p-adic Differential Operators on Automorphic Forms and Applications

by

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Dedicated to
my grandmother,
Marianne Bodenheimer

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ABSTRACT

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We construct certain C^∞-differential operators and their p-adic analogues, which act on (vector- or scalar-valued) automorphic forms on the unitary groups $U(n,n)$. We study properties of these operators, and we prove some arithmeticity theorems using them. These differential operators are a generalization to the p-adic case of the C^∞-differential operators first studied by H. Maass and later studied extensively by M. Harris and G. Shimura. They are a generalization to the vector-valued situation of the p-adic differential operators constructed in the one-dimensional setting by N. Katz. They should be useful in the construction of certain p-adic L-functions, in particular p-adic L-functions attached to p-adic families of automorphic forms on the unitary groups $U(n) \times U(n)$.

CHAPTER I

Introduction

1.1 Motivation

Though the topic of this thesis is differential operators, the motivation comes from the study of L-functions, in particular p-adic interpolation of special values of L-functions attached to families of automorphic forms on the unitary groups $U(n, n)$. We now give some motivation leading up to an explanation of what these operators do and why they are useful.

Among the most useful objects in number theory are L-functions. To various objects X of arithmetic signifcance, one can attach an L-function $L(s, X)$, an analytic function that encodes arithmetic data. At certain special points, the values of many L-functions are algebraic, up to a period, a well-determined transcendental factor. Furthermore, special values of many L-functions satisfy a much stronger property: their values at special points can be p-adically interpolated (after normalizing by a period) by p-adic analytic functions, called appropriately enough *p-adic L-functions*. These p-adic L-functions not only provide a useful tool for studying special values of classical L-functions, but they also encode significant arithmetic data.

A useful feature of the p-adic theory is the possibility that the object X lies in a p-adic family. The philosophy is, roughly speaking, that as X varies in a family the

L-values $L(s, X)$ should vary p-adically. For example, in 1986, H. Hida showed that certain classical modular forms — specifically, p-ordinary (slope 0) Hecke eigenforms — lie in p-adic families (*Hida families*) indexed by weight; that is, the eigenforms in the family vary p-adic analytically as the weight varies p-adic analytically. Later, Hida ([Hid93]) showed that given such a family $\{f_k\}$ of eigenforms, there is a two-variable p-adic L-function that interpolates the values of $L_{f_k}(s, \chi)$ as s and k vary p-adically. More recently, using techniques from rigid analytic geometry, R. Coleman and B. Mazur showed that overconvergent positive slope modular forms also lie in p-adic analytic families indexed by weight. In 2003, by extending the methods used by Hida, A. Panchishkin ([Pan03]) constructed two-variable p-adic L-functions interpolating the values of $L_{f_k}(s, \chi)$ for families of positive slope eigenforms. Both Hida's and Panchishkin's p-adic L-functions have applications to the Birch and Swinnerton-Dyer Conjecture. In recent years, Hida's construction has been generalized ([Hid98], [TU99], [Hid02], [Hid04]) to families of ordinary automorphic forms on more general groups, and Coleman's construction has also been generalized to certain more general groups, such as definite unitary groups ([Che04], [Buz04]). In [HLS06], M. Harris, J.-S. Li, and C. Skinner initiated a project to attach a p-adic L-function $L(s, f)$ to Hida families on unitary groups.

Currently, there are limits on the extent to which the above constructions of p-adic L-functions can be generalized to allow constructions of more general 2- or 3-variable p-adic L-functions attached to families of automorphic forms. The differential operators constructed in this thesis are expected to ease these restrictions. Here are some examples of advances that should be possible with the application of the differential operators in this thesis:

1. As the introduction of [HLS06] notes (in the subsection titled "Why the present

construction is not altogether satisfactory"), they can only interpolate the L-function at a fixed point s_0; removal of this restriction requires the differential operators that are the topic of this thesis.

2. No one has constructed p-adic L-functions attached to vector-valued automorphic forms, only scalar-valued automorphic forms. The differential operators are expected to make this generalization possible.

3. To date, no one has constructed p-adic L-functions attached to families of *non*-ordinary automorphic forms other than modular forms. The differential operators should also make this possible. In fact, this project originated from my attempt to construct (two- and three-variable) p-adic L-functions attached to certain families of overconvergent automorphic forms on the unitary groups $U(n,n)$. This thesis can be viewed as a step in that project, which has turned out to be more widely applicable.

The L-functions (and p-adic L-functions) discussed above are intimately tied to certain Eisenstein series. In the one-dimensional case (i.e. Hecke characters), Damarell's formula expresses special values of L-functions attached to Hecke characters in terms of a finite sum of special values of Eisenstein series. This is the approach taken by Katz in [Kat78]. In higher dimensions, one can use the doubling method[1] to construct L-functions; this is the approach taken in [HLS06]. Through the doubling method, one can express special values of L-functions in terms of a finite sum of special values of Eisenstein series. Thus, if one can show that each of the finitely many terms in the sum is algebraic (or p-integral, or lies in a desired ring) up to a period, then one has shown that the special values of the L-function are also algebraic (or p-integral, or lie in a specific ring).

[1]The doubling method – studied extensively by G. Shimura – is a higher-dimensional analogue of Damarell's formula and is a special case of the pullback methods discovered by P. Garrett ([Gar84]).

In [Ser73], J.-P. Serre observed the possibility of using Eisenstein series to p-adically interpolate special values of the Riemann zeta function and, more generally, of L-functions attached to totally real fields. Many constructions of p-adic L-functions since then have also relied on p-adic interpolation of special values of Eisenstein series, including [Kat77], [Kat78], [Pan03], and [HLS06]. In the case of holomorphic Eisenstein series, the p-adic interpolation often takes place through p-adic interpolation of Fourier coefficients. This approach is in general not sufficient, though, because of the following issue: most special values of L-functions come from non-holomorphic Eisenstein series, which do not have Fourier expansions. This is closely related to the reason that the L-functions $L(s, f)$ in [HLS06] are only p-adically varied at a fixed point $s = s_0$ (as mentioned above).

For the case of the unitary groups $U(n, n)$, I have constructed p-adic differential operators that can be used to solve this issue.[2] These differential operators are a p-adic analogue of a class of C^∞-differential operators first studied by H. Maass ([Maa56], [Maa71]) and later studied extensively by G. Shimura ([Shi94], [Shi90], [Shi84], [Shi81], [Shi81], [Shi84], [Shi00]) and M. Harris ([Har86], [Har81]). (In the case of modular forms on the upper half plane, these C^∞-differential operators are the widely used operators $g \mapsto y^{-k}(\frac{\partial}{\partial z})(y^k g)$ that map a weight k modular form g to a weight $k+2$ modular form. More generally, they map a vector- or scalar-valued automorphic function to an automorphic function of a different weight.) These C^∞-differential operators play an important role in Shimura's proofs of algebraicity properties of Eisenstein series and L-functions. Shimura's proofs, however, do not provide insight into p-adic properties.

For Hilbert modular forms (which are scalar-valued), N. Katz ([Kat78]) reformu-

[2] The material in this thesis also generalizes almost immediately to give a similar construction of differential operators and similar results in the case of the symplectic groups $Sp(n)$, and it should be relatively straightforward to generalize to $U(m, n)$.

lates Shimura's C^∞-differential operators in terms of the Gauss-Manin connection and the Kodaira-Spencer isomorphism. This algebraic-geometric approach is useful because it allows Katz to construct a p-adic analogue of the C^∞-differential operators for Hilbert modular forms. An algebraic geometric argument then shows that for each of the p-adic differential operators D and each holomorphic automorphic form f with p-integral Fourier coefficients, Df is p-integral (up to a period) at points corresponding to abelian varieties with complex multiplication (viewing automorphic functions as sections of a sheaf on a Shimura variety, which is a certain moduli space of abelian varieties); these are the only points whose values matter in the construction of L-functions. Katz also studies the coefficients of a q-expansion DE obtained by applying a p-adic differential operator D to a classical Eisenstein series E; this is useful for using the coefficients of the q-expansion of E to prove that DE can be p-adically interpolated. I have generalized [Kat78] to the setting of automorphic forms on $U(n,n)$ and symplectic groups, including the more general case of *vector*-valued forms. This should be useful for constructing some of the more general p-adic L-functions mentioned above.

The p-adic interpolation of special values of L-functions is dependent in part upon the following remarkable fact, which is an easy consequence of theorems we prove using the differential operators: The differential operators allow one to show that the values of a certain *p-adic* – in general non-algebraic – function at CM points over $\mathcal{O}_{\mathbb{C}_p}$ are in fact not only algebraic but also the *same* as the values of a closely related C^∞ – in general non-holomorphic – function at CM points over $\mathcal{O}_{\mathbb{C}_p}$. Thus, special values of pairs of seemingly unrelated functions (namely, a C^∞ non-holomorphic function and p-adic function) are meaningfully compared and shown to be equal.

The starting point for my construction and proofs is [Kat78]; however, the higher-dimensional, vector-valued situation is more complicated and involves several obstacles not encountered in the one-dimensional case considered in [Kat78]. My generalization involves a more delicate use of the Kodaira-Spencer morphism (which, for the unitary case, is no longer an isomorphism) than in Katz's situation. Also, unlike in [Kat78], the action of the operator on q-expansions is no longer in terms of a derivation on a commutative ring, but rather a map (in general, not a derivation) on a ring that is in general non-commutative; I formulate the precise action of this map on coefficients of vector-valued q-expansions so that the description of the resulting coefficients might be used to study p-adic interpolation. (Similarly, there are other instances in which a commutative ring in [Kat78] is replaced with a non-commutative one in my situation.)

I have aimed to express the material in such a way here that it can easily be applied to generalize some of the constructions of L-functions mentioned above. For example, I have given formulas for the action of the differential operators on q-expansions, which is crucial information if one is to try to p-adically interpolate q-expansion coefficients (in order to p-adically interpolate L-functions, as described above). Along the way, we obtain a higher-dimensional, vector-valued analogue of Ramanujan's operator $q\frac{d}{dq}$. I have also provided a detailed discussion of the Kodaira-Spencer isomorphism in coordinates. To date, this does not appear elsewhere in the literature, but it is important for understanding the action of the differential operators at the level of coordinates (rather than just as abstract maps). Furthermore, this thesis provides a user's guide to q-expansions and the "Mumford object," a generalization of the Tate curve to the higher dimensional setting. Unlike the situation for Tate curves, which are used explicitly in computations and described in detail in coordinates over $\mathbb{C}$, the current literature ([Lan08]) on Mumford objects and algebraic q-expansions is at the level of existence statements. Since our intended applications, as well as other unrelated projects ([SU09]) require a more explicit description of the Mumford object, we provide one here. While it is in some ways simpler, the simplicity of the one-dimensional case (i.e. Tate curves, as discussed in [Kat78] and [Kat73b]) obscures the larger picture. In fact, many details of the one-dimensional case become more transparent in the arbitrary-dimension situation.

1.2 Notation

We now introduce some notation that we will use throughout the paper. Our setup is exactly the same as the setup in Sections 0 and 1 of [HLS06], though our notation is not always the same. We have tried to be as consistent as possible with the notation of [Shi00], [Har81], [Kat78], [HLS06], and [Hid04]. Absolute consistency is frequently impossible, though, since the notation often varies from one source to the next.

Throughout the paper, fix a quadratic imaginary extension $\mathcal{K}$ of $\mathbb{Q}$, and let $\mathcal{O}_{\mathcal{K}}$ denote the ring of integers in $\mathcal{K}$. Fix a CM type Σ of $\mathcal{K}$, i.e. an embedding

$$\mathcal{K} \hookrightarrow \mathbb{C}. \tag{1.1}$$

Throughout this paper, we associate $\mathcal{K}$ with its image in $\mathbb{C}$ under the embedding (1.1). Let $\delta_{\mathcal{K}}$ denote the discriminant of $\mathcal{K}$. Unless otherwise noted, we will always use R_0 to denote an $\mathcal{O}_{\mathcal{K}}$-algebra. Let α be a generator of $\mathcal{O}_{\mathcal{K}}$ over $\mathbb{Z}$. The reason for fixing this element α is that it makes various examples later on more transparent. In cases where it is possible, we note that it is easiest to follow the examples if one takes α to be purely imaginary. One may also find it helpful to keep in mind the case where $\alpha = i = \sqrt{-1}$.

Let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and write

$$\mathrm{incl}_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

to denote the given embedding of $\bar{\mathbb{Q}}$ in $\mathbb{C}$.

Fix a prime ideal (p) in $\mathbb{Z}$ that splits completely in K. We write $\mathbb{A}_f$ to denote the finite adeles of $\mathbb{Q}$, and we write $\mathbb{A}_f^p$ to denote the restricted product $\prod' \mathbb{Q}_l$ over finite primes $l \neq p$. Let $\mathbb{C}_p$ denote the completion of an algebraic closure of $\mathbb{Q}_p$, and fix an embedding

$$\mathrm{incl}_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p.$$

Given an embedding

$$\sigma : \mathcal{K} \hookrightarrow \bar{\mathbb{Q}},$$

we write σ_∞ to denote the embedding

$$\mathrm{incl}_\infty \circ \sigma : \mathcal{K} \hookrightarrow \mathbb{C},$$

and we write σ_p to denote the embedding

$$\mathrm{incl}_p \circ \sigma : \mathcal{K} \hookrightarrow \mathbb{C}_p.$$

We write $\bar{\sigma}$ to denote the composition of the embedding σ with complex conjugation.

We now establish some notation for modules. For any module M, we denote $\sum_{e=0}^{\infty} M^{\otimes e}$ by $T(M)$. From now on, for any module M, we associate

$$\mathrm{Sym}(M) = \sum_e \mathrm{Sym}^e M$$

with its image in

$$T(M) = \sum_e T^e(M)$$

via the inclusions

$$\operatorname{Sym}^e(M) \hookrightarrow M^{\otimes e} \tag{1.2}$$

$$x_1 \cdot \cdots \cdot x_e \mapsto \sum_{s \in S_e} x_{s(1)} \otimes \cdots \otimes x_{s(e)},$$

where S is the group of permutations of $1, \ldots, e$. Let r be a positive integer, and let V be a vector space containing vectors v_i indexed by a subscript i. For any r-tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$ of integers, we use the notation v_λ to denote the tensor product $v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_r}$ of r vectors $v_{\lambda_1}, \ldots, v_{\lambda_r}$. We write v^λ to denote the symmetric product $v_1^{\lambda_1} \cdots v_n^{\lambda_n}$. We write ρ_{st} or simply st to denote the standard representation of $GL(V)$ on a vector space V.

In parts of the paper dealing with a complex analytic approach, we primarily use Shimura's notation (used throughout his papers, e.g. as discussed in the Notation and Terminology section of [Shi00]). We review some of it here. For a ring R and positive integers r and c, we write R_c^r to denote the R-module of $r \times c$-matrices with entries in R, i.e. a matrix with r rows and c columns. When we want to be careful about distinguishing between column and row vectors, we shall take advantage of this notation. For a matrix z with entries in $\mathbb{C}$, we write ${}^t z$ to denote the transpose of z and z^* to denote the complex conjugation ${}^t \bar{z}$ of ${}^t z$.

Throughout the paper, fix a positive integer n and set

$$g = 2n.$$

We write 1_n to mean the $n \times n$ identity matrix. As in [Shi00], we define

$$\begin{aligned} \mathcal{H}_n &= \{z \in \mathbb{C}_n^n \mid i(z^* - z) > 0\} \\ \eta_n &= \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}. \end{aligned}$$

Though it may seem arbitrary at this point in the paper, we will also find the following notation helpful. If A is a matrix, we write A^+ to denote A and A^- to denote ${}^tA$.

Given a subgroup G of $GL_n(\mathbb{R})$, we denote by G^+ the subgroup of G consisting of elements of positive determinant.

We now establish some conventions for schemes. For any morphism of schemes $\pi : Y \to Z$, let $(\Omega^{\bullet}_{Y/Z}, d)$ denote the complex of sheaves of relative differentials on Y/Z (where d is the usual differentiation map). The **de Rham cohomology** $H^i_{DR}(Y/Z)$ is defined to be the hypercohomology $\mathbb{R}^i\pi_*(\Omega^{\bullet}_{Y/Z})$. Given a scheme X over a scheme S and a scheme T over S, we denote by X_S the scheme $X \times_S T$. When working with a separated scheme S of finite type over $\mathbb{C}$, we write S^{an} to denote the associated complex analytic space. We then write $\mathcal{O}_S^{\mathrm{hol}}$ or $\mathcal{O}_S(\mathrm{hol})$ (resp. $\mathcal{O}_S^{C^\infty}$ or $\mathcal{O}_S(C^\infty)$) to denote the sheaf of holomorphic (resp. C^∞) functions on S^{an}.

CHAPTER II

Certain abelian varieties of PEL type and automorphic forms

In this chapter, following the perspectives of [Shi00], [Kat78], and [Hid04], we discuss automorphic forms on the unitary groups $U(n,n)$ and certain abelian varieties with PEL structure.

2.1 Unitary groups

In order to discuss automorphic forms on unitary groups and abelian varieties of PEL type, we need first to establish conventions for unitary groups. In this section, we recall the notation and conventions concerning unitary groups given in Section (0.1) of [HLS06]; all the material in Section (0.1) of [HLS06] applies to our situation. Let V be an n-dimensional vector space over $\mathcal{K}$, and let $\langle\bullet,\bullet\rangle_V$ be a non-degenerate hermitian pairing on V relative to the extension $\mathcal{K}/\mathbb{Q}$. We write $-V$ to denote the vector space V over $\mathcal{K}$ with hermitian pairing $\langle\bullet,\bullet\rangle_{-V}$ defined by

$$\langle\bullet,\bullet\rangle_{-V} = -\langle\bullet,\bullet\rangle_V.$$

We write $2V$ to denote the $\mathcal{K}$-vector space $V\oplus V$ with the hermitian pairing

$$\langle\bullet,\bullet\rangle_{2V}$$

defined by

$$\begin{aligned}\langle (v_1, v_2), (w_1, w_2)\rangle_{2V} &= \langle v_1, w_1\rangle_V + \langle v_2, w_2\rangle_{-V}\\ &(= \langle v_1, w_1\rangle_V - \langle v_2, w_2\rangle_V)\end{aligned}$$

for all vectors v_1, v_2, w_1, w_2 in V.

The Hermitian pairing $\langle \bullet, \bullet\rangle_V$ defines an involution c on $\mathrm{End}(V)$ via

$$\langle gv, v'\rangle = \langle v, c(g)v'\rangle$$

for all g in $\mathrm{End}(V)$ and $v, v' \in V$. Note that for any $\mathbb{Q}$-algebra R, the involution c extends to an involution of $V \otimes_{\mathbb{Q}} R$.

For any vector space W with hermitian pairing $\langle \bullet, \bullet\rangle_W$ and $\mathbb{Q}$-algebra R, we define the following unitary groups over $\mathbb{Q}$:

$$\begin{aligned}&U(W)(R) = U(W, \langle \bullet, \bullet\rangle_W)(R)\\ &= \{g \in GL\left(W \otimes_{\mathbb{Q}} R\right) \mid \langle gv, gv'\rangle = \langle v, v'\rangle, \text{ for all } v, v' \in W\}\\ &GU(W)(R) = GU(W, \langle \bullet, \bullet\rangle_W)(R)\\ &= \left\{g \in GL\left(W \otimes_{\mathbb{Q}} R\right) \mid \text{ for all } v, v' \in W, \langle gv, gv'\rangle = \nu(g)\langle v, v'\rangle \text{ with } \nu(g) \in R^\times\right\}.\end{aligned}$$

Then

$$\begin{aligned}U(2V)(R) &\cong U(n,n)(R) = \{g \in GL_{2n}(K \otimes_{\mathbb{Q}} R) \mid g\eta_n g^* = \eta_n\}\\ GU(2V)(R) &\cong GU(n,n)(R) = \left\{g \in GL_{2n}(K \otimes_{\mathbb{Q}} R) \mid g\eta_n g^* = \nu(g)\eta_n \text{ some } \nu(g) \in R^\times\right\}.\end{aligned}$$

2.2 Certain abelian varieties of PEL type

In this section, we review certain abelian varieties of PEL type.

Our situation is similar to the setup in Sections 1.2 through 1.4 of [HLS06]. We review here the most important features of the setup in [HLS06], following [HLS06] closely. Our notation in this section is not entirely the same as the notation in [HLS06]. (For details on the material covered in this section, the reader may also find it helpful to look at chapters 1 and 2 of [Lan08] and at chapters 6 and 8 of [Mil04].)

Let $G = GU(V)$. Fix a compact open subgroup $K = K_p \times K^p$ of $GU(V)(\mathbb{A}_f)$, with $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial maximal open compact subgroup of $G(\mathbb{Q}_p)$ and K^p in $G(\mathbb{A}_f^p)$. (In applications, the maximal compacts of interest will be those used in [HLS06].) We recall the functor ${}_K\mathbb{A}_V$ from schemes S over $\mathbb{Q}$ to the category of sets that is given in (1.3.1) of [HLS06].

$$S \mapsto \{(A, \lambda, \iota, \alpha)\} \tag{2.1}$$

where

- A is an abelian scheme over S, up to isogeny
- $\lambda : A \to A^\vee$ is a polarization
- $\iota : \mathcal{K} \to \mathrm{End}_S(A) \otimes \mathbb{Q}$ is an embedding of $\mathbb{Q}$-algebras
- $\alpha : V \otimes \mathbb{A}_f \xrightarrow{\sim} \prod_l T_l(A) \otimes \mathbb{Q}$ is an isomorphism of $\mathcal{K}$-spaces, modulo the action of K.

The above data are required to satisfy the Rosati condition, i.e. the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\iota(\bar{a})"'] & A^\vee \arrow[d, "\iota^\vee(a)"] \\ A \arrow[r, "\lambda"'] & A^\vee. \end{tikzcd}$$

Furthermore, the isomorphism α must identify the Hermitian pairing on V with a multiple of the one coming from the Weil pairing associated to λ. We call the tuples $(A, \lambda, \iota, \alpha)$ abelian varieties of PEL type.

Given a vector space W over $\mathcal{K}$ with a non-degenerate hermitian pairing on W, one can canonically associate to the group $G = GU(W)$ a Shimura datum (G, X) and a Shimura variety $Sh(W) = Sh(G, X)$. The complex-valued points of $Sh(G, X)$ are given by

$$Sh(G,X)(\mathbb{C}) = \lim_{K} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K,$$

where the limit is over all open compact subgroups of $G(\mathbb{A}_f)$. Let

$${}_K Sh =_K Sh(V) =_K Sh(G,X)$$

be the variety whose complex points are given by $G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$; the complex points of this variety classify complex abelian varieties satisfying the above moduli problem. If U is a compact closed subgroup of $GU(\mathbb{A}_f)$, we use the notation ${}_U Sh(V)$ to mean the tower of the varieties ${}_K Sh =_K Sh(V)$ with $K \subset U$ a compact open.

We remind the reader of Theorem (1.3.2) in [HLS06], which is originally due to Shimura:

Theorem II.1. *Whenever K is sufficiently small ("neat", in the sense of [Lan08], suffices), the functor ${}_K\mathbb{A}_V$ is representable by a quasi-projective scheme ${}_K\mathcal{M}$ over $\mathbb{Q}$. The scheme ${}_K\mathcal{M}$ is the canonical model for ${}_K Sh(V)$. As K varies, the natural maps between the above functors induce the natural maps between the varieties ${}_K Sh(V)$. The action of $GU(V)(\mathbb{A}_f)$ on ${}_K Sh(V)$ preserves the $\mathbb{Q}$-rational structure.*

Now we consider a similar but slightly different moduli problem that is discussed in Section (1.6) of [HLS06]; it will be useful for the p-adic theory, which we will discuss later. Fix a sufficiently small compact open subgroup $K = K_p \times K^p \subset G(\mathbb{A}_f)$ as above. Consider the above moduli problem with ι replaced by an injection

$$(\mathcal{O}_{\mathcal{K}})_{(p)} \hookrightarrow \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)}.$$

As explained in [Kot92], this moduli problem is represented by a smooth integral scheme ${}_K\mathbb{S}(G,X)$ over $\mathbb{Z}_{(p)}$, which is a smooth integral model for ${}_K Sh(G,X)$. This moduli problem is closely related to the moduli problem we now describe. Let ${}_{K^p}A^p$ be the functor $S \mapsto \{(A,\lambda,\iota,\alpha^p)\}$ with A an abelian scheme over S up to prime-to-p isogeny, λ a polarization of degree prime to p, $\iota : (\mathcal{O}_{\mathcal{K}})_{(p)} \to \mathrm{End}_S(A) \otimes \mathbb{Z}_{(p)}$ an embedding of $\mathbb{Z}_{(p)}$-algebras, and $\alpha^p : V(\mathbb{A}_f^p) \xrightarrow{\sim} V^{f,p}(A)$ a prime-to-p $(\mathcal{O}_{\mathcal{K}})_{(p)}$-linear level structure modulo K^p. This functor is representable over $\mathbb{Z}_p$ by a scheme also denoted ${}_K\mathbb{S}(G,X)$. The forgetful map gives an isomorphism ${}_K\mathbb{A}_V \xrightarrow{\sim}{}_{K^p} A^p$.

