result may be that

$$\text{dom}\, K^{i-1} \not\subseteq \text{dom}\,(G^i \cup K^{i-1} - \text{dom}\, G^i)$$

in which case we must find the $K_1^i \neq \emptyset$ in equation (11). This situation for a given h is shown in Figures 11, 12, and 13, which picture A^s, A^n, and A^i respectively. In these figures, the heavy lines show the region that contains (but is not necessarily equal to) $G^i \cup K^{i-1}$-dom G^i, the light broken lines show the region J(h) that contains the undominated imputations, and the heavy broken lines show the additional part $K_1^i(h)$

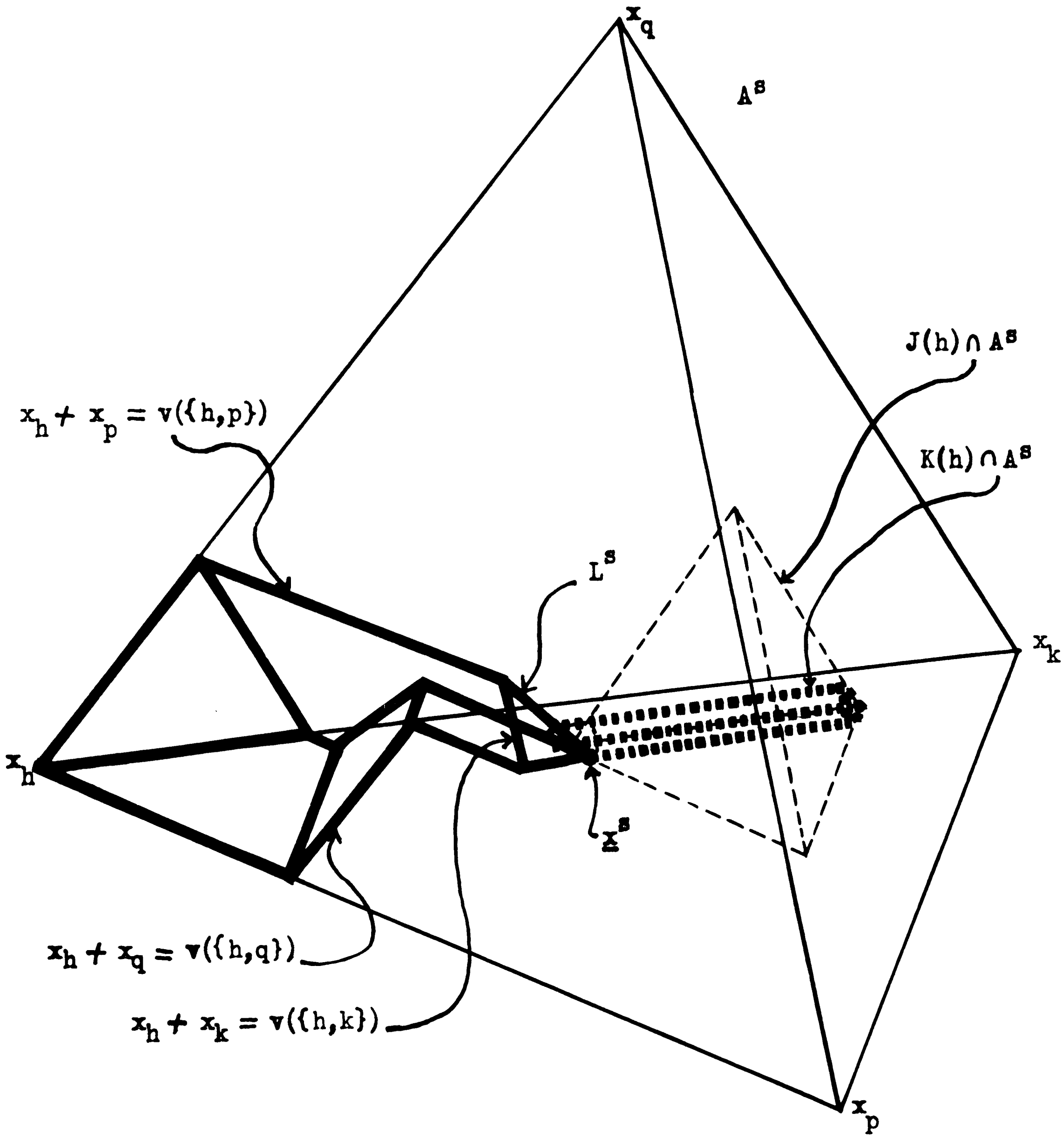

Figure 11. Exceptional Case on A^s.

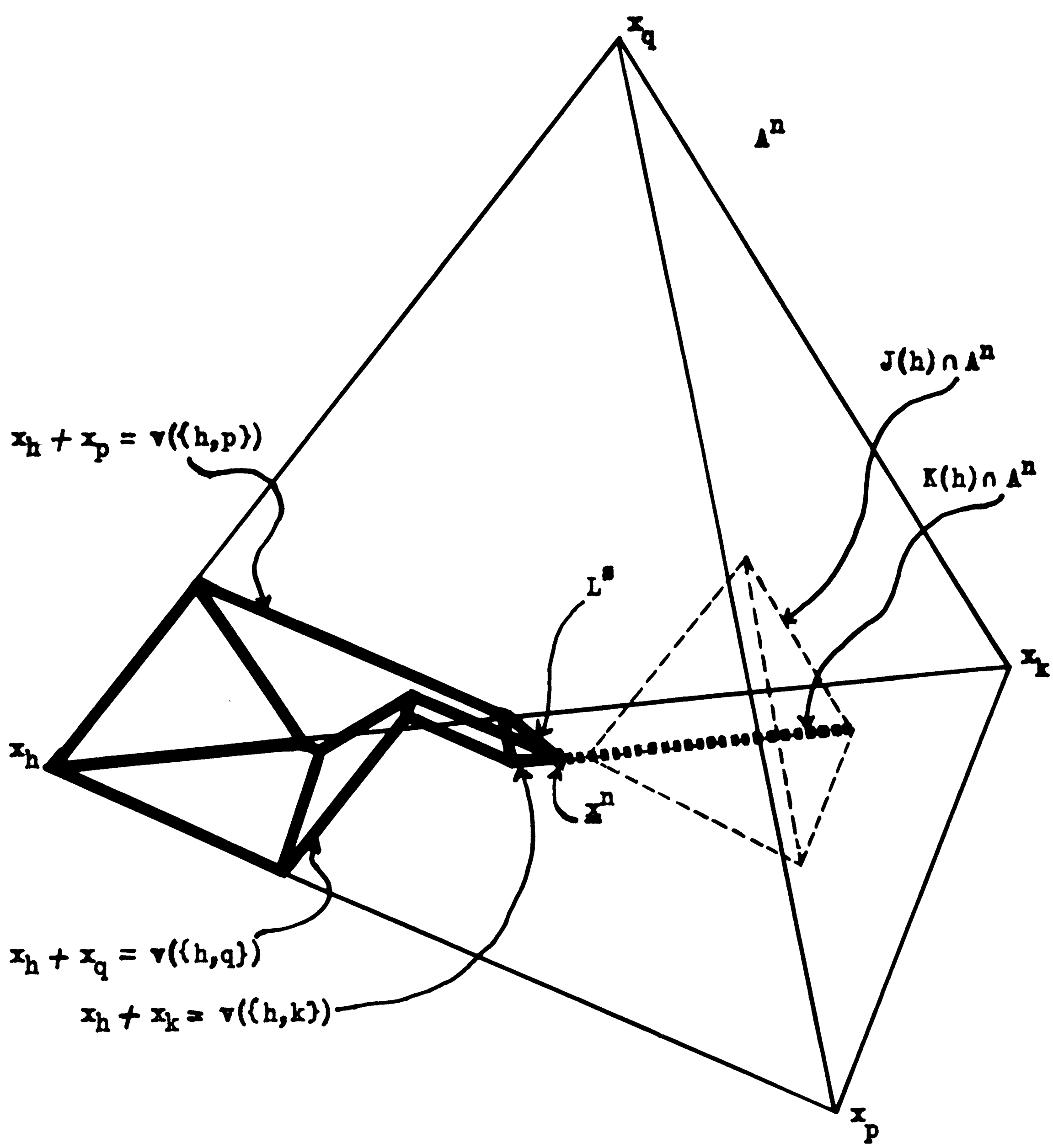

Figure 12. Exceptional Case on A^n.

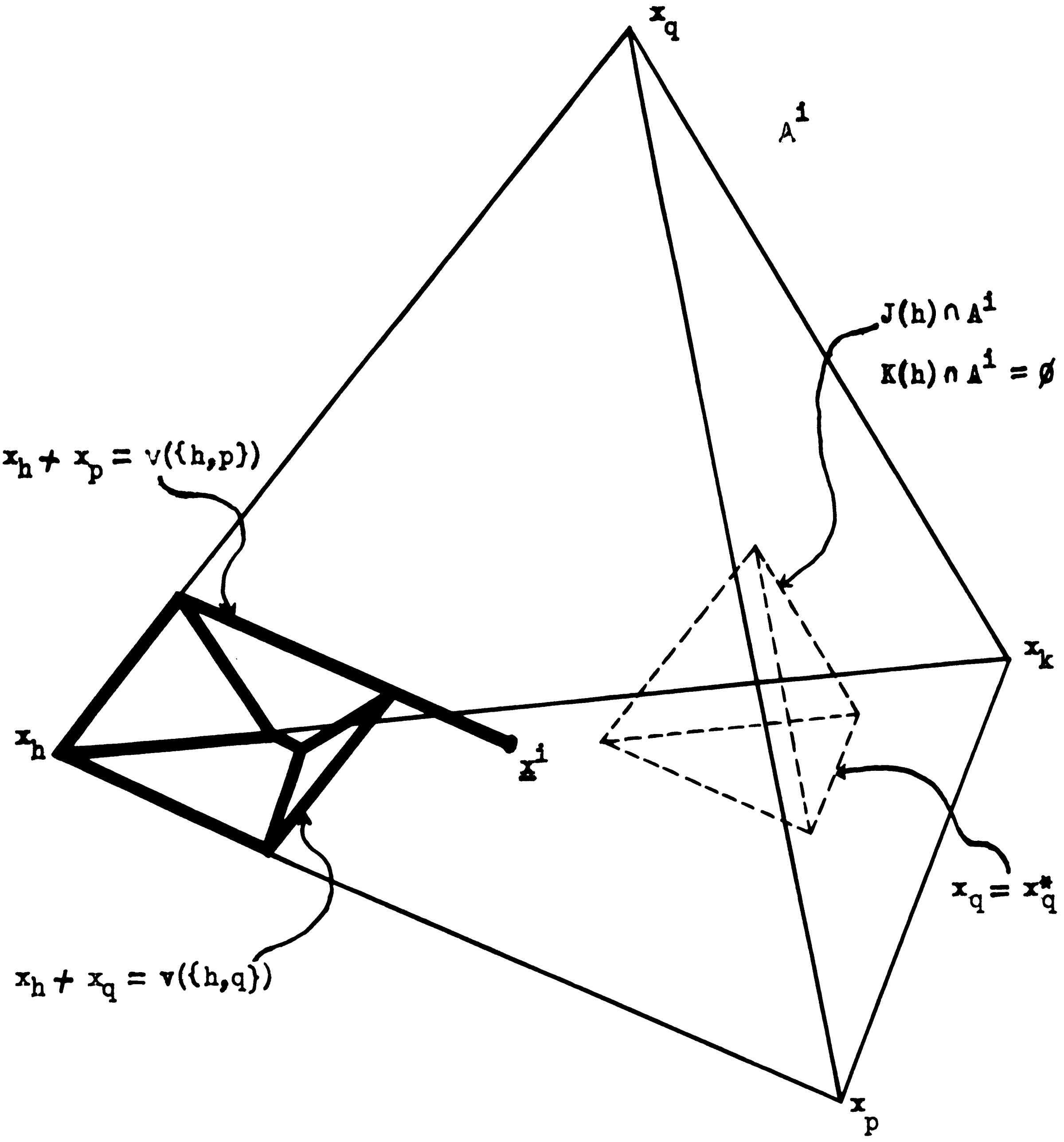

Figure 13. Exceptional Case on A^i.

