Rank functors and representation rings of quivers

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2009

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To my wife Kelly.

ACKNOWLEDGEMENTS

I would like to thank, first and foremost, my advisor Harm Derksen for guiding me along the path to becoming a mathematical researcher. The many hours we spent in discussion and his creative suggestions were absolutely essential for completion of this project. I would also like to thank John Stembridge for reading this dissertation and for some helpful discussions relating to future directions; specifically, a conversation at MSRI on graph minors, and later directing me to some references on minuscule posets and representation theory. Additional thanks are due to Hugh Thomas for providing many useful comments about Chapter 3 when it was in its preprint form. He also has suggested some interesting connections to other work, as mentioned in the final section. I thank Andreas Blass for his extremely meticulous reading of the thesis which yielded a substantial improvement of the final product, and Achilleas Anastasopoulos and Farkhod Eshmatov for supporting me by being on my dissertation committee.

I am indebted to the University of Michigan Department of Mathematics community in general for sustaining a stimulating environment in which to work, and preparing me for a career in mathematics. In particular, I thank Toby Stafford for his advice and support at a critical time when I was choosing an advisor and research topic. My research interests have been influenced by so many graduate students and faculty that I will not name them individually, but I will say that it has been quite an extraordinary place to become a mathematician over the past six years. Besides those that have helped me develop as a researcher, I would like to thank Gavin LaRose, Karen Rhea, and Dale Winter for helping me improve as an instructor. The work of the department staff, especially Warren Noone and Tara McQueen, is deeply appreciated for making my experience at the university a better one.

I would like thank several professors at the University of Kansas for getting me interested in research while I was an undergraduate. I thank Estela Gavosto, who advised me during my four years there, helped prepare me for graduate school, and got me started on my first mathematical research project. I thank Craig Huneke for his excellent courses which drew me to algebra, and whose encouragement and enthusiasm made me understand the difference between knowing something and knowing it *well*. I thank Jeffrey Olafsen for giving me my first research experience, a magnetically driven granular dynamics project, and for his solid advice as I sought out the right field of study for myself as an undergraduate.

I thank my loving wife Kelly for her patience and unconditional support that carried me through the more difficult parts of the past six years. I thank my parents David and Pam for their care, encouragement, and understanding in everything, and especially for nurturing my interest in mathematics from a young age. Numerous educators in the Oklahoma City Public School System influenced me in a positive way. I want to specially thank Randy Auxier for his philosophical guidance, notably his emphasis on the principle of "Don't say that you know something that you don't know," which has proven quite valuable in mathematics, among other areas of life.

Chapter 2 is based on my article [38], and Chapter 3 on my preprint [37].

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CHAPTER 1

Introduction

A **quiver** is a directed graph $Q = (Q_\bullet, Q_\to, t, h)$, where $Q_\bullet$ is a vertex set, $Q_\to$ is an arrow set, and t, h are functions from $Q_\to$ to $Q_\bullet$ giving the tail and head of an arrow, respectively. We assume $Q_\bullet$ and $Q_\to$ are finite in this thesis. For any quiver Q and field K, there is a category $\mathrm{Rep}_K(Q)$ of representations of Q over K. An object V of $\mathrm{Rep}_K(Q)$ is an assignment of a finite dimensional K-vector space V_x to each vertex $x \in Q_\bullet$, and an assignment of a K-linear map $V_a \colon V_{ta} \to V_{ha}$ to each arrow $a \in Q_\to$. For any path p in Q, we get a K-linear map V_p by composition. Morphisms in $\mathrm{Rep}_K(Q)$ are given by linear maps at each vertex which form commutative diagrams over each arrow. The category $\mathrm{Rep}_K(Q)$ is abelian, and in fact equivalent to the category of finite dimensional modules over a noncommutative ring called the **path algebra** of Q. Hence we have familiar notions such as direct sum of representations and kernels of morphisms (cf. §2.1.1 for more a more detailed account). The field K will be fixed and arbitrary throughout the thesis, except where noted, and hence we omit it from notation when possible.

There is also a natural tensor product of quiver representations, induced by the tensor product in the category of vector spaces (cf. §2.1.2). Tensor products of quiver representations have been studied by Strassen [58] in relation to orbit closure degenerations [2, 1, 8, 7, 49, 59], and Herschend has a general discussion of tensor product operations on quivers in [33]. Concretely, we use the "pointwise" tensor product of representations defined by

$$(V \otimes W)_x := V_x \otimes W_x$$

for each vertex x, and similarly for arrows.

The category $\mathrm{Rep}(Q)$ has the **Krull-Schmidt property** [5, Theorem I.4.10], meaning that V has an essentially unique expression

$$V \simeq \bigoplus_{i=1}^{n} V_i$$

as a direct sum of indecomposable representations V_i. That is, given any other expression $V \simeq \bigoplus \tilde{V}_i$ with each $\tilde{V}_i$ indecomposable, there is a permutation σ of $\{1, \cdots n\}$ such that $\tilde{V}_i \simeq V_{\sigma i}$ for all i. This property naturally leads to the **Clebsch-Gordan problem** for quiver representations: given $V, W \in \mathrm{Rep}(Q)$, what are the indecomposable summands occurring in $V \otimes W$, and what are their multiplicities? Clebsch and Gordan answered this problem for representations of the semi-simple Lie algebra $\mathfrak{sl}_2$, but the question is natural to ask in many representation theoretic settings. Since the tensor product distributes over direct sum, we can assume without loss of generality that V and W are indecomposable, and so a good starting point is to have a description of indecomposable objects. We review some contexts in which the Clebsch-Gordan problem has been studied historically.

The indecomposable representations of the classical groups (and their Lie algebras) can be explicitly described in terms of weights and root systems. See [23, p. 424] and the references therein. For example, the indecomposable representations V_λ of $GL_n(\mathbb{C})$ are indexed by partitions λ with no more than n parts, and one can study tensor products of these representations via combinatorics of partitions. In this case, it turns out that

$$V_\lambda \otimes V_\mu \simeq \bigoplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}},$$

where $c^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients. There is a wealth of combinatorics relating to these mutliplicities [22].

Indecomposable representations of the symmetric group $\mathfrak{S}_n$ are indexed by partitions of n. The tensor product of two indecomposable representations $V_\lambda \otimes V_\mu$ (with diagonal action of $\mathfrak{S}_n$) decomposes into a direct sum of indecomposable representations of $\mathfrak{S}_n$, and in this case it is an open problem to describe the summands and their multiplicities. The problem naturally generalizes to any finite group.

A representation of the loop quiver

$$\tilde{A}_0 = \bullet \circlearrowright$$

is simply a vector space with an endomorphism, so the category $\mathrm{Rep}_K(\tilde{A}_0)$ is equivalent to the category of finite dimensional $K[x]$-modules. When K is algebraically closed, we have the Jordan normal form for representations of $\tilde{A}_0$. This Clebsch-Gordan problem was studied (over $\mathbb{C}$) as far back as the 1930's by Aitken [3] and Roth [51], whose results were generalized and simplified by D.E. Littlewood [42]. These tensor product multiplicities have been independently rediscovered a number of times since then in various contexts and levels of generality. Martsinkovsky and Vlassov considered the above situation in the language of $K[x]$-modules, and they also worked with another multiplicative structure on representations [44]. Herschend gave a solution to the problem for not only the loop quiver, but all quivers of extended A-type (i.e. with underlying graph

$$\tilde{A}_n = \begin{tikzcd}[arrows=dash] & \bullet \arrow[dl] \arrow[drr] & & \\ \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \end{tikzcd}$$

and any orientation) [30]. There has been recent work on the problem for K of positive characteristic by Iima and Iwamatsu [35].

For a quiver Q which is not of Dynkin or extended Dynkin type, the problem of classifying its indecomposable representations is unsolved and very difficult, to say the least. Such a quiver is said to be of "wild representation type" [19] and has families of indecomposable representations depending on arbitrarily large numbers of parameters in the base field. In fact, classifying the representations of any wild quiver would encompass a solution to the classical problem of classifying pairs of matrices up to simultaneous conjugation. This limits the effectiveness of an enumerative approach to studying tensor products of quiver representations, in contrast with the cases surveyed above. Alternatively, we approach the problem by placing the representations of Q inside a ring $R(Q)$, in which addition corresponds to direct sum and multiplication corresponds to tensor product. Analyzing the properties of $R(Q)$ (e.g. ideals, idempotents, nilpotents) gives a way of stating and approaching problems involving tensor products of quiver representations even in the absence of an explicit description of the isomorphism classes in $\mathrm{Rep}(Q)$.

More precisely, direct sum and tensor product endow the set of isomorphism classes in $\mathrm{Rep}(Q)$ with a semiring structure. There is an associated ring $R(Q)$, called the **representation ring** of Q, which is commutative with identity element $\mathbb{I}_Q \in \mathrm{Rep}(Q)$, where we define $(\mathbb{I}_Q)_x := K$ and $(\mathbb{I}_Q)_a := id$ for all vertices x and arrows a (cf. §2.1.3). Grothendieck introduced this technique, applied in a slightly modified form to the category of coherent sheaves on an algebraic variety, to prove his generalization of the Riemann-Roch theorem [10]. Similar constructions have since appeared in equivariant K-theory [52], and the study of both Lie groups [53] [23, § 23.2] and finite groups [54, §12]. For example, the representation ring of $GL_n(\mathbb{C})$ is naturally isomorphic to the ring of symmetric functions in n variables, localized at $s_{(1^n)}$ (the character of the determinant representation) [23, p. 379]. The isomorphism is given by the character map, which sends the indecomposable representation V_λ to the Schur function s_λ.

Additionally, we can define **Schur functors** on $\mathrm{Rep}(Q)$ by letting the standard Schur functors for vector spaces act pointwise at each vertex, inducing a λ-ring structure on $R(Q)$ (cf. §2.1.5). This structure was studied for representation rings of simply connected Lie groups in [29], and D.E. Littlewood has given generating functions for the decomposition of any Schur functor applied to a complex representation of the quiver $\tilde{A}_0$ in [42]. We will examine these functors only briefly in this thesis, giving one result about them. The author is unaware of any other work on Schur functors of quiver representations.

We restrict our discussion to representation rings of quivers from here on. The structure of $R(Q)$ has been determined by Herschend for Q of type $A_n, D_n, \tilde{A}_n$, and E_6. In the Dynkin cases [34, 32], direct computation allows for a solution of the problem since there are only finitely many indecomposable representations of a given Dynkin quiver (Theorem 2.1). For the extended Dynkin quiver $\tilde{A}_n$, direct computation gave one solution [30] (the indecomposables are still easily described), and it was shown in [31] how part of the solution can be streamlined using Galois coverings.

A general way to get information about $R(Q)$ for an arbitrary quiver is to study homomorphisms from $R(Q)$ to other rings (usually $\mathbb{Z}$). For example, when

$$Q = A_2\colon\ \bullet \longrightarrow \bullet\ ,$$

the objects of the category $\mathrm{Rep}(Q)$ are pairs of vector spaces with a linear map between them: $V_1 \xrightarrow{V_a} V_2$. In this case, the map

$$R(Q) \to \mathbb{Z}^3 \qquad V \mapsto (\dim V_1, \dim V_2, \operatorname{rank} V_a)$$

is an isomorphism of rings because these three numbers completely determine an object of $\operatorname{Rep}(Q)$ up to isomorphism, and each respects direct sum and tensor product.

One way of constructing such homomorphisms for more general quivers is through **tensor functors**, which are essentially functors which commute with direct sum and tensor product (cf. §2.1.4). For example, let p be a path in some quiver Q, and denote by $K\text{-}\operatorname{mod}$ the category of finite dimensional K-vector spaces (or sometimes, $\operatorname{Rep}_K(A_1)$, where A_1 is the quiver with one vertex and no arrows). Then the functor

$$\operatorname{Im}_p\colon \operatorname{Rep}(Q) \to K\text{-}\operatorname{mod} \qquad V \mapsto \operatorname{Im} V_p$$

is a tensor functor, and the dimension of the output (i.e. $\operatorname{rank} V_p$) is a numerical invariant of V which multiplies with respect to tensor product. In particular, the trivial path ε_x at a vertex x corresponds to the functor $\operatorname{Im}_{\varepsilon_x}(V) = V_x$, which picks out the vector space assigned to x by V. But in general these are not the only tensor functors on Q. Motivated by this example, we will call a tensor functor from $\operatorname{Rep}(Q)$ to $K\text{-}\operatorname{mod}$ a **rank functor** on Q.

A tensor functor $F\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q')$ induces a ring homomorphism $f\colon R(Q) \to R(Q')$, which is a homomorphism of λ-rings in characteristic 0 (Theorem 2.16). In particular, rank functors on Q induce ring homomorphisms $R(Q) \to R(A_1) = \mathbb{Z}$ via the isomorphism $R(A_1) \cong \mathbb{Z}$ which identifies a vector space with its dimension.

Given any quiver Q, we will construct in Section 2.2.1 a **global tensor functor** $\mathscr{R}_Q$ whose properties we summarize here:

Theorem 1.1. *Let Q by any connected quiver. The functor*

$$\mathscr{R}_Q\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q)$$

has the following properties:

(a) *It commutes with direct sum, tensor product, Schur functors (when* $\operatorname{char} K = 0$*), and duality; we also have that* $\mathscr{R}_Q(\mathbb{I}_Q) = \mathbb{I}_Q$*. (Proposition 2.22, Theorem 2.16, Theorem 2.36)*

(b) *Let* $V \in \operatorname{Rep}(Q)$*. Then for every arrow* a *of* Q*, the linear map* $(\mathscr{R}_Q(V))_a$ *is an isomorphism. Hence the isomorphism class of the functor*

$$\begin{aligned} \operatorname{rank}_Q \colon \operatorname{Rep}(Q) &\to K\text{-mod} \\ V &\mapsto \mathscr{R}_Q(V)_x \end{aligned}$$

is independent of the vertex x*; we call this functor the* ***global rank functor*** *of* Q*. (Proposition 2.22)*

(c) *When* Q *is a tree and* $V \in \operatorname{Rep}(Q)$*, the representation* $\mathscr{R}_Q(V)$ *is isomorphic to a direct summand of* V*. More precisely, for any indecomposable representation* W *of* Q*, we have*

$$\mathscr{R}_Q(W) = \begin{cases} W & \text{if } W \simeq \mathbb{I}_Q \\ 0 & \text{if } W \not\simeq \mathbb{I}_Q \end{cases}.$$

(Theorem 2.32)

Via restriction, the global rank functors of subquivers $P \subseteq Q$ give more rank functors on Q; for example, when P is a path p, considered as a subquiver of Q, we get the functors Im_p described above. More generally, for any map of directed graphs $\alpha \colon Q' \to Q$, the global rank functor of Q' can be pushed forward along α (cf. §2.3.1) to give a rank functor on Q. Sometimes, a well-chosen α gives a rank functor on Q that does not come from the global rank functor of any subquiver of Q (cf. §2.3.2). The functions on $R(Q)$ induced by such functors are called **rank functions**.

In Chapter 3, we will apply the above tools to a particular class of quivers for which we can explicitly compute all rank functions. A **rooted tree quiver** is a directed graph Q, whose underlying graph is a tree, and which has a unique sink called the **root** of Q. It should also be noted that the main results of this thesis hold (with minor changes in terminology) for a quiver Q which is a tree with a unique *source*. In that case, Q^{op} (the quiver obtained by switching the heads and tails of all arrows) is a tree with a unique sink, and the standard duality between representations of Q and representations of Q^{op} induces a ring isomorphism $R(Q) \cong R(Q^{op})$. Global rank functors commute with duality also, so the methods used for the unique sink case can be applied in a straightforward way to treat the unique source case.

The main result on rooted trees has two equivalent formulations, the first of which (Theorem 3.39) can be stated in a simplified form here:

Theorem. *When Q is a rooted tree quiver, $R(Q)_{red}$ is generated as a $\mathbb{Z}$-module by a finite set of explicit representations of Q. (Here A_{red} is the reduction of a ring A, that is, A modulo its ideal of nilpotent elements.)*

This theorem has an equivalent formulation (Theorem 3.40) as a splitting principle for large tensor powers $V^{\otimes n}$ of a fixed representation V, which makes no mention of the representation ring.

Chapter 3 is organized as follows. Section 3.1 establishes basic tools for studying representations of a rooted tree quiver Q via maps of directed graphs. This leads to the construction of various distinct rank functors on Q within a combinatorial framework. The focus is shifted from rank functors to representations of Q in Section 3.2, by constructing a set of "reduced" representations that are in some sense dual to the rank functors of the previous section. Then the combinatorics of these rank functors can be utilized to obtain information about tensor products of reduced representations, and morphisms between them. In Section 3.3, we make use of the properties of reduced representations to study representation rings. First, we give a product decomposition of $R(Q)$ that allows us to reduce questions about tensor products on any quiver (not just rooted trees) to tensor products of representations with full support. Then by introducing a property of representations of rooted tree quivers which generalizes the support of a representation, we refine this direct product decomposition of $R(Q)$.

The two theorems mentioned directly above on the representation rings of rooted tree quivers are stated and proven in Section 3.4. First, we show the equivalence of the two theorems, then prove the result by induction on the "complexity" of a rooted tree (cf. §3.4.2). There are two cases in the proof: one is essentially combinatorial, using the representation ring form of the result as the induction hypothesis; the other is essentially computational, using the splitting principle form of the result as the induction hypothesis. In Section 3.5, we introduce the name **finite multiplicative type** for a quiver whose reduced representation ring is module finite over $\mathbb{Z}$. Having demonstrated the existence of a large class of such quivers (the main result), it is then natural to try to classify all of them. To this end, we show that the class of quivers of finite multiplicative type is minor closed, and can only include trees; but we also give an example of a tree quiver (of tame representation type even) which is not of finite multiplicative type. By an application of Kruskal's Tree Theorem, we will see that this property can be characterized by a *finite* set of forbidden minors.

CHAPTER 2

Rank functors of quivers

2.1 Preliminaries

2.1.1 Basic definitions

Throughout, $Q = (Q_\bullet, Q_\to, t, h)$ is a quiver on a finite vertex set $Q_\bullet$ with finite arrow set $Q_\to$. The maps

$$t, h : Q_\to \to Q_\bullet$$

give the "tail" and "head" of an arrow, respectively. We allow Q to have oriented cycles, but for simplicity we will assume that Q is connected. A **subquiver** $P \subseteq Q$ will also always be assumed to be connected. We fix a field K of any characteristic. A **representation** of a quiver Q is a collection of finite dimensional K-vector spaces $\{V_x\}_{x\in Q_\bullet}$, and linear maps $\{V_a : V_{ta} \to V_{ha}\}_{a\in Q_\to}$. When $p = a_n \cdots a_2 a_1$ is a **path** in Q (where paths are read from right to left), we write $V_p := V_{a_n} \cdots V_{a_2} V_{a_1}$. A **morphism** $\varphi\colon V \to W$ between representations of a quiver Q is given by specifying a linear map at each vertex

$$\{\varphi_x\colon V_x \to W_x\}_{x\in Q_\bullet}$$

such that these maps commute with the maps assigned to the arrows in V and W, that is,

$$\varphi_{ha} \circ V_a = W_a \circ \varphi_{ta}$$

for $a \in Q_\to$. We denote by $\operatorname{Rep}(Q)$ the category of representations of Q. The **dimension vector** of a representation V, written $\underline{\dim}\, V \in \mathbb{N}^{Q_\bullet}$, is defined by $(\underline{\dim}\, V)_x := \dim_K V_x$, and the **support** of V is the set

$$\operatorname{supp} V := \{x \in Q_\bullet \mid V_x \neq 0\}.$$

We say that Q is a **tree** if the underlying undirected graph is a tree (i.e., if removing any edge makes the graph disconnected). The **opposite quiver** Q^{op} of a quiver $Q = (Q_\bullet, Q_\to, t, h)$ is given by reversing the orientation of all arrows, so $Q^{op} = (Q_\bullet, Q_\to, h, t)$. The categories $\operatorname{Rep}(Q)^{op}$ and $\operatorname{Rep}(Q^{op})$ are equivalent. Vector space duals are denoted by superscript *, and the **duality functor** on $\operatorname{Rep}(Q)$ by

$$D\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q^{op})$$

$$(DV)_x = V_x^* \qquad (DV)_a = V_a^*$$

for $V \in \operatorname{Rep}(Q)$, $x \in Q_\bullet$ and $a \in Q_\to$.

A quiver Q defines a K-algebra KQ, called the **path algebra** of Q (over K). As a vector space, it is free on the paths in Q, and multiplication of two paths p, q is given by by the path pq if the head of p is equal to the tail of q, and 0 otherwise. Note that we must include "trivial paths" ε_x for each vertex x, which have both head and tail x and traverse no arrows. These are idempotents in KQ. There is a standard equivalence of categories between $\operatorname{Rep}_K(Q)$ and KQ-mod, the category of finite dimensional right KQ-modules. With this, it is immediate that $\operatorname{Rep}(Q)$ is an abelian category, and it is easy to see that the kernel and image of a morphism $\varphi = \{\varphi_x\}_{x\in Q_\bullet}$ in the categorial sense can be taken "pointwise." That is, $\ker\varphi = \{\ker\varphi_x\}_{x\in Q_\bullet}$ and similarly for images. An isomorphism of quiver representations is a morphism which is an isomorphism of vector spaces at each vertex. Of particular importance in this thesis will be direct sums of representations, which are also taken pointwise:

$$(V \oplus W)_x := V_x \oplus W_x \qquad x \in Q_\bullet$$

$$(V \oplus W)_a := V_a \oplus W_a \qquad a \in Q_\to.$$

Suppose we have a quiver representation explicitly described by matrices with respect to some basis at each vertex. Then decomposing it as a direct sum of two representations amounts to finding a change of basis at each vertex which puts all the matrices in block forms, subject to some compatibility conditions on the sizes of the blocks. A representation V is said to be **indecomposable** if whenever it is isomorphic to a direct sum of finitely many represenations, one of those representations is isomorphic to V itself.

The following theorem of Gabriel initiated the study of interactions between quiver representations and combinatorics.

Theorem 2.1 (Gabriel [24])**.** *Let K be any field, and Q a connected quiver. Then Q has only finitely many indecomposable representations if and only if Q is a Dynkin quiver (i.e. the underlying undirected graph of Q is a Dynkin diagram of A, D, or E type). Furthermore, the map sending a representation V to its dimension vector $\underline{\dim}\, V \in \mathbb{N}^{Q_\bullet} \subset \mathbb{R}^{Q_\bullet}$ restricts to a bijection between the indecomposable representations of Q and the positive roots of the root system associated to Q.*

In particular, the simple root labeled by a vertex x of Q corresponds to the simple representation $S(x)$ of Q concentrated at x. That is, $S(x) \in \mathrm{Rep}(Q)$ is given by

$$(S(x))_y := \begin{cases} K & y = x \\ 0 & \text{else} \end{cases}$$

for each vertex y, and of course each map in $S(x)$ must be zero. For Dynkin quivers, the indecomposable representations can all be presented using only matrices with entries 0 and 1, which makes it easy to see that their classification does not depend on the base field.

Less than ten years later, Kac generalized Gabriel's theorem to arbitrary quivers. We state the theorem for K algebraically closed of characteristic 0, for simplicity.

Theorem 2.2 (Kac [36])**.** *Let Q be a quiver without oriented cycles, K an algebraically closed field of characteristic 0, and Δ the root system of the Kac-Moody Lie algebra associated to the underlying graph of Q. Then for any dimension vector α, we have:*

a) If α is not a root, there is no indecomposable representation of Q of dimension α.

b) There exists a unique indecomposable representation of Q (over K) of dimension α if and only if α is a positive real root.

c) There exist infinitely many nonisomorphic indecomposable representations of dimension α when α is a positive imaginary root; the number of K-parameters in this family of representations is independent of the orientation of Q.

The relevance of this theorem in our context is the interpretation that, at least when K is finite or algebraically closed, some of the fundamental additive structure of $\mathrm{Rep}_K(Q)$ is independent of the orientation of the arrows in Q.

2.1.2 Tensor products of quiver representations

Definition 2.3. The **tensor product** of two quiver representations $V, W \in \mathrm{Rep}(Q)$ is defined pointwise:

$$(V \otimes W)_x := V_x \otimes W_x \qquad x \in Q_\bullet$$

$$(V \otimes W)_a := V_a \otimes W_a \qquad a \in Q_\to.$$

An interesting, more general discussion of the tensor product of quiver representations can be found in [33]. We introduce the following notation.

Definition 2.4. For any quiver Q, we define the **identity representation** $\mathbb{I}_Q$ of Q by

$$(\mathbb{I}_Q)_x = K \qquad\qquad (\mathbb{I}_Q)_a = id_K$$

for all $x \in Q_\bullet$ and $a \in Q_\to$. (The subscript Q is often omitted.)

Note that $\mathbb{I} \otimes V \cong V$ for any representation V. For the reader's convenience, we list some similarities and differences between the tensor product of quiver representations and tensor product of vector spaces. A more complete treatment, in the language of tensor categories, will be given in Section 2.1.4.

(a) Quiver tensor product is a bifunctor

$$\otimes\colon \operatorname{Rep}(Q) \times \operatorname{Rep}(Q) \to \operatorname{Rep}(Q)$$

which is associative, commutative, and distributes over direct sum of quiver representations.

(b) If we fix a representation V, the functor $T_V\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q)$ defined by setting $T_V(W) = V \otimes W$ is exact. This follows from the fact that exactness of a sequence of morphisms of quiver representations can be checked at each vertex. However, T_V is not faithful, in general. For example, when Q has no oriented cycles, the tensor product of two non-isomorphic simple representations is 0.

(c) Quiver tensor product commutes with duality: $D(V \otimes W) \cong DV \otimes DW$.

We will see in the next examples that the tensor product structure of $\operatorname{Rep}(Q)$ *does* depend on the orientation of Q.

Notation 2.5. We often denote a representation of a quiver by its dimension vector, drawn schematically in the shape of the quiver, when that representation is in the unique indecomposable isomorphism class of that dimension.

Example 2.6. Let Q be the 3-subspace quiver,

$$Q = \begin{tikzcd} & \bullet & \\ \bullet \arrow[ur] & \bullet \arrow[u] & \bullet \arrow[ul] \end{tikzcd}$$

and let V be the indecomposable representation

$$V = \begin{tikzcd} & K^2 & \\ K \arrow[ur, "A"] & K \arrow[u, "B"] & K \arrow[ul, "C"'] \end{tikzcd} \qquad A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then we can calculate

$$V \otimes V \simeq \begin{smallmatrix} & 1 & \\ 1 & 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & 0 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \end{smallmatrix}$$

since $\{m, n\}$ linearly independent in some vector space L implies that $\{m \otimes m,\ n \otimes n,\ (m+n) \otimes (m+n)\}$ is linearly independent in $L \otimes L$.

Example 2.7. However, if we change the orientation by flipping one of the arrows,

$$Q = \begin{tikzcd} \bullet \arrow[dr] & & \\ & \bullet \arrow[r] & \bullet \\ \bullet \arrow[ur] & & \end{tikzcd}$$

and let W be the indecomposable representation of the same dimension vector as before,

$$W = \begin{tikzcd} K \arrow[dr, "A"] & & \\ & K^2 \arrow[r, "C"] & K \\ K \arrow[ur, "B"] & & \end{tikzcd} \qquad A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

then in this case we find that

$$W \otimes W \simeq W \oplus \begin{smallmatrix} 0 & & \\ & 1 & 0 \\ 0 & & \end{smallmatrix} \oplus \begin{smallmatrix} 0 & & \\ & 1 & 0 \\ 0 & & \end{smallmatrix}.$$

This can be directly calculated by writing down matrices, taking their tensor products, then finding the correct change of basis to put them in block form. However, in Section 2.3.2, we will use rank functors to determine this decomposition without calculating any tensor products or change of basis.

2.1.3 The representation ring of a quiver

We now give the precise definition of the representation ring of a quiver.

Definition 2.8. Let $[V]$ denote the isomorphism class of a representation V. Then define $R(Q)$ to be the free abelian group generated by isomorphism classes of representations of Q, modulo the subgroup generated by all $[V \oplus W] - [V] - [W]$. The operation

$$[V] \cdot [W] := [V \otimes W] \qquad \text{for } V,\ W \in \mathrm{Rep}(Q)$$

induces a well-defined multiplication on $R(Q)$, making $R(Q)$ into a commutative ring with identity $[\mathbb{I}_Q]$, called the **representation ring** of Q.

The ring $R(Q)$ generally depends on the base field K, but we omit the K from the notation. Also we usually omit the brackets [ ] and just refer to representations of Q as elements of $R(Q)$.

Although we introduce virtual representations to form this ring, every element $r \in R(Q)$ can be written as a formal difference

$$r = V - W \qquad \text{with } V,\ W \in \mathrm{Rep}(Q).$$

Then any additive (resp. multiplicative) relation $z = x + y$ (resp. $z = xy$) can be rewritten to give some isomorphism of actual representations of Q (see (a) below).

The Grothendieck group of $\mathrm{Rep}(Q)$ [5, p. 87] is $R(Q)/\mathfrak{a}$, where $\mathfrak{a}$ is the subgroup generated by elements

$$[V] - [U] - [W]$$

for all short exact sequences

$$0 \to U \to V \to W \to 0.$$

We do not work modulo short exact sequences, because this loses too much information in our setting. For example, the Grothendieck group of a quiver Q without oriented cycles is always isomorphic to $\mathbb{Z}^{Q_\bullet}$, and the image of a representation in the Grothendieck group is just its dimension vector.

We note some properties of the representation ring.

(a) By the Krull-Schmidt property of $\mathrm{Rep}(Q)$, the ring $R(Q)$ is free as a $\mathbb{Z}$-module, with basis given by the isomorphism classes of indecomposable representations of Q.

(b) By Theorem 2.1, $R(Q)$ is a finitely generated $\mathbb{Z}$-module if and only if Q is a Dynkin quiver. In this case,

$$\operatorname{rank}_{\mathbb{Z}} R(Q) = \#\{\text{indecomposables in } \mathrm{Rep}(Q)\} = \#\{\text{positive roots of } Q\}$$

is explicitly known from the theory of root systems.

(c) Write $[\mathrm{Rep}(Q)] := \{[V] \in R(Q) \mid V \in \mathrm{Rep}(Q)\}$ for the semiring of "actual representations" in $R(Q)$. The ring $R(Q)$ has the following universal property: if A is a ring, and $f\colon [\mathrm{Rep}(Q)] \to A$ a map of semirings, then f uniquely extends to a ring homomorphism $f\colon R(Q) \to A$ (which we usually denote by the same symbol). That is, the functor sending the semiring $[\mathrm{Rep}(Q)]$ to the ring $R(Q)$ is left adjoint to the forgetful functor from rings to semirings.

Remark 2.9. One can generalize the theory of quiver representations and study quivers with relations [5, §II.2]. A **relation** on a quiver is a K-linear combination of paths which start and end at the same vertex, and an **ideal of relations**, usually denoted $\mathcal{I}$, is a collection of such relations which form an ideal in the path algebra KQ. One usually imposes some technical conditions on the ideal and only considers "admissible" ideals to get a nice theory. The pair $(Q, \mathcal{I})$ is called a **bound quiver**, and a representation of $(Q, \mathcal{I})$ is just a representation of Q which satisfies the relations in $\mathcal{I}$. More precisely, $V \in \operatorname{Rep}(Q)$ **satisfies** a relation $\sum \lambda_i p_i$ (where each $\lambda_i \in K$ and each p_i is a path in Q) if $\sum \lambda_i V_{p_i} = 0$ as a linear map. The category $\operatorname{Rep}(Q, \mathcal{I})$ of representations of $(Q, \mathcal{I})$ is a full subcategory of $\operatorname{Rep}(Q)$.

This setup is quite versatile. For an arbitrary associative algebra A which is finite dimensional over an algebraically closed field K, denote by mod-A the category of finite dimensional right A-modules. Then one can construct a bound quiver $(Q, \mathcal{I})$ such that

$$\operatorname{Rep}_K(Q, \mathcal{I}) \cong \text{mod-}A,$$

where '$\cong$' denotes equivalence of categories here [5, §II.3].

Now if V, W are representations of a bound quiver $(Q, \mathcal{I})$, then $V \otimes W$ might *not* be a representation of $(Q, \mathcal{I})$. For example, let $(Q, \mathcal{I})$ be the bound quiver

$$Q = \bullet \overset{\xrightarrow{a}}{\underset{\xrightarrow{c}}{\xrightarrow{b}}} \bullet \qquad \mathcal{I} = \langle a + b - c \rangle$$

and assume char $K \neq 2$. Then for the representation

$$V = K \overset{\xrightarrow{(1)}}{\underset{\xrightarrow{(2)}}{\xrightarrow{(1)}}} K$$

we have that $V_c^{\otimes 2} = (4) \neq V_a^{\otimes 2} + V_b^{\otimes 2} = (1) + (1) = (2)$, so $V \otimes V \notin \operatorname{Rep}(Q, \mathcal{I})$.

However, if $\mathcal{I}$ is generated by commutativity relations (that is, relations of the form $p - q$ for paths p, q) then the representations of $(Q, \mathcal{I})$ do generate a subalgebra of $R(Q)$. If $\mathcal{I}$ is generated by zero relations (relations of the form $p = 0$ for p a path), then representations of $(Q, \mathcal{I})$ do not generate a subalgebra because the identity element $\mathbb{I} \notin \mathrm{Rep}(Q, \mathcal{I})$. But because the tensor product of any map with a zero map is zero, these representations generate an ideal in $R(Q)$.

2.1.4 Tensor categories

One way to construct homomorphisms between representation rings of quivers is through functors between their categories of representations which commute with direct sum and tensor product. Precise definitions and properties of these functors can be stated concisely using the language of tensor categories, which we summarize here. Although this subsection is necessary for technical purposes, we hope that it will not obscure the main ideas of the thesis, which can be understood purely in terms of quiver representations.