We denote by $\mathcal{M}$ a moduli space over an $\mathcal{O}_{\mathcal{K}}$-algebra in the situation where we want to remain ambiguous about the level structure or any other details about the moduli problem; we also use this notation when it is clear from context which moduli space we mean. We write A_{univ} to denote the universal abelian variety over $\mathcal{M}$:

$$\begin{array}{c} A_{univ} \\ \big\downarrow{\scriptstyle \pi} \\ \mathcal{M}. \end{array}$$

We define

$$\underline{\omega} := \pi_*(\Omega_{A_{univ}/\mathcal{M}})$$

$$H^1_{DR} := H^1_{DR}(A_{univ}/\mathcal{M}).$$

Note that we will *always* take $\mathcal{M}$ to be over an $\mathcal{O}_{\mathcal{K}}$-algebra.[1] Working over $\mathcal{O}_{\mathcal{K}}$ affords us the following convenient splittings. The embedding

$$\iota : \mathcal{K} \to \mathrm{End}_S(A) \otimes \mathbb{Q}$$

[1] Although much of the material in this paper can be developed over other schemes, for example over $\mathbb{Q}$, the construction of the Mumford object (discussed in Section IV) and the compactification of Shimura varieties takes place over $(\mathcal{O}_{\mathcal{K}})_{(p)}$. In addition, it will be convenient – though not necessary – to take advantage of the splittings of $\underline{\omega}$ and H^1_{DR} given above.

makes $\underline{\omega}$ and H^1_{DR} into $\mathcal{O}_\mathcal{K}$-modules through the action defined by

$$a \cdot v = \iota(a)^*(v)$$

for each $a \in \mathcal{O}_\mathcal{K}$. Similarly, since A lies over an $\mathcal{O}_\mathcal{K}$-scheme S, the action on $\underline{\omega}$ and H^1_{DR} induced by the composition of morphisms of structure sheaves

$$\mathcal{O}_\mathcal{K} \to \mathcal{O}_S \to \mathcal{O}_A$$

also makes $\underline{\omega}$ and H^1_{DR} into $\mathcal{O}_\mathcal{K}$-modules. The isomorphism

$$\mathcal{O}_\mathcal{K} \otimes_\mathbb{Q} \mathcal{O}_\mathcal{K} \xrightarrow{\sim} \mathcal{O}_\mathcal{K} \oplus \mathcal{O}_\mathcal{K}$$

$$a \otimes b \mapsto (ab, \bar{a}b)$$

extends to an isomorphism

$$\mathcal{O}_\mathcal{K} \otimes_\mathbb{Q} \mathcal{O}_S \xrightarrow{\sim} \mathcal{O}_S \oplus \mathcal{O}_S \tag{2.2}$$

$$a \otimes b \mapsto (ab, \bar{a}b).$$

So there is a splitting over $\mathcal{O}_S$

$$\underline{\omega} = \underline{\omega}^+ \oplus \underline{\omega}^-,$$

where $\underline{\omega}^+$ is the $\mathcal{O}_S$-module defined by

$$\underline{\omega}^+ := \{w \in \underline{\omega} | \iota(a)^* w = aw, a \in \mathcal{O}_\mathcal{K}\}$$

and $\underline{\omega}^-$ is given by

$$\underline{\omega}^- := \{w \in \underline{\omega} | \iota(a)^* w = \bar{a}w, a \in \mathcal{O}_\mathcal{K}\}.$$

There is a similarly defined splitting

$$H^1_{DR} = {H^1_{DR}}^+ \oplus {H^1_{DR}}^-$$

of H^1_{DR}. We shall sometimes denote ${H^1_{DR}}^\pm$ by $H^\pm$.

2.3 The complex analytic viewpoint (Shimura's perspective)

In this section, for a fixed open compact K, let Γ be the congruence subgroup of $GU(\eta_n)$ defined by

$$\Gamma = K \cap G(\mathbb{Q}).$$

2.3.1 Transcendental description of abelian varieties of PEL type

In this section, we give a transcendental description of complex abelian varieties of PEL type, following the approach of Shimura in [Shi00] and [Shi98].

Let

$$\underline{A} = (A, \lambda, \iota, \alpha)$$

be a complex abelian variety of PEL type. Then A^{an} is a complex torus $\mathbb{C}^{2n}/\mathcal{L}$, for some $\mathbb{Z}$-lattice $\mathcal{L}$ in $\mathbb{C}^{2n}$, which can be obtained as follows. Let $\{\omega_i^+\}_{i=1}^n$ be a basis for $\underline{\omega}^+_{A/\mathbb{C}}$, and let $\{\omega_i^-\}_{i=1}^n$ be a basis for $\underline{\omega}^-_{A/\mathbb{C}}$. Then, we define the $\mathbb{Z}$-lattice $L(A, \{\omega_i^+\}_{i=1}^n, \{\omega_i^-\}_{i=1}^n)$ to be

$$L(A, \{\omega_i^+\}_{i=1}^n, \{\omega_i^-\}_{i=1}^n) = \left\{ \left. \begin{pmatrix} \int_\gamma \omega_1^- \\ \vdots \\ \int_\gamma \omega_n^- \\ \int_\gamma \omega_1^+ \\ \vdots \\ \int_\gamma \omega_n^+ \end{pmatrix} \right| \gamma \in H_1(A, \mathbb{Z}) \right\}.$$

The complex abelian variety A^{an} is isomorphic to $\mathbb{C}^g / L(A, \{\omega_i^+\}_{i=1}^n, \{\omega_i^-\}_{i=1}^n)$. The polarization λ on A corresponds to a Riemann form on $L(A, \{\omega_i^+\}_{i=1}^n, \{\omega_i^-\}_{i=1}^n)$. The morphism

$$\iota : \mathcal{K} \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A)$$

corresponds to the action ι of $\mathcal{K}$ on $L(A, \{\omega_i^+\}_{i=1}^n, \{\omega_i^-\}_{i=1}^n)$ given by

$$i(a)v = \Psi(a) \cdot v$$

for all a in $\mathcal{K}$ and v in $L(A, \{\omega_i^+\}_{i=1}^n, \{\omega_i^-\}_{i=1}^n)$, where

$$\Psi : \mathcal{K} \to \mathbb{C}_{2n}^{2n}$$

is given by

$$a \mapsto diag[\bar{a} \cdot 1_n, a \cdot 1_n].$$

The level structure

$$\alpha : V(\mathbb{A}_f) \xrightarrow{\sim} V^f(A) \mod K$$

corresponds to a map

$$V(\mathbb{A}_f) \xrightarrow{\sim} L(A, \{\omega_i^+\}_{i=1}^n, \{\omega_i^-\}_{i=1}^n) \otimes \mathbb{A}_f \mod K,$$

with a compatibility of pairings as above.

2.3.2 Families of complex abelian varieties of PEL type

We now recall Shimura's construction (Section 4 of [Shi00]) of some families of complex abelian varieties of PEL type. Throughout this section, fix a $\mathbb{Z}$-lattice L in $\mathcal{K}_{2n}^1$.

For each $z \in \mathcal{H}_n$ and each row vector x in $\mathcal{K}_{2n}^1$, let $p_z(x)$ be the vector in $\mathbb{C}^{2n}$ defined by

$$p_z(x) = \left([z \quad 1_n] x^*, \quad [{}^t z \quad 1_n] \cdot {}^t x \right).$$

The function $p_z(x)$ is holomorphic in z.

We define

$$\mathcal{L}_L(z) = p_z(L).$$

Then $p_z(L)$ is a lattice in $\mathbb{C}^g$.

Let A_z be the complex torus defined by

$$A_z = \mathbb{C}^{2n}/p_z(L).$$

Let C_z be the polarization on A_z given by the Riemann form E_z defined by

$$E_z\left(p_z(x), p_z(y)\right) = \operatorname{tr}_{(\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{R})/\mathbb{R}}\left(x\eta_n y^*\right)$$

Every complex-valued point $(A, \lambda, \iota, \alpha)$ of the Shimura variety ${}_KSh(V)$ is isomorphic for some z to A_z with polarization C_z and action of $\mathcal{K}$ given by

$$\iota_z(a)\cdot v = diag[\bar{a}\cdot 1_n, a\cdot 1_n]\cdot v$$

(for $v \in \mathcal{L}_L(z)$ and $a \in \mathcal{K}$).

To specify a finite ordered set of points $t_1(z)\ldots, t_s(z)$ of finite order on A_z, it is equivalent to specify a finite set of elements $u_1, \ldots, u_s$ in $L\otimes\mathbb{Q}/L$ such that

$$t_i(z) = p_z(u_i).$$

We conclude this section by giving a more explicit classification of analytic families of complex abelian varieties of PEL type. Fix a finite set of points $\{u_1, \ldots, u_s\}$ in $\mathcal{K}^1_{2n}$. Consider the quintuple

$$\Omega = \left\{\mathcal{K}, \Psi, L, \eta_n, \{u_i\}_{i=1}^s\right\}. \tag{2.3}$$

Such a quintuple is called a PEL-type. Let

$$P = (A, C, \iota; \{t_i\}_{i=1}^s)$$

be a tuple consisting of a complex abelian variety A, a polarization C, a set of points $t_1, \ldots, t_s$ of finite order on A, and a ring injection

$$\iota : \mathcal{K} \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A)$$

that is stable under the involution of $\mathrm{End}_{\mathbb{Q}}(A)$ determined by C. We say that P is of type Ω if the following holds: there is a $\mathbb{Z}$-lattice Λ; a homomorphism $\xi : \mathbb{C}^g \to A$; an $\mathbb{R}$-linear isomorphism

$$q : \mathbb{C}^1_g \to \mathbb{C}^g$$

such that

$$q(ax) = \Psi(a)q(x) \qquad \text{and}$$

$$\iota(a) \circ \xi = \xi \circ \Psi(a)$$

for all $a \in \mathcal{K}$ and $x \in \mathbb{C}^1_g$ and such that $q(u_i) = t_i$ for each i; a Riemann form E determined by C that satisfies

$$E(q(x), q(y)) = \mathrm{tr}_{K/\mathbb{Q}}(x\eta_n y^*);$$

and a commutative diagram (Figure (4.3) in [Shi00])

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & \mathbb{C}^1_g & \longrightarrow & \mathbb{C}^1_g/L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \\
0 & \longrightarrow & \Lambda & \longrightarrow & \mathbb{C}^g & \xrightarrow{\xi} & A & \longrightarrow & 0.
\end{array}$$

The classification of complex abelian varieties of type Ω, which we now make precise, will be useful for understanding how the classical (analytic) definition of automorphic forms motivates the algebraic-geometric definition of automorphic forms. For each z in $\mathcal{H}_n$, let

$$P_z = (A_z, C_z, \iota_z; \{t_i(z)\}_{i=1}^s)$$

with A_z, C_z, ι_z, and $\{t_i(z)\}_{i=1}^s$ defined as above. Then, P_z is of type Ω for each z in $\mathcal{H}_n$. Furthermore, Theorem II.2 (Theorem 4.8 in [Shi00]) classifies abelian varieties of type Ω. An element

$$\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(\eta_n)$$

acts on $\mathcal{H}_n$ by

$$\alpha z = (Az + B)(Cz + D)^{-1}.$$

Theorem II.2. *The tuple P_z is of type Ω for each z in $\mathcal{H}_n$, and every tuple of type Ω is isomorphic to P_z for some z in $\mathcal{H}_n$. Tuples P_z and P_w are isomorphic if and only if there is an element γ in the group*

$$\Gamma = \{\alpha \in U(\eta_n) | L\alpha = L \text{ and } u_i\alpha - u_i \in L \text{ for each } i\}$$

that satisfies

$$w = \gamma z.$$

Remark II.3. Taking L to be the lattice in $\mathcal{K}^1_{2n}$ generated by the standard basis vectors $e_1, \ldots, e_{2n}$ and the vectors $\alpha \cdot e_1, \ldots, \alpha \cdot e_{2n}$ (with α a generator of $\mathcal{K}$ over $\mathbb{Q}$), we see that there is an analytic family of abelian varieties A_{univ} over $\mathcal{H}_n$ such that the fiber of A_{univ} over each point $z = (z_{ij})$ in $\mathcal{H}_n$ is the abelian variety

$$A_z := \mathbb{C}^n / L_z,$$

where L_z is the $\mathbb{Z}$-lattice in $\mathbb{C}^{2n}$ generated by:

(2.4)

$z_j = (z_{1j}, \ldots, z_{nj}, z_{j1}, \ldots, z_{jn})$,

(2.5)

$e_j =$ vector with 1 in the j-th and $j+n$-th positions and zeroes everywhere else

(2.6)

$z'_j = (\alpha^* z_{1j}, \ldots, \alpha^* z_{nj}, \alpha z_{j1}, \ldots, \alpha z_{jn})$

(2.7)

$e'_j =$ vector with α^* in j-th position, α in $j+n$-th position, and zeroes everywhere else,

with $j = 1, \ldots, n$. We will work with this family of abelian varieties in examples in future sections.

2.3.3 Complex analytic automorphic forms

In this section, we remind the reader of the classical definition of automorphic forms over $\mathbb{C}$, following the perspective of Shimura ([Shi00]).

For $\alpha = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(\eta_n)$ and $z \in \mathcal{H}_n$, we define

$$M_\alpha(z) = M(\alpha, z) = (\mu(\alpha, z), \lambda(\alpha, z)),$$

where

$$\mu(\alpha, z) = Cz + D$$

and

$$\lambda(\alpha, z) = \bar{C} \cdot {}^t z + \bar{D}.$$

Let X be a finite-dimensional vector space, and fix a representation

$$\omega : GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \to GL(X).$$

For any map

$$f : \mathcal{H}_n \to X$$

and $\alpha \in U(\eta_n)$, define

$$f\|_\omega \alpha : \mathcal{H}_n \to X$$

by

$$(f\|_\omega \alpha)(z) = \omega\left(M_\alpha(z)\right)^{-1} f(\alpha z).$$

Definition II.4. Let Γ be a congruence subgroup of $U(\eta_n)$. A (holomorphic) automorphic form of weight ω with respect to Γ is a holomorphic function

$$f : \mathcal{H}_n \to X$$

that satisfies

$$f\|_\gamma = f \tag{2.8}$$

for each $\gamma \in \Gamma$. When $n = 1$, we also require that f is holomorphic at the cusps.

We denote the space of all (holomorphic) automorphic forms of weight ω with respect to Γ by $\mathcal{M}_\omega(\Gamma)$, and we set

$$\mathcal{M}_\omega = \cup_\Gamma \mathcal{M}_\omega(\Gamma).$$

Definition II.5. A C^∞-automorphic form of weight ω with respect to a congruence subgroup Γ is a C^∞-function

$$f : \mathcal{H}_n \to X$$

that satisfies (2.8) for each $\gamma \in \Gamma$.

2.4 Automorphic forms from another perspective

In this section we reformulate the definition of automorphic forms given in Section 2.3.3 in terms of functions on certain lattices (with additional structure) and, finally, in terms of functions on abelian varieties of type Ω. Our discussion here is a generalization to abelian varieties (of type Ω) of the situation for elliptic curves discussed in Appendix A1.1 of [Kat73b]. Although the situation for modular forms (viewed as functions on lattices in $\mathbb{C}$, or equivalently, as functions on genus 1 abelian varieties) is explained in [Kat73b], the case $g = 1$ is so simple as to obscure the situation for more general automorphic forms.

Let Ω be as in (2.3). We begin by defining a *lattice of type* Ω. Let $(\mathcal{L}, E, \{t_i\}_{i=1}^s)$ be a tuple consisting of a lattice $\mathcal{L}$ in $\mathbb{C}^g$ such that there is an $\mathbb{R}$-linear isomorphism

$$q : \mathbb{C}_g^1 \to \mathbb{C}^g \tag{2.9}$$

satisfying

$$q(L) = \mathcal{L}$$

and

$$q(ax) = \Psi(a)q(x)$$

for all $a \in \mathcal{K}$ and $x \in L$, a finite set of points $\{t_1, \ldots, t_s\}$ in $\mathcal{L} \otimes \mathbb{Q}$ such that $q(u_i) = t_i$ for all i, and a Riemann form E on $\mathbb{C}^g$ relative to $\mathcal{L}$ such that

$$E(q(x), q(y)) = \mathrm{tr}_{K/\mathbb{Q}}(x\eta_n y^*) \tag{2.10}$$

for all $x, y \in \mathbb{C}^g$. We call such a tuple $(\mathcal{L}, E, \{t_i\}_{i=1}^s)$ a *lattice of type* Ω. We define an action of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ on the set of tuples $(\mathcal{L}, E, \{t_i\}_{i=1}^s)$ of type Ω via

$$\alpha \cdot (\mathcal{L}, E, \{t_i\}_{i=1}^s) = (\alpha\mathcal{L}, E^\alpha, \{\alpha t_i\}_{i=1}^s),$$

where E^{α} is defined by

$$E^{\alpha}(\alpha z, \alpha w) = E(z, w).$$

(Note that by $\alpha\mathcal{L}$, we mean

$$\alpha\mathcal{L} = \{\alpha v | v \in \mathcal{L}\}.)$$

Observe that, modulo the action of $GL_n \times GL_n$ on lattices of type Ω, there is a natural correspondence between lattices of type Ω and isomorphism classes of abelian varieties of type Ω.

From now on, given an $\mathbb{R}$-linear isomorphism $q : \mathbb{C}^g \to \mathbb{C}^g$, we write E_q to denote the Riemann form on $\mathbb{C}^g$ defined by

$$E_q(q(x), q(y)) = \operatorname{tr}_{K/\mathbb{Q}}(x\eta_n y^*).$$

Let

$$f : \mathcal{H}_n \to X$$

be an automorphic form of weight ω with respect to a congruence subgroup Γ of $U(\eta_n)$. Fix a $\mathbb{Z}$-lattice L in $\mathcal{K}^1_{2n}$. We now associate to the pair (f, L) a function f_L of lattices $\mathcal{L} \subset \mathbb{C}^g$ of type Ω, which we will later use to reformulate our definition of automorphic forms in terms of functions on abelian varieties of type Ω.

Theorem II.6. *Fix a CM type Ω. Let L be a $\mathbb{Z}$-lattice in $\mathbb{C}^g$, and let f be an automorphic form of weight ω with respect to a congruence subgroup Γ of $U(\eta_n)$ containing the group*

$$\Gamma' = \{\alpha \in U(\eta_n) | L\alpha = L \text{ and } u_i\alpha - u_i \in L \text{ for each } i\}. \tag{2.11}$$

There exists a unique function f_L of lattices $(\mathcal{L} \subset \mathbb{C}^g, E, \{t_i\}_{i=1}^s)$ of type Ω with $Nu_i \in \mathcal{L}$ for all i, such that for each $\alpha \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$,

$$f_L(\alpha \cdot (\mathcal{L}, E, \{t_i\}_{i=1}^s)) = \omega({}^t\alpha)^{-1} f_L(\mathcal{L}, E, \{t_i\}_{i=1}^s) \tag{2.12}$$

and such that

$$f_L(p_z(L), E_z, \{p_z(u_i)\}_{i=1}^s) = f(z) \tag{2.13}$$

for all z in $\mathcal{H}_n$.

Proof. Let $(\mathcal{L} \subset \mathbb{C}^g, E, \{t_i\}_{i=1}^s)$ be a lattice of type Ω such that $Nt_i \in \mathcal{L}$ for all i, and let q be as in (2.9) through (2.10). Then, as explained in the first paragraph of Theorem 4.8 of [Shi00], there is a diagonal matrix $S \in GL_{2n}(\mathbb{C})$ and an element $z \in \mathcal{H}_n$ such that

$$q = S \cdot p_z, \tag{2.14}$$

i.e. such that

$$(\mathcal{L} \subset \mathbb{C}^g, E, \{t_i\}_{i=1}^s) = S \cdot (p_z(L), E_{p_z}, p_z(u_i)).$$

Therefore, if the function f_L exists, then by (2.12) and (2.13), its value at $\mathcal{L}$ *must* be $\omega({}^tS)^{-1} f(z)$. So if the function f_L exists, it is unique.

Now we show that the function f_L exists. For this, it suffices to show that if there exist matrices S and T in $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ and elements z and w in $\mathcal{H}_n$ such that

$$(\mathcal{L} \subset \mathbb{C}^g, E, \{t_i\}_{i=1}^s) = S \cdot (p_z(L), E_{p_z}, p_z(u_i)) \tag{2.15}$$

and

$$(\mathcal{L} \subset \mathbb{C}^g, E, \{t_i\}_{i=1}^s) = T \cdot (p_w(L), E_{p_w}, p_w(u_i)), \tag{2.16}$$

then

$$\omega({}^tT)^{-1}f(w) = \omega({}^tS)^{-1}f(z). \tag{2.17}$$

For the remainder of the proof, suppose that both (2.15) and (2.16) hold. Then the abelian varieties of type Ω attached to $(p_z(L), E_{p_z}, p_z(u_i))$ and $(p_w(L), E_{p_w}, p_w(u_i))$ are both isomorphic to the abelian variety of type Ω attached to $(\mathcal{L} \subset \mathbb{C}^g, E, \{t_i\}_{i=1}^s)$. Therefore, by Theorem II.2,

$$w = \gamma z$$

for some

$$\gamma \in \Gamma' \subset \Gamma \subset U(\eta_n).$$

By line (4.31) of [Shi00],

$$p_z(x\alpha) = {}^tM(\alpha, z)p_{\alpha z}(x) \tag{2.18}$$

for all $x \in \mathbb{C}^g$ and $\alpha \in U(\eta_n)$. Since $L = L\alpha$ for all α in Γ', (2.18) shows that

$$p_z(L) = {}^tM(\gamma, z)p_w(L).$$

Since $u_i - \alpha u_i \in L$ for each α in Γ',

$$p_z(u_i\alpha) = p_z(u_i)$$

for each α in Γ'. Since $\gamma \in U(\eta_n)$,

$$\gamma\eta_n\gamma^* = \eta_n,$$

Hence, it immediately follows from the definition of E_z that

$$E_{p_z}(p_z(x\alpha), p_z(y\alpha)) = E_{p_z}(p_z(x), p_z(y)).$$

So

$$(p_z(L), E_{p_z}, p_z(u_i)) = {}^tM(\gamma, z) \cdot (p_w(L), E_{p_w}, p_w(u_i))$$

Therefore

$$T = S \cdot {}^tM(\gamma, z),$$

so the right hand side of (2.17) is equal to

$$\omega({}^tT)^{-1}\omega(M(\gamma, z))f(z) = \omega({}^tT)^{-1}f(\gamma z). \tag{2.19}$$

Since $w = \gamma z$, the right hand side of (2.19) is equal to the left hand side of (2.17). So Equation (2.17) holds, which proves the existence of the function f_L. ■

Having reformulated the definition of automorphic forms in terms of functions of lattices of type Ω, we now reformulate it again in terms of functions of complex abelian varieties of type Ω. Observe that giving an ordered basis $\{\omega_i^\pm\}_{i=1}^n$ of $\underline{\omega}^\pm$ is equivalent to giving an element of the module

$$\mathcal{E}_{\underline{A}}^\pm = \mathrm{Isom}_{\mathbb{C}}(\mathbb{C}^n, \underline{\omega}_{\underline{A}}^\pm).$$

(This equivalence is via $\langle e_i \mapsto \omega_i^\pm\rangle_{i=1}^n \leftrightarrow \{\omega_i^\pm\}_{i=1}^n$, with e_i standard basis vectors in $\mathbb{C}^n$.) The group $GL_n(\mathbb{C})$ acts on $\mathcal{E}_{\underline{A}}\pm$ via

$$(\alpha \cdot \lambda)(v) := \lambda({}^t\alpha \cdot v) \tag{2.20}$$

for all $v \in \mathbb{C}^n$ and $\alpha \in GL_n(\mathbb{C})$. We define $\mathcal{E}_{\underline{A}}$ by

$$\mathcal{E}_{\underline{A}} := \mathcal{E}_{\underline{A}}^- \oplus \mathcal{E}_{\underline{A}}^+.$$

Then the action of $GL_n(\mathbb{C})$ on $\mathcal{E}_{\underline{A}}^\pm$ given in (2.20) induces an action of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathcal{E}_{\underline{A}}$, and to give an element of $\mathcal{E}_{\underline{A}}$ is equivalent to specifying an ordered basis of $\underline{\omega}^+$ and an ordered basis of $\underline{\omega}^-$.

Let $\underline{\mathcal{L}}(\underline{A})$ be the lattice of type Ω attached to $\underline{A}$ as in Section 2.3.1. Then we see that for each $\alpha \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ and $\lambda \in \mathcal{E}_{\underline{A}}$,

$$\alpha \cdot \underline{\mathcal{L}} = \underline{\mathcal{L}}(\underline{A}, \alpha \cdot \lambda). \tag{2.21}$$

Let (ω, V) be a finite-dimensional representation of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$, and let

$$f : \mathcal{H}_n \to V$$

be an automorphic form of weight ω with respect to some congruence subgroup Γ of $U(\eta_n)$ containing Γ' (with Γ' defined as in (2.11)). We define F_L to be the unique function from pairs $(\underline{A}, \lambda)^2$ (where λ is an element of $\mathcal{E}_{\underline{A}}$) to V satisfying both

$$F_L(\underline{A}, \alpha\lambda) = \omega({}^t\alpha)^{-1} F_L(\underline{A}, \lambda)$$

and

$$F_L(\underline{A}, \lambda) = f_L(\underline{\mathcal{L}}(\underline{A}, \lambda)).$$

Thus, an automorphic form f of weight ω on Γ corresponds to a function F from pairs $(\underline{A}, \lambda)$ to V satisfying

$$F(\underline{A}, \alpha\lambda) = \omega({}^t\alpha)^{-1} F(\underline{A}, \lambda). \tag{2.22}$$

Now we explain how to view functions F satisfying (2.22) as certain functions on abelian varieties $\underline{A}$ of type Ω.

Given a ring R and a group B that acts on R-modules V_1 and V_2, we define the contracted product $V_1 \times^B V_2$ to be $V_1 \oplus V_2$ modulo the relation $(v_1, v_2) \sim (bv_1, bv_2)$. Let $H^\omega := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ act on $\mathcal{E}_{\underline{A}}$ by the action induced by (2.20), and let H^ω act on V via

$$v \mapsto \omega({}^t h)^{-1} v.$$

[2] To avoid confusion with the notation λ for the polarization, the notation $(\underline{A}, \lambda)$ will always refer to an an abelian variety $\underline{A}/R$ in ${}_K\mathbb{A}_V(R)$ for some $\mathcal{O}_{\mathcal{K}}$-algebra R and an element $\lambda \in \mathcal{E}_{\underline{A}}$.

