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Technical Report

ON THE CENTRALIZER OF A GALOIS RING

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TABLE OF CONTENTS

ABSTRACT

Let V and W be a pair of dual vector spaces over a division ring D. There is an associated weak topology on V, a subbase at zero consisting of the kernels of the functionals in W. The resulting topological vector space V is said to be weakly topologized. The ring $A = \mathcal{L}(V,W)$ of all continuous linear transformations on V is called a continuous transformation ring. When V is one dimensional A is a division ring. When V is a finite dimensional then A becomes a typical simple ring with minimum condition. And when W is the conjugate space of V then A is a completely primitive ring.

The classical Galois theory consists in studying the one to one correspondence between the groups of automorphisms of a field and the subfields of invariant elements. Similar theories have been developed for division rings by Cartan and Jacobson, for simple rings with minimum condition by Hochschild and Nakayama, for completely primitive rings by Dieudonné and Nakayama, and for continuous transformation rings by Rosenberg and Zelinsky. In all these cases, except when A is a field, the Galois correspondence does not pair off an arbitrary subgroup with an intermediate subring.

One of the main theorems of classical Galois theory states that the intermediate field E is Galois over the base field E_0 if and only if the Galois group, Γ, of A over E is normal in the Galois group, Γ_0, of A over E_0. In proving a generalization of this theorem in the case of continuous transformation rings, Rosenberg and Zelinsky had to make the ad hoc hypothesis that the centralizer of E in A is a semi-simple ring.

The principal result established in this thesis is that this hypothesis is not necessary. In fact, one can prove the following theorem.

Let (V,W) be a pair of dual vector spaces over a division ring D. Let $A = \mathcal{L}(V,W)$ be the ring of all continuous transformations on V which is topologized weakly by W, and let E be a subring of A which is also a continuous transformation ring. Denote by $\mathcal{S}(A)$ the socle of A (i.e., the sum of the irreducible left ideals of A) and by $\mathcal{S}(E)$ the socle of E. Suppose that

(i) $\mathcal{S}(E)V = V$

(ii) $W\mathcal{S}(E)^* = W$

(iii) $\mathcal{V}(E) \subseteq \mathcal{V}(A)$

(iv) V is a finitely generated $\mathcal{V}(E)$-module,

(v) G is a group of automorphisms of A with $[\Lambda : DT] < \infty$ where Λ is the group of all semi-linear transformations on V belonging to G and T is the group of all linear transformations on V contained in Λ. Suppose further that

(vi) E is the fixed ring under G. Then $\mathcal{L}_A(E)$, the centralizer of E in A, is semi-simple.

INTRODUCTION

Let V and W be a pair of dual vector spaces over a division ring D. There is an associated weak topology on V, a subbase at zero consisting of the kernels of the functionals in W. The resulting topological vector space V is said to be weakly topologized. Let $A = \mathcal{L}(V,W)$ be the ring of all continuous linear transformations on V. Such a ring after Jacobson [9] is called a continuous transformation ring. When V is one dimensional, A is a division ring. When V is a finite dimensional, then A becomes a typical simple ring with minimum condition. And when W is the conjugate space of V, then A is a completely primitive ring.

The usual Galois theory consists in studying the one to one correspondence between the groups of automorphisms of a field and the subfields of invariant elements (see [1]). Similar theories have been developed for division rings by Cartan [3] and Jacobson [8], for simple rings with minimum condition by Hochschild [7] and Nakayama [15], for completely primitive rings by Dieudonne [5] and Nakayama [15], and for continuous transformation rings by Rosenberg and Zelinsky [16]. In all these cases, except when A is a field, the Galois correspondence does not pair off an arbitrary subgroup with an intermediate subring.

One of the main theorems of classical Galois theory states that the intermediate field E is Galois over the base field E_0 if and only if the Galois group, Γ, of A over E is normal in the Galois group, Γ_0, of A over E_0. In proving a generalization of this theorem in the case of continuous transformation rings, Rosenberg and Zelinsky [16] had to make the ad hoc hypothesis that the centralizer of E in A is a semi-simple ring. The principal result established here is that this hypothesis is not necessary. In fact, we prove the following theorem:

Main theorem: Let V and W be a pair of dual vector spaces over a division ring D, and let E be a subring of $A = \mathcal{L}(V,W)$ which is also a continuous transformation ring. Denote by $\mathcal{Y}(A)$ the socle of A and by $\mathcal{Y}(E)$ the socle of E. Suppose that

(i) $\mathcal{Y}(E)V = V$,

(ii) $W\mathcal{Y}(E)^* = W$,

(iii) $\mathcal{Y}(E) \subseteq \mathcal{Y}(A)$,

(iv) V is a finitely generated $\mathcal{Y}(E)$-module,

(v) G is a group of automorphisms of A with $[\Lambda : DT] < \infty$, where Λ is the group of all semi-linear transformations on V belonging to G and T is the group of all linear transformations on V contained in Λ. Suppose further that

(vi) E is the fixed ring under G.

Then $\mathcal{L}_A(E)$, the centralizer of E in A, is semi-simple.

In Chapter 1, we will prove some lemmas which will be used in proving the main theorem. The proof of the main theorem will appear in Chapter 2 and Chapter 3.

The author is very much indebted for many valuable suggestions to Professor Jack E. McLaughlin, under whose direction this thesis was written.

PRELIMINARIES

Let R be a ring.

Definition 0.1. A left R-module is a system consisting of an additive abelian group M, and a function defined on the product set $R \times M$ having values in M, such that if ax denotes the element in M determined by $x \in M$, $a \in R$, then

$$(a+b)x = ax + bx$$
$$(ab)x = a(bx)$$
$$a(x+y) = ax + ay$$

hold for any a, b in R and x, y in M.

The concept of a right module is defined in a similar fashion. Henceforth, the term "module" without modifier will always mean left module.

Definition 0.2. An R-module M is said to be unitary if $RM = M$.

Definition 0.3. A subgroup N of M is said to be an R-submodule of M if $RN \subseteq N$.

Definition 0.4. If N is an R-submodule of M, the factor group M/N can be turned into an R-module by defining

$$a(x+N) = ax + N.$$

We call this module the difference module of M relative to N, and it will also be denoted by M/N.

Definition 0.5. An R-module M is said to be finitely generated if there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of M such that every element in M can be written in the form

$$m_1x_1 + m_2x_2 + \ldots + m_nx_n + a_1x_1 + a_2x_2 + \ldots a_nx_n$$

where the m_i are integers, and the a_i are in R.

Definition 0.6. A module M satisfies the ascending chain condition for submodules if for any increasing sequence of submodules

$$N_1 \subseteq N_2 \subseteq \ldots$$

there exists an integer n such $N_n = N_{n+1} = \ldots$.

Proposition 0.1. A module M satisfies the ascending chain condition for submodules if and only if every submodule of M is finitely generated (see [10]).

Definition 0.7. A module M satisfies the descending chain condition for submodules if for any decreasing sequence of submodules

$$N_1 \supseteq N_2 \supseteq \ldots$$

there exists an integer n such that $N_n = N_{n+1} = \ldots$.

Proposition 0.2. If R is a ring that satisfies the descending chain condition for left (right) ideals, then any finitely generated unitary R-module satisfies the descending chain condition for submodules (right submodules) (see [10]).

Definition 0.8. Let M and M' be R-modules. A group homomorphism

Θ of M into M' is called an R-<u>homomorphism</u> if for all $x \in M$ and all $a \in R$,

$$\Theta(ax) = a\Theta(x).$$

If Θ is a one to one mapping, it is called an R-<u>isomorphism</u>. If there exists an R-isomorphism of M onto M', then M and M' are called isomorphic modules and we write $M \cong M'$. If $M = M'$, an R-homomorphism of M into M itself is called an R-<u>endomorphism</u>.

We will denote by $\mathrm{End}_R M$ the ring of R-endomorphisms on M, and denote by $\mathrm{Hom}_R(M,M')$ the group of R-homomorphisms of M into M'.

<u>Definition</u> <u>0.9</u>. An R-module M is called <u>irreducible</u> if $RM \neq \{0\}$ and there is no proper R-submodule of M other than $\{0\}$.

<u>Definition</u> <u>0.10</u>. An R-module M is called <u>completely</u> <u>reducible</u> if it is a sum of irreducible R-submodules of M.

<u>Definition</u> <u>0.11</u>. An R-module M is called a direct sum of the family $\{M_\lambda | \lambda \in \Lambda\}$ of R-submodules of M and we write $M = \sum_\lambda \oplus M_\lambda$ if any x in M can be written in one and only one way in the form $\sum_\lambda x_\lambda$ with $x_\lambda \in M_\lambda$.

Proposition 0.3. If $M = \sum_{\lambda \in \Lambda} M_\lambda$ where $\{M_\lambda | \lambda \in \Lambda\}$ is a family of irreducible R-submodules of M, then $M = \sum_{\delta \in \Delta} \oplus M_\delta$ where $\{M_\delta | \delta \in \Delta\}$ is a subfamily of $\{M_\lambda | \lambda \in \Lambda\}$ (see [12]).

If $M = \sum_\lambda \oplus M_\lambda = \sum_\mu \oplus N_\mu$ where M_λ and N_μ are irreducible R-submodules of M, then the cardinal number of $\{M_\lambda\}$ equals the cardinal number of $\{N_\mu\}$.