The category $\mathrm{Rep}(Q)$ is an abelian K-category [5, p. 407–409], so the spaces $\mathrm{Hom}_Q(V, W)$ are K-vector spaces, and we will be interested in functors that preserve this structure. We will use the following characterization: a functor

$$F\colon\ \mathrm{Rep}(Q) \to \mathrm{Rep}(Q')$$

is **additive** if and only if it **preserves direct sums**, meaning that each direct sum in $\mathrm{Rep}(Q)$ with insertion maps i_V, i_W and projection maps p_V, p_W,

$$\begin{tikzcd} V \arrow[r, shift left, "i_V"] & V \oplus W \arrow[l, shift left, "p_V"] \arrow[r, shift right, "p_W"'] & W \arrow[l, shift right, "i_W"'] \end{tikzcd},$$

is taken to an isomorphism in $\mathrm{Rep}(Q')$

$$F(V) \oplus F(W) \cong F(V \oplus W)$$

with insertion maps $F(i_V)$, $F(i_W)$ and projection maps $F(p_V)$, $F(p_W)$ [46, p. 67]. Subfunctors and quotient functors of additive functors are additive [46, p. 81].

The additive bifunctor $\otimes$ (along with identity object $\mathbb{I}$) endows $\mathrm{Rep}(Q)$ with the structure of a (relaxed) symmetric monoidal category [43, VII], or simply a **tensor category** [17]. This amounts to saying that the tensor product is functorial, satisfying some associativity and commutativity conditions, and has an identity object. We summarize the definition of a tensor category, following [17, p. 104–105], but omitting some technicalities relating to associativity. Consider a pair $(\mathcal{C}, \otimes)$, where $\mathcal{C}$ is a category and $\otimes$ is a functor

$$\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \qquad (X, Y) \mapsto X \otimes Y.$$

Now let ϕ and ψ be functorial isomorphisms

$$\phi_{X,Y,Z}\colon X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z \qquad \psi_{X,Y}\colon X \otimes Y \xrightarrow{\cong} Y \otimes X.$$

We say that ϕ is an **associativity constraint** for $(\mathcal{C}, \otimes)$ if ϕ satisfies a "pentagon axiom", and that ψ is a **commutativity constraint** if $\psi_{Y,X} \circ \psi_{X,Y} = id_{X \otimes Y}$. Such constraints are compatible with one another if they satisfy a "hexagon axiom". The axioms omitted here are that certain diagrams of functorial isomorphisms involving ϕ and ψ are commutative. An **identity object** for $(\mathcal{C}, \otimes)$ is an object U of $\mathcal{C}$ and an isomorphism

$$u\colon U \to U \otimes U$$

such that the functor

$$T_U\colon \mathcal{C} \to \mathcal{C} \qquad X \mapsto U \otimes X$$

is an equivalence of categories. A system $(\mathcal{C}, \otimes, \phi, \psi)$ as above will be called a **tensor category** if the constraints are compatible and there is an identity object.

The category of finite dimensional vector spaces, with the standard tensor product and standard associativity and commutativity isomorphisms, is a tensor category.

For an arbitrary quiver Q, let us note once and for all that a morphism $\alpha = \{\alpha_x\}_{x \in Q_\bullet}$ is an isomorphism if and only if each α_x is an isomorphism, and similarly, commutativity of diagrams can be checked at each vertex of Q. Then since our tensor product $\otimes$ is just the standard tensor product of finite dimensional vector spaces at each vertex, defining associativity and commutativity constraints pointwise equips $\mathrm{Rep}(Q)$ with the structure of a tensor category (with identity object $\mathbb{I}$).

A **tensor functor** [17, p. 113–114] is a pair (F, c) consisting of an additive functor

$$F\colon \mathrm{Rep}(Q) \to \mathrm{Rep}(Q')$$

and a functorial isomorphism

$$c_{X,Y}\colon F(X) \otimes F(Y) \cong F(X \otimes Y)$$

such that:

(a) The isomorphism c is compatible with associativity, expressed by another "hexagon axiom".

(b) c is compatible with commutativity, that is, $c_{Y,X} \circ \psi_{F(X),F(Y)} \cong F(\psi_{X,Y}) \circ c_{X,Y}$.

(c) $F(\mathbb{I}_Q) \cong \mathbb{I}_{Q'}$.

If we relax the condition that c be an isomorphism by simply requiring the existence of either

$$c_{X,Y}\colon F(X) \otimes F(Y) \to F(X \otimes Y) \qquad \text{or} \qquad c_{X,Y}\colon F(X \otimes Y) \to F(X) \otimes F(Y)$$

satisfying (a), (b), and (c) (appropriately modified), we will say that (F, c) is a **weak tensor functor**. Note that the definition of a tensor functor covers both covariant and contravariant functors, since all the maps appearing in the definition are isomorphisms. For example, the duality functor D is a tensor functor.

Remark 2.10. The technical axioms involving the associativity and commutativity constraints give us the following facts about tensor categories and tensor functors [43, VII.2], [17, p. 106–108, 113–115]:

(a) There is an essentially unique way to extend the tensor product to any finite family of objects of $\mathcal{C}$,

$$\bigotimes_{i=1}^{k} \colon \mathcal{C}^k \to \mathcal{C} \qquad (X_i) \mapsto \bigotimes_{i=1}^{k} X_i.$$

(b) There is an action of the symmetric group $\mathfrak{S}_k$ on tensor products

$$\sigma \cdot \bigotimes_{i=1}^{k} X_i := \bigotimes_{i=1}^{k} X_{\sigma(i)} \qquad \sigma \in \mathfrak{S}_k.$$

(c) For any tensor functor F, there is an isomorphism of functors

$$\bigotimes_{i=1}^{k} F(X_i) \cong F\left(\bigotimes_{i=1}^{k} X_i\right)$$

which is $\mathfrak{S}_k$-equivariant, that is,

$$F\left(\sigma \cdot \bigotimes_{i=1}^{k} X_i\right) \cong \sigma \cdot \bigotimes_{i=1}^{k} F\left(X_i\right).$$

Definition 2.11. A **rank functor** on Q is a tensor functor from $\mathrm{Rep}(Q)$ to $\mathrm{Rep}(A_1) = K\text{-}\mathrm{mod}$.

A tensor functor $F\colon \mathrm{Rep}(Q) \to \mathrm{Rep}(Q')$ between categories of quiver representations induces a homomorphism of representation rings $f\colon R(Q) \to R(Q')$, defined on representations by the value of F and extended by linearity. In particular, a rank functor on Q induces a ring homomorphism $R(Q) \to \mathbb{Z}$. Such homomorphism can be useful in determining the structure of $R(Q)$.

2.1.5 Schur functors on quiver representations

For any partition λ, there is an associated functor S^λ from the category of $GL_n(K)$ representations to itself, called a **Schur functor** [23, p. 76]. These functors exist and can be described independently of the field K. If $K = \mathbb{C}$ and $V \simeq \mathbb{C}^n$ is the standard representation of $G = GL_n(\mathbb{C})$, then the irreducible representations of G are precisely all $S^\lambda V$ as λ varies over all partitions with less than or equal to n parts. (We omit the parentheses around the argument of S^λ when possible).

The Schur functors act on quiver representations pointwise:

$$\begin{aligned} \left(S^\lambda V\right)_x &:= S^\lambda\left(V_x\right) \qquad && x \in Q_\bullet \\ \left(S^\lambda V\right)_a &:= S^\lambda\left(V_a\right) \qquad && a \in Q_\to \end{aligned}$$

for $V \in \operatorname{Rep}(Q)$. From the functoriality of S^λ on vector spaces, it follows that this defines a functor from $\operatorname{Rep}(Q)$ to $\operatorname{Rep}(Q)$.

Example 2.12. Using V from Example 2.6, we get

$$S^2 V \simeq \begin{smallmatrix} 1 \\ 100 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 010 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 001 \end{smallmatrix} \qquad \text{and} \qquad S^{(1,1)} V = \bigwedge^2 V \simeq \begin{smallmatrix} 1 \\ 000 \end{smallmatrix}.$$

In particular, note that $S^2 V$ is not even indecomposable, in contrast with the situation for GL_n representations.

Now suppose that $\operatorname{char} K = 0$ in the remainder of this subsection. For a vector space V_x, we have the Schur decomposition of a tensor power of V_x [23, p. 87]:

$$\bigotimes^k V_x \simeq \bigoplus_{\lambda \vdash k} S^\lambda(V_x) \otimes G_\lambda \tag{2.1}$$

where $\lambda \vdash k$ means that λ is a partition of k, and G_λ is the irreducible representation of the symmetric group $\mathfrak{S}_k$ corresponding to λ. Because this is functorial in V_x, we expect to be able to utilize this decomposition for quiver representations.

Definition 2.13. A **Q-$\mathfrak{S}_k$-representation** is a representation $V \in \mathrm{Rep}(Q)$ such that each V_x is a representation of $\mathfrak{S}_k$ and each map V_a is $\mathfrak{S}_k$-equivariant. A morphism of Q-$\mathfrak{S}_k$-representations is just a morphism of quiver representations $\{\varphi_x\}$ such that each φ_x is $\mathfrak{S}_k$-equivariant.

The Q-$\mathfrak{S}_k$-representations form a category, so we have notions of subrepresentations, irreducible objects, and so forth.

Example 2.14. Any linear representation V of $\mathfrak{S}_k$ gives rise to a Q-$\mathfrak{S}_k$-representation V^Q by setting $V_x^Q = V$ for all vertices x, and $V_a^Q = id$ for all arrows a. Note that if V is an irreducible $\mathfrak{S}_k$ representation, V^Q is an irreducible Q-$\mathfrak{S}_k$-representation, but V^Q is not even necessarily indecomposable in $\mathrm{Rep}(Q)$ or as a representation of $\mathfrak{S}_k$. In fact, we have decompositions

$$V^Q \simeq \bigoplus_{i=1}^{\dim V} \mathbb{I} \qquad \text{and} \qquad V^Q \simeq \bigoplus_{i=1}^{\# Q_\bullet} V \tag{2.2}$$

as a representation of Q, and as a representation of $\mathfrak{S}_k$, respectively.

If V is any representation of Q, then $\bigotimes^k V$ becomes a Q-$\mathfrak{S}_k$-representation by letting $\mathfrak{S}_k$ permute the factors.

Proposition 2.15. *Let Q be a quiver and $V \in \mathrm{Rep}(Q)$. Denote by G_λ the irreducible linear representation of $\mathfrak{S}_k$ corresponding to λ. Then we have a direct sum decomposition*

$$\bigotimes^k V \simeq \bigoplus_{\lambda \vdash k} S^\lambda V \otimes G_\lambda^Q \tag{2.3}$$

as Q-$\mathfrak{S}_k$-representations, which is functorial in V.

Proof. The isomorphism is defined at each vertex by (2.1). Functoriality of (2.1) in V_x implies that (2.3) is an isomorphism in $\mathrm{Rep}(Q)$. Since $\mathfrak{S}_k$ acts trivially on each

$S^\lambda V_x$, and the identity map is $\mathfrak{S}_k$-equivariant, the induced maps $\bigoplus_{\lambda \vdash k} S^\lambda V_a \otimes id_{G_\lambda}$ are evidently $\mathfrak{S}_k$-equivariant, so the right hand side is in fact a Q-$\mathfrak{S}_k$-representation. Then because (2.1) is an isomorphism of $\mathfrak{S}_k$ representations, (2.3) is an isomorphism of Q-$\mathfrak{S}_k$-representations. Functoriality in V follows from the functoriality at each vertex of (2.1). □

Theorem 2.16. *Let* $\operatorname{char} K = 0$. *Then any tensor functor* $F\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q)$ *commutes with the Schur functors. That is, there is an isomorphism of functors*

$$F \circ S^\mu \cong S^\mu \circ F.$$

Proof. We demonstrate the isomorphism for an object $V \in \operatorname{Rep}(Q)$; functoriality in V will be clear at each step of the proof, without explicit mention. Let $k = |\mu|$. By Remark 2.10, there is an isomorphism of Q-$\mathfrak{S}_k$-representations

$$F(\bigotimes^k V) \cong \bigotimes^k FV. \tag{2.4}$$

To prove the theorem, we just write down the Schur decomposition of each side, then try to match up the correct pieces.

Applying Proposition 2.15 to the left hand side of (2.4), we have isomorphisms of Q-$\mathfrak{S}_k$-representations

$$F\left(\bigotimes^k V\right) \cong F\left(\bigoplus_{\lambda \vdash k} S^\lambda V \otimes G_\lambda^Q\right) \cong \bigoplus_{\lambda \vdash k} F\left(S^\lambda V\right) \otimes F(G_\lambda^Q) \cong \bigoplus_{\lambda \vdash k} F\left(S^\lambda V\right) \otimes G_\lambda^Q.$$

Note that $F(G_\lambda^Q) = G_\lambda^Q$ by the first isomorphism of (2.2).

Then applying Proposition 2.15 to the right hand side of (2.4), we have an isomorphism of Q-$\mathfrak{S}_k$-representations

$$\bigotimes^k FV \cong \bigoplus_{\lambda \vdash k} S^\lambda (FV) \otimes G_\lambda^Q$$

and then an isomorphism of Q-$\mathfrak{S}_k$-representations

$$\bigoplus_{\lambda \vdash k} F\left(S^\lambda V\right) \otimes G_\lambda^Q \cong \bigoplus_{\lambda \vdash k} S^\lambda(FV) \otimes G_\lambda^Q.$$

Each G_λ-isotypic component of the left hand side must map to the G_λ-isotypic component of the right hand side, so by dimension count we get isomorphisms of the summands on each side indexed by the same partition λ. This gives an isomorphism in $\mathrm{Rep}(Q)$

$$\bigoplus_{\dim G_\mu} F\left(S^\mu V\right) \simeq F\left(S^\mu V\right) \otimes G_\mu^Q \cong S^\mu(FV) \otimes G_\mu^Q \simeq \bigoplus_{\dim G_\mu} S^\mu(FV)$$

and so by the Krull-Schmidt property of $\mathrm{Rep}(Q)$, we have

$$F\left(S^\mu V\right) \cong S^\mu(FV). \qquad \square$$

The notion of λ-rings was introduced by Grothendieck in [28]. The idea is to define unary operations λ^i on a commutative ring with identity, called "λ-operations", which have the formal properties of exterior power operations. Essentially, one wants to express $\lambda^i(x+y)$, $\lambda^i(xy)$, and $\lambda^i(\lambda^j(x))$ as some universal polynomials in the values of $\lambda^k(x)$ and $\lambda^k(y)$. The reader is referred to [39] and [6] and [23, p. 380] for the definitions and basic properties. For example, the ring of integers is a λ-ring under the operations

$$\lambda^i \colon \mathbb{Z} \to \mathbb{Z} \qquad n \mapsto \binom{n}{i}.$$

In the quiver setting, we can define λ-operations on $R(Q)$ by setting

$$\lambda^i V = \bigwedge^i V$$

for $V \in \mathrm{Rep}(Q)$. For example, this gives the same λ-ring structure as above on $R(A_1) \cong \mathbb{Z}$, since $\dim \bigwedge^i V = \binom{\dim V}{i}$. In the general case, because the exterior power operations on a quiver representation act as the standard exterior powers at each vertex, it is immediate that these operations give $R(Q)$ the structure of a λ-ring.

A homomorphism of λ-rings is just a ring homomorphism that commutes with the λ-operations. In Theorem 2.16 we saw that when $\operatorname{char} K = 0$, tensor functors commute with the Schur operations on quiver representations, so in particular they commute with exterior powers. Hence a homomorphism of representation rings induced from a tensor functor (in characteristic 0) is a λ-ring homomorphism.

2.2 The global tensor functor of a quiver

2.2.1 Construction

We can construct a canonical tensor functor $\mathscr{R}_Q\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q)$ for any quiver Q. First, we introduce some new properties of quiver representations that will be useful in constructing $\mathscr{R}$.

Definition 2.17. Call a representation V of a quiver Q **epimorphic** (resp. **monomorphic**) if V_a is surjective (resp. injective) for each arrow $a \in Q_\to$.

Example 2.18. If Q has no oriented cycles, then injective representations are epimorphic and projective representations are monomorphic. Let P be the projective representation corresponding to a vertex x, and let a be an arrow in Q. For each vertex y, the vector space P_y has a basis given by all paths from x to y, and the maps $P_a\colon P_{ta} \to P_{ha}$ are given by composition with a [5, p. 79]. If p, q are distinct paths from x to ta, then ap, aq are distinct paths from x to ha, so P_a takes the standard basis of P_{ta} to a subset of the standard basis of P_{ha}. Hence each map P_a is injective. The case of injective representations is similar.

For $V \in \operatorname{Rep}(Q)$, the sum of any collection of epimorphic subrepresentations of V is epimorphic; hence V has a unique maximal epimorphic subrepresentation $\mathscr{E}_Q(V)$.

Dually, V also has a maximal monomorphic quotient $\mathscr{M}_Q(V)$, and these are related by a canonical isomorphism of Q^{op} representations

$$D\mathscr{M}_Q(V) \cong \mathscr{E}_{Q^{op}}(DV). \tag{2.5}$$

Example 2.19. Let Q and V be as in Example 2.7. Then we get $\mathscr{E}_Q(V) = 0$, essentially because $\operatorname{Im} A \cap \operatorname{Im} B = 0$, and $\mathscr{M}_Q(V) \simeq \begin{smallmatrix} 1 & & \\ & 1 & 1 \\ 1 & & \end{smallmatrix}$, given by $K^2/\ker C$ at the branch point.

Example 2.20. Let Q be the single loop quiver, so $V \in \operatorname{Rep}(Q)$ is given by a vector space V_0 together with an endomorphism A. Then if V is *indecomposable*, we have

$$\mathscr{E}_Q(V) = \mathscr{M}_Q(V) = \begin{cases} V & \text{if } A \text{ is an isomorphism} \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that for any quiver Q, both $\mathscr{E}_Q$ and $\mathscr{M}_Q$ are covariant functors from $\operatorname{Rep}(Q)$ to $\operatorname{Rep}(Q)$, and (2.5) is a natural isomorphism of functors. Furthermore, $\mathscr{E}_Q$ (resp. $\mathscr{M}_Q$) is a sub- (resp. quotient-) functor of the identity functor; hence each is additive, and there is a natural transformation given by the composition

$$\Phi\colon \mathscr{E}_Q \hookrightarrow id_{\operatorname{Rep}(Q)} \twoheadrightarrow \mathscr{M}_Q.$$

Definition 2.21. The **global tensor functor** of Q is defined to be the image functor of this natural transformation:

$$\mathscr{R}_Q := \operatorname{Im}(\Phi) : \operatorname{Rep}(Q) \to \operatorname{Rep}(Q).$$

(The subscript Q and parentheses around the input are often omitted from all of the above functors.)

We can immediately note some useful properties of $\mathscr{R}_Q$.

Proposition 2.22. *The global tensor functor is additive and commutes with duality. So we have natural isomorphisms of functors*

$$\mathscr{R}_Q V \oplus \mathscr{R}_Q W \cong \mathscr{R}_Q(V \oplus W) \qquad \textit{and} \qquad D \circ \mathscr{R}_Q \cong \mathscr{R}_{Q^{op}} \circ D.$$

Furthermore, for any $V \in \mathrm{Rep}(Q)$, the representation $\mathscr{R}_Q V$ is both epimorphic and monomorphic, i.e., the linear maps $(\mathscr{R}_Q V)_a$ are isomorphisms for all $a \in Q_\to$.

Proof. The functor $\mathscr{R}$ is additive because it is a quotient of the additive functor $\mathscr{E}$. That $\mathscr{R}$ commutes with duality follows from (2.5) and the universal property of an image. For the last statement, we have maps

$$\mathscr{E}V \twoheadrightarrow \mathscr{R}V \hookrightarrow \mathscr{M}V.$$

The conditions to be a morphism of quiver representations imply that a quotient of an epimorphic representation is epimorphic, and a subrepresentation of a monomorphic representation is monomorphic, so $\mathscr{R}V$ is both. □

In particular, the dimension of $(\mathscr{R}_Q V)_x$ is independent of $x \in Q_\bullet$ when Q is connected (as all quivers in this thesis are). We will later prove (Theorem 2.36) that $\mathscr{R}_Q$ is actually a tensor functor, which leads us to define:

Definition 2.23. The **global rank functor** of a quiver Q is

$$\mathrm{rank}_Q := \mathrm{Im}_{\varepsilon_x} \circ \mathscr{R}_Q \colon \mathrm{Rep}(Q) \to K\text{-}\mathrm{mod}\,.$$

If F is any rank functor on a quiver Q, then the induced ring homomorphism

$$f \colon R(Q) \to \mathbb{Z}$$

is called the **rank function** associated to F. Thus the **global rank function** of a connected quiver Q is given by $r_Q(V) = \dim_K(\mathscr{R}_Q V)_x$ for $V \in \mathrm{Rep}(Q)$, and extended by linearity to $R(Q)$. By the remark above, this is independent of the choice of $x \in Q_\bullet$.

We can compute some simple examples of the global tensor functor.

Example 2.24. Let Q be equioriented of type A_3,

$$Q = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet\,.$$

Then for a representation $V = V_1 \xrightarrow{A} V_2 \xrightarrow{B} V_3$, one can compute from the definitions that

$$\mathscr{R}_Q V = \frac{V_1}{\ker BA} \xrightarrow{\overline{A}} \frac{\operatorname{Im} A}{\ker B \cap \operatorname{Im} A} \xrightarrow{\overline{B}} \operatorname{Im} BA$$

so the global rank functor is $\mathrm{rank}_Q \cong \mathrm{Im}_{ba}$, and the associated rank function is $r_Q(V) = \mathrm{rank}(BA)$. This easily generalizes to a quiver of type equioriented A_n.

The global rank function does not always correspond to the rank of some map, but can still be easily described sometimes.

Example 2.25. When Q is the two subspace quiver

$$Q\colon\ \bullet \longrightarrow \bullet \longleftarrow \bullet$$

we can again explicitly compute the global tensor functor. If $V = V_1 \xrightarrow{A} V_2 \xleftarrow{B} V_3$, then

$$\mathscr{R}_Q V = \frac{A^{-1}(\operatorname{Im} B)}{\ker A} \xrightarrow{\overline{A}} \operatorname{Im} A \cap \operatorname{Im} B \xleftarrow{\overline{B}} \frac{B^{-1}(\operatorname{Im} A)}{\ker B}$$

and so $r_Q(V) = \dim_K(\operatorname{Im} A \cap \operatorname{Im} B)$.

When Q has many sinks and sources, the global rank function becomes more cumbersome to write down explicitly:

Example 2.26. Let Q be of type A_4 and the alternating orientation

$$Q = \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet\,.$$

If we write a representation as

$$V = \; V_1 \xrightarrow{A} V_2 \xleftarrow{B} V_3 \xrightarrow{C} V_4$$

then we can compute from the definitions that

$$r_Q(V) = \dim_K \left(\frac{\operatorname{Im} A \cap \operatorname{Im} B}{\operatorname{Im} A \cap B(\ker C)} \right) = \dim_K \left(\frac{B^{-1}(\operatorname{Im} A)}{\ker B + \ker C} \right).$$

Remark 2.27. Since $\mathscr{R}V$ is both a monomorphic quotient of $\mathscr{E}V$, and an epimorphic subrepresentation of $\mathscr{M}V$, the universal properties yield natural transformations

$$\mathscr{M} \circ \mathscr{E} \twoheadrightarrow \mathscr{R} \hookrightarrow \mathscr{E} \circ \mathscr{M},$$

Neither of these are necessarily isomorphisms: let Q and V be as in Example 2.19. Then

$$\mathscr{R}V = 0 \qquad \text{but} \quad \mathscr{E}\left(\mathscr{M}V\right) \simeq {}^{1}_{1}11\,.$$

Dualize to get an analogous example for the other map. (We will not be interested in compositions of $\mathscr{E}, \mathscr{R}$, and $\mathscr{M}$ with one another.)

Remark 2.28. The definitions of the functors $\mathscr{E}, \mathscr{M}$, and $\mathscr{R}$ are natural enough to make sense in a much more general context. We give an outline of the idea in this remark, though we will only apply them in the setting of quiver representations in this thesis. Suppose that $\mathscr{A}$ is an abelian category which has arbitrary sums and intersections of subobjects of fixed object. That is, for an object $X \in \mathscr{A}$ and a family of subobjects $\{X_i \hookrightarrow X\}_{i \in I}$, both

$$\bigcap_{i \in I} X_i \qquad \text{and} \qquad \sum_{i \in I} X_i$$

exist in $\mathscr{A}$ (defined by their usual universal properties). For example, $\mathscr{A}$ could be the category of all modules over some ring, or the category of coherent sheaves on a Noetherian scheme.

Let $\mathscr{C}$ be an arbitrary (small) category. Then the category $\mathrm{Fun}(\mathscr{C}, \mathscr{A})$ of functors from $\mathscr{C}$ to $\mathscr{A}$ is abelian and has sums and intersections of subobjects, given by taking these operations on the output (i.e. "pointwise"). A functor F is epimorphic if $F(\alpha)$ is an epimorphism for every morphism α in $\mathscr{C}$, and similarly we get a notion of monomorphic functors.

Lemma 2.29. *Let $\mathscr{C}$ be a small category and $\mathscr{A}$ and abelian category. For an arbitrary collection of epimorphic subfunctors $\{F_i \hookrightarrow F\}_{i\in I}$ of some functor $F \in \mathrm{Fun}(\mathscr{C}, \mathscr{A})$, we have that $\sum_i F_i$ is also epimorphic, whenever it exists.*

Proof. Let $g\colon X \to Y$ be a morphism in $\mathscr{C}$, so that $F_i(g)$ is an epimorphism for all i. We have the following commutative diagram in $\mathscr{A}$, for each i:

$$\begin{tikzcd} F_i(X) \arrow[r, hook, "\alpha_i"] \arrow[d, "F_i(g)"] & \sum_i F_i(X) \arrow[d, "\sum_i F_i(g)"] \\ F_i(Y) \arrow[r, hook, "\beta_i"] & \sum_i F_i(Y) \end{tikzcd} .$$

Let $\gamma\colon \sum_i F_i(Y) \to C$ be the cokernel of $\sum_i F_i(g)$. Then $\gamma \circ \sum_i F_i(g) = 0$ by definition, and so $\gamma \circ \sum_i F_i(g) \circ \alpha_i = 0$ also. By commutativity of the diagram, we have $\gamma \circ \beta_i \circ F_i(g) = 0$, and then $\gamma \circ \beta_i = 0$ since $F_i(g)$ is an epimorphism. Now this holds for all i, so the universal property of $\sum$ guarantees that $\gamma = 0$, and thus $\sum_i F_i(g)$ is an epimorphism. □

The dual statement that

$$(\forall i) \left(\frac{F}{F_i} \text{ monomorphic} \right) \implies \frac{F}{\bigcap_i F_i} \text{ monomorphic}$$

(whenever the right hand side exists) can be shown *mutatis mutandis*. Then the assumption on sums and intersections in $\mathscr{A}$ above guarantees that any functor F has a unique maximal epimorphic subfunctor $\mathscr{E}(F)$ and a unique maximal monomorphic quotient functor $\mathscr{M}(F)$, and we can view

$$\mathscr{E}, \mathscr{M}\colon \mathrm{Fun}(\mathscr{C}, \mathscr{A}) \to \mathrm{Fun}(\mathscr{C}, \mathscr{A})$$

as endofunctors on the functor category. The definition of $\mathscr{R}\colon \operatorname{Fun}(\mathscr{C},\mathscr{A}) \to \operatorname{Fun}(\mathscr{C},\mathscr{A})$ is the same as before.

Even more generalization is possible, though we will not go into details here. It should be noted in particular that the functors $\mathscr{E}, \mathscr{M}$, and $\mathscr{R}$ exist for categories of (finite dimensional) representations of quivers with relations (cf. Remark 2.9) by restriction from the associated unbound quiver. The appropriate conditions on sums and intersections hold since these categories have both ascending and descending chain conditions on subobjects, although these categories are not necessarily equivalent to any category of the form $\operatorname{Fun}(\mathscr{C},\mathscr{A})$.

2.2.2 Global tensor functors of trees

A quiver Q generates a category $\mathscr{Q}$, called the **free category** on Q [43, §II.7] by taking the objects of $\mathscr{Q}$ to be the vertices of Q, and the morphisms of $\mathscr{Q}$ to be the paths in Q:

$$\operatorname{Ob}\mathscr{Q} := Q_\bullet \qquad \operatorname{Mor}_{\mathscr{Q}}(x,y) := \{\text{paths from } x \text{ to } y\}.$$

The trivial path at a vertex x is the identity morphism for x, and composition of morphisms is composition of paths. A representation V of Q is the same thing as a functor from $\mathscr{Q}$ to K-mod, and a morphism of representations is a natural transformation of the corresponding functors. In other words, a quiver representation is a **diagram** of type $\mathscr{Q}$ in K-mod.

Taking the categorical limit and colimit of such a diagram V, we get vector spaces $\varprojlim V$ and $\varinjlim V$, respectively, with natural maps

$$\alpha_x\colon \varprojlim V \to V_x \qquad \text{and} \qquad \beta_x\colon V_x \to \varinjlim V$$

for each $x \in Q_\bullet$. These maps satisfy $\alpha_{ha} = V_a \circ \alpha_{ta}$ and $\beta_{ta} = \beta_{ha} \circ V_a$ for every arrow $a \in Q_\to$, and therefore $\eta_V := \beta_x \circ \alpha_x$ does not depend on x. When Q is a tree, we will see that the functors $\mathscr{E}, \mathscr{M}$, and $\mathscr{R}$ can be constructed using limits, and have a nice connection to Hom spaces in $\mathrm{Rep}(Q)$.

Proposition 2.30. *There are functorial isomorphisms of vector spaces*

$$\varprojlim V \cong \mathrm{Hom}_Q(\mathbb{I}, V) \qquad \textit{and} \qquad \varinjlim V \cong \mathrm{Hom}_Q(V, \mathbb{I})^*,$$

so $\varprojlim V$ *is representable by* $\mathbb{I}$*. The natural map* $\eta_V \colon \varprojlim V \to \varinjlim V$ *corresponds to the pairing*

$$\mathrm{Hom}_Q(\mathbb{I}, V) \times \mathrm{Hom}_Q(V, \mathbb{I}) \to \mathrm{Hom}_Q(\mathbb{I}, \mathbb{I}) \cong K \qquad (f, g) \mapsto g \circ f.$$

Proof. Fix a compatible basis $\{e_x\}_{x \in Q_\bullet}$ of $\mathbb{I}$, that is, vectors $e_x \in \mathbb{I}_x \simeq K$ such that

$$\mathbb{I}_a(e_{ta}) = e_{ha}$$

for every arrow a. Now given $v \in \varprojlim V$, define $f_v \in \mathrm{Hom}_Q(\mathbb{I}, V)$ by $(f_v)_x(e_x) = \alpha_x(v)$. It is easy to check from definitions and universal properties that f_v is a Q-morphism, that $v \mapsto f_v$ gives a vector space isomorphism $\varprojlim V \xrightarrow{\sim} \mathrm{Hom}_Q(\mathbb{I}, V)$, and that this isomorphism is natural in V.

By applying this to Q^{op} and dualizing, we get $\varinjlim V \cong \mathrm{Hom}_Q(V, \mathbb{I})^*$. With this, it is routine to check that η_V corresponds to the stated natural pairing. □

The following proposition relates these spaces to the global tensor functor when Q is a tree.

Proposition 2.31. *If* Q *is a tree, we can construct the functors* $\mathscr{E}$ *and* $\mathscr{M}$ *from limits and colimits:*

$$(\mathscr{E}V)_x = \operatorname{Im} \alpha_x \qquad \textit{and} \qquad (\mathscr{M}V)_x = \frac{V_x}{\ker \beta_x}$$

where α_x and β_x are defined for $V \in \mathrm{Rep}(Q)$ above, and the maps $(\mathscr{E}V)_a$, $(\mathscr{M}V)_a$ are induced from V_a. Thus, for each $x \in Q_\bullet$, we have

$$(\mathscr{R}V)_x = \frac{\mathrm{Im}\,\alpha_x}{\ker \beta_x \cap \mathrm{Im}\,\alpha_x} \cong \mathrm{Im}\,\eta_V \qquad \textit{and so} \quad r_Q(V) = \mathrm{rank}\,\eta_V.$$

Proof. For each arrow $a \in Q_\to$, the universal property of $\varprojlim$ gives a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Im}\alpha_{ta} \arrow[dd, dashed] & \subseteq & V_{ta} \arrow[dd, "V_a"] \\
\varprojlim V \arrow[ur, two heads, "\alpha_{ta}"] \arrow[dr, two heads, "\alpha_{ha}"'] & & & \\
 & \mathrm{Im}\alpha_{ha} & \subseteq & V_{ha}
\end{tikzcd}$$

which shows that $N := \bigoplus_{x \in Q_\bullet} \mathrm{Im}\,\alpha_x$ is an epimorphic subrepresentation of V. We will show that any epimorphic subrepresentation of V is contained in N.

Now let $E \subseteq V$ be an arbitrary epimorphic subrepresentation of V. For any vertex x, and $v \in E_x$, we claim that there is some $f \in \mathrm{Hom}_Q(\mathbb{I}, E)$ such that $v \in \mathrm{Im}\,f$. This is proved by induction on the number of vertices of Q, using the notation of Proposition 2.30. If Q has one vertex, the situation trivial. Otherwise, choose a vertex $y \neq x$ such that there is precisely one arrow $a \in Q_\to$ with $y = ta$ or $y = ha$. This is possible because a tree always has at least two such vertices. Let $P \subset Q$ be the subquiver of Q obtained by removing y and a, so that $x \in P_\bullet$ and $v \in (E|_P)_x$, where $E|_P$ denotes the restriction of E to P. Then by induction, there exists $f \in \mathrm{Hom}_P(\mathbb{I}_P, E|_P)$ such that $v \in \mathrm{Im}\,f$.

We can extend f to Q: if $y = ha$, then simply set $f(e_y) = E_a(f(e_{ta}))$. If $y = ta$, then since E is epimorphic, there exists some $w \in E(y)$ such that $V_a(w) = f(e_{ha})$. In this case, set $f(e_y) = w$; it is immediate from the definition that in either situation $f \in \mathrm{Hom}_Q(\mathbb{I}_Q, E)$.