We define $\mathcal{E}_{\underline{A},V,\omega}$ to be the contracted product

$$\mathcal{E}_{\underline{A},V,\omega} := \mathcal{E}_{\underline{A}} \times^{H^\omega} V.$$

To give a function F from pairs $(\underline{A}, \lambda)$ to V satisfying (2.22) is equivalent to giving a function $\tilde{F}$ from abelian varieties $\underline{A}$ of type Ω to $\mathcal{E}_{\underline{A},V,\omega}$. This equivalence is via

$$\tilde{F}(A) = (\lambda, F(\underline{A}, \lambda)).$$

Letting A^{an}_{univ} be the universal family of abelian varieties of type Ω over $\Gamma\backslash\mathcal{H}_n$, we see that giving a holomorphic automorphic form f is equivalent to giving a section of the $\mathcal{O}^{\mathrm{hol}}_{\mathcal{H}_n}$-module

$$\mathcal{E}^{\mathrm{an}}_{V,\omega} := \mathcal{E}_{A^{\mathrm{an}}_{univ},V,\omega} \otimes \mathcal{O}^{\mathrm{hol}}_{\mathcal{H}_n}.$$

Similarly, giving a C^∞-automorphic form f is equivalent to giving a section of the $\mathcal{O}_{\mathcal{H}_n}(C^\infty)$-module

$$\mathcal{E}_{V,\omega}(C^\infty) := \mathcal{E}_{A^{\mathrm{an}}_{univ},V,\omega} \otimes \mathcal{O}_{\mathcal{H}_n}(C^\infty).$$

2.5 Algebraic geometric approach to automorphic forms on unitary groups

Having discussed several equivalent definitions of automorphic forms from an analytic perspective, we now approach automorphic forms from an algebraic geometric perspective. As the reader will see, over $\mathbb{C}$, we recover our earlier definition of automorphic forms from the algebraic geometric definition. Our approach here is similar to the one taken in Section 1.2 of [Kat78].

Fix an $\mathcal{O}_{\mathcal{K}}$-algebra R_0. Let V be an R_0-module. For any R_0-algebra R, we denote by V_R the R-module $V \otimes_{R_0} R$ obtained by extension of scalars $R_0 \to R$. Let (ρ, V) be an algebraic representation of $GL_n \times GL_n$ that is defined over R_0. That is, for each R_0-algebra R, ρ defines a homomorphism

$$\rho_R : GL_n(R) \times GL_n(R) \to GL(V_R)$$

that commutes with extension of scalars $R \to R'$ of R_0-algebras.

Fix a compact open subgroup K of $G = GU(n,n)$. We denote by Γ the congruence subgroup $G(\mathbb{Q}) \cap K$. For each abelian variety $\underline{A} = (A, \lambda, \iota, \alpha)$ in ${}_K\mathbb{A}_V(R)$ over an R_0-algebra R, we now define modules $\mathcal{E}^+_{\underline{A}/R}$, $\mathcal{E}^-_{\underline{A}/R}$, and $\mathcal{E}_{\underline{A}/R}$. Similarly to in Section 2.4, we define

$$\mathcal{E}^{\pm}_{\underline{A}/R} = \mathrm{Isom}_R(R^n, \underline{\omega}^{\pm}_{A/R})$$
$$\mathcal{E}_{\underline{A}/R} = \mathcal{E}^-_{\underline{A}/R} \oplus \mathcal{E}^+_{\underline{A}/R}.$$

To give an element of $\lambda \in \mathcal{E}^{\pm}_{\underline{A}/R}$ is equivalent to specifying an ordered basis $\omega_1, \ldots, \omega_n$ of $\underline{\omega}^{\pm}_{A/R}$; the equivalence is via

$$\lambda \in \mathcal{E}^{\pm}_{\underline{A}/R} \leftrightarrow \lambda(e_1), \ldots, \lambda(e_n) \in \underline{\omega}^{\pm}_{A/R}.$$

So to giving element of $\mathcal{E}_{\underline{A}/R}$ is equivalent to specifying an ordered basis of $\underline{\omega}^-$ and an ordered basis of $\underline{\omega}^+$. The group $GL_n(R)$ acts on $\mathcal{E}^{\pm}_{\underline{A}/R}$ via

$$(\alpha \cdot \lambda)(v) := \lambda({}^t\alpha v). \tag{2.23}$$

The action of $GL_n(R)$ given in (2.23) induces an action of $GL_n(R) \times GL_n(R)$ on $\mathcal{E}_{\underline{A}/R}$.

Let $H^\rho = GL_n(R) \times GL_n(R)$ act on V_R via $v \mapsto \rho({}^t\alpha)^{-1}v$ and act on $\mathcal{E}_{\underline{A}/R}$ through the action induced by (2.23). Similarly to in Section 2.4, we denote by $\mathcal{E}_{(\underline{A},V,\rho)/R}$ the R-module

$$\mathcal{E}_{\underline{A}/R} \times^{H^\rho} V.$$

Observe that formation of $\mathcal{E}_{(\underline{A},V,\rho)/R}$ commutes with extension of scalars $R \to R'$ of R_0-algebras.

Definition II.7. An automorphic form of weight ρ, defined over R_0, is a function f from the set of pairs $(\underline{A}, \lambda)$, consisting of $\underline{A}$ in ${}_KA_V(R)$ over an R_0-algebra R and an element λ in $\mathcal{E}_{\underline{A}/R}$, to V_R such that all of the following hold:

1. The element $f(\underline{A}, \lambda)$ depends only on the R-isomorphism class of $(\underline{A}, \lambda)$.
2. The formation of $f(\underline{A}, \lambda) \in V_R$ commutes with extension of scalars $R \to R'$ of R_0-algebras, i.e.
$$f(\underline{A} \times_{\mathrm{Spec}R} R', \lambda \otimes_R R') = f(\underline{A}, \lambda) \otimes_R 1 \in V \otimes_R R'$$
.
3. For each $(\underline{A}, \lambda)$ over R and $\alpha \in H^\rho(R)$,
$$f(\underline{A}, \alpha\lambda) = \rho({}^t\alpha)^{-1} f(\underline{A}, \lambda).$$

We write $M_\rho(R_0)$ to denote the R_0-module of automorphic forms of weight ρ defined over R.

Similarly to in Section 2.4, we now give an equivalent definition of automorphic forms of weight ρ.

Definition II.8. An automorphic form of weight ρ defined over R_0, is a rule $\tilde{f}$ that assigns to each $\underline{A}$ in ${}_KA_V(R)$ over an R_0-algebra R an element of $\mathcal{E}_{(\underline{A},V,\rho)/R}$ such that both of the following conditions hold:

1. The element$\tilde{f}(\underline{A})$ in $\mathcal{E}_{(\underline{A},V,\rho)/R}$ depends only on the R-isomorphism class of $\underline{A}$.
2. The formation of $\tilde{f}(\underline{A})$ commutes with extension of scalars $R \to R'$ of R_0-algebras, i.e.
$$\tilde{f}(\underline{A} \times_{\mathrm{Spec}R} \mathrm{Spec}R') = \tilde{f}(\underline{A}) \otimes_R 1 \in \mathcal{E}_{(\underline{A},V,\rho)/R} \otimes_R R'.$$

The equivalence of these Definition II.7 with Definition II.8 is through

$$\tilde{f}(\underline{A}) = (\lambda, f(\underline{A}, \lambda)).$$

The perspective of Definition II.8 leads us to another (equivalent) formulation of the definition of automorphic forms in the case where ${}_K A_V(R)$ is representable (i.e. in the case where K is sufficiently small). In this case, consider the scheme

$$\begin{array}{c} \mathcal{M} := \mathcal{M}_R(K) :={}_K Sh(V) \times_{\mathcal{O}_\mathcal{K}} R \\ \downarrow \\ \mathrm{Spec} R \end{array}$$

We denote by $\mathcal{E}^\pm$ the locally free sheaf

$$\mathcal{E}^\pm = \underline{\mathrm{Isom}}_{\mathcal{O}_\mathcal{M}}(\mathcal{O}^n_\mathcal{M}, \underline{\omega}^\pm)$$

of $\mathcal{O}_\mathcal{M}$-modules on $\mathcal{M}$, and we denote by $\mathcal{E}$ the locally free sheaf

$$\mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$$

of $\mathcal{O}_\mathcal{M}$-modules on $\mathcal{M}$. We denote by $\mathcal{E}_{V,\rho}$, the locally free sheaf $\mathcal{E} \times^{H^\rho} V$. Then an automorphic form of weight ρ is a global section of the sheaf $\mathcal{E}_{V,\rho}$ on $\mathcal{M}_R(K)$. Note that for any representation (ρ, V) that can be decomposed as a direct sum $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$, the map

$$\mathcal{E}_{V,\rho} \longrightarrow \mathcal{E}_{V_1,\rho_1} \oplus \mathcal{E}_{V_2,\rho_2} \tag{2.24}$$
$$(\lambda, v) \mapsto ((\lambda, v_1), (\lambda, v_2))$$

is an isomorphism. (Its inverse is $((\lambda, v_1), (\alpha\lambda, \alpha v_2)) \mapsto (\alpha\lambda, \alpha(v_1, v_2))$.) Therefore, to give an automorphic form of weight ρ is equivalent to giving an automorphic form of weight ρ_1 and an automorphic form of weight ρ_2.

Since GL_n is reductive, each finite dimensional representation ρ can be written as a direct sum of irreducible representations

$$\rho = \rho_1 \oplus \cdots \oplus \rho_m$$

for some m. Every irreducible representation of GL_n can be realized as a subrepresentation of one of the representations constructed as follows.[3] For each set Λ of ordered integers $\lambda_1 \geq \ldots \geq \lambda_n$, there is a representation $(\rho_\Lambda, V_\Lambda)$ of highest weight Λ. The representation $(\rho_\Lambda, V_\Lambda)$ can be realized explicitly by taking

$$V_\Lambda = \mathrm{Sym}^{(\lambda_1-\lambda_2)}(R^n) \otimes \mathrm{Sym}^{(\lambda_2-\lambda_3)}(\wedge^2 R^n) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n}(\wedge^n R^n),$$

and letting ρ_Λ be the GL_n-action on V_Λ induced by the standard representation of $GL_n(R)$ on R^n. If λ_n is negative, then by $\mathrm{Sym}^{\lambda_n}(\wedge^n R^n)$, we mean the dual representation of $\mathrm{Sym}^{-\lambda_n}(\wedge^n R^n)$, which is just the representation in which each $g \in GL_n$ acts on each $v \in R$ by $v \mapsto \det g^{\lambda_n} v$. (Note that the highest weight vector in V_Λ is $(e_1)^{(\lambda_1-\lambda_2)} \otimes (e_1 \wedge e_2)^{(\lambda_2-\lambda_3)} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_n)^{\lambda_n}$.) Every irreducible representation of $GL_n \times GL_n$ is of the form $\rho^- \otimes \rho^+$ with $\rho^\pm$ irreducible representations of GL_n.

Let W be a free rank n R-module. We write W^{ρ_Λ} to denote $\mathrm{Sym}^{(\lambda_1-\lambda_2)}(W) \otimes \mathrm{Sym}^{(\lambda_2-\lambda_3)}(\wedge^2 W) \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n}(\wedge^n W)$. When ρ is an arbitrary representation whose decomposition into irreducible representations is $\rho_{\Lambda_1} \oplus \cdots \oplus \rho_{\Lambda_m}$, we denote by W^ρ the module $W^{\rho_{\Lambda_1}} \oplus \cdots \oplus W^{\rho_{\Lambda_m}}$. Given another free-module W_0 of rank n and a representation $\rho = \rho_{\Lambda_1} \otimes \rho_{\Lambda_2}$, we write $(W \otimes W_0)^{\rho_{\Lambda_1} \otimes \rho_{\Lambda_2}}$ to denote the module $W^{\rho_{\Lambda_1}} \otimes W_0^{\rho_{\Lambda_2}}$. Given an arbitrary representation ρ whose decomposition into irreducible representations is $\rho_1 \oplus \cdots \oplus \rho_m$, we write $(W \otimes W_0)^\rho$ to denote the module $W^{\rho_1} \oplus \cdots \oplus W^{\rho_m}$.

[3] To see that not all of these representations are irreducible, observe that the representation of GL_3 corresponding to $(2,1,0)$ is the representation $st \otimes (st^\vee \otimes \det)$, and $st \otimes st^\vee = Ad^0 \oplus \mathbb{1}$.

Let Λ be an ordered set of integers $\lambda_1 \geq \ldots \geq \lambda_n$, corresponding to the representation of GL_n of highest weight Λ. Each $\lambda^{\pm} \in \mathcal{E}^{\pm}$ induces an isomorphism

$$\lambda^{\pm,\rho_\Lambda} : V_\Lambda = (R^n)^{\rho_\Lambda} \to (\underline{\omega}^{\pm})^{\rho_\Lambda}$$

defined by

$$v_1 \cdot v_2 \cdot \cdots \cdot v_m \mapsto \lambda(v_1) \cdot \lambda(v_2) \cdot \cdots \cdot \lambda(v_m)$$

where each v_i is in R^n and each $\cdot$ denotes the symmetric, tensor, or alterating product (according to V_Λ). Observe that

$$(\alpha \cdot \lambda)^{\pm\rho_\Lambda} = \lambda^{\rho_\Lambda}(\rho_\Lambda({}^t\alpha)v).$$

So given a representation $\rho_{\Lambda_-} \otimes \rho_{\Lambda_+}$ of $GL_n \times GL_n$, each isomorphism $(\lambda^-, \lambda^+) \in \mathcal{E} = \mathcal{E}^- \oplus \mathcal{E}^+$ induces an isomorphism

$$\lambda^{\rho_{\Lambda_-} \otimes \rho_{\Lambda_+}} : V_{\rho_{\Lambda_-}} \otimes V_{\rho_{\Lambda_+}} = (R^n)^{\rho_{\Lambda_-}} \otimes (R^n)^{\rho_{\Lambda_+}} \xrightarrow{\sim} (\underline{\omega}^-)^{\rho_{\Lambda_-}} \otimes (\underline{\omega}^+)^{\rho_{\Lambda_+}}$$

via

$$v^- \otimes v^+ \mapsto \lambda^{\rho_{\Lambda_-}}(v^-) \otimes \lambda^{\rho_{\Lambda_+}}(v^+).$$

Observe that

$$(\alpha \cdot \lambda)^{\rho_{\Lambda_-} \otimes \rho_{\Lambda_+}}(v) = \lambda^{\rho_{\Lambda_-} \otimes \rho_{\Lambda_+}}(\rho_{\Lambda_-} \otimes \rho_{\Lambda_+}({}^t\alpha)v).$$

Therefore, there is an isomorphism

$$\mathcal{E}_{V_{\Lambda_-} \otimes V_{\Lambda_+}, \rho_{\Lambda_-} \otimes \rho_{\Lambda_+}} \xrightarrow{\sim} (\underline{\omega}^-)^{\rho_{\Lambda_-}} \otimes (\underline{\omega}^+)^{\rho_{\Lambda_+}}, \tag{2.25}$$

defined by

$$(\lambda, v) \mapsto \lambda^{\rho_{\Lambda_-} \otimes \rho_{\Lambda_+}}(v).$$

Thus, at least in the case in which K is sufficiently small (i.e. when the moduli problem ${}_K A_V$ is representable), automorphic forms of weight $\tilde{\rho}$ (with $\tilde{\rho}$ a subrepresentation of $\rho = \rho_{\Lambda_-} \otimes \rho_{\Lambda_+}$) are sections of $(\underline{\omega}^- \otimes \underline{\omega}^+)^\rho$. This last perspective (i.e. viewing automorphic forms as sections of $(\underline{\omega}^- \otimes \underline{\omega}^+)^\rho$ will be particularly useful to us when defining the differential operators.

When working over $\mathbb{C}$, the following theorem, which relates algebraic automorphic forms to holomorphic ones, is useful.

Theorem II.9. *If $n > 1$, then $f \mapsto f^{\mathrm{an}}$ gives an isomorphism*

$$M^{alg}(\mathbb{C})(\rho, \Gamma) \to M^{\mathrm{an}}(\rho, \Gamma).$$

This fact involves the existence of toroidal compactifications and the analytic Koecher principle. The reader may see [FC90] for additional details.

2.6 C^∞-automorphic forms

For any sheaf $\mathcal{F}$ on $\mathcal{M}$, we let $\mathcal{F}(C^\infty)$ be the sheaf obtained by tensoring $\mathcal{F}$ with the C^∞-structural sheaf of $\mathcal{M}^{\mathrm{an}}$.

Note that we have inclusions (analogous to (1.8.1) of [Kat78])

$$M^{alg} = H^0(\mathcal{M}_\mathbb{C}, \mathcal{E}_{V,\rho}) \subset M^{\mathrm{hol}} = H^0(\mathcal{M}^{\mathrm{an}}, \mathcal{E}_{V,\rho} \otimes \mathcal{O}^{\mathrm{hol}}_{\mathcal{M}}) \subset M^{C^\infty} = H^0(\mathcal{M}, \mathcal{E}_{V,\rho} \otimes \mathcal{O}^{C^\infty}_{\mathcal{M}})$$

Over $\mathcal{M}^{\mathrm{an}}$, we have the Hodge decomposition

$$H^1_{DR}(C^\infty) = \underline{\omega}(C^\infty) \oplus \overline{\underline{\omega}(C^\infty)} \tag{2.26}$$

of H^1_{DR} into the sheaf $\underline{\omega}(C^\infty)$ of holomorphic differentials and the sheaf $\overline{\underline{\omega}(C^\infty)}$ of anti-holomorphic differentials. The fiber $H^1_{DR}(C^\infty)_z$ of $H^1_{DR}(C^\infty)$ over a point $z \in \mathcal{H}_n$ is the de Rham cohomology of that fiber. The splitting (2.26) induces the Hodge decomposition of $H^1_{DR}(C^\infty)$ at each fiber, i.e. the splitting into holomorphic and antiholomorphic submodules.

CHAPTER III

The Gauss-Manin connection and the Kodaira-Spencer isomorphism

In this chapter, we review the Gauss-Manin connection and the Kodaira-Spencer morphism, two maps that are important for the construction of the differential operators. We follow the construction of each map with an explicit example in terms of coordinates over $\mathbb{C}$.

Throughout this section, let $\pi : X \to S$ and $\pi' : Y \to S$ be smooth, proper morphisms of schemes, and suppose that S is a smooth scheme over a scheme T. (For our upcoming construction of the C^∞-operators, it will be useful to keep in mind the case where $T = Spec(\mathbb{C})$, $S = \mathcal{M}$, and $X = A_{\mathrm{univ}}$.) As noted in Section 2.6, we move freely between the algebraic and analytic perspectives.

3.1 The Gauss-Manin connection

In this section, we briefly review the construction of the Gauss-Manin connection ([Kat70], [KO68], [kan], [Ked08]), from which the differential operators will be constructed. Consider the decreasing filtration of $(\Omega^\bullet_{X/T}, d)$ defined by

$$\begin{aligned} F^i &= Fil^i(\Omega^\bullet_{X/T}) \\ &= \mathrm{Image}(\pi^*\Omega^i_{S/T} \otimes_{\mathcal{O}_X} \Omega^{\bullet-i}_{X/T} \to \Omega^\bullet_{X/T}), \end{aligned} \tag{3.1}$$

where the morphism in (3.1) is the canonical one. The associated graded complex is $Gr(\Omega^\bullet) = \oplus_{p\geq 0} Gr^p$, with $Gr^p = F^p/F^{p+1}$. As explained in [KO68] and [Kat70], there is a spectral sequence (which converges to $\mathbb{R}^{p+q}\pi_*(\Omega^\bullet_{X/T}) = H^{p+q}_{DR}(X/S)$) with E_1 term given by

$$E_1^{p,q} = \mathbb{R}^q\pi_*(Gr^p).$$

The Gauss-Manin connection is the differential

$$d_1 : E_1^{0,q} \to E^{1,q}.$$

We denote the Gauss-Manin connection by ∇.

Observe that

$$Gr^i \cong \Omega^{\bullet - i}_{X/S} \otimes_{\mathcal{O}_X} \pi^*\Omega^i_{S/T}$$

for all i. In particular, we see that

$$\begin{aligned} E_1^{0,q} &\cong H^q_{DR}(X/S) \\ E_1^{1,q} &\cong H^q_{DR}(X/S) \otimes_{\mathcal{O}_S} \Omega^1_{S/T}. \end{aligned}$$

So the Gauss-Manin connection is the map

$$\nabla = d_1 : H^q_{DR}(X/S) \to H^q_{DR}(X/S) \otimes_{\mathcal{O}_S} \Omega^1_{S/T}.$$

We will always take $q = 1$ when applying the Gauss-Manin connection.

Observe that by construction of ∇, if f is an endomorphism of X over S, then

$$\nabla(f^*(v)) = (f^* \otimes \mathrm{Id})(\nabla(v)) \tag{3.2}$$

for each v in $H^1_{DR}(X/S)$. As a consequence of (3.2), we see that if A is an abelian variety of type (2.1) over an $\mathcal{O}_\mathcal{K}$-scheme S and v is in $H^1_{DR}(A/S)^+$, then

$$(\iota(a)^* \otimes \mathrm{Id})(\nabla(v)) = \nabla(\iota(a)^* v) = \nabla(a \cdot v) = a\nabla(v) = (a \otimes \mathrm{Id})\nabla(v),$$

so

$$\nabla \left(H^1_{DR}(A/S)^+ \right) \subseteq H^1_{DR}(A/S)^+ \otimes \Omega \tag{3.3}$$

Similarly,

$$\nabla \left(H^1_{DR}(A/S)^- \right) \subseteq H^1_{DR}(A/S)^- \otimes \Omega. \tag{3.4}$$

3.1.1 An important example

If we trace through the map given by the Gauss-Manin connection, we see that it involves lifting a relative form to an absolute form, differentiating the absolute form, and then projecting down to an element of $H^1_{DR}(X/S) \otimes \Omega^1_{S/T}$. This idea is best made clear in an example over $\mathbb{C}$, which we will now provide. This example not only explicitly illustrates how the Gauss-Manin connection acts in one of the main cases that interests us (the other example being over a p-adic base); it also will be useful later when we explicitly describe the action of our C^∞-differential operators and when we relate our C^∞-operators to the ones in [Shi00]. This example is strongly inspired by sections 4.0-4.2 of [Har81], and the construction here is directly analogous to and closely follows what Harris does for symplectic modular forms. Our example is the $U(n,n)$ analogue of the example for symplectic groups in sections 4.0-4.2 of [Har81].

As noted in Section II, we will often work over the underlying C^∞-manifold of our moduli space. We take this approach right now. In our example, we will consider A_{univ} over $\mathcal{H}_n$ over $\mathbb{C}$, as in Section 2.3. The sheaf $H^1_{DR}(C^\infty)$ has a splitting

$$H^1_{DR}(C^\infty) \cong \underline{\omega}(C^\infty) \oplus \mathrm{Split}(C^\infty), \tag{3.5}$$

where $\underline{\omega}(C^\infty)$ is the space of holomorphic one-forms (which is the C^∞ vector bundle corresponding to the sheaf of relative one-forms $\underline{\omega} = \pi_* \Omega_{A_{\text{univ}}/\mathcal{H}_n}$) and $\mathrm{Split}(C^\infty)$

is the space of anti-holomorphic one-forms. Also, recall that the fiber $H^1_{DR}(C^\infty)_z$ of $H^1_{DR}(C^\infty)$ over a point $z \in \mathcal{H}_n$ is the de Rham cohomology of that fiber (i.e. A_z, in the notation of Section 2.3) and that the splitting (3.5) induces the Hodge decomposition of $H^1_{DR}(C^\infty)$ at each fiber.

Let $u_1, \ldots, u_{2n}$ denote standard coordinates in $\mathbb{C}^{2n}$. Then the global relative 1-forms $du_1, \ldots, du_{2n}$ form a basis of the fiber of $\underline{\omega}$ over each point $z \in H$.

We now define some global relative 1-forms that have constant periods across the fibers of $A_{\text{univ}}/\mathcal{H}_n$. We define them to be dual to the one-cycles (in homology) defined in terms of the basis for L_z given in Equations (2.4-2.7). We consider the $\mathbb{R}$-linear global relative one forms (for $i = 1, \ldots, n$) which are given over

$$z = (z_{ij}) \text{ in } \mathcal{H}_n$$

by

$$\alpha_i\left(\sum_{j=1}^n a_je_j + \sum_{j=1}^n b_jz_j + \sum_{j=1}^n a'_je'_j + \sum_{j=1}^n b'_jz'_j\right) = a_i \tag{3.6}$$

$$\beta_i\left(\sum_{j=1}^n a_je_j + \sum_{j=1}^n b_jz_j + \sum_{j=1}^n a'_je'_j + \sum_{j=1}^n b'_jz'_j\right) = b_i \tag{3.7}$$

$$\alpha'_i\left(\sum_{j=1}^n a_je_j + \sum_{j=1}^n b_jz_j + \sum_{j=1}^n a'_je'_j + \sum_{j=1}^n b'_jz'_j\right) = a'_i \tag{3.8}$$

$$\beta'_i\left(\sum_{j=1}^n a_je_j + \sum_{j=1}^n b_jz_j + \sum_{j=1}^n a'_je'_j + \sum_{j=1}^n b'_jz'_j\right) = b'_i, \tag{3.9}$$

for each a_i, b_i, a'_i, b'_i in $\mathbb{R}$. (Here we are using the notation for the basis of L_z given in Equations (2.4-2.7).)

Observe that these forms have constant periods along the fibers. Therefore,

$$\nabla(\alpha_i) = \nabla(\beta_i) = \nabla(\alpha'_i) = \nabla(\beta'_i) = 0 \quad \text{for } i = 1, \ldots, n. \tag{3.10}$$

That is, the sections $\alpha_i, \beta_i, \alpha'_i, \beta'_i$ are *horizontal*.