If both sets are finite, the proof of the result can be found, for example, in [2]. For the infinite case, the proof is available in [12]. From this fact, we can define the dimension of a completely reducible module as follows:

Definition 0.12. If M is expressed as a direct sum of irreducible R-submodules, the cardinal number of direct summands is called the dimension of M over R and is denoted by $\dim_R M$.

In a particular case, if B is a ring with unit element and D a division subring with the same unit element, then B is a completely reducible D-module whose dimension we write $[B : D]_\ell$.

Proposition 0.4. Every R-submodule N of a completely reducible R-module M has a complement N'; that is, an R-submodule N' of M exists such that $M = N \oplus N'$ (see [12]).

Proposition 0.5. Every R-homomorphic image and every R-submodule of a completely reducible R-module is completely reducible (see [12]).

Proposition 0.6. Let M be a completely reducible R-module, $E = \operatorname{End}_R M$. Then M is completely reducible as E-module (see [12]).

Definition 0.13. Let M be an R-module and let $\{M_\lambda \mid \lambda \in \Lambda\}$ be the family of all irreducible R-modules of M, then $\sum_{\lambda\in\Lambda} M_\lambda$ is called the socle of M and it will be denoted by $\gamma(M)$.

Definition 0.14. Let M be an R-module. The sum $\sum_{\delta\in\Delta} M_\delta$ of all the irreducible R-submodules of M R-isomorphic to a given irreducible R-submodule N of M is called the <u>homogeneous component</u> of the socle determined by N.

It is easy to see that the socle of M and its homogeneous components are fully invariant in the sense that they are mapped into themselves by every endomorphism of M.

<u>Definition 0.15</u>. If R is a ring, the socle of R as an R-module is called the <u>socle</u> of R.

Note that the socle $\mathcal{Y}(R)$ of a ring R is the sum of irreducible left ideals of R and so $\mathcal{Y}(R)$ is a left ideal. Every right multiplication $x \to xa$ is an endomorphism of R as an R-module. This maps $\mathcal{Y}(R)$ into itself; hence $\mathcal{Y}(R)$ is also a right ideal.

<u>Definition 0.16</u>. An R-module M is said to be <u>faithful</u> if $aM \neq 0$ for each $a \neq 0$ in R.

<u>Definition 0.17</u>. A ring R is called <u>primitive</u> (right primitive) if it admits a faithful irreducible module (right module).

<u>Definition 0.18</u>. A ring R is called <u>completely primitive</u> if it is isomorphic to the ring of all linear transformations on a vector space over a division ring.

Clearly, a completely primitive ring is primitive.

<u>Proposition 0.7</u>. Let R be a primitive ring and let I_1 and I_2 be non-zero ideals in R. Then $I_1 I_2 \neq \{0\}$ (see [12]).

<u>Definition 0.19</u>. Let R be an arbitrary ring and let $\mathfrak{M}$ be the set of irreducible R-modules. Then the ideal

$$J(R) = \bigcap_{M \in \mathfrak{M}} \{a \in R \mid aM = 0\}$$

is called the (Jacobson) radical of R.

Proposition 0.8. Every element z in J(R) is left quasi-regular; i.e., there exists an element z' in R such that

$$z + z' - z'z = 0$$

(see [12]).

Definition 0.20. A ring R is called semi-simple if the radical J(R) of R is {0}.

Definition 0.21. A ring R is called simple if there are no proper ideals in R other than {0}, and $R^2 \neq \{0\}$.

Proposition 0.9. (Wedderburn's Theorem). Any simple ring R satisfying the minimum condition for left ideals is isomorphic to the complete ring of linear transformations on a finite dimensional vector space over a division ring (see [2]).

Definition 0.22. Let V be a left vector space over a division ring D and let W be a right vector space over D. A mapping f of the product set $V \times W$ into D is called bilinear form on V and W if for all v, v_1, v_2 in V, w, w_1, w_2 in W and d in D, we have

$$f(v_1+v_2,w) = f(v_1,w) + f(v_2,w),$$

$$f(v,w_1+w_2) = f(v,w_1) + f(v,w_2),$$

$$f(dv,w) = df(v,w),$$

and

$$f(v,wd) = f(v,w)d$$

The bilinear form f is called <u>non-degenerate</u> if $f(v,w) = 0$ for all $v \in V$ implies $w = 0$ and $f(v,w) = 0$ for all $w \in W$ implies $v = 0$.

If there exists a non-degenerate bilinear form f on V and W, then (V,W) is called <u>a pair of dual vector spaces relative to</u> f <u>over</u> D.

In dealing with a single bilinear form, it is convenient to use the abbreviation (v,w) for $f(v,w)$ and we simply say that (V,W) is a pair of dual vector spaces over D. We shall do this from now on.

<u>Definition 0.23</u>. Let V be a left vector space over a division ring D. f is called a <u>linear function</u> on V if f is a linear transformation on V into the left vector space D over D.

The set V^* of all linear functions on V is a right vector space over D relative to the laws of composition:

$$(f+g)v = fv + gv,$$

$$(fd)v = d(fv)$$

for all $v \in V$, $f,g \in V^*$, and $d \in D$. The right vector space V^* is called the <u>conjugate space</u> of V.

Let V be a left vector space over D and let V^* be the conjugate space of V. Then (V,V^*) is a pair of dual vector spaces over D with the bilinear form given by $(v,f) = fv$, for $f \in V^*$, $v \in V$.

Conversely, if (V,W) is a pair of dual vector spaces, then there is a natural isomorphism Θ of W into V^* given by

$$\Theta(w)v = (v,w),$$

for all $v \in V$. Therefore, we can consider W as a vector subspace of V^*.

Definition 0.24. Given a pair of dual vector spaces V and W, there is an associated topology on V, a subbase at zero consisting of the kernels $f^{\perp}$ of the linear functions f in W. The resulting topological vector space V after Dieudonné [4] is said to be weakly topologized by W.

W can be then retrieved from the topology as the set of all continuous linear transformations of V into the vector space D over D with D carrying the descrete topology.

Proposition 0.10. Let V be weakly topologized by W. If U is an open subspace of V, then U is also closed and has finite codimension. Conversely, if a closed subspace U of V has a finite codimension, then U is also open (see [15]).

Let V and W be a pair of dual vector spaces. For any subset S of V and T of W, we use the notation $S^{\perp}$ for $\{f \in W \mid (v,f) = 0 \text{ for all } v \in S\}$ and $T^{\perp}$ for $\{v \in V \mid (v,f) = 0 \text{ for all } f \in T\}$.

Proposition 0.11. Let (V,W) be a pair of dual vector spaces over a division ring D. If U is a vector subspace of V, then the closure of U in the weak topology of V is $U^{\perp\perp}$. Hence, a vector subspace U of V is closed in the weak topology if and only if $U^{\perp\perp} = U$ (see [12]).

If (V,W) is a pair of dual vector spaces over a division ring, we will denote by $\mathcal{L}(V,W)$ the ring of all continuous linear transformations on the weakly topologized vector space V.

Proposition 0.12. A linear transformation a on V is continuous if and only if for every f in W the linear function $v \to (av,f)$ is again an element of W (see [9]). If this is the case, denote this function by fa^* so that $(av,f) = (v,fa^*)$ and a^* becomes a linear transformation on W.

Definition 0.25. Let V be a left vector space over a division ring D. A ring R of linear transformations on V is said to be a dense subring of the ring of all linear transformations on V if for any natural number k, and any linearly independent vectors $v_1,\ldots,v_k$, and any k vectors $u_1, u_2,\ldots,u_k$, there exists an $a \in R$ such that

$$av_i = u_i \qquad i = 1,2,\ldots,k.$$

Definition 0.26. Let V be a left vector space over division ring D. A linear transformation a on V is said to be of finite rank if the dimension of the image, aV, is finite.

Suppose (V,W) is a pair of dual vector spaces. Then we denote by $\mathcal{F}(V,W)$ the set of all continuous linear transformations of finite rank on the weakly topologized space V.

Proposition 0.13. The following three conditions on a ring R are equivalent:

(1) R is a primitive ring with non-zero socle.

(2) R is isomorphic to a dense subring of the ring of linear transformations of a left vector space V over a division ring D con-

taining non-zero linear transformations of finite rank.

(3) There exists a pair of dual vector spaces (V,W) over a division ring D such that R is isomorphic to a subring of $\mathcal{L}$(V,W) containing $\mathcal{F}$(V,W).

If R is represented as in (2), its socle is the set of linear transformations of finite rank contained in this ring. If R is represented as in (3), then its socle is $\mathcal{F}$(V,W). Moreover, the socle of R is a simple ring which is contained in every non-zero ideal of R (see [12]).

Proposition 0.14. Let V and W be a pair of dual vector spaces over a division ring D. Then any finite subset of $\mathcal{F}$(V,W) can be embedded in a subring of $\mathcal{F}$(V,W) which is isomorphic to a matrix ring D_n (see [13]).

Proposition 0.15. Suppose R is a ring of linear transformations on a left vector space V over D and R is a primitive ring with minimal left ideals (abbreviated to P.M.I. ring). Then V is a homogeneous completely reducible R-module if and only if V = $\mathcal{Y}$(R)V (see [16]).

Throughout this thesis if B and C are subrings of a ring A, we use $\mathcal{L}_B$(C) denote the centralizer of C in B: the set of all elements in B which commute with every element of C.