of the solution for the given h. J(h) and $K_1^i(h)$ will meet the other simplices $A^j (j \neq s,n,$ nor $i)$ in similar regions. Then the union of all such $K_1^i(h)$ that occur at the ith step of our induction gives the K_1^i in equation (11).

In order to describe J(h) analytically, we introduce the following notation. On each A^j there is the line segment

$$L^j = \left\{ \underline{x} \epsilon A^j \,|\, x_k + x_h = v(\{k,h\}), x_p + x_h = v(\{p,h\}) \right\}.$$

And let

$$x_q^* = \max\left\{ x_q \,|\, \underline{x} \epsilon G^i \text{ and } x_q + x_h \leq v(\{q,h\}) \right\}.$$

On each A^j consider the point where $x_q = x_q^*$ meets L^j. These points have coordinates

$$x_p^j = |P^j| - x_q^* - v(\{k,h\})$$

$$x_q^j = x_q^*$$

$$x_k^j = |P^j| - x_q^* - v(\{p,h\})$$

$$x_h^j = x_q^* + v(\{p,h\}) + v(\{k,h\}) - |P^j|.$$

Note that $x_q^* \leq |P^i| - v(\{p,h\})$, that is where $x_p + x_h = v(\{p,h\})$ meets $x_k = 0$ on A^i. Assume x_q^* exists, because $E^{i-1} \cap A^i \cap \left\{ \underline{x} \,|\, x_q + x_h \leq v(\{q,h\}) \right\}$ is closed and if it is empty, then dom $G^i_{\{q,h\}} = \emptyset$ and so $J(h) = \emptyset$. Also assume $x_q^* > 0$ for if not, then dom $G^i_{\{q,h\}} = \emptyset$ and we are again back in the case (vii a) of the last section. And we can assume that the points $\underline{x}^j$ are on A^j. For if $x_q = x_q^*$ does not meet L^i on A^i, then either $v(\{k,h\})$ or $v(\{p,h\})$ is negative and the corresponding domination does not enter and so we are in case (vii a), or $v(\{k,h\}) > |P^s|$ or $v(\{p,h\}) > |P^n|$ which is impossible for $v(\{j,h\}) \leq F_P(\{k,h\})+F_P(\{p\})+ F_P(\{q\}) \leq |P^s|$ where $P = \{\{p\}, \{q\}, \{k,h\}\}$, or x_q^* is too small for $\operatorname{dom}_{\{q,h\}} G^i$ to contain the $\{k,h\}$-effective imputations in $G^s - \bigcup_{j=s+1}^{n-1} \operatorname{dom} G^j$ which have $x_p+x_h \geq v(\{p,h\})$ and large values of x_q and again $J(h) = \emptyset$ as in (vii a) above.

The undominated imputations for the case we are considering are contained in the region $J(h)$ which is bounded as follows. If $\underline{x} \epsilon J(h)$, then

$$
\begin{aligned}
&x_p \geq x_p^n \\
&x_q \geq x_q^* = x_q^j \text{ for all } j \\
&x_k \geq x_k^n, \text{ and} \\
&x_k \geq x_k^s \text{ when } x_k + x_h < v(\{k,h\}).
\end{aligned} \tag{13}
$$

Because if $a_p < x_p^n$ then let $S = \{p,h\}$ and $M = \{q,h\}$ in the argument in (vii a) and then the coalition $\{k,h\}$ causes no trouble in finding the $\underline{a}'$ in (vii a). See the Figures. And, if $a_k < x_k^n$ and $a_k+a_h \geq v(\{k,h\})$ then let $S = \{p,h\}$ and $M = \{q,h\}$ in (vii a) and then the coalition $\{k,h\}$ causes no trouble in finding the $\underline{a}'$ in (vii a). Also, if $a_k < x_k^s(> x_k^n)$

and $a_k+a_h < v(\{k,h\})$, then the coalition $\{q,h\}$ causes no trouble in finding $\underline{a}'$ in (vii a). To show $x_q \geq x_q^*$ assume that $x_p+x_h < v(\{p,h\})$ or $x_k+x_h < (\{k,h\})$ for if not then $\underline{x} \notin \mathrm{dom}_{\{k,h\}} G^s \cup \mathrm{dom}_{\{p,h\}} G^n$ so that $\underline{x}$ is not in the undominated area mentioned in the first paragraph of this section. So if $x_j+x_h < v(\{j,h\})$ for $j = p$ or k and $x_q < x_q^* = x_q^j$, then $x_h < v(\{j,h\})-x_j \leq v(\{j,h\})-x_j^m \leq v(\{p,h\})-x_p^i = x_h^i$ (where $m=n$ if $j = p$ and $m = s$ if $j = k$), and so $\underline{x} \in \mathrm{dom}_{\{q,h\}} \underline{x}^i \subseteq \mathrm{dom}_{\{q,h\}} G^i$. Thus $\underline{x}$ is not an undominated imputation.

Before describing the additional part $K_1^i(h)$ that needs to be added in $J(h)$ in order to have a solution in this part, we will discuss which coalitions M satisfy the conditions

(14)
M is in a partition P with $|P| > |P^i|$, and
M is effective in some part of $J(h)$.

For the other coalitions will not have to be considered in finding $K_1^i(h)$. Note that if $E^{i-1} \cap A^i = \emptyset$, then we already have a solution for all of R before the ith step in the induction, so assume $E^{i-1} \cap A^i \neq \emptyset$. Clearly, $N \notin P$ if $|P| > |P^i|$ and $E^{i-1} \cap A^i \neq \emptyset$. Also, if $\{p,q\}$ or $\{q,k\}$ are in a P with $|P| > |P^i|$ and $E^{i-1} \cap A^i \neq \emptyset$, then $v(\{p,q\})+v(\{k,h\}) \leq |P^i|$ or $v(\{q,k\})+v(\{p,h\}) \leq |P^i|$. (See Lemma 5.) If $\underline{x} \in J(h)$, then $x_p+x_q \geq x_p^n+x_q^n = |P^n|-x_k^n-x_h^n = |P^n|-v(\{k,h\}) = |P^i| - \Delta_{ni}-v(\{k,h\}) > |P^i| -v(\{k,h\}) \geq v(\{p,q\})$ where $\Delta_{ni} = |P^n|-|P^i|$, and so $\underline{x}$ is $\{p,q\}$- ineffective. If $\underline{x} \in J(h)$, then $x_q+x_k \geq x_q^n+x_k^n = |P^n|-x_p^n-x_h^n = |P^n|-v(\{p,h\}) >$

$|P^i|-v(\{p,h\}) \geq v(\{q,k\})$, and so $\underline{x}$ is $\{q,k\}$-ineffective. Next, note that if $\{p,k\}\epsilon P$ with $|P| > |P^i|$, then $\underline{a}\epsilon\ E^{i-1}\cap A^i$ has $\sum_{j\epsilon N} a_j = |P^i|$

and $a_p+a_k \geq v(\{p,k\})$. Then $\underline{x}^i$ has $x_p^i+x_k^i > v(\{p,k\})$. So if $\underline{x}\epsilon J(h)$, we get $x_p+x_k \geq x_p^n+x_k^n > x_p^i+x_k^i \geq v(\{p,k\})$, and so $\underline{x}$ is $\{p,k\}$-ineffective. Also, if $\{p,q,k\}\epsilon P$ with $|P| > |P^i|$, then $\underline{a}\epsilon E^{i-1}\cap A^i$ has $\sum_{j\epsilon N} a_j = |P^i|$

and $a_p+a_q+a_k \geq v(\{p,q,k\})$. Then $\underline{x}^i$ has $x_p^i+x_q^i+x_k^i \geq v(\{p,q,k\})$. So if $\underline{x}\epsilon J(h)$, then $x_p+x_q+x_k \geq x_p^n+x_q^n+x_k^n > x_p^i+x_q^i+x_k^i \geq v(\{p,q,k\})$, and so $\underline{x}$ is $\{p,q,k\}$-ineffective. In summary, if N, $\{p,q\}$, $\{q,k\}$, $\{p,k\}$ or $\{p,q,k\}$ are in a P with $|P| > |P^i|$, then they are ineffective in J(h). The $K_1^i(h)$ that we will describe in the next paragraph will have $K_1^i(h)\cap A^j = \emptyset$ for all $j > n$. So in determining $K_1^i(h)$, only domination via $\{k,h\}$, $\{p,h\}$, $\{q,k,h\}$, $\{p,k,h\}$ and $\{p,q,h\}$ need be considered.