Now we express $du_1, \ldots, du_{2n}$ and $d\bar{u}_1, \ldots, d\bar{u}_{2n}$ in terms of $\alpha_i, \beta_i, \alpha_i', \beta_i'$. Using the definitions of e_i, z_i, e_i', z_i' (as in Equations (2.4-2.7)) and $\alpha_i, \beta_i, \alpha_i', \beta_i'$ (as above), we see that for $i = 1, \ldots, n$,

$$du_i = \alpha_i + \sum_{j=1}^{n} z_{ij}\beta_j + \bar{\alpha}\alpha_i' + \bar{\alpha}\sum_{j=1}^{n} z_{ij}\beta_j' \tag{3.11}$$

$$d\bar{u}_i = \alpha_i + \sum_{j=1}^{n} \bar{z}_{ij}\beta_j + \alpha\alpha_i' + \alpha\sum_{j=1}^{n} \bar{z}_{ij}\beta_j'; \tag{3.12}$$

and for $i = n+1, \ldots, 2n$, we have

$$du_i = \alpha_i + \sum_{j=1}^{n} z_{j,i-n}\beta_j + \alpha\alpha_i' + \alpha\sum_{j=1}^{n} z_{j,i-n}\beta_j' \tag{3.13}$$

$$d\bar{u}_i = \alpha_i + \sum_{j=1}^{n} \bar{z}_{j,i-n}\beta_j + \bar{\alpha}\alpha_i' + \bar{\alpha}\sum_{j=1}^{n} \bar{z}_{j,i-n}\beta_j' \tag{3.14}$$

Since ∇ is a connection, it satisfies:

$$\nabla(a \cdot v) = a\nabla(v) + v \otimes da$$

for any $a \in \mathcal{O}_{\mathcal{H}_n}$ and any $v \in H^1_{DR}$. This, combined with Equations (3.11) - (3.14) shows that for $i = 1, \ldots, n$,

$$\begin{aligned} \nabla(du_i) &= \sum_{j=1}^{n} \beta_j \otimes dz_{ij} + \bar{\alpha}\sum_{j=1}^{n} \beta_j' \otimes dz_{ij} \\ &= \sum_{j=1}^{n} \left(\beta_j + \bar{\alpha}\beta_j'\right) \otimes dz_{ij} \end{aligned} \tag{3.15}$$

$$\begin{aligned} \nabla(du_{i+n}) &= \sum_{j=1}^{n} \beta_j \otimes dz_{ji} + \alpha\sum_{j=1}^{n} \beta_j' \otimes dz_{ji} \\ &= \sum_{j=1}^{n} \left(\beta_j + \alpha\beta_j'\right) \otimes dz_{ji} \end{aligned} \tag{3.16}$$

One can similarly compute $\nabla\left(d\bar{u}_i\right)$. For our purposes, however, we will only be interested in the application of ∇ to the holomorphic differentials. (This is because automorphic forms are associated with the *holomorphic* differentials.)

From Equations (3.11) - (3.14), we see that

$$(3.17)\qquad \begin{pmatrix} \beta_1 + \alpha\beta_1' \\ \vdots \\ \beta_n + \alpha\beta_n' \end{pmatrix} = (z^t - \bar{z})^{-1} \begin{pmatrix} du_{n+1} - d\bar{u}_1 \\ \vdots \\ du_{2n} - d\bar{u}_n \end{pmatrix}$$

$$(3.18)\qquad \begin{pmatrix} \beta_1 + \bar{\alpha}\beta_1' \\ \vdots \\ \beta_n + \bar{\alpha}\beta_n' \end{pmatrix} = (z - z^*)^{-1} \begin{pmatrix} du_1 - d\bar{u}_{n+1} \\ \vdots \\ du_n - d\bar{u}_{2n} \end{pmatrix}$$

Remark III.1. We revisit (3.2) in the context of our example. Consider an endomorphism $f : A_{\text{univ}} \to A_{\text{univ}}$ over $\mathcal{H}_n$. Then $f(L_z) \subset L_z$, and so f^* maps the horizontal sections $\alpha_i, \beta_i, \alpha_i', \beta_i'$ to horizontal sections. (i.e. Forms with constant periods are mapped to forms with constant periods.) Let v be an element of H^1_{DR}. Then we can write $v = \sum f_i\gamma_i$ for some horizontal sections γ_i and some sections $f_i \in \mathcal{O}_{\mathcal{H}_n}$. Since f is a morphism over $\mathcal{H}_n$, we see that $f^*(v) = \sum f_i f^*\gamma_i$. So

$$\nabla (f^*(v)) = f^*\gamma_i \otimes df_i = (f^* \otimes \mathrm{Id})\, \nabla \left(\sum f_i\gamma_i\right) = (f^* \otimes \mathrm{Id})\, \nabla (v)$$

This completes our example (for now) regarding the action of the Gauss-Manin connection. This perspective will be useful again when we define the C^∞-differential operators.

3.1.2 Remark about some related connections

From the Gauss-Manin connection, we can construct connections on $(H^1_{DR})^{\otimes m}$, $\wedge H^1_{DR}$, and $\mathrm{Sym} H^1_{DR}$ through the product rule. For example, for any v and w in

H^1_{DR}, we set

$$\nabla (v \otimes w) = \sigma (\nabla (v) \otimes w) + v \otimes \nabla (w), \tag{3.19}$$

where σ is the canonical isomorphism switching the order of the last two components of the tensor product:

$$\sigma : H^1_{DR} \otimes \Omega \otimes H^1_{DR} \xrightarrow{\sim} H^1_{DR} \otimes H^1_{DR} \otimes \Omega$$

$$v_1 \otimes v_2 \otimes v_3 \mapsto v_1 \otimes v_3 \otimes v_2$$

We similarly define ∇ on higher tensor powers of H^1_{DR} inductively.

Connections constructed from the Gauss-Manin connection through the product rule play a role in the construction of the differential operators in later sections.

3.2 The Kodaira-Spencer isomorphism

We now briefly review the Kodaira-Spencer isomorphism, which will be essential in our construction of the differential operators. For some more details (from the perspective of deformations), see [FC90]. For a much more thorough treatment (also from the perspective of deformations), see [Lan08].

From here on, we will restrict our discussion to the case where X is an abelian scheme. In our construction of the differential operators, we need to apply the Gauss-Manin connection iteratively. For this, we need to relate $\Omega_{S/T}$ to a submodule of $H^1_{DR}(X/S)$. (We will relate $\Omega_{X/S}$ to $\Omega_{S/T}$.) We do this via the Kodaira-Spencer morphism.

Let

$$\omega_X := \pi_* \Omega_{X/S}$$

$$\omega_{X^\vee} := \pi_* \Omega_{X^\vee/S},$$

where $X^\vee$ denotes the dual of X.

Hypercohomology gives a canonical exact sequence

$$0 \to \omega_X \hookrightarrow H^1_{DR}(X/S) \to R^1\pi_*(\mathcal{O}_X) \tag{3.20}$$

We then have canonical isomorphisms

$$\begin{aligned} H^1_{DR}(X/S)/(\pi_*\Omega_{X/S}) &\xrightarrow{\sim} R^1\pi_*(\mathcal{O}_X) \\ &\xrightarrow{\sim} (\omega_{X^\vee})^\vee, \end{aligned}$$

Define

$$KS' : \omega_X \to (\omega_{X^\vee})^\vee \otimes \Omega_{S/T} \tag{3.21}$$

to be the composition of canonical maps

$$\omega \hookrightarrow H^1_{DR}(X/S) \xrightarrow{\nabla} H^1_{DR}(X/S) \otimes \Omega_{S/T} \twoheadrightarrow R^1\pi_*(\mathcal{O}_X) \otimes \Omega_{S/T} \xrightarrow{\sim} (\omega_{X^\vee})^\vee \otimes \Omega_{S/T}. \tag{3.22}$$

Tensoring each side with of 3.21 with $\omega_{X^\vee}$, we obtain a morphism

$$KS : \omega_X \otimes \omega_{X^\vee} \to \Omega_{S/T},$$

making the Diagram (3.23) commute.

$$\begin{tikzcd} \omega_X \otimes \omega_{X^\vee} \arrow[rr, "KS"] \arrow[dr, "KS'\otimes \mathrm{Id}"'] & & \Omega_{S/T} \\ & (\omega_{X^\vee})^\vee \otimes \Omega_{S/T} \otimes \omega_{X^\vee} \arrow[ur, "f\otimes g\otimes h \mapsto f(h)\otimes g"'] & \end{tikzcd} \tag{3.23}$$

In Subsection 3.2.1, we explicitly describe the Kodaira-Spencer morphism in coordinates in an example over $\mathbb{C}$. In our example, we will be able to explicitly give the kernel of KS. For more general cases, the kernel of KS is provided in [Lan08]; we provide the relevant result from [Lan08] in Subsection 3.3. However, while the abstract result from [Lan08] is useful, it is also important (for our particular situation) to keep in mind the example in coordinates that we work out in Subsection 3.2.1.

3.2.1 Useful Example over $\mathbb{C}$

We now discuss the Kodaira-Spencer isomorphism in in detail in coordinates over $\mathbb{C}$, in order to provide the reader with a more explicit understanding. In this example, like in the example in Subsection 3.1.1, we will consider A_{univ} over $\mathcal{H}_n$ over $\mathbb{C}$. To describe KS, it will be helpful first to describe the polarization (for z in $\mathcal{H}_n$)

$$\lambda := \lambda_z : A_z \to A_z^\vee$$

following section 3.3 of [Shi98] and section 4 of [Shi00].

Define $\langle , \rangle$ to be the non-degenerate symmetric $\mathbb{R}$-bilinear pairing on $\mathbb{C}^{2n}$ defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i + \bar{x}_i y_i$$

for all vectors $x = (x_i)$ and $y = (y_i)$ in $\mathbb{C}^{2n}$. Let $z_j^*, e_j^*, z_j'^*, e_j'^*$[1] denote the elements of $\mathbb{C}^{2n}$ such that for any vector $v \in \mathbb{C}^{2n}$

$$\begin{aligned}
\langle v, z_j^* \rangle &= \beta_j(v) \\
\langle v, z_j'^* \rangle &= \beta_j'(v) \\
\langle v, e_j^* \rangle &= \alpha_j(v) \\
\langle v, e_j'^* \rangle &= \alpha_j'(v),
\end{aligned}$$

where $\beta_j, \alpha_j, \beta_j', \alpha_j'$ are defined as in Equations (3.6) through (3.9).

Let $(,)$ be the pairing on $\mathbb{C}^{2n}$ defined by

$$(x, y) = \sum_{i=1}^{n} \bar{x}_i y_i$$

[1] We remark that the notation $*$ here has a different meaning from in Section 2.3. We use this notation in both places in order to be consistent with Shimura's notation in [Shi00] and [Shi98], since he also uses this notation in both ways. It should be clear from context in this paper which meaning $*$ has.

for all vectors $x = (x_i)$ and $y = (y_i)$ in $\mathbb{C}^{2n}$. Then we can write $\alpha_i, \beta_i, \alpha'_i, \beta'_i$ as the sum of its $\mathbb{C}$-anti-linear and $\mathbb{C}$-linear pieces as

$$\begin{aligned}
\alpha_i(\bullet) &= (\bullet, e_j^*) + (e_j^*, \bullet) \\
\beta_i(\bullet) &= (\bullet, z_j^*) + (z_j^*, \bullet) \\
\alpha'_i(\bullet) &= (\bullet, e_j'^*) + (e_j'^*, \bullet) \\
\beta'_i(\bullet) &= (\bullet, z_j'^*) + (z_j'^*, \bullet)
\end{aligned}$$

Consider the Riemann form E_z on $\mathbb{C}^{2n}$ defined on the lattice L_z by

$$E_z(p_z(x), p_z(y)) := \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(x\eta_n y^*)$$

for all $x, y \in \mathcal{O}_{\mathcal{K}}^{2n}$, where $p_z(\bullet)$ is defined as in section 2.3.

Viewing $\check{A_z}$ as

$$\mathbb{C}^{2n}/L_z^*, \tag{3.24}$$

where L_z^* is the lattice spanned by $z_i^*, e_i^*, z_i'^*, e_i'^*$, we have that λ is the $\mathbb{C}$-linear map defined by

$$\langle \lambda(u), v \rangle = E_z(u, v)$$

for all u, v in $\mathbb{C}^{2n}$. In particular, we see that

$$\begin{aligned}
\lambda(e_i) &= 2z_i^* + \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\bar{\alpha}) z_i'^* \\
\lambda(e'_i) &= \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\alpha) z_i^* + 2\alpha\bar{\alpha} z'_i * .
\end{aligned}$$

So for $u_j \in \mathbb{C}$, we have that for $j = 1, \ldots, n$,

$$\begin{aligned}
\lambda((0, \ldots, u_j, \ldots, 0)) &= u_j \lambda\left(\frac{1}{\alpha - \bar{\alpha}}(\alpha e_j - e'_j)\right) \\
&= \frac{u_j}{\alpha - \bar{\alpha}} \left(\left(2\alpha - \mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\alpha)\right) z_j^* + \alpha\left(\mathrm{tr}_{\mathbb{C}/\mathbb{R}}(\bar{\alpha}) - 2\bar{\alpha}\right) z_j'^*\right) \\
&= u_j \left(z_j^* + \alpha z_j'^*\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}\lambda((0,\ldots,u_{j+n},\ldots,0)) &= u_j\lambda\left(\frac{1}{\bar{\alpha}-\alpha}(\bar{\alpha}e_j - e_j')\right)\\ &= u_j(z_j^* + \bar{\alpha}z_j'^*).\end{aligned}$$

Now let $w_1,\ldots w_n, w_{n+1},\ldots,w_{2n}$ be coordinates on $\mathbb{C}^{2n}$ in terms of the vectors

$$z_1^* + \alpha z_1'^*,\ldots,z_n^* + \alpha z_n'^*, z_1^* + \bar{\alpha}z_1'^*,\ldots,z_n^* + \bar{\alpha}z_n'^*,$$

respectively. Then for $1 \le j \le n$,

$$\lambda^*(dw_j) = du_j$$

We are now in a position to look at the action of the Kodaira-Spencer morphism KS on the basis $du_i \otimes dw_j$, $i, j = 1,\ldots 2n$ for $\omega_{A_z} \otimes \omega_{A_z^\vee}$. We will do this by tracing $du_i \otimes dw_j$ step-by-step through the composition of maps (3.22) and (3.23). For $i = 1,\ldots,n$ and $j = n+1,\ldots,2n$,

$$du_i \otimes dw_j \overset{\text{(incl.)}}{\longmapsto} du_i \otimes dw_j \overset{\nabla\otimes\mathrm{Id}}{\longmapsto} \left(\sum_{k=1}^{n}(\beta_k + \bar{\alpha}\beta_k')\otimes dz_{ik}\right)\otimes dw_j \tag{3.25}$$

$$\overset{\text{mod } \omega_{A_z}}{\longmapsto} \sum_{k=1}^{n}((\bullet, z_k^* + \bar{\alpha}z_k'^*)\otimes dz_{ik})\otimes dw_j \tag{3.26}$$

$$\mapsto \sum_{k=1}^{n}\big((dw_k)^\vee \otimes dz_{ik}\big)\otimes dw_j \mapsto dz_{i,j-n}. \tag{3.27}$$

(Note that in lines (3.24), (3.26), and (3.27), we are implicitly associating the following via their canonical identifications: $\mathbb{C}^{2n}$, the tangent space of $A_z^\vee$, $\omega_{A_z^\vee}^\vee$, $\mathrm{Hom}_{\mathbb{C}\text{-anti-lin}}(\omega_{A_z},\mathbb{C})$, and $\bar{\omega}_{A_z}$.)

So from lines (3.25) through (3.27), we see that

$$KS(du_i \otimes dw_j) = dz_{i,j-n} \qquad \text{for } 1 \le i \le n \text{ and } n+1 \le j \le 2n. \tag{3.28}$$

Similarly, by tracing $du_i \otimes dw_j$ through the composition of maps (3.22) and (3.23), one finds that

$$KS(du_i \otimes dw_j) = \begin{cases} dz_{j,i-n} & \text{if } n+1 \leq i \leq 2n \text{ and } 1 \leq j \leq n \\ 0 & \text{if } 1 \leq i, j \leq n \\ 0 & \text{if } n+1 \leq i, j \leq 2n \end{cases} \tag{3.29}$$

We thus find that

$$I_{KS} := \ker(KS)$$

is spanned by

(3.30)

$$\{du_i \otimes dw_j - du_j \otimes dw_i | 1 \leq i, j \leq 2n\} \cup \{\, du_i \otimes dw_j | 1 \leq i, j \leq n \text{ or } n+1 \leq i, j \leq 2n\}\, .$$

This is a special case of the more general result given in Lemma III.4. In Section VIII, the above description of I_{KS} will be important in our consideration of the action of ∇ on I_{KS} (defined through the product rule).

For our construction of the differential operators and our comparison of our differential operators to the C^∞-operators in [Shi00], the following lemma (which follows from our work above) will be useful.

Lemma III.2. *KS induces an isomorphism (the "Kodaira-Spencer isomorphism")*

$$KS : \omega_{A_{\mathrm{univ}}} \otimes \omega_{A^\vee_{\mathrm{univ}}} / I_{KS} \xrightarrow{\sim} \Omega_{\mathcal{H}/\mathbb{C}}.$$

Associating $\Omega_{\mathcal{H}/\mathbb{C}}$ with the complex vector space $\mathbb{C}^n_n$ via

$dz_{ij} \leftrightarrow e_{ij} :=$ *the $n \times n$ matrix with 1 in the ij-th position and zeroes everywhere else,*

we have that for all $\gamma \in K^c \subset GL_{2n}(\mathbb{C})$, $h \in \omega_A \otimes \omega_{A^\vee}$, and $g \in \Omega_{\mathcal{H}/\mathbb{C}}$,

$$KS\left((\rho_{St} \otimes \rho_{St})(\gamma \otimes \gamma)(h)\right) = \tau(\gamma) g. \tag{3.31}$$

The reader may find it instructive to compare the above lemma with the description of the Kodaira-Spencer isomorphism over $Mum_L(q)$ given in Section 4.2.1.

Proof. The isomorphism follows directly from the above example. Equation (3.31) also follows from the above example, combined with the definitions of ρ_{St} and τ from Section 2.3. ∎

For convenience, we often associate ω_A with $\omega_{A^\vee}$ via the isomorphism

$$\lambda^* : \omega_{A^\vee} \to \omega_A$$

$$dw_i \mapsto du_i$$

coming from the polarization λ.

Lemma III.3.

$$\nabla(I_{KS}) \mod (\mathrm{Split}(C^\infty) \otimes \Omega(C^\infty))$$

is contained in

$$I_{KS} \otimes \Omega(C^\infty).$$

Proof. We prove this lemma by showing that

$$\nabla(v) \subset I \otimes \Omega \tag{3.32}$$

for all the elements v in the basis for I_{KS} given in (3.30).

Using the product rule given in (3.19), we have that for all x and y in $\underline{\omega}(C^\infty)$,

$$\nabla(x \otimes y - y \otimes x) = \sigma(\nabla(x) \otimes y) + x \otimes \nabla(y) - \sigma(\nabla(y) \otimes x) - y \otimes \nabla(x). \tag{3.33}$$

By (3.30), we know that for all x and y in $\underline{\omega}(C^\infty)$

$$x \otimes y - y \otimes x$$

lies in I_{KS}. Therefore, for all z and w in $\underline{\omega}(C^\infty)$,

$$z \otimes \nabla(w) - \sigma(\nabla(w) \otimes z) \mod (\mathrm{Split}(C^\infty) \otimes \Omega(C^\infty))$$

lies in $I_{KS} \otimes \Omega$. Consequently, from Equation (3.33), we see that

$$\nabla(x \otimes y - y \otimes x)$$

lies in $I_{KS} \otimes \Omega \mod (\mathrm{Split}(C^\infty) \otimes \Omega(C^\infty))$ for all x and y in $\underline{\omega}(C^\infty)$. In particular, for all i and j,

$$\nabla(du_i \otimes du_j - du_j \otimes du_i) \mod (\mathrm{Split}(C^\infty) \otimes \Omega(C^\infty))$$

lies in $I_{KS} \otimes \Omega$.

Now, we check that

$$\nabla(du_i \otimes du_j) \mod (\mathrm{Split}(C^\infty) \otimes \Omega(C^\infty))$$

lies in $I_{KS} \otimes \Omega(C^\infty)$ whenever $1 \leq i, j \leq n$ or $n+1 \leq i, j \leq 2n$. From Equations (3.15) through (3.18), we see that for $1 \leq i, j \leq n$,

$$\nabla(du_i) \mod \mathrm{Split}(C^\infty) \otimes \Omega(C^\infty)$$

is contained in the submodule of $\underline{\omega}(C^\infty) \otimes \Omega(C^\infty)$ generated by the set of elements

$$\{du_k \otimes w | 1 \leq k \leq n \text{ and } w \in \Omega(C^\infty)\};$$

and similarly,

$$\nabla(du_{i+n}) \mod \mathrm{Split}(C^\infty) \otimes \Omega(C^\infty)$$

is contained in the submodule of $\underline{\omega}(C^\infty) \otimes \Omega(C^\infty)$ generated by the set of elements

$$\{du_k \otimes w | n+1 \leq k \leq 2n \text{ and } w \in \Omega(C^\infty)\}.$$

Therefore, we see that for $1 \leq i, j \leq n$,

$$\nabla(du_i \otimes du_j) \mod \mathrm{Split}(C^\infty) \otimes \Omega(C^\infty)$$

is contained in the submodule of $\underline{\omega}(C^\infty) \otimes \underline{\omega}(C^\infty) \otimes \Omega(C^\infty)$ generated by

$$\{du_k \otimes du_l \otimes w | 1 \leq k, l \leq n \text{ and } w \in \Omega(C^\infty)\},$$

which is a submodule of $I_{KS} \otimes \Omega(C^\infty)$ (by 3.30). Similarly, we see that for $1 \leq i, j \leq n$,

$$\nabla(du_{i+n} \otimes du_{j+n}) \mod \mathrm{Split}(C^\infty) \otimes \Omega(C^\infty)$$

is contained in the submodule of $\underline{\omega}(C^\infty) \otimes \underline{\omega}(C^\infty) \otimes \Omega(C^\infty)$ generated by

$$\{du_{k+n} \otimes du_{l+n} \otimes w | n+1 \leq k+n, l+n \leq 2n \text{ and } w \in \Omega(C^\infty)\},$$

which is a also submodule of $I_{KS} \otimes \Omega(C^\infty)$ (by 3.30).

Since we have now shown that (3.32) holds for all v in the basis for I_{KS} given in (3.30), the proof of the lemma is complete. ■

3.3 The kernel of the Kodaira-Spencer isomorphism

We try as much as possible in this section to be consistent with the notation of [Lan08]. Throughout this section, let S_0 be the base scheme over which ${}_K\mathbb{S}$ is defined, and let $(A, \lambda, \iota, \alpha)$ be the tuple associated to a morphism $S \to_K \mathbb{S}$ (by the universal property). As usual, $(A, \lambda, \iota, \alpha)$ consists of the following data:

1. an abelian scheme A over S
2. a prime-to-p polarization $\lambda : A \to A^\vee$ (if p is the characteristic of S)
3. an endomorphism $\iota : \mathcal{O}_\mathcal{K} \to End_S(A)$
4. a level structure α

parametrized by ${}_K\mathbb{S}$ over S_0, where the characteristic of S_0 is 0 or p.

Lemma III.4 ([Lan08], part of Proposition 2.3.4.2)**.** *The kernel I_{KS} of*

$$KS : \omega_{A/S} \otimes \omega_{A^\vee/S} \to \Omega_{S/S_0}$$

contains the submodule J_{KS} of $\omega_{A/S} \otimes \omega_{A^\vee/S}$ generated by the set of elements

$$\left\{\lambda^*(y) \otimes x - \lambda^*(x) \otimes y | x, y \in \omega_{A^\vee/S}\right\} \cup$$
$$\left\{(i(b)^*x) \otimes y - x \otimes ((i(\check{b}))^* y) | x \in \omega A/S, y \in \omega_{A^\vee/S}, b \in \mathcal{O}_{\mathcal{K}}\right\}, \tag{3.34}$$

Furthermore, if $S \to \mathcal{M}$ is étale, then the map

$$KS : \omega_{A/S} \otimes \omega_{A^\vee/S}/J_{KS} \to \Omega_{S/S_0}$$

is an isomorphism.

Since λ is a prime-to-p polarization, the morphism

$$\lambda^* : \omega_{A^\vee/S} \to \omega_{A/S}$$

is an isomorphism.

Therefore, we obtain the following Corollary of Lemma III.4.

Corollary III.5. *Suppose $S \to \mathcal{M}$ is étale. Then, the Kodaira-Spencer morphism KS induces an isomorphism, which by abuse of notation we also denote KS:*

$$KS : Sym^2(\omega_{A/S})/J_{KS} \xrightarrow{\sim} \Omega_{S/S_0}. \tag{3.35}$$

(Here, we associate J_{KS} with its image in $Sym^2(\omega_{A/S})$.)

Remark III.6. We shall mainly be applying Corollary III.5 in the case where $S = \mathcal{M}$ and the map $\mathcal{M} \to \mathcal{M}$ is the identity (so the abelian scheme associated to $S \to \mathcal{M}$ is the universal abelian scheme A_{univ}). Since the identity map is étale, we can indeed apply Corollary III.5 in this situation.

We note, however, that in the case where S is an arbitrary scheme over S_0 and $S \to \mathcal{M}$ is not étale, I_{KS} can be strictly larger than J_{KS}. For example, consider the case where $\mathcal{M} = \mathcal{H}_n$ over $S_0 = \mathbb{C}$, $S = \mathbb{C}$, and A is the abelian variety corresponding to a morphism $S := Spec(\mathbb{C}) \to \mathcal{H}_n/\mathbb{C}$. In this case, $\Omega_{S/S_0} = 0$, but $\mathrm{Sym}^2(\omega_{A/S})/J_{KS} \neq 0$.