Proposition 0.16. Let $\mathcal{C}$ be the class of all subrings E of A = $\mathcal{L}$(V,W) satisfying the conditions

(1) E is a continuous transformation ring with socle $\mathcal{Y}$(E),

(2) $\mathcal{Y}$(E)V = V,

(3) W$\mathcal{Y}(E)^*$ = W,

(4) $\mathcal{Y}(E) \subseteq \mathcal{Y}(A)$.

Let $\mathcal{B}$ be the class of all subrings B of $\mathcal{E}$ = End(V,+) containing D and satisfying

(1*) B is completely primitive with socle $\mathcal{Y}(B)$

(2*) $\mathcal{Y}(B)V = V$

(3*) The left D-dimension of a minimal left ideal of B is finite.

Then the correspondence $E \to \mathcal{L}_{\mathcal{E}}(E)$, $B \to \mathcal{L}_{\mathcal{E}}(B)$ is a one to one correspondence between $\mathcal{C}$ and $\mathcal{B}$. If E in $\mathcal{C}$ and B in $\mathcal{B}$ correspond, then

(I) Every endomorphism of V commuting with E is continuous.

(II) If B consists of all linear transformations on a vector space of dimension $\aleph$ then $\dim_E V = \aleph$.

(III) V is an irreducible BE-module

(IV) If $v \in V$, there is an idempotent e in $\mathcal{Y}(E)$ with $ev = v$

(V) Every E-submodule of V is closed (see [16]).

<u>Proposition</u> <u>0.17</u>. Under the hypothesis of Proposition 0.16, if B is a simple ring with minimum condition, then $[B : D]_\ell < \infty$ (see [16]).

<u>Definition</u> <u>0.27</u>. A mapping λ of a left vector space V_1 over D_1 into a left vector space V_2 over D_2 is called a <u>semi-linear</u> <u>transformation</u> if

(1) λ is a homomorphism of $(V_1,+)$ into $(V_2,+)$,

(2) there exists an isomorphism σ of D_1 onto D_2 such that for all $v \in V_1$ and $d \in D_1$, we have

$$\lambda(dv) = \sigma(d)\lambda v.$$

We call σ the isomorphism associated with λ.

<u>Proposition</u> <u>0.18</u>. Let (V,W) be a pair of dual vector spaces over a division ring D, and let $A = \mathcal{L}(V,W)$. Then every automorphism g of A is of the form $a \to \lambda^{-1}a\lambda$ with λ and λ^{-1} continuous semi-linear transformations on V; and conversely (see [11]). In this case, we say λ is a semi-linear transformation belonging to g.

Clearly, if λ belongs to g, $D\lambda = \lambda D$ is exactly the set of all semi-linear transformations which belong to g, and, moreover, if G is a group of automorphisms of A, then the set Λ of all semi-linear transformations on V belonging to some g in G form a multiplicative group. We will call Λ the group of semi-linear transformations on V belonging to G.

Now we define the tensor product of two modules.

<u>Definition</u> <u>0.28</u>. Let V be a right module and W a left module over a ring R. Let F be the free abelian group generated by the pairs (v,w) with $v\in V$, $w\in W$, and let K be the subgroup of F generated by elements of the form

$$(v,w+w') - (v,w) - (v,w'),$$

$$(v+v',w) - (v,w) - (v',w),$$

$$(vr,w) - (v,rw), \qquad (r\in R).$$

Then the tensor product $V\otimes_R W$ of V and W is defined as the quotient group F/K, regarded as an abelian group.

<u>Definition</u> <u>0.29</u>. Let R be an arbitrary ring and let D be a <u>commutative</u> ring with identity. Then we shall say that R is an algebra over D if a composition $(\alpha,x)\to\alpha x$ of the product set $D\times R$ into R is de-

fined such that

(i) $(R,+)$ is a unitary left D-module relative to the composition $(\alpha,x)\to\alpha x$,

(ii) for all $\alpha\in D$ and $x,y\in R$

$$\alpha(xy) = (\alpha x)y = x(\alpha y).$$

Definition 0.30. If V and W are algebras over a communitative ring D with unit element, then the tensor product module $V\otimes_D W$ is an algebra relative to the multiplication composition

$$\sum_i v_i w_i \sum_j v_j' w_j' = \sum_{i,j} v_i v_j' w_i w_j'$$

with $v_i, v_j' \in V$, $w_i', w_j \in W$.

We call this algebra the tensor product of the algebras V and W.

Proposition 0.19. Let X be a primitive algebra over a field K having a non-zero socle $\mathcal{Y}(X)$. Assume that $\{x_1,\ldots,x_n\}$ is a finite linearly independent subset of X. Then there exists an element $s\in\mathcal{Y}(X)$ such that $\{sx_1,\ldots,sx_n\}$ is linearly independent (see [12]).

Proposition 0.20. Let B be an algebra over a field K which contains a central simple ideal S and C an algebra over K with a unit element. Assume that, if $\{b_1,\ldots,b_m\}$ is a linearly independent subset of B then there exists an element $x\in B_r B_\ell$ such that $\{xb_1,\ldots,xb_m\}$ is a linearly independent subset of S. Assume, moreover, that (1) $bc = cb$ for all $b\in B$, $c\in C$, (2) $Sc = 0$ for $c\in C$ implies $c = 0$. Then BC

$= B \otimes_K C$ (see [12]).

Finally, we will define quasi-Frobenius rings and list some of their properties.

Let A be a ring with identity which satisfies the minimum condition on left and right ideals. If S is a subset of A, we denote, respectively, by $r(S)$ and $\ell(S)$ the right and left annihilators of S in A.

Definition 0.31. If for each left ideal L and each right ideal R in A

$$\ell(r(L)) = L \quad , \quad r(\ell(R)) = R,$$

then A is called a quasi-Frobenius ring.

Definition 0.32. If for each proper left ideal L and proper right ideal R in A,

$$r(L) \neq 0 \quad , \quad \ell(R) \neq 0,$$

then A is called Kasch ring.

Clearly, every quasi-Frobenius ring is a Kasch ring.

Definition 0.33. Let S be a ring with ring A as two-sided operator domain. A mapping Θ of S into A is called an operator-homomorphism of S into A if

$$\Theta(as) = a\Theta(s),$$
$$\Theta(sa) = \Theta(s)a$$

hold for all $a \in A$, $s \in S$.

Definition 0.34. An operator-homomorphism Θ of S into A is called a Frobenius homomorphism if there are no non-zero left ideals and non-zero right ideals of S contained in the kernel of Θ.

Definition 0.35. If there is a Frobenius homomorphism which maps S onto A with A as two-sided operator ring of S, and, moreover, if A is a Kasch ring, then S is called a Frobenius extension of A.

Proposition 0.21. A Frobenius extension of a quasi-Frobenius ring is a quasi-Frobenius ring (see [14]).

Proposition 0.22. If $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional algebras over a field Z and if both $\mathcal{X}$ and $\mathcal{Y}$ are semi-simple, then $\mathcal{X} \otimes_Z \mathcal{Y}$ is a quasi-Frobenius ring (see [6]).

CHAPTER 1

SOME LEMMAS

In this section, we prove some lemmas that will be used in the proof of our main theorem.

Lemma 1.1. Let R be a ring and let M be a faithful completely reducible R-module. Then R is semi-simple.

Proof: If R is not semi-simple, then by Definition 0.20 the radical $J = J(R)$ of R is not $\{0\}$. Since M is faithful, JM will be a non-zero R-submodule of M. Write

$$M = \sum_{\alpha \in \mathfrak{A}} S_\alpha,$$

where S_α are irreducible R-submodules of M. Hence, there exists an irreducible R-submodule, say S, of M so that $JS \neq 0$, and so there exists an $s \in S$ such that $Js \neq 0$. Since Js is an R-submodule of M and is contained in the irreducible R-submodule S of M, we have $Js = S$. Thus, $s = js$ for some j in J. Now be Proposition 0.8, a $k \in J$ exists such that

$$j + k - kj = 0.$$

It would follow that

$$s = s - (j+k-kj)s = (s-js) - k(s-js) = 0,$$

a contradiction. Therefore, R is semi-simple.

Lemma 1.2. Let (V,W) be a pair of dual vector spaces over a division ring D, let $A = \mathcal{L}(V,W)$, and let E be a subring of A which is also a continuous transformation ring with

$$\mathcal{Y}(E)V = V \ , \quad W\mathcal{Y}(E)^* = W \ , \quad \text{and } \mathcal{Y}(E) \subseteq \mathcal{Y}(A).$$

Then if V is completely reducible as a DE-module, $\tilde{\mathcal{L}}_A(E)$ is semi-simple.

Proof: By Proposition 0.16, $\mathrm{End}_{DE}\, V = \tilde{\mathcal{L}}_A(E)$. Then according to Proposition 0.6, V is completely reducible as an $\tilde{\mathcal{L}}_A(E)$-module. Hence from Lemma 1.1, $\tilde{\mathcal{L}}_A(E)$ is semi-simple.

Lemma 1.3. Let V be a left vector space over a division ring D and let E be a continuous transformation ring which is a subring of $A = \mathrm{End}_D V$. If $\mathcal{Y}(E)V = V$, $\mathcal{Y}(E) \subseteq \mathcal{Y}(A)$, and, furthermore, V is a finitely generated $\mathcal{Y}(E)$-module, then $\mathcal{Y}(E)$ is a finite dimensional algebra over the center Z of A.