Regarding $f \in \mathrm{Hom}_Q(\mathbb{I}_Q, V)$ via the inclusion $E \subseteq V$, the explicit formulation of the isomorphism $\varprojlim V \cong \mathrm{Hom}_Q(\mathbb{I}, V)$ in the proof of Proposition 2.30 shows that

$v \in \operatorname{Im} \alpha_x$, and so $v \in N$. Hence $E \subseteq N$. So N must be maximal epimorphic, that is, $N = \mathscr{E}V$.

The equation for $\mathscr{M}$ follows by dualizing, then the equation for $\mathscr{R}$ from the other two equations and its definition. Since these are equalities as a subrepresentation and quotient representation of V, respectively, we get the following commutative diagram:

$$\begin{tikzcd}
\varprojlim V \arrow[rr, "\alpha_x"] \arrow[dr, two heads] \arrow[rrrr, bend left, "\eta_V"] & & V_x \arrow[rr, "\beta_x"] \arrow[dr, two heads] & & \varinjlim V \\
& \mathrm{Im}\alpha_x = (\mathscr{E}V)_x \arrow[ur, hook] \arrow[dr, two heads] & & \mathrm{Im}\beta_x \cong (\mathscr{M}V)_x \arrow[ur, hook] & \\
& & (\mathscr{R}V)_x \arrow[ur, hook] & &
\end{tikzcd}$$

In particular note that $(\mathscr{R}V)_x \cong \operatorname{Im} \eta_V$. $\square$

This characterization of $\mathscr{R}_Q$ allows us to see that $\mathscr{R}_Q V$ is isomorphic to a direct summand of V when Q is a tree.

Theorem 2.32. *Let Q be a tree, and V an indecomposable representation of Q. Then*

$$\mathscr{R}V \neq 0 \iff V \simeq \mathbb{I}.$$

In particular, if we write

$$V \simeq \bigoplus_j V_j$$

where $V_j \subseteq V$ are indecomposable subrepresentations of V, then

$$\mathscr{R}V \simeq \bigoplus_{V_j \simeq \mathbb{I}} V_j$$

Proof. Suppose V is an indecomposable representation of Q. If $\mathscr{R}V = 0$, then certainly $V \not\simeq \mathbb{I}$ because $\mathscr{R}\mathbb{I} = \mathbb{I}$. If $\mathscr{R}V \neq 0$, then by Proposition 2.31, $\operatorname{Im} \eta_V \neq 0$.

Then Proposition 2.30 implies that there is a pair $(f, g) \in \mathrm{Hom}_Q(\mathbb{I}, V) \times \mathrm{Hom}_Q(V, \mathbb{I})$ such that

$$\begin{tikzcd} \mathbb{I} \arrow[r, "f"'] \arrow[rr, bend left, "id"] & V \arrow[r, "g"'] & \mathbb{I} \end{tikzcd}$$

so V has a direct summand isomorphic to $\mathbb{I}$. But we took V to be indecomposable, so $V \simeq \mathbb{I}$. The second statement follows from additivity of $\mathscr{R}$ (Proposition 2.22). □

Example 2.33. Let Q be the following quiver, and V an indecomposable representation of Q.

$$Q = \begin{tikzcd} \bullet \arrow[dr, "a_1"] & & \\ \bullet \arrow[r, "a_2", "a_3"'] & \bullet \arrow[r, "b"] & \bullet \\ \bullet \arrow[ur] & & \end{tikzcd}$$

Also suppose that V is not simple, so that we can regard each V_{a_i} as an inclusion and assume V_b is surjective. Denoting the branch vertex of Q by x, one can calculate from the definitions that

$$(\mathscr{E}V)_x = \bigcap_{i=1}^{3} V_{a_i} \qquad\qquad (\mathscr{M}V)_x = \frac{V_x}{\ker V_b}$$

so by Theorem 2.32 we have

$$V \simeq \mathbb{I} \iff (\mathscr{R}_Q V)_x \neq 0 \iff \frac{\bigcap_{i=1}^{3} V_{a_i} + \ker V_b}{\ker V_b} \neq 0 \iff \bigcap_{i=1}^{3} V_{a_i} \nsubseteq \ker V_b.$$

One can check for some other quivers which are not trees, for example when Q is the Kronecker quiver

$$Q\colon \ \bullet \rightrightarrows \bullet\,,$$

that $\mathscr{R}_Q V = 0$ when V is indecomposable and some V_a is not an isomorphism. This is easy for this particular example because Q is of tame representation type, so we have nice descriptions of the indecomposable representations of Q. In this case we can still say that $\mathscr{R}_Q$ "picks out" the indecomposable summands for which the map over every arrow is an isomorphism, although these representations are not necessarily isomorphic to $\mathbb{I}_Q$. The following example, however, shows that this property does not hold for all quivers.

Example 2.34. This example shows that $\mathscr{R}_Q V$ is not necessarily isomorphic to a direct summand of V. Let Q be the generalized Kronecker quiver with three arrows a, b, c.

$$Q\colon \bullet \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}} \bullet$$

Let V be the representation with dimension vector $\alpha = (2,3)$ and maps given by

$$V_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad V_b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad V_c = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $\mathscr{E}V \simeq \mathbb{I}$, given by the subspace Ke_1 at each vertex: this is an epimorphic subrepresentation, and V has no subrepresentations of dimension $(2,1)$ or $(2,2)$, so this subrepresentation must be maximal epimorphic. Since each map is already injective, $\mathscr{M}V = V$, and so $\mathscr{R}V \simeq \mathbb{I}$.

The subrepresentation $\mathscr{E}V$ is unique of dimension $(1,1)$, but is not a direct summand: both uniqueness and the fact that $\mathscr{E}$ has no complementary subrepresentation follow from the linear independence of the second columns of the above matrices. So $\mathscr{R}V$ is not isomorphic to a direct summand of V. In fact, there are no direct summands of any other dimension, and V is actually indecomposable.

2.2.3 Tensor product and the global tensor functor

We return to the study of an arbitrary quiver Q. Since the tensor product of two surjective maps is surjective, and likewise for injective maps, the universal properties of $\mathscr{E}$ and $\mathscr{M}$ induce natural transformations θ and ζ giving us commutative diagrams of functors

$$
\begin{tikzcd}
& V \otimes W \\
\theta\colon \mathscr{E}V \otimes \mathscr{E}W \arrow[ur, hook] \arrow[r, hook] & \mathscr{E}(V \otimes W) \arrow[u, hook]
\end{tikzcd}
\qquad
\begin{tikzcd}
V \otimes W \arrow[d, two heads] \arrow[dr, two heads] \\
\zeta\colon \mathscr{M}(V \otimes W) \arrow[r, two heads] & \mathscr{M}V \otimes \mathscr{M}W
\end{tikzcd}
.
$$

These natural transformations satisfy $D \circ \zeta = \theta \circ D$, and give $\mathscr{E}$ and $\mathscr{M}$ the structure of weak tensor functors. To show that $\mathscr{E}$ is symmetric, that is, the structure map θ commutes with the commutativity constraint ψ for $\mathscr{E}$, one can verify that the following diagram (of natural transformations) is commutative:

$$
\begin{tikzcd}
\mathscr{E}V \otimes \mathscr{E}W \arrow[rrr, "\psi"] \arrow[dd, hook, "\theta"] \arrow[dr, hook] & & & \mathscr{E}W \otimes \mathscr{E}V \arrow[dd, hook, "\theta"] \arrow[dl, hook'] \\
& V \otimes W \arrow[r, "\psi"] & W \otimes V & \\
\mathscr{E}(V \otimes W) \arrow[rrr, "\mathscr{E}(\psi)"] \arrow[ur, hook] & & & \mathscr{E}(W \otimes V) \arrow[ul, hook']
\end{tikzcd}
.
$$

The left and right triangles commute because $\mathscr{E}V \otimes \mathscr{E}W$ is a subspace of $\mathscr{E}(V \otimes W)$, by the universal property, so θ commutes with the monomorphisms of these functors to the identity functor on $\operatorname{Rep}(Q)$. The lower trapezoid commutes because $\mathscr{E}$ is a subfunctor of $id_{\operatorname{Rep}(Q)}$, and the upper trapezoid commutes because of the same statement, along with the fact that bifunctoriality of $\otimes$ forces $id_{\operatorname{Rep}(Q)} \otimes id_{\operatorname{Rep}(Q)} = id_{\operatorname{Rep}(Q)}$. Checking the other conditions for $\mathscr{E}$ and $\mathscr{M}$ to be weak tensor functors is similar. However, the next example shows that neither $\mathscr{E}$ nor $\mathscr{M}$ is a tensor functor.

Example 2.35. The maps θ and ζ are not in general isomorphisms. For example, take $Q = \bullet \rightrightarrows \bullet$ with representations

$$
V = \begin{tikzcd} K \arrow[r, shift left, "1"] \arrow[r, shift right, "0"'] & K \end{tikzcd} \qquad W = \begin{tikzcd} K \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & K \end{tikzcd} .
$$

Then $\mathscr{E}V = \mathscr{E}W = 0$, but $\mathscr{E}(V \otimes W) = K \rightrightarrows 0$, so θ is not an isomorphism (dualize to get an analogous example for $\mathscr{M}$).

For linear maps A and B, rank is multiplicative in the sense that $\operatorname{rank}(A \otimes B) = \operatorname{rank} A \cdot \operatorname{rank} B$. Although we have just seen that neither $\mathscr{E}$ nor $\mathscr{M}$ commutes with tensor product, the global tensor functor $\mathscr{R}$ does.

Theorem 2.36. *There is a natural isomorphism of bifunctors*

$$\mathscr{R}_Q V \otimes \mathscr{R}_Q W \cong \mathscr{R}_Q(V \otimes W) \tag{2.6}$$

giving $\mathscr{R}_Q$ the structure of a tensor functor.

Before proving the theorem, we need to establish a technical lemma. As usual, since Q is fixed we omit this subscript.

Lemma 2.37. *Consider the natural transformation of bifunctors defined by the composition*

$$\sigma \colon \mathscr{E}(V \otimes W) \subseteq V \otimes W \twoheadrightarrow V \otimes \mathscr{M}W$$

where the second map is $\mathrm{id}_V \otimes q_W$, writing $q_W \colon W \to \mathscr{M}W$ for the canonical quotient. Then

$$\operatorname{Im} \sigma \subseteq \mathscr{E}V \otimes \mathscr{M}W.$$

Proof. We check this as maps of vector spaces at each vertex z. Using the natural isomorphism of vector spaces

$$\operatorname{Hom}_K(\mathscr{E}(V \otimes W)_z, V_z \otimes_K \mathscr{M}(W)_z) \cong \operatorname{Hom}_K(\mathscr{E}(V \otimes W)_z \otimes_K \mathscr{M}(W)_z^*, V_z)$$

we can identify σ_z with the map

$$\pi_z \colon \mathscr{E}(V \otimes W)_z \otimes \mathscr{M}(W)_z^* \to V_z \qquad \Big(\sum v_i \otimes w_i\Big) \otimes f \mapsto \sum f(w_i) v_i$$

which we want to show takes image in $\mathscr{E}(V)_z$.

We claim that the subspace $M := \oplus_z \operatorname{Im} \pi_z$ is an epimorphic subrepresentation of V. To see this, let $a \in Q_\to$ and set $x = ta,\ y = ha$. Given

$$\sum f(w_i)v_i = \pi_x \left[\left(\sum v_i \otimes w_i \right) \otimes f \right] \in \operatorname{Im} \pi_x$$

where $(\sum v_i \otimes w_i) \in \mathscr{E}(V \otimes W)_x$ and $f \in \mathscr{M}(W)^*_x$, we want to show first that V_a maps this element into $\operatorname{Im} \pi_y$. Now from (2.5), we have $\mathscr{M}(W)^*_x = \mathscr{E}(DW)_x$, so there exists $g \in \mathscr{E}(DW)_y$ such that $f = W_a^* g$. Then

$$\begin{aligned} V_a \left(\sum f(w_i)v_i \right) &= \sum f(w_i)V_a(v_i) = \sum g\left(W_a(w_i)\right) V_a(v_i) \\ &= \pi_y \left[\left(\sum V_a(v_i) \otimes W_a(w_i) \right) \otimes g \right] \in \operatorname{Im} \pi_y \end{aligned}$$

showing that M is a subrepresentation of V.

To see that this subrepresentation is epimorphic, a similar argument works. Given

$$\pi_y \left[\left(\sum s_j \otimes t_j \right) \otimes g \right] \in \operatorname{Im} \pi_y$$

where $(\sum s_j \otimes t_j) \in \mathscr{E}(V \otimes W)_y$ and $g \in \mathscr{M}(W)^*_y$, there exists $\sum v_i \otimes w_i \in \mathscr{E}(V \otimes W)_x$ such that

$$(V_a \otimes W_a) \left(\sum v_i \otimes w_i \right) = \sum V_a(v_i) \otimes W_a(w_i) = \sum s_j \otimes t_j.$$

Thus we have

$$\begin{aligned} V_a \left(\pi_x \left[\left(\sum v_i \otimes w_i \right) \otimes W_a^* g \right] \right) &= V_a \left(\sum g\left(W_a(w_i)\right)\, v_i \right) = \sum g\left(W_a(w_i)\right) V_a(v_i) \\ &= \pi_y \left[\left(\sum V_a(v_i) \otimes W_a(w_i) \right) \otimes g \right] = \pi_y \left[\left(\sum s_j \otimes t_j \right) \otimes g \right]. \end{aligned}$$

So we see that $V_a|_M$ is surjective for each arrow a; hence $M \subseteq \mathscr{E}V$ by the universal property of $\mathscr{E}$. $\square$

Proof of Theorem 2.36. The lemma establishes that $\operatorname{Im} \sigma \subseteq \mathscr{E}V \otimes \mathscr{M}W$, giving the dashed arrow in the diagram

$$\begin{tikzcd}
\mathscr{E}(V \otimes W) \arrow[r, dashed, "\sigma"] \arrow[d, hook] & \mathscr{E}V \otimes \mathscr{M}W \arrow[r, two heads] \arrow[d, hook] & \mathscr{R}V \otimes \mathscr{M}W \arrow[d, hook] \\
V \otimes W \arrow[r, two heads] & V \otimes \mathscr{M}W \arrow[r, two heads] & \mathscr{M}V \otimes \mathscr{M}W
\end{tikzcd} .$$

Here, every arrow represents a canonical natural transformation of bifunctors, but we will simply say "map" throughout the proof to avoid this cumbersome phrase. Thus the map

$$\mathscr{E}(V \otimes W) \to \mathscr{M}V \otimes \mathscr{M}W$$

factors through $\mathscr{R}V \otimes \mathscr{M}W$. But by applying the same reasoning it must also factor through $\mathscr{M}V \otimes \mathscr{R}W$; hence it factors through the intersection (as subfunctors of $\mathscr{M}V \otimes \mathscr{M}W$)

$$(\mathscr{M}V \otimes \mathscr{R}W) \cap (\mathscr{R}V \otimes \mathscr{M}W) = \mathscr{R}V \otimes \mathscr{R}W.$$

So we have a natural map

$$\alpha \colon \mathscr{E}(V \otimes W) \twoheadrightarrow \mathscr{R}V \otimes \mathscr{R}W$$

which is surjective because the subrepresentation $\mathscr{E}V \otimes \mathscr{E}W$ already surjects onto the right hand side, by definition of the global tensor functor. Applying the same argument with Q^{op}, then dualizing, we get a map

$$\beta \colon \mathscr{R}V \otimes \mathscr{R}W \hookrightarrow \mathscr{M}(V \otimes W).$$

We summarize this with the commutative diagram

$$\begin{tikzcd}
\mathscr{E}V \otimes \mathscr{E}W \arrow[r, two heads] \arrow[d, hook, "\theta"'] & \mathscr{R}V \otimes \mathscr{R}W \arrow[r, hook] \arrow[dr, hook, "\beta"] & \mathscr{M}V \otimes \mathscr{M}W \\
\mathscr{E}(V \otimes W) \arrow[r, two heads] \arrow[ur, two heads, "\alpha"] & \mathscr{R}(V \otimes W) \arrow[r, hook] & \mathscr{M}(V \otimes W) \arrow[u, two heads, "\zeta"']
\end{tikzcd}$$

which shows that the natural map

$$\mathscr{E}(V \otimes W) \hookrightarrow V \otimes W \twoheadrightarrow \mathscr{M}(V \otimes W)$$

factors through $\mathscr{R}V \otimes \mathscr{R}W$, and the uniqueness of the image of a map gives an isomorphism

$$\mathscr{R}V \otimes \mathscr{R}W \cong \mathscr{R}(V \otimes W).$$

We already know that $\mathscr{R}(\mathbb{I}) \cong \mathbb{I}$, and to show that $\mathscr{R}$ satisfies the other conditions to be a tensor functor is straightforward. □

In particular, we have finally shown that in fact $\mathrm{rank}_Q\colon \mathrm{Rep}(Q) \to \mathrm{Rep}(A_1)$ is a rank functor.

Remark 2.38. It seems plausible that Theorem 2.36 would also generalize along the lines of Remark 2.28. The same proof might work with only formal adjustments when $\mathscr{A}$ is, say, a rigid abelian tensor category [17] (for which $\mathscr{E}$ and $\mathscr{M}$ exist).

2.3 Rank Functions and the Representation Ring

For a given quiver Q, we can use the global rank functions of other quivers to construct rank functions on Q. For example, if $P \subset Q$ is any connected subquiver, we can define a homomorphism from $R(Q) \to \mathbb{Z}$ by restricting V to P then applying the rank function of P. This will be denoted by $r_P(V)$, with the restriction being understood. More generally, for any map of directed graphs $\alpha\colon Q' \to Q$, we can push forward a rank function on Q' to act on representations of Q.

2.3.1 Pushforward and pullback of representations

Maps of directed graphs and covering quivers have been used in works such as [48, 25, 9, 16, 20, 26] to study representations of quivers, or more generally, of finite dimensional algebras. In this thesis, we will not be interested in maps that are topological coverings of some base quiver, but rather maps that encode combinatorial data about the base quiver. By definition, a map of directed graphs $\alpha\colon Q' \to Q$

sends vertices to vertices and arrows to arrows, and satisfies $t\alpha(a) = \alpha(ta)$ and $h\alpha(a) = \alpha(ha)$ for each arrow $a \in Q'_\to$. A **quiver over** Q is a pair (Q', α) where Q' is a quiver, and $\alpha\colon Q' \to Q$ a map of directed graphs called the **structure map** of (Q', α). To simplify the notation, we consider the maps V_a of a representation V to be defined on the total vector space $\bigoplus_{x \in Q_\bullet} V_x$ by taking $V_a(v) = 0$ for $v \in V_y$, when $y \neq ta$. The **pullback** $\alpha^* W \in \operatorname{Rep}(Q')$ of a representation $W \in \operatorname{Rep}(Q)$ along a map of directed graphs $\alpha\colon Q' \to Q$ is given by

$$(\alpha^* W)_x := W_{\alpha(x)} \qquad x \in Q'_\bullet \qquad\qquad (\alpha^* W)_a := W_{\alpha(a)} \qquad a \in Q'_\to,$$

and the **pushforward** $\alpha_* V \in \operatorname{Rep}(Q)$ of a representation $V \in \operatorname{Rep}(Q')$ is given by

$$(\alpha_* V)_x := \bigoplus_{y \in \alpha^{-1}(x)} V_y \qquad x \in Q_\bullet \qquad\qquad (\alpha_* V)_a := \sum_{b \in \alpha^{-1}(a)} V_b \qquad a \in Q_\to.$$

From the categorical viewpoint of quiver representations (cf. §2.2.2), a map of directed graphs $\alpha\colon Q' \to Q$ induces a functor $\alpha\colon \mathscr{Q}' \to \mathscr{Q}$ between the associated free categories. Then pullback is just the composition of functors

$$\alpha^* V \colon \mathscr{Q}' \xrightarrow{\alpha} \mathscr{Q} \xrightarrow{V} K\text{-}\operatorname{mod},$$

and so pullback along α defines a tensor functor

$$\alpha^* \colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q').$$

Thus, any rank functor F on Q' pushes forward to a rank functor on Q by composition

$$\alpha_* F := F \circ \alpha^* \colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q') \to K\text{-}\operatorname{mod}.$$

In terms of representation rings, α induces a ring homomorphism

$$\alpha^* \colon R(Q) \to R(Q')$$

so that when f is a rank function on Q', we get a pushforward rank function $(\alpha_* f)(V) := f(\alpha^* V)$ on Q.

2.3.2 Examples

The example of applying the rank function of a subquiver to the restriction of a representation is just the case when we take α to be an inclusion map $P \hookrightarrow Q$. The following example illustrates how to use this technique to construct rank functions which do not come from any subquiver.

Example 2.39. Consider the following two quivers, where the numbers at the vertices are just labels rather than dimension vectors:

$$Q' = \begin{tikzcd} 1 \arrow[r, "a"] & 3 \arrow[dr, "c"] & \\ & & 4 \\ 2 \arrow[r, "b"] & 3 \arrow[ur, "c"'] & \end{tikzcd} \qquad Q = \begin{tikzcd} 1 \arrow[dr, "a"] & & \\ & 3 \arrow[r, "c"] & 4 \\ 2 \arrow[ur, "b"] & & \end{tikzcd}.$$

We have a map of directed graphs $\alpha\colon Q' \to Q$ by identifying vertices and arrows of the same label. A representation

$$V = \begin{tikzcd} V_1 \arrow[dr, "A"] & & \\ & V_3 \arrow[r, "C"] & V_4 \\ V_2 \arrow[ur, "B"] & & \end{tikzcd} \in \operatorname{Rep}(Q)$$

pulls back to a representation

$$\alpha^* V = \begin{tikzcd} V_1 \arrow[r, "A"] & V_3 \arrow[dr, "C"] & \\ & & V_4 \\ V_2 \arrow[r, "B"] & V_3 \arrow[ur, "C"'] & \end{tikzcd} \in \operatorname{Rep}(Q')$$

and it is apparent that α^* commutes with direct sum and tensor product. Then we can compute the pushforward of the global rank function of Q':

$$\alpha_* r_{Q'}(V) = r_{Q'}(\alpha^* V) = \dim_K(\operatorname{Im} CA \cap \operatorname{Im} CB).$$

This function is distinct from the global rank function of Q, which can be computed from the definitions to be

$$r_Q(V) = \operatorname{rank}\left(C|_{\operatorname{Im} A \cap \operatorname{Im} B}\right).$$

Note that $\alpha_* r_{Q'}(V) = 0$ if $\operatorname{supp} V \neq Q_\bullet$.

In Example 2.7, it was claimed that we could find the decomposition of $W \otimes W$ in a different way than direct computation. The multiplicativity of r_Q and $\alpha_* r_{Q'}$ give us:

$$\alpha_* r_{Q'}(W \otimes W) = \alpha_* r_{Q'}(W)^2 = 1 \tag{2.7}$$

$$r_Q(W \otimes W) = r_Q(W)^2 = 0 \tag{2.8}$$

where the values of $\alpha_* r_{Q'}(W)$ and $r_Q(W)$ are computed using linear algebra with our description of W. Then (2.7), along with additivity of $\alpha_* r_{Q'}$, implies that $W \otimes W$ has an indecomposable summand Z such that $\operatorname{supp} Z = Q_\bullet$. But (2.8) implies that $\mathbb{I}_Q$ is not a direct summand of $W \otimes W$. Since $\mathbb{I}_Q$ and W are the only two indecomposable representations of Q supported on all of Q, we get that W must be a direct summand of $W \otimes W$, and the other indecomposable summands must be simple by dimension count.

We can think of this carefully chosen pushforward function as "distinguishing" W from other indecomposable representations. In the following examples, we show how to apply this idea to find structure in $R(Q)$. These techniques can be used for many quivers (of finite, tame, or wild type) to study their representation rings. We continue with examples of finite type in order to simplify the demonstrations.

Example 2.40. Continuing the previous example, there are 11 connected subquivers $P \subseteq Q$, each of which gives a rank function

$$r_P \colon R(Q) \to \mathbb{Z} \qquad P \subseteq Q.$$

We also have $\alpha_* r_{Q'} \colon R(Q) \to \mathbb{Z}$. These maps define a ring homomorphism to the product

$$\Delta \colon R(Q) \to \mathbb{Z}^{12}.$$

This is actually an isomorphism, which is easy to check: We already know that as an abelian group, $R(Q) \simeq \mathbb{Z}^{\oplus 12}$, with the indecomposable representations of Q as a basis. One can simply compute the value of $\Delta(V)$ for each indecomposable representation of Q, using explicit descriptions of the rank functors. Then we verify that the image vectors form a $\mathbb{Z}$-module basis for $\mathbb{Z}^{12}$.

Thus the isomorphism class of a representation of Q is completely determined by the values of these 12 explicitly given functions, and since this is a map of algebras, we can simplify the problem of computing tensor products of representations of Q by just multiplying in $\mathbb{Z}^{12}$.

We cannot expect such simple structure for the representation rings of all quivers. We saw with Examples 2.6 and 2.7 that the tensor product structure depends on orientation, and now we will see that the isomorphism class of $R(Q)$ depends on the orientation of Q.

Example 2.41. When Q is the three subspace quiver of Example 2.6, we again have a surjective ring homomorphism

$$\Gamma = \prod_{P \subseteq Q} r_P \colon R(Q) \to \mathbb{Z}^{11}$$

as in the previous example. But one cannot find by inspection any other distinct rank function. The analogous push-forward function for this orientation is actually equal to r_Q. This might lead one to believe that $R(Q)$ has a non-reduced factor, which cannot be detected by homomorphisms to $\mathbb{Z}$.

Again, we can calculate the values $\Gamma(V)$ for each indecomposable V, and use linear algebra to see that Γ is surjective with kernel generated by $E - F$, where

$$E = \left(\begin{smallmatrix} & 2 & \\ 1 & 1 & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & 0 & 0 \end{smallmatrix} \right) \qquad F = \left(\begin{smallmatrix} & 1 & \\ 1 & 0 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \end{smallmatrix} \oplus \begin{smallmatrix} & 1 & \\ 0 & 0 & 1 \end{smallmatrix} \right).$$

To compute powers of $E-F$, the only nontrivial multiplication we need is $\begin{smallmatrix} & 2 & \\ 1 & 1 & 1 \end{smallmatrix} \otimes \begin{smallmatrix} & 2 & \\ 1 & 1 & 1 \end{smallmatrix}$, which was done in Example 2.6. Using this, it is easy to verify that $(E-F)^2 = 0$ in $R(Q)$, and then that Γ lifts to an isomorphism of $\mathbb{Z}$-algebras

$$\tilde{\Gamma} \colon R(Q) \xrightarrow{\sim} \mathbb{Z}^{10} \times \frac{\mathbb{Z}[\varepsilon]}{(\varepsilon^2)}$$

where the map to the first factor is given by $\prod_{P \subsetneq Q} r_P$, and $\tilde{\Gamma}(E-F) = \varepsilon$.

So we see by example how rank functions can be used to give some basic structure of $R(Q)$, although there are not always enough tensor functors to completely determine $R(Q)$.

When Q is not of finite representation type, one does not have the luxury of writing down a nice $\mathbb{Z}$-basis of $R(Q)$ on which we can easily compute the values of rank functions, so the above examples cannot be simply imitated. However, one can use more complicated techniques with rank functions to study $R(Q)$. For example, there are natural inequalities among certain rank functions, and one can try to use these to get additional information about $R(Q)$. We will carry out this type of analysis for a class of quivers called "rooted trees" in the next chapter.

CHAPTER 3

Rank functions on rooted trees

3.1 Rank Functors on Rooted Trees

3.1.1 Rooted tree quivers

A **rooted tree quiver** is a directed graph Q which is a tree (i.e. removing any edge leaves a disconnected graph), and which has a unique sink σ, called the **root** of Q. We sometimes write (Q, σ) if we want to specify the root. Equivalently, one may give a graph which is a tree and specify a root vertex, with the convention that all edges are oriented towards this root. Rooted trees have connections to chemistry, through counting isomers, and to computer science, through data structures. They have been studied as far back as the 1857, by Cayley [12, 13, 14, 15, 45]. It should be mentioned that from the algebraic perspective, the path algebra of a rooted tree quiver is hereditary and right serial; conversely, the ordinary quiver of any associative, basic, hereditary, right serial K-algebra is a rooted tree [5, Thm. 2.6]. (Hereditary means that any nonzero submodule of a projective module is projective, and right serial means that every indecomposable projective right module has a unique composition series.) We will not need this terminology, however.

A map of rooted tree quivers $f\colon (Q', \sigma') \to (Q, \sigma)$ is said to be **root preserving** if $f(\sigma') = \sigma$. We will show that pushforward is left adjoint to pullback of representations along a root preserving map of rooted tree quivers (cf. §2.3.1).

Proposition 3.1. *Let $f\colon (Q',\sigma') \to (Q,\sigma)$ be a root preserving map between rooted tree quivers. Then the pushforward functor f_* is left adjoint to the pullback functor f^*.*

Proof. For $X, Y \in \mathrm{Rep}(Q)$, the map of vector spaces

$$\bigoplus_{z\in Q_\bullet} \mathrm{Hom}_K(X_z, Y_z) \xrightarrow{c_X^Y} \bigoplus_{a\in Q_\to} \mathrm{Hom}_K(X_{ta}, Y_{ha})$$
$$(\phi_z)_{z\in Q_\bullet} \mapsto (Y_a\phi_{ta} - \phi_{ha}X_a)_{a\in Q_\to}$$

has kernel precisely $\mathrm{Hom}_Q(X,Y)$, by the definition of morphisms in $\mathrm{Rep}(Q)$, and similarly for Q'. We use this to give a natural isomorphism $\mathrm{Hom}_Q(f_*V, W) \cong \mathrm{Hom}_{Q'}(V, f^*W)$.

For $V \in \mathrm{Rep}(Q')$ and $W \in \mathrm{Rep}(Q)$, there is a natural isomorphism of vector spaces

$$\begin{aligned}\bigoplus_{x\in Q'_\bullet} \mathrm{Hom}_K(V_x, (f^*W)_x) &\cong \bigoplus_{y\in Q_\bullet} \mathrm{Hom}_K((f_*V)_y, W_y)\\ (\phi_x)_{x\in Q'_\bullet} &\mapsto \left(\sum_{x\in f^{-1}(y)} \phi_x\right)_{y\in Q_\bullet}.\end{aligned} \tag{3.1}$$

Then we can compute

$$c_V^{f^*W}(\phi_x) = \sum_{b\in Q_\to}\sum_{c\in f^{-1}(b)} W_b\phi_{tc} - \sum_{b\in Q_\to}\sum_{c\in f^{-1}(b)} \phi_{hc}V_c = \sum_{c\in Q'_\to} W_{f(c)}\phi_{tc} - \sum_{c\in Q'_\to}\phi_{hc}V_c$$

and using the identification (3.1) we also compute

$$c_{f_*V}^{W}(\phi_x) = \sum_{b\in Q_\to}\sum_{x\in f^{-1}(tb)} W_b\phi_x - \sum_{b\in Q_\to}\sum_{y\in f^{-1}(hb)}\sum_{\substack{c\in f^{-1}(b)\\ hc=y}} \phi_y V_c = \sum_{\substack{(x,b)\in Q'_\bullet\times Q_\to\\ x=f^{-1}(tb)}} W_b\phi_x - \sum_{c\in Q'_\to}\phi_{hc}V_c.$$

Now for each pair $(x,b) \in Q'_\bullet \times Q_\to$ such that $f(x) = tb$, there exists a unique arrow $c \in Q'_\to$ such that $f(c) = b$ and $tc = x$. This is because, in a rooted tree quiver, every vertex except the sink has a unique outgoing arrow, and the assumption that

$f(\sigma') = \sigma$ guarantees that $x \neq \sigma'$. This allows us to identify the terms $W_{f(c)}\phi_{tc}$ with the terms $W_b\phi_x$ in the sums above. Thus, the isomorphism (3.1) induces a natural isomorphism $\operatorname{Im} c^W_{f_*V} \cong \operatorname{Im} c^{f^*W}_V$, and so the kernels of $c^{f^*W}_V$ and $c^W_{f_*V}$ are naturally isomorphic also. □

We sketch a more elegant proof of the preceding proposition using sheaves, which was provided by an anonymous referee of [37]. A rooted tree quiver Q can be viewed as a poset by declaring $x \leq y$ for any two vertices x, y such that there exists a path from x to y. Then we can define a topology on $Q_\bullet$ by taking the open sets to be the order ideals of Q (cf. §3.1.2). The category of sheaves of finite dimensional vector spaces on this topological space is equivalent to the category $\operatorname{Rep}(Q^{op})$, where Q^{op} denotes the quiver Q with the orientations of the arrows reversed. The stalk of a sheaf at some vertex corresponds the vector space associated to that vertex in a representation.

A continuous map $f\colon Q' \to Q$ under this topology on rooted tree quivers is just a map of directed graphs, and when f is root preserving we find that pushforward and pullback of sheaves agree with the corresponding notions for quiver representations under the equivalence. So the adjointness of f^* and f_* for sheaves implies that $f^*\colon \operatorname{Rep}(Q^{op}) \to \operatorname{Rep}(Q'^{op})$ is left adjoint to $f_*\colon \operatorname{Rep}(Q'^{op}) \to \operatorname{Rep}(Q^{op})$, and dualizing we get the adjoint pair (f_*, f^*) of the proposition.

Let (Q, σ) be a rooted tree quiver. Then any $Q' \xrightarrow{f} Q$ induces a rank functor on Q by composing pullback along f with the global rank functor of Q'. This gives the possibility of constructing infinitely many rank functors on Q, *a priori*, but it is possible for distinct quivers over Q to give isomorphic rank functors. It turns out that when Q is a rooted tree, there is a finite set of "reduced" quivers over Q which give all rank functors on Q that can be obtained in the way just described.

Furthermore, there is a natural partial ordering on this set of rank functors.