As a direct consequence of Corollary III.5, we have isomorphisms (which are crucial in our construction of the differential operators)

$$\Omega_{\mathcal{M}/S_0} \xrightarrow{\sim} Sym^2(\underline{\omega})/J_{KS} \tag{3.36}$$

$$\xrightarrow{\sim} \underline{\omega}^+ \otimes \underline{\omega}^- \tag{3.37}$$

Note that depending on the situation, we will work sometimes with isomorphism (3.36) and sometimes with (3.37).

CHAPTER IV

Algebraic and analytic q-expansions

In this chapter, we briefly discuss algebraic q-expansions, which will be important in later proofs.

4.1 Fourier expansions

In this section, we discuss the complex-analytic theory of q-expansions, closely following section 5 of [Shi00].

For $c \in \mathbb{C}$ and $X \in \mathbb{C}_n^n$, we let

$$\mathbf{e}(\mathbf{c}) = \exp(2\pi i c)$$

$$\mathbf{e^n}(\mathbf{X}) = \mathbf{e}(\mathrm{tr}\,(\mathbf{X}))$$

$$S = S^n = \{\sigma \in K_n^n | \sigma^* = \sigma\}.$$

Let $\Gamma \subset GU(\eta_n)$ be a congruence subgroup. Then there exists a $\mathbb{Z}$-lattice M in S such that $\begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix}$ is in Γ for each σ in M. Let

$$L' = \{h \in S | tr(hM) \subset \mathbb{Z}\}.$$

Remark IV.1. Shimura denotes the $\mathbb{Z}$-lattice L' by L ([Shi00]). Since we (and Shimura) use L to denote another lattice as well, we introduce the notation L' rather than labeling both lattices (which appear in the same contexts but are not equal) the same.

Let f be a holomorphic automorphic form with respect to Γ that takes values in a vector-space X. Then because $f(z+\sigma) = f(z)$ for each $\sigma \in M$, f has a *Fourier expansion*, i.e. there exist elements $c(h) \in X$ such that

$$f(z) = \sum_{h \in L'} c(h)\mathbf{e}^{\mathbf{n}}(\mathbf{hz}).$$

We write the Fourier expansion of f as

$$f(z) = \sum_{h \in S} c(h)\mathbf{e}^{\mathbf{n}}(\mathbf{hz}).$$

The elements $c(h)$ are called the Fourier coefficients of f. If $n > 1$, then $c(h) \neq 0$ only if h is nonnegative definite.

4.2 A user's guide: The algebraic theory of q-expansions and the Mumford object

In this section, we discuss the algebraic theory of q-expansions. This section should be viewed as a user's guide to algebraic q-expansions in the PEL moduli problem (i.e. the situation of this paper). The situation for the $Sp(n)$ moduli problem is similar.

The algebraic theory of q-expansions relies upon the existence of what we shall call "Mumford objects" or "Mumford abelian varieties." Mumford ojects are the higher dimensional generalization of Tate elliptic curves. Like Tate curves, Mumford abelian varieties arise naturally from a certain semiabelian scheme over toroidal compactifications of the moduli scheme $\mathcal{M} = Sh(2V)$ over $(\mathcal{O}_{\mathcal{K}})_{(p)}$. For each cusp of $\mathcal{M}$, there is a corresponding Mumford object, which lies over the compactification of $Sh(2V)$ at that cusp. The details of the construction of toroidal compactifications of PEL Shimura varieties is given in [Lan08].[1] For Hilbert modular forms, the corresponding toroidal compactifications were constructed in [Rap78]. For symplectic modular forms, this is discussed is [FC90]. For details on the discussion in [FC90], the reader is advised to see [Lan08].

Tate curves are often used explicitly in computations and described in detail in coordinates over $\mathbb{C}$. The current literature on Mumford objects and algebraic q-expansions, however, does not provide a similarly explicit description. The current literature ([Lan08]) does, however, provide a q-expansion principle analogous to the q-expansion principle for Tate curves.

Since our intended applications, as well as other unrelated projects ([SU09]) require a more explicit description of the Mumford object, we provide one here. The reader wishing, on the other hand, to learn the details of toroidal compactifications of $\mathcal{M}$ should consult [Lan08].

While it is in some ways simpler, the simplicity of the one-dimensional case (i.e. Tate curves, as discussed in [Kat78] and [Kat73b]) obscures the larger picture. In fact, many details of the one-dimensional case become more transparent in the arbitrary-dimension situation.

4.2.1 The Mumford object

Let

$$V = \mathcal{W} \oplus \mathcal{W}',$$

where

$$\mathcal{W}, \mathcal{W}' = \mathcal{K}^n.$$

[1] Though it is not in [Lan08], Lan shows in [Lan09] that the q-expansions defined algebraically in [Lan08] agree with the analytic Fourier expansions.

Note that

$$J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

induces a pairing Ψ on V. The pairing Ψ induces an isomorphism

$$\mathcal{W}' \xrightarrow{\sim} \mathrm{Hom}(\mathcal{W}, \mathcal{K}).$$

Define a pairing

$$\Psi' : V \times V \to \mathbb{Q}$$

by

$$(x, y) \mapsto \Psi'(x, y) = \mathrm{tr}\,\left(\delta_{\mathcal{K}}^{-1} \Psi\,(x, y)\right)$$

for each $x, y \in V$. Let L be the lattice inside V defined by

$$L = \mathcal{O}_{\mathcal{K}}^n \oplus \mathcal{O}_{\mathcal{K}}^n.$$

Then the restriction

$$\Psi' : L \times L \to \mathbb{Z}$$

of Ψ' to $L \times L$ is a perfect pairing. We shall write W and W' to denote $\mathcal{W} \cap L$ and $\mathcal{W}' \cap L$, respectively.

Let U be an open compact subgroup of $GU(n, n)(\mathbb{A}_f) = G(\mathbb{A}_f)$. Then

$$Sh_G(U) = G(\mathbb{Q})^+ \backslash \mathcal{H}_n \times G(\mathbb{A}_f)/U \supseteq \Gamma \backslash \mathcal{H}_n,$$

with

$$\Gamma = G^+(\mathbb{Q}) \cap U.$$

Let P be the stabilizer of W' in $GU(n,n)$, where $GU(n,n)$ acts on W' (viewed as row vectors) on the right. Then each matrix in P is of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

Let N be the unipotent radical of P, and let

$$H = N \cap \Gamma.$$

Then H is an upper-triangular, unipotent subgroup of $GU(n,n)(K)$. It is a simple computation to show that N is contained in the group of matrices of the form

$$\begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}$$

with B a Hermitian matrix with entries in K. Thus, we can choose the lattice M used to construct Fourier expansions in Subsection 4.1 so that

$$H = \begin{pmatrix} 1_n & M \\ 0 & 1_n \end{pmatrix}.$$

Note that H maps W to W'.

Let $H^\vee$ be the dual lattice of H. That is, each element of $H^\vee$ may be viewed as a $\mathbb{Z}$-linear map $H \to \mathbb{Z}$ given by

$$h \mapsto \mathrm{tr}\ (gh) \subseteq \mathbb{Z}.$$

for some (non-unique) matrix g. A simple computation shows that we may associate $H^\vee$ with the lattice L' in K_n^n, where L' is defined in terms of M as above.

The data $L = W \oplus W'$ above is called the (zero-dimensional) "cusp at infinity." The zero-dimensional cusps are in one-to-one correspondence with the elements of

$P(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$. There is not a canonical way to associate a lattice to each g in $P(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$, but a systematic way is the following. Write

$$G(\mathbb{A}_f) = \coprod_i G(\mathbb{Q}) g_i K$$

$$g = \gamma g_i k.$$

Define the lattice at the cusp corresponding to g to be

$$L_g = (L \otimes \hat{\mathbb{Z}}) g_i \cap V$$

$$W_g = W\gamma \cap L$$

$$W'_g = W'\gamma \cap L.$$

Then H is defined accordingly, corresponding to our new choice of cusp. Note that the symmetric space is $\coprod_{\Gamma_i} \mathcal{H}_n$. Each g_i tells us which Γ_i and γ says which Borel. In the following discussion of Mumford objects and q-expansions, one can consider any of the cusps g, even though we write our discuss in terms of the notation for the cusp at infinity; the reader wishing to work with a different cusp g should simply replace L with L_g, W with W_g, etc.

As explained in [Lan08], for a toroidal compactification of $\mathcal{M}$, the completion along the boundary stratum for the zero-dimensional cusp $[g]$ lies over $\mathrm{Spf} R$, where R is the ring

$$\left(\mathcal{O}_{\mathcal{K}}\right)_{(p)}[[q, H^\vee_{\geq 0}]]^{\Gamma_g} = \left\{ \sum_{h \in H^\vee_{\geq 0}} a_h q^h \mid h \geq 0,\ a_h \in \left(\mathcal{O}_{\mathcal{K}}\right)_{(p)},\ \text{and } a(\gamma h \gamma^*) \text{ for all } (\gamma, \gamma^*) \in \Gamma_g \right\}.$$

There is a semiabelian scheme over the toroidal compactification of $\mathcal{M}$. By passing to $\left(\mathcal{O}_{\mathcal{K}}\right)_{(p)}((q, H^\vee_{\geq 0}))$, we obtain an abelian variety $\mathcal{G}_H$ over $\mathcal{M}/\mathrm{Spec}(\left(\mathcal{O}_{\mathcal{K}}\right)_{(p)}((q, H^\vee_{\geq 0})))$, which gives the Mumford object at the cusp H, a semi-abelian scheme lying over

$$\left(\mathcal{O}_{\mathcal{K}}\right)_{(p)}((q, H^\vee_{\geq 0})) = \left\{ \sum_{h \in H^\vee} a_h q^h \mid a_h \in \left(\mathcal{O}_{\mathcal{K}}\right)_{(p)} \text{ and } a_h = 0 \text{ if } h << 0 \right\}$$

and denoted $\mathrm{Mum}_L(q)$.

We now briefly discuss the analytic situation over $\mathbb{C}$ to provide some context and motivation for our upcoming (algebraic) description of Mumford objects. Recall that for an elliptic curve E, the analytic construction

$$E = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \xrightarrow{\exp} \mathbb{C}^\times/q(\mathbb{Z}),$$

where q is the map

$$\begin{aligned} q : \mathbb{Z} &\to \mathbb{C}^\times \\ n &\mapsto e^{2\pi i n}, \end{aligned}$$

gives a map from an elliptic curve to the complex points of the Tate curve. We now do the analogue of this in our situation.

Recall the notation

$$\mathcal{H}_n = \{Z \in M_n(\mathbb{C}) \mid i\,(Z^* - Z) > 0\}.$$

Let $Z = X + iY$ be in $\mathcal{H}_n$. Then setting

$$\begin{aligned} X &= \frac{Z + Z^*}{2} \\ Y &= \frac{-i\,(Z - Z^*)}{2}, \end{aligned}$$

we have $Z = X + iY$, with $Y > 0$. Now we express $\mathcal{H}_n$ in terms of H:

$$\mathcal{H}_n = H \otimes \mathbb{R} + (H \otimes \mathbb{R})_{>0}\, i \subset H \otimes \mathbb{C}.$$

(For a set S of Hermitian matrices, we write $S_{>0}$ to denote the set of positive definite matrices in S.) Let $\tau \in \mathcal{H}_n \subset H \otimes \mathbb{C}$. Then we may express the abelian variety A_τ as

$$W' \otimes \mathbb{C}/L_\tau,$$

where L_τ is the $\mathbb{Z}$-lattice generated by $W' \otimes 1$ and $\tau(W) \subset W' \otimes \mathbb{C}$. So we have a commutative diagram of Lie groups (where the map $\underline{q}$ is defined implicitly through the exp map, and $\underline{q}(H^\vee) = \exp(\tau(W))$)

$$\begin{tikzcd}
W' \otimes \mathbb{C}/L_\tau \arrow[r, "\exp"'] \arrow[d] & W' \otimes \mathbb{C}^\times/\underline{q}(H^\vee) = W' \otimes \mathbb{G}_m(\mathbb{C})/\underline{q_\tau}(H^\vee) \arrow[d] \\
\mathcal{H}_n \subset H \otimes \mathbb{C} \arrow[r, "\exp"'] & H \otimes \mathbb{C}^\times = \operatorname{Spec}\left((\mathcal{O}_\mathcal{K})_{(p)}((q, H^\vee_{\geq 0})) \otimes_{\mathcal{O}_\mathcal{K}} \mathbb{C}\right)^{\mathrm{an}}
\end{tikzcd}$$

The quotient $W' \otimes \mathbb{G}_m(\mathbb{C})/\underline{q}(H^\vee)$ above is the set of complex points of an abelian variety.

We now describe the Mumford object more explicitly, analogous to the description of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ in [Kat78].

Remark IV.2. For the reader trying to understand [Kat78] in the context of our description of the general situation, we provide the following dictionary between Katz's notation and the notation we will use for the general situation.

$$\mathfrak{a} \leftrightarrow W'$$

$$\mathfrak{b} \leftrightarrow W$$

$$\mathfrak{a}^{-1}\mathfrak{b}^{-1} \leftrightarrow H$$

$$\mathfrak{a}\mathfrak{b} \leftrightarrow H^\vee.$$

We define a $\mathbb{Z}$-linear morphism

$$\underline{q} : W \to W' \otimes \mathbb{G}_m \tag{4.1}$$

from W to the torus $W' \otimes \mathbb{G}_m$ lying over $\mathcal{O}_\mathcal{K}((q, H^\vee))$ to be the composition of morphisms

$$W \xrightarrow{w \mapsto \mathrm{eval}_w} \mathrm{Hom}_\mathbb{Z}(H, W') \xrightarrow{\sim} H^\vee \otimes W \to W' \otimes \mathbb{G}_m.$$

(By eval_w, we mean the map $h \mapsto h(w)$ in $\mathrm{Hom}_\mathbb{Z}(H, W')$.)

The "Mumford abelian variety at the cusp H" is the algebraification of the rigid analytic quotient

$$\underline{q}(W)\backslash\left((W')^\vee \otimes \mathbb{G}_m\right). \tag{4.2}$$

We denote the Mumford abelian variety by $\mathrm{Mum}_L(q)$ or $\mathrm{Mum}_H(q)$. The construction of the Mumford abelian variety is discussed in [Mum72] and in [Lan08].

The Mumford abelian variety $\mathrm{Mum}_L(q)$ has a canonical PEL structure. The canonical endomorphism

$$\iota_{can} : \mathcal{O}_{\mathcal{K}} \to End_{(\mathcal{O}_{\mathcal{K}})_{(p)}((q,H^\vee_{\geq 0}))}(\mathrm{Mum}_L(q))$$

is defined by

$$\alpha : L \to L$$

$$l \mapsto \alpha \cdot l$$

for each element α of $\mathcal{O}_{\mathcal{K}}$. The dual abelian variety is

$$\mathrm{Mum}_L(q)^\vee = \mathrm{Mum}_{W'^\vee \oplus W^\vee}(q),$$

i.e. the algebraification of the rigid analytic quotient

$$\underline{q}(W'^\vee)\backslash\left(W^\vee \otimes \mathbb{G}_m\right).$$

The canonical isomorphism

$$W' \xrightarrow{\sim} W^\vee$$

induced by the pairing Ψ induces a canonical polarization

$$\lambda_{can} : \underline{q}(W)\backslash\left(W' \otimes \mathbb{G}_m\right) \to \underline{q}(W'^\vee)\backslash\left(W^\vee \otimes \mathbb{G}_m\right)$$

of $\mathrm{Mum}_L(q)$.

The natural exact sequence

$$0 \to W' \otimes \prod_l \varprojlim_n \mu_{l^n} \to \prod_l T_l(A) \to W \otimes \hat{\mathbb{Z}} \to 0$$

induces a canonical level K structure

$$\alpha_{can} : V \otimes \mathbb{A}_f \xrightarrow{\sim} \prod_l T_l(A) \otimes \mathbb{Q}$$

modulo the action of K.

At times, we shall write $\mathrm{Mum}_L(q)$ to mean the tuple $(\mathrm{Mum}_L(q), \lambda_{can}, \iota_{can}, \alpha_{can})$.

Observe that there is a canonical isomorphism

$$\omega_L : \mathrm{Lie}(\mathrm{Mum}_L(q)) \xrightarrow{\sim} \mathrm{Lie}(W' \otimes \mathbb{G}_m) = W' \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0}))$$

Dualizing the morphism ω_L, we obtain a canonical isomorphism

$$\omega_{can} : W'^{\vee} \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0})) \xrightarrow{\sim} \underline{\omega}, \tag{4.3}$$

which gives a canonical element of $\mathcal{E}$. There is a similar isomorphism on $\mathrm{Mum}_L(q)^{\vee}$:

$$W^{\vee} \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0})) = \mathrm{Lie}(\mathbb{G}_m \otimes W^{\vee}) \xrightarrow{\sim} \mathrm{Lie}(\mathrm{Mum}_L(q)^{\vee}) \tag{4.4}$$

We now revisit the Kodaira-Spencer morphism in the context of $\mathrm{Mum}_L(q)$. Recall that for an abelian variety A the Kodaira-Spencer isomorphism identifies derivations of $\mathcal{O}_{\mathcal{M}}$ with pairings of $\underline{\omega}_A$ and $\mathrm{Lie}(A^{\vee})$, i.e. with elements of $\mathrm{Lie}(A) \otimes \mathrm{Lie}(A^{\vee})$. We provide a concise reminder of the Kodaira-Spencer isomorphism for an abelian variety A, reviewing precisely the details that we will need in our discussion of the situation for $\mathrm{Mum}_L(q)$. Recall the exact sequence

$$0 \to \underline{\omega} \to H^1_{DR} \to \mathrm{Lie}(A^{\vee}) \to 0$$

and the Gauss-Manin connection (from which the Kodaira-Spencer morphism is constructed)

$$\nabla : H^1_{DR} \to H^1_{DR} \otimes \Omega.$$

Each

$$D \in T_{\mathcal{M}/\mathcal{O}_\mathcal{K}} = \underline{\mathrm{Der}}\left(\mathcal{O}_\mathcal{M}, \mathcal{O}_\mathcal{M}\right)$$

defines a morphism

$$\nabla(D) : H^1_{DR} \to H^1_{DR},$$

which induces a morphism

$$KS(D) : \underline{\omega} \to \mathrm{Lie}(A^\vee)$$

defined to be the composition of maps

$$\underline{\omega} \hookrightarrow H^1_{DR} \xrightarrow{\nabla(D)} H^1_{DR} \twoheadrightarrow \mathrm{Lie}(A^\vee).$$

The map

$$D \mapsto KS(D)$$

defines the Kodaira-Spencer morphism

$$T_{\mathcal{M}/\mathcal{O}_\mathcal{K}} \to \mathrm{Hom}_{\mathcal{O}_\mathcal{M}}\left(\underline{\omega}, \mathrm{Lie}(A^\vee)\right) \cong \mathrm{Lie}(A) \otimes \mathrm{Lie}(A^\vee).$$

Dualizing gives

$$\Omega \leftarrow \mathrm{Lie}(A) \otimes \mathrm{Lie}(A^\vee).$$

On $\mathrm{Mum}_L(q)$, the Kodaira-Spencer map is

$$\begin{aligned} KS : \mathrm{Der}\left(\left(\mathcal{O}_\mathcal{K}\right)_{(p)}((q, H^\vee_{\geq 0})), \left(\mathcal{O}_\mathcal{K}\right)_{(p)}((q, H^\vee_{\geq 0}))\right) &\to \mathrm{Lie}\left(\mathrm{Mum}_{W \oplus W'}(q)\right) \otimes \mathrm{Lie}\left(\mathrm{Mum}_{W'^\vee \oplus W^\vee}(q)\right) \\ &\cong W' \otimes W^\vee \otimes \left(\mathcal{O}_\mathcal{K}\right)_{(p)}((q, H^\vee_{\geq 0})) \end{aligned}$$

Given $\gamma \in H \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{K}}$, define

$$D(\gamma) \in \mathrm{Der}\left((\mathcal{O}_{\mathcal{K}})_{(p)}\,((q, H^{\vee}_{\geq 0})), (\mathcal{O}_{\mathcal{K}})_{(p)}\,((q, H^{\vee}_{\geq 0})) \right)$$

by

$$D(\gamma)\left(\sum_{\alpha \in H^{\vee}} a_{\alpha} q^{\alpha} \right) = \sum_{\alpha \in H^{\vee}} \mathrm{tr}\,(\alpha\gamma) a_{\alpha} q^{\alpha}.$$

Note that there is a natural $\mathbb{Z}$-linear morphism

$$\phi_H : H \to W^{\vee} \otimes_{\mathbb{Z}} W'.$$

Then

$$KS(D(\gamma)) = \phi_H(\gamma) \otimes 1$$

in $W' \otimes W^{\vee} \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}\,((q, H^{\vee}_{\geq 0}))$.

For $w \in W'^{\vee}$, let $\omega(w)$ denote the image in $\underline{\omega}$ of $w \otimes 1$ under the morphism (4.3). For each $v \in W^{\vee}$, let $l(v)$ denote the image of $v \otimes 1$ under the morphism (4.4). Then for each $\gamma \in H$,

$$\nabla\,(D(\gamma))\,(\omega(w)) \equiv l(w \cdot \gamma) \mod \underline{\omega}.$$

The notation ω and l has been chosen to be similar to similar maps in [Kat78]. We further denote by

$$\omega^{\pm}(w)$$

the projection of $\omega(w)$ onto $\underline{\omega}^{\pm}$. We then have that the Kodaira-Spencer isomorphism is the map

$$\omega^{+}(e_i) \otimes \omega^{-}(e_j) \mapsto (D(e_{ij}))^{\vee},$$

with e_i, e_j, e_{ij} standard basis vectors in W and $H \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{K}} = (\mathcal{O}_{\mathcal{K}})^n_n$, respectively.

Note that by extending scalars, we may consider $\mathrm{Mum}_L(q)$ over $R \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}\left((q, H^{\vee}_{\geq 0})\right)$ for each $\mathcal{O}_{\mathcal{K}}$-algebra R, and we can extend the above maps to the case of $\mathrm{Mum}_L(q)$ over $R \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}\left((q, H^{\vee}_{\geq 0})\right)$.

4.2.2 Algebraic q-expansions

The algebraic theory of Fourier-Jacobi expansions is discussed in detail in [Lan08]. We now discuss the key features for our situation. For a general, in-depth discussion of Fourier-Jacobi expansions, the reader is referred to [Lan08].

Definition IV.3. Let f be an automorphic form of weight (ρ, V) over R. We define the **q-expansion of f at the cusp H** to be

$$f\left((\mathrm{Mum}_L(q), \alpha_{can}, \iota_{can}, \lambda_{can}) \otimes R, \omega_{can} \otimes R\right)$$

As noted at the beginning of the section on the Mumford object, the q-expansions of f lie inside $V \otimes_R R \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}[[q, H^{\vee}_{\geq 0}]]$.

Furthermore, when working over $\mathbb{C}$, the analytically defined Fourier coefficients of the function f^{an} on $\mathcal{H}_n$ (where we associate the function f of abelian varieties with the function $f^{an} : \mathcal{H}_n \to \mathbb{C}$ as in Sections 2.3 through 2.5) at the cusp L are the same as the algebraically defined q-expansion coefficents at the cusp L. That is, if

$$f\left((\mathrm{Mum}_L(q), \alpha_{can}, \iota_{can}, \lambda_{can}) \otimes \mathbb{C}, \omega_{can} \otimes \mathbb{C}\right) = \sum_{h \in H^{\vee}} c(h) q^h,$$

then the h-Fourier coefficient of f^{an} for each $h \in H^{\vee}$ is also $c(h)$.

In Proposition 7.1.2.15 of [Lan08], Lan proves the Fourier-Jacobi Principle for automorphic forms on PEL Shimura varieties. The q-expansion principle for modular forms is a special case of this.

Theorem IV.4 (q-expansion Principle, special case of Proposition 7.1.2.15 of [Lan08]). *Let f be an automorphic form on $U(n,n)$ over an $\mathcal{O}_\mathcal{K}$-algebra R of weight ρ with values in an R-module $R \otimes_{\mathcal{O}_\mathcal{K}} X$ for some $\mathcal{O}_\mathcal{K}$-module X.*

1. *If $f(Mum_L(q)) = 0$ at one cusp on each connected component of $\mathcal{M}$, then $f = 0$.*
2. *Let $R_0 \hookrightarrow R$ be an $\mathcal{O}_\mathcal{K}$-subalgebra of R. If $f(Mum_L(q)) \in (\mathcal{O}_\mathcal{K})_{(p)}((q, H^\vee_{\geq 0})) \otimes_{\mathcal{O}_\mathcal{K}} R \otimes X$ actually lies in $(\mathcal{O}_\mathcal{K})_{(p)}((q, H^\vee_{\geq 0})) \otimes_{\mathcal{O}_\mathcal{K}} R_0 \otimes X \hookrightarrow (\mathcal{O}_\mathcal{K})_{(p)}((q, H^\vee_{\geq 0})) \otimes_{\mathcal{O}_\mathcal{K}} R \otimes X$ for one cusp in each component of $\mathcal{M}$, then there is a unique automorphic form of weight ρ on $U(n,n)$ defined over R_0 which becomes f after the extension of scalars $R_0 \hookrightarrow R$.*

CHAPTER V

p-adic automorphic forms and the Igusa tower

In this chapter, we review p-adic automorphic forms, following the viewpoints of [Hid04], [Hid05] and [HLS06]. We also introduce the results that will be necessary for our construction of the p-adic differential operators. Many of these results are the analogue for $U(n,n)$ of the results in sections 1.9 through 1.12 of [Kat78].

Note that our notation is neither exactly that of [Hid04] nor that of [HLS06]; those sources use different notation from each other, and so we've chosen the notation we found most appropriate for our situation.