Proof: Let

$$W = V^*\mathcal{Y}(E)^*.$$

(1) V and W are a pair of dual vector spaces. It is trivial that $(v,f) = 0$ for all $v \in V$ implies $f = 0$. On the other hand, if $v \in V$ and $(v,f) = 0$ for all $f \in W$, then $(v,fs^*) = 0$ for all $f \in V^*$, $s \in \mathcal{Y}(E)$ and so $(sv,f) = 0$ for all $f \in V^*$, $s \in \mathcal{Y}(E)$ which implies that $sv = 0$ for all $s \in \mathcal{Y}(E)$ and hence $\mathcal{Y}(E)v = 0$. Since $\mathcal{Y}(E)V = V$, by Proposi-

tion 0.15, V is a homogeneous completely reducible E-module. Write

$$V = \sum_i \oplus M_i$$

where the M are irreducible E-modules. Thus,

$$v = \sum_i m_i \qquad \text{with } m_i \in M_i.$$

and hence

$$0 = \gamma(E)v = \sum_i \gamma(E)m_i,$$

so

$$\gamma(E)m_i = 0 \qquad \text{for all } i.$$

Let

$$N_i = \{n_i \in M_i \mid \gamma(E)n_i = 0\}.$$

Then N_i being an E-submodule of M_i either equals $\{0\}$ or M_i. If $N_i = M_i$ for some i, it would imply that $\gamma(E)M_i = 0$ for all i since the M_i are E-isomorphic, and then $\gamma(E)V = 0$, a contradiction. Therefore, we have $N_i = 0$ for all i. So $m_i = 0$ for all i. Hence, $v = \sum_i m_i = 0$. This proves that V and W are a pair of dual vector spaces.

(2) Let $A_o = \mathcal{L}(V,W)$. Then $E \subseteq A_o$. If $a \in E$, then there exists $a^* \in E^*$ so that $(av,f) = (v,fa^*)$ for all $v \in V$, $f \in V^*$. We will show that $fa^* \in W$ for any $f \in W$.

Write

$$f = \sum_i f_i s_i^*, \quad f_i \in V^*, \quad s_i \in \gamma(E).$$

Then we have

$$fa^* = (\sum_i f_i s_i^*)a^* = \sum_i f_i(s_i^* a^*).$$

Since, $s_i^* a^* \in \mathcal{Y}(E)^*$, $fa^* \in V^* \mathcal{Y}(E)^* = W$. Hence, by Proposition 0.12, $a \in A_0$.

(3) $W\mathcal{Y}(E)^* = W$. Since E is a primitive ring, $\mathcal{Y}(E)^2$ is a non-zero ideal of E by Proposition 0.7. By Proposition 0.13, $[\mathcal{Y}(E)]^2 \supseteq \mathcal{Y}(E)$ and hence $[\mathcal{Y}(E)]^2 = \mathcal{Y}(E)$. Thus, $[\mathcal{Y}(E)^*]^2 = [\mathcal{Y}(E)^2]^* = \mathcal{Y}(E)^*$. Therefore,

$$W\mathcal{Y}(E)^* = V^*\mathcal{Y}(E)^*\mathcal{Y}(E)^* = V^*\mathcal{Y}(E)^* = W.$$

(4) $\mathcal{Y}(E) \subseteq \mathcal{Y}(A_0)$. If $s \in \mathcal{Y}(E)$, then $s \in \mathcal{Y}(A)$ and hence by Proposition 0.13, $\dim_D sV < \infty$. Therefore, $s \in \mathcal{Y}(A_0)$.

Now from (1)—(4), we see that V, W, A_0 and E satisfy all assumptions of Proposition 0.16. Hence $B = \mathcal{L}'_{\mathcal{E}}(E)$ is completely primitive, where $\mathcal{E} = \mathrm{End}\ (V,+)$, so B is the complete ring of linear transformation on a vector space V' over a division ring D'. But by Proposition 0.16,

$$\dim_{D'} V' = \dim_E V.$$

Since, $\dim_E V < \infty$, $\dim_{D'} V' < \infty$, and hence by Proposition 0.9, B is a simple ring with minimum condition. By Proposition 0.17, $[B : D]_\ell < \infty$. It is clear that $\mathcal{L}_A(E) \subseteq B$. Let U be D-subspace of B generated by $\mathcal{L}_A(E)$. Then $[U : D]_\ell < \infty$, say $t_1, \ldots, t_n$ is a basis for U over D. Thus for any $t \in \mathcal{L}_A(E)$, $t = \sum d_i t_i$ with $d_i \in D$ and $dt = td$ for all $d \in D$.

This implies that $dd_i = d_i d$ for all $d \in D$ and all i. Hence each $d_i \in Z$ and $\{t_i\}$ spans $\tilde{\mathcal{L}}_A(E)$ over Z, thus $[\tilde{\mathcal{L}}_A(E) : Z] < \infty$. This completes the proof.

CHAPTER 2

PROOF OF THE MAIN THEOREM

The main theorem has been stated in the introduction. Now to prove this, we suppose contrarily that $\mathcal{L}_A(E)$ is not semi-simple. Then according to Lemma 1.2,

(vii) V is not a completely reducible DE-module.

Therefore, there is a sequence $\{V_i\}$ of DE-submodules so that

$$0 = V_0 \subset V_1 \subset V_2 \subset \ldots .$$

where the V_i/V_{i-1} is the DE-socle of V/V_{i-1}

We shall show first that this sequence of DE-submodules terminates. By Proposition 0.1, it will be sufficient to show that every E- submodule of V is finitely generated.

Since V is a completely reducible E-module, V can be expressed as $\sum\limits_{\gamma\in\Gamma} \oplus M_\gamma$ with each M_γ an irreducible E-module. By (iv)

$$V = \mathcal{Y}(E)v_1 + \mathcal{Y}(E)v_2 + \ldots + \mathcal{Y}(E)v_m$$

and since for each i

$$v_i \in \sum_{\gamma\in\Gamma_i \oplus} M_\gamma$$

where Γ_i is a finite subset of Γ. Thus

$$V = \sum_{i=1}^{m} \sum_{\gamma \in \Gamma_i} M_\gamma = \sum_{\gamma \in \bigcup_{i=1}^{m} \Gamma_i} M_\gamma$$

and hence $\dim_E V < \infty$. Therefore, $\dim_E U < \infty$ for any E-submodule U of V; that is, any E-submodule is finitely generated. Hence, we have

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subsetneq V_n = V$$

for some integer $n \geqq 2$ where V_i/V_{i-1} is the DE-socle of V/V_{i-1}.

Our next step is to establish an $\bar{h} \neq 0$ in Hom $(V/V_{n-1}, V_1)$ so that

$$\bar{h}\lambda\bar{v} = \lambda\overline{\bar{h}\bar{v}} \quad \text{for all } \lambda \in \Lambda \quad , \quad \bar{v} \in V/V_{n-1}.$$

We remark that since the socle of a module is fully invariant, each V_i is an $\mathcal{L}_A(\Lambda)$-submodule of V. Now suppose we have found such an $\bar{h}$. Then define $h \in \mathrm{End}_D V$ as follows:

Write

$$V = V_{n-1} \oplus U \qquad \text{(D-direct)}$$

and set

$$h(u) = \bar{h}(\bar{u}) \qquad \text{for } u \in U$$

and

$$hV_{n-1} = 0.$$

For $\lambda \in \Lambda$, $v_{n-1} \in V_{n-1}$, $u \in U$, we have then

$$\lambda h(v_{n-1}+u) = \lambda\overline{\bar{h}\bar{u}} = \bar{h}\lambda\bar{u} = \bar{h}(\overline{\lambda u}) = h(\lambda u) = h\lambda(v_{n-1}+u),$$

so $\lambda h = h\lambda$ for all $\lambda\epsilon\Lambda$ and $h \neq 0$.

Consider the set

$$X = \{a\epsilon E \mid aU \subseteq V_{n-1} \text{ and } aV_{n-1} = 0\}.$$

It is easy to see that X is a left ideal of E and $X^2 = 0$. Since E is primitive, by Proposition 0.7, X = 0, so $h \notin E$. Hence there exists $u\epsilon U$ with $hu \neq 0$. Also by Proposition 0.16, there exists an idempotent e in $\gamma(E)$ such that $eu = u$. Thus $he \neq 0$. But

$$heV_{n-1} \subseteq hV_{n-1} = 0.$$

By the same argument as above, we have $he \notin E$.

Since $e\epsilon\gamma(E)$, $\dim_D eV < \infty$. Let $\{v_1, v_2, \ldots, v_t\}$ be a D-basis for eV. By the density of $\gamma(A)$ in $\mathrm{End}_D V$, there is an s in $\gamma(A)$ so that

$$sv_i = hv_i \quad i = 1,2,\ldots,t.$$

Hence, for any $v\epsilon V$, $ev = \sum_{i=1}^{t} d_i v_i$

with $d_i \epsilon D$, and then

$$hev = h\sum_{i=1}^{t} d_i v_i = \sum_{i=1}^{t} d_i hv_i$$

$$= \sum_{i=1}^{t} d_i sv_i = s\sum_{i=1}^{t} d_i v_i = sev.$$

Thus, $he = se\epsilon\gamma(A)$ and hence $he\epsilon A$. Furthermore, for any $\lambda\epsilon\Lambda$,

$$(he)\lambda = h(e\lambda) = h(\lambda e) = (h\lambda)e = (\lambda h)e = \lambda(he),$$

so $he\in\mathcal{L}_A(\Lambda)$. But $he\notin E$, whence $\mathcal{L}_A(\Lambda)\supset E$ properly, or E is not the fixed ring under G which contradicts our assumption (vi). This proves our main theorem except that we have yet to show the existence of such an $\bar{h} \neq 0$ in $\mathrm{Hom}_D(V/V_{n-1},V_1)$ with $\bar{h}\lambda\bar{v} = \lambda\bar{h}\bar{v}$ for all $\lambda\in\Lambda$, $\bar{v}\in V/V_{n-1}$.