The technique of many proofs in this thesis is induction on the number of vertices of Q, using the observation that every connected subquiver of a rooted tree quiver is again a rooted tree quiver. A **subquiver** P of a quiver Q is given by a subset of vertices $P_\bullet \subseteq Q_\bullet$ and a subset of arrows $P_\to \subseteq Q_\to$ with the same orientation as in Q. We will usually assume that subquivers are connected, so that the global rank functor of a subquiver is defined. We can build any rooted tree quiver by two fundamental processes, which we call "extension" and "gluing".

We say that a rooted tree quiver (Q, σ) is obtained from a subquiver $(P, \tau) \subset Q$ by **extension** if

$$Q_\bullet = P_\bullet \cup \{\sigma\} \qquad \text{and} \qquad Q_\to = P_\to \cup \{\alpha\}$$

where $t\alpha = \tau$ and $h\alpha = \sigma$. Note that if such a P exists (in a rooted tree quiver), it is unique.

In any rooted tree quiver which is not an extension of a subquiver, in the above sense, there exists a unique maximal collection of subquivers $\{Q_i \subsetneq Q\}_{i\in I}$ such that Q is obtained by **gluing** the Q_i at their sinks: that is, $Q = \bigcup_i Q_i$, and $Q_i \cap Q_j = \sigma$ for $i \neq j$. In this case, we write

$$Q = \coprod_{i\in I}^{\sigma} Q_i.$$

Each of these Q_i is an extension of a unique subquiver $P_i \subsetneq Q_i$. The notion of Q being glued from any two subquivers $P, S \subset Q$ at their sinks is similarly defined.

Many proofs in this thesis will use induction on the number of vertices of (Q, σ). If there is a unique arrow in Q whose head is σ, then Q is obtained from a quiver with fewer vertices by extension; if there is more than one arrow with head σ, then Q is obtained by gluing two quivers P and S at σ, each with fewer vertices than Q.

Hence, to recursively define a construction depending on Q, or to prove any property of Q by induction, we just need to start with the rooted tree quiver with one vertex, and then show how to proceed for quivers obtained by extension or gluing. The base case is usually trivial and will be omitted. In the gluing case, *a priori* a definition or property could depend on the choice of P and S in Q. However, this will not be the case for any properties or definitions of this thesis, which can be easily seen in any particular case by working with the unique maximal collection of Q_i's as in the definition of gluing above. So, for simplicity, in the gluing case we will always use the setup of two subquivers P and S with it implicit that the result does not depend on the choice of P and S used. In the course of such proofs, the notation above will be assumed to be in place unless explicitly stated otherwise.

Our first task will be to establish a connection between the global rank functor of a rooted tree quiver, and the global rank functors of its subquivers.

Lemma 3.2. *If (Q, σ) is a rooted tree quiver, then $(\mathscr{M}_Q V)_\sigma = V_\sigma$, and hence*

$$\operatorname{rank}_Q V = (\mathscr{E}_Q V)_\sigma \subseteq V_\sigma.$$

Proof. For $x \in Q_\bullet$, denote by $V_{x\to\sigma}$ the linear map V_p for p the unique path from x to σ. Then

$$W_x := \frac{V_x}{\ker V_{x\to\sigma}}$$

defines a monomorphic quotient representation of V. But any intermediate quotient $V \twoheadrightarrow W' \twoheadrightarrow W$ is not monomorphic, because by definition of W there must be some path p such that W'_p has non-trivial kernel. Hence $\mathscr{M}_Q V = W$, and since $V_{\sigma\to\sigma}$ is the identity map, we get the first statement. Then the second statement follows from the definition of $\operatorname{rank}_Q V$. $\square$

If $P \subset Q$ is a subquiver, and $V \in \mathrm{Rep}(Q)$, denote by $V|_P$ the restriction of V to P. The previous lemma can be used to inductively construct global rank functors.

Lemma 3.3. *If (Q, σ) is obtained by extension from (P, τ) via an arrow α, then the global rank functor of Q can be calculated as*

$$\operatorname{rank}_Q V = V_\alpha(\operatorname{rank}_P(V|_P))$$

where we consider $\operatorname{rank}_P(V|_P) \subseteq V_\tau$. If Q is obtained by gluing P and S, then we have

$$\operatorname{rank}_Q V = \operatorname{rank}_P(V|_P) \cap \operatorname{rank}_S(V|_S)$$

where the intersection is taken in $V_\sigma = (V|_P)_\sigma = (V|_S)_\sigma$.

Proof. Extension: Define an epimorphic subrepresentation $E \subseteq V$ by

$$E_x = \begin{cases} \mathscr{E}_P(V|_P)_x & x \in P_\bullet \\ V_\alpha(E_\tau) & x = \sigma \end{cases}.$$

If $M \subseteq V$ is any epimorphic subrepresentation, then we have an inclusion $M|_P \subseteq \mathscr{E}_P(V|_P) = E|_P$ by the universal property of $\mathscr{E}_P$, and the epimorphic property gives

$$M_\sigma \subseteq V_\alpha(M_\tau) \subseteq V_\alpha(E_\tau) = E_\sigma.$$

Hence any epimorphic subrepresentation $M \subseteq V$ is contained in E, so we have $E = \mathscr{E}_Q V$ by the universal property of $\mathscr{E}_Q$. Using Lemma 3.2, we get

$$\operatorname{rank}_Q V = (\mathscr{E}_Q V)_\sigma = E_\sigma = V_\alpha(E_\tau) = V_\alpha(\mathscr{E}_P(V|_P)_\tau) = V_\alpha(\operatorname{rank}_P(V|_P)).$$

Gluing: We want to show that $\operatorname{rank}_Q V = Z$, where

$$Z := \operatorname{rank}_P(V|_P) \cap \operatorname{rank}_S(V|_S).$$

By Lemma 3.2, we have $\operatorname{rank}_Q V = (\mathscr{E}_Q V)_\sigma$ and $Z = \mathscr{E}_P(V|_P)_\sigma \cap \mathscr{E}_S(V|_S)_\sigma$.

$\supseteq$: First we construct an epimorphic subrepresentation $M \subseteq V$ such that $M_\sigma = Z$. Retaining the notation $V_{x\to\sigma}$ from the previous lemma, define M by

$$M_x := \begin{cases} (V_{x\to\sigma})^{-1}(Z) \cap \mathscr{E}_P(V|_P)_x & x \in P_\bullet \\ (V_{x\to\sigma})^{-1}(Z) \cap \mathscr{E}_S(V|_S)_x & x \in S_\bullet \end{cases}.$$

It is straightforward to check that M is an epimorphic subrepresentation of V, so M is contained in $\mathscr{E}_Q V$. Hence $Z = M_\sigma \subseteq (\mathscr{E}_Q V)_\sigma = \operatorname{rank}_Q V$.

$\subseteq$: The universal properties of $\mathscr{E}_P$ and $\mathscr{E}_S$ give

$$(\mathscr{E}_Q V)|_P \subseteq \mathscr{E}_P(V|_P) \qquad \text{and} \qquad (\mathscr{E}_Q V)|_S \subseteq \mathscr{E}_S(V|_S)$$

so again using Lemma 3.2 we have

$$\operatorname{rank}_Q V = (\mathscr{E}_Q V)_\sigma \subseteq Z. \qquad \square$$

Our goal is to construct as many distinct rank functors on Q as possible. We motivate the general procedure with two examples.

Example 3.4. The three subspace quiver Q can be obtained from three A_2 quivers

$$Q_1 = \begin{tikzcd} & \sigma \\ 1 \arrow[ur] & \end{tikzcd} \qquad Q_2 = \begin{tikzcd} \sigma \\ 2 \arrow[u] \end{tikzcd} \qquad Q_3 = \begin{tikzcd} \sigma & \\ & 3 \arrow[ul] \end{tikzcd}$$

by gluing them at their sinks.

$$Q = \begin{tikzcd} & \sigma & \\ 1 \arrow[ur] & 2 \arrow[u] & 3 \arrow[ul] \end{tikzcd}$$

On each Q_i, the global rank functor is just given on a representation $V = V_i \xrightarrow{A_i} V_\sigma$ by

$$\operatorname{rank}_{Q_i}(V) = \operatorname{Im} A_i \subseteq V_\sigma,$$

and $F_i(V) = V_\sigma$ is another rank functor on Q_i.

To any subset $J \subseteq \{1, 2, 3\}$ and $V \in \mathrm{Rep}(Q)$, we can use the rank functors on the subquivers Q_i to get a vector space

$$\mathrm{rank}_J(V) = \bigcap_{j \in J} \mathrm{rank}_{Q_j}(V|_{Q_j}) \subseteq V_\sigma$$

and in fact this defines a rank functor on Q. This gives an ordering reversing correspondence between the lattice of subsets of $B_3 = \{1, 2, 3\}$, and a set of rank functors on Q, with the latter ordered by inclusion of functors. But not only are these rank functors built from rank functors on smaller quivers, the partial order on them also comes from these smaller quivers, in the following sense. Let $\mathbf{2} = \{\hat{0}, \hat{1}\}$ be ordered by $\hat{0} < \hat{1}$. If, for each i, we associate $\hat{0}$ with F_i and $\hat{1}$ with rank_{Q_i}, then the isomorphism of posets $B_3 \simeq \mathbf{2} \times \mathbf{2} \times \mathbf{2}$ induces the ordering on rank functors of Q by associating a product of elements on the right hand side with intersection of the associated functors inside V_σ. The idea is illustrated by the following diagram.

$$\begin{tikzcd}
& V_\sigma & & & & & & & \\
\mathrm{Im}A_1 \arrow[ur, hook] & \mathrm{Im}A_2 \arrow[u, hook] & \mathrm{Im}A_3 \arrow[ul, hook] & & V_\sigma & & V_\sigma & & V_\sigma \\
\mathrm{Im}A_1 \cap \mathrm{Im}A_2 \arrow[u, hook] \arrow[ur, hook] & \mathrm{Im}A_1 \cap \mathrm{Im}A_3 \arrow[ul, hook] \arrow[ur, hook] & \mathrm{Im}A_2 \cap \mathrm{Im}A_3 \arrow[ul, hook] \arrow[u, hook] & \longleftrightarrow & \mathrm{Im}A_1 \arrow[u, hook] & \times & \mathrm{Im}A_2 \arrow[u, hook] & \times & \mathrm{Im}A_3 \arrow[u, hook] \\
& \mathrm{Im}A_1 \cap \mathrm{Im}A_2 \cap \mathrm{Im}A_3 \arrow[ul, hook] \arrow[u, hook] \arrow[ur, hook] & & & & & & &
\end{tikzcd}$$

Example 3.5. Now write the three subspace quiver as P below, and let Q be the extension of P from its sink.

$$P = \begin{tikzcd} & \tau & \\ 1 \arrow[ur] & 2 \arrow[u] & 3 \arrow[ul] \end{tikzcd} \qquad Q = \begin{tikzcd} 1 \arrow[dr] & & \\ 2 \arrow[r] & \tau \arrow[r, "\alpha"] & \sigma \\ 3 \arrow[ur] & & \end{tikzcd}$$

Given any rank functor rank_J on P from the previous example, the image functor

$$V_\alpha(\mathrm{rank}_J(V|_P)) \subseteq V_\sigma$$

is a rank functor on Q. Now we make the simple observation that for any linear map between vector spaces $A : U \to W$, and two subspaces $X, Y \subset U$, the containment $A(X) \cap A(Y) \supseteq A(X \cap Y)$ is not necessarily an equality. Thus, any *collection* of subsets $\{J_i\} \subseteq B_3$ induces a rank functor on Q given by

$$V \mapsto \bigcap_i V_\alpha(\mathrm{rank}_{J_i}(V|_P)) \subseteq V_\sigma$$

which is not in general the image of any one rank functor on P. But because there are inclusions among these rank functors, some collections will be redundant: for example, if any $J_i = \{1,2,3\}$ then the intersection simplifies to $V_\alpha(\mathrm{rank}_{\{1,2,3\}} V)$, since this space is contained in $V_\alpha(\mathrm{rank}_{J'} V)$ for any $J' \subseteq \{1,2,3\}$. To avoid redundancy, we must consider collections of *incomparable* elements of B_3.

These two examples capture the essence of the combinatorics we will use to index a nice set of rank functors on a given rooted tree quiver.

3.1.2 A combinatorial construction

Before undertaking an analysis of the rank functors on Q, we introduce an auxiliary combinatorial framework which will organize the connection between quivers over Q, rank functors on Q, and the representation ring of Q.

First, we recall some definitions which can be found in Stanley's book [55]. We assume that all posets here are finite. An **(order) ideal** in a poset A is a subset $I \subseteq A$ such that $y \leq x$ and $x \in I$ implies that $y \in I$ also. In particular, both $\emptyset$ and A are ideals of A. For any subset $\{x_1, \ldots, x_n\} \subseteq A$, we denote by $\langle x_1, \ldots, x_n \rangle$ the smallest ideal of A containing all of the x_i. The set of all order ideals in A is denoted by $J(A)$, and is partially ordered by inclusion. It is even a distributive lattice, with join operator $\vee$ corresponding to union of ideals and meet operator $\wedge$ corresponding to intersection of ideals. An order-preserving map of posets $f \colon A \to B$ induces a map $f\colon J(A) \to J(B)$ which sends an ideal $I \subseteq A$ to the ideal generated by the image $f(I) \subseteq B$. The **product** of two posets A and B is their usual product $A \times B$ as sets, ordered by $(x, y) \leq (z, w)$ if and only if both $x \leq z$ and $y \leq w$. If both A and B are distributive lattices, then so is $A \times B$ with the meet and join operations carried out in each coordinate. We will always consider A to be a sublattice of $A \times B$ via the inclusion

$$A \simeq \{(a, \hat{0}) \mid a \in A\} \subseteq A \times B, \tag{3.2}$$

and similarly for B. We denote the minimal and maximal elements of a finite lattice L by $\hat{0}$ and $\hat{1}$, respectively, using subscripts to clarify the role of L if necessary.

A set of pairwise incomparable elements $\{x_1, \ldots, x_k\}$ in a poset A is called an **antichain** in A. The map

$$\max\colon J(A) \xrightarrow{\sim} \{\text{antichains in } A\}$$

sending an ideal of A to its set of maximal elements is a bijection between the set of ideals of A and the set of antichains in A. The inverse associates to an antichain C the order ideal generated by C.

Now we proceed to define, for each vertex $x \in Q_\bullet$, a finite, distributive lattice L_Q^x. If Q has a single vertex σ, then L_Q^σ is just the lattice with one element. For Q with more than one vertex, we define L_Q^x recursively. For any vertex $x \in Q_\bullet$, there is a unique maximal connected subquiver $Q_{\geq x}$ for which x is the sink. Its vertices are

$$(Q_{\geq x})_\bullet := \{y \in Q_\bullet \mid \text{there exists a path from } y \text{ to } x\}.$$

If $x \neq \sigma$, then we can assume that $L^x_{Q_{\geq x}}$ is already defined, and we take

$$L_Q^x := L^x_{Q_{\geq x}}.$$

If $x = \sigma$, then removing the vertex σ and all arrows attached to σ leaves a disjoint union of rooted tree quivers (Q_i, σ_i). We inductively define

$$L_Q^\sigma := \prod_i J\left(L_{Q_i}^{\sigma_i}\right).$$

In particular, when (Q, σ) is an extension of (P, τ), we have

$$L_Q^\sigma = J(L_P^\tau),$$

and if Q is obtained by gluing subquivers P and S, then we have

$$L_Q^\sigma = L_P^\sigma \times L_S^\sigma.$$

We define the set L_Q as the disjoint union

$$L_Q := \coprod_{x \in Q_\bullet} L_Q^x.$$

3.1.3 Reduced quivers over Q

Now for each $M \in L_Q^x$, we will construct a rooted tree quiver (Q_M, σ_M) and a map of directed graphs

$$c_M \colon Q_M \to Q$$

such that $c_M(\sigma_M) = x$ (and hence $c_M^{-1}(x) = \{\sigma_M\}$ since c_M preserves heads and tails of arrows). When Q has one vertex σ, the lattice L_Q^σ has one element $\hat{1}$ for which we define $Q_{\hat{1}} = Q$ and

$$c_{\hat{1}} \colon Q \xrightarrow{id} Q.$$

If Q has more than one vertex, we make the definition recursively. For $M \in L_Q^x$ such that $x \neq \sigma$, we defined $L_Q^x = L_{Q_{\geq x}}^x$, so we can assume that we already have (Q_M, c_M), a quiver over $Q_{\geq x}$, which we regard as a quiver over Q via the inclusion

$$c_M \colon Q_M \to Q_{\geq x} \subset Q.$$

We use the same notation whether considering Q_M as a quiver over Q or $Q_{\geq x}$, with the context making the target clear. For $M \in L_Q^\sigma$, there are two cases. As always, we retain the notation from §3.1.1.

Extension: Let $M \in L_Q^\sigma = J(L_P^\tau)$ be an order ideal. If $M = \emptyset$ is the empty ideal, then $M = \hat{0}$ in the lattice L_Q^σ, and we take Q_M to have one vertex σ_M. Define $c_M \colon Q_M \to Q$ to map this vertex to σ. Now if $M \neq \emptyset$, let $\max(M) = \{S_1, \cdots S_m\} \subset L_P^\tau$. First suppose $m = 1$, and set $S := S_1$. We define $(Q_{\langle S \rangle}, \sigma_{\langle S \rangle})$ as the one point extension of (P_S, τ_S), which we can assume is already defined. There is a unique map $c_{\langle S \rangle} \colon Q_{\langle S \rangle} \to Q$ which extends $c_S \colon P_S \to P$, necessarily sending $\sigma_{\langle S \rangle}$ to σ.

For the general case $m \geq 1$, we have already rooted tree quivers (P_{S_i}, τ_i) and structure morphisms

$$c_{S_i} \colon P_{S_i} \to P$$

for each i. Each of these gives

$$c_{\langle S_i \rangle} \colon Q_{\langle S_i \rangle} \to Q$$

from the case $m = 1$. We form Q_M by gluing the collection $\{Q_{\langle S_i \rangle}\}$ at their sinks $\sigma_{\langle S_i \rangle}$. This induces a unique map

$$c_M \colon Q_M \to Q$$

which restricts to $c_{\langle S_i \rangle}$ on each $Q_{\langle S_i \rangle} \subset Q_M$.

Gluing: Let $M \in L_Q^\sigma$. In the gluing case, by definition we have $L_Q^\sigma = L_P^\sigma \times L_S^\sigma$, so we can write $M = (X, Y)$ for some $X \in L_P^\sigma$ and $Y \in L_S^\sigma$. To define (Q_M, c_M) recursively, we can assume that we have

$$c_X \colon P_X \to P \qquad \text{and} \qquad c_Y \colon S_Y \to S$$

defined already. Then Q_M is given by gluing P_X and S_Y at their sinks, and $c_M \colon Q_M \to Q$ is the unique map which restricts to c_X and c_Y on P_X and S_Y, respectively.

Definition 3.6. The quivers Q_M and structure maps $c_M \colon Q_M \to Q$ constructed above for $M \in L_Q$ are the **reduced** quivers over Q.

We will usually just refer to Q_M alone as a quiver over Q, with the structure map c_M being understood. Also, when $M \in L_Q^\sigma$, we will sometimes employ a slight abuse of notation by denoting the sink of Q_M also as σ. This is unlikely to result in any confusion in the representation theory, since by definition both $(c_M^* V)_{\sigma_M} = V_\sigma$ and $(c_{M*}W)_\sigma = W_{\sigma_M}$ for any $V \in \mathrm{Rep}(Q)$ and $W \in \mathrm{Rep}(Q_M)$.

Let $\Lambda \xrightarrow{\lambda} Q$ and $\Gamma \xrightarrow{\gamma} Q$ be any rooted tree quivers over Q. A map of directed graphs $f \colon \Lambda \to \Gamma$ is a **morphism of quivers over** Q if it commutes with the structure maps, that is, if

$$\begin{tikzcd} \Lambda \arrow[rr, "f"] \arrow[dr, "\lambda"'] & \arrow[d, phantom, "\circlearrowleft"] & \Gamma \arrow[dl, "\gamma"] \\ & Q & \end{tikzcd}$$

is a commutative diagram of maps of directed graphs. We write $f \in \mathrm{Hom}_{\downarrow Q}(\Lambda, \Gamma)$. The following lemma motivates our interest in morphisms between quivers over Q by relating them to rank functors.

Lemma 3.7. *Let $\Lambda \xrightarrow{\lambda} Q$ and $\Gamma \xrightarrow{\gamma} Q$ be rooted tree quivers over (Q, σ), and $f \in \mathrm{Hom}_{\downarrow Q}(\Lambda, \Gamma)$. Assume that f, λ, γ are all root preserving, and call all of the root vertices of the quivers here σ for simplicity. Then f induces an injective natural transformation $f^* \colon \mathrm{rank}_\Gamma \circ \gamma^* \hookrightarrow \mathrm{rank}_\Lambda \circ \lambda^*$ such that for any $V \in \mathrm{Rep}(Q)$ the diagram*

$$\begin{tikzcd} & (\gamma^* V)_\sigma = (\lambda^* V)_\sigma = V_\sigma \\ \mathrm{rank}_\Gamma(\gamma^* V) \arrow[ur, hook] \arrow[r, hook, "f^*"] & \mathrm{rank}_\Lambda(\lambda^* V) \arrow[u, hook] \end{tikzcd}$$

commutes.

Proof. By Theorem 2.32, there is a decomposition $\gamma^*V \simeq U \oplus W$ for some subrepresentations $U, W \subseteq \gamma^*V$ such that $U_\sigma = \mathrm{rank}_\Gamma(\gamma^*V)$ and $U \simeq \mathbb{I}_\Gamma^{\oplus d}$. This gives a decomposition

$$\lambda^*V \simeq f^*\gamma^*V \simeq f^*U \oplus f^*W$$

which, by additivity of rank functors, gives $\mathrm{rank}_\Lambda(\lambda^*V) = \mathrm{rank}_\Lambda(f^*U) \oplus \mathrm{rank}_\Lambda(f^*W)$. Since $f^*U \simeq \mathbb{I}_\Lambda^{\oplus d}$, this gives the inclusion

$$\mathrm{rank}_\Lambda(f^*U) = (f^*U)_\sigma = f^*(U_\sigma) = f^*(\mathrm{rank}_\Gamma(\gamma^*V)) \subseteq \mathrm{rank}_\Lambda(\lambda^*V). \qquad \square$$

For elements M, N of a poset A, we write $M \prec N$ when N **covers** M in A, which is by definition when $M < N$ and there does not exist any $Z \in A$ with $M < Z < N$.

Proposition 3.8. *The construction above defines an injective map of sets from L_Q to the set of quivers over Q. Furthermore, if we fix a vertex $x \in Q_\bullet$, the following statements hold for any $M, N \in L_Q^x$.*

(a) Reduced quivers over Q admit no nontrivial endomorphisms, i.e. $\mathrm{Hom}_{\downarrow Q}(Q_M, Q_M) = \{id\}$.

(b) For each cover relation $M \prec N$, there exists a unique morphism $\mathrm{Hom}_{\downarrow Q}(Q_M, Q_N) = \{\rho_{M,N}\}$.

(c) If $M \not\leq N$, then $\mathrm{Hom}_{\downarrow Q}(Q_M, Q_N) = \emptyset$.

In particular, we have that

$$\mathrm{Hom}_{\downarrow Q}(Q_M, Q_N) \neq \emptyset \iff M \leq N.$$

Proof. The map given by sending $M \in L_Q$ to Q_M can inductively be seen to be injective by gluing and extension. The three statements about morphisms will be verified by induction on the number of vertices of Q. When $x \neq \sigma$, by definition

Q_M and Q_N are also reduced quivers over $Q_{\geq x} \subsetneq Q$, so the proposition holds by induction. So assume $x = \sigma$.

Extension: Let $\max(M) = \{S_1, \ldots, S_m\}$. Then $f \in \mathrm{Hom}_{\downarrow Q}(Q_M, Q_M)$ restricts to a morphism

$$\overline{f} \in \mathrm{Hom}_{\downarrow P}\Big(\coprod_{i=1}^{m} P_{S_i}, \coprod_{i=1}^{m} P_{S_i}\Big) = \prod_{i=1}^{m} \mathrm{Hom}_{\downarrow P}\Big(P_{S_i}, \coprod_{i=1}^{m} P_{S_i}\Big)$$

over P. Since the S_i are pairwise incomparable in L_P^τ, the induction hypothesis implies that

$$\mathrm{Hom}_{\downarrow P}(P_{S_i}, P_{S_j}) = \begin{cases} id & i = j \\ \emptyset & i \neq j \end{cases}.$$

Hence $\overline{f}$ is the identity on $\coprod_{i=1}^{m} P_{S_i}$, and $f = id_{Q_M}$ is the only way this can extend as a morphism of quivers over Q.

For b), suppose that $M \prec N$ in $L_Q^\sigma = J(L_P^\tau)$, and let $f \in \mathrm{Hom}_{\downarrow Q}(Q_M, Q_N)$. Then there exists $T \in L_P^\tau$ such that $N = M \cup \{T\}$ as ideals in L_P^τ. First consider the case that M is a principal ideal, say $M = \langle S \rangle$ for some $S \in L_P^\tau$, so that Q_M is the one point extension of P_S from its sink. Since N is an ideal, it must be that either $T \succ S$ or that T and S are incomparable. If $T \succ S$, then by the induction hypothesis we have a unique morphism $P_S \xrightarrow{\rho_{S,T}} P_T$ of quivers over P, which must be the restriction of f over P. Since $N = M \cup \{T\} = \langle T \rangle$, by construction Q_N is extended from P_T, so f is uniquely determined as being the extension of $\rho_{S,T}$ to a morphism $Q_M \to Q_N$ over Q. On the other hand, if T and S are incomparable, then $\max(N) = \{S, T\}$ and so Q_N is glued from $Q_{\langle S \rangle}$ and $Q_{\langle T \rangle}$. Now by considering the restriction of f over P again, the induction hypotheses implies that f must be the inclusion $Q_M = Q_{\langle S \rangle} \subseteq Q_N$.

Now if M is not principal, say $\max(M) = \{S_1, \ldots, S_m\}$, then by definition Q_M is obtained by gluing the $Q_{\langle S_i \rangle}$ at their sinks, so to give a map from Q_M it is enough to define it on each $Q_{\langle S_i \rangle}$ for $1 \leq i \leq m$. But for each i, again we have either $S_i \prec T$ or S_i and T are incomparable. If $S_i \prec T$, then necessarily $T \in \max(N)$, so f must restrict to the unique map $Q_{\langle S_i \rangle} \to Q_{\langle T \rangle} \subseteq Q_N$ from the principal case. If they are incomparable, then again $S_i \in \max(N)$ and $Q_{\langle S_i \rangle}$ is a subquiver of Q_N. In this case f must restrict to the inclusion from $Q_{\langle S_i \rangle}$ to Q_N.

Now suppose $M \npreceq N$, and let

$$\max(M) = \{S_1, \ldots, S_m\} \qquad \text{and} \qquad \max(N) = \{T_1, \ldots, T_n\}.$$

To be compatible with the structure maps, any morphism $f \colon Q_M \to Q_N$ over Q necessarily satisfies $f^{-1}(\sigma_N) = \{\sigma_M\}$, hence would restrict to a morphism

$$\overline{f} \in \mathrm{Hom}_{\downarrow P}(\coprod_{i=1}^{m} P_{S_i}, \coprod_{j=1}^{n} P_{T_j})$$

over P. But $M \npreceq N$ implies that there exists some i such that $S_i \npreceq T_j$ for all j, so by induction $\mathrm{Hom}_{\downarrow P}(P_{S_i}, P_{T_j}) = \emptyset$ for all j. Hence such an f does not exist over Q, and we get $\mathrm{Hom}_{\downarrow Q}(Q_M, Q_N) = \emptyset$.

Gluing: We can write $M = (X, Y)$ and $N = (Z, W)$ for some $X, Z \in L_P^{\sigma}$ and $Y, W \in L_S^{\sigma}$. By induction, we immediately get (a). For b), suppose $M \prec N$ and that $f \in \mathrm{Hom}_{\downarrow Q}(Q_M, Q_N)$. Then by switching P and S if necessary, we can assume that $X \prec Z$ and $Y = W$. Then by induction f must restrict to the unique map $P_X \xrightarrow{\rho_{X,Z}} P_Z$ over P, and over S to the identity on S_Y. This defines $\rho_{M,N}$ uniquely over Q. Now suppose $M \npreceq N$, so without loss of generality $X \npreceq Z$. Any $f \in \mathrm{Hom}_{\downarrow Q}(Q_M, Q_N)$ would restrict to some $\overline{f} \in \mathrm{Hom}_{\downarrow P}(P_X, P_Z)$, but by induction such a morphism does not exist. Hence there is no such f, and so $\mathrm{Hom}_{\downarrow Q}(Q_M, Q_N) = \emptyset$. $\square$

3.1.4 Induced rank functors

For each $M \in L_Q^x \subset L_Q$, we get a rank functor rank_M on Q by pulling back a representation along c_M, then applying the global rank functor of Q_M:

$$\operatorname{rank}_M V := \operatorname{rank}_{Q_M}(c_M^* V) \subseteq V_x.$$

The vector space $\operatorname{rank}_M V$ is naturally a subspace of V_x because Lemma 3.2 gives an inclusion

$$\operatorname{rank}_{Q_M}(c_M^* V) \subseteq (c_M^* V)_{\sigma_M} = V_{c_M(\sigma_M)} = V_x.$$

We call these subspaces **rank spaces** of V, always considering them as subspaces of some appropriate V_x without explicit mention.

The inductive definition of L_Q^σ, along with Lemma 3.3, provides the following inductive description of rank_M: suppose that removing σ from Q leaves a disjoint union of rooted trees (Q_i, σ_i), so that $M = (M_i)$ for some ideals $M_i \subseteq L_{Q_i}^{\sigma_i}$. Let α_i be the unique arrow from σ_i to σ. Then we have that

$$\operatorname{rank}_M V = \bigcap_i \bigcap_{N \in \max(M_i)} V_{\alpha_i}(\operatorname{rank}_N(V|_{Q_i})).$$

Example 3.9. For each vertex $x \in Q_\bullet$, the reduced quiver $Q_{\hat{0}_x}$ corresponding to the minimal element $\hat{0}_x \in L_Q^x$ is the inclusion of the vertex x as a subquiver of Q, and hence $\operatorname{rank}_{\hat{0}_x}(V) = V_x$. For the maximal element $\hat{1}_x$, we get $Q_{\hat{1}_x} = Q_{\geq x}$ and the structure map is inclusion, hence $\operatorname{rank}_{\hat{1}_x}(V) = \operatorname{rank}_{Q_{\geq x}}(V|_{Q_{\geq x}})$. In fact, Lemma 3.3 and induction imply that for every connected subquiver $P \subseteq Q$, the rank functor rank_P (applied to the restriction $V|_P$) appears among the functors $\{\operatorname{rank}_M\}_{M \in L_Q}$.

The relations between induced rank functors given by Lemma 3.7 motivate the following definition.

Definition 3.10. Let $f\colon (Q', \sigma') \to Q$ be a quiver over Q. We will say that $Q' \xrightarrow{f} Q$ is **rank equivalent** to Q_M (for some $M \in L_Q$) when there exist morphisms

$$Q_M \xrightarrow{g} Q' \xrightarrow{h} Q_M$$

as quivers over Q such that $h \circ g = id_{Q_M}$.

By Lemma 3.7, if $Q' \xrightarrow{f} Q$ is rank equivalent to Q_M then it induces the same rank functor on Q:

$$\operatorname{rank}_{Q'}(f^*(V)) = \operatorname{rank}_M V \subseteq V_{f(\sigma')}.$$

Furthermore, in such a rank equivalence g is injective on vertices and arrows, so the number of arrows in Q_M is less than or equal to the number of arrows in Q', and if these numbers are equal then g is an isomorphism of quivers over Q. This explains the terminology "reduced" quivers over Q. Now to see that for any representation V, rank functors give an order reversing map from the lattice L_Q^x to the collection of subspaces of V_x, partially ordered by inclusion.

Proposition 3.11. *Let $M, N \in L_Q^x$ and $V \in \operatorname{Rep}(Q)$. Then we have:*

(a) If $N \geq M$, then $\operatorname{rank}_N V \subseteq \operatorname{rank}_M V$ as subspaces of V_x.

(b) Join in L_Q^x corresponds to intersection in V_x, i.e.

$$\operatorname{rank}_{M \vee N} V = \operatorname{rank}_M V \cap \operatorname{rank}_N V.$$

Proof. Part (a) follows from Proposition 3.8 and Lemma 3.7. Part (b) is proven by induction.

Extension: Considering M, N as ideals in L_P^τ, let $\max(M) = \{S_1, \dots, S_m\}$ and $\max(N) = \{T_1, \dots, T_n\}$. Then we can apply Lemma 3.3, keeping in mind the gluing construction of Q_M and Q_N, to get

$$\operatorname{rank}_M V \cap \operatorname{rank}_N V = \left(\bigcap_{i=1}^{m} \operatorname{rank}_{\langle S_i\rangle} V\right) \cap \left(\bigcap_{i=1}^{n} \operatorname{rank}_{\langle T_i\rangle} V\right).$$

By part (a), we only need to intersect over the maximal elements of L_P^τ appearing here. Now since $\langle S_1, \ldots, S_m\rangle \cup \langle T_1, \ldots, T_n\rangle = M \vee N$, we get that

$$\operatorname{rank}_M V \cap \operatorname{rank}_N V = \bigcap_{A \in \max(M \vee N)} \operatorname{rank}_{\langle A\rangle} V = \operatorname{rank}_{M\vee N} V.$$

Gluing: If we write $M = (X, Y)$ and $N = (Z, W)$ for some $X, Z \in L_P^\sigma$ and $Y, W \in L_S^\sigma$, then we have $M \vee N = (X \vee Z, Y \vee W)$, so the result follows from Lemma 3.3 and the induction hypothesis. $\square$

Now we will see that the reduced quivers over Q constructed above give all possible rank functors induced by pullback to a rooted tree.