We now describe how to get the additional part $K_1^i(h)$ of the solution in J(h). We proceed to take maximal elements in successive $A^j\cap J(h)$ as we did before in the A^j where $j = 1,2,\ldots$. The verification then of the following assertions is similar to the corresponding results done in Sections 3 and 4. First, $K^1(h) = G^1(h) = E^1\cap A^1\cap J(h)$ is a solution on $(J(h)\cap A^1)\cup(J(h)-E^1)$. Second, let $G^2(h)$ be the elements in $E^1\cap A^2\cap J(h)$ that are maximal with respect to $\underset{M^2}{\text{dom}}$ where M^2 is the coalition in P^2 that can be effective in J(h). There is only one such M^2 by the preceeding paragraph. Then $K^2(h) = G^2(h)\cup K^1(h)\text{-dom }G^2(h)$ is a solution for $[J(h)\cap(A^1\cup A^2)]\cup[J(h)-E^2]$. Continuing, let $G^j(h)$ be the elements in $E^{j-1} \cap A^j \cap J(h)$ that are maximal with respect to $\underset{M^j}{\text{dom}}$ where $M^j \epsilon P^j$ may be effective in $J(h)$. Then $K^j(h) = G^j(h) \cup K^{j-1}(h)\text{-dom } G^j(h)$ is a solution for $[J(h) \cap (A^1 \cup \ldots \cup A^j)] \cup [J(h)-E^j]$. Pick the first t such that $J(h) \subseteq (A^1 \cup \ldots \cup A^t) \cup (J(h)-E^t)$. Then $K_1^i(h) = K^t(h)$ is a solution in $J(h)$. The above process terminates for some $t \leq n$. Because if $\underline{x} \epsilon A^j \cap J(h)$ for $j > n$, then $\sum_{u \epsilon N} x_u = |P^j| < |P^n|$ and $x_p + x_q \geq x_p^n + x_q^n$, and

so $x_k + x_h \leq |P^j| - x_p^n - x_q^n < |P^n| - x_p^n - x_q^n = v(\{k,h\})$. But such $\underline{x}$ are in dom $K^w(h)$ for some $w \leq s$. Since $t \leq n < i$, the exceptional case (vii b) that we are discussing in this section will never occur in finding $K_1^i(h)$. The

heavy broken lines in Figures 11, 12 and 13 show $K_1^i(h)$ on A^s, A^n and A^i for the case where domination via $\{q,k,h\}$, $\{p,k,h\}$ and $\{p,q,h\}$ do not enter in determining $K_1^i(h)$, for example, if they are all in a P with $|P| < |P^n|$.

The K_1^i in equation (11) is the union of all such $K_1^i(h)$ that may come in at the ith step of our induction, that is,

$$K_1^i = \bigcup_h K_1^i(h)$$

where $\{h,q\} \epsilon P^i$, $\{h,k\} \epsilon P^s$ and $\{h,p\} \epsilon P^n$ for $s < n < i$. Since $G^i \cup K^{i-1}$ -dom G^i was the desired solution for all but part of the $J(h)$ regions and the $K_1^i(h)$ are solutions in these regions, it follows that

$$K^i \cup \text{dom } K^i = R.$$

To prove that

$$K^i \cap \operatorname{dom} K^i = \emptyset$$

it is sufficient to prove that

(15) $K_1^i(h) \cap \operatorname{dom} (G^i \cup K^{i-1} - \operatorname{dom} G^i) = \emptyset,$

(16) $(G^i \cup K^{i-1} - \operatorname{dom} G^i) \cap \operatorname{dom} K_1^i(h) = \emptyset$

and

(17) $K_1^i(h) \cap \operatorname{dom} K_1^{i'}(h') = \emptyset$

for the four possible non empty J(h) and J(h') regions. We can prove these equations by again considering several cases described in the following.

First, to prove (15), we need to show that

(15') $K_1^i(h) \cap \operatorname{dom}_M (G^i \cup K^{i-1} - \operatorname{dom} G^i) = \emptyset$

for all M in a P with $|P| \geq |P^i|$. From (14) above if M = N, {q,k}, {p,k}, {p,q} or {p,q,k} and MϵP with $|P| > |P^i|$ then M is ineffective in J(h) so (15') is true for such M. From the definition of x_q^*, (15') holds for M = {q,h}. (15') holds for M = {p,h} and {q,h} because J(h) was determined by the elements not in $\operatorname{dom}_M (G^i \cup K^{i-1} - \operatorname{dom} G^i)$ for M = {p,h} and {q,h}. If M = {p,k,h} ϵP^t for some t < i, then $\underline{x} \in E^{i-1} \cap A^i$ has $x_p + x_k + x_h \geq v(\{p,k,h\})$ and so $x_q^* \leq |P^i| - v(\{p,k,h\}) < |P^t| - v(\{p,k,h\}),$

and so if $\underline{a}$ is M-effective and realizable, then $a_q > x_q^*$. In this case, the set $G^t(h)$ of maximal elements in $E^{t-1} \cap A^t \cap J(h)$ with respect to $\mathrm{dom}_{\{p,k,h\}}$ is contained in the previously obtained set G^t of maximal elements in $E^{t-1} \cap A^t$ and $\mathrm{dom}\ G^t \cap (K^t \cup [E^t \cap (A^t \cup \ldots \cup A^{15})]) = \emptyset$. Since K^{i-1} -dom $G^i \subseteq K^t \cup [E^t \cap (A^t \cup \ldots \cup A^{15})]$ and $\mathrm{dom}_{\{p,k,h\}}\ K_1^i(h) = \mathrm{dom}_{\{p,k,h\}}\ G^t(h)$, we have (15') for $M = \{p,k,h\}$. If $M = \{p,q,h\} \in P^j$ and $j < i$, then consider the two cases $j < n$ and $j > n$. If $j < n$ define the points $\underline{z}^j \in A^j$ which satisfy the equations $x_k + x_h = v(\{k,h\})$, $x_p + x_h = v(\{p,h\})$, $x_p + x_q + x_h = v(\{p,q,h\})$ and $\sum_{u \in N} x_u = |P^j|$. Then it is easy to show that, if $z_q^n = x_q^*$, then we are in a case similar to (vii a), and if $z_q^n \leq x_q^*$, we are in a case similar to $M = \{p,k,h\}$ that is done just above. In the case $j > n$, if $\underline{x} \in G^j$ and $\underline{x}$ is M-effective, then $x_p + x_h \geq v(\{p,h\})$ or $x_q + x_k \leq |P^j| - v(\{p,h\})$, and if $\underline{a} \in J(h)$, then $a_q + a_k \geq x_q^n + x_k^n \geq |P^n| - v(\{p,h\}) > |P^j| - v(\{p,h\})$, and so $\underline{x}\ \mathrm{dom}_M\ \underline{a}$ is impossible. This proves (15') for $M = \{p,q,h\}$. Replace p by k in this last argument and we get (15') for $M = \{q,k,h\}$. This proves equation (15).

Next, to prove (16), we need to show

$$(16') \quad (G^i \cup K^{i-1} \text{- dom } G^i) \cap \mathrm{dom}_M\ K_1^i(h) = \emptyset$$

for all $M \in P^j$ for $j \leq n$. By the paragraph containing equation (14), we then have $\mathrm{dom}_M\ K_1^i(h) = \emptyset$ for $M = N$, $\{p,q\}$, $\{p,k\}$, $\{q,k\}$ and $\{p,q,k\}$ since such M are ineffective in J(h). Also, $\underset{\{q,h\}}{\text{dom}}\ K_1^i(h) = \emptyset$ for $\{q,h\}$ is first realized on A^i and $i > n$. The remaining M that have to be considered all contain h and have their domination cones away from the x_h vertex and toward $x_h = 0$. But it is easy to see that in taking the successive maximal elements $G^t(h)$ $(t \leq n)$ in $E^{t-1} \cap A^t \cap J(h)$ in the process of obtaining $K_1^i(h)$ that $G^t(h)$ is contained in the previous maximal elements G^t in $E^{t-1} \cap A^t$ which were obtained in the process leading to $G^i \cup K^{i-1}\text{-dom } G^i$, with the only possible exception being the last non empty $G^u(h)$ $(u \leq n)$. Since $G^i \cup K^{i-1}\text{-dom } G^i \subseteq K^t \cup [E^t \cap (A^{t+1} \cup \ldots \cup A^{15})]$, $\text{dom } G^t(h) \subseteq \text{dom } G^t$, $\underset{M^t}{\text{dom}}\ K_1^i(h) = \underset{M^t}{\text{dom}}\ G^t(h)$ and $\text{dom } G^t \cap (K^t \cup [E^t \cap (A^{t+1} \cup \ldots \cup A^{15})]) = \emptyset$, we get $\underset{M}{\text{dom}}\ G^t(h) \cap (G^i \cup K^{i-1}\text{- dom } G^i) = \emptyset$ for all $G^t(h)$ but the exceptional case $G^u(h)$. By considering cases of all the remaining M, we can show that the elements in $G^u(h)$ are either in G^u or are "less maximal" than those in G^u and so also $\underset{M^u}{\text{dom}}\ G^u(h) \cap (G^i \cup K^{i-1}\text{-dom } G^i) = \emptyset$. This proves (16).

Finally, to prove (17), we need

$$(17') \quad K_1^i(h) \cap \underset{M}{\text{dom}}\ K_1^{i'}(h') = \emptyset$$

for all M. But the boundaries of the regions J(h) or J(h') are determined by equations like (13) which are related by the definitions of the $\underline{x}^j$ and L^j to the v(M) for certain two element coalitions M containing h or h'. On the A^j with $A^j \cap E^{j-1} \neq \emptyset$, that is, on the A^j with $A^j \cap K^j \neq \emptyset$, Lemma 5 separates the regions on $A^1 \cup \ldots \cup A^j$ on which complimentary two element coalitions can be effective, and using equation (13) thus separates the regions $J(h)$ and $J(h')$. And by considering cases of the few remaining M that are effective in both $J(h)$ and $J(h')$, it is easy to see that $J(h)$ and $\underset{M}{\mathrm{dom}}\, J(h')$ are also separated by certain of the $x_m + x_n = v(\{m,n\})$ planes, and so $K_1^i(h) \cap \mathrm{dom}\, K_1^{i'}(h') = \emptyset$. This proves (17) and completes the discussion of the exceptional case (vii b).

6. REMARKS

Our process for finding K terminates at the first K^f where $E^f \cap A^{f+1} = \emptyset$, for example if $P^f = \{N\}$ or if $|P^{f+1}| < 0$. Also note that a two player coalition M can be realized on the two possible A^j which are obtained from the corresponding partitions of type (2,2) or (2,1,1). It is clear in our process of taking successive maximal elements that we need only consider domination via M off the A^j with the higher $|P^j|$. From these remarks, it is clear, that for our solution K obtained above to hold, condition (10) about distinct imputation simplices A^j need not be satisfied by all A^j. The simplices A^j need not be distinct for the partitions P^j that have $|P^j| < 0$, $|P^j| < |P^f|$ or $|P^j| < |P^0|$ where $P^0 = \{N\}$, or even when these conditions are not true but A^j is realized only by some of the partitions $\{\{p,q\}, \{k,h\}\}$, $\{\{p\}, \{q\}, \{k,h\}\}$ and $\{\{p,q\}, \{k\}, \{h\}\}$. Also, certain A^j need not be distinct if the $v(M)$ for the M realizable on this A^j are small enough so that the corresponding M-strictly effective regions on A^j are disjoint.