Let R be an $\mathcal{O}_{\mathcal{K}}$-algebra that is separated for the p-adic topology, i.e. satisfying

$$R \hookrightarrow \varprojlim_m R/p^m R.$$

Let R_0 be the p-adic completion of R. Let $K = K_p \times K^p \subset G(\mathbb{A}_f)$ be a compact open subgroup with $K_p \subseteq G(\mathbb{Q}_p)$ a hyperspecial maximal compact and $K^p \subseteq G(\mathbb{A}_f^p)$.

Let v be the prime of $\mathcal{K}$ over p determined by the embedding incl_p. Let $W = (\mathcal{O}_{\mathcal{K}})_v$, and let $W_m = W/p^m W$ for all nonnegative integers m. Fix a toroidal compactification ${}_K\mathbb{S}$ of ${}_K\mathbb{S}(G,X)$ over W. (Recall that ${}_K\mathbb{S}(G,X)$ is a smooth integral model for the prime-to-p moduli problem[1] discussed in Section 2.2.) The theory of p-adic automorphic forms is independent of the choice of toroidal compactification.

[1] Recall from Section 2.2 that ${}_K\mathbb{S}(G,X)$ classifies abelian varieties with additional structure, including a *prime-to-p* level structure.

Let $\tilde{H}$ be a lift of a power of the Hasse invariant H to ${}_K\mathbb{S}$. For m a positive integer, let

$$M_m = {}_K\mathbb{S} \times_W W_m,$$

and let

$$S_m = M_m \left[\frac{1}{\tilde{H}}\right]$$

be the nonvanishing locus of $\tilde{H}$, i.e. the ordinary locus. Let

$$S_0 = M \left[\frac{1}{\tilde{H}}\right],$$

and let S_∞ be the formal completion $\varprojlim_m S_m$ of S_0 along S_1. Note that S_m is independent of the choice of $\tilde{H}$ as long as p is nilpotent in W_m for all positive m.

For $m \geq 0$, let $P_{m,r}$ be the rank g p-adic étale sheaf

$$P_{m,r} = A_{\text{univ}}[p^r]^{\text{ét}} = A_{\text{univ}}[p^r]/A_{\text{univ}}[p^r]^0$$

over S_m. We define $T_{m,r}$ to be the finite étale S_m-scheme

$$\begin{array}{c} T_{m,r} = Isom_{S_m}(P_{m,r}, (\mathcal{O}_\mathcal{K}/p^r\mathcal{O}_\mathcal{K})^n) \\ \Big\downarrow{\scriptstyle \pi_{m,r}} \\ S_m \end{array}$$

representing the functor

$$(\pi : X \to S_m) \mapsto \left\{ \text{isomorphisms } \Psi : P_{m,r} \xrightarrow{\sim} (\mathcal{O}_\mathcal{K}/p^r\mathcal{O}_\mathcal{K})^n \text{ over } X \right\}.$$

Details about this scheme are given in [Hid04] and [Hid05]; of particular interest to us will be the fact that $\pi_{m,r}$ is an *affine* morphism. Define

$$T_{m,\infty} = \varprojlim_r T_{m,r}$$

and

$$T_{\infty,\infty} = \varprojlim_m T_{m,\infty}.$$

Note that the formal scheme $T_{\infty,\infty}$ is an étale cover of S_∞.

Note that $T_{m,r}$ classifies quintuples

$$\underline{A} = \left(A, \lambda, \iota, \alpha^p, X[p^r]^{\text{ét}} \xrightarrow{\sim} (\mathcal{O}_\mathcal{K}/p^r\mathcal{O}_\mathcal{K})^n\right),$$

where $(A, \lambda, \iota, \alpha^p)$ is the abelian variety with prime-to-p structure corresponding to a point of ${}_K\mathbb{S}(G, X)$. Note that

$$(\mathcal{O}_\mathcal{K}/p^r\mathcal{O}_\mathcal{K})^n \cong (\mathbb{Z}/p^r\mathbb{Z})^g.$$

Therefore, the prime-to-p polarization λ and the isomorphism

$$X[p^r]^{\text{ét}} \xrightarrow{\sim} (\mathcal{O}_\mathcal{K}/p^r\mathcal{O}_\mathcal{K})^n$$

induce isomorphisms

$$A[p^r]^0 \xrightarrow{\sim} A^\vee[p^r]^0 \xrightarrow{\sim} \mu_{p^r}^g,$$

which induces an inclusion

$$\alpha_p : \mu_{p^r}^g \hookrightarrow A.$$

Let $\underline{\omega}_{m,r}$ be the pullback of $P_{m,r}$ to $T_{m,r}$, i.e.

$$\underline{\omega}_{m,r} = \left(\pi_{m,r}^* P_{m,r}\right) \otimes \mathcal{O}_{T_{m,r}}.$$

For $r \geq m$, there's a universal isomorphism (i.e. the universal object over $T_{m,r}$)

$$\Psi_{univ} : \pi_{m,r}^* P_{m,r} \xrightarrow{\sim} (\mathcal{O}_\mathcal{K}/p^r\mathcal{O}_\mathcal{K})^n,$$

which induces an isomorphism

$$\omega_{can} = \Psi_{univ} \otimes \mathrm{Id} : \underline{\omega}_{m,r} \xrightarrow{\sim} (\mathcal{O}_{\mathcal{K}}/p^r\mathcal{O}_{\mathcal{K}})^n \otimes_{\mathbb{Z}_p} \mathcal{O}_{T_{m,r}}.$$

For $r \geq m$, the sheaf $\underline{\omega}_{m,r}$ *is* just the pullback of the sheaf of differentials $\underline{\omega}$ to $T_{m,r}$. Indeed, the pullback of $\underline{\omega}$ to $T_{m,r}$ is canonically identified with $\mathrm{Lie}(A^{\vee}_{\mathrm{univ}}) \otimes W_m$; the isomorphism with $\underline{\omega}_{m,r}$ now follows from the canonical isomorphisms

$$\mathrm{Lie}(A^{\vee}_{\mathrm{univ}}) \otimes W_m \xrightarrow{\sim} \mathrm{Lie}(A^{\vee}_{\mathrm{univ}}[p^r]^0) \otimes W_m \xrightarrow{\sim} \mathrm{Lie}(\mu_{p^r})^n \otimes W_r \xrightarrow{\sim} (\mathcal{O}_{\mathcal{K}}/p^r\mathcal{O}_{\mathcal{K}})^n \otimes \mathcal{O}_{T_{m,r}}.$$

Note that $\underline{\omega}_{can}$ induces isomorphisms $\underline{\omega}^+_{can}$ and $\underline{\omega}^-_{can}$ of $\underline{\omega}^-$ and $\underline{\omega}^+$ with $\mathcal{O}_{\mathcal{K}_v}{}^n \otimes \mathcal{O}_{T_{m,r}} \cong \mathcal{O}^n_{T_{m,r}}$ and $\mathcal{O}_{\mathcal{K}_{\bar{v}}}{}^n \otimes \mathcal{O}_{T_{m,r}} \cong \mathcal{O}^n_{T_{m,r}}$, respectively. So for the p-adic situation (in contrast to the situation over $\mathbb{C}$), $\underline{\omega}_{can}$ provides a canonical element of the sheaf $\mathcal{E}$ introduced earlier.

A p-adic automorphic form of weight $\rho_- \otimes \rho_+$ is defined to be a global section of $\left(\mathcal{O}^n_{T_{\infty,\infty}}\right)^{\rho_-} \otimes \left(\mathcal{O}^n_{T_{\infty,\infty}}\right)^{\rho_+}$, where the action of $\mathcal{O}_{\mathcal{K}}$ on each copy of $\left(\mathcal{O}^n_{T_{\infty,\infty}}\right)$ is induced by the action on $\underline{\omega}_-^{\rho_-} \otimes \underline{\omega}_+^{\rho_+}$ (identifying the two sheaves via $\underline{\omega}_{can}$). When we want to eliminate ambiguity about the identification, we shall write $\left(\mathcal{O}^-_{T_{\infty,\infty}}\right)^n$ or $\left(\mathcal{O}^+_{T_{\infty,\infty}}\right)^n$ to mean $\underline{\omega}_{can}$ $(\underline{\omega}^{\mp})$, respectively. We write $V(\rho, R_0)$ to denote the space of p-adic automorphic forms of weight ρ over R_0.

Above, we have used notation similar to that for the Igusa tower in [Hid04], [Hid05], and [HLS06]. To emphasize the analogy with [Kat78], we shall sometimes use the notation

$$\underline{\omega}(p\text{-adic})^{\pm} = \underline{\omega}^{\pm}_{T_{\infty,\infty}}$$

$$\mathcal{O}_{\mathcal{M}}(p\text{-adic}) = \mathcal{O}_{T_{\infty,\infty}}$$

$$\mathcal{M}(p\text{-adic}) = T_{\infty,\infty}.$$

We shall denote the space of p-adic automorphic forms of weight $\rho = \rho^- \otimes \rho^+$ over a p-adically complete and separated $\mathcal{O}_{\mathcal{K}}$-algebra R_0 by $\mathbf{M}^{p\text{-adic}}$.

5.1 q-expansions of p-adic automorphic forms

In this section, we discuss q-expansions of p-adic automorphic forms and the q-expansion principle for p-adic automorphic forms. This is also covered in [HLS06], which cites [Hid04].

By extending scalars from $(\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0}))$ to the p-adic completion $R_0 \hat{\otimes}_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0}))$ of $R_0 \otimes_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0}))$, we obtain the Mumford object $(\mathrm{Mum}_L(q), \lambda_{can}, \alpha_{can}, \iota_{can})$ over the p-adic ring $R_0 \hat{\otimes}_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0}))$. (Note that by construction, the isomorphism $\underline{\omega}_{can}$ from the previous section, viewed over $\mathrm{Mum}_L(q)$ is the same as the isomorphism ω_{can} in (4.3).

So we obtain a q-expansion homomorphism[2] FJ from the space of p-adic automorphic forms with values in an R_0-module X to $R_0 \hat{\otimes}_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0})) \otimes_{R_0} X$.

$$f \mapsto f\left(\left(\mathrm{Mum}_L(q), \lambda_{can}, \alpha_{can} = \alpha^p_{can} \times (\alpha_{can})_p, \iota_{can}\right)_{R_0 \hat{\otimes}_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0}))}, \omega_{can}\right)$$

Definition V.1. When R_0 has no p-torsion, we define the space of p-adic automorphic forms defined over $R_0 \otimes \mathbb{Q}$ to be $V(\rho, R_0) \otimes_{\mathcal{O}_{\mathcal{K}_v}} K_v$.

The q-expansion homomorphism extends to a q-expansion homomorphism

$$FJ' : V(\rho, R_0) \otimes_{\mathcal{O}_{\mathcal{K}_v}} K_v \to R_0 \hat{\otimes}_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0})) \otimes K_v \otimes_{R_0} X.$$

We now state the q-expansion principle for p-adic automorphic forms. This is the analogue of Theorem 1.9.9 of [Kat78] and Corollary 1.9.17 of [Kat78].

Theorem V.2 (Theorem 2.3.3 of [HLS06], which cites [Hid04])**.** *The q-expansion homomorphisms have the following properties.*

1. *The q-expansion homomorphisms FJ and FJ' are injective, and when R_0 has no p-torsion, the cokernel of FJ and FJ' has no p-torsion.*
2. $FJ'^{-1}(R_0 \otimes X) = V(\rho, R_0)$.

[2] The notation FJ stands for Fourier-Jacobi.

5.1.1 Map from automorphic forms over a p-adic ring R_0 to p-adic automorphic forms over R_0

Theorem V.3 (((2.2.7) in [HLS06]), analogue of Theorem 1.10.15 of [Kat78])**.** *The homomorphism*

$$f \mapsto \tilde{f}$$

from the space of weight ρ level α automorphic forms to the space of weight ρ level α^p p-adic automorphic forms defined by

$$\tilde{f}(X, \lambda, \iota, \alpha) = f(X, \lambda, \iota, \alpha, \omega_{can})$$

preserves q-expansions.

5.1.2 Frobenius and the unit root splitting

In this section, we give a splitting

$$H^1_{DR} = \underline{\omega} \oplus \underline{U} \tag{5.1}$$

over $T_{\infty,\infty}$ analogous to the C^∞-splitting

$$H^1_{DR} = \underline{\omega} \oplus \bar{\underline{\omega}}.$$

The splitting (5.1) will be indispensable for the construction of the p-adic differential operators.

Most of the material in the section is essentially covered in section 1.11 of [Kat78], with trivial generalizations. However, we have provided details not given in [Kat78].

Let X be an abelian variety of PEL type over an $\mathcal{O}_{\mathcal{K}}$-algebra R in which p is nilpotent, and suppose that each of the geometric fibers of X/R is ordinary. So there is an inclusion

$$\alpha_p : \mu_p^g \hookrightarrow X.$$

Let $\hat{X}$ denote the formal group of X, and let H_{can} be the canonical subgroup of X, i.e. the kernel of multiplication of by p in $\hat{X}$. Then $H_{can} = \alpha_p\left(\mu_p^g\right)$.

Let

$$X' = X/H_{can},$$

and let

$$\pi : X \to X'$$

be the projection map. When $p = 0$ in R, $X' = X^{(p)}$, where $X^{(p)}$ denotes the scheme over R obtained from X by extension of scalars $F_{abs} : R \to R$, and π is the relative Frobenius morphism:

$$\begin{tikzcd}
X \arrow[dr, "\pi"] \arrow[drr, "F_{abs}"] \arrow[ddr] & & \\
& X^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
& R \arrow[r, "F_{abs}"'] & R
\end{tikzcd}$$

Given a morphism

$$\alpha_p : \mu_{p^\infty}^g \hookrightarrow X,$$

we define

$$\alpha_p' : \mu_{p^\infty}^g \hookrightarrow X'$$

to be the morphism that makes the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & \mu_p^g \arrow[r] \arrow[d, "\cong"'] & \mu_{p^\infty}^g \arrow[r, "p"] \arrow[d, hook, "\alpha_p"] & \mu_{p^\infty}^g \arrow[r] \arrow[d, dotted, "\alpha_p'"] & 0 \\
0 \arrow[r] & H_{can} \arrow[r] & X \arrow[r] & X' \arrow[r] & 0
\end{tikzcd}$$

Note that a prime-to-p level structure α^p induces a prime-to-p level structure $\alpha^{p\prime}$ on X'. We let

$$\iota^{p\prime} : \mathcal{O}_{\mathcal{K}(p)} \hookrightarrow End(A') \otimes \mathbb{Z}_{(p)}$$

be the embedding induced by

$$\iota^p : \mathcal{O}_{\mathcal{K}(p)} \hookrightarrow End(A) \otimes \mathbb{Z}_{(p)}.$$

As Katz explains in Lemma 1.11.6 of [Kat78], if (X, λ) is in $T_{\infty,r}$, then there is a unique polarization λ' that reduces $\mod p$ to the polarization $\lambda^{(p)}$ on $X^{(p)}$. We shall now also use π to denote the morphism

$$(X, \lambda, \iota, \alpha^p, \alpha_p) \mapsto (X', \lambda', \iota', \alpha^{p\prime}, \alpha_p')$$

induced by π.

Note that, by construction, π is compatible with change in base

$$R/p^m R \to R/p^{m-1}R$$

induced by projection. So the morphisms π induce a morphism

$$A_{\mathrm{univ}}/W \to A'_{\mathrm{univ}}/W \tag{5.2}$$

(where $W = \mathrm{Spec}R$) over the p-adic ring R. Since this is not explicitly mentioned in [Kat78] and could be somewhat confusing to the reader, we note that the map π

in (5.2) is defined over R, not over $T_{\infty,\infty}$, though A_{univ} and A'_{univ} lie over $T_{\infty,\infty}$. So there is a unique isomorphism

$$F : T_{\infty,\infty} \to T_{\infty,\infty}$$

such that A'_{univ} is the fiber product

$$\begin{array}{ccc} A'_{\mathrm{univ}} & \longrightarrow & A_{\mathrm{univ}} \\ \downarrow & & \downarrow \\ T_{\infty,\infty} & \xrightarrow{F} & T_{\infty,\infty} \end{array}$$

We now describe the action of F on q-expansions, which we will use in the proof of Lemma V.9.

Lemma V.4. *For any q-expansion homomorphism*

$$f \mapsto f(q),$$

the action of F on f satisfies

$$(Ff)(q) = f(q^p),$$

and so, if

$$f(q) = \sum_{h \in H^\vee} \left(c(h)q^h\right),$$

then

$$(Ff)(q) = \sum_{h \in H^\vee} \left(c(h)q^{ph}\right).$$

Lemma V.5. *The abelian variety $Mum_L(q)'$ and the morphism*

$$\pi : Mum_L(q) \to Mum_L(q)'$$

a priori defined over $(\mathcal{O}_\mathcal{K})_v(q, L)$ are in fact defined over $\mathcal{O}_\mathcal{K}(q, L)$.

Proof. Since $\mathrm{Mum}_L(q)'$ is obtained from $\mathrm{Mum}_L(q)$ by extension of scalars $q \mapsto q^p$, which is defined over $\mathcal{O}_{\mathcal{K}}(q, L)$, $\mathrm{Mum}_L(q) = Mum_L(q^p)$ is defined over $\mathcal{O}_{\mathcal{K}}(q, L)$. It follows from the definition of π that π is the map making the following diagram commute (where the vertical maps are projection onto the quotient):

$$\begin{tikzcd}
W' \otimes \mathbb{G}_m \arrow[r, "\times p"] \arrow[d] & W' \otimes \mathbb{G}_m \arrow[d] \\
W' \otimes \mathbb{G}_m \arrow[r, "\pi"] & W' \otimes \mathbb{G}_m / p \cdot q(H^\vee) = W' \otimes \mathbb{G}_m / q(p \cdot H^\vee)
\end{tikzcd}$$

∎

Remark V.6. Since π and $\mathrm{Mum}_L(q)'$ are defined over $\mathcal{O}_{\mathcal{K}}$, we can extend scalars and consider the map π over $\mathbb{C}$. In this case, observe that π corresponds to the map on lattices

$$p_z(L) \to p_{pz}(L),$$
$$l \mapsto pl$$

i.e. the map

$$\mathbb{C}/p_z(L) \to \mathbb{C}/p_{pz}(L)$$
$$x \mapsto px.$$

The morphism F corresponds to the morphism

$$\mathcal{H}_n \to \mathcal{H}_n$$
$$z \mapsto pz.$$

Define Fr to be the morphism

$$Fr = \pi^* : F^*(H^1_{DR}) \to H^1_{DR}.$$

Note that Fr defines a (F-linear) morphism of H^1_{DR}.

The higher dimensional analogue of Lemma (A2.1) in [Kat73b] is the following.

Lemma V.7.

$$\pi^* \left(F^* \underline{\omega}\right) = p\underline{\omega}.$$

As in [Kat78], we have the following powerful proposition, which is essential in the construction of the p-adic differential operators.

Proposition V.8. *There is a unique splitting*

$$H^1_{DR} = \underline{\omega} \oplus \underline{U} \tag{5.3}$$

over $\mathcal{O}_{\mathcal{M}}$ such that $\pi^* F^*$ is an isomorphism on $\underline{U}$ when tensored with $\mathbb{Q}$ and such that

$$\nabla(\underline{U}) \subseteq \underline{U} \otimes \Omega$$

The splitting (5.3) is called the *unit root splitting* and $\underline{U}$ the unit root submodule of H^1_{DR}, as in [Kat78] and [Kat73a].

[Kat78] notes simply that the version of Proposition V.8 in [Kat78] (Theorem 1.11.27) is explained in [Kat73a]. If $T_{\infty,\infty}$ were affine (which it is not), then all but the uniqueness statement would follow immediately from [Kat78]. Since $T_{\infty,\infty}$ is not affine and since the reader may not see immediately why Proposition V.8 holds in this instance, we explain here. Let $\{U_i\}_i$ be an affine cover of $T_{\infty,\infty}$ such that each U_i is sufficiently small so that the locally free sheaves H^1_{DR}, $\underline{U}$, and $\underline{\omega}$ are free on U_i. Then by Lemma 4.1 of [Kat73a], for each i, there is a splitting

$$H^1_{DR}|_{U_i} = \underline{\omega}|_{U_i} \oplus \underline{U}_i \tag{5.4}$$

such that

$$\pi^* : F^*\underline{U} \to \underline{U}_i \tag{5.5}$$

is an isomorphism on $\underline{U}_i$ and such that

$$\nabla(\underline{U}_i) \subseteq \underline{U}_i \otimes \Omega.$$

If the splittings (5.4) are unique, then they glue together to give the desired splitting on $T_{\infty,\infty}$. So it suffices to show that the splittings are (5.4) are unique. Since (5.5) is an isomorphism and since Lemma V.7 holds, π^* is of the form

$$\begin{pmatrix} pA & 0 \\ 0 & D \end{pmatrix}$$

with A and D $g \times g$ matrices with entries in $\mathcal{O}_{\mathcal{M}}$ with D invertible, with respect to fixed bases for $\underline{\omega}|_{U_i}$ and $\underline{U}_i$, respectively. Suppose $\underline{U}'_i$ is another sheaf on U_i such that

$$H^1_{DR}|_{U_i} = \underline{\omega}|_{U_i} \oplus \underline{U}'_i$$

such that

$$\pi^* : F^*\underline{U} \to \underline{U}'_i$$

is an isomorphism on $\underline{U}'_i$ and such that

$$\nabla(\underline{U}'_i) \subseteq \underline{U}'_i \otimes \Omega.$$

Then π^* is of the form

$$\begin{pmatrix} pA & 0 \\ 0 & D' \end{pmatrix}$$

with D' invertible, with respect to fixed bases for $\underline{\omega}$ and $\underline{U}'_i$. Therefore, since p is not invertible in $\mathcal{O}_{\mathcal{M}}$,

$$\underline{U} = \underline{U}',$$

i.e. the splitting is unique.

5.1.3 Unit root splitting for the Mumford abelian variety

In this section, we study the unit root splitting over $\mathrm{Mum}_L(q)$, which is important in the proof of Theorem IX.3.

Lemma V.9. *(Analogue of [Kat78] Key Lemma (1.12.7)) Upon extension of scalars to* $(\mathcal{O}_{\mathcal{K}})_v \otimes_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^\vee_{\geq 0}))$*, the elements* $\nabla(D(\gamma))(\omega(w))$ *lie in* $\underline{U} \subseteq H^1_{DR}$ *for each* $\gamma \in H$ *and* $w \in (W')^\vee$*.*

Proof. By Lemma V.5, F, $\mathrm{Mum}_L(q)'$, and

$$\pi : \mathrm{Mum}_L(q) \to \mathrm{Mum}_L(q)'$$

are defined over $\mathcal{O}_{\mathcal{K}}(q, L)$. By Lemmas V.7 and V.8, π^* has the form

$$\begin{pmatrix} pA & 0 \\ 0 & D' \end{pmatrix}$$

with respect to fixed bases for $\underline{\omega}$ and $\underline{U}$ for some $g \times g$ matrixes A and D with entries in $\mathcal{O}_{\mathcal{M}}$ and D invertible. So it suffices to show that

$$\pi^* \left(F^* \left(\nabla \left(D(\gamma)\right) \left(\omega(w)\right)\right)\right) = \nabla \left(D(\gamma)\right) \left(\omega(w)\right) \tag{5.6}$$

for all $\gamma \in H$ and $w \in (W')^\vee$. So it is sufficient to extend scalars to $\mathbb{C}$ and check (5.6) over $\mathbb{C}$.

In our proof, we shall work with $L = \mathcal{O}_{\mathcal{K}}^{2n}$, i.e. the cusp at ∞, and we note that the proof at other cusps is similar. (We choose $L = \mathcal{O}_{\mathcal{K}}^{2n}$ because working in the context of our explicit examples over $\mathbb{C}$ - which all used this lattice - provides the most insight.)

Over $\mathbb{C}$,

$$H^1_{DR} = \mathrm{Hom}_{\mathbb{Z}} \left(p_z(L), \mathbb{C}\right),$$

and $\omega(w)$ is a $\mathbb{C}$-linear combination of the elements du_i. So we are now reduced to proving an assertion about maps of lattices.

By Remark V.6,

$$(\pi^* l)\,\gamma = l(p\gamma)$$

for each $l \in F^* H^1_{DR}$ and $\gamma \in L_z$.

By (3.15) and (3.16), $\nabla(D(\gamma))(du_i)$ lies in the subspace of H^1_{DR} generated by elements of the form

$$\beta_j + \alpha\beta_j' \tag{5.7}$$

or

$$\beta_j + \bar{\alpha}\beta_j' \tag{5.8}$$

$1 \le j \le n$ (in the notation of (3.15) and (3.16)). Therefore, it suffices to show that $\pi^* \circ F^*$ fixes each element of the form (5.7) and each element of the form (5.8).

By Remark V.6 and the definition of β_j, we see that

$$F^*(\beta_j) : p_{pz}(L) \to \mathbb{C}$$

is the $\mathbb{Z}$-linear map defined by

$$\begin{aligned} p \cdot z_j &\mapsto 1 \\ pz_i &\mapsto 0, i \neq j \\ e_i, e_i', pz_i' &\mapsto 0, \text{for all } i. \end{aligned}$$

Similarly,

$$F^*(\beta_j) : p_{pz}(L) \to \mathbb{C}$$

is the $\mathbb{Z}$-linear map defined by

$$p \cdot z'_j \mapsto 1$$

$$p \cdot z'_i \mapsto 0, i \neq j$$

$$e_i, e_i, p \cdot z_i \mapsto 0, \text{for all } i.$$

So

$$\begin{aligned}\pi^* F^* \left(\beta_j + c\beta'_j\right)(l) &= F^* \left(\beta_j + c\beta'_j\right)(pl) \\ &= \left(\beta_j + c\beta'_j\right)(l)\end{aligned}$$

for each $l \in p_z(L)$ and each $c \in \mathcal{O}_{\mathcal{K}}$, in particular for $c = \alpha, \bar{\alpha}$. Therefore,

$$\pi^* F^* \nabla \left(D(\gamma)\right)(du_i) = \nabla \left(D(\gamma)\right)(du_i)$$

for $1 \leq i \leq 2n$. ∎

CHAPTER VI

C^∞-differential operators from the perspective of Shimura

In this chapter, we review C^∞-differential operators (acting on automorphic forms on unitary groups) from the perspective of Shimura, as discussed for example in [Shi00]. In later sections, we reformulate Shimura's differential operators algebreo-geometrically, and then we construct and discuss a p-adic analogue of the C^∞-differential operators. In Proposition VIII.5, we show that the C^∞-differential operators we construct algebreo-geometrically in Chapter VIII are the same as Shimura's differential operators that we discuss in this chapter.