We will proceed to do this by mathematical induction on n under the assumptions (i)—(v) and (vii).

The proof of the case $n = 2$ will be postponed to Section 3. We suppose now that $n > 2$. Consider the vector space V/V_1 and $V_1^{\perp}$. The bilinear form on V/V_1 and $V_1^{\perp}$ given by $(\bar{v},f) = (v,f)$ for $\bar{v}\in V/V_1$, $f\in V_1^{\perp}$ is non-degenerate since $(\bar{v},f) = 0$ for all $f\in V_1^{\perp}$ implies $v\in V_1^{\perp\perp} = V_1$ or $\bar{v} = \bar{0}$ in V/V_1 and $(\bar{v},f) = 0$ for all $\bar{v}\in V/V_1$ implies $(v,f) = 0$ for all $v\in V$ so $f = 0$. Hence, $(V/V_1,V_1^{\perp})$ is a pair of dual vector spaces over D.

Let $A_1 = \mathcal{L}(V/V_1,V_1^{\perp})$. Then for any $a^*\in E^*$, $(V_1,V_1^{\perp}a^*)$ $= (aV_1,V_1^{\perp})\subseteq(V_1,V_1^{\perp}) = 0$. This means that $V_1^{\perp}a^*\subseteq V_1^{\perp}$, so $E\subseteq A_1$.

Now we are ready to verify the conditions (i)—(v) and (vii) for V/V_1 and $V_1^{\perp}$.

(i) $\mathcal{J}(E)V/V_1 = V/V_1$. Clearly, $\mathcal{J}(E)V/V_1\subseteq V/V_1$. On the other hand, since $\mathcal{J}(E)V = V$, for any $v\in V$, there exists $v_i\in V$, $s_i\in\mathcal{J}(E)$ $(i = 1,\ldots,k)$, so that

$$v = \sum_{i=1}^{k} s_i v_i .$$

Thus, $\bar{v} = \sum_{i=1}^{k} s_i v_i$ in V/V_1, and hence $V/V_1\subseteq\mathcal{J}(E)V/V_1$.

(ii) $V_1^{\perp}(E)^* = V_1^{\perp}$. Since $W = W\mathcal{Y}(E)^*$, for any $f\epsilon W$,

$f = \sum_{i=1}^{r} f_i s_i^*$ with $f_i \epsilon W$, $s_i \epsilon \mathcal{Y}(E)$. By proposition 0.14, there exists an idempotent element e in $\mathcal{Y}(E)$ such that $s_i e = s_i$ and $s_i^* = s_i^* e^*$ for all i. Therefore,

$$fe^* = \sum_{i=1}^{r} f_i s_i^* e^* = \sum_{i=1}^{r} f_i s_i^* = f,$$

so

$$V_1^{\perp} = V_1^{\perp}\mathcal{Y}(E)^*.$$

(iii) $\mathcal{Y}(E) \subseteq \mathcal{Y}(A_1)$. If $s\epsilon\mathcal{Y}(E)$, then $s\epsilon\mathcal{Y}(A)$, so, by Proposition 0.16, $\dim_D sV < \infty$. So $\dim_D s(V/V_1) < \infty$ and $s\epsilon A_1$. This means that $s\epsilon\mathcal{Y}(A_1)$.

(iv) V/V_1 is finitely generated $\mathcal{Y}(E)$-module. It is an immediate consequence from the assumption that V is a finitely generated $\mathcal{Y}(E)$-module.

(v) Let Λ_1 be the set of all semi-linear transformations on V/V_1 induced from Λ; i.e., $\lambda_1 \epsilon \Lambda_1$ is given by

$$\lambda_1(\bar{v}) = \lambda(v+V_1) = \lambda v + V_1 = \overline{\lambda v}$$

for all $v\epsilon V$ and some $\lambda\epsilon\Lambda$.

Let $G_1 = \{g_1 | g_1(a_1) = \lambda_1^{-1} a_1 \lambda_1,\ \forall a_1 \epsilon A_1$ and some $\lambda_1 \epsilon \Lambda_1\}$. Evidently, G_1 is a group of automorphisms of A_1 and $[\Lambda_1 : DT_1] < \infty$, where the T_1 is the group of all linear transformations on V/V_1 contained in Λ_1.

(vii) V/V_1 is not a completely reducible DE-module since $n > 2$.

Hence, by the induction hypothesis, there exists $p \neq 0$ in $\mathrm{Hom}_D(V/V_{n-1}, V_2/V_1)$ so that $p\lambda_1\overline{v} = \lambda_1 p\overline{v}$ for all $\lambda_1 \in \Lambda_1$ and $\overline{v} \in V/V_{n-1}$. Now choose Q with $V_2 \supseteq Q \supset V_1$ so that the image of V/V_{n-1} under p is Q/V_1. Certainly, Q admits Λ, it might not admit E. Let

$$P = \sum_{a \in E} aQ.$$

Then P is a DE-submodule of V_2 and P/V_1 is a completely reducible DE-submodule of V_2/V_1 by Proposition 0.5. It is easy to see that P and $W/P^{\perp}$ are a pair of dual vector spaces with bilinear form given by

$$(v,\overline{f}) = (v,f)$$

for $f \in W/P^{\perp}$, $v \in P$.

Now let $A_2 = \mathcal{L}(P, W/P^{\perp})$. Since $(W/P^{\perp})E^{*} \subseteq W/P^{\perp}$, E can be considered as a subring of A_2. Next, we shall again verify the conditions (i)—(v) and (vii) for P and $W/P^{\perp}$.

(i) $\gamma(E)P = P$. Since $\gamma(E)V = V$, V is a homogeneous completely reducible E-module (by Proposition 0.15), hence, by Proposition 0.5, P is. Applying Proposition 0.15 again, we have $\gamma(E)P = P$.

(ii) $(W/P^{\perp})\gamma(E)^{*} = W/P^{\perp}$. We see that for any $f \in W$, there is $e = e^2$ in $\gamma(E)$ so that $f = fe^{*}$. Hence

$$\overline{f}e^{*} = \overline{fe^{*}} = \overline{f} \quad \text{for all } \overline{f} \in W/P^{\perp}$$

and so

$$(W/P^{\perp})\gamma(E)^{*} = W/P^{\perp}.$$

(iii) $\gamma(E) \subseteq \gamma(A_2)$. If $s \in \gamma(E)$, then $s \in \gamma(A)$, so $\dim_D sV < \infty$, so $\dim_D sP < \infty$. But $s \in A_2$, so $s \in \gamma(A_2)$.

(iv) P is finitely generated $\gamma(E)$-module. Since V is completely reducible E-module, V is completely reducible $\gamma(E)$-module. By Proposition 0.4, $V = P \oplus P'$ ($\gamma(E)$-direct). P is then a finitely generated $\gamma(E)$-module since V is.

(v) Let

$$\Lambda_2 = \{\lambda_2 \mid \lambda_2 v = \lambda v \quad \forall v \in P \text{ and some } \lambda \in \Lambda\}.$$

Then Λ_2 is a group of semi-linear transformations on P since Λ admits Q and so admits P.

Let

$$G_2 = \{g_2 : a_2 \to \lambda_2^{-1} a_2 \lambda_2, \ \forall a_2 \in A_2 \text{ and some } \lambda_2 \in \Lambda_2\}.$$

Evidently, G_2 is a group of automorphisms of A_2 and $[\Lambda_2 : DT_2] < \infty$ where T_2 is the group of all linear transformations on P contained in Λ_2.

(vii) P is not completely reducible DE-module since $P \neq V_1$. Therefore, we can use the induction hypothesis again to obtain a $q \neq 0$ in $\mathrm{Hom}_D(P/V_1, V_1)$ so that

$$q\lambda_2\bar{v} = \lambda_2 q\bar{v} \quad \forall \lambda_2 \in \Lambda_2, \ \bar{v} \in P/V_1.$$

Now since $q \neq 0$ and $P/V_1 = \sum_{a \in E} a(Q/V_1)$, there exists an $a \in E$ such that $qa(Q/V_1) \neq 0$. Consider the mapping qap in $\mathrm{Hom}_D(V/V_{n-1}, V_1)$,

$$qap(V/V_{n-1}) = qa(Q/V_1) \neq 0$$

Also, for any

$$\lambda \epsilon \Lambda, \ \overline{v} \epsilon V/V_{n-1},$$

$$\lambda qap\overline{v} = \lambda_2 qap\overline{v} = q\lambda_2 ap\overline{v} = qa\lambda_2 p\overline{v}$$

$$= qa\lambda p\overline{v} = qa\lambda_1 p\overline{v} = qap\lambda_1\overline{v} = qap\lambda\overline{v}.$$

Therefore, qap may be taken as the desired $\overline{h}$.