Our method of obtaining the solution K is constructive. From Lemma 4 and the fact that the dominion of a polyhedron is a polyhedron, we get that K is polyhedral. It is also conjectured that K is the unique solution but this has not been proved. In games in partition function form with distinct imputation simplices, the solutions obtained so far are not of the discriminatory nature and do not contain the bargaining curves that appear in the von Neumann-Morgenstern theory

CHAPTER 4

n-PERSON GAMES WITH ONLY (n), (n-1,1) AND (1,1,...,1) TYPE PARTITIONS

1. INTRODUCTION

In this chapter, we will give some solutions for the n-person games where only partitions of the type (n), (n-1,1) and (1,1,...,1) enter into the problem. These games are a direct generalization of the 3-person games and thus the notation of this chapter will be similar to that of Chapter 2. The set of players is

$$N = \{1,\ldots,n\}.$$

Consider the following partitions of N.

$$P^0 = \{N\}$$

$$P^i = \{N - \{i\}, \{i\}\} \qquad i = 1,2,\ldots,n$$

$$P^{n+1} = \{\{1\}, \{2\},\ldots,\{n\}\}$$

$$Q = \{Q_1,\ldots,Q_t\} = \text{any other partition of } N.$$

Assume each partition P has an outcome function F_P defined on it, and let

$$F_{P^0}(N) = c$$

$$F_{P^i}(\{i\}) = d_i,\ F_{P^i}(N-\{i\}) = e_i \qquad i = 1,2,\ldots,n$$

$$F_{P^{n+1}}(\{i\}) = g_i \qquad i = 1,2,\ldots,n.$$

Also let

$$c_i = e_i + d_i \qquad i = 1,2,\ldots,n$$

$$g = g_1 + g_2 + \ldots + g_n.$$

In this chapter, we will consider games that have

$$(18) \quad |Q| = \sum_{i=1}^{t} F_Q(Q_i) < \max(0,c)$$

for all $Q \neq P^i (i = 0,1,2,\ldots,$ or $n+1)$. From assumption (18) and Theorems 1, 2 and 4 in Chapter 1, we need only consider the partitions $P^i (i = 0,1,2,\ldots,n)$ in discussing solutions, and P^0 is only used to determine which $P^i (i = 1,\ldots,n)$ have $|P^i| \geq |P^0| = c$. Assuming our games in normal form under S-equivalence (Section 4 of Chapter 1) gives

$$d_i \geq 0, \; e_i \leq c_i$$

$$g_i \geq 0, \; g \geq 0.$$

And by equation (1), the values of the coalitions used to find a solution K then are

$$v(N) = c$$

$$v(N - \{i\}) = e_i$$

$$v(\{i\}) = v_i = 0.$$

For $N-\{i\}$ to be effective means that $\sum_{-i} x_j \leq e_i$ or equivalently that $x_i \geq d_i$ where we use $-i$ to mean that the sum is over all $j\epsilon N-\{i\}$. And an imputation $\underline{x}$ is realized by $N-\{i\}$ only if $\underline{x}\epsilon A(c_i)$.

2. THE CONSTANT SUM CASE

Consider the case

$$(19) \quad c_1 = c_2 = \ldots = c_n.$$

First, if $c_1 < c$, then $A(c) \cup [A(g)\text{-}\underset{N}{\text{dom}}\, A(c)]$ is the unique solution by Theorems 3 and 4. So we will assume that $c_1 \geq c$. Next, if $c_1 < 0$ or $c_1 = 0$, then the unique solution is $A(g)$ or $A(g) \cup \{(0,0,\ldots,0)\}$ respectively. So we will further assume that $c_1 > 0$. Finally, a solution K for $A(c_1)$ can then be extended to a solution for all of R by Theorems 2 and 4. Therefore, we will only consider solutions K for $A(c_1)$ where $c_1 \geq c$ and $c_1 > 0$.

Theorem 9. A solution for the case given in (19) is

$$\begin{aligned} K = \bigcup_{r=0}^{[n/2]} \bigcup_{\sigma_r} \{\underline{x}\epsilon A(c_1) \mid & x_p \geq d_p,\ p = i_1, i_2,\ldots,i_{2r}; \\ & x_q \leq d_q,\ q = i_{2r+1}, i_{2r+2},\ldots,i_n; \\ & x_{i_{s-1}}-d_{i_{s-1}} = x_{i_s}-d_{i_s},\ s = 2,4,\ldots,2r\} \end{aligned}$$

where $[n/2]$ is the greatest integer in $n/2$ and each inner union is taken over the $\dfrac{n!}{(n-2r)!r!2^r}$ permutations $\sigma_r = (i_1, i_2,\ldots, i_n)$ of $(1,2,\ldots,n)$ which give distinct terms.

In other words, $\underline{a}\epsilon A(c_1)$ is in K if and only if all a_p-d_p that are positive are equal to each other in pairs. Figures 14 and 15 show K for $n = 3$ when $\sum_{j \in N} d_j > c_1$ and $\sum_{j \in N} d_j < c_1$, respectively. Figure 16 shows K for $n = 4$ and $\sum_{j \in N} d_j < c_1$. For $r = 0$, we get the term

$$C = \{\underline{x} \epsilon A(c_1) \mid x_q \leq d_q,\ q = 1,2,\ldots,n\}$$

which is the core. For $r = [n/2]$, we get the term

$$\begin{aligned} K_{[n/2]} = \bigcup_{\sigma_{[n/2]}} \{\underline{x} \epsilon A(c_1) \mid x_p \geq d_p,\ p &= i_1, i_2, \ldots, i_{2[n/2]}; \\ x_{i_n} &\leq d_{i_n} \text{ if } n \text{ is odd;} \\ x_{i_{s-1}} - d_{i_{s-1}} &= x_{i_s} - d_{i_s}, \\ s &= 2,4,\ldots,2[n/2]\}. \end{aligned}$$

Let

$$Z = \{\underline{x} \epsilon A(c_1) \mid x_p \geq d_p,\ p = 1,2,\ldots,n\}.$$

Then $K_Z = K_{[n/2]} \cap Z$ is a "translation" of the solution to the (n,k) simple majority games when $k = n-1$ which was given by Bott in [1]. So our solution K is the natural generalization of Bott's solution when $k = n-1$. Note that if $\sum_{j \in N} d_j < c_1$, then $C = \emptyset$ and $Z \neq \emptyset$, and if $\sum_{j \in N} d_j > c_1$, then $C \neq \emptyset$ and $Z = \emptyset$, and if $\sum_{j \in N} d_j = c_1$, then $C = Z = \{(d_1, d_2, \ldots, d_n)\}$.

Geometrically, we have a simple game in the interior part Z of $A(c_1)$ and n truncated pyramid games (see page 81 of [7]) in the regions $S_h = \{\underline{x} \epsilon A(c_1) \mid x_h \leq d_h\}$ which extend off each of the faces of Z. A

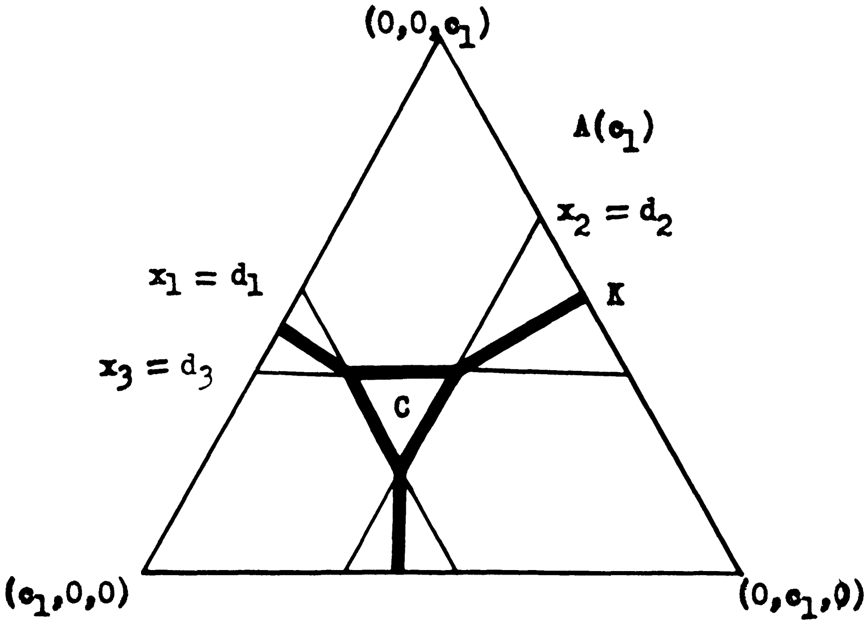

Figure 14. Solution for n = 3 and C $\neq \emptyset$.

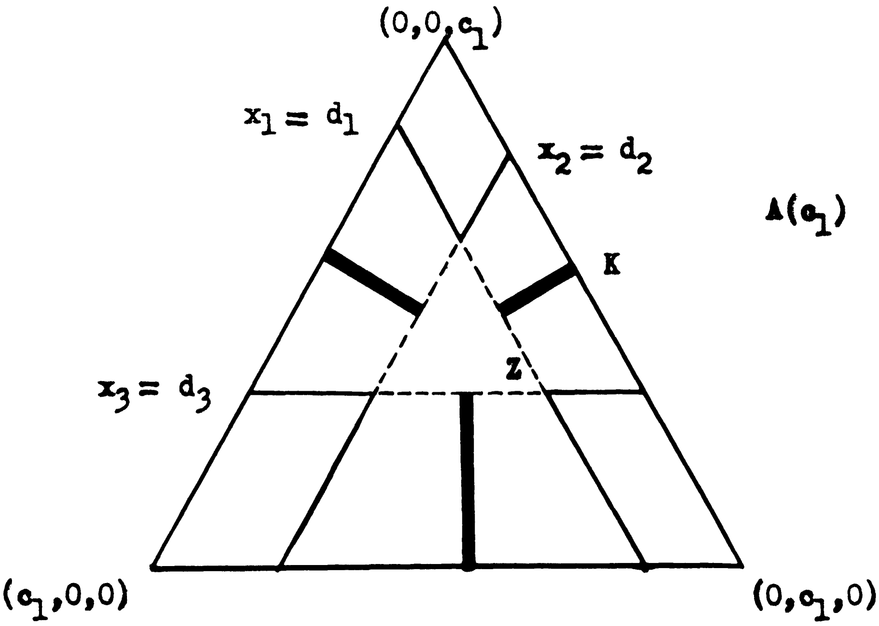

Figure 15. Solution for n = 3 and Z $\neq \emptyset$.