Theorem 3.12. *Let (Q, σ) be a rooted tree, and $f\colon (Q', \sigma') \to Q$ a rooted tree over Q. Then there exists $M \in L_Q$ such that $Q' \xrightarrow{f} Q$ is rank equivalent to Q_M.*

Proof. The idea is to use induction with Lemma 3.3. When Q has one vertex, a representation of Q is a vector space and the identity functor is the only rank functor on Q. If Q has more than one vertex, assume that the proposition holds for any quiver with fewer vertices. By induction on the number of vertices of Q, we can reduce to the case that $f(\sigma') = \sigma$.

Extension: First, assume that Q' is a one point extension of (P', τ') by an arrow α', so $f(\sigma') = \sigma$ implies that $f(\alpha') = \alpha$ and $f(\tau') = \tau$. The induction hypothesis gives $S \in L_P^\tau$ and morphisms of quivers over P

$$P_S \xrightarrow{\tilde{g}} P' \xrightarrow{\tilde{h}} P_S$$

such that $\tilde{h} \circ \tilde{g} = id$. These morphisms uniquely extend to maps $Q_{\langle S\rangle} \xrightarrow{g} Q' \xrightarrow{h} Q_{\langle S\rangle}$ over Q, satisfying $h \circ g = id$.

Now an arbitrary Q' can be written as $Q' = \coprod_{i\in I}^{\sigma} Q'_i$, where each Q'_i is a one point extension of some (P'_i, τ'_i). Using the previous case, for each $i \in I$ we have $S_i \in L_P^{\tau}$ and morphisms of quivers over Q

$$Q_{\langle S_i\rangle} \xrightarrow{\tilde{g}_i} Q'_i \xrightarrow{\tilde{h}_i} Q_{\langle S_i\rangle}$$

such that $\tilde{h}_i \circ \tilde{g}_i = id$. The elements $\{S_i\}_{i\in I}$ generate an ideal $M \subseteq L_P^{\tau}$ which we claim corresponds to the desired reduced quiver over Q. The ideal M has maximal elements $\max(M) = \{S_i\}_{i\in J}$ for some (not necessarily unique) subset $J \subseteq I$. For $i \in J$, define $\rho_i \colon Q_{\langle S_i\rangle} \xrightarrow{id} Q_{\langle S_i\rangle}$. For $i \notin J$ we can choose some (not necessarily unique) $j(i) \in J$ such that $S_i \leq S_{j(i)}$, and so by Proposition 3.8 there exists a morphism $\rho_i \colon Q_{\langle S_i\rangle} \to Q_{\langle S_{j(i)}\rangle}$ of quivers over Q. The collection $\{\rho_i\}_{i\in I}$ induces a map

$$\rho \colon \coprod_{i\in I}^{\sigma} Q_{\langle S_i\rangle} \to \coprod_{i\in J}^{\sigma} Q_{\langle S_i\rangle} = Q_M$$

of quivers over Q. Then we get a commutative diagram of quivers over Q

$$\begin{tikzcd}
\coprod_{i\in I}^{\sigma} Q_{\langle S_i\rangle} \arrow[r, "\coprod^{\sigma} \tilde{g}_i"] & Q' \arrow[r, "\coprod^{\sigma} \tilde{h}_i"] \arrow[dr] & \coprod_{i\in I}^{\sigma} Q_{\langle S_i\rangle} \arrow[d, "\rho"] \\
Q_M \arrow[u, hook] \arrow[ur] \arrow[rr, "id"] & & Q_M
\end{tikzcd}$$

with the lower triangle giving a rank equivalence between Q' and Q_M.

Gluing: If Q is a gluing of P and S, then Q' is a gluing of $P' := f^{-1}P$ and $S' := f^{-1}S$. By restricting f, we see that P' and S' are quivers over P and S, respectively. Then by the induction hypothesis, we have $X \in L_P^{\sigma}$ and $Y \in L_S^{\sigma}$ and morphisms

$$P_X \xrightarrow{\tilde{g}_P} P' \xrightarrow{\tilde{h}_P} P_X \qquad \text{and} \qquad S_Y \xrightarrow{\tilde{g}_S} S' \xrightarrow{\tilde{h}_S} P_Y$$

that satisfy $\tilde{h}_P \circ \tilde{g}_P = id$ and $\tilde{h}_S \circ \tilde{g}_S = id$. These induce a rank equivalence between Q' and $Q_M = Q_{(X,Y)}$. $\square$

This shows that there are only finitely many distinct rank functors on Q induced by global rank functors of rooted trees over Q.

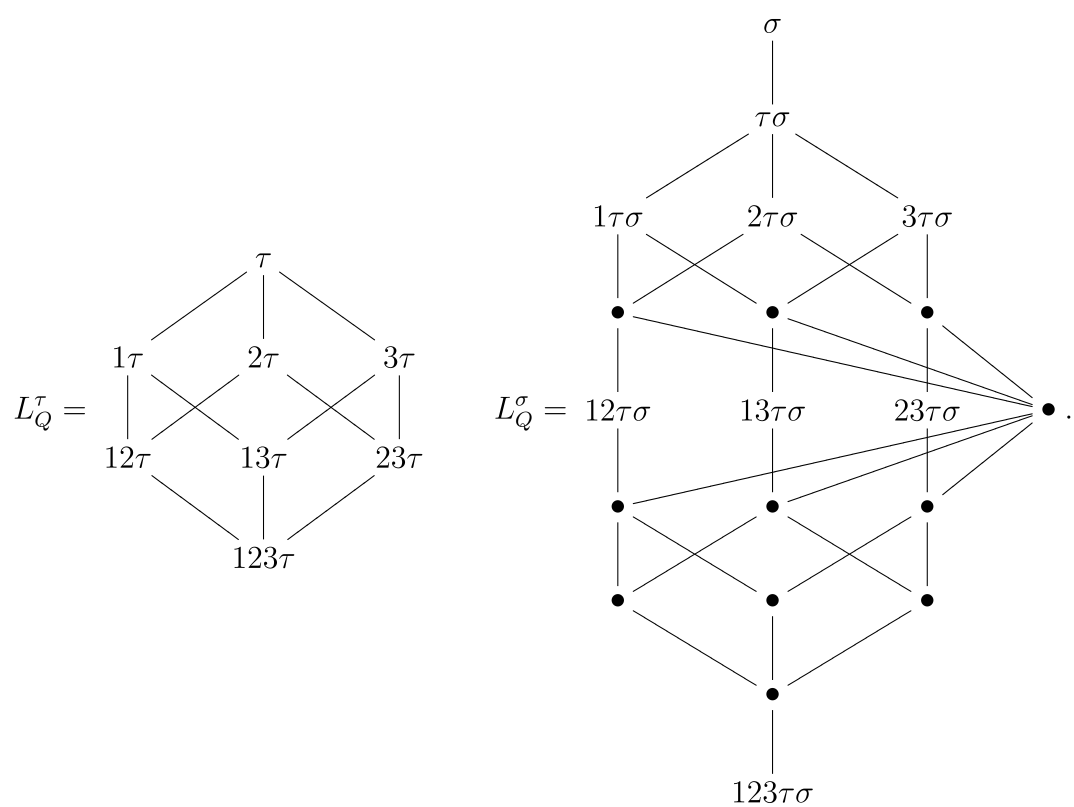
Example 3.13. Generalizing Example 3.4: let (Q, σ) be the n-subspace quiver, labeled as

$$\begin{tikzcd} & & \sigma & \\ 1 \arrow[urr, "a_1"] & 2 \arrow[ur, "a_2"'] & \cdots & n \arrow[ul, "a_n"'] \end{tikzcd}.$$

Then L_Q^x has one element for $x \neq \sigma$, and L_Q^σ is the lattice B_n of subsets of $\{1, \ldots, n\}$. The reduced quivers over Q are exactly the connected subquivers of Q. The rank functor corresponding to $J \subseteq \{1, \ldots, n\}$ sends $V \in \operatorname{Rep}(Q)$ to

$$\operatorname{rank}_J V = \bigcap_{j \in J} \operatorname{Im} V_{a_j}.$$

Example 3.14. Continuing with Q as in Example 3.5, we have that $L_Q^\tau = L_P^\tau = B_3$, and $L_Q^\sigma = J(B_3)$. The Hasse diagrams (with smaller elements drawn towards the top) are illustrated below:



The elements of the lattices that have been labeled are those whose corresponding rank functor is the global rank functor of some subquiver of Q. The label is then the set of vertices in the corresponding subquiver.

It is interesting to note that the lattices L_Q^x are always self-dual. This can easily seen by induction, though a formal proof will not be given since this fact is not used in this thesis.

3.2 Reduced Representations of Q

3.2.1 Construction and first properties

We turn our focus to studying a set of representations of Q, also indexed by L_Q, which are in some sense dual to the rank functors.

Definition 3.15. Let Q be a rooted tree quiver and $M \in L_Q^x$. Define a representation of Q by

$$\Omega_M := c_{M*}\mathbb{I}_{Q_M}$$

which will be called a **reduced representation** of Q.

Example 3.16. Continuing with Q of type $\tilde{D}_4$ from Example 3.5, and taking $M \in L_Q^\sigma$ to be the unique coatom (see diagram above), we get

$$Q = \begin{tikzcd}[row sep=small] 1 \arrow[dr] & & \\ 2 \arrow[r] & \tau \arrow[r] & \sigma \\ 3 \arrow[ur] & & \end{tikzcd} \qquad Q_M = \begin{tikzcd}[row sep=tiny] 1 \arrow[dr] & & & & \\ 2 \arrow[r] & \tau \arrow[dr] & & & 2 \arrow[dl] \\ & & \sigma & \tau \arrow[l] & \\ 1 \arrow[r] & \tau \arrow[ur] & & & 3 \arrow[ul] \\ 3 \arrow[ur] & & & & \end{tikzcd}$$

with the label of a vertex in Q_M denoting the label of its image in Q under c_M. The associated reduced representation of Q is

$$\Omega_M = \begin{tikzcd}[row sep=small] K^2 \arrow[dr, "A"] & & \\ K^2 \arrow[r, "B"', "C"] & K^3 \arrow[r, "(1\,1\,1)"] & K \\ K^2 \arrow[ur] & & \end{tikzcd} \qquad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Such a representation is by definition a **tree module** in the sense of [50], defined over $\mathbb{Z}$ by 0-1 matrices. In general, let $Q' \xrightarrow{c} Q$ be a quiver over Q, and $\{v_y\}_{y \in Q'_\bullet}$ a vector space basis for $\mathbb{I}_{Q'}$ such that $(\mathbb{I}_{Q'})_a(v_{ta}) = v_{ha}$ for every arrow a of Q'. Then this gives the representation $c_*\mathbb{I}_{Q'}$ a vector space basis $\{v_y\}_{y \in Q'_\bullet}$ such that at each vertex $x \in Q_\bullet$, we have

$$(c_*\mathbb{I}_{Q'})_x = \bigoplus_{y \in c_M^{-1}(x)} K v_y.$$

We call such a basis $\{v_y\}$ a **standard basis** for $c_*\mathbb{I}_{Q'}$.

It is natural to ask what kind of morphisms exist between these reduced representations. We saw in Proposition 3.8 that the existence of morphisms between reduced quivers over Q, whose sinks lie over a common vertex $x \in Q_\bullet$, correspond to relations in the poset L_Q^x. One might wonder if the same phenomenon holds for reduced representations of Q, that is, if all homomorphisms between reduced representations are induced from morphisms of quivers over Q. This is not exactly true. The basic problem with this comes from homomorphisms which are 0 at x, the sink of their supports. To get a nice correspondence we must "stabilize" the hom spaces, as defined below.

Definition 3.17. For $M, N \in L_Q^x$, the **stable hom space** between Ω_M and Ω_N is

$$\overline{\mathrm{Hom}}_Q(\Omega_M, \Omega_N) := \frac{\mathrm{Hom}_Q(\Omega_M, \Omega_N)}{\{f \mid f_x = 0\}}.$$

For a set S, let $K\langle S\rangle$ denote the free vector space on the elements of S.

Theorem 3.18. *Let $\Lambda \xrightarrow{\lambda} Q$ and $\Gamma \xrightarrow{\gamma} Q$ be rooted tree quivers over Q. A root preserving map $f\colon \Lambda \to \Gamma$ of quivers over Q induces a morphism $\tilde{f}\colon \lambda_*\mathbb{I}_\Lambda \to \gamma_*\mathbb{I}_\Gamma$ in* $\mathrm{Rep}(Q)$ *such that $\tilde{f}_x \neq 0$, where $x = \lambda(\sigma_\Lambda) = \gamma(\sigma_\Gamma)$. This gives, for $M, N \in L_Q^x$, a surjective map of vector spaces*

$$K\langle \mathrm{Hom}_{\downarrow Q}(Q_M, Q_N)\rangle \twoheadrightarrow \overline{\mathrm{Hom}}_Q(\Omega_M, \Omega_N).$$

In particular, $\overline{\mathrm{Hom}}_Q(\Omega_M, \Omega_N) \neq 0$ *if and only if* $M \leq N$.

Proof. Let $\{v_y\}$ and $\{w_y\}$ be standard bases for $\lambda_* \mathbb{I}_\Lambda$ and $\gamma_* \mathbb{I}_\Gamma$, respectively. Define $\tilde{f}(v_y) := w_{f(y)}$, which gives a map of vector spaces from $\lambda_* \mathbb{I}_\Lambda$ to $\gamma_* \mathbb{I}_\Gamma$. To see that this is a morphism of quiver representations, suppose that $a \in Q_\rightarrow$ is an arrow in Q, and fix $y \in \lambda^{-1}(ta)$. We need to show that

$$\tilde{f} \circ (\lambda_* \mathbb{I}_\Lambda)_a(v_y) = (\gamma_* \mathbb{I}_\Gamma)_a \circ \tilde{f}(v_y).$$

The vertex ta is not a sink, so the assumption that f is root preserving implies that y is not a sink either. Then since Λ is a rooted tree, there is a unique arrow b with $tb = y$. Then by definition, we have $\tilde{f} \circ (\lambda_* \mathbb{I}_\Lambda)_a(v_y) = \tilde{f}(v_{hb}) = w_{f(hb)}$. On the other hand, $\tilde{f}(v_y) = w_{f(y)}$ is by definition. Similarly to the situation above, there is a unique arrow in Γ with tail $f(y)$, and this arrow must be $f(b)$ since b has nowhere else to map to. The head of $f(b)$ must be $f(hb)$ since f is a map of directed graphs. Hence we have $(\gamma_* \mathbb{I}_\Gamma)_a(w_{f(y)}) = w_{f(hb)}$, so $\tilde{f}$ is a map of quiver representations. To simplify the notation, we will drop the tilde throughout the rest of proof. Since f is root preserving, it takes the sink σ_Λ of Λ to the sink σ_Γ of Γ. Then $f_x(v_{\sigma_\Lambda}) = w_{\sigma_\Gamma}$ holds by definition, so f_x is nonzero.

Now take $\Lambda = Q_M$ and $\Gamma = Q_N$ to be reduced quivers over Q, with $\{v_y\}$ and $\{w_y\}$ still standard bases as above. We show that the induced map is surjective, by induction. Given $\overline{f}$ in the stable hom space, choose a representative $f \in \mathrm{Hom}_Q(\Omega_M, \Omega_N)$ of $\overline{f}$. Define $\kappa \in K$ by $f_\sigma(v_\sigma) = \kappa w_\sigma$, which does not depend on the representative f chosen. We can assume $\kappa \neq 0$.

Extension: Let $\max(M) = \{S_1, \dots, S_m\}$ and $\max(N) = \{T_1, \dots, T_n\}$. Then f restricts over P to

$$f|_P = \sum_{ij} f_{ij} \in \mathrm{Hom}_P\left(\bigoplus_i \Omega_{S_i}, \bigoplus_j \Omega_{T_j}\right)$$

with each $f_{ij} \in \mathrm{Hom}_P(\Omega_{S_i}, \Omega_{T_j})$. By induction, there exists for each pair (i,j) a collection

$$g_{ij}^k \in \mathrm{Hom}_{\downarrow P}(P_{S_i}, P_{T_j}) \qquad 1 \leq k \leq d(i,j)$$

and scalars $\lambda_{ij}^k \in K$ such that

$$\overline{f_{ij}} = \sum_{k=1}^{d(i,j)} \lambda_{ij}^k \overline{g_{ij}^k}.$$

Standard bases of Ω_M and Ω_N restrict to standard bases of each Ω_{S_i} and Ω_{T_j}. After normalizing the maps induced by the g_{ij}^k to be compatible with these standard bases at τ, we can assume that

$$\sum_j \sum_{k=1}^{d(i,j)} \lambda_{ij}^k = \kappa. \tag{3.3}$$

We can identify $\mathrm{Hom}_{\downarrow Q}(Q_M, Q_N) = \mathrm{Hom}_{\downarrow P}(\coprod_i P_{S_i}, \coprod_j P_{T_j})$, because any element of the right hand side extends uniquely over the extending arrow α to a morphism on the left hand side. Thus, we can give an element of $\mathrm{Hom}_{\downarrow Q}(Q_M, Q_N)$ in terms of quivers over P by specifying an m-tuple $(j_1, \ldots, j_m)$ with $1 \leq j_i \leq n$, and a sequence of k_i's with $1 \leq k_i \leq d(i, j_i)$

$$(g_{1j_1}^{k_1}, \ldots, g_{mj_m}^{k_m}) \in \mathrm{Hom}_{\downarrow Q}(Q_M, Q_N).$$

We will show that

$$\overline{f} = \kappa^{1-m} \sum_{(j_1,\ldots,j_m)} \sum_{(k_1,\ldots,k_m)} \left[\left(\prod_{i=1}^m \lambda_{ij_i}^{k_i} \right) \overline{(g_{1j_1}^{k_1}, \ldots, g_{mj_m}^{k_m})} \right] \tag{3.4}$$

where the first two sums are indexed as above. The key is to see that

$$\sum_{(j_1,\ldots,j_m)} \sum_{(k_1,\ldots,k_m)} \left(\prod_{i=1}^m \lambda_{ij_i}^{k_i} \right) = \kappa^m \tag{3.5}$$

holds in the field K, which can be shown by an easy induction on m, using the assumption of equation (3.3). Now over P, the map induced on representations by

$(g^{k_1}_{1j_1}, \ldots, g^{k_m}_{mj_m})$ restricts to the sum

$$\sum_i g^{k_i}_{ij_i} \in \mathrm{Hom}_P \left(\bigoplus_i \Omega_{S_i}, \bigoplus_j \Omega_{T_j} \right).$$

Using this we can compute the coefficient of $\overline{g^c_{ab}}$ in the restriction of the right hand side of equation (3.4) to P. By factoring out λ^c_{ab}, which always appears when $\overline{g^c_{ab}}$ does, we get

$$\kappa^{1-m}\lambda^c_{ab} \sum_{(j_1,\ldots,\hat{j_a},\ldots,j_m)} \sum_{(k_1,\ldots,\hat{k_a},\ldots,k_m)} \prod_{i\neq a} \lambda^{k_i}_{ij_i} = \kappa^{1-m}\lambda^c_{ab}\kappa^{m-1} = \lambda^c_{ab}$$

where $\hat{j_a}$ and $\hat{k_a}$ mean to omit those indices. Then the next to last equality follows from equation (3.5). Hence the right hand side of equation (3.4) agrees with $\overline{f}$ over P. But at σ, each map $(g^{k_1}_{1j_1}, \ldots, g^{k_m}_{mj_m})$ sends v_σ to w_σ, and so the right hand side of equation (3.4) maps v_σ to

$$\kappa^{1-m} \sum_{(j_1,\ldots,j_m)} \sum_{(k_1,\ldots,k_m)} \left(\prod_{i=1}^m \lambda^{k_i}_{ij_i} \right) w_\sigma = \kappa^{1-m}\kappa^m w_\sigma = \kappa w_\sigma$$

which agrees with $\overline{f}$ also. So equation (3.4) holds, expressing $\overline{f}$ as a linear combination of morphisms induced by maps of quivers.

Gluing: Let $M = (X, Y)$ and $N = (Z, W)$. By induction, we can write

$$\overline{f}|_P = \sum_i \lambda_i \overline{g_i} \qquad\qquad \overline{f}|_S = \sum_j \mu_j \overline{h_j} \qquad \lambda_i, \mu_j \in K$$

for some collection of $g_i \in \mathrm{Hom}_{\downarrow P}(P_X, P_Z)$ and $h_j \in \mathrm{Hom}_{\downarrow S}(S_Y, S_W)$. Note that, since each g_i and h_j sends v_σ to w_σ, we have that

$$\sum_i \lambda_i = \sum_j \mu_j = \kappa$$

in order for the restrictions over P and S to be equal at σ. Let

$$(g_i, h_j) \in \mathrm{Hom}_{\downarrow Q}(Q_M, Q_N) = \mathrm{Hom}_{\downarrow P}(P_X, P_Z) \times \mathrm{Hom}_{\downarrow S}(S_Y, S_W)$$

be the morphism of quivers over Q defined by g_i over P and h_j over S. Then it is straightforward to check that

$$\overline{f} = \kappa^{-1} \sum_{ij} \lambda_i \mu_j \overline{(g_i, h_j)}$$

which completes the proof. □

One may note the similarity in spirit of this theorem to a theorem of Crawley-Boevey [16] on morphisms between tree modules over zero-relation algebras. As a corollary, we get an alternative characterization of reduced quivers over Q.

Corollary 3.19. *If $Q' \xrightarrow{c} Q$ is a rooted tree quiver over (Q, σ), then $c_* \mathbb{I}_{Q'}$ is indecomposable if and only if Q' is a reduced quiver over Q. Thus, the reduced representations Ω_M are indecomposable and pairwise non-isomorphic.*

Proof. Suppose that $Q' \xrightarrow{c} Q$ is not reduced. Then by Theorem 3.12, there is some $M \in L_Q$ and there are maps $Q_M \xrightarrow{g} Q' \xrightarrow{h} Q_M$ of quivers over Q such that $h \circ g = id$. By Theorem 3.18, this induces $\Omega_M \xrightarrow{\tilde{g}} c_* \mathbb{I}_{Q'} \xrightarrow{\tilde{h}} \Omega_M$ such that $\tilde{h} \circ \tilde{g} = id$, and so Ω_M is a direct summand of $c_*(\mathbb{I}_{Q'})$.

Now suppose that $Q' = Q_M$ is reduced. To prove that Ω_M is indecomposable, we use induction and the theorem.

Extension: Let $\max(M) = \{S_1, \ldots, S_m\}$. If $\Omega_M \simeq V \oplus W$ for some subrepresentations $V, W \subseteq \Omega_M$, then since $\dim_K(\Omega_M)_\sigma = 1$, without loss of generality we can assume $W_\sigma = 0$. By construction, there is a decomposition $\Omega_M|_P \simeq \bigoplus_{i=1}^m U_i$ for some subrepresentations $U_i \simeq \Omega_{S_i}$, and by the induction hypothesis each of these summands is indecomposable. Using a standard basis for Ω_M, we can even take these U_i such that $(\Omega_M)_\alpha((U_i)_\tau) \neq 0$ for all i. But using the decomposition $\Omega_M \simeq V \oplus W$, and the fact that the indecomposable summands are uniquely determined, we can also find a decomposition $\Omega_M|_P \simeq \bigoplus_i X_i$ for some subrepresentations $X_i \simeq \Omega_{S_i}$, such that (after perhaps renumbering) $X_i \subseteq V|_P$ for $1 \leq i \leq k$, and $X_i \subseteq W|_P$ for $k+1 \leq i \leq m$.

Then an isomorphism $\Omega_M \simeq V \oplus W$ would give a commutative diagram

$$\begin{tikzcd} \bigoplus_i (U_i)_\tau \arrow[r, "A"] \arrow[d, "B"] & K \arrow[d, "C"] \\ \bigoplus_i (X_i)_\tau \arrow[r, "D"] & K \end{tikzcd}$$

over the extending arrow α such that both vertical maps are isomorphisms. The theorem implies that, since the elements of $\{S_1, \ldots, S_m\}$ are pairwise incomparable,

$$\overline{\mathrm{Hom}}_P(\Omega_{S_i}, \Omega_{S_j}) = \begin{cases} K & i = j \\ 0 & i \neq j \end{cases}.$$

Hence the matrix giving B is diagonal, say with entries λ_i. But since W is a direct summand and $W_\sigma = 0$, we get $D \circ B\left((U_{k+1})_\tau\right) = D\left((X_{k+1})_\tau\right) = 0$, whereas $C \circ A\left((U_{k+1})_\tau\right) = C\left(K\right) = K$. Since the diagram is supposed to commute, this is a contradiction; hence no nontrivial direct sum decomposition of Ω_M exists.

Gluing: If $\Omega_M \simeq V \oplus W$ is a nontrivial decomposition, then it must restrict to a nontrivial decomposition over either P or S. But Ω_M restricts to some reduced representation on both P and S, which by induction is indecomposable. Hence Ω_M has no nontrivial decomposition. Now pairwise non-isomorphic follows since $\overline{\mathrm{Hom}}_Q(\Omega_M, \Omega_N) \neq 0$ implies that $\overline{\mathrm{Hom}}_Q(\Omega_N, \Omega_M) = 0$ for $M \neq N$. $\square$

3.2.2 Combinatorial adjunctions and reduced representations

Our goal is to gain some understanding of the structure of the representation ring $R(Q)$ through the representation rings $R(Q_M)$. We have seen that the order relations in the lattices L_Q^σ encode a lot of information about morphisms between quivers over Q, and morphisms between the reduced representations of Q. So in order to connect the representation theory of Q and Q_M, it is natural to seek some combinatorial connection between the lattices L^σ_Q and $L^\sigma_{Q_M}$. Summarily, we will see that for any $M \in L^\sigma_Q$, there is an *adjunction* (sometimes called a *Galois connection*) between the lattices $L^\sigma_{Q_M}$ and L^σ_Q. Simply put, an **adjunction** is a pair of maps

$$A \underset{\rho}{\overset{\lambda}{\rightleftarrows}} B$$

between posets A and B, such that

$$a \leq \rho(b) \iff \lambda(a) \leq b$$

holds for all $a \in A, b \in B$. We say that (λ, ρ) are an adjoint pair with λ the lower adjoint, and ρ the upper adjoint. The following proposition extracts from [21, Prop. 3,4] a summary of what we'll need to use from the theory of adjunctions.

Proposition 3.20. *Suppose $\lambda\colon A \to B$ is any order preserving map between finite lattices. If λ preserves joins, then it has a unique upper adjoint given by*

$$\rho(b) = \max\{a \in A \mid \lambda(a) \leq b\} \tag{3.6}$$

which, furthermore, preserves meets.

Now fix $M \in L^\sigma_Q$. Then for any $A \in L^\sigma_{Q_M}$, the composition of structure maps

$$(Q_M)_A \xrightarrow{c_A} Q_M \xrightarrow{c_M} Q$$

gives a quiver over Q. By Theorem 3.12, this quiver over Q is rank equivalent to a unique reduced quiver over Q, which we'll denote by $Q_{\pi_*(A)}$. This gives a map of sets

$$\pi_*\colon L^\sigma_{Q_M} \to L^\sigma_Q$$

such that

$$\mathrm{rank}_{\pi_*(A)} = \mathrm{rank}_A \circ c^*_M$$

for $A \in L^\sigma_{Q_M}$.

Proposition 3.21. *Let (Q,σ) be a rooted tree quiver and $M \in L^\sigma_Q$. Then there is an adjunction*

$$L^\sigma_{Q_M} \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} L^\sigma_Q\ .$$

Proof. To simplify the notation, let $c := c_M$, $L' := L^\sigma_{Q_M}$, and $L := L^\sigma_Q$. If $A \leq B$ in L', then there exists a map $\rho\colon (Q_M)_A \to (Q_M)_B$ of quivers over Q_M, by Proposition 3.8. Composing with the structure map c gives a map between them as quivers over Q, so using the definition of rank equivalence we get a sequence of maps over Q

$$Q_{\pi_*(A)} \longrightarrow (Q_M)_A \xrightarrow{\rho} (Q_M)_B \longrightarrow Q_{\pi_*(B)}$$

and hence $\pi_*(B) \geq \pi_*(A)$. So π_* is order preserving. Now we can simply calculate

$$\begin{aligned}\mathrm{rank}_{\pi_*(A\vee B)} &= \mathrm{rank}_{A\vee B}\circ c^* = \mathrm{rank}_A\circ c^* \cap \mathrm{rank}_B \circ c^*\\ &= \mathrm{rank}_{\pi_*(A)} \cap \mathrm{rank}_{\pi_*(B)} = \mathrm{rank}_{\pi_*(A)\vee\pi_*(B)}\end{aligned}$$

which shows that π_* preserves the join operation. By Proposition 3.20,

$$\pi^*(N) = \max\{A \mid \pi_*(A) \leq N\}$$

is the upper adjoint to π_*. $\square$

One can show that $(Q_M)_{\pi^*(N)}$ is the fiber product of Q_N and Q_M over Q. This fact won't be proven, however, since it won't be needed in this thesis. Now in order to use this adjunction to inductively study the representation theory of Q, we need show that it is compatible with gluing and extension in an appropriate sense.

Lemma 3.22. *Fix $M \in L^\sigma_Q$ and let (π_*, π^*) be the adjunction of Proposition 3.21. Suppose Q is obtained from P by extension, and that* $\max(M) = \{S_1, \ldots, S_m\} \subseteq L^\tau_P$. *For each $1 \leq i \leq m$, let (π^i_*, π^*_i) be the adjunction of Proposition 3.21 corresponding* *to* (P,τ) *and* S_i*. Then, if we write* $A \in L^\sigma_{Q_M}$ *as a product of ideals* $(A_1, \dots, A_m) \in \prod_{i=1}^m J(L^\tau_{P_{S_i}})$*, we have*

$$\pi_*(A) = \bigcup_{i=1}^{m} \pi^i_*(A_i) \tag{3.7}$$

(on the right hand side π^i_* *is the induced map on ideals as in §3.1.2). Similarly, for* $N \in L^\sigma_Q$ *we get*

$$\pi^*(N) = (\pi^*_1(N), \dots, \pi^*_m(N)) \tag{3.8}$$

where N *is considered as an ideal of* L^τ_P *on the right hand side.*

When Q *is glued together from* P *and* S*, and we write* $M = (X,Y) \in L^\sigma_P \times L^\sigma_S$*, then we have adjunctions* (π^P_*, π^*_P) *and* (π^S_*, π^*_S) *over* P *and* S*, respectively. In this case, for* $(A,B) \in L^\sigma_{P_X} \times L^\sigma_{S_Y} = L^\sigma_{Q_M}$ *we have*

$$\pi_*((A,B)) = (\pi^P_*(A), \pi^S_*(B)), \tag{3.9}$$

and similarly, for $N = (Z,W)$ *we find*

$$\pi^*(N) = (\pi^*_P(Z), \pi^*_S(W)). \tag{3.10}$$

Proof. As usual, we proceed by induction, retaining the notation from the statement of the lemma.

Extension: First consider the case that $M = \langle S \rangle$ is principal, so Q_M is a one point extension of (P_S, τ), and $L^\sigma_{Q_M} = J(L^\tau_{P_S})$. For any principal ideal $\langle T \rangle \subseteq L^\tau_{P_S}$, we know $\operatorname{rank}_{\langle T \rangle} V = V_\alpha(\operatorname{rank}_T V|_{P_S})$ from Lemma 3.3, so we can simply compute

$$\begin{aligned}\operatorname{rank}_{\pi_*(\langle T \rangle)} V &= \operatorname{rank}_{\langle T \rangle}(c^*_M V) = V_\alpha(\operatorname{rank}_T(c^*_S(V|_P))) \\ &= V_\alpha(\operatorname{rank}_{\pi^1_*(T)}(V|_P)) = \operatorname{rank}_{\langle \pi^1_*(T) \rangle} V\end{aligned}$$

which shows that $\pi_*(\langle T \rangle) = \langle \pi^1_*(T) \rangle$. We know that π_* commutes with join and π^P_* commutes with union of ideals, and these two operations correspond with one another in the identification $L^\sigma_Q = J(L^\tau_P)$. Thus equation (3.7) holds for an arbitrary element of $L^\sigma_{Q_M}$ when M is principal.

In general, suppose that $\max(M) = \{S_1, \dots, S_m\}$, so we have the identification

$$L^\sigma_{Q_M} = \prod_{i=1}^{m} L^\sigma_{Q_{\langle S_i \rangle}}.$$

Under natural embeddings $L^\sigma_{Q_{\langle S_i \rangle}} \hookrightarrow L^\sigma_{Q_M}$ of (3.2), every element of $L^\sigma_{Q_M}$ can be written as a join of elements in the images of these, so again the correspondence between join and union shows that equation (3.7) holds in general.

Gluing: For any $(A, B) \in L^\sigma_{Q_M} = L^\sigma_{P_X} \times L^\sigma_{S_Y}$, we can compute

$$\begin{aligned} \operatorname{rank}_{\pi_*(A,B)} &= \operatorname{rank}_{(A,B)} \circ c^* = \operatorname{rank}_A \circ c^*|_P \cap \operatorname{rank}_B \circ c^*|_S \\ &= \operatorname{rank}_{\pi^P_*(A)} \cap \operatorname{rank}_{\pi^S_*(B)} = \operatorname{rank}_{(\pi^P_*(A), \pi^S_*(B))} \end{aligned}$$

which shows that $(\pi^P_*(A), \pi^S_*(B)) = \pi_*(A, B)$.

A routine argument using uniqueness of upper adjoints from Proposition 3.20 gives the formulas for π^*. $\square$

In light of this discussion, we will often omit notation indicating what base quiver an adjunction is over, letting the context make it clear. We use this compatibility to inductively prove some combinatorial properties of such an adjunction. Recall that a **coatom** of a finite lattice is an element immediately preceding $\hat{1}$, that is, an element that $\hat{1}$ covers.