For a matrix z, we use the notation from [Shi00]

$$\Xi(z) = (i(\bar{z} - {}^t z), i(z^* - z)).$$

Let $T = \mathbb{C}_n^n$, and let $\mathcal{H}_n$ be the irreducible hermitian symmetric space of non-compact type defined by

$$\mathcal{H}_n = \{z \in T | i\,(z^* - z) > 0\}.$$

Let $\{\epsilon_\nu\}$ be an $\mathbb{R}$-rational basis of T over $\mathbb{C}$. For $u \in T$, let u_ν be defined by

$$u = \sum_\nu u_\nu \epsilon_\nu.$$

Similarly, for $z \in \mathcal{H}_n$, define $z_\nu \in \mathbb{C}$ by

$$z = \sum_\nu z_\nu \epsilon_\nu.$$

Let $(\rho, V) = (\rho_- \otimes \rho_+, V_- \otimes V_+)$ be a finite-dimensional representation of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. Let e be a positive integer. For finite-dimensional vector spaces X and Y, define $S_e(Y, X)$ to be the vector space of degree e homogeneous polynomial maps of Y into X, i.e. the space of maps h from Y to X such that

$$h(a \cdot y) = a^e h(y)$$

for each $a \in \mathbb{C}$ and $y \in Y$. We let $S_e(Y)$ denote $S_e(Y, \mathbb{C})$. From here on, we identify $S_e(Y, X)$ with $S_e(Y) \otimes X$ via

$$h(u) \otimes x \mapsto h(u)x,$$

for each function h in $S_e(Y) = S_e(Y, \mathbb{C})$ and x in X. Let $Ml_e(Y, X)$ denote the vector space of all $\mathbb{C}$-multilinear maps

$$\underbrace{Y \times \cdots \times Y}_{e \text{ times}} \to X.$$

An element of $Ml_e(Y, X)$ is called *symmetric* if

$$g\left(y_{\pi(1)}, \ldots, y_{\pi(e)}\right) = g(y_1, \ldots, y_e)$$

for each permutation π of $\{1, \ldots, e\}$. As explained in Lemma 12.4 of [Shi00], for each h in $S_e(Y, X)$, there is a unique symmetric element h_* of $Ml_p(Y, X)$ such that

$$h(y) = h_*(y, \ldots, y)$$

for all y in Y. We shall associate $S_e(Y, X)$ with a subspace of $Ml_e(Y, X)$ in this way. We define a representation $(\tau^e, Ml_e(T, \mathbb{C}))$ of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ as follows: Given

$(a,b) \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ and $h \in Ml_e(T,\mathbb{C})$,

$$[\tau^e(a,b)h](u_1,\ldots,u_e) = h({}^t a u_1 b,\ldots,{}^t a u_e b).$$

Thus, we obtain a representation $\rho^e \otimes \tau$ of $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$ on $Ml_e(T,\mathbb{C}) \otimes X = Ml_e(T,X)$ via

$$[\rho \otimes \tau^e(g)](h(u) \otimes x) = \tau^e(g)h \otimes \rho(g)x$$

for each $g \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$, $h \in Ml_e(T,\mathbb{C})$, and $x \in X$. We also write $\rho \otimes \tau^e$ to denote the restriction of this representation to $S_e(T,X)$.

For $f \in C^\infty(\mathcal{H}_n, V)$, define operators

$$C, D : C^\infty(\mathcal{H}_n, V) \to C^\infty(\mathcal{H}_n, S_1(T,V)) \tag{6.1}$$

by

$$\begin{aligned}
(Df)(u) &= \sum_\nu u_\nu \frac{\partial f}{\partial z_\nu},\\
(Cf)(u) &= (\tau^1(\Xi)Df)(u) = (Df)({}^t\xi u \eta),
\end{aligned}$$

respectively. For $e > 1$, we write $D^e f$ and $C^e f$ to denote $D(D^{e-1}f)$ and $C(C^{e-1}f)$, respectively. The functions $D^e f$ and $C^e f$ have symmetric elements of $Ml_e(T,V)$ as their values, which allows us – as explained in Section 12.1 of [Shi00] – to view them as elements of $C^\infty\left(\mathcal{H}_n, S_e(T,V)\right)$. Therefore, the operators C^e and D^e can be viewed as maps

$$C^\infty(\mathcal{H}_n, V) \to C^\infty(\mathcal{H}_n, S_e(T,V)).$$

In general, the operators C^e and D^e do not map automorphic forms to automorphic forms. They are, however useful for constructing a map from the space of automorphic forms of weight ρ to the space of automorphic forms of weight $\rho \otimes \tau$. Define

$$\begin{aligned}(D_\rho f)(u) &= \rho(\Xi)^{-1} D[\rho(\Xi) f](u) \\ &= (\rho \otimes \tau)\,(\Xi)^{-1} C[\rho(\Xi) f]\end{aligned}$$

and, more generally,

$$(D_\rho^e f) = (\rho \otimes \tau^e)(\Xi)^{-1} C^e[\rho(\Xi) f].$$

The operator D_ρ^e satisfies the following properties ([Shi00]):

$$\begin{aligned}D_\rho^{e+1} &= D_{\rho\otimes\tau} D_\rho^e = D_{\rho\otimes\tau}^e D_\rho \\ D_\rho^e(f\|_\rho \alpha) &= (D_\rho^e f)\|_{\rho\otimes\tau^e}\alpha, \end{aligned} \tag{6.2}$$

for α in G. From (6.2), we see that D_ρ^e maps automorphic forms of weight ρ to automorphic forms of weight $\rho \otimes \tau^e$.

Example VI.1. In the case where $n = 1$ and the representations ρ under consideration are powers of the determinant representation, the operators D_ρ are the usual weight-raising Maass differential operators, as we see in the following example.

Let $n = 1$. Then $\mathcal{H}_n$ is the usual upper half plane in the complex plane

$$\mathcal{H}_n = \mathcal{H}_1 = \{z = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}, y > 0\},$$

and

$$T = \mathbb{C}.$$

We shall choose the element $1 \in \mathbb{C}$ as a basis for $\mathbb{C}$. Let ρ be the representation of $GL_1(\mathbb{C}) \times GL_1(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ given by

$$(a, b) \mapsto (ab)^k.$$

Then for any weight ρ automorphic form with values in $\mathbb{C}$ and $u \in \mathbb{C} = T$,

$$\begin{aligned}(D_\rho f)(u) &= \rho(2y, 2y)^{-1} \frac{\partial}{\partial z} (\rho(2y, 2y) f) \cdot u \\ &= \left((2y)^{-2k} \frac{\partial f}{\partial z} - 2kiyf \right) \cdot u.\end{aligned}$$

Note that we may identify the one-dimensional complex vector space $S_1(T)$ with $\mathbb{C}$ by identifying each element $h \in S_1(T)$ with $h(1)$. Therefore we may view D_ρ as the map

$$f \mapsto (2y)^{-2k} \frac{\partial f}{\partial z} - 2kiyf.$$

Then $D_\rho f$ maps $\mathbb{C}$-valued automorphic forms of weight ρ to $\mathbb{C}$-valued automorphic forms of weight ρ', where ρ' is the representation of $GL_1(\mathbb{C}) \times GL_1(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ given by

$$(a, b) \mapsto (ab)^{k+1}.$$

Let Z be a $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$-stable quotient of $S_e(T)$, and let ϕ_Z denote the projection of $S_e(T) \otimes X$ onto $Z \otimes X$. Then the operator

$$D_\rho^Z = \phi_Z D_\rho^e$$

is a map from the space of automorphic forms of weight ρ to the space of automorphic forms of weight $\rho \otimes \tau_Z$, where τ_Z denotes the restriction of τ to Z.

CHAPTER VII

Some purely algebraic differential operators

In this chapter, we introduce some algebraic differential operators and maps that are key ingredients in the construction of the C^∞- and the p-adic-differential operators.

7.1 Some algebraic differential operators

In this section, we define some purely algebraic differential operators, which will be used to construct the p-adic and C^∞-differential operators. Let S be an $\mathcal{O}_{\mathcal{K}}$-scheme. The notation here is the same as in Section 3.3. We also assume throughout this section:

$$S \to \mathcal{M} \text{ is an } \textit{étale} \text{ morphism.}$$

We also try in this section to be consistent with the notation of [Kat78].

Recall that by (3.3) and (3.4),

$$\nabla(H^{\pm}(A/S)) \subseteq H^{\pm}(A/S) \otimes \Omega_{S/T}. \tag{7.1}$$

So the Gauss-Manin connection induces a connection (through the product rule (3.19) and the fact that (7.1) holds)

$$\nabla : T^{\bullet}(H^{\pm}(A/S)) \to T^{\bullet}(H^{\pm}(A/S)) \otimes \Omega_{S/T}.$$

Let $\rho = \rho_+ \otimes \rho_-$ be a quotient of $\rho_{st}^{\otimes d_1} \otimes \rho_{st}^{\otimes d_2}$ for some d_1 and d_2. Applying the product rule (3.19) again, we get a connection

$$\nabla : H^1_{DR}(A/S)^\rho \otimes T^\bullet(H^+(A/S) \otimes H^-(A/S)) \to H^1_{DR}(A/S)^\rho \otimes T^\bullet(H^+(A/S) \otimes H^-(A/S)) \otimes \Omega_{S/T}.$$

We define a differential operator

$$D^\rho_{A/S} : V_{A/S} := H^1_{DR}(A/S)^\rho \otimes T^\bullet(H^+(A/S) \otimes H^-(A/S)) \to H^1_{DR}(A/S)^\rho \otimes T^{\bullet+1}(H^+(A/S) \otimes H^-(A/S))$$

to be the composition of maps:

(7.2)

$$\begin{tikzcd}
V_{A/S} := H^1_{DR}(A/S)^\rho \otimes T^\bullet(H^+(A/S) \otimes H^-(A/S)) \arrow[r, "\nabla"] \arrow[dddr, "D^\rho_{A/S}"'] & V_{A/S} \otimes \Omega_{S/T} \arrow[d, "id\otimes KS"] \\
& V_{A/S} \otimes \underline{\omega}^+(A/S) \otimes \underline{\omega}^-(A/S) \arrow[d, "\iota"] \\
& V_{A/S} \otimes H^+(A/S)) \otimes H^-(A/S) \arrow[d, "="] \\
& H^1_{DR}(A/S)^\rho \otimes T^{\bullet+1}(H^+(A/S) \otimes H^-(A/S))
\end{tikzcd}$$

Remark VII.1. Observe that we can similarly construct an algebraic differential operator

$$\tilde{D}^\rho_{A/S} : H^1_{DR}(A/S)^\rho \otimes \mathrm{Sym}^\bullet(H^+(A/S) \otimes H^-(A/S)) \to H^1_{DR}(A/S)^\rho \otimes \mathrm{Sym}^{\bullet+1}(H^+(A/S) \otimes H^-(A/S))$$

(essentially by replacing $T^\bullet$ with $\mathrm{Sym}^\bullet$ in the definition of $D^\rho_{A/S}$).

In the case where $A = A_{\mathrm{univ}}$ and $S = \mathcal{M}_{R_0}/R_0$, we define

$$D^\rho := D^\rho_{A_{\mathrm{univ}}/\mathcal{M}_{R_0}}$$

We denote by D the morphism

$$D : T^\bullet(H^1_{DR}(A/S)) \to T^{\bullet+2}(H^1_{DR}(A/S))$$

whose restriction to $T^r(H^1_{DR}(A/S))$ is $D^{\otimes r}$.

We write $(D^{\rho}_{A/S})^d$ (resp. $(\tilde{D}^{\rho}_{A/S})^d$) to denote $D^{\rho}_{A/S}$ (resp. $\tilde{D}^{\rho}_{A/S}$) composed with itself d times.

Now we give a formula for the action of D in terms of the basis of invariant one-forms $\alpha_i, \beta_i, \alpha'_i, \beta'_i$ in $H^1_{DR}(C^\infty)$ over $\mathbb{C}$. So that we can consider all representations of interest simultaneously, we consider the representation ρ as a subrepresentation of the tensor algebra. So we may view D^ρ in terms of the restriction to $H^1_{DR}(C^\infty)$ of the morphism D, which is a morphism

$$T^\bullet(H^1_{DR}(C^\infty)) \mapsto T^{\bullet+2}(H^1_{DR}(C^\infty))$$

that is homogeneous of degree two in the sense that D maps $T^r(H^1_{DR}(C^\infty))$ to $T^{r+2}(H^1_{DR}(C^\infty))$. The sheaf $T(H^1_{DR}(C^\infty))$ *is* the sheaf $\mathcal{R}$ of graded non-commutative $\mathcal{O}_{\mathcal{M}}(C^\infty)$-algebras generated by the horizontal sections $\alpha_i, \beta_i, \alpha'_i, \beta'_i$ (defined in Subsection 3.1.1) with no relations other than those in the commutative ring $\mathcal{O}_{\mathcal{M}}(C^\infty)$. Lemma VII.2 gives the action of D on $\mathcal{R}$ explicitly.

Lemma VII.2. *Viewed as a morphism on $\mathcal{R}$, the action of D is defined for all sections f of $\mathcal{O}_{\mathcal{M}}(C^\infty)$ and nonnegative integers $\kappa_i, \kappa'_i, \lambda_i, \lambda'_i$ by*

$$D\left(\left(\prod_{1\le l\le n} \alpha_l^{\kappa_l} \alpha_l'^{\kappa'_l} \beta_l^{\lambda_l} \beta_l'^{\lambda'_l}\right) f\right) =$$

$$\prod_{1\le l\le n} \alpha_l^{\kappa_l} \alpha_l'^{\kappa'_l} \beta_l^{\lambda_l} \beta_l'^{\lambda'_l} \sum_{1\le i,j\le n} \frac{\partial f}{\partial z_{ij}} \cdot (Q_j \cdot P_i)\,, \tag{7.3}$$

with P_i and Q_j the elements of $\mathcal{R}$ defined by

$$P_i = \alpha_i + \sum_{k=1}^n z_{ik}\beta_k + \bar{\alpha}\alpha'_i + \bar{\alpha}\sum_{k=1}^n z_{ik}\beta'_k$$

and

$$Q_j = \alpha_i + \sum_{k=1}^n z_{kj}\beta_k + \alpha\alpha'_i + \alpha\sum_{k=1}^n z_{kj}\beta'_k.$$

For all v and w in $\mathcal{R}$,

$$D(v+w) = D(v) + D(w). \tag{7.4}$$

Proof. Equation (7.4) follows immediately from the definition of D. Since $\alpha_i, \beta_i, \alpha'_i, \beta'_i$ are horizontal,

$$\nabla\left(\left(\prod_{1\le l\le n} \alpha_l^{\kappa_l}\alpha_l'^{\kappa'_l}\beta_l^{\lambda_l}\beta_l'^{\lambda'_l}\right) f\right) = \prod_{1\le l\le n} \alpha_l^{\kappa_l}\alpha_l'^{\kappa'_l}\beta_l^{\lambda_l}\beta_l'^{\lambda'_l} \sum_{1\le i,j\le n} \frac{\partial f}{\partial z_{ij}} \cdot dZ_{ij}.$$

Recall from (3.28) that the injection

$$\Omega \hookrightarrow T^2(H^1_{DR})$$

defined by the Kodaira-Spencer isomorphism is given by

$$dZ_{ij} \mapsto du_{j+n} \otimes du_i.$$

By (3.11)–(3.14), we see that in $\mathcal{R}$,

$$du_i = P_i$$

$$du_{j+n} = Q_j,$$

for $1 \le i, j \le n$. So now the lemma follows from the definition of D. ∎

Note that since

$$\nabla(H^{\pm}) \subseteq H^{\pm} \otimes \Omega,$$

restriction of D_ρ to $(H^{\pm})^{\rho_\pm}$ gives maps

$$(H^{\pm})^{\rho_\pm} \to (H^{\pm})^{\rho_\pm \otimes \rho_{st}} \otimes (H^{\mp})^{\rho_{st}}.$$

So through the product rule, D induces morphisms

$$(H^1_{DR}(A/S)^+)^{\rho_+} \otimes (H^1_{DR}(A/S)^-)^{\rho_-} \longrightarrow (H^1_{DR}(A/S)^+)^{\rho_+\otimes\rho_{st}} \otimes (H^1_{DR}(A/S)^-)^{\rho_-\otimes\rho_{st}}.$$

7.2 Some algebraically defined maps on automorphic forms

The maps defined in this section will be used in the proofs of Theorems VIII.2 and IX.2, algebraicity theorems about the differential operators defined in Sections 8.1 and 9.1. Our construction is completely analogous to the one in [Kat78], and we follow [Kat78] closely. Our construction here is a general vector-valued construction that generalizes the scalar-valued one in [Kat78] to our higher-dimensional setting.

We work over an $\mathcal{O}_{\mathcal{K}}$-algebra R_0. Let R be an R_0-algebra, and let x be an R-valued point of the moduli scheme $\mathcal{M}_{R_0}$ over R_0, corresponding to a morphism

$$\mathrm{Spec}(R) \to \mathcal{M}$$

over R_0. Let $\underline{X}$ denote the associated abelian variety X with the associated PEL structure. Let λ be an element of $\mathcal{E}_{X/R}$.

Suppose that we are given an R-sub-module

$$\mathrm{Split}(X/R)$$

in $H^1_{DR}(X/R)$ such that the natural map (induced by the inclusions)

$$\omega_{X/R} \oplus \mathrm{Split}(X/R) \rightarrow H^1_{DR}(X/R) \tag{7.5}$$

is an isomorphism and such that

$$H^1_{DR}(X/R)^{\pm} \subseteq \Omega^{\pm}_{X/R} \oplus \mathrm{Split}(X/R).$$

(For example, when $R = \mathbb{C}$ and we work in the C^{∞}-category, the Hodge decomposition gives us a splitting in which we can take $\mathrm{Split}(X/R)$ to be the sheaf of anti-holomorphic one-forms.)

As earlier, we let $\mathbf{M}_\rho(\alpha)(R_0)$ denote the space of automorphic forms over R_0 of weight

$$(\rho = \rho^- \otimes \rho^+, V = V^- \otimes V^+)$$

and level α. Let e and d be positive integers. In this section, we define an R_0-linear map

$$\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d) : \mathbf{M}_{\rho\otimes\tau^e}(R_0) \to V^- \otimes V^+ \otimes (R^n \otimes R^n)^{\otimes e+d}.$$

We identify each automorphic form f in $\mathbf{M}_{\rho\otimes\tau^e}(R_0)$ with the corresponding global section of $(\underline{\omega}^- \otimes \underline{\omega}^+)^\rho \otimes (\Omega^{\otimes e}_{\mathcal{M}/R_0})$, as in Section 2.5.[1] The canonical inclusion

$$\underline{\omega}(X/R)^\pm \hookrightarrow H^1_{DR}(X/R)^\pm$$

and the Kodaira-Spencer map (3.37) induce inclusions

$$(7.6)\quad (\underline{\omega}^-_{X/R} \otimes \underline{\omega}^+_{X/R})^\rho \otimes (\Omega^{\otimes e}_{\mathcal{M}/R_0}) \hookrightarrow ({H^1_{DR}}^- \otimes {H^1_{DR}}^+)^{\rho_-\otimes\rho_+} \otimes ({H^1_{DR}}^+ \otimes {H^1_{DR}}^-)^{\otimes e}.$$

Associate f with its image in $(H^1_{DR} \otimes H^1_{DR})^\rho \otimes ({H^1_{DR}}^+ \otimes {H^1_{DR}}^-)^{\otimes e}$ via the inclusion (7.6). Then for each integer d, $(D^\rho)^d(f)$ is a global section of $({H^1_{DR}}^+ \otimes {H^1_{DR}}^-)^\rho \otimes ({H^1_{DR}}^+ \otimes {H^1_{DR}}^-)^{\otimes e+d}$. Thus,

$$((D^\rho)^d(f))(x) \in ({H^1_{DR}}^-(X/R) \otimes {H^1_{DR}}^+(X/R))^{\rho_-\otimes\rho_+} \otimes \left(({H^1_{DR}}^+(X/R) \otimes {H^1_{DR}}^-(X/R)\right)^{\otimes d+e}.$$

The choice of λ gives isomorphisms

$$\lambda^\pm : \omega^\pm_{X/R} \xrightarrow{\sim} R^n,$$

which induce isomorphisms

$$(\omega^-_{X/R} \otimes \omega^+_{X/R})^{\rho_-\otimes\rho_+} \otimes (\underline{\omega}^-_{X/R} \otimes \underline{\omega}^+_{X/R})^{\otimes e+d} \xrightarrow{\sim} V^- \otimes V^+ \otimes (R^n \otimes R^n)^{\otimes e+d}$$

[1] Recall that we write $(\underline{\omega}^- \otimes \underline{\omega}^+)^{\rho_-\otimes\rho_+}$ to mean $(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}$.

The splitting (7.5) gives projections

$$H^1_{DR}(X/R)^{\pm} \to \omega^{\pm}_{X/R}, \tag{7.7}$$

which induce projections

$$(\omega^{\pm} \oplus \mathrm{Split}(X/R))^{\rho_{\pm}} \to (\omega^{\pm}_{X/R})^{\rho_{\pm}}.$$

The projection (7.7) also induces a projection

$$(H^1_{DR}(X/R)^{-} \otimes H^1_{DR}(X/R)^{+})^{\otimes d+e} \to (\omega^{-}_{X/R} \otimes \omega^{+}_{X/R})^{\tau^{\otimes(d+e)}}$$

and a projection

$$(\omega^{-}_{X/R} \oplus \mathrm{Split}(X/R))^{\rho_-} \otimes (\omega^{+}_{X/R} \oplus \mathrm{Split}(X/R))^{\rho_+} \to (\omega^{-}_{X/R})^{\rho_-} \otimes (\omega^{+}_{X/R})^{\rho_+}.$$

We now define the R_0-linear map

$$\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d) : \mathbf{M}_{\rho\otimes\tau^e}(R_0) \to V^- \otimes V^+ \otimes (R^n \otimes R^n)^{\otimes e+d},$$

as follows. We define $\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)(f)$ to be the image of $(D^{\rho^+\otimes\rho^-})^d(f) \in ({H^1_{DR}}^+)^{\rho^+\otimes\rho^d_{st}} \otimes ({H^1_{DR}}^-)^{\rho^-\otimes\rho^d_{st}}$ under the composition of morphisms given by the diagonal map in the commutative diagram (7.8):

$$\begin{tikzcd}
({H^1_{DR}}^+)^{\rho^+\otimes\rho^d_{st}} \otimes ({H^1_{DR}}^-)^{\rho^-\otimes\rho^d_{st}} \arrow[r, "g\mapsto g(x)"] \arrow[ddr, dotted] & (\omega^{-}_{X/R} \oplus \mathrm{Split}(X/R))^{\rho_-\otimes\rho^d_{st}} \otimes (\omega^{+}_{X/R} \oplus \mathrm{Split}(X/R))^{\rho_+\otimes\rho^d_{st}} \arrow[d, "\mathrm{mod}\ \mathrm{Split}(X/R)"] \\
 & (\underline{\omega}^{-}_{X/R})^{\rho^-\otimes\rho^d_{st}} \otimes (\underline{\omega}^{+}_{X/R})^{\rho^+\otimes\rho^d_{st}} \arrow[d, "\lambda"] \\
 & V^- \otimes V^+ \otimes (R^n \otimes R^n)^{\otimes e+d}
\end{tikzcd} \tag{7.8}$$

For each R-submodule Z that is a $GL_n(R) \otimes GL_n(R)$-stable quotient of $(\underline{\omega}^{-}_{X/R})^{\rho^- \otimes \rho^d_{st}} \otimes (\underline{\omega}^{+}_{X/R})^{\rho^+ \otimes \rho^d_{st}}$, define ϕ_Z to be the projection of $(\underline{\omega}^{-}_{X/R})^{\rho^- \otimes \rho^d_{st}} \otimes (\underline{\omega}^{+}_{X/R})^{\rho^+ \otimes \rho^d_{st}}$ onto Z. Identify Z with $\lambda(Z)$. Then we define

$$\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)^Z = \phi_Z \circ \partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d).$$

CHAPTER VIII

The C^∞-differential operators

8.1 Construction of the C^∞ differential operators

First we construct the C^∞ differential operators. Later, we will define p-adic differential operators through a similar construction.