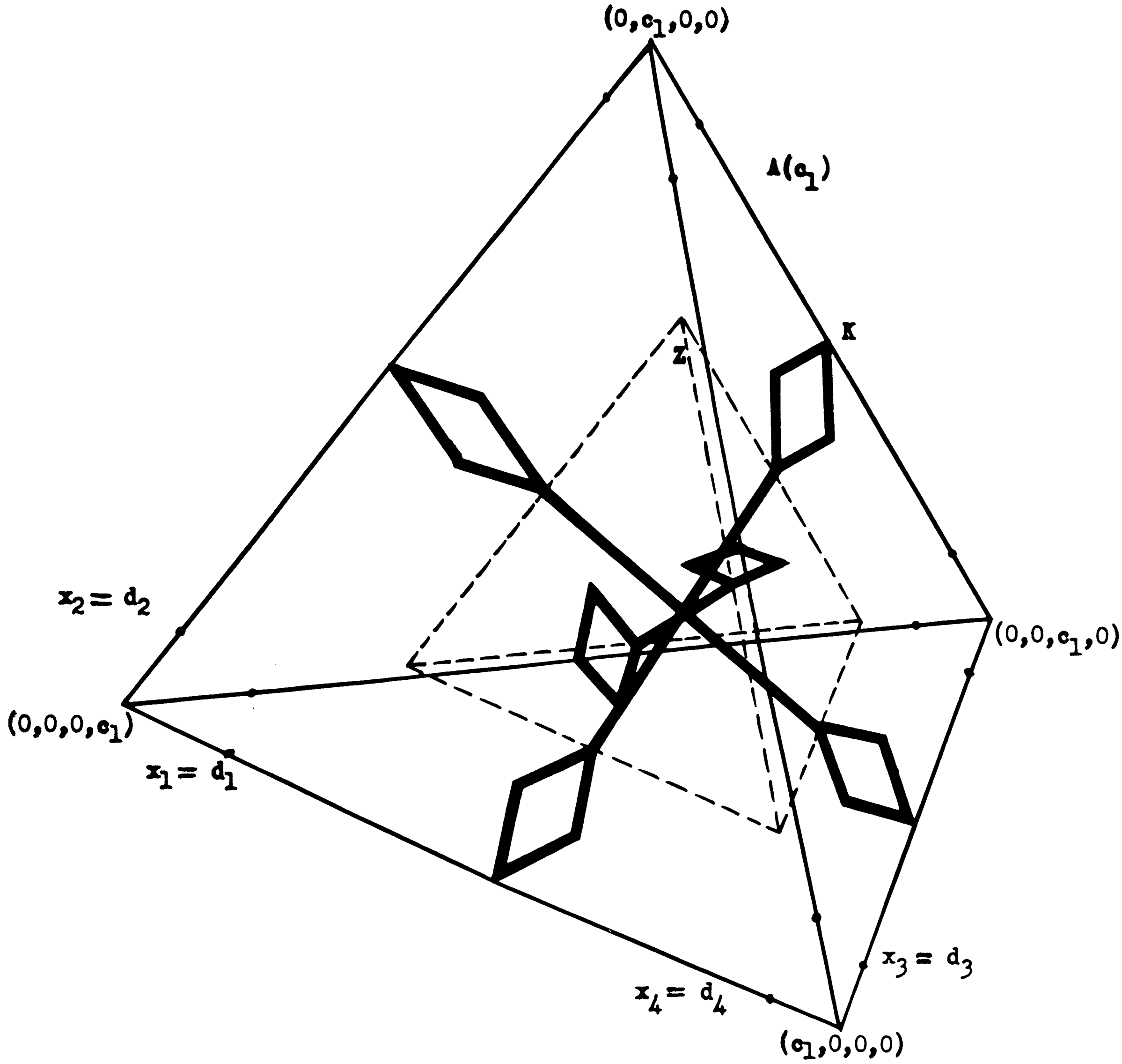

Figure 16. Solution for $n = 4$ and $Z \neq \emptyset$.

trace, x_h = constant, in S_h gives an (n-1)-person game of the type being considered and this trace of K is the corresponding solution for this new game. In Z, the solution K is symmetric with respect to all permutations of the $x_i - d_i$. In S_h, the solution K is symmetric with respect to all permutations of $x_i - d_i$ with $i \neq h$. Note that if $Z \neq \emptyset$, the dimension of K is smallest in the interior part Z of $A(c_1)$ and the dimension increases as one goes more toward the exterior parts, that is, as more $x_i \leq d_i$.

The following two lemmas are useful in the proof of the theorem.

Lemma 6. If

$$\begin{array}{ccccccccccccccccccc}
X_1 & & X_2 & = & X_3 & & & = & X_{2t-1} & & X_{2t} & = & X_{2t+1} & & & & X_{2m} & = & X_1 \\
\wedge & & \vee & & \vee & & \cdots & & \vee & & \vee & & \vee & & \cdots & & \vee & & \\
Y_1 & = & Y_2 & & Y_3 & = & & & Y_{2t-1} & = & Y_{2t} & & Y_{2t+1} & = & & = & Y_{2m} & &
\end{array}$$

and $X_{2m+j} > Y_{2m+j}$ for $j = 1,2,\ldots,n-2m$, then

$$\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} Y_i.$$

Proof. Observe that

$$X_{2t} > Y_{2t} \qquad t = 1,2,\ldots,m$$

$$X_{2t+1} = X_{2t} > Y_{2t} = Y_{2t-1} \qquad t = 1,2,\ldots,m-1$$

$$X_1 = X_{2m} > Y_{2m} = Y_{2m-1}$$

$$X_{2m+j} > Y_{2m+j} \qquad j = 1,2,\ldots,n-2m.$$

Summing over all these gives the desired result.

Lemma 7. If

$$\begin{array}{cccccccccccc}
0 \leq X_1 & X_2 = X_3 & & = X_{2t-1} & X_{2t} = & X_{2t+1} & & = X_{2m+1} \\
\wedge & \vee \quad \vee & \cdots & \vee & \vee & \vee & \cdots & \vee \\
Y_1 = Y_2 & Y_3 = & & Y_{2t-1} = Y_{2t} & & Y_{2t+1} = & & Y_{2m+1} \leq 0
\end{array}$$

and $X_{2m+1+j} > Y_{2m+1+j}$ for $j = 1,2,\ldots,n\text{-}2m\text{-}1$, then

$$\sum_{i=1}^{n} X_i > \sum_{i=1}^{n} Y_i.$$

Proof. Observe that

$$X_{2t} > Y_{2t} \qquad\qquad t = 1,2,\ldots,m$$

$$X_{2t+1} = X_{2t} > Y_{2t} = Y_{2t-1} \qquad\qquad t = 1,2,\ldots,m\text{-}1$$

$$X_1 \geq 0 \geq Y_{2m+1}$$

$$X_{2m+1+j} > Y_{2m+1+j} \qquad\qquad j = 1,2,\ldots,n\text{-}2m\text{-}1.$$

Summing over all these gives the desired result.

Proof of the theorem. First, we prove $K \cap \text{dom } K = \emptyset$. Since $K \cap \text{dom } K = [(K\text{-}C) \cup C] \cap \text{dom}[(K\text{-}C) \cup C] = [(K\text{-}C) \cap \text{dom}(K\text{-}C)] \cup [(K\text{-}C) \cap \text{dom } C] \cup [C \cap \text{dom } K]$, it is sufficient to prove that

(20) $K \cap \text{dom } C = \emptyset$

(21) $C \cap \text{dom } K = \emptyset$, and

(22) $(K\text{-}C) \cap \text{dom } (K\text{-}C) = \emptyset$.

If (20) fails, then there exists $\underline{a}\in C$ and $\underline{b}\in K$ such $\underline{a} \operatorname{dom}_{-k} \underline{b}$ for some $k\in N$. Since $N-\{k\}$ is effective, $\sum_{j\in N} a_j = \sum_{j\in N} b_j$ and $a_i > b_i$ for all $i \neq k$, we get $d_k \leq a_k < b_k$ which implies $\underline{b}\notin C$. Since $\underline{a}\in C$, we also get $b_i < a_i \leq d_i$ for $i \neq k$. It follows that $\underline{b}$ has exactly one coordinate with $b_j > d_j$ ($j = k$) which then implies $\underline{b}\notin K-C$. Thus $\underline{b}\notin K$ which is a contradiction.

If (21) fails, then there exists $\underline{a}\in K$ and $\underline{b}\in C$ such that $\underline{a} \operatorname{dom}_{-k} \underline{b}$. Since $\sum_{j\in N} a_j = \sum_{j\in N} b_j$, $a_i > b_i$ for all $i \neq k$, and $N-\{k\}$ is effective we get $b_k > a_k \geq d_k$. This implies $\underline{b}\notin C$, which is a contradiction.

Assume that (22) fails. Then there exists $\underline{a},\underline{b}\in K-C$ such that $\underline{a} \operatorname{dom}_{-k} \underline{b}$. Since $\underline{b}\notin C$ there exists an i such that $b_i - d_i > 0$. But $\underline{b}\in K-C$ implies that all $b_i - d_i$ that are positive are equal to each other in pairs. So after permuting the subscripts, we can assume $\underline{b}$ is of the form

$$b_{s-1} - d_{s-1} = b_s - d_s > 0, \qquad s = 2,4,\ldots,2r$$

$$b_j \leq d_j \qquad j = 2r+1,\ 2r+2,\ldots,n$$

where $r \geq 1$. $\underline{a}$ is $N-\{k\}$ effective, so $d_k \leq a_k < b_k$, which implies $k\in\{1,2,\ldots,2r\}$. Thus we can take distinct $k,\ i_2,\ i_3,\ldots,\ i_{2m+1}\in \{1,2,\ldots,2r\}$ such that

$$\begin{array}{ccccccccc}
0 \leq a_k - d_k & & a_{i_2} - d_{i_2} & = & \cdots & & a_{i_{2m}} - d_{i_{2m}} & = & a_{i_{2m+1}} - d_{i_{2m+1}} \\
\wedge & & \vee & & & & \vee & & \vee \\
b_k - d_k & = & b_{i_2} - d_{i_2} & & \cdots & = & b_{i_{2m}} - d_{i_{2m}} & & b_{i_{2m+1}} - d_{i_{2m+1}}
\end{array}$$

where eventually either

(i) $i_{2m+1} = k$ (in which case disregard the ">" in the last term),

or

(ii) $b_{i_{2m+1}} - d_{i_{2m+1}} \leq 0$.

In cases (i) and (ii), Lemmas 6 and 7, respectively, imply that $\sum_{i=1}^{n} (a_i - d_i) > \sum_{i=1}^{n} (b_i - d_i)$ or $\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i$, which is a contradiction and (22) follows. This completes the proof that $K \cap \text{dom } K = \emptyset$.