Lemma 3.23. *The maps π_* and π^* of Proposition 3.21 have the following properties:*

(a) $\pi_*(\hat{1}) = M$ and $\pi^*(M) = \hat{1}$,

(b) $\pi^*(N) = \pi^*(M \wedge N)$ and $\pi_* \circ \pi^*(N) = M \wedge N$,

(c) they restrict to inverse bijections between the sets of coatoms of $L^\sigma_{Q_M}$ and $\langle M \rangle$.

Proof. By definition of rank_M, we have that $\pi_*(\hat{1}) = M$. Then the expression for an upper adjoint in equation (3.6) implies that $\pi^*(M) = \hat{1}$, and from the last statement of the same proposition we get

$$\pi^*(N) = \pi^*(N) \wedge \hat{1} = \pi^*(N) \wedge \pi^*(M) = \pi^*(M \wedge N).$$

To see that $\pi_* \circ \pi^*(N) = M \wedge N$, we proceed by induction using Lemma 3.22, retaining the notation from the statement of the lemma.

Extension: In this case, the induction hypothesis implies that $\pi^i_* \circ \pi^*_i(T) = S_i \wedge T$ for $T \in L^\tau_P$, so the induced map on ideals sends $N \in L^\sigma_Q$ to $\langle S_i \rangle \wedge N$. Then applying equations (3.7) and (3.8) gives

$$\pi_* \circ \pi^*(N) = \bigcup_{i=1}^{m} \pi^i_* \circ \pi^*_i(N) = \bigcup_{i=1}^{m} \langle S_i \rangle \wedge N = M \wedge N.$$

Gluing: By the induction hypothesis, $\pi^P_* \circ \pi^*_P(Z) = X \wedge Z$, and similarly over S. So applying equations (3.9) and (3.10) to $N = (Z, W)$ gives

$$\begin{aligned}\pi_* \circ \pi^*(N) &= \pi_* \left((\pi^*_P(Z), \pi^*_S(W)) \right) = (\pi^P_* \circ \pi^*_P(Z), \pi^S_* \circ \pi^*_S(W)) \\ &= (X \wedge Z, Y \wedge W) = (X, Y) \wedge (Z, W) = M \wedge N.\end{aligned}$$

The third item is proven by induction.

Extension: Let $\max(M) = \{S_1, \ldots, S_m\}$, so that the coatoms of $\langle M \rangle$ are the ideals

$$I_j = M \setminus \{S_j\}$$

obtained by removing one maximal element from $\langle M \rangle$. The coatoms of $\prod_i J(L^{\tau_i}_{P_{S_i}})$ are of the form $(\hat{1}, \ldots, \hat{1}, C_i, \hat{1}, \ldots, \hat{1})$, where $C_i := L^{\tau_i}_{P_{S_i}} \setminus \{\hat{1}\}$ is the unique coatom of $J(L^{\tau_i}_{P_{S_i}})$. The formulas of the previous lemma, along with the previous two items of this lemma, give the desired bijection.

Gluing: If $M = (X, Y)$, then $\langle M\rangle = \langle X\rangle \times \langle Y\rangle$, and the coatoms of $\langle M\rangle$ are of the form (C, Y) and (X, C) with C a coatom of L_P^σ or L_S^σ as appropriate. The coatoms of $L_{Q_M}^\sigma$ are of the form $(C', \hat{1})$ or $(\hat{1}, C')$, where C' is a coatom of $L_{P_X}^\sigma$ or $L_{P_Y}^\sigma$. If we assume that the lemma holds over P and S, by induction, then the compatibility of π^* and π_* with gluing implies the lemma for Q. $\square$

With these facts in hand, we can realize this combinatorial adjunction in a representation theoretic setting. A technical lemma will clean up the proof of the theorem giving this realization; first we recall another notion from combinatorics which will be necessary for the lemma. A **quasi-order** on a set X is a binary relation $\preceq$ which is both reflexive and transitive. That is, $x \preceq x$ for every $x \in X$ and if $x \preceq y$ and $y \preceq z$, then $x \preceq z$. A quasi-order for which $x \preceq y \preceq x$ implies that $x = y$ is precisely the definition of a partial-order. So, defining an equivalence relation on a quasi-ordered set X by

$$x \sim y \iff x \preceq y \preceq x$$

induces a natural partial-order the set of $\sim$-equivalence classes in X.

Lemma 3.24. *Suppose that the rooted tree quiver (Q, σ) is obtained from (P, τ) by extension along an arrow $\tau \xrightarrow{\alpha} \sigma$, and that $V \in \mathrm{Rep}(Q)$ is a representation of Q with $\dim_K V_\sigma = 1$. Assume that the restriction of V to P decomposes as $V|_P \simeq \bigoplus_{i\in I} U_i \oplus \tilde{U}$ for some subrepresentations $U_i, \tilde{U} \subset V|_P$, with $\dim_K (U_i)_\tau = 1$ and $\dim_K \tilde{U}_\tau = 0$. Furthermore, assume that V_α restricts to an isomorphism*

$$V_\alpha \colon (U_i)_\tau \xrightarrow{\sim} V_\sigma$$

on each summand U_i.

The set $\{U_i\}$ is quasi-ordered by the relation

$$U_i \preceq U_j \iff \text{there exists } f\colon U_i \to U_j \text{ such that } f_\tau \neq 0$$

which induces an equivalence relation $\sim$ on $\{U_i\}$ as described above. Let $J \subseteq I$ be such that $\{U_j\}_{j \in J}$ contains exactly one element of each maximal equivalence class, with respect to the induced partial order.

Then V has a direct sum decomposition $V \simeq X \oplus Y$, with

$$X|_P \simeq \bigoplus_{j \in J} U_j \qquad \text{and} \qquad X_\sigma = V_\sigma$$

(and hence $Y_\sigma = 0$).

Proof. Fix a basis $\{u_i\}$ of V_τ such that each $u_i \in U_i$, and $V_\alpha(u_i) = V_\alpha(u_j)$ for all i, j (for instance, fix a nonzero $v_\sigma \in V_\sigma$ and set each u_i to be the preimage of v_σ in U_i). Then for each $i \notin J$, there exists some (not necessarily unique) $j(i) \in J$ and $f_i \in \mathrm{Hom}_P(U_i, U_{j(i)})$ with $(f_i)_\tau \neq 0$. Multiplying by a scalar, if necessary, we can assume $f_i(u_i) = u_{j(i)}$. Now define

$$\varphi := id - \sum_{i \notin J} f_i \in \mathrm{End}_P\left(\bigoplus_{i \in I} U_i\right)$$

so that $f_i(U_j) = 0$ for all $i \notin J$ and $j \in J$. Then φ restricts to the identity on the subrepresentation $\bigoplus_{j \in J} U_j \subset V|_P$, hence φ splits the inclusion of these summands. Furthermore, $\ker \varphi \simeq \bigoplus_{i \notin J} U_i \in \mathrm{Rep}(P)$ by the Krull-Schmidt theorem. Since $(\ker \varphi)_\tau$ is generated by

$$\{u_i - u_{j(i)} \mid i \notin J\} \subset \ker V_\alpha,$$

the subrepresentation $\ker \varphi$ extends by 0 to a representation Y of Q satisfying the conclusion of the theorem. □

Theorem 3.25. *Let $M \in L_Q^\sigma$ and define $c := c_M$. Let (π_*, π^*) be the corresponding adjunction of Proposition 3.21. Then for any $N \in L_Q^\sigma$, we have*

$$c^* \Omega_N \simeq \Omega_{\pi^* N} \oplus U$$

where U is a direct sum of reduced representations of Q_M such that $\sigma \notin \operatorname{supp} U$. Similarly, for $A \in L^{\sigma}_{Q_M}$ we have

$$c_* \Omega_A \simeq \Omega_{\pi_* A} \oplus U'$$

with U' a direct sum of reduced representations of Q and $\sigma \notin \operatorname{supp} U$.

Proof. The proof is by induction on the number of vertices of Q.

Extension: First consider the case that $M = \langle S \rangle$ is principal, so Q_M is a one point extension of (P_S, τ), and there is an adjunction

$$L^{\tau}_{P_S} \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} L^{\tau}_{P} \ .$$

Let $\max(N) = \{T_1, \ldots, T_n\} \subset L^{\tau}_P$, so by the induction hypothesis we get

$$c^* \Omega_N|_{P_S} = c^*_S \left(\bigoplus_{i=1}^{n} \Omega_{T_i} \right) \simeq \bigoplus_{i=1}^{n} \Omega_{\pi^*(T_i)} \oplus \tilde{U}$$

with $\tilde{U}$ a direct sum of reduced representations of P_S, and $\tau \notin \operatorname{supp} \tilde{U}$ by dimension count. Since $(c^*\Omega_N)_\alpha = (\Omega_N)_\alpha$ restricts to an isomorphism on each summand $\Omega_{\pi^*(T_i)}$, the representation $c^*\Omega_N$ with the decomposition above satisfies the hypothesis of the previous lemma. By Theorem 3.18, a maximal subset of $\{\Omega_{\pi^*(T_i)}\}$ (with respect to the ordering in the previous lemma) is given by taking those representations indexed by the set of maximal elements of $\{\pi^*(T_i)\}$. Then the lemma gives a direct sum decomposition

$$c^* \Omega_N \simeq W \oplus U$$

with U a direct sum of reduced representations, $U_\sigma = 0$, and $W|_P \simeq \bigoplus \Omega_{\pi^*(T_i)}$, the sum taken over maximal elements of $\{\pi^*(T_1), \ldots, \pi^*(T_n)\}$. By construction of the reduced representations, $W \simeq \Omega_{\tilde{N}}$, where $\tilde{N} := \langle \pi^*(T_1), \ldots, \pi^*(T_n) \rangle$. The compatibility of π^* with extension from Lemma 3.22 then gives that $\tilde{N} = \pi^*(N)$.

For a general M, say with $\max(M) = \{S_1, \ldots, S_m\}$, we have adjunctions

$$L^{\tau_i}_{P_{S_i}} \underset{\pi_i^*}{\overset{\pi_*^i}{\rightleftarrows}} L^{\tau}_P$$

and by the case above we know that $c^*\Omega_N|_{Q_{\langle S_i\rangle}} \simeq \Omega_{\pi_i^*(N)} \oplus U_i$, with U_i a direct sum of reduced representations. But, since $Q = \coprod_i^\sigma Q_{\langle S_i\rangle}$, taking $U := \bigoplus U_i$ we get

$$c^*\Omega_N \simeq \Omega_{(\pi_1^*(N),\ldots,\pi_m^*(N))} \oplus U = \Omega_{\pi^*(N)} \oplus U$$

where the last equality follows from the compatibility of π^* with gluing.

The proof for pushforward is similar, and σ is not in the support of U by dimension count.

Gluing: Write $M = (X, Y)$ and $N = (Z, W)$ in $L^\sigma_Q = L^\sigma_P \times L^\sigma_S$. Over P, we have an adjunction

$$L^\sigma_{P_X} \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} L^\sigma_P \, .$$

By the induction hypothesis, we have

$$c^*\Omega_N|_{P_X} = c_X^*\Omega_Z \simeq \Omega_{\pi^*(Z)} \oplus U_P \qquad \text{and} \qquad c^*\Omega_N|_{S_Y} = c_Y^*\Omega_W \simeq \Omega_{\pi^*(W)} \oplus U_S.$$

Since $\sigma \notin \operatorname{supp} U_P \cup \operatorname{supp} U_S$ by dimension reasons, if we set $U := U_P \oplus U_S$ we get that

$$c^*\Omega_N \simeq \Omega_{(\pi^*(Z),\pi^*(W))} \oplus U = \Omega_{\pi^*(N)} \oplus U.$$

The proof for pushforward is similar, and $\sigma \notin \operatorname{supp} U'$ by dimension reasons again.

$\square$

The following lemma, valid for any quiver Q (not just rooted trees), gives an essential connection between the tensor product in $\mathrm{Rep}(Q)$ and quivers over Q.

Lemma 3.26. *Let Q be any quiver, $c\colon Q' \to Q$ any quiver over Q, and $V \in \mathrm{Rep}(Q)$. Then there is an isomorphism*

$$(c_*\mathbb{I}_{Q'}) \otimes V \cong c_*c^*V.$$

Proof. For each $y \in Q_\bullet$ we have an isomorphism of vector spaces

$$(c_* \mathbb{I}_{Q'} \otimes V)_y = \left(\bigoplus_{x \in c^{-1} y} \mathbb{I}_x \right) \otimes V_y \cong \bigoplus_{x \in c^{-1} y} V_y = \bigoplus_{x \in c^{-1} y} (c^* V)_x = (c_* c^* V)_y$$

such that for each arrow $a \in Q_\rightarrow$, the maps over a are identified under this isomorphism:

$$(c_* \mathbb{I}_{Q'} \otimes V)_a = \sum_{b \in c^{-1} a} \mathbb{I}_b \otimes V_a = \sum_{b \in c^{-1} a} \mathbb{I}_b \otimes (c^* V)_b = \sum_{b \in c^{-1} a} (c^* V)_b = (c_* c^* V)_a. \qquad \square$$

This lemma allows us to compute the tensor product of reduced representations as a corollary of the theorem.

Corollary 3.27. *For $M, N \in L_Q^x$, there is an isomorphism*

$$\Omega_M \otimes \Omega_N \simeq \Omega_{M \wedge N} \oplus U$$

where U is a direct sum of reduced representations without x in its support.

Proof. Let $c := c_M$, so that by Theorem 3.25 and Lemma 3.26, we have

$$\Omega_M \otimes \Omega_N \simeq c_* c^* (\Omega_N) \simeq c_* (\Omega_{\pi^* N} \oplus U') \simeq \Omega_{\pi_*(\pi^*(N))} \oplus c_* U' = \Omega_{M \wedge N} \oplus U$$

with the last equality following from Lemma 3.23. That U is a direct sum of reduced representations also follows from Theorem 3.25, and x cannot be in the support of U because the other three representations appearing in the formula have dimension one at x. $\square$

For $M \in L_Q^x$, define the corresponding **rank function** on $V \in \mathrm{Rep}(Q)$ by

$$r_M(V) := \dim_K(\mathrm{rank}_M V) \in \mathbb{Z}_{\geq 0}.$$

Corollary 3.28. *Evaluation of rank functions on reduced representations is given by the zeta function of L_Q^x. That is, for $M, N \in L_Q^x$, we have*

$$r_M(\Omega_N) = \zeta(M, N) := \begin{cases} 1 & M \leq N \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. From Theorem 2.32, $r_M(\Omega_N)$ counts the number of direct summands of $c_M^*\Omega_N$ which are isomorphic to $\mathbb{I}_{Q_M}$. By Theorem 3.25, this number is nonzero if and only if $\pi^*(N) = \hat{1}$, which is if and only if $M \leq N$ by Lemma 3.23. This number can be at most 1 since $\dim_K(\Omega_N)_\sigma = 1$. □

This corollary strengthens part (a) of Proposition 3.11 to: "$N \geq M$ in L_Q^x if and only if there is an inclusion of functors $\mathrm{rank}_N \subseteq \mathrm{rank}_M$".

Remark 3.29. With a little more work, the previous corollary can be extended to show that the relation over L_Q given by

$$M \leq N \iff r_M(\Omega_N) \neq 0$$

for all $M, N \in L_Q$ is a partial ordering. This partial order on all of L_Q can also be obtained in a purely combinatorial way, without reference to rank functors or representations, by "patching together" the lattices L_Q^x. More precisely, we start with the partial order on L_Q inherited from its definition as the disjoint union of all L_Q^x, and add certain relations for each arrow of Q as follows. Let a be an arrow of Q. By construction, we have an inclusion

$$J(L_Q^{ta}) \subseteq L_Q^{ha} \subseteq L_Q.$$

Then for every $S \in L_Q^{ta}$, we add the relation $S \leq \langle S \rangle$ in L_Q, and refine the inherited partial order on L_Q to include these additional relations. Then the cover relations in this partial ordering of L_Q correspond to more morphisms between rank functors and between reduced representations. However, L_Q is not a lattice if Q has more than one vertex, and most of our correspondences between combinatorial properties of L_Q and representation theoretic statements do not generalize as neatly as Corollary 3.28. For example, the map sending $M \in L_Q$ to $\Omega_M \in R(Q)$ does not induce a ring isomorphism between the Möbius algebra of L_Q and the subalgebra of $R(Q)$ generated by $\{\Omega_M\}_{M\in L_Q}$. It is much easier to work with the individual lattices L_Q^x, and this ordering on the entire set L_Q will not be needed in any proofs in this thesis; hence we omit formal proofs of the statements in this remark.

3.3 Factors of the Representation Ring

We now have enough tools to start an analysis of the representation ring of a rooted tree quiver. Recall that the representation ring $R(Q)$ of a quiver Q (Section 2.1.3) is defined as the free abelian group on the isomorphism classes of representations of Q, modulo the subgroup generated by elements $[V]+[W]-[V\oplus W]$, with multiplication given by $[V][W]=[V\otimes W]$.

3.3.1 The Support Algebra of a Quiver

In this subsection, Q can be any quiver, possibly disconnected or with oriented cycles and parallel arrows. The set $\mathscr{S}$ of connected subquivers of Q is partially ordered by inclusion. We will show that the Möbius algebra of $\mathscr{S}$ over $\mathbb{Z}$ is naturally a subalgebra of the representation ring of Q, decomposing $R(Q)$ into a product of rings. This generalizes some results of [34, §3], while at the same time putting them in a natural combinatorial setting.

The **Möbius algebra** (over $\mathbb{Z}$) of a poset P [27], written $A(P,\mathbb{Z})$, is defined as the free $\mathbb{Z}$-module on the elements of P, with multiplication of two of these basis elements $x,y\in P$ given by

$$x\cdot y := \sum_{s\in P}\left[\sum_{s\le t\le x,y}\mu(s,t)\right]s \tag{3.11}$$

where μ is the Möbius function of P. In case the meet of x and y exists, this simplifies to $x\cdot y = x\wedge y$, so if P is a meet semi-lattice then $A(P,\mathbb{Z})$ is just $\mathbb{Z}[L;\wedge]$, the semi-group ring of L with respect to the meet operator. The multiplication defined in (3.11) is precisely the structure that gives a $\mathbb{Z}$-basis for $A(P,\mathbb{Z})$ of orthogonal idempotents

$$\delta_x := \sum_{y \leq x} \mu(y,x)y \qquad x \in P$$

via Möbius inversion. The original basis elements can be recovered as $y = \sum_{x \leq y} \delta_x$.

The poset $\mathscr{S}$ is a meet semi-lattice if and only if Q is a tree. When Q is not a tree, we can still view the multiplication in $A := A(\mathscr{S},\mathbb{Z})$ in a more intuitive way than one might expect from the expression in (3.11). Given $P, S \in \mathscr{S}$, let $\{M_i\}$ be the set of maximal connected subquivers (i.e., the connected components) of $P \cap S$. If U is connected and contained in both P and S, then U is contained in some *unique* M_i. Furthermore, any connected T containing U but also contained in both P and S will itself also be contained in M_i. In other words, for each $U \leq P, S$ there exists a unique i such that

$$\{T \in \mathscr{S} \mid U \leq T \leq P, S\} = \{T \in \mathscr{S} \mid U \leq T \leq M_i\}.$$

So the bracketed sum in equation (3.11) can be computed for $P, S \in \mathscr{S}$ to be

$$\sum_{U \leq T \leq P,S} \mu(U,T) = \sum_{U \leq T \leq M_i} \mu(U,T) = \begin{cases} 1 & U = M_i \\ 0 & U \neq M_i \end{cases}$$

using the standard property of the Möbius function [55, § 3.7]. Hence, the product of two connected subquivers in A is the sum of the connected components of their intersection $P \cap S$:

$$P \cdot S = \sum_i M_i.$$

There is an injective map $\phi\colon \mathscr{S} \to R(Q)$ given by $\phi(P) = \mathbb{I}_P$. Since each $P \in \mathscr{S}$ is connected, $\mathbb{I}_P$ is indecomposable, so the image of ϕ is a set of $\mathbb{Z}$-linearly independent elements of $R(Q)$. Hence ϕ uniquely extends to map of $\mathbb{Z}$-modules $\tilde{\phi}\colon A \hookrightarrow R(Q)$. From the definition of quiver tensor product, it is easy to see that

$$\mathbb{I}_P \otimes \mathbb{I}_S = \bigoplus_i \mathbb{I}_{M_i}$$

where again $\{M_i\}$ is the set of connected components of $P \cap S$. This shows that $\tilde{\phi}$ is a ring homomorphism, so we can regard A as a subalgebra of $R(Q)$ by identifying a connected subquiver of Q with the identity representation of that subquiver.

Definition 3.30. We call $A(\mathscr{S}, \mathbb{Z})$ (or its natural image in $R(Q)$) the **support algebra** of Q.

In particular, we can define for each $P \in \mathscr{S}$ an element

$$e_P := \sum_{P' \leq P} \mu(P', P)\mathbb{I}_{P'}$$

of the support algebra such that $\mathbb{I}_P = \sum_{P' \leq P} e_{P'}$, and $\{e_P\}_{P \in \mathscr{S}}$ is a set of orthogonal idempotents in $R(Q)$. From this discussion, and the fact that orthogonal idempotents whose sum is 1 give a direct product decomposition of a ring, the following proposition is immediate.

Proposition 3.31. *For any quiver Q, the support algebra gives a decomposition of $R(Q)$ into a product of rings*

$$R(Q) \cong \prod_{P \in \mathscr{S}} e_P R(Q)$$

*called the **decomposition of** $R(Q)$ **by supports**.*

The next proposition gives a first entry in the dictionary between $R(Q)$ and $\mathrm{Rep}(Q)$.

Proposition 3.32. *For a connected subquiver $Q' \subseteq Q$, the following hold:*

(a) For any $V \in \mathrm{Rep}(Q)$, we have that $Q' \nsubseteq \operatorname{supp} V$ implies $e_{Q'}V = 0$.

(b) The set of images $\{e_{Q'}V\}$ freely generates $e_{Q'}R(Q)$ as a $\mathbb{Z}$-module, where V ranges over isomorphism classes of indecomposable representations with support exactly Q'. In particular, if V is indecomposable and $\operatorname{supp} V = Q'$*, then $e_{Q'}V \neq 0$.*

Proof. (a) Let $P := \operatorname{supp} V$, so we have that $V = \mathbb{I}_P V = \sum_{P' \leq P} e_{P'}V$. Then

$$e_{Q'}V = e_{Q'}\mathbb{I}_P V = \begin{cases} e_{Q'}V & Q' \subseteq P \\ 0 & Q' \nsubseteq P \end{cases}$$

by orthogonality.

(b) This part reduces to the case $Q' = Q$ by induction on the number of vertices.

Since $R(Q)$ is generated as a $\mathbb{Z}$-module by indecomposable representations, the images of the indecomposables generate the factor ring $e_Q R(Q)$. But if $\operatorname{supp} V \nsupseteq Q$ (i.e. $\operatorname{supp} V \neq Q$ since Q is the maximal subquiver of itself), then $e_Q V = 0$ by (a), so in fact $e_Q R(Q)$ is generated by the images of indecomposables with support exactly Q.

Now suppose there is a relation

$$\sum_i n_i e_Q V_i = 0$$

where $\{V_i\}$ are pairwise non-isomorphic indecomposables with $\operatorname{supp} V_i = Q$ and $n_i \in \mathbb{Z}$. Then substituting the expression

$$e_Q = \mathbb{I}_Q + \sum_{P<Q} \mu(P,Q)\mathbb{I}_P$$

and using that $\mathbb{I}_Q V_i = V_i$ for all i, we would get

$$\sum_i n_i V_i = -\sum_i \sum_{P<Q} n_i \mu(P,Q)\mathbb{I}_P V_i.$$

Every term $\mathbb{I}_P V_i$ on the right hand side has support smaller than Q, and the terms on the left hand side are indecomposable with support Q. Since indecomposables freely generate $R(Q)$ by definition, it must be that each $n_i = 0$. $\square$

It should be noted again that idea of giving an orthogonal decomposition of $R(Q)$ in order to simplify inductive proofs is due to Herschend, and that some parts of the above propositions appear in the work cited above, under additional assumptions (e.g. Q is a tree, Q is Dynkin).

3.3.2 A finer notion of support

Now suppose that (Q, σ) is a rooted tree quiver, and let $R := R(Q)$ be the representation ring of Q. We will use the reduced representations Ω_M and lattices L_Q^x to further decompose each factor $e_P R$. The main theorem of this chapter will be that, after this, no further decomposition is possible. More precisely, we will show that L_Q indexes a complete set of orthogonal idempotents in R.

In the remainder of the thesis, we will simplify some of the constructions and proofs by ignoring direct summands of representations which don't have σ in their support, assuming that we know about these summands by some induction. Working in an appropriate factor ring of R is the technical tool that allows us to do this rigorously. Writing $\mathscr{S}_\sigma := \{P \subseteq Q \mid \sigma \in P_\bullet\}$ for the collection of all connected subquivers of Q containing σ, we define

$$R_\sigma = \prod_{P \in \mathscr{S}_\sigma} e_P R.$$

Then R_σ is naturally both an ideal in R, and a factor ring of R with identity $\sum_{P \in \mathscr{S}_\sigma} e_P$. For $r \in R$, we denote by

$$\overline{r} := \left(\sum_{P \in \mathscr{S}_\sigma} e_P \right) r$$

the image of r in R_σ. By Proposition 3.32, R_σ is freely generated as a $\mathbb{Z}$-module by the images of all indecomposable representations of Q with σ in their support. The following lemma justifies why induction reduces the study of R to that of R_σ, and interprets passage to R_σ in terms of the representation theory of Q.

Lemma 3.33. *Let $X \subset Q_\bullet$ be the set of vertices x of Q for which there exists an arrow from x to σ. Then*

$$R \cong R_\sigma \times \prod_{x \in X} R(Q_{\geq x})$$

and for $V \in \operatorname{Rep}(Q)$, we have $\overline{V} = 0$ in R_σ if and only if $\sigma \notin \operatorname{supp} V$.

Proof. The first statement holds because if $\sigma \notin \operatorname{supp} P$ for some connected $P \subset Q$, then $P \subseteq Q_{\geq x}$ for a unique x. The second statement is a corollary of Proposition 3.32. □

Now consider the Möbius algebra of L_Q^σ over $\mathbb{Z}$, which we will denote by

$$A_\sigma := A(L_Q^\sigma, \mathbb{Z}).$$

Since L_Q^σ is a lattice, this is the semigroup algebra of L_Q^σ with respect to the meet operator $\wedge$. The map $\psi \colon L_Q^\sigma \to R_\sigma$ given by $\psi(M) = \overline{\Omega_M}$ extends by linearity to a map of $\mathbb{Z}$-modules

$$\tilde{\psi} \colon A_\sigma \hookrightarrow R_\sigma.$$

Proposition 3.32 implies that this map is injective, since the reduced representations Ω_M for $M \in L_Q^\sigma$ are indecomposable and pairwise non-isomorphic by Corollary 3.19. Using Corollary 3.27, in R_σ we have

$$\overline{\Omega_M \Omega_N} = \overline{\Omega_{M \wedge N}}$$

so in fact ψ is a ring homomorphism. Thus A_σ is a subalgebra of R_σ, and as before we get orthogonal idempotents

$$f_M := \sum_{M' \leq M} \mu(M', M)\overline{\Omega_{M'}}.$$

in R_σ which give a direct product decomposition

$$R_\sigma \cong \prod_{M \in L_Q^\sigma} f_M R_\sigma.$$

In particular, note that

$$f_M \Omega_N = f_M \overline{\Omega_N} = f_M \, \zeta(M, N) \tag{3.12}$$

where ζ is again the zeta function of the poset L_Q^σ, as in Corollary 3.28. This simply follows from substituting the expression $\overline{\Omega_N} = \sum_{N' \leq N} f_{N'}$.

In Proposition 3.32, we related the images of a representation in the factor rings $e_P R$ to a basic representation theoretic property, namely the support of a representation. Our goal now is to add another entry to the dictionary between $R(Q)$ and $\mathrm{Rep}(Q)$ by doing something analogous for the factor rings $f_M R_\sigma$.

Definition 3.34. For $V \in \mathrm{Rep}(Q)$ such that $\overline{V} \neq 0$ in R_σ, let

$$\mathcal{F}_V = \{M \in L_Q^\sigma \mid \overline{\Omega_M V} = \overline{V}\}.$$

We define the **fine support** of V to be

$$\text{f-supp}\, V := \min \mathcal{F}_V \in L_Q^\sigma.$$

The set has a unique minimal element because $\mathcal{F}_V$ is a meet semi-lattice of L_Q^σ. That is, if both $\overline{\Omega_M V} = \overline{V}$ and $\overline{\Omega_N V} = \overline{V}$, then

$$\overline{\Omega_{M \wedge N} V} = \overline{\Omega_M \Omega_N V} = \overline{V}.$$

Furthermore, every $M \geq \text{f-supp}\, V =: N$ has the property that $\overline{\Omega_M V} = \overline{V}$, since

$$\overline{\Omega_M V} = \overline{\Omega_M \left(\Omega_N V\right)} = \overline{\left(\Omega_M \Omega_N\right) V} = \overline{\Omega_{M \wedge N} V} = \overline{\Omega_N V} = \overline{V}.$$

In other words, $\mathcal{F}_V$ is always a principal filter (or dual order ideal) in L_Q^σ, and by definition $\text{f-supp}\, V$ is its generator. Now suppose that $\overline{W} \neq 0$ for every indecomposable summand W of V. Then by considering the dimension at σ, an alternative characterization of fine support is that $N \geq \text{f-supp}\, V$ if and only if $\Omega_N \otimes V \simeq V \oplus U$ for some $U \in \text{Rep}(Q)$. The next proposition gives the basic properties of fine support. Note that the first four properties are analogous to properties of the support of a representation.

Proposition 3.35. *In the statements below, assume that every representation appearing has nonzero image in R_σ, so the fine support is defined. Then the following properties hold:*

(F1) For a direct sum decomposition $V \simeq \bigoplus_{i=1}^n V_i$, we get

$$\text{f-supp}\, V = \bigvee_{i=1}^{n} \text{f-supp}\, V_i.$$

(F2) Tensor product can only decrease the fine support of a representation. More precisely, we have

$$\text{f-supp}(V \otimes W) \leq \text{f-supp}\, V \wedge \text{f-supp}\, W.$$

(F3) If $\text{f-supp}\, V \not\geq M$, then $f_M V = 0$.

(F4) The ring $f_M R$ is freely generated as a $\mathbb{Z}$-module by

$$\{f_M V \mid V \text{ indecomposable with } \text{f-supp}\, V = M\}.$$

In particular, if V is indecomposable and $\text{f-supp}\, V = M$, then $f_M V \neq 0$.

(F5) Whenever $\operatorname{rank}_M V \neq 0$*, we have that* $M \leq \text{f-supp}\, V$*.*

(F6) Suppose V *is indecomposable and* $\text{f-supp}\, V = N$*. Then* $\operatorname{rank}_N V \neq 0$ *implies* $V \simeq \Omega_N$*.*

Proof. Let $N := \text{f-supp}\, V$ throughout the proof.

(F1) Let $M := \bigvee_{i=1}^{n} \text{f-supp}\, V_i$, so we want so show $M = N$. Since $M \geq \text{f-supp}\, V_i$ for each i, we know that $\overline{\Omega_M V_i} = \overline{V_i}$ for each i. Thus we can compute

$$\overline{\Omega_M V} = \overline{\Omega_M \sum_i V_i} = \sum_i \overline{\Omega_M V_i} = \sum_i \overline{V_i} = \overline{V}$$

and so $N \leq M$.

For the reverse inequality, it is enough to show that $N \geq \text{f-supp}\, V_i$ for each i, and without loss of generality we can take each V_i to be indecomposable. Let $\{W_{ij}\}$ be the set of indecomposable summands of $\Omega_N \otimes V_i$ that have σ in their support, so $\overline{\Omega_N V_i} = \sum_{j=1}^{d(i)} \overline{W_{ij}}$. Then by assumption, we get an equality

$$\sum_i \overline{V_i} = \overline{V} = \overline{\Omega_N V} = \overline{\Omega_N} \sum_i \overline{V_i} = \sum_i \overline{\Omega_N V_i} = \sum_{ij} \overline{W_{ij}}.$$

Since the V_i and W_{ij} are indecomposable, Proposition 3.32 implies that each $d(i) = 1$ and there is a permutation $\pi \in \mathfrak{S}_n$ such that $\overline{\Omega_N V_i} = \overline{V_{\pi i}}$ for all i. But since $\overline{\Omega_N}$ idempotent in R_σ, the associated permutation π is also, and so π is the identity permutation. Thus $\overline{\Omega_N V_i} = \overline{V_i}$ for all i, which implies that $N \geq \text{f-supp}\, V_i$ for all i, and finally that $N \geq M$.

(F2) If we write $M := \text{f-supp}\, W$, then we can use Corollary 3.27 to compute

$$\overline{\Omega_{N \wedge M}}(\overline{V \otimes W}) = \overline{\Omega_N \Omega_M V W} = \overline{\Omega_N V}\ \overline{\Omega_M W} = \overline{VW}.$$

So by definition we get that $\text{f-supp}(V \otimes W) \leq N \wedge W$.

(F3) If $N \ngeq M$, then equation (3.12) implies that $f_M V = f_M(\overline{\Omega_N V}) = 0$.

(F4) The ring R_σ is generated as a $\mathbb{Z}$-module by the images of indecomposable representations with σ in their support, and $f_M R$ is a factor ring of R_σ, so the image of this set generates $f_M R$ also. For V indecomposable with $\sigma \in \operatorname{supp} V$, we can write

$$\overline{\Omega_M V} = \sum_{i \in I} \overline{V_i} \neq 0$$

where each V_i is indecomposable and has σ in its support. Then for each i, it must be that $\text{f-supp}\, V_i \leq M$, since properties (F1) and (F2) give that

$$\text{f-supp}\, V_i \leq \text{f-supp} \left(\bigoplus_i V_i \right) = \text{f-supp}\, (\Omega_M \otimes V) \leq M \wedge \text{f-supp}\, V \leq M.$$

Now (F3) implies that $f_M V_i = 0$ when $\text{f-supp}\, V_i$ is strictly less than M, so we can use equation (3.12) to get

$$f_M V = (f_M \Omega_M) V = f_M(\Omega_M V) = f_M \sum_{i \in J} V_i$$

where $J := \{ i \in I \mid \text{ f-supp}\, V_i = M \}$. Hence $f_M R$ is generated by images of indecomposables with fine support exactly M.