CHAPTER 3

PROOF OF THE MAIN THEOREM (CONTINUED)

In this chapter, we will consider the case $n = 2$. In this case, V_1 and V/V_1 are completely reducible DE-modules. Write $V = V_1 \oplus U$ (D-direct) and let $\bar{A} = \mathrm{End}_D V$ and

$$\bar{A}_{V_1} = \{a \in \bar{A} \mid aV_1 \subseteq V_1\}.$$

It is obvious that $\bar{A}_{V_1}$ is a subring of $\bar{A}$ containing E, and we have maps

$$\Theta : \bar{A}_{V_1} \to \mathrm{End}_D V_1 = B$$

$$\phi : \bar{A}_{V_1} \to \mathrm{End}_D U = C$$

$$\delta : \bar{A}_{V_1} \to \mathrm{Hom}_D(U, V_1) = H$$

given by

$$av = \Theta(a)v$$

$$au = \phi(a)u + \delta(a)u$$

for $a \in \bar{A}_{V_1}$, $u \in U$, $v \in V_1$. Then Θ and ϕ are ring epimorphisms and δ is a Θ-ϕ derivation, i.e.,

$$\delta(ab) = \Theta(a)\delta(b) + \delta(a)\phi(b) \quad , \quad a, b \in \bar{A}_{V_1}.$$

We shall show that Θ and ϕ are one to one maps on E.

Let $\mathfrak{A} = \{a \in E \mid \Theta(a) = 0\}$. If $\mathfrak{A} \neq 0$, then, as an ideal of E, $\mathfrak{A} \supseteq \gamma(E)$. It would follow that $\gamma(E)V_1 = 0$, a contradiction. Hence,

$\mathcal{O}\!\!\mathcal{L} = 0$, Θ is a one to one mapping from E onto $\Theta(E)$, and hence $\Theta(E)$ is a subring of B. Since B can be considered as a sub-algebra of $\bar{A}$, and $\gamma(E) \subseteq \gamma(\bar{A})$, by Lemma 1.3, $\mathcal{L}_{\bar{A}}(E)$ is a finite dimensional algebra over Z. Hence $\mathcal{X} = \mathcal{L}_B(\Theta(E))$ being a sub-algebra of $\mathcal{L}_{\bar{A}}(E)$, is a finite dimensional algebra over Z.

Moreover, since V_1 is a completely reducible DE-module by Proposition 0.6, V_1 is a completely reducible $\mathcal{X}$-module so $\mathcal{X}$ is semi-simple.

Now, let $\mathcal{G} = \{a \in E \mid \phi(a) = 0\}$. If $\mathcal{G} \neq 0$, then, as an ideal of E, $\mathcal{G} \supseteq \gamma(E)$ and this would imply that

$$V = \gamma(E)V \subseteq \mathcal{G}V \subseteq V_1$$

a contradiction. Hence, $\mathcal{G} = 0$, so ϕ is an isomorphism from E onto $\phi(E)$.

Let Φ be the projection on U along V_1. Then, for any $a \in E$ and $v \in V$,

$$\begin{aligned} v &= v_1 + u \text{ with } v_1 \in V_1,\ u \in U, \\ \Phi(av) &= \Phi(a(v_1+u)) = \Phi(au) = \Phi(\phi(a)u + \delta(a)u) \\ &= \phi(a)u = \phi(a)\Phi(v) \end{aligned}$$

so

$$\begin{aligned} U &= \Phi(V) = \Phi(\gamma(E)V) = \phi(\gamma(E))\Phi(V) \\ &= \gamma(\phi(E))\Phi(V) = \gamma(\phi(E))U. \end{aligned}$$

Now, we claim that U is a finitely generated $\gamma(\phi(E))$-module. To see this, let $\{v_1,\ldots,v_m\}$ generate V over $\gamma^L(E)$. Then for any $u \in U$, $u = a_1v_1+\ldots+a_mv_m$ with $a_i \in \gamma^L(E)$ and hence

$$
\begin{aligned}
u &= \Phi(u) = \Phi(a_1v_1+\ldots+a_mv_m) \\
&= \phi(a_1)\Phi(v_1) + \ldots + \phi(a_m)\Phi(v_m).
\end{aligned}
$$

This means that U is generated by $\{\Phi(u_1),\ldots,\Phi(u_m)\}$ over $\gamma^L(\phi(E))$.

We assert that $\gamma^L(\phi(E)) \subseteq \gamma^L(C)$. Indeed, for any $s \in \gamma^L(E)$, $\dim_D sV < \infty$, hence $\dim_D \Phi(sV) < \infty$. However, $\Phi(sV) = \phi(s)\Phi(V) = \phi(s)U$, so $\dim_D \phi(s)U < \infty$ and hence $\phi(s) \in \gamma^L(C)$. Thus $\gamma^L(\phi(E))$ $= \phi(\gamma^L(E)) \subseteq \gamma^L(C)$.

By Lemma 1.3, $\mathcal{U} = \mathcal{L}_C(\phi(E))$ is a finite dimensional algebra over Z.

Also, since U, D-isomorphic to V/V_1, is a completely reducible $D\phi(E)$-module, by Proposition 0.6. U is a completely reducible $\mathcal{U}$-module and hence by Lemma 1.1, $\mathcal{U}$ is semi-simple.

Now, let $\mathcal{A}$ = End H. We have the mapping $B \to \mathcal{A}$ given by

$$(bh)u = b(hu), \quad h \in H.$$

Since $HU = V_1$, the above mapping is an injection. We will think of B as a subring of $\mathcal{A}$.

Likewise, we have the mapping $C \to \mathcal{A}$ given by

$$(ch)u = h(cu), \quad h \in H,$$

and again this mapping has zero kernel but it switches the multiplication so we will think of C^o, the reciprocal ring of C, as a subring of $\mathcal{Y}$.

Now, we want to show that the subring generated by $\mathcal{X}$ and $\mathcal{Y}^o$ is isomorphic to $\mathcal{X} \otimes_Z \mathcal{Y}^o$. It will suffice to show that $B \otimes_Z \mathcal{Y}^o \cong B\mathcal{Y}^o$. To see this, we will apply Proposition 0.20. Obviously B is an algebra over Z, $\gamma(B)$ is central simple ideal of B, and $[B, \mathcal{Y}^o] = 1$. Also $\gamma(B)z^o = 0$ with z in $\mathcal{Y}$ would imply $\gamma(B)Hz = 0$ in H or $\gamma(B)V_1 = 0$ contradicting the fact that $\gamma(B)V_1 = V_1$. Hence, $\gamma(B)z^o = 0$ implies $z^o = 0$. Moreover, by Proposition 0.19, $\{b_1, \ldots, b_m\}$ being a linear independent set in B over Z assures the existence an x in $B_r B_\ell$ such $\{xb_1, \ldots, xb_m\}$ is a linear independent subset of $\gamma(B)$. Therefore, by Proposition 0.20, $B \otimes_Z \mathcal{Y}^o = B\mathcal{Y}^o$. Since $\mathcal{X}$ and $\mathcal{Y}^o$ both are semi-simple, by Proposition 0.22, $\mathcal{X}\mathcal{Y}^o \cong \mathcal{X} \otimes_Z \mathcal{Y}^o$ is a quasi-Frobenius ring.

Since $\Lambda V_1 \subseteq V_1$, we can define the mapping Θ. ϕ and δ on Λ as before:

Θ : $\Lambda \to$ the group of semi-linear transformations on V_1,

ϕ : $\Lambda \to$ the group of semi-linear transformations on U,

δ : $\Lambda \to$ the set of semi-linear transformations from U to V_1,

given by

$$\lambda v = \Theta(\lambda)v$$

$$\lambda u = \phi(\lambda)u + \delta(\lambda)u$$

for $u \in U$, $v \in V_1$ and $\lambda \in \Lambda$.

Clearly, the mapping

$$\rho(\lambda) : h \to \Theta(\lambda)h\phi(\lambda^{-1})$$

is an endomorphism on H.

First, we shall prove the following lemma:

Lemma 3.1. Under the assumptions of our main theorem with $n = 2$, suppose that $\lambda_1,\ldots,\lambda_t$ are representatives for the left cosets of DT in Λ. If

$$\alpha_1\rho(\lambda_1) + \ldots + \alpha_t\rho(\lambda_t) = 0 \qquad (1)$$

in End H with $\alpha_i \epsilon BC^{\circ}$, then $\alpha_i = 0$, $i = 1,\ldots,t$.

Proof: Let $\sigma_1,\ldots,\sigma_t$ be the automorphisms of D associated with $\lambda_1,\ldots,\lambda_t$ respectively. Certainly, we may assume that $\lambda_1 = 1$. Suppose α_i are not all zero and I is the group of inner automorphisms on D. We claim that $\sigma_i \neq \sigma_j$ mod I for $i \neq j$. Indeed, if $\sigma_i\sigma_j^{-1}(x) = dxd^{-1}$ for all $x\epsilon D$ with $d\epsilon D$ then, for all x in D, all v in V,

$$d^{-1}(\lambda_i\lambda_j^{-1})(xv) = d^{-1}(\sigma_i\sigma_j^{-1}(x))\lambda_i\lambda_j^{-1}(v)$$
$$= d^{-1}dxd^{-1}\lambda_i\lambda_j^{-1}v = xd^{-1}(\lambda_i\lambda_j^{-1})v$$

and it would imply that

$$d^{-1}\lambda_i\lambda_j^{-1}\epsilon T \qquad \text{or} \qquad \lambda_i = dt\lambda_j$$

for some $t\epsilon T$ a contradiction.