It remains for us to prove that $K \cup \text{dom } K = A(c_1)$. Assume that $\underline{b} \in A(c_1) - K$. Since $\underline{b} \notin C \subseteq K$, there exists i such that $b_i - d_i > 0$. Also, there exists k such that $0 < b_k - d_k \neq b_j - d_j$ for an odd number of $j \neq k$, because if all such positive $b_j - d_j$ could be set equal in pairs, then $\underline{b} \in K - C$. By permuting subscripts, we can assume that the components of $\underline{b} - \underline{d}$ are ordered as

$$b_j - d_j \geq b_{j+1} - d_{j+1} \qquad j = 1,2,\ldots,n-1$$

$$b_k - d_k > 0$$

$$b_k - d_k > b_{k+1} - d_{k+1} \tag{23}$$

$$b_q - d_q \geq 0 > b_{q+1} - d_{q+1}$$

where k is odd and $k \leq q$. The following three cases will be considered.

(i) $q \geq 3$ is odd

(ii) $q = 1$, and

(iii) q is even

In case (i) let

$$(n-1)\epsilon_1 = (b_k - d_k) - \max(b_{k+1} - d_{k+1}, 0) > 0$$

$$\epsilon_2 = -(b_{q+1} - d_{q+1}) > 0$$

$$\epsilon = \min(\epsilon_1, \epsilon_2) > 0$$

$$(q-1)\delta = (b_1 - d_1) - \sum_{i=1}^{(q-1)/2} \left[(b_{2i} - d_{2i}) - (b_{2i+1} - d_{2i+1})\right]$$

$$- (n-1)\epsilon \geq 0.$$

Next define $\underline{a}$ by

$$a_1 - d_1 = 0$$

$$a_{2i} - d_{2i} = a_{2i+1} - d_{2i+1}$$

$$= b_{2i} - d_{2i} + \epsilon + \delta, \qquad i = 1,2,\ldots,(q-1)/2$$

$$a_j = b_j + \epsilon \qquad j = q+1, q+2, \ldots, n.$$

Then $\underline{a} \epsilon A(c_1)$, because $a_1 = d_1 \geq 0$ and $a_i - d_i > b_i - d_i$ implies $a_i > b_i \geq 0$ for all $i \neq 1$, and because

$$\sum_{i\epsilon N} (a_i - d_i) = 0 + 2 \sum_{i=1}^{(q-1)/2} (b_{2i} - d_{2i}) + \sum_{j=q+1}^{n} (b_j - d_j)$$

$$+ (q-1)\delta + (n-1)\epsilon = \sum_{i\epsilon N} (b_i - d_i)$$

and so $\sum_{i\epsilon N} a_i = \sum_{i\epsilon N} b_i = c_1$. And $\underline{a}\epsilon K$ since the positive $a_i - d_i$ are equal in pairs. Also, $\underline{a}$ dom$_{-1}$ $\underline{b}$, because $a_i - d_i > b_i - d_i$ for all $i \neq 1$, and $a_1 = d_1$ so $N-\{1\}$ is effective. Thus $\underline{b}\epsilon$ dom K. If one had to permute the subscripts of the components of $\underline{b}$-$\underline{d}$ to get it in form (23), then the inverse permutation will get the corresponding $\underline{a}$ which is clearly still in K. This completes the proof of (i).

Now, consider case (ii) where $q = 1$. Then define $\underline{a}$ by

$$a_1 - d_1 = 0$$

$$a_2 - d_2 = b_2 - d_2 + \epsilon + \delta_2$$

$$a_3 - d_3 = b_3 - d_3 + \epsilon + \delta_3$$

$$a_j = b_j + \epsilon \qquad j = 4,5,\ldots,n$$

where ϵ is the same as in case (i) and δ_2 and δ_3 are defined by

$$\delta_2 + \delta_3 = (b_1 - d_1) - (n-1)\epsilon \geq 0, \text{ and}$$

$$a_2 - d_2 = a_3 - d_3 \text{ if } \delta_2 + \delta_3 \geq (b_2 - d_2) - (b_3 - d_3), \text{ or}$$

$$\delta_2 = 0 \text{ if } \delta_2 + \delta_3 < (b_2-d_2) - (b_3-d_3).$$

Then $\underline{a}\epsilon A(c_1)$, because $a_1 = d_1 \geq 0$ and $a_i-d_i > b_i-d_i$ implies $a_i > b_i \geq 0$ for all $i \neq 1$, and because

$$\sum_{i\epsilon N} (a_i-d_i) = \sum_{i=2}^{n} (b_i-d_i) + (n-1)\epsilon + (b_1-d_1) - (n-1)\epsilon$$

$$= \sum_{i\epsilon N} (b_i-d_i).$$

And $\underline{a}\epsilon K$ since a_2-d_2 and a_3-d_3 are either equal or non positive and all other $a_i-d_i \leq 0$. Also $\underline{a} \text{ dom}_{-1} \underline{b}$. Thus $\underline{b}\epsilon$ dom K.

In case (iii) where q is even, let

$$n\epsilon_1 = (b_k-d_k) - (b_{k+1}-d_{k+1}) > 0$$

$$\epsilon_2 = -(b_{q+1}-d_{q+1}) > 0$$

$$\epsilon = \min(\epsilon_1, \epsilon_2) > 0$$

$$2\delta = (b_1-d_1) - \sum_{i=1}^{(q-1)/2} [(b_{2i}-d_{2i}) - (b_{2i+1}-d_{2i+1})]$$

$$- n\epsilon \geq 0.$$

And define $\underline{a}$ by

$$a_1 - d_1 = a_q - d_q = b_q - d_q + \epsilon + \delta$$

$$a_{2i} - d_{2i} = a_{2i+1} - d_{2i+1}$$

$$= b_{2i} - d_{2i} + \epsilon \qquad i = 1,2,\dots,(q-2)/2$$

$$a_j = b_j + \epsilon \qquad j = q+1, q+2,\dots,n.$$

Then $\underline{a}\epsilon A(c_1)$, because clearly $a_i \geq 0$, and because

$$\sum_{i\epsilon N} (a_i - d_i) = 2\sum_{i=1}^{q/2} (b_{2i} - d_{2i}) + \sum_{j=q+1}^{n} (b_j - d_j) + n\epsilon + 2\delta$$

$$= \sum_{i\epsilon N} (b_i - d_i).$$

And $\underline{a}\epsilon K$ since the positive $a_i - d_i$ are equal in pairs. Also, $\underline{a}$ dom_{-1} $\underline{b}$, because $a_1 - d_1 = a_q - d_q \geq 0$ so $\underline{a}$ is $N-\{1\}$ effective and $a_i - d_i > b_i - d_i$ for all $i \neq 1$. So again $\underline{b}\epsilon\text{dom } K$, which proves (iii). This completes the proof that $K \cup \text{dom } K = A(c_1)$. Thus the theorem is proved.

3. THE DISTINCT SIMPLICES CASE

We will now consider the case

$$(24) \qquad \begin{aligned} &c_1 > c_2 > \dots > c_m \geq c \text{ where } m \leq n, \text{ and} \\ &c > c_{m+j} \text{ for } j = 1,\dots,n-m \text{ if } m < n. \end{aligned}$$

In this case, the imputation simplices $A(c_j)$ $(j = 1,2,\dots,m)$ are distinct and there is a unique solution K which is given in the following theorem. As before, we will not describe the parts of the solution on $A(c)$ and $A(g)$. For $n = 3$ and $m = 1$, 2 and 3, we get the solutions in Genus 1, Genus 2A and Genus 3A, respectively, given in Section 2 of Chapter 2.

Theorem 10. The unique solution for any game satisfying (18) and (24) is

$$K = \bigcup_{i=1}^{m} A(c_i) - \bigcup_{j=1}^{m} \left\{ \underline{x} \mid \sum_{-j} x_k < e_j;\ x_p < d_p - \Delta_{pj},\ p = 1,\ldots,j-1 \right\}.$$

Proof. First, we must show that $K \cup \text{dom}\, K = \bigcup_{j=1}^{m} A(c_j)$. But it is easy to show, that except for one case, that $V^j = \left\{ \underline{x} \in A(c_j) \mid \sum_{-j} x_k = e_j;\ x_p \leq d_p - \Delta_{pj},\ p = 1,\ldots,j-1 \right\}$ is contained in K and $\text{dom}_{-j}\, V^j \supseteq \left\{ x \mid \sum_{-j} x_k < e_j;\ x_p < d_p - \Delta_{pj},\ p = 1,\ldots,j-1 \right\}$ where $j = 1,\ldots,m$. The exceptional case occurs when $j = m = n$ and $\sum_{i \in N} d_i < c_n$ because then $V^n = \emptyset$. But in this case, $V_0^n = \bigcup_{p=1}^{n-1} \{\underline{x} \in A(c_n) \mid x_p = d_p - \Delta_{pn};\ x_q \leq d_q - \Delta_{qn},\ q = 1,\ldots,n-1\}$ is contained in K and $\text{dom}_{-n}\, V_0^n \supseteq \left\{ \underline{x} \mid \sum_{-n} x_k < e_n;\ x_p < d_p - \Delta_{pn},\ p = 1,\ldots,n-1 \right\}$.

Thus dom K will contain all those imputations that were subtracted from $\bigcup_{j=1}^{m} A(c_j)$ to obtain K. In fact, we only need to dominate with those imputations in K that are exactly effective except for the exceptional case.

Next, we must prove that $K \cap \text{dom}\, K = \emptyset$. Let $\underline{x} \in K$ be $N-\{j\}$ effective and realizable. Then $\underline{x} \in K \cap \left\{ \underline{x} \mid \sum_{-j} x_k \leq e_j \right\} \cap A(c_j)$, that is, $\underline{x} \in K$, $x_j \geq d_j$ and $\sum_{i \in N} x_i = c_j$. From our definition of K, it is then clear that $x_p \leq d_p - \Delta_{pj}$ for $p = 1,\ldots,$ and $j-1$, and $\sum_{-j} x_k \leq e_j$. Such $\underline{x}$ can only dominate via $N-\{j\}$ those $\underline{z}$ with $z_p < d_p-\Delta_{pj}$ for $p = 1,\ldots,j-1$, and $\sum_{-j} z_k < e_j$. But from the definition of K, such $\underline{z}\notin K$. This proves that

$K\cap \underset{-j}{\mathrm{dom}}\ K = \emptyset$ for $j = 1,\ldots,m$, which is sufficient for $K\cap \mathrm{dom}\ K = \emptyset$.
Therefore, K is a solution.