Suppose we had a relation of the form

$$\sum_i n_i f_M V_i = 0 \qquad n_i \in \mathbb{Z}$$

where each $V_i \in \operatorname{Rep}(Q)$ has fine support exactly M, and the V_i are pairwise non-isomorphic. Then substituting the expression

$$f_M = \overline{\Omega_M} + \sum_{M' < M} \mu(M', M) \overline{\Omega_{M'}}$$

into the previous equation, we use the assumption that $\overline{\Omega_M V_i} = \overline{V_i}$ for each i to get

$$\sum_i n_i \overline{V_i} + \sum_i \sum_{M' < M} n_i \mu(M', M) \overline{\Omega_{M'} V_i} = 0.$$

Now the double sum lies in the span of images of indecomposables with fine support strictly less than M, by property (F2), and the first sum consists of images of indecomposables with fine support exactly M. Since the images of indecomposables with σ in their support freely generate R_σ, we get that $n_i = 0$ for all i. This shows that $f_M R$ is *freely* generated by the images of the indecomposables with fine support M.

(F5) Recalling that we defined $N = \text{f-supp}\, V$, we have that $\overline{\Omega_N V} = \overline{V}$ holds by definition, and so there exist $X, Y \in \mathrm{Rep}(Q)$ with σ not in either of their supports and such that

$$X \oplus (\Omega_N \otimes V) \simeq V \oplus Y.$$

Since σ is not in the support of X, it must be that $\mathrm{rank}_M X \subseteq X_\sigma = 0$, and similarly $\mathrm{rank}_M Y = 0$. Hence we can compute

$$\begin{aligned}(\mathrm{rank}_M \Omega_N) \otimes_K (\mathrm{rank}_M V) = \mathrm{rank}_M(\Omega_N \otimes V) &= \mathrm{rank}_M(X \oplus (\Omega_N \otimes V)) \\ &= \mathrm{rank}_M(V \oplus Y) = \mathrm{rank}_M V \neq 0\end{aligned}$$

which implies that $\mathrm{rank}_M \Omega_N \neq 0$. Then by Corollary 3.28 we get that $M \leq N = \text{f-supp}\, V$.

(F6) Let $c := c_N$. If $\mathrm{rank}_N V = \mathrm{rank}_{Q_N}(c^* V) \neq 0$, then by Theorem 2.32 there is a decomposition $c^* V \simeq \mathbb{I}_{Q_N} \oplus U$ for some $U \in \mathrm{Rep}(Q_N)$. Pushing back down to Q we get

$$c_* c^* V \simeq c_* \mathbb{I}_{Q_N} \oplus c_* U = \Omega_N \oplus c_* U.$$

By Lemma 3.26, the left hand side is isomorphic to $\Omega_N \otimes V$. If V is indecomposable and $\text{f-supp}\, V = N$, then V itself is the only indecomposable direct summand of $\Omega_N \otimes V$ with σ in its support, by considering dimension at the vertex σ. By comparison with the right hand side, it must be that $V \simeq \Omega_N$. $\square$

The rank spaces and fine support of a representation V give information about morphisms between reduced representations and V.

Lemma 3.36. *For any $M \in L_Q^\sigma$, the vectors in $\operatorname{rank}_M V \subseteq V_\sigma$ are precisely the vectors contained in the image at σ of some morphism from Ω_M to V. More precisely, if we fix a nonzero vector $v_\sigma \in (\Omega_M)_\sigma$, then there is a natural map of vector spaces*

$$\Psi\colon \operatorname{Hom}_Q(\Omega_M, V) \twoheadrightarrow \operatorname{rank}_M V$$

given by

$$\Psi(f) = f(v_\sigma)$$

for $f\colon \Omega_M \to V$.

Proof. Let $c := c_M$. Then (c_*, c^*) is an adjoint pair by Proposition 3.1, so there is an isomorphism

$$\operatorname{Hom}_Q(\Omega_M, V) = \operatorname{Hom}_Q(c_*\mathbb{I}, V) \cong \operatorname{Hom}_{Q_M}(\mathbb{I}, c^*V).$$

From Propositions 2.30 and 2.31 there is a surjective linear map

$$\operatorname{Hom}_{Q_M}(\mathbb{I}, c^*V) \twoheadrightarrow \mathscr{E}_{Q_M}(c^*V)_\sigma$$

sending a morphism f to the vector $f(v_\sigma)$. Using Lemma 3.2, the right hand side is equal to $\operatorname{rank}_M V$. $\square$

Lemma 3.37. *If $V \in \operatorname{Rep}(Q)$ is such that $\operatorname{f-supp} V \leq M$, then any linear functional on the vector space V_σ lifts to a morphism of representations from V to Ω_M. More* *precisely, if we fix an isomorphism* $(\Omega_M)_\sigma \simeq K$*, then there is a natural map of vector spaces*

$$\Phi\colon \operatorname{Hom}_K(V_\sigma, K) \hookrightarrow \operatorname{Hom}_Q(V, \Omega_M)$$

such that for $g\colon V_\sigma \to K$*,*

$$\Phi(g)_\sigma = g.$$

Proof. Let $c := c_M$. Then $c^*\Omega_M \simeq \mathbb{I}_{Q_M} \oplus U$ with $\sigma \notin \operatorname{supp} U$ by Theorem 3.25. There is a natural isomorphism of vector spaces

$$\operatorname{Hom}_K((c^*V)_\sigma, K) \cong \operatorname{Hom}_{Q_M}(c^*V, \mathbb{I}_{Q_M})$$

coming from the fact that $\mathbb{I}_{Q_M}$ is the injective representation of Q_M associated to the vertex σ, [5, Lem. III.2.11]. Now since $V_\sigma = (c^*V)_\sigma$, we use the decomposition of $c^*\Omega_M$ above to get an embedding

$$\operatorname{Hom}_K(V_\sigma, K) \hookrightarrow \operatorname{Hom}_{Q_M}(c^*V, c^*\Omega_M).$$

Since $\operatorname{f-supp} V \leq M$, there is a decomposition $c_*c^*V \simeq V \oplus W$ with $\sigma \notin \operatorname{supp} W$. The adjoint pair (c_*, c^*) gives an isomorphism

$$\operatorname{Hom}_{Q_M}(c^*V, c^*\Omega_M) \cong \operatorname{Hom}_Q(c_*c^*V, \Omega_M) = \operatorname{Hom}_Q(V, \Omega_M) \oplus \operatorname{Hom}_Q(W, \Omega_M).$$

which, projected to the first summand on the right hand side, gives Φ as stated. $\square$

In general, suppose we have a quiver representation V, and that we know the isomorphism class of $V|_P$ for every proper subquiver $P \subset Q$. Then we cannot deduce the isomorphism class of V in $\operatorname{Rep}(Q)$ without further information regarding how to glue together the restricted representations. In our case, we are essentially facing this problem when we try to inductively study representations of Q via gluing and extension of rooted tree quivers. The base change lemma below is a tool that allows us to utilize information in the rank functors to address this problem.

We will introduce some new notation. If $U \subseteq V$ are representations of Q, and $Z \subseteq V_\sigma$ a vector subspace, then the notation

$$U \simeq \Omega_M \otimes_K Z$$

means that $U \simeq \Omega_M^{\oplus \dim_K Z}$ and $U_\sigma = Z$. This does not uniquely identify U as a subrepresentation of V. In this case, for any $W \in \mathrm{Rep}(Q)$ there is an isomorphism

$$\mathrm{Hom}_Q(W, \Omega_M) \otimes_K U_\sigma \cong \mathrm{Hom}_Q(W, \Omega_M \otimes_K U_\sigma) = \mathrm{Hom}_Q(W, U) \tag{3.13}$$

given by sending an indecomposable tensor $f \otimes u$ on the right hand side to the morphism $w \mapsto f(w) \otimes u$. It is easy to see that the map is injective, so isomorphism follows from the fact that both spaces have the same dimension. Similarly, we get that

$$\mathrm{Hom}_Q(U, W) \cong \mathrm{Hom}_Q(\Omega_M, W) \otimes_K U_\sigma^*. \tag{3.14}$$

Now we can state and prove the base change lemma that will be our key to getting gluing data from the rank functors.

Lemma 3.38. *Suppose $U, W \subset V$ are subrepresentations such that $V \simeq U \oplus W$ and $U \simeq \Omega_M \otimes_K U_\sigma$ for some $M \in L_Q^\sigma$.*

(a) Suppose $\text{f-supp}\, W \leq M$*, and $\tilde{Z} \subseteq V_\sigma$ is a subspace such that $V_\sigma = U_\sigma \oplus \tilde{Z}$. Then there exists a subrepresentation $Z \subset V$ such that $Z_\sigma = \tilde{Z}$ and $V \simeq U \oplus Z$.*

(b) Now suppose W is arbitrary, and $\tilde{Z} \subseteq \mathrm{rank}_M V \subset V_\sigma$ is such that $V_\sigma = \tilde{Z} \oplus W_\sigma$. Then there exists a subrepresentation $Z \subset V$ such that $Z_\sigma = \tilde{Z}$ and $V \simeq Z \oplus W$.

Proof. (a) There is a short exact sequence of vector spaces

$$0 \to \tilde{Z} \xrightarrow{i} U_\sigma \oplus W_\sigma \xrightarrow{(\tilde{f}\ \tilde{g})} U_\sigma \to 0$$

where i is the subspace inclusion, and $\tilde{f}$ is invertible since $V_\sigma = U_\sigma \oplus \tilde{Z}$. By composing the last map with $\tilde{f}^{-1}$, and replacing $\tilde{g}$ with $\tilde{f}^{-1} \circ \tilde{g}$, we can assume $\tilde{f} = id$ without loss of generality. By Lemma 3.37 and the isomorphism (3.13), we have an injective linear map

$$\mathrm{Hom}_K(W_\sigma, U_\sigma) \cong W_\sigma^* \otimes_K U_\sigma \hookrightarrow \mathrm{Hom}_Q(W, \Omega_M) \otimes_K U_\sigma \cong \mathrm{Hom}_Q(W, U)$$

which gives a morphism $g \in \mathrm{Hom}_Q(W, U)$ such that $g_\sigma = \tilde{g}$. Thus we can define $Z \subset V$ by the split exact sequence in $\mathrm{Rep}(Q)$

$$0 \to Z \to U \oplus W \xrightarrow{(id\ g)} U \to 0.$$

(b) Similarly, we can take a short exact sequences of vectors spaces of the form

$$0 \to \tilde{Z} \xrightarrow{i} U_\sigma \oplus W_\sigma \xrightarrow{(\tilde{f}\ -id)} W_\sigma \to 0$$

and again we want to lift $\tilde{f}$ to some $f \in \mathrm{Hom}_Q(U, W)$. Let π_U, π_W be the projections given by the decomposition $V \simeq U \oplus W$. Since $\tilde{Z} \subseteq \mathrm{rank}_M V$ and rank_M is additive, we get that

$$\pi_W(\tilde{Z}) \subseteq \pi_W(\mathrm{rank}_M V) = \mathrm{rank}_M W.$$

Now the projections give a decomposition $\tilde{Z} = \pi_U(\tilde{Z}) \oplus \pi_W(\tilde{Z})$, and then exactness of the sequence implies that $(\tilde{f} \circ \pi_U - \pi_W)(\tilde{Z}) = 0$, or

$$\tilde{f} \circ \pi_U(\tilde{Z}) = \pi_W(\tilde{Z}).$$

Since W_σ is the kernel of the projector π_U restricted to V_σ, and $\tilde{Z} \cap W_\sigma = 0$, we find that $\pi_U(\tilde{Z}) = U_\sigma$ and so $\tilde{f} \in \mathrm{Hom}_Q(U_\sigma, \mathrm{rank}_M W)$. By Lemma 3.36, we have a surjection $\mathrm{Hom}_Q(\Omega_M, W) \twoheadrightarrow \mathrm{rank}_M W$ that we can tensor with U_σ^* to get

$$\mathrm{Hom}_Q(U, W) \cong \mathrm{Hom}_Q(\Omega_M, W) \otimes_K U_\sigma^* \twoheadrightarrow \mathrm{rank}_M W \otimes_K U_\sigma^* \cong \mathrm{Hom}_K(U_\sigma, \mathrm{rank}_M W)$$

using (3.14). Thus we can lift $\tilde{f}$ to some $f \in \mathrm{Hom}_Q(U, W)$ such that $f_\sigma = \tilde{f}$, and we can again define $Z \subset V$ by the split exact sequence of representations

$$0 \to Z \to U \oplus W \xrightarrow{(f\ -id)} W \to 0. \qquad \square$$

3.4 Structure of the Representation Ring of a Rooted Tree

3.4.1 Statements of the structure theorems

The main result of this chapter has two equivalent formulations: firstly, as a property of the representation ring of a rooted tree quiver, and secondly, as a "splitting principle" for representations of such a quiver. Since rank functors commute with tensor product, we have an equality of vector spaces $\mathrm{rank}_M(V^{\otimes n}) = (\mathrm{rank}_M V)^{\otimes n}$ inside $V_x^{\otimes n}$ for any $M \in L_Q^x$ and $n \in \mathbb{Z}_{\geq 1}$. Consequently, we omit the parentheses in this situation.

Theorem 3.39. *Let Q be a rooted tree quiver, so that its representation ring $R := R(Q)$ has a finite decomposition*

$$R \cong \prod_{M \in L_Q} f_M R$$

as a direct product of rings (cf. §3.3.2). Then any indecomposable representation $V \not\cong \Omega_M$, with f-supp $V = M$*, has nilpotent image in the factor $f_M R$. Consequently, each factor has a $\mathbb{Z}$-module decomposition*

$$f_M R \cong \mathbb{Z} f_M \oplus f_M \mathcal{N}$$

where $\mathcal{N}$ is the nilradical of R.

The "consequently" part follows from the first statement because $f_M R$ is freely generated as a $\mathbb{Z}$-module by the indecomposable representations of V with fine support M, and $f_M \Omega_M = f_M$. Since $\mathbb{Z}$ is reduced (has no nilpotent elements), the rank function $r_M : R \to \mathbb{Z}$ restricts to the projection

$$r_M : f_M R \to \mathbb{Z} f_M = (f_M R)_{red}$$

so that, for any representation V, we have

$$f_M V = r_M(V) f_M + n \qquad\qquad n \in \mathcal{N}.$$

Theorem 3.40. *Let $N \in L_Q^\sigma$ be maximal such that* $\operatorname{rank}_N V \neq 0$*. Then for some $l > 0$, there exists a subrepresentation $U \simeq \Omega_N \otimes_K (\operatorname{rank}_N V^{\otimes l}) \subseteq V^{\otimes l}$ such that*

$$V^{\otimes l} \simeq U \oplus W$$

for some $W \subseteq V^{\otimes l}$, necessarily with $\operatorname{rank}_N W = 0$.

Note that if the conclusion holds for some l, then it also holds for all $l' > l$. We will refer to this theorem as the **splitting principle** for representations of rooted tree quivers. Strictly speaking, we will only need to use that Theorem 3.39 implies the splitting principle in order to prove the theorems. However, we will show that the two theorems are equivalent in order to expand our dictionary between $R(Q)$ and $\operatorname{Rep}(Q)$.

Proof of equivalence of Theorems 3.39 and 3.40. Suppose that Theorem 3.39 is true, and let V be a representation of Q. Then for sufficiently large l, in each factor $f_M R$ we have

$$f_M V^l = 0 \iff r_M(V) = 0.$$

For a fixed l such that this holds, choose any maximal $N \in L_Q^\sigma$ such that $f_N V^l \neq 0$. Then write $V^{\otimes l} \simeq (\bigoplus_i U_i) \oplus W$ where each U_i is indecomposable with $r_N(U_i) \neq 0$, and $r_N(W) = 0$. Then for each i, $\operatorname{f-supp} U_i \geq N$ by (F5). Now take any $N' \geq N$ maximal such that $V^{\otimes l}$ has a direct summand with fine support N'. Properties (F3) and (F4) imply that $f_{N'}V^l \neq 0$, so $N' = N$ by maximality of N. Thus $V^{\otimes l}$ has no direct summands with fine support strictly greater than N, so $\operatorname{f-supp} U_i = N$. By property (F6), each $U_i \simeq \Omega_N$. This gives a decomposition as in Theorem 3.40.

Now assume that Theorem 3.40 is true, and let V be as in the hypotheses of Theorem 3.39. We proceed by induction on the order of

$$\mathcal{U}_V := \{M' \in L_Q^\sigma \mid \operatorname{rank}_{M'} V \neq 0\}.$$

Since V has fine support $M \in L_Q^\sigma$, we have that $V_\sigma \neq 0$, so $\#\mathcal{U}_V \geq 1$. Now choose any maximal element N of $\mathcal{U}_V$. Applying Theorem 3.40, we get

$$V^{\otimes l} \simeq U \oplus W$$

with $U \simeq \Omega_N \otimes_K (\operatorname{rank}_N V^{\otimes l})$. Property (F5) implies that $N \leq \operatorname{f-supp} V = M$. Since we assumed $V \not\simeq \Omega_M$, property (F6) implies that $\operatorname{rank}_M V = 0$, so $N < M$. Thus $f_M U = 0$ by equation (3.12), so $f_M V^l = f_M W$.

By properties (F1) and (F2), every direct summand of W has fine support less than or equal to M. Then (F3) and (F4) imply that $f_M W = \sum_i f_M X_i$ where X_i are the indecomposable summands of W with fine support exactly M. Since $\operatorname{rank}_M X_i \subseteq \operatorname{rank}_N X_i \subseteq \operatorname{rank}_N W = 0$, we know that no $X_i \not\simeq \Omega_M$. Now by the induction hypothesis, each $f_M X_i$ is nilpotent, and hence $f_M W = f_M V^l$ is nilpotent. □

We illustrate the splitting principle with an example.

Example 3.41. Let Q be the five subspace quiver labeled as below, and α a dimension vector for Q:

$$Q = \begin{tikzcd} & & \sigma & & \\ 1 \arrow[urr] & 2 \arrow[ur] & 3 \arrow[u] & 4 \arrow[ul] & 5 \arrow[ull] \end{tikzcd} \qquad \alpha = \begin{matrix} 3 \\ 22222 \end{matrix}.$$

Let $V \in \operatorname{Rep}(Q)$ be an indecomposable of dimension α, so it can be thought of as a three dimensional vector space with a collection of five specified planes $\{V_i \mid 1 \leq i \leq 5\}$. Assume V is general in the sense that the planes V_i are in general position, with pairwise intersection of dimension one and the intersection of any three planes being zero.

In the notation of Example 3.13, the element $J = \{1, 2\} \in L_Q^\sigma$ is maximal such that the corresponding rank space, $\operatorname{rank}_J V = V_1 \cap V_2$, is nonzero. So the splitting principle (Theorem 3.40) says that for some $l > 0$ we have $V^{\otimes l} \simeq U \oplus W$, where U is an indecomposable direct summand such that

$$U_\sigma = \left(\bigcap_{i \in J} V_i^{\otimes l} \right) \qquad \text{and} \qquad \underline{\dim}\,(U) \simeq \begin{matrix} 1 \\ 11000 \end{matrix}.$$

Furthermore, we necessarily have that $W_1 \cap W_2 = 0$ since rank functors are additive and multiplicative. Repeating this process with other pairs $J' = \{i, j\}$, for large m we eventually get a decomposition

$$V^{\otimes m} \simeq \bigoplus \Omega_{i,j} \oplus Z$$

where the sum is taken over all two element subsets $\{i, j\} \subset \{1, 2, 3, 4, 5\}$, and $\Omega_{i,j}$ is the analogous representation to U above but whose support is the vertex set $\{i, j, \sigma\}$. Furthermore, Z must be a representation of Q given by a collection of subspaces with all pairwise intersections zero.

Theorem 3.39 will be proven by induction on the "complexity" of Q (see below). When $M < \hat{1}_Q$, we can use the combinatorics of Section 3.2 to utilize the induction hypothesis. The case that $M = \hat{1}_Q$ is essentially computational, utilizing an inductive application of the splitting principle.

3.4.2 The case $M < \hat{1}_Q$

Using the connection between reduced representations of Q and those of Q_M given by Theorem 3.25, we can easily handle this case by induction. However, Q_M might have more vertices than Q, so we must find another ordering to use induction on. It turns out that the set of rooted tree quivers, $\mathscr{T}$, can be well-ordered such that a reduced quiver Q_M over Q is less than or equal to Q, with equality only for $Q_M = Q$. This **complexity ordering** turns out to be essentially lexicographical order, if we encode rooted trees in the right notation.

A sequence of rooted tree quivers $(\Gamma_1, \ldots, \Gamma_n)$ defines a rooted tree quiver by extending each Γ_i from its sink, then gluing these extensions together at their sinks:

$$(\Gamma_1, \ldots, \Gamma_n) := \begin{array}{c} \Gamma_1 \searrow \\ \vdots \qquad \sigma \\ \Gamma_n \nearrow \end{array} .$$

Any rooted tree arises in this way. We recursively define a well-ordering on $\mathscr{T}$. Let $\mathscr{T}_i \subset \mathscr{T}$ be the collection of rooted tree quivers which have a path of length i, and no longer path. Then $\mathscr{T}_0$ has one element, the rooted tree with one vertex, and for $k \geq 1$,

$$\mathscr{T}_k = \{(\Gamma_1, \ldots, \Gamma_n) \mid \Gamma_1 \in \mathscr{T}_{k-1} \text{ and } \Gamma_i \in \bigcup_{j=0}^{k-1} \mathscr{T}_j\}.$$

The set $\{\mathscr{T}_k\}$ is a partition of $\mathscr{T}$. Now let Γ, Λ be arbitrary rooted tree quivers, say with $\Gamma \in \mathscr{T}_k$ and $\Lambda \in \mathscr{T}_l$. If $k < l$, then define $\Gamma < \Lambda$. If $k = l$, then we can write

$$\Gamma = (\Gamma_1, \ldots, \Gamma_n), \qquad \Lambda = (\Lambda_1, \ldots, \Lambda_m) \tag{3.15}$$

with $\{\Gamma_i\}$ and $\{\Lambda_i\}$ contained in $\bigcup_{j=0}^{k-1} \mathscr{T}_j$, which we can assume is well-ordered. By requiring that the sequences in (3.15) be weakly decreasing, these expressions are unique. To simplify the order condition below, we will assume $n = m$ by filling out the shorter sequence with symbols $\emptyset$ which we take to be less than all rooted trees.

Now define $\Gamma < \Lambda$ if and only if $\Gamma_i < \Lambda_i$ for the smallest i such that $\Gamma_i \neq \Lambda_i$. In other words, the ordering in $\mathscr{T}_k$ is lexicographical with respect to the ordering on $\bigcup_{j=0}^{k-1} \mathscr{T}_j$.

To see that this is a well-ordering, consider a descending chain of rooted tree quivers $Q_1 \geq Q_2 \geq \cdots$ and use induction. By truncating the sequence if necessary, we can assume that $Q_i \in \mathscr{T}_n$ for all i and some $n > 0$. Write Q_i as a sequence

$$Q_i = (P_{i,1}, P_{i,2}, \dots)$$

as above, with only finitely many $P_i \neq \emptyset$. Then comparing the first positions gives a descending chain $P_{1,1} \geq P_{2,1} \geq \cdots$ in the set of rooted trees with smaller maximal path length. We can assume this set is well ordered, so this chain stabilizes to some value S_1. For i large enough that the first position has stabilized, the second position $P_{i,2}$ gives a descending chain which we can assume stabilizes to some S_2, and continuing in this way we get a stable value $S_k \in \{\emptyset\} \cup \bigcup_{j=0}^{n-1} \mathscr{T}_j$ for each k.

Without loss of generality we can replace the chain of Q_i's with a subchain (and rename) so that the first i positions of Q_i are stable:

$$Q_i = (S_1, S_2, \dots, S_i, P_{i,i+1}, P_{i,i+2}, \dots).$$

But the descending chain $S_1 \geq S_2 \geq \cdots$ must also stabilize, say $S_N = S_{N+1} = \cdots$. Then we have

$$Q_N = (S_1, \dots, S_N, P_{N,N+1}, \dots) \geq Q_{N+1} = (S_1, S_2, \dots, S_N, S_{N+1}, \dots),$$

and hence $S_N \geq P_{N,N+1} \geq S_{N+1} = S_N$ must be equalities. Continuing along the positions of Q_N, we get that $P_{N,m} = S_N$ for all $m > N$, and hence the chain (Q_i) must be stable from Q_N onward (and $S_N = \emptyset$).

Proposition 3.42. *If $Q' \xrightarrow{c} Q$ is a reduced quiver over Q, then $Q' \leq Q$ in the complexity ordering, with equality if and only if $c = id$.*

Proof. Let $Q' = (Q'_1, \dots, Q'_m)$ and $Q = (Q_1, \dots, Q_n)$ in the notation above. Then the structure map c sends each Q'_i into some $Q_{\varphi(i)}$, defining a function

$$\varphi\colon [1, \dots, m] \to [1, \dots, n] \qquad c(Q'_i) \subseteq Q_{\varphi(i)}.$$

Each Q'_i is a reduced quiver over $Q_{\varphi(i)}$, by the construction of §3.1.3. We prove the proposition by induction on the number of vertices of Q. By the induction hypothesis and our notational convention, $Q'_1 \leq Q_{\varphi(1)} \leq Q_1$. If any of these inequalities is strict, then $Q' < Q$ and we are done. If these are equalities, then by induction c restricts to the identity on Q'_1, and the reducedness assumption for Q' implies that no other Q'_i maps into $Q_{\varphi(1)}$. Hence $(Q'_2, \dots, Q'_m)$ is a reduced quiver over $(Q_2, \dots, Q_n)$, with structure map the restriction of c. By the induction hypothesis, $(Q'_2, \dots, Q'_m) \leq (Q_2, \dots, Q_n)$, with equality if and only if c restricts to the identity on $(Q'_2, \dots, Q'_m)$. Hence $Q' \leq Q$ with equality if and only if c is the identity on all of Q'. $\square$

Proposition 3.43. *For $M \in L^{\sigma}_{Q}$ with $M < \hat{1}_Q$, let $c := c_M$. Then the composition*

$$c_* \circ c^*\colon R \to R(Q_M) \to R$$

is the identity on $f_M R$.

Proof. Let $V \in \operatorname{Rep}(Q)$ with $\operatorname{f-supp} V = M$. Since both c_* and c^* are additive, so is the composition. For any N, (F2) implies that

$$\operatorname{f-supp}(\Omega_N \otimes V) \leq M$$

and so $c_* c^*(\overline{\Omega_N V}) = \overline{\Omega_M}(\overline{\Omega_N V}) = \overline{\Omega_N V}$ by Lemma 3.26 and Corollary 3.27. Since

c_* and c^* are both additive, we can use the definition of f_M to compute

$$c_*c^*(f_M V) = c_*c^* \left(\sum_{N \le M} \mu(N,M)\overline{\Omega_N V} \right) = \sum_{N \le M} \mu(N,M) c_*c^* \left(\overline{\Omega_N V}\right)$$
$$= \sum_{N \le M} \mu(N,M)\overline{\Omega_N V} = f_M V.$$

By property (F4), $f_M R$ is generated as a $\mathbb{Z}$-module by $\{f_M V \mid \text{f-supp}\, V = M\}$; hence c_*c^* is the identity on $f_M R$. $\square$

This shows that c^* gives an embedding of the factor $f_M R$ into the representation ring of Q_M. Now we'll see that the image lies in the "top" factor of $R(Q_M)$.

Proposition 3.44. *For any $M \in L_Q^\sigma$, $c_M^* f_M = f_{\hat{1}}$ in $R(Q_M)$.*

Proof. Let A be any finite lattice, and $\delta := \sum_{x \in A} \mu(x, \hat{1})x$ in the Möbius algebra of A. There is a factorization

$$\delta = \prod_{x \prec \hat{1}} (\hat{1} - x)$$

(which is used in the proof of [55, Cor. 3.9.4], for example). By Theorem 3.25, we know that $c_M^*(\overline{\Omega_N}) = \overline{\Omega_{\pi^*(N)}}$ in R_σ, the Möbius algebra of L_Q^σ. Lemma 3.23 gives that $\pi^*(M) = \hat{1}$, and $N \prec M$ if and only if $\pi^*(N) \prec \hat{1}$. Since c_M^* is a ring homomorphism, we can use the factorization of $f_{\hat{1}}$ given by the formula above to get

$$c_M^*(f_M) = \prod_{N \prec M} \left(c_M^*(\overline{\Omega_M}) - c_M^*(\overline{\Omega_N}) \right) = \prod_{A \prec \hat{1}} \left(\overline{\Omega_{\hat{1}}} - \overline{\Omega_A} \right) = f_{\hat{1}}. \qquad \square$$

Proof of Theorem 3.39 for $M < \hat{1}$. Assume that Theorem 3.39 holds for rooted tree quivers which are less complex than Q (e.g. Q_M). The last two propositions and the induction hypothesis give an injective ring homomorphism

$$c^* \colon f_M R \hookrightarrow f_{\hat{1}} R(Q_M).$$

If $V \not\simeq \Omega_M$ is indecomposable with fine support M, then $\operatorname{rank}_{Q_M}(c^* V) = \operatorname{rank}_M V = 0$ by property (F6) of fine support, so $c^* V$ has no direct summands of $\Omega_{\hat{1}} = \mathbb{I}_{Q_M}$.

Thus $c^*(f_M V) = f_{\hat{1}} c^* V$ is nilpotent by the induction hypothesis, and so $f_M V$ is also nilpotent since c^* is injective. □

3.4.3 The case $M = \hat{1}_Q$

This case is essentially computational, and will require a number of technical lemmas. We will need a few facts from linear algebra. The following two lemmas roughly say that subspaces of a vector space become "more spread out" as we take tensor powers.

Lemma 3.45. *Let V and W be finite dimensional vector spaces. If $A_1, A_2, X \subset V$ and $B_1, B_2, Y \subset W$ are subspaces such that $X \cap A_1 = Y \cap B_2 = 0$, then*

$$\Big(X \otimes Y\Big) \cap \Big(A_1 \otimes B_1 + A_2 \otimes B_2\Big) = 0.$$

Proof. Choose projections $\pi_X \colon V \to X$ and $\pi_Y \colon W \to Y$ such that $A_1 \subseteq \ker \pi_X$ and $B_2 \subseteq \ker \pi_Y$. Then

$$(\pi_X \otimes \pi_Y)(A_1 \otimes B_1 + A_2 \otimes B_2) \subseteq (\pi_X \otimes \pi_Y)(A_1 \otimes B_1) + (\pi_X \otimes \pi_Y)(A_2 \otimes B_2) = 0$$

and so the intersection is zero. □

Lemma 3.46. *Let $\{V_i\}_{i=1}^n$ and W be a collection of subspaces of a finite dimensional vector space V, and suppose that $W \cap V_i = 0$ for all i. Then $W^{\otimes s} \cap \sum V_i^{\otimes s} = 0$ for $s \geq n$.*

Proof. First note that it is enough to show that the intersection is 0 for $s = n$, since for $s > n$,

$$W^{\otimes s} \cap \sum V_i^{\otimes s} \subseteq \big(W^{\otimes n} \otimes V^{\otimes s-n}\big) \cap \Big(\sum V_i^{\otimes n} \otimes V^{\otimes s-n}\Big) \subseteq \Big(W^{\otimes n} \cap \sum V_i^{\otimes n}\Big) \otimes V^{\otimes s-n} = 0.$$

Use induction on n, the base case being $n = 2$. For this, take $X = Y = W$, $A_i = V_1$, and $B_i = V_2$ in the above lemma. For the induction step, take another subspace

V_{n+1} such that $W \cap V_{n+1} = 0$, and assume that

$$W^{\otimes s} \cap \sum_{i=1}^{n} V_i^{\otimes s} = 0$$

for $s \geq n$. Now using the previous lemma again, with

$$X = W^{\otimes s}, \quad Y = W, \quad A_1 = \sum_{i=1}^{n} V_i^{\otimes s}, \quad B_1 = V, \quad A_2 = V_{n+1}^{\otimes s}, \quad B_2 = V_{n+1}$$

we find that

$$W^{\otimes s+1} \cap \sum_{i=1}^{n+1} V_i^{\otimes s+1} \subseteq W^{\otimes s} \otimes W \cap \left(\left(\sum_{i=1}^{n} V_i^{\otimes s} \right) \otimes V + V_{n+1}^{\otimes s} \otimes V_{n+1} \right) = 0. \quad \square$$

Lemma 3.47. *Let $M, N \in L_Q^\sigma$ and $V \in \mathrm{Rep}(Q)$ with* f-supp $V \leq N$. *Choose u_M, u_N some fixed nonzero vectors in $(\Omega_M)_\sigma$ and $(\Omega_N)_\sigma$, respectively. Then there exists a morphism*

$$\theta\colon \Omega_M \otimes V \to \Omega_N \otimes V$$

such that $\theta_\sigma(u_M \otimes v) = u_N \otimes v$ for all $v \in V_\sigma$.