Now let $\{\xi_j\}$ be a D-basis for V_1. Set $S_i : V_1 \to V_1$ given by

$$S_i(\sum_k d_k\xi_k) = \sum_k \sigma_i(d_k)\xi_k, \; d_k\epsilon D$$

and

$$T_i \in \mathrm{End}_D V_1$$

by

$$T_i \xi_k = \sum_j s_{ikj} \xi_j$$

if

$$\Theta(\lambda_i)\xi_k = \sum_j s_{ikj} \xi_j$$

with s_{ikj} in D. Then

$$T_i S_i \left(\sum_k d_k \xi_k\right) = T_i \sum_k \sigma_i(d_k)\xi_k = \sum_{j,k} \sigma_i(d_k) s_{ikj} \xi_j,$$

and

$$\Theta(\lambda_i) \sum_k d_k \xi_k = \sum_k \sigma_i(d_k)\Theta(\lambda_i)\xi_k = \sum_{k,j} \sigma_i(d_k) s_{ikj} \xi_j,$$

so

$$\Theta(\lambda_i) = T_i S_i$$

for all i.

Let $\{C_k\}$ be a Z-basis for C and

$$\alpha_i = \sum_k (b_{ik})_\ell (C_k)_r$$

with b_{ik} in B. By the above assertion, we can form the equation (1) as

$$\sum_{i,k} b_{ik} S_i h\, \phi(\lambda_i^{-1}) C_k = 0 \tag{2}$$

in H for all $h \in H$, and also we can assume that $b_{11} \neq 0$.

Since $Bb_{11}B$ is a non-zero ideal in B, we have

$$Bb_{11}B \supseteq \gamma^{\iota}(B).$$

We then choose $b_{11}^{\prime}$ in $\gamma^{\iota}(B)$ with $b_{11}^{\prime} \neq 0$ and

$$b_{11}^{\prime} = \sum b_f b_{11} b_f^{\prime} \quad , \quad b_f, b_f^{\prime} \epsilon B.$$

Multiply the identity (2) by b_f from left. We obtain

$$\sum_i \sum_k b_f b_{ik} S_i h\phi(\lambda_i^{-1}) C_k = 0.$$

Replacing h then by $b_f^{\prime}h$, we have

$$\sum_{i,k} b_f b_{ik} S_i b_f^{\prime} h\phi(\lambda_i^{-1}) C_k = 0,$$

Sum over f. We get

$$\sum_{i,k} b_{ik}^{\prime} S_i h\phi(\lambda_i^{-1}) C_k = 0 \tag{3}$$

for all h in H, where

$$b_{ik}^{\prime} = \sum_f b_f b_{ik} S_i b_f^{\prime} S_i^{-1}$$

in B.

Let $e \epsilon B$ given by $e\xi_1 = \xi_1$ and $e\xi_j = 0$ for $j \neq 1$. Then, obviously, $e \epsilon \gamma^{\iota}(B)$.

Now we select $b_{11}^{\prime} = e$ and assume that the expression (3) has the smallest number of non-zero $b_{ik}^{\prime}$. Then by (3)

$$\sum_{i,k} (1-e) b_{ik}^{\prime} S_i h\phi(\lambda_i^{-1}) C_k = 0.$$

in which the first term is

$$(1-e)eS_1h\phi(\lambda_1^{-1})C_k = 0$$

since $e^2 = e$, so

$$(1-e)b'_{ik} = 0$$

for all i and k.

Now if

$$b'_{ik}\xi_j = \sum_{\ell} d_{ikj\ell}\xi_\ell$$

then

$$0 = (1-e)b'_{ik}\xi_j = \sum_{\ell}(1-e)d_{ikj\ell}\xi_\ell = \sum_{\ell} d_{ikj\ell}(1-e)\xi_\ell.$$

But

$$(1-e)\xi_\ell = \xi_\ell \qquad \text{for all } \ell \neq 1,$$

hence

$$d_{ikj\ell} = 0 \qquad \text{for all } \ell \neq 1,$$

and

$$b'_{ik}V_1 \subseteq \langle \xi_1 \rangle$$

the D-space generated by ξ_1, for all i and k.

We note that $S_i(1-e) = (1-e)S_i$ for all i, since

$$S_i(1-e) \sum_{k} d_k\xi_k = S_i \sum_{k\neq 1} d_k\xi_k = \sum_{k\neq 1} \sigma_i(d_k)\xi_k$$

and

$$(1-e)S_i \sum_{k} d_k\xi_k = (1-e) \sum_{k} \sigma_i(d_k)\xi_k$$

$$= \sum_{k\neq 1} \sigma_i(d_k)\xi_k.$$

Now in (3) replacing h by $(1-e)h$, we obtain

$$\sum_{i,k} b'_{ik}S_i(1-e)h\phi(\lambda_i^{-1})C_k = 0$$

for all h in H, or

$$\sum_{i,k} b'_{ik}(1-e)S_ih\phi(\lambda_i^{-1})C_k = 0 \tag{4}$$

for all h in H. From (3) and (4), we have by subtraction

$$\sum_{(i,k)\neq(1,1)} [(i-e)b'_{ik}-b'_{ik}(1-e)]S_ih\phi(\lambda_i^{-1})C_k = 0$$

for all h in H, in which the first term is equal to zero. Hence

$$(1-e)b'_{ik} - b'_{ik}(1-e) = 0$$

for all i and k.

But we have seen before that

$$(1-e)b_{ik}' = 0$$

so

$$b_{ik}'(1-e) = 0 \quad \text{or} \quad b_{ik}' = b_{ik}'e$$

for all i and k. Thus, for $j \neq 1$,

$$b_{ik}'\xi_j = b_{ik}'e\xi_j = 0$$

and

$$b_{ik}'\xi_1 = d_{ik}\xi_1$$

for some $d_{ik} \neq 0$ in D.

Now for each $d \neq 0$ in D, let T_d be the linear transformation given by

$$T_d\xi_k = d\xi_k$$

In identity (3), replace h by $T_d h$ and then multiply $T_{d^{-1}}$ from the left in both sides of the obtained identity. We obtain

$$\sum_{i,k} T_{d^{-1}}b_{ik}S_iT_dh\phi(\lambda_i^{-1})C_k = 0 \tag{5}$$

for all h in H.

Remark that since for all j

$$(S_i T_d S_i^{-1})\xi_j = S_i T_d \xi_j = S_i d\xi_j = \sigma_i(d)\xi_j$$

$$S_i T_d = T_{\sigma_i(d)} S_i.$$

From identity (5), we have

$$\sum_{i,k} T_{d^{-1}} b'_{ik} T_{\sigma_i(d)} S_i h\phi(\lambda_i^{-1}) C_k = 0 \tag{6}$$

for all $h \in H$.

From (3) and (6) by subtraction, we have

$$\sum_{i,k} (T_{d^{-1}} b'_{ik} T_{\sigma_i(d)} - b'_{ik}) S_i h\phi(\lambda_i^{-1}) C_k = 0$$

for all $h \in H$, in which the first term is

$$T_{d^{-1}} b'_{11} T_{\sigma_i(d)} - b'_{11} = 0,$$

so

$$b'_{ik} = T_{d^{-1}} b'_{ik} T_{\sigma_i(d)}$$

for all i and k. It follows that

$$\begin{aligned} d_{ik}\xi_1 &= b'_{ik}\xi_1 = (T_{d^{-1}} b'_{ik} T_{\sigma_i(d)})\xi_1 \\ &= T_{d^{-1}} b'_{ik}\sigma_i(d)\xi_1 = T_{d^{-1}}\sigma_i(d) b'_{ik}\xi_1 \\ &= T_{d^{-1}}\sigma_i(d) d_{ik}\xi_1 = \sigma_i(d) d_{ik} d^{-1}\xi_1, \end{aligned}$$

so

$$d_{ik} = \sigma_i(d)d_{ik}d^{-1}$$

or

$$\sigma_i(d) = d_{ik}dd_{ik}^{-1}$$

for all i and k. This means $\sigma_i \equiv \sigma_1$ mod I and hence $t = 1$, so $b_{11} = 0$ a contradiction. This completes the proof of the lemma. Notice that the lemma generalizes the classical lemma of Dedekind.

Now we go back to our proof of the main theorem. Let

$$F = \sum_{i=1}^{t} \mathcal{X}\mathcal{Y}^{o}\rho(\lambda_i)$$

where $\{\lambda_1,\ldots,\lambda_t\}$ is again a set of representatives for the cosets of DT in Λ and $\lambda_1 = 1$. Then for any i and j, there are $d\epsilon D$, $t\epsilon T$ and a λ_k so that

$$\lambda_i\lambda_j = dt\lambda_k.$$

Also, for $h\epsilon H$

$$\begin{aligned}\rho(\lambda_i)\rho(\lambda_j)h &= \Theta(dt\lambda_k)h\phi(\lambda_k^{-1}t^{-1}d^{-1}) \\ &= \Theta(t)\Theta(\lambda_k)h\phi(\lambda_k^{-1}t^{-1}) \\ &= \Theta(t)\rho(\lambda_k)h\phi(t^{-1})\end{aligned}$$

where t commutes with elements of E so $\Theta(t)\epsilon\mathcal{X}$ and $\phi(t^{-1})\epsilon\mathcal{Y}$. Hence

$$\rho(\lambda_i)\rho(\lambda_j)\in\mathcal{X}\mathcal{Y}^o\rho(\lambda_k)\subseteq F.$$

Since

$$\Theta(\lambda_i)x\Theta(\lambda_i^{-1})\in\mathcal{X}$$

and

$$\phi(\lambda_i^{-1})y\phi(\lambda_i)\in\mathcal{Y},$$

we have

$$\Theta(\lambda_i)\mathcal{X}=\mathcal{X}\,\Theta(\lambda_i)$$

and

$$\phi(\lambda_i^{-1})\mathcal{Y}=\mathcal{Y}\,\phi(\lambda_i^{-1})$$

so F is a subring of End H containing $\mathcal{X}\mathcal{Y}^o$ and F is a finitely generated $\mathcal{X}\mathcal{Y}^o$-module.