Finally, we will prove that K is the unique solution by showing that an imputation in $\bigcup_{i=1}^{m} A(c_i)-K$ cannot be in any solution K'. Let $\underline{a}\in\left\{\underline{x} \mid \sum_{-1} x_k < e_1\right\}$, that is, consider the term $j = 1$ in the equation above for K. The proof that $\underline{a}$ is not in any solution K' is similar to the proof of Theorem 5 except that the set $\{2,3\}$ is replaced by $\{2,3,\ldots,n\}$ and so we omit it. But now we will give the proof for an arbitrary j, that is, we will prove that if $\underline{a}$ is in

$$D^j = \left\{\underline{x} \mid \sum_{-j} x_k < e_j;\quad x_p < d_p-\Delta_{pj},\ p = 1,\ldots,j-1\right\}$$

then $\underline{a}$ cannot be in any solution K'. Assume that $0 < e_j \leq c_j$, for if $e_j \leq 0$, then there is nothing to prove, and define

$$B^r = \left\{\underline{x}\in D^j \cap A(c_j) \mid \sum_{-j} x_k \leq d^r\right\}$$

where $d^r = \min\{e_j,\ r\Delta_{j,j-1}\}$ and $\Delta_{pq} = c_p-c_q$.

We will show that $B^1\subseteq K'\cup \underset{-j}{\mathrm{dom}}\ K'$. Since K' is a solution, $B^1\subseteq K'\cup \mathrm{dom}\ K'$. But from (18), (24) and Theorem 1, we get that $B^1\cap \underset{M}{\mathrm{dom}}\ K' = \emptyset$ unless $M = N-\{j\}$ where $j = 1,\ldots,m$. But $B^r\cap \underset{-k}{\mathrm{dom}}\ K' =$

$\emptyset$ when $k < j$ because if $\underline{a}\epsilon B^r \subseteq D^j$, then $a_k < d_k - \Delta_{kj}$ and so $\sum_{-k} a_i = c_j$ $- a_k > c_j - d_k + \Delta_{kj} = c_k - d_k = e_k = v(N-\{k\})$ and thus $\underline{a}$ is ineffective for $N-\{k\}$. And $B^1 \cap \underset{-k}{\mathrm{dom}}\, K' = \emptyset$ when $m \geq k > j$; because if $\underline{a}\epsilon B^1$ and the intersection is non empty, then there exists $\underline{b}\epsilon\, K' \cap A(c_k)$ such that $\underline{b} \underset{-k}{\mathrm{dom}}\, \underline{a}$ which implies that $\sum_{i\epsilon N} b_i > b_j > a_j > c_j - d^1 \geq c_j - \Delta_{j,j+1}$ $= c_{j+1} \geq c_k$ which says that $\underline{b}$ is not realized by $N-\{k\}$ and thus contradicts $\underline{b} \underset{-k}{\mathrm{dom}}\, \underline{a}$. Therefore $B^1 \cap \underset{-k}{\mathrm{dom}}\, K' = \emptyset$ when $k \neq j$, and $B^1 \subseteq K' \cup \underset{-j}{\mathrm{dom}}\, K'$. Then, it is clear that $D^j \cap \left\{ \underline{x} \mid \sum_{-j} x_k < d^1 \right\} \subseteq \underset{-j}{\mathrm{dom}}\, B^1 \subseteq \underset{-j}{\mathrm{dom}}\, K'$ and so $D^j \cap \left\{ \underline{x} \mid \sum_{-j} x_k < d^1 \right\} \cap K' = \emptyset$. If $d^1 = e_j$, then we have proved the uniqueness part of our theorem.

If $d^1 < e_j$, then $d^1 = \Delta_{j,j+1}$ and $\left\{ \underline{x} \mid \sum_{-j} x_k < \Delta_{j,j+1} \right\} \cap K' = \emptyset$. So if $\underline{b}\epsilon K' \cap A(c_k)$, then $\sum_{-j} b_i \geq \Delta_{j,j+1}$ and $b_j \leq c_k - \Delta_{j,j+1}$. Now we can show that $B^2 \subseteq K' \cup \underset{-j}{\mathrm{dom}}\, K'$. Because $B^2 \cap \underset{-k}{\mathrm{dom}}\, K' = \emptyset$ when $k < j$, since if $\underline{a}\epsilon B^2 \subseteq D^j$, then $\underline{a}$ is ineffective for $N-\{k\}$ as proved in the paragraph above. And because $B^2 \cap \underset{-k}{\mathrm{dom}}\, K' = \emptyset$ when $m \geq k > j$, since if not then there exists $\underline{a}\epsilon B^2$ and $\underline{b}\epsilon K' \cap A(c_k)$ such that $\underline{b} \underset{-k}{\mathrm{dom}}\, \underline{a}$ which implies that $b_j > a_j = c_j - \sum_{-j} a_i \geq c_j - d^2 \geq c_j - 2\Delta_{j,j+1}$ and $\sum_{-j} b_i$

$= c_k - b_j < c_k - c_j + 2\Delta_{j,j+1} = \Delta_{kj} + 2\Delta_{j,j+1} \leq \Delta_{j,j+1}$ which is a contradiction. Therefore, $B^2 \subseteq K' \cup \mathrm{dom}_{-j}\, K'$. Next, it is then clear that $D^j \cap \left\{\underline{x} \mid \sum_{-j} x_k < d^2\right\} \subseteq \mathrm{dom}_{-j}\, B^2 \subseteq \mathrm{dom}_{-j}\, K'$ and so $D^j \cap \left\{\underline{x} \mid \sum_{-j} x_k < d^2\right\} \cap K' = \emptyset$. If $d^2 = e_j$, then we have proved the uniqueness part of our theorem.

If $d^2 < e_j$, then we continue as above to get $B^r \subseteq K' \cup \mathrm{dom}_{-j}\, K'$ for $r = 3, 4, \ldots, r_0$ where finally $d^{r_0} = e_j$. Then $D^j \cap \left\{\underline{x} \mid \sum_{-j} x_k < d^{r_0} = e_j\right\} = D^j \subseteq \mathrm{dom}_{-j}\, B^{r_0} \subseteq \mathrm{dom}_{-j}\, K'$. So $D^j \cap K' = \emptyset$. Since $j = 1, \ldots,$ or m, this proves the uniqueness of the solution K, and completes the proof of Theorem 10.

4. A SEPARATION THEOREM

The following separation theorem will be used in the next section to show that for each strict inequality $c_q > c_{q+1}$ in $c_1 \geq c_2 \geq \ldots \geq c_m \geq c$ we can partition $\bigcup_{i=1}^{m} A(c_i)$ into three parts. The first part is contained in every solution. In the second part, only domination via $N-\{j\}$ $(j = 1, \ldots, q)$ need be considered in finding any solution. And in the third part, only domination via $N-\{j\}$ $(j = q+1, \ldots, m)$ need be considered in finding any solution.

Theorem 11. If $c_1 \geq \ldots \geq c_q > c_{q+1} \geq \ldots \geq c_m \geq c > c_{m+j}$ for $j = 1, \ldots, n-m$, and K is any solution for a game satisfying (18), then

$$\{\underline{x} \mid x_j > d_j - \Delta_{j,q+1},\ j = 1, \ldots, \text{ or } q\} \cap \mathrm{dom}_{-i}\, K = \emptyset$$

for $i = q+1,\ldots,m$, and where again $\Delta_{kp} = c_k - c_p$.

Proof. Let

$$D^i = \{\underline{x}\epsilon A(c_i) \mid x_j > d_j - \Delta_{j,q+1} \text{ for } j = 1,\ldots, \text{ or } q\}.$$

We will first show that $D^i \cap K = \emptyset$ when $i = q+1,\ldots,$ and m. For let $\underline{a}\epsilon D^i$, that is, $\underline{a}\epsilon A(c_i)$ and $a_j > d_j - \Delta_{j,q+1}$ for $j = 1,\ldots,$ or q, and then $\sum_{-j} a_k = c_i - a_j < c_j - \Delta_{ji} - d_j + \Delta_{j,q+1} \leq e_j = v(N-\{j\})$. Then define $\underline{b}$ by $b_k = a_k + \epsilon$ for all $k \neq j$ and $b_j = a_j + \Delta_{j,q+1} - (n-1)\epsilon$ where $n\epsilon = \min\left\{ e_j - \sum_{-j} a_k, \Delta_{j,q+1} \right\} > 0$. And then $\underline{b} \underset{N}{>} \underline{a}$ and $\underline{b} \operatorname{dom}_{-j} \underline{a}$, because ϵ and $\Delta_{j,q+1} - (n-1)\epsilon > 0$, $\sum_{-j} b_k = \sum_{-j} a_k + (n-1)\epsilon < e_j = v(N-\{j\})$, and $\sum_{p\epsilon N} b_p = \sum_{p\epsilon N} a_p + \Delta_{j,q+1} = c_j$. But $\underline{b}\epsilon K \cup \operatorname{dom} K$. If $\underline{b}\epsilon K$, then $\underline{a}\epsilon \operatorname{dom}_{-j} \underline{b} \subseteq \operatorname{dom} K$. If $\underline{b}\epsilon \operatorname{dom} K$, then $\underline{a}\epsilon \operatorname{dom} K$ by Lemma 1. In either case, $\underline{a}\notin K$, and therefore $D^i \cap K = \emptyset$ when $i = q+1,\ldots,$ and m. That is, if $\underline{b}\epsilon A(c_i) \cap K$ ($i = q+1,\ldots,$ or m), then $b_j \leq d_j - \Delta_{j,q+1}$ for $j = 1,\ldots,$ and q. So if $\underline{x}\epsilon \operatorname{dom}_{-i} K$ for $i = q+1,\ldots,$ or m, then $x_j < d_j - \Delta_{j,q+1}$ for $j = 1,\ldots,$ and q. This completes the proof of the theorem.

5. INTERMEDIATE CASES

We will now summarize some partial results for those remaining games that are intermediate to the constant sum case of Section 2 and the distinct simplices case of Section 3. Thus, in addition to (18) we will assume that

$$
\begin{aligned}
c_1 &= \ldots = c_{r_1} > c_{r_1+1} = \ldots = c_{r_2} > \ldots > c_{r_{q-1}+1} \\
&= \ldots = c_{r_q} \\
(25) \qquad &\geq c > c_{r_q+j} \text{ for } j = 1,\ldots,n-r_q.
\end{aligned}
$$

We will let $r_q = m$ as in previous usage. The solutions obtained so far for such games are closely related to those in Sections 2 and 3. As usual, we will not discuss that part of the solutions on $A(c)$ and $A(g)$.