Proof. The map θ will be the following composition, to be explained one step at a time:

$$\theta\colon \Omega_M \otimes V \xrightarrow{id \otimes f} \Omega_M \otimes \Omega_N \otimes V \xrightarrow{g \otimes id} \Omega_{M\wedge N} \otimes V \xrightarrow{\rho \otimes id} \Omega_N \otimes V.$$

Since f-supp $V \leq N$, there is a decomposition $\Omega_N \otimes V \simeq V \oplus W$ for some W, giving a map $f\colon V \hookrightarrow \Omega_N \otimes V$ which sends $v \mapsto u_N \otimes v$ at σ. Tensoring this with $\Omega_M \xrightarrow{id} \Omega_M$ gives the first map. From Corollary 3.27, there is a map $g\colon \Omega_M \otimes \Omega_N \to \Omega_{M\wedge N}$ which is an isomorphism at σ. Tensoring this with the identity on V gives the second map. By Theorem 3.18, we can choose a map $\rho\colon \Omega_{M\wedge N} \to \Omega_N$ such that $\rho_\sigma\left(g(u_M \otimes u_N)\right) = u_N$. Tensoring this with the identity on V gives the third map, so that the composition maps $u_M \otimes v$ to $u_N \otimes v$ at σ. $\square$

Lemma 3.48. *Let* $\{S_1, \ldots, S_n\} \subseteq L_Q^\sigma$ *be an arbitrary subset, and* $V \in \mathrm{Rep}(Q)$ *such that* $\operatorname{f-supp} V \leq T \in L_Q^\sigma$. *Let* $u_i \in (\Omega_{S_i})_\sigma$ *be nonzero, and* $z \in \Omega_T$ *also nonzero. Define*

$$Y := (\Omega_T \otimes V) \oplus \big(\bigoplus_i \Omega_{S_i} \otimes V\big)$$

so that $Y_\sigma = (Kz \otimes V_\sigma) \oplus (\bigoplus_i Ku_i \otimes V_\sigma)$, *and set* $w_i := u_i - z$.

Then Y *has a direct sum decomposition* $Y \simeq (\Omega_T \otimes V) \oplus W$, *where* $W \subset Y$ *is a subrepresentation such that*

$$W_\sigma = \bigoplus_i Kw_i \otimes V_\sigma.$$

Proof. For each i, the previous lemma gives a morphism

$$\theta_i \colon \Omega_{S_i} \otimes V \to \Omega_T \otimes V$$

such that $(\theta_i)_\sigma(u_i \otimes v) = z \otimes v$ for all $v \in V_\sigma$. Writing $X := \bigoplus_i \Omega_{S_i} \otimes V$, we get a map

$$\phi := (id_X - \sum_i \theta_i) \colon X \to Y.$$

Since the image of $\sum_i \theta_i$ is contained in $\Omega_T \otimes V$, the map ϕ is injective and $\operatorname{Im} \phi \cap \Omega_T \otimes V = 0$. Hence $W := \operatorname{Im} \phi$ is a complementary subrepresentation to $\Omega_T \otimes V$, and $\phi_\sigma(u_i \otimes v) = (u_i - z) \otimes v = w_i \otimes v$ for $v \in V_\sigma$. $\square$

To make the induction step, we will actually need a stronger version of the splitting principle, which is unfortunately quite technical. We will show, however, that this stronger version follows as a corollary of Theorem 3.40, so that we can apply it under the induction hypothesis.

Corollary 3.49 (to Theorem 3.40)**.** *Let* $Z \subseteq V_\sigma$ *be a subspace such that both* $Z \subseteq \operatorname{rank}_M V$ *for some* $M \in L_Q^\sigma$, *and* $Z \cap \operatorname{rank}_N V = 0$ *for* $N \not\leq M$. *Then for some* $l \geq 0$,

$V^{\otimes l}$ has a subrepresentation $U \simeq \Omega_M \otimes_K Z^{\otimes l}$ such that

$$V^{\otimes l} \simeq U \oplus W$$

for some $W \subseteq V^{\otimes l}$.

Proof. Assume that we've proven Theorem 3.40 for a quiver Q. Note that the corollary holds as stated if and only if it holds after replacing Z and V with $Z^{\otimes k}$ and $V^{\otimes k}$ in the hypotheses, for any $k > 0$. Similarly, a power $V^{\otimes k}$ can be replaced with a direct summand $W \subset V^{\otimes k}$ containing $Z^{\otimes k}$ at any point in the proof, since direct summands of $W^{\otimes l}$ are also direct summands of $V^{\otimes kl}$.

First we reduce to the case that $\operatorname{rank}_N V \neq 0$ only for N which are comparable to M. To this end, suppose that there exists some $N \ngeq M$ such that $\operatorname{rank}_N V \neq 0$, and take a maximal N with this property. By the theorem, we get

$$V^{\otimes l} \simeq U \oplus W$$

with $U \simeq \Omega_N \otimes_K (\operatorname{rank}_N V^{\otimes l})$. Hence,

$$Z^{\otimes l} \subseteq \operatorname{rank}_M V^{\otimes l} = \operatorname{rank}_M U \oplus \operatorname{rank}_M W = \operatorname{rank}_M W$$

where $\operatorname{rank}_M U = 0$ since $M \nleq N$. Then we can work in W, with $\operatorname{rank}_N W = 0$. Repeating this process, we can assume all nonzero rank spaces are comparable to M.

Now by Lemma 3.46, we can replace V and Z with some tensor powers of themselves and assume that $Z \cap X = 0$, where

$$X := \left(\sum_{N \nleq M} \operatorname{rank}_N V \right).$$

Choose a vector space projection $\pi \colon V_\sigma \twoheadrightarrow Z$ such that $X \subseteq \ker \pi$ and $\pi|_Z = id_Z$.

Now we claim that for large enough l, $V^{\otimes l} \simeq A \oplus B$ with $A_\sigma \subseteq \ker(\pi^{\otimes l})$ and M maximal such that $\operatorname{rank}_M B \neq 0$: suppose for contradiction that this is not possible, and take a decomposition as above such that $A_\sigma \subseteq \ker(\pi^{\otimes l})$ and the quantity

$$\#\{M' \not\leq M \mid \operatorname{rank}_{M'} B \neq 0\} \tag{3.16}$$

is minimal. If this quantity is 0, then the claim is verified. If not, let $N \not\leq M$ be maximal such that $\operatorname{rank}_N B \neq 0$, and apply the theorem to get $B^{\otimes l'} \simeq C \oplus D$ with $C_\sigma = \operatorname{rank}_N B^{\otimes l'}$ and $\operatorname{rank}_N D = 0$. Then

$$\left(V^{\otimes l}\right)^{\otimes l'} \simeq A^{\otimes l'} \oplus \cdots \oplus B^{\otimes l'} \simeq A^{\otimes l'} \oplus \cdots \oplus C \oplus D \tag{3.17}$$

and $\{M' \not\leq M \mid \operatorname{rank}_{M'} D \neq 0\} \subsetneq \{M' \not\leq M \mid \operatorname{rank}_{M'} B \neq 0\}$. All the summands represented by $\cdots$ have at least one tensor factor of A, so each is also in $\ker(\pi^{\otimes ll'})$. Also $C_\sigma \subseteq \operatorname{rank}_N V^{\otimes ll'} \subseteq \ker(\pi^{\otimes ll'})$, since $N \not\leq M$, and thus

$$\pi^{\otimes ll'}\left(A_\sigma^{\otimes l'} \oplus \cdots \oplus C_\sigma\right) \subseteq \pi^{\otimes ll'}(A_\sigma^{\otimes l'}) + \pi^{\otimes ll'}(\cdots) + \pi^{\otimes ll'}(C_\sigma) = 0$$

and so (3.17) contradicts the minimality of the quantity in (3.16).

Now, again replacing $V^{\otimes l}$ with V, and $\pi^{\otimes l}$ with π, we have a decomposition $V \simeq A \oplus B$ with $A_\sigma \subseteq \ker \pi$ and M maximal such that $\operatorname{rank}_M B \neq 0$. We can apply the theorem to B to get

$$V^{\otimes l} \simeq A^{\otimes l} \oplus \cdots B^{\otimes l} \simeq A^{\otimes l} \oplus \cdots \oplus C \oplus D = A' \oplus C \oplus D$$

where the last equality is just collecting summands. Here we have that $A'_\sigma \subseteq \ker(\pi^{\otimes l})$, just as in the argument of the last paragraph, and $C \simeq \Omega_M \otimes_K C_\sigma$. But also $\operatorname{rank}_M D = 0$, so

$$Z^{\otimes l} \subseteq \operatorname{rank}_M V^{\otimes l} = \operatorname{rank}_M A' \oplus \operatorname{rank}_M C \oplus \operatorname{rank}_M D = \operatorname{rank}_M A' \oplus \operatorname{rank}_M C$$

and so we can work in $A' \oplus C$. Replacing $A' \oplus C$ with V and $Z^{\otimes l}$ with Z, we can now assume that V has a decomposition $V \simeq C \oplus A'$ with $C \simeq \Omega_M \otimes_K C_\sigma$ and

$A'_\sigma \subseteq \ker \pi$. The last containment implies that

$$\ker \pi = \ker \pi \cap (C_\sigma \oplus A'_\sigma) = (\ker \pi \cap C_\sigma) \oplus A'_\sigma.$$

Via a linear combination of scalar endomorphisms, we can write $C \simeq C' \oplus C''$, with $\dim_K C'_\sigma = \dim_K Z$ and $C''_\sigma = \ker \pi \cap C_\sigma$, and then let $U := C'$ and $W := A' \oplus C''$. Now $W_\sigma = \ker \pi$, by dimension count, and since $\pi|_Z = id_Z$, it must be that $Z \cap W_\sigma = 0$. By Lemma 3.38, we can make a change of basis in V to get $U_\sigma = Z$. This proves the corollary. $\square$

Proof of Theorem 3.39 for $M = \hat{1}_Q$. We are assuming that Theorem 3.40 holds for quivers which are less complex than Q. Suppose V is indecomposable and $\text{f-supp}\, V = \hat{1}_Q$, but $V \not\simeq \mathbb{I}_Q$. Then $\operatorname{rank}_Q V = 0$ by Theorem 2.32, and we need to show that $f_{\hat{1}_Q} V$ is nilpotent. We are assuming that the theorem holds for rooted tree quivers which are less complex than Q; in particular, it holds for subquivers of Q, so we can use the usual extension and gluing cases.

Extension: If $\operatorname{rank}_P V|_P \neq 0$, then by Theorem 2.32 we have a direct sum decomposition

$$V|_P \simeq (\mathbb{I}_P \otimes_K \operatorname{rank}_P V) \oplus W$$

with $\operatorname{rank}_P W = 0$. But because $\operatorname{rank}_Q V = 0$, Lemma 3.3 implies that $\operatorname{rank}_P V|_P \subseteq \ker V_\alpha$. Then W would extend to a direct summand of V over Q by setting $W_\sigma = V_\sigma$, contradicting the indecomposability of V. Hence $\operatorname{rank}_P V = 0$.

Then by induction, $f_{\hat{1}_P} V \in R(P) \subset R(Q)$ is nilpotent, so $f_{\hat{1}_P} V^l = 0$ for large l. This implies that there is a direct sum decomposition of the restriction

$$V|_P^{\otimes l} \simeq \bigoplus_i V_i \oplus Z$$

such that each V_i is indecomposable with $(V_i)_\tau \neq 0$ and f-supp $V_i < \hat{1}_P$, and $Z_\tau = 0$. The lattice $L_Q^\sigma = J(L_P^\tau)$ has a unique coatom

$$C = L_P^\tau \setminus \{\hat{1}_P\} = \langle C_1, \ldots, C_k \rangle$$

where $\{C_i\}$ is the set of coatoms of L_P^τ. We will show that for such an l as above, f-supp $V^{\otimes l} \leq C$.

First, restricting to P we get:

$$(\Omega_C \otimes V^{\otimes l})|_P = (\Omega_C)|_P \otimes (V^{\otimes l})|_P = \big(\bigoplus_j \Omega_{C_j}\big) \otimes \big(\bigoplus_i V_i \oplus Z\big) = \bigoplus_i Y_i \oplus \tilde{Z}$$

where $Y_i := \big(\bigoplus_j \Omega_{C_j}\big) \otimes V_i$, and $\tilde{Z} = \big(\bigoplus_j \Omega_{C_j}\big) \otimes Z$. For each indecomposable summand V_i, choose a coatom $T_i \in \{C_1, \ldots, C_k\}$ such that $T_i \geq$ f-supp V_i. At least one such T_i exists because f-supp $V_i < \hat{1}$. Let $u_j \in (\Omega_{C_j})_\sigma$ be nonzero such that $(\Omega_C)_\alpha(u_j) = u_\sigma \in (\Omega_C)_\sigma$ for all j (that is, the u_j and u_σ are part of a standard basis for Ω_C). Then apply Lemma 3.48 with $\{S_1, \ldots, S_n\} = \{C_1, \ldots, C_k\} \setminus \{T_i\}$ to write each

$$Y_i = (\Omega_{T_i} \otimes V_i) \oplus \big(\bigoplus_{C_j \neq T_i} \Omega_{C_j} \otimes V_i\big) \simeq (\Omega_{T_i} \otimes V_i) \oplus W_i$$

with $(W_i)_\tau = \bigoplus_j K(u_j - u_i) \otimes (V_i)_\tau \subseteq \ker(\Omega_C \otimes V^{\otimes l})_\alpha$. Then since f-supp $V_i \leq T_i$, each $\Omega_{T_i} \otimes V_i \simeq V_i \oplus U_i$ with $(U_i)_\tau = 0$, and by setting $X := \big(\bigoplus_i W_i \oplus U_i\big) \oplus \tilde{Z}$ we get

$$(\Omega_C \otimes V^{\otimes l})|_P \simeq \big(\bigoplus_i V_i\big) \oplus X \simeq V^{\otimes l}|_P \oplus X$$

with $X_\sigma \subseteq \ker(\Omega_C \otimes V^{\otimes l})_\alpha$. Since this isomorphism sends $u_i \otimes v$ to $(v, 0)$ for $v \in V_\tau^{\otimes l}$, the map $(\Omega_C \otimes V^{\otimes l})_\alpha$ acts on the right hand side just as $V_\alpha^{\otimes l}$, so the decomposition extends to give $\Omega_C \otimes V^{\otimes l} \simeq V^{\otimes l} \oplus X$ with $\sigma \notin \operatorname{supp} X$. Hence f-supp $V^{\otimes l} \leq C$, and so $f_{\hat{1}_Q} V^l = 0$.

Gluing: Suppose Q is a gluing of P and S, as usual, but also take P to be a one point extension of some smaller quiver (so there is a unique arrow $a \in P_\to$ with

$ha = \sigma$). Denote by $\hat{1}_P$ and $\hat{1}_S$ the maximal elements of L_P^σ and L_S^σ, respectively. We show that $f_{\hat{1}_Q} V$ is nilpotent by induction on $d = \#\{M \in L_S^\sigma \mid \operatorname{rank}_{(\hat{1}_P, M)} V \neq 0\}$.

If $d = 0$, then $\operatorname{rank}_P V = \operatorname{rank}_{(\hat{1}_P, \hat{0}_S)} V = 0$, and the induction hypothesis gives (just as in the extension case)

$$\text{f-supp}\, V|_P^{\otimes l} \leq C$$

for large l, where C is the unique coatom of L_P^σ. But then, regardless of how $V|_S^{\otimes l}$ decomposes, we get that $\Omega_{(C, \hat{1}_S)} \otimes V^{\otimes l} = V^{\otimes l} \oplus U$, so $\text{f-supp}\, V^{\otimes l} \leq (C, \hat{1}_S) < \hat{1}_Q$. Hence $f_{\hat{1}_Q} V^l = 0$.

If $d > 0$, choose any $M \in L_S^\sigma$ maximal such that $Z := \operatorname{rank}_{(\hat{1}_P, M)} V \neq 0$. Then $M \neq \hat{1}_S$ since $\operatorname{rank}_{(\hat{1}_P, \hat{1}_S)} V = \operatorname{rank}_Q V = 0$ by hypothesis. Now by maximality, $Z \subseteq \operatorname{rank}_M V$ and $Z \cap \operatorname{rank}_N V = 0$ for $N \not\leq M$ in L_S^σ. An inductive application of Corollary 3.49 over S then gives subrepresentations $U, W \subset V|_S^{\otimes l}$ such that $U \simeq \Omega_M \otimes_K Z^{\otimes l}$, and $V|_S^{\otimes l} = U \oplus W$.

But over P, since $Z^{\otimes l} \subseteq \operatorname{rank}_P V|_P^{\otimes l}$, Theorem 2.32 gives subrepresentations $X, Y \subset V|_P^{\otimes l}$ such that $X = \mathbb{I}_P \otimes_K Z^{\otimes l}$ and $V|_P^{\otimes l} = X \oplus Y$– in particular, $X_\sigma = Z^{\otimes l} = U_\sigma$. By Lemma 3.38, we can take $Y_\sigma = W_\sigma$. So we have subrepresentations $A, B \subset V^{\otimes l}$ defined by

$$A := \begin{cases} X & \text{over } P \\ U & \text{over } S \end{cases} \qquad B := \begin{cases} Y & \text{over } P \\ W & \text{over } S \end{cases}$$

giving a direct sum decomposition $V^{\otimes l} \simeq A \oplus B$, since this holds over both P and S. Now since

$$A \simeq \Omega_{(\hat{1}_P, M)} \otimes_K (\operatorname{rank}_{(\hat{1}_P, M)} V^{\otimes l})$$

we know that both $f_{\hat{1}_Q} A = 0$ and $\operatorname{rank}_{(\hat{1}_P, M)} B = 0$. By the induction hypothesis in this gluing case, $f_{\hat{1}_Q} B$ is nilpotent, so $f_{\hat{1}_Q}(A + B) = f_{\hat{1}_Q} V^l$ is nilpotent. $\square$

The completes the proof of the main result. The commentary after the statement of Theorem 3.39 gives as an immediate corollary:

Corollary 3.50. *The rank functions on Q give an isomorphism*

$$R(Q)_{red} \xrightarrow{\sim} \prod_{M\in L_Q} \mathbb{Z} \qquad\qquad \overline{V} \mapsto \big(r_M(V)\big).$$

In particular, $R(Q)_{red}$ is a finitely generated $\mathbb{Z}$-module.

Example 3.51. Continuing Examples 3.5 and 3.14, we can calculate that

$$\operatorname{rank}_{\mathbb{Z}} R(Q)_{red} = \sum_{x\in Q_0} \#L_Q^x = 31$$

where $\operatorname{rank}_{\mathbb{Z}}$ is the rank as an abelian group. This is because each of the three source vertices contribute 1, and we can count $\#L_Q^\tau = 8$ and $\#L_Q^\sigma = 20$ from the diagrams in Example 3.14.

Corollary 3.52. *For a fixed $V \in \operatorname{Rep} Q$, only finitely many indecomposable representations appear as direct summands of the representations $\{V^{\otimes i}\}_{i\geq 0}$. In other words, there exists a finite set of indecomposables $\{V_k\}$ such that $\{V^{\otimes i}\}_{i\geq 0}$ is contained in the subcategory additively generated by $\{V_k\}$.*

Proof. The ring R_{red} is a finitely generated $\mathbb{Z}$-module, hence integral over $\mathbb{Z}$. This implies that R is integral over $\mathbb{Z}$ also, and hence the subalgebra $\mathbb{Z}[V]$ is too. Then $\mathbb{Z}[V]$ is a finitely generated $\mathbb{Z}$-module, which gives the conclusion. □

Since $R(Q)$ itself is a finitely generated $\mathbb{Z}$-module if and only if Q is Dynkin, it is natural to ask for which choice of the root on a Dynkin quiver is $R(Q) = R(Q)_{red}$. Hugh Thomas noted that this is the case precisely when the root is chosen to be a **minuscule node** of the Dynkin diagram. This can be easily checked by counting L_Q in each case, and comparing it to the number of positive roots in the corresponding root system. Furthermore, in these cases the corresponding poset L_Q^σ is exactly the minuscule poset corresponding to that vertex [47, §4]. We list these cases by type, for reference. We denote by $[n]$ the poset $\{1, 2, \ldots, n\}$ under the usual order of integers.

Type A: For any choice of root, we get $R(Q) = R(Q)_{red}$. If we number the vertices of the graph as

$$A_n:\ 1 \;\text{——}\; 2 \;\text{——}\; 3 \;\text{——}\; \cdots \;\text{——}\; n$$

and take $\sigma = j$ to be the root (without loss of generality assume $j \neq n$), then $L_Q^\sigma \cong [j] \times [n-j]$.

Type D: If root is any extremal vertex, that is, a vertex connected to only one edge, the we get $R(Q) = R(Q)_{red}$. Labeling the vertices as

$$D_n:\ \begin{array}{l} 1 \diagdown \\ \;\; 3 \;\text{——}\; \cdots \;\text{——}\; n \\ 2 \diagup \end{array}$$

we find that for $\sigma = 1$, we get $L_Q^\sigma \cong J([2] \times [n-2])$ (and similarly for $\sigma = 2$). Taking $\sigma = n$, we get instead $L_Q^\sigma \cong J^{n-3}([2] \times [2])$.

Type E: For E_6, if the root σ is either vertex 1 or 6 in the labeling below, then we get $R(Q) = R(Q)_{red}$.

$$E_6:\ \begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & & | & & & & \\ 1 & \text{——} & 3 & \text{——} & 4 & \text{——} & 5 & \text{——} & 6 \end{array}$$

In either case, we get $L_Q^\sigma \cong J^2([2] \times [3])$, but again the labeling by positive roots is not the standard order.

For E_7, we only get $R(Q) = R(Q)_{red}$ for $\sigma = 7$ in the labeling below.

$$E_7:\ \begin{array}{ccccccccccc} & & & & 2 & & & & & & \\ & & & & | & & & & & & \\ 1 & \text{——} & 3 & \text{——} & 4 & \text{——} & 5 & \text{——} & 6 & \text{——} & 7 \end{array}$$

For E_8, no choice of the root gives $R(Q) = R(Q)_{red}$.

For any rooted tree quiver (Q, σ), each element $M \in L_Q$ has an associated indecomposable $\Omega_M \in \operatorname{Rep}(Q)$, whose dimension vector is a positive root in the root system of type Q (Theorem 2.2). This correspondence gives a labeling of the elements of L_Q by positive roots. In type A, for any choice of the root σ, this labeling gives an isomorphism between L_Q^σ and the subposet of the positive roots supported at σ. In type D, these labels do *not* give an isomorphism between L_Q^σ and the subposet of the positive roots supported at σ, although these lattices are abstractly isomorphic. For example, in type D_4 (with the root at the end of any branch) we have

$$\begin{smallmatrix}1\\1\end{smallmatrix}21 \; < \underline{\dim}\,\mathbb{I} = \begin{smallmatrix}1\\1\end{smallmatrix}11$$

in L_Q^σ (because $\Omega_{\hat{1}} = \mathbb{I}$), but

$$\begin{smallmatrix}1\\1\end{smallmatrix}11 \; < \; \begin{smallmatrix}1\\1\end{smallmatrix}21$$

in the root lattice. For the other cases above in which L_Q^σ is minuscule (equiv. $R(Q) = R(Q)_{red}$), there is again an abstract lattice isomorphism between L_Q^σ and the lattice of roots supported at σ, but the labeling of L_Q^σ by positive roots is not compatible with this isomorphism.

The correspondences above motivate the search for a connection between tensor products of quiver representations and the representation theory of simple Lie algebras, in these cases. The interested reader may consult [47, 56, 57] for more information on minuscule posets.

3.5 Future directions

If Q is any quiver now, and $R(Q)_{red}$ is a finitely generated $\mathbb{Z}$-module, say that Q is of **finite multiplicative type** (over K). Since the ring $R(Q)$ generally depends on the field K, this property could also. Then so far we know that Dynkin quivers of any orientation and rooted tree quivers are of finite multiplicative type over any field.

A natural question to ask is then, "What other quivers are of finite multiplicative type?". The first observation we make regarding this question is that if Q is of finite multiplicative type over K, then so is any **minor** of Q; that is, any quiver obtained from Q by contracting edges and removing any combination of vertices and edges [18, §1.7].

Proposition 3.53. *The set of finite multiplicative type quivers over a given field K is minor closed.*

Proof. Let Q be of finite multiplicative type. If Q' is obtained from Q by contracting an edge, then there is an injective homomorphism of rings $R(Q') \hookrightarrow R(Q)$ given on representations by assigning the identity map to the contracted edge [32, §6]. If Q' is obtained from Q by removing some vertex or edge, then $R(Q')$ is isomorphic to the quotient of $R(Q)$ by the ideal generated by e_P for all connected subquivers P containing that vertex or edge. □

It is easy to see that the loop quiver $\tilde{A}_0$ is not of finite multiplicative type over an infinite field K. The trace of an endomorphism is additive with respect to direct sum, and multiplicative with respect to tensor product, hence extends to a ring homomorphism

$$\mathrm{Tr}\colon R(\tilde{A}_0) \twoheadrightarrow K.$$

A field K is infinite if and only if it is not a finitely generated $\mathbb{Z}$-module. Since a field is reduced, the trace map factors through $R(\tilde{A}_0)_{red}$, so $\tilde{A}_0$ is not of finite multiplicative type when K is infinite. The case that K is finite can be handled by a more sophisticated argument, provided by an anonymous referee to [37]. The map sending an endomorphism to its characteristic polynomial can be used to map the Grothendieck ring of $\mathrm{Rep}_K(\tilde{A}_0)$ into $W(K)$, the ring of universal Witt vectors over K (see [41, p. 330] or [11, Ch. IX] for Witt vectors, and [4] for the relation to K-theory). That $W(K)$ is reduced follows easily from the definition of multiplication in $W(K)$ and that K has no nilpotents. Since the Grothendieck ring is a quotient of $R(\tilde{A}_0)$, and its image in $W(K)$ is not finitely generated as a $\mathbb{Z}$-module, we get that $\tilde{A}_0$ is not of multiplicative finite type over a finite field either.

Since a graph is a tree if and only if it doesn't have a loop as a minor, the proposition above implies that a quiver of finite multiplicative type must be a tree. The following example shows that not every tree is of finite multiplicative type, when K is infinite.

Example 3.54. Let Q be the quiver of type $\tilde{D}_4$, oriented to have "crossing paths":

$$Q = \begin{tikzcd}[row sep=small] \bullet \arrow[dr] & & \bullet \\ & \bullet \arrow[ur] \arrow[dr] & \\ \bullet \arrow[ur] & & \bullet \end{tikzcd}.$$

Now consider the quiver

$$Q' = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \bullet \arrow[l] \arrow[u] \end{tikzcd}$$

of type $\tilde{A}_3$. There is a functor $f^*\colon \operatorname{Rep}(Q) \to \operatorname{Rep}(Q')$ given by

$$\begin{tikzcd}[row sep=small] V_1 \arrow[dr, "V_a"] & & V_2 \\ & V_3 \arrow[ur, "V_c"] \arrow[dr, "V_d"] & \\ V_4 \arrow[ur, "V_b"] & & V_5 \end{tikzcd} \longmapsto \begin{tikzcd} V_1 \arrow[r, "V_cV_a"] \arrow[d, "V_dV_a"] & V_2 \\ V_5 & V_4 \arrow[l, "V_dV_c"] \arrow[u, "V_cV_b"'] \end{tikzcd}$$

which, from the categorical viewpoint, is the composition of a representation $V\colon \mathscr{Q} \to K\text{-}\mathrm{mod}$ with a certain functor $f\colon \mathscr{Q}' \to \mathscr{Q}$ (cf. §2.2.2). Equivalently, if we relax our definition of maps of directed graphs to allow arrows to be mapped to *paths* in the target, then f^* is still a pullback along a certain map $f\colon Q' \to Q$. Applying the global tensor functor $\mathscr{R}_{Q'}$ to a representation of Q' gives a representation in which all maps over the arrows of Q' are isomorphisms (Proposition 2.22). Such a representation induces a representation of $\tilde{A}_0$ by taking the underlying vector space to be the space at any vertex of Q', and the endomorphism to be given by traversing around the cycle once, say clockwise. This gives a functor $\mathscr{L}\colon \operatorname{Rep}^\circ(Q') \to \operatorname{Rep}(\tilde{A}_0)$, where the domain is defined as the full subcategory of $\operatorname{Rep}(Q')$ consisting of representations which have an isomorphism at each arrow of Q'. So, summarily, we compose functors:

$$\operatorname{Rep}(Q) \xrightarrow{f^*} \operatorname{Rep}(Q') \xrightarrow{\mathscr{R}_{Q'}} \operatorname{Rep}^\circ(Q') \xrightarrow{\mathscr{L}} \operatorname{Rep}(\tilde{A}_0).$$

Each of these functors respects direct sum and tensor product, and preserves $\mathbb{I}$, hence we have a ring homomorphism $R(Q) \to R(\tilde{A}_0)$.

For $\lambda \in K$ and $n \in \mathbb{Z}$, define a representation

$$V_\lambda(n) := \begin{tikzcd} K^n \arrow[dr, "A"] & & K^n \\ & K^{2n} \arrow[ur, "C"] \arrow[dr, "D"'] & \\ K^n \arrow[ur, "B"'] & & K^n \end{tikzcd}$$

with the maps given by the block form matrices

$$A = \begin{pmatrix} id \\ 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ id \end{pmatrix} \qquad C = \begin{pmatrix} id & id \end{pmatrix} \qquad D = \begin{pmatrix} id & J_\lambda(n) \end{pmatrix},$$

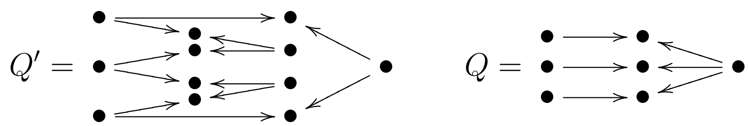
where $J_\lambda(n)$ is the Jordan block of size n with eigenvalue λ.

Then the global tensor functor of Q', applied to $f^*(V_\lambda(n))$, gives the representation

$$\mathscr{R}_{Q'}(f^* V_\lambda(n)) = \begin{tikzcd} K^n \arrow[r, "id"] \arrow[d, "id"] & K^n \\ K^n & K^n \arrow[l, "J_\lambda(n)"] \arrow[u, "id"'] \end{tikzcd}$$

when $\lambda \neq 0$, and the 0 representation when $\lambda = 0$. Applying the functor $\mathscr{L}$ gives the endomorphism $J_\lambda(n)$ of K^n for $\lambda \neq 0$, and hence the image of the induced map $R(Q) \to R(\tilde{A}_0)$ contains the representations which have all eigenvalues nonzero. The reduction of this image is not a finitely generated $\mathbb{Z}$-module, by a slight modification of the above argument for the loop quiver, and hence $R(Q)_{red}$ cannot be a finitely generated $\mathbb{Z}$ module either. This shows that the tree quiver Q is not of finite multiplicative type.

A similar argument with the pair of quivers



shows that this Q is also not of finite multiplicative type. To approach the classification of all quivers of finite multiplicative type, we suggest an analogy to the classification of quivers of finite representation type. It is well-known that a quiver Q is of finite representation type if and only if Q is a Dynkin diagram, of any orientation. This is equivalent to saying that Q does not have a minor of type $\tilde{A}_0$, $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$ (of any orientation). There are only finitely many orientations of a given graph, so the finite set of quivers of these five types gives a finite set of "obstructions" to a quiver being of finite representation type; these are sometimes called **forbidden minors**. The Tree Theorem of J.B. Kruskal can be applied to show that finitely many forbidden minors are enough to characterize the finite multiplicative type property, as well (with respect to a fixed K). An upper-ideal U in a quasi-ordered set X (cf. §3.2.2) is a subset for which $x \in U$ and $x \le y$ imply that $y \in U$. Then X is said to be **well-quasi-ordered** if every upper ideal has a finite generating set. All trees below will be assumed to be finite.

Theorem 3.55 (Kruskal [40])**.** *The set of tree quivers is well-quasi-ordered under the minor relation.*

Proof. Kruskal proves that the set $T(X)$ of "structured trees" over a well-quasi-ordered set X is itself well-quasi-ordered with respect to a certain ordering. He also shows that if $h\colon Y \to Z$ is an epimorphism of quasi-ordered sets, and Y is well-quasi-ordered, then so is Z. In our case, h will be a "forgetful" map from the highly structured set $T(X)$ to the set of tree quivers, quasi-ordered by the minor relation.

We take $X = \{I, O\}$ to be a set of two incomparable symbols (short for 'in' and 'out'), which is finite and hence well-quasi-ordered. A structured tree over X is essentially a rooted tree with a linear ordering on the edges, in which the vertices have been labeled by elements of X. The ordering on $T(X)$ is the homeomorphic embedding ordering (i.e. topological minor ordering) on the underlying trees, with some further conditions about preserving the additional structure.

We define the forgetful map h from $T(X)$ to the set of tree quivers. Given a structured tree A over X, the underlying graph of $h(A)$ is the same. Since this graph is a tree, each vertex other than the root of A has a unique edge which is in the direction of the root. If the vertex is labeled I, we orient the corresponding edge in $h(A)$ towards the root, and otherwise we orient it away from the root. Since each edge has precisely one endpoint which is farther from the root, this uniquely defines an orientation of the underlying tree. For example, $h(A)$ is a rooted tree quiver in the sense of this thesis, with the same root specified by the structure of A, if and only if each vertex other than the root is labeled by I. Sorting out Kruskal's ordering conditions on $T(X)$, one can see that if $A \leq B$ in $T(X)$ then $h(A) \leq h(B)$ in the minor ordering. It is clear that h is onto, since given any tree quiver Q we can choose an arbitrary root, arbitrary linear ordering on the vertices, etc, and label vertices appropriately to get a structured tree which h maps to Q. Then the Tree Theorem implies that $T(X)$ is well-quasi-ordered and the Epimorphism Theorem implies that the set of tree quivers is well-quasi-ordered. □

Corollary 3.56. *For any field K, the set of quivers of finite multiplicative type over K can be characterized by a* finite *set of forbidden minors.*

Proof. A quiver of finite multiplicative type must be a tree, by discussion above; forbidding the loop quiver as a minor guarantees this condition. Now apply the theorem to the upper ideal of tree quivers which are not of finite multiplicative type. $\square$

These results indicate that a forbidden minor characterization of the multiplicative finite type property could be tractable over some particular fields. As it stands now, there are no other known examples of finite multiplicative type quivers besides the Dynkin and rooted tree ones. There is computational evidence, however, suggesting that there are more (of types $\tilde{E}_6$ and $\tilde{E}_7$, specifically). It seems likely that tree modules will be useful in constructing idempotents in the representation rings of more general quivers, and some other techniques of this paper might be useful in the more general setting. However, there will be many formidable difficulties to overcome, such as the correct induction process to get results for general classes of quivers rather than just isolated examples.

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