Now consider the mapping μ from F onto $\mathcal{X}\mathcal{Y}^o$ given by

$$\mu\left(\sum_{i=1}^{t}\sum_{j=1}^{i_m}x_{ij}y_{ij}^o\rho(\lambda_i)\right)=\sum_{j=1}^{i_m}x_{1j}y_{1j}^o$$

This mapping is uniquely defined since by Lemma 1.4 each element in F has a unique expression as a linear combination on $\rho(\lambda_i)$'s over $\mathcal{X}\mathcal{Y}^o$.

Moreover, for any x in $\mathcal{X}$, y in $\mathcal{Y}$

$$\mu\left(xy^{o}\sum_{i=1}^{t}\sum_{j=1}^{i_m}x_{ij}y^{o}_{ij}\rho(\lambda_i)\right)$$

$$=\sum_{j=1}^{i_1}xy^{o}x_{ij}y^{o}_{ij}$$

$$=xy^{o}\mu\sum_{i=1}^{t}\sum_{j=1}^{i_m}x_{ij}y^{o}_{ij}\rho(\lambda_i)$$

and there are no one-sided ideals of F other than zero contained in the kernel of μ. Therefore, μ is a Frobenius homomorphism from F onto $\mathfrak{X}\mathfrak{Y}^{o}$. Thus, F is a Frobenius extension of $\mathfrak{X}\mathfrak{Y}^{o}$, and by Proposition 0.21, F is quasi-Frobenius ring.

Now, we know that $[\Lambda,E] = 1$ and $\Lambda V_1 \subseteq V_1$, so $[\Theta(\Lambda),\Theta(E)] = 1$ and $[\phi(\Lambda),\phi(E)] = 1$. Also, for $\lambda \epsilon \Lambda$, $a \epsilon E$,

$$\delta(a\lambda) = \delta(\lambda a),$$

or

$$\delta(a)\phi(\lambda) + \Theta(a)\delta(\lambda) = \delta(\lambda)\phi(a) + \Theta(\lambda)\delta(a),$$

or

$$\Theta(\lambda)\delta(a) - \delta(a)\phi(\lambda) = \Theta(a)\delta(\lambda) - \delta(\lambda)\phi(a),$$

where

$$\delta(a)\epsilon \mathrm{Hom}_D(U,V_1)$$

Let

$$L = \{\eta \in F \,|\, \exists\, h(\eta) \in H \quad \text{with} \quad \eta\delta(a) = \Theta(a)h - h\phi(a), \forall a \in E\}.$$

If $\xi \in \mathcal{X}$, $\zeta \in \mathcal{Y}$, $\eta \in L$, then

$$\xi\zeta^{\circ}\rho(\lambda_i)\eta\ (a) = \xi\rho(\lambda_i)\eta\ (a)\zeta$$

$$= \xi\Theta(\lambda_i)[\Theta(a)h - h\phi(a)]\phi(\lambda_i^{-1})\zeta$$

$$= \xi\Theta(a)\Theta(\lambda_i)h\phi(\lambda_i^{-1})\zeta - \xi\Theta(\lambda_i)h\phi(\lambda^{-1})\phi(a)\zeta$$

$$= \Theta(a)\xi\Theta(\lambda_i)h\phi(\lambda_i^{-1})\zeta - \xi\Theta(\lambda_i)h\phi(\lambda_i^{-1})\zeta\phi(a)$$

so L is a left ideal in F.

Furthermore, $L \neq F$ since $1 \in L$ would imply the existence of $h \in H$ with

$$\delta(a) = \Theta(a)h - h\phi(a)$$

for all a in E; i.e., $\delta|E$ is inner $\Theta\phi$-derivation. Consider

$$X = \{u - h(u) \,|\, u \in U\} \subseteq V.$$

X is a D-module and

$$X \cap V_1 = \{0\}$$

since

$$w = u - h(u) \in V_1$$

implies

$$u = h(u) + w \in U \cap V_1 = \{0\}$$

so $h(u) = 0$ and hence $w = 0$. Moreover, for any $v \in V$, $v = u+w$ with u in U, w in V_1 and then

$$v = u - h(u) + h(u) + w \in X + V_1.$$

Hence

$$V = X \oplus V_1 \qquad \text{(D-direct)}.$$

Now we assert that X is a DE-module. In fact, for $a \in E$ and $u \in U$,

$$\begin{aligned} a(u-h(u)) &= \theta(a)u - \theta(a)hu \\ &= \theta(a)u - \delta(a)u - h\phi(a)u \\ &= \phi(a)u - h\phi(a)u \in X. \end{aligned}$$

This fact contradicts our assumption that V_1 is the DE-socle of V. Therefore, $1 \notin L$ and $L \neq F$.

Now since F is quasi-Frobenius ring and $L \neq F$, the set

$$\{x \in F \mid Lx = 0\} \neq 0,$$

so there exists $f \neq 0$ in F so that $Lf = 0$, and so a $\mathfrak{p}$ in H exists with

$h = fp \neq 0$ and $Lh = 0$. However,

$$\begin{aligned}
&\Theta(\lambda^{-1})[\Theta(\lambda)-\phi^{o}(\lambda)]\delta(a) \\
&= \Theta(\lambda^{-1})[\Theta(\lambda)\delta(a)-\delta(a)\phi(\lambda)] \\
&= \Theta(\lambda^{-1})[\Theta(a)\delta(\lambda)-\delta(\lambda)\phi(a)] \\
&= \Theta(a)[\Theta(\lambda^{-1})\delta(\lambda)] - [\Theta(\lambda^{-1})\delta(\lambda)]\phi(a),
\end{aligned}$$

and

$$\Theta(\lambda^{-1})[\Theta(\lambda\ \ )-\phi^{o}(\lambda)] = 1 - \rho(\lambda^{-1})\epsilon F,$$

and

$$\Theta(\lambda^{-1})\delta(\lambda)\epsilon H,$$

so

$$\Theta(\lambda^{-1})[\Theta(\lambda)-\phi^{o}(\lambda)]\epsilon L \qquad \forall\lambda\epsilon\Lambda.$$

Hence

$$\Theta(\lambda^{-1})[\Theta(\lambda)-\phi^{o}(\lambda)]h \doteq 0 \qquad \forall\lambda\epsilon\Lambda$$

or

$$\Theta(\lambda)h - h\phi(\lambda) = 0 \qquad \forall\lambda\epsilon\Lambda.$$

This h induces an $\overline{h} \neq 0$ in $\mathrm{Hom}_D(V/V_1,V_1)$ with $\overline{h}\lambda\overline{v} = \lambda\overline{h\overline{v}}$, $\forall\lambda\epsilon\Lambda$, $\overline{v}\epsilon V/V_1$. This completes the proof.

BIBLIOGRAPHY

1. Artin, E., Galois Theory, Second Edition, Notre Dame (1959).

2. Artin, E., Nesbitt, C., and Thrall, R., Rings with Minimum Condition. University of Michigan (1944).

3. Cartan, H., "Théorie de Galois pour les corps non commutatifs," Ann. École Norm. 64 (1947), 59-77.

4. Dieudonné, J., "Sur le socle d'un anneau et les anneaux simples infinis," Bull. Soc. Math. France, 62 (1942), 46-75.

5. __________, "La théorie de Galois des anneaux simples et semi-simples," Comment. Math. Helv., 21 (1948), 154-184.

6. Eilenberg, S., and Nakayama, T., "On the dimension of modules and algebras II," Nagoya Math. J. 9 (1955), 1-16.

7. Hochschild, G., "Automorphisms of simple algebras," Trans. Amer. Math. Soc., 69 (1950), 292-301.

8. Jacobson, N., "A note on division rings," Amer. J. Math., 64 (1947), 27-36.

9. __________, "On the theory of primitive rings," Ann. of Math., 48 (1947), 8-21.

10. __________, Lectures in abstract algebra, I, Basic Concept, Van Nostrand, New York (1951).

11. __________, Lectures in abstract algebra, II, Linear algebra, Van Nostrand, New York (1953).

12. __________, "Structure of rings," Amer. Math. Soc. Colloq. Publ., 37, New York (1956).

13. Jacobson, N., and Richart, C.E., "Jordan Homomorphisms of rings," Trans. Amer. Math. Soc. 69 (1950), 479-502.

14. Kasch, F., "Grundlagen einer Theorie der Frobeniuserweiterungen," Math. Ann. 127 (1954), 453-474.

BIBLIOGRAPHY (concluded)

15. Nakayama, T., "Galois theory of simple rings," Trans. Amer. Math. Soc. 73 (1952), 276-292.

16. Rosenberg, A., and Zelinsky, D., "Galois theory of continuous transformation rings," Trans. Amer. Math. Soc. 79 (1955), 429-452.