As a result of Theorem 11, we get that for each $c_{r_i} > c_{r_i+1}$ in (25) there is a partitioning of $V = \bigcup_{j=1}^{m} A(c_j)$ into the polyhedrons

$$T_i = \left\{ \underline{x} \in V \,\middle|\, \sum_{-j} x_i < e_j \text{ for } j = 1,\ldots, \text{ or } r_i \right\},$$

$$F_i = \left\{ \underline{x} \in V \,\middle|\, \begin{array}{l} \sum_{-j} x_i \geq e_j \text{ for } j = 1,\ldots, \text{ and } r_i; \\ x_k \geq d_k - \Delta_{k,r_i+1} \text{ for } k = 1,\ldots, \text{ or } r_i \end{array} \right\}$$

and

$$L_i = \left\{ \underline{x} \in V \,\middle|\, \begin{array}{l} \sum_{-j} x_i \geq e_j \text{ for } j = 1,\ldots, \text{ and } r_i; \\ x_k < d_k - \Delta_{k,r_i+1} \text{ for } k = 1,\ldots, \text{ and } r_i \end{array} \right\}.$$

We also define F_0 to be those elements undominated via all $N-\{i\}$,

$$F_0 = \{\underline{x} \in V \mid x_j \leq d_j \text{ for } j = 1,\ldots, \text{ and } n\},$$

that is, F_0 is the core if the values of all coalitions with less than n-1 players are non positive. In T_i any imputation is N-$\{j\}$ strictly effective for $j = 1,\ldots,$ or r_i; and from Theorem 11, no imputation in

$$T_i \cap \bigcup_{j=1}^{r_i} A(c_j) \text{ is in } \underset{-k}{\text{dom}}\ K \text{ for } k = r_i+1,\ldots,m \text{ where } K \text{ is any solution.}$$

It is also clear from Theorem 11 that F_i is contained in any K for $i = 0,1,\ldots,$ and q. (But in general $F = \bigcup_{j=0}^{q} F_j$ need not contain the intersection of all solutions even when m = n.) And in L_i, only domination via N-$\{k\}$ for $k = r_i+1,\ldots,m$ need be considered in finding K.

To find a solution K for a game satisfying (25), we consider K in different parts of V. First, we give a solution for $T_1 \cup F$. Consider the solution given in Theorem 9 for the constant sum case when $d_k \geq c_k$ (that is $e_k \leq 0$) for $k = r_1+1,\ldots,$ and n. Let K_0 be this solution intersected with T_1. Then $K_0 \cup F$ will be a solution for $T_1 \cup F$ where in $T_1 \cup F$ we only consider domination off of $K_0 \cup F$ via N-$\{j\}$ for $j = 1,\ldots,$ and r_1. Second, let us assume that we can find a solution $F \cup K_{q-1}$ in $F \cup L_{q-1}$ for all those games where the r_{q-1} in the definition of L_{q-1} could be 1,2, $\ldots$,m-1. In doing this, the case where m < n follows easily from the case where m = n and $c_m = c_{m+1} = \ldots = c_n$ by setting $d_{m+1} = \ldots = d_n = c_m$. That is, we can assume that $r_q = m = n$. Third, we can then get a solution for the $F \cup (T_i - T_{i-1})$ by taking $F \cup (K_{q-1} \cap T_{i-1})$ when i = q-1 and $d_k \geq c_k$ for $k = r_{q-1}+1,\ldots,n$ where only domination via N-$\{j\}$ for $j = r_{i-1}+1,\ldots,r$ need be considered. Therefore, the problem reduces to finding a solution $F \cup K_{q-1}$ in $F \cup L_{q-1}$ where $r_{q-1} = 1,2,\ldots,$ and n-1.

The problem of finding a solution in $F \cup L_{q-1}$ can be reduced somewhat as follows. If K is any solution in V, then $L_{q-1} \cap \underset{-k}{\mathrm{dom}}\, K = \emptyset$ unless $k = r_{q-1}+1,\ldots,$ or n. But the only imputations in L_{q-1} that are realizable by $N-\{k\}$ for $k = r_{q-1}+1,\ldots,$ or n are on the imputation simplex $A(c_{r_{q-1}+1}) = \ldots = A(c_n)$. So a solution in $(F \cup L_{q-1}) \cap A(c_n)$ can be extended to one in all of $F \cup L_{q-1}$ by simply adding the undominated imputations in $(F \cup L_{q-1}) \cap A(c_i)$ for $i = 1,\ldots,r_{q-1}$. And it can also be shown that $\bigcup_{k=r_{q-1}+1}^{n} \underset{-k}{\mathrm{dom}}\, F$ contains exactly those elements in $L_{q-1} \cap A(c_n)$ that are not in

$$W = \{\underline{x} \epsilon A(c_n) \cap L_{q-1} \mid x_j + x_k \leq d_j + d_k - \Delta_{j,r_q} \text{ where } j = 1,\ldots,r_{q-1},\ k = r_{q-1}+1,\ldots,n\}$$
$$- \bigcup_{i=r_{q-1}+1}^{n} \{\underline{x} \epsilon A(c_n) \cap L_{q-1} \mid x_k < d_k \text{ for } k = r_{q-1}+1,\ldots,\hat{i},\ldots,n\}.$$

The subtracted terms in the definition of W are the imputations contained in $(F_0 \cup \mathrm{dom}\, F_0) \cap (A(c_n) \cup L_{q-1})$. Therefore, our problem of finding a solution for games satisfying (25) reduced to finding a solution in the region W for the cases $r_{q-1} = 1,2,\ldots,$ and n-1.

Solutions in region W have been found for a large number of games for various values of n and r_{q-1}, including all games with $n \leq 4$ and all games with arbitrary n and $r_{q-1} = n-2$ and n-1. But few of the results for small n seem to generalize without some modification. Work is continuing on these remaining problems.

CHAPTER 5

UNSOLVED PROBLEMS

There are many conjectures and open questions in the theory of n-person games in partition function form. We will now list some of the more immediate ones for which we are presently seeking answers.

In Chapter 3, we gave a solution for each 4-person game that has distinct imputation simplices. As mentioned there we conjecture that this solution is unique. This could probably be proved in a manner similar to the uniqueness discussion for 3-person games given in Section 3 of Chapter 2. However, such an approach for the 4-person games would involve a large number of cases each involving an inductive proof and thus seems impractical. Hence, we are looking for a more direct or local type proof for uniqueness in the 3-person and 4-person games with distinct imputation simplices. If successful, then we will try to generalize the treatment to handle n-person games with distinct simplices.

Solutions have also been found for all 4-person games in which only partitions of type (4), (3,1) and (1,1,1,1) can have large outcomes (in the sense of equation (18)). Such games are covered in Chapter 4 when $n = 4$. By using the separation property in Lemma 5 and considering a large number of cases, we can also give solutions for all 4-person games in which only partitions of type (4), (2,2), (2,1,1) and (1,1,1,1) can have large outcomes. In addition solutions have been found for many other 4-person games in which both two and three player coalitions are used in the domination. But there are too many games of this latter type to prove the general existence of solutions by just considering cases. Experience to date indicates that there is little difficulty in finding a solution for any given 4-person game by merely combining the methods for the various cases discussed above, but this has not yet been proved. Therefore, work is continuing on the proof of a general existence theorem for all 4-person games.

We would like to answer the question of existence of solutions for all of the intermediate type games given in Section 5 of Chapter 4. Solutions in the region W (defined at the end of Chapter 4) are being sought for 5- and 6-person games in hope that some of these solutions will generalize. The solutions found so far in the region W are made up of parts of the solutions given in Section 2 of Chapter 4 (the constant sum case) along with certain maximal elements with respect to domination via certain coalitions $N - \{i\}$. Furthermore, the games in Chapter 4 are just a special case ($k = 1$) of those games in which only partitions of type (n), $(n-k,k)$ and $(1,1,\ldots,1)$ have large outcomes for some given $k = 1,2,\ldots,$ or $[n/2]$, where $[n/2]$ again stands for the greatest integer in $n/2$. We will attempt to find solutions for such games when $k > 1$. If successful then we will try to combine the results for various k values in order to get solutions for some games in which partitions of type (n), $(n-1,1)$, $(n-2,2)$,$\ldots$,$(n-[n/2],[n/2])$ are significant. Theorem 9 generalizes the solutions given by Bott in [1] when $k = 1$. Perhaps there are symmetric solutions for the constant sum case when $k > 1$ which will also generalize the corresponding solutions by Bott when $k > 1$.

The most outstanding problem in the theory is the proof of a general existence theorem for solutions of all n-person games or the demonstration of a counter example. Since Thrall games are a generalization of the von Neumann-Morgenstern games, the solution of this problem would solve the classical case also. More modestly, we will attempt to prove that there is a polyhedral solution (perhaps unique) for n-person games when the imputation simplices A(P) corresponding to different partitions P of N are distinct. We hope to obtain the solution in a manner similar to that in Chapter 3 where $n = 4$. That is, we will take maximal elements with respect to "dom" on the successive simplices A(P) as the values of $|P|$ decrease; and whenever the maximal elements dominate away too much from a previous solution for part of R, then we repeat the process of taking successive maximal elements in the undominated parts. This process should terminate in a finite number of steps. It is clear from the large number of cases considered in Chapter 3 that a more direct proof will need to be developed when $n > 4$. We have no example yet in which this process of taking successive maximal elements has not led to a solution for a particular game, but we note that when $n \geq 5$ then Lemma 5 fails and when $n \geq 6$ then there can be three non-trivial coalitions in one partition. However, if we are successful in our search for a solution in the distinct simplex case, then the results should give some insight into the other games where the simplices are not all distinct. It is hoped that this would lead to some finite constructive process for solving the more difficult problem. But the solutions for the latter problem are not just a simple limiting situation of the distinct simplex case. We also remark that for the von Neumann-Morgenstern games Shapley [4] has conjectured that there exists a solution which is contained in $\left\{\underline{x} \,\middle|\, \sum_{i \in P_j} x_i \geq v(P_j) \text{ for all } P_j \epsilon P\right\}$ where P is an arbitrary partition of N. The corresponding conjecture for Thrall games would be only for any P that has $|P| = \max \{|Q| \mid Q \epsilon \Pi\}$. This is consistent with our approach of taking successive maximal elements on the A(P), starting with a largest $|P|$.

In Section 4 of Chapter 1 we stated that it is easy to prove that "$\underset{M}{\text{dom}}$" is preserved under S-equivalence, and thus "dom" and solution sets are also preserved. It would be of interest to know if the converse is true, that is, whether a one to one map of the imputation spaces of two games which preserves "$\underset{M}{\text{dom}}$" is an S-equivalence.

Finally, we will just mention some less specific problems of possible interest. Several changes could be made in the basic definitions and assumptions in the Thrall theory to see what effect they have on our present results. The experimental work on n-person games could be reviewed in light of this approach, and possible new experiments designed. Additional studies could also be made into the relationship of Thrall's theory to some of the other recent approaches to n-person game theory (for example, see [3]).

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