Let

$$H^1_{DR}(C^\infty) = \underline{\omega}(C^\infty) \oplus \mathrm{Split}(C^\infty) \tag{8.1}$$

be the canonical splitting of the Hodge filtration corresponding to the holomorphic and anti-holomorphic one-forms. Here, $\mathrm{Split}(C^\infty)$ is the sheaf $\overline{\underline{\omega}(C^\infty)}$ of anti-holomorphic one forms. Note that for each derivation $D \in Der(\mathcal{O}_{\mathcal{M}}^{C^\infty}, \mathcal{O}_{\mathcal{M}}^{C^\infty})$,

$$\nabla(D)(\overline{\underline{\omega}(C^\infty)}) \subset \overline{\underline{\omega}(C^\infty)}.$$

Since

$$H^1_{DR}(C^\infty)^\pm \subseteq \underline{\omega}(C^\infty)^\pm \oplus \mathrm{Split}(C^\infty),$$

the splitting (8.1) induces projections

$$H^1_{DR}(C^\infty)^\pm \to \underline{\omega}(C^\infty)^\pm,$$

which induces a projection

(8.2)

$$H^1_{DR}(C^\infty)^\rho \otimes T^\bullet({H^1_{DR}}^+(C^\infty) \otimes {H^1_{DR}}^-(C^\infty)) \to \underline{\omega}(C^\infty)^\rho \otimes T^\bullet(\underline{\omega}^+(C^\infty) \otimes \underline{\omega}^-(C^\infty))$$
$$\xrightarrow{\sim} \underline{\omega}(C^\infty)^\rho \otimes T^\bullet(\Omega(C^\infty)).$$

As usual, we associate $\underline{\omega}^\pm$ with its image in $(H^1_{DR})^\pm$ under the inclusion coming from hypercohomology

$$\underline{\omega} \hookrightarrow H^1_{DR}. \tag{8.3}$$

As in (7.6), the inclusion (8.3) and the Kodaira-Spencer isomorphism (3.37) induce inclusions

$$(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes (\Omega^{\otimes e}_{\mathcal{M}/R_0}) \hookrightarrow (H^1_{DR})^{\rho_- \otimes \rho_+} \otimes ({H^1_{DR}}^- \otimes {H^1_{DR}}^+). \tag{8.4}$$

Restricting D^ρ to the image of (8.4), we get a map

$$(D^\rho)^d|_{(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes (\Omega^{\otimes e}_{\mathcal{M}/R_0})} : (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes (\Omega^{\otimes e}_{\mathcal{M}/R_0}) \to H^1_{DR}(C^\infty)^\rho_- \otimes \rho_+ \otimes ({H^1_{DR}}^+ \otimes {H^1_{DR}}^-)^{\otimes e+d}.$$

We define the C^∞-differential operator $\partial(\rho, C^\infty, d)$ to be the map

$$\partial(\rho, C^\infty, d) : (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes (\Omega(C^\infty)^{\otimes \bullet}) \to (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes (\Omega(C^\infty))^{\otimes \bullet + d})$$

that is the composition of maps in the following commutative diagram:

$$\begin{tikzcd}
(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes (\Omega(C^\infty)^{\otimes \bullet}) \arrow[r, "(D^\rho)^d"] \arrow[ddr, dotted, "\partial(\rho, C^\infty, d)"'] & H^1_{DR}(C^\infty)^\rho \otimes ({H^1_{DR}}^+ \otimes {H^1_{DR}}^-)^{\bullet + d} \arrow[d, "\mathrm{mod\ Split}(C^\infty)"] \\
& (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes (\underline{\omega}^+ \otimes \underline{\omega}^-)^{\otimes \bullet + d} \arrow[d, "\cong"] \\
& (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes (\Omega(C^\infty))^{\bullet + d}
\end{tikzcd}$$

Remark VIII.1. For the reader who worries that defining the differential operators on $(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}$ (instead of $\mathcal{E}_{V,\rho}$) is too *ad hoc* or non-canonical, we note that the differential operators can equivalently be defined as morphisms from $\mathcal{E}_{V,\rho}$ to $\mathcal{E}_{V,(\rho_+\otimes\rho_{st})\otimes(\rho_-\otimes\rho_{st})}$. Indeed, the composition of maps

$$\begin{tikzcd}
\mathcal{E}_{V,\rho} \arrow[rr, dotted] \arrow[d, "(\lambda,v)\mapsto\lambda(v)"'] & & \mathcal{E}_{(V_-\otimes\mathbb{C}^n)\otimes(V_+\otimes\mathbb{C}^n),(\rho_-\otimes\rho_{st})\otimes(\rho_+\otimes\rho_+)} \\
(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \arrow[r, "\partial(\rho,C^\infty,d)"'] & (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes \Omega \arrow[r, "KS"'] & (\underline{\omega}^-)^{\rho_-\otimes\rho_{st}} \otimes (\underline{\omega}^+)^{\rho_+\otimes\rho_{st}} \arrow[u, "v\mapsto(\lambda,\lambda^{-1}(v))"']
\end{tikzcd}$$

gives an equivalent expression of our differential operator as an operator from $\mathcal{E}_{V,\rho}$ to $\mathcal{E}_{(V\otimes(R^n)\otimes(R^n)),(\rho_+\otimes\rho_{st})\otimes(\rho_-\otimes\rho_{st})}$.

Let Z be a $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$-stable quotient of

$$(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes (\Omega_{A_{\mathrm{univ}}/\mathcal{M}}(C^\infty))^{\bullet+d},$$

and let ϕ_Z be the projection of $(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes (\Omega_{A_{\mathrm{univ}}/\mathcal{M}}(C^\infty))^{\bullet+d}$ onto Z. We define the differential operator $\partial(\rho, C^\infty, 1)^Z$ by

$$\phi_Z \circ \partial(\rho, C^\infty, d).$$

8.2 Algebraicity theorem for C^∞-differential operators

The following algebraicity theorems (Theorems VIII.2 and VIII.3) are important for our intended applications. The statement of Theorem VIII.2 and the idea of the proof are essentially the same as what is done in Section 2.4 of [Kat78]; the new parts are our generalizations from Katz's special scalar-valued case to the arbitrary (often vector-valued) case and to the case of projection onto subrepresentations.

Throughout this section, fix an $\mathcal{O}_{\mathcal{K}}$-algebra R with an inclusion

$$\iota_R : R \hookrightarrow \mathbb{C}.$$

In the special case $R = \bar{\mathbb{Q}}$, the statement of Theorem VIII.2 is essentially the same as Theorem 14.9 (2) of [Shi00]. However, the methods or the proof of Theorem VIII.2 are different from the proof in [Shi00]; the proof we present is similar to what is done in Section 2.4 of [Kat78].

We associate each automorphic form f in $\mathbf{M}_{\rho\otimes\tau^e}(R)$ with its image in $\mathbf{M}_{\rho\otimes\tau^e}(\mathbb{C})$ via the extension of scalars induced by ι_R. We also associate each automorphic form in $\mathbf{M}_{\rho\otimes\tau^e}(\mathbb{C})$ with the corresponding holomorphic section of $(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes \underline{\Omega}(C^\infty)^{\otimes e}$ on $\mathcal{M}(C^\infty)$.

Let $x = \underline{X}$ be an R-valued point of $\mathcal{M}_R$, and let λ be an element of $\mathcal{E}_{X/R}$. Suppose there is a splitting over R

$$\mathrm{Split}(X/R) \oplus \underline{\omega}_{X/R} \xrightarrow{\sim} H^1_{DR}(X/R).$$

Then we have an inclusion

$$\mathrm{Split}(X/R) \hookrightarrow H^1(X^{an}_{\mathbb{C}}, \mathbb{C}) \tag{8.5}$$

coming from the composition of maps

$$\begin{tikzcd}
\mathrm{Split}(X/R) \arrow[r, hook, "\text{ext. of scalars via } \iota_R"] \arrow[drr, dotted] & \mathrm{Split}(X/R) \otimes \mathbb{C} \arrow[r, hook, "(8.5)\otimes \mathrm{Id}"] & H^1_{DR}(X/R) \otimes \mathbb{C} \arrow[d, "\cong"] \\
 & & H^1(X^{an}_{\mathbb{C}}, \mathbb{C})
\end{tikzcd}$$

We say that the the pair $(x, \mathrm{Split}(X/R))$ satisfies the condition (†)if the following holds:

The image of $\mathrm{Split}(X/R)$ under the inclusion (8.5) is the antiholomorphic subspace

$$H^{0,1} \subset H^1({X^{an}}_{\mathbb{C}}, \mathbb{C}),$$

$$\text{i.e. } \mathrm{Split}(X/R) \otimes \mathbb{C} = \mathrm{Split}(C^\infty)(x_{\mathbb{C}}). \tag{†}$$

Note that this condition is essentially the same as condition (2.4.2) in [Kat78].

For our intended applications, the only points that will interest us are certain ordinary CM points. We shall see later that for each such ordinary CM point, there is indeed a splitting satisfying condition (†).

Note that in general, we only know that the values of $\partial(\rho, C^\infty, d)(f)$ at points (x, λ) lie in a $\mathbb{C}$-vector space. (This is because even if $f \in \mathbf{M}_\rho(R)$, we only know that $\partial(\rho, C^\infty, d)(f)$ is a C^∞-function and nothing about where its values at arbitrary points lie.) We see in theorem VIII.2, however, that we can say much more about the values of $\partial(\rho, C^\infty, d)(f)$ at points satisfying (†).

Theorem VIII.2. *Suppose that* $(x, \mathrm{Split}(X/R))$ *is an* R*-valued point of* $\mathcal{M}_R$ *that satisfies condition* (†)*. Let* f *be an automorphic form in* $\mathbf{M}_{\rho\otimes\tau^e}(R)$ *with values in an* R*-module* V*. Then*

$$(\partial(\rho, C^\infty, d)f)(x, \lambda)_{\mathbb{C}} = \iota_R(\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)f). \tag{8.6}$$

Therefore,

$$(\partial(\rho, C^\infty, d)f)(x, \lambda)_{\mathbb{C}} \in V \otimes_R (R^n \otimes R^n)^{\otimes d}.$$

The proof we provide is similar to Katz's proof of Theorem 2.4.5 in [Kat78].

Proof. As it was defined in Section 7.2, $\iota_R(\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)f)$ lies in

$$V \otimes_R (R^n \otimes R^n)^{\otimes d}.$$

So to prove the theorem, it suffices to prove that Equation (8.6) holds.

By the extension of scalars from R to $\mathbb{C}$ given by ι_R, we associate the automorphic form f with its image $f_{\mathbb{C}}$ in $\mathbf{M}_{\rho\otimes\tau^e}(\mathbb{C})$, the R-valued point x with R-basis λ with x with basis $\lambda_{\mathbb{C}}$, and $\mathrm{Split}(X/R)$ with its image in $\mathrm{Split}(X/R) \otimes \mathbb{C}$. Then, we see that

$$\iota_R(\partial(\rho, e, x_{\mathbb{C}}, \lambda_{\mathbb{C}}, \mathrm{Split}(X/R)_{\mathbb{C}}, d)f_{\mathbb{C}}) = \iota_R(\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)f)$$

is $V \otimes_R (R^n \otimes R^n)^{\otimes d}$-valued. So it suffices to show that Equation (8.6) holds in the case $R = \mathbb{C}$, which we will now do.

Consider the following commutative diagram (8.7). (Note that commutativity in (8.7) follows from the hypothesis that $(x, \mathrm{Split}(X/R))$ satisfies condition (†).)

(8.7)

$$\begin{tikzcd}
(H^1_{DR})^{\rho} \otimes ({H^1_{DR}}^{+} \otimes {H^1_{DR}}^{-})^{\otimes e+d} \arrow[d, "\text{mod Split}(C^{\infty})"'] \arrow[dr, "\text{take fiber at } x"] & \\
(\underline{\omega}^{-})^{\rho_-} \otimes (\underline{\omega}^{+})^{\rho_+}(C^{\infty}) \otimes (\underline{\omega}^{+}(C^{\infty}) \otimes \underline{\omega}^{-}(C^{\infty}))^{\otimes e+d} \arrow[dr, "\text{take fiber at } x"'] & (H^1_{DR}(X/R))^{\rho} \otimes (({H^1_{DR}}^{+} \otimes {H^1_{DR}}^{-})(X/R))^{\otimes e+d} \arrow[d, "\text{mod Split}(X/R)"] \\
& ((\underline{\omega}^{+} \otimes \underline{\omega}^{-})(X/R))^{\rho} \otimes ((\underline{\omega}^{-} \otimes \underline{\omega}^{+})(X/R))^{\otimes e+d} \arrow[d, "\cong \text{ from choice of } \lambda"] \\
& (V \otimes_R \mathbb{C}) \otimes_{\mathbb{C}} (\mathbb{C}^n \otimes \mathbb{C}^n)^{\otimes d}
\end{tikzcd}$$

As usual, we associate f with its image in $(H^1_{DR})^{\rho} \otimes (H^{+} \otimes H^{-})^{\otimes e}$. Then $(\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)f)(x)$ is obtained by applying $(D^{\rho})^d$ to f and composing with the maps along the right side of the commutative diagram (8.7). Similarly, $(\partial(\rho, C^{\infty}, d)f)(x)$ is obtained by applying $(D^{\rho})^d$ to f and composing with the maps along the left side of the commutative diagram (8.7). Therefore, (8.6) holds for all x admitting a splitting satisfying (†). ∎

We also obtain the following generalization of Theorem VIII.2:

Theorem VIII.3. *Suppose that $(x, \mathrm{Split}(X/R))$ is an R-valued point of $\mathcal{M}_R$ that satisfies condition (†). Let f be an automorphic form in $\mathbf{M}_{\rho\otimes\tau^e}(R)$. Let Z be a $GL_n(R) \times GL_n(R)$-stable R-quotient of $\underline{\omega}_R^{\rho} \otimes (\Omega_{A_{\mathrm{univ}}/\mathcal{M}_R})^{\bullet+d}$, and let ϕ_Z be the projection of $\underline{\omega}(C^{\infty})^{\rho} \otimes (\Omega_{A_{\mathrm{univ}}/\mathcal{M}}(C^{\infty}))^{\bullet+d}$ onto $Z \otimes_R \mathbb{C}$. Let Z_x be the fiber of Z at x. Then*

$$(\partial(\rho, C^{\infty}, d)^{Z} f)((X, \lambda, \iota, \alpha), \lambda)_{\mathbb{C}} = \iota_R(\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)^{Z_x} f).$$

Therefore,

$$(\partial(\rho, C^{\infty}, d)^{Z} f)(x, \lambda)_{\mathbb{C}}$$

actually takes values in the R-module $\lambda(Z)$.

Proof. The proof is similar to the proof of Theorem VIII.2, except that we replace the commutative diagram 8.7 with the commutative diagram

(8.8)

$$\begin{tikzcd}
(H^1_{DR})^{\rho} \otimes ({H^1_{DR}}^{+} \otimes {H^1_{DR}}^{-})^{\otimes e+d} \arrow[d, "\text{mod Split}(C^{\infty})"'] \arrow[dr, "\text{take fiber at } x"] & \\
((\underline{\omega}^{-})^{\rho_{-}} \otimes (\underline{\omega}^{+})^{\rho_{+}})(C^{\infty}) \otimes ((H^{+} \otimes H^{-})(C^{\infty}))^{\otimes e+d} \arrow[d, "\phi_Z"'] & (H^1_{DR}(X/R))^{\rho} \otimes ((H^{+} \otimes H^{-})(X/R))^{\otimes e+d} \arrow[d, "\text{mod Split}(X/R)"] \\
Z \arrow[dr, "\text{take fiber at } x"'] & ((\underline{\omega}^{-})^{\rho_{-}} \otimes (\underline{\omega}^{+})^{\rho_{+}}(X/R))^{\rho} \otimes ((\underline{\omega}^{+} \otimes \underline{\omega}^{-})(X/R))^{\otimes e+d} \arrow[d, "\phi_{Z_X}"] \\
 & Z_X \arrow[d, "\cong \text{ from choice of } \lambda"] \\
 & \mathbb{C}^{rank(Z_X)}
\end{tikzcd}$$

Now, the proof goes in the same way as the proof of Theorem VIII.2. ■

Note that similarly to in Remark VIII.1, restriction to Z yields a canonical map

$$\mathcal{E}_{V,\rho} \to \mathcal{E}_{Z}.$$

8.3 Some properties of the C^{∞}-differential operators

In this section, we give some fundamental properties of the C^{∞}-differential operators $\partial(\rho, C^{\infty}, d)$.

We denote by $(\partial(\rho, C^\infty, 1))^d$ the composition of $\partial(\rho, C^\infty, d)$ with itself d times.

Theorem VIII.4. *The differential operators satisfy the following properties.*

1.

$$\partial(\rho, C^\infty, d) = (\partial(\rho, C^\infty, 1))^d \tag{8.9}$$

for each positive integer d.

2. Associating the space of automorphic forms of weight $\rho \otimes \tau^f$ with $(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes \Omega(C^\infty)^{\otimes f}$ via the natural isomorphism induced by the Kodiara-Spencer isomorphism, we have that $\partial(\rho, C^\infty, d)$ is the same as $\partial(\rho \otimes \tau^f, C^\infty, d)$, i.e.

$$\begin{aligned} &\partial(\rho, C^\infty, d)|_{(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes \Omega_{A_{\mathrm{univ}}/\mathcal{M}}(C^\infty)^{\otimes e}} \\ &\qquad = \partial(\rho \otimes \tau^f, C^\infty, d)|_{(\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(C^\infty) \otimes \Omega_{A_{\mathrm{univ}}/\mathcal{M}}(C^\infty)^{\otimes e}} \end{aligned} \tag{8.10}$$

for all non-negative integers $f \leq e$.

3.

$$\partial(\rho \otimes \tau^{d-1}, C^\infty, 1) \circ \cdots \circ \partial(\rho \otimes \tau, C^\infty, 1) \circ \partial(\rho, C^\infty, 1) = \partial(\rho, C^\infty, d) \tag{8.11}$$

for all positive integers $d \geq 2$.

Proof. Recall that $\mathrm{Split}(C^\infty)$ is horizontal with respect to ∇, in the sense that

$$\nabla(\mathrm{Split}(C^\infty)) \subseteq \mathrm{Split}(C^\infty) \otimes \Omega_{A_{\mathrm{univ}}/\mathcal{M}}(C^\infty).$$

So from the definition of D^ρ, we see that

$$D^\rho(\mathrm{Split}(C^\infty)) \subseteq \mathrm{Split}(C^\infty),$$

and hence, for all positive integers d,

$$(D^\rho)^d(\mathrm{Split}(C^\infty)) \subseteq \mathrm{Split}(C^\infty).$$

So

$$(D^\rho)^d \circ (\text{“} \mod \mathrm{Split}(C^\infty)\text{”}) = (D^\rho \circ (\text{“} \mod \mathrm{Split}(C^\infty)\text{”}))^d,$$

where $(D^\rho \circ (\text{“} \mod \mathrm{Split}(C^\infty)\text{”}))^d$ denotes $D^\rho \circ (\text{“} \mod \mathrm{Split}(C^\infty)\text{”})$ composed with itself d times. Therefore, it follows directly from the definition of $\partial(\rho, C^\infty, d)$ that

$$\partial(\rho, C^\infty, d) = (\partial(\rho, C^\infty, 1))^d.$$

So (8.9) holds.

Equation (8.10) follows directly from the definition of τ, our earlier explicit description of the Kodaira-Spencer isomorphism, and the definition of the map $\partial(\rho, C^\infty, d)$ for any representation ρ.

Now we prove that (8.11) holds. Note that for any ρ, $\partial(\rho, C^\infty, 1)$ has image in $\underline{\omega}(C^\infty)^\rho \otimes \sum_{e=1}^{\infty} \Omega^{\otimes e}(C^\infty)$. So (9.2) is a corollary of (8.9) and (8.10). ■

8.4 Explicit formulas for the C^∞-differential operators

In this section, we present explicit formulas for the C^∞-differential operators and compare them to the formulas in [Shi00]. The reader familiar with [BSY92], which handles the symplectic case, can compare our formulas with [BSY92] to see that we have constructed essentially the same C^∞-operators that [BSY92] defines combinatorially. The form in which we have chosen to write our formulas should help the reader see the connection between our operators and the ones in [BSY92]. In particular, the reader may wish to compare our formulas with Formula (2.1) of [BSY92].

Recall that a holomorphic (resp. C^∞) automorphic form of weight $(\rho, V) = (\rho_- \otimes \rho_+, V_- \otimes V_+)$ and level $\Gamma \subseteq GU(\eta_n)$ may be viewed as a holomorphic (resp. C^∞)

function

$$f : \mathcal{H}_n \to V = V_- \otimes V_+$$

(where $V_\pm$ are complex vector spaces) that satisfies

$$f((A\tau + B)(C\tau + D)^{-1}) = \rho(C\tau + D)f(\tau) \tag{8.12}$$

for all

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma.$$

Let D_ρ be Shimura's differential operator discussed in the introduction (cf. section 12.1 of [Shi00]). In Proposition VIII.5, we will show that our C^∞-differential operator $\partial(\rho, C^\infty, d)$ is the *same* as Shimura's C^∞-differential operator D_ρ^d.

Proposition VIII.5. *Let $f : \mathcal{H}_n \to V = V_- \otimes V_+$ be a C^∞-function. Let $\lambda \in \mathcal{E}$. Then*

$$\partial(\rho, C^\infty, 1)\,(\lambda(f)) = \lambda(D_\rho f). \tag{8.13}$$

(In Equation (8.13), λ refers to the induced map from $V_\pm$ to $\underline{\omega}_\pm^{\rho_\pm}$.)

Proof. Define

$$v^\pm_{\lambda_\pm} = \lambda^{-1}(du^\pm_{\lambda_\pm})$$

for each tuple $\lambda_\pm$. Writing f in terms of the basis $v_{\lambda_\pm}$ for $V^\pm$, we have

$$f = \sum_{\lambda_-, \lambda_+} f_{\lambda_-,\lambda_+}\lambda(v^-_{\lambda_-} \otimes v^+_{\lambda_+}), \tag{8.14}$$

for some C^∞ complex valued functions f_{λ_-,λ_+}. The sum in Equation (8.14) is, as usual, over all tuples $\lambda_\pm$ so that $v_{\lambda_\pm}$ is in $V^\pm$. Note that

(8.15)
$$\partial(\rho, C^\infty, 1)(\lambda(f(z))) = \partial(\rho, C^\infty, 1)\left(\sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}\lambda(v^-_{\lambda_-}\otimes v^+_{\lambda_+})\right)$$

$$= \partial(\rho, C^\infty, 1)\left(\sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}(du^- - d\bar{u}^-)_{\lambda_-}\otimes (du^+ - d\bar{u}^+)_{\lambda_+}\right). \tag{8.16}$$

We get from (8.15) to (8.16) by recalling that $\partial(\rho, C^\infty, 1)(\underline{\bar{\omega}}) = 0$, since $\underline{\bar{\omega}}$ is holomorphically horizontal with respect to ∇.

Applying Equations (3.17) and (3.18) to Equation (8.16), we see that

$$\partial(\rho, C^\infty, 1)(\lambda(f(z)))$$
$$= \partial(\rho, C^\infty, 1)\left(\rho(\Xi(z))\left(\sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}(\beta+\bar{\alpha}\beta)_{\lambda_-}\otimes(\beta+\alpha\beta)_{\lambda_+}\right)\right).$$

Now recall that the sections $\beta_i + \bar{\alpha}\beta_i$ and $\beta_i + \alpha\beta_i$ are horizontal for the Gauss-Manin connection. So their image under $\partial(\rho, C^\infty, 1)$ is zero.

Therefore

$$\partial(\rho, C^\infty, 1)(\lambda(f(z))) = D\left(\rho(\Xi(z))\left(\sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}(\beta+\bar{\alpha}\beta)_{\lambda_-}\otimes(\beta+\alpha\beta)_{\lambda_+}\right)\right) \mod \mathrm{Split}(C^\infty),$$

where D is the map (6.1). Applying (3.17) and (3.18) again, we obtain

$$\partial(\rho, C^\infty, 1)(\lambda(f(z)))$$
$$= \rho^{-1}(\Xi(z))D\left(\rho\left(\Xi(z)\right)\left(\sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}(du^- - d\bar{u}^-)_{\lambda_-}\otimes(du^+ - d\bar{u}^+)_{\lambda_+}\right)\right) \mod \mathrm{Split}(C^\infty)$$
$$= \rho^{-1}(\Xi(z))D\left(\rho\left(\Xi(z)\right)\left(\sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}(du^-)_{\lambda_-}\otimes(du^+)_{\lambda_+}\right)\right)$$
$$= \rho^{-1}(\Xi(z))D\left(\rho\left(\Xi(z)\right)\lambda(f)\right).$$

$\blacksquare$

Let Z be a $GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$-stable quotient of

$$\underline{\omega}(C^\infty)^\rho \otimes (\Omega_{A_{\mathrm{univ}}/\mathcal{M}}(C^\infty))^{\bullet+d},$$

and let

$$\mathcal{Z} = H^0(\mathcal{H}_n, Z).$$

Let $\phi_{\mathcal{Z}}$ denote projection onto $\mathcal{Z}$.

Recall that Shimura defines a differential operator $D_\rho^{\mathcal{Z}}$ (see e.g. [Shi00]) by

$$D_\rho^{\mathcal{Z}} = \phi_{\mathcal{Z}} D^d.$$

So as a corollary of Proposition VIII.5, we obtain the following

Corollary VIII.6.

$$D_\rho^{\mathcal{Z}} = \partial(\rho, C^\infty, 1)^Z.$$

From the proof of the above propositions, we see that

$$\text{Image}(\partial(\rho, C^\infty, d)) \subset (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes \mathrm{Sym}^d(\Omega(C^\infty)) \subset (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+} \otimes \Omega(C^\infty)^{\otimes d}, \tag{8.17}$$

where we associate $\mathrm{Sym}^d(\Omega(C^\infty))$ with its image in $\Omega(C^\infty)^{\otimes d}$ as in (1.2). Note that we could have defined another C^∞-differential operators with similar properties to those of $\partial(\rho, C^\infty, d)$, by replacing D^ρ with $\tilde{D}^\rho$ in the construction of $\partial(\rho, C^\infty, d)$. By (8.17), we see that in fact, the differential operators constructed from $\tilde{D}^\rho$ are the *same* differential operators as $\partial(\rho, C^\infty, d)$.

We have seen above that our C^∞-differential operators are the same as the C^∞-differential operators Shimura defines in [Shi00]. For our later applications, as well as a reference for the reader who wishes to apply the differential operators, we now provide explicit formulas for the action of some of the differential operators.

Lemma VIII.7. *Let*

$$f = \sum_{1 \le \lambda_-, \lambda_+ \le n} f_{\lambda_-,\lambda_+} du^-_{\lambda_-} \otimes du^+_{\lambda_+}$$

be an automorphic form of weight $\rho_{st} \otimes \rho_{st}$, viewed as an element of $\underline{\omega}^- \otimes \underline{\omega}^+$. Then

(8.18)

$$\partial(\rho, C^\infty, 1)(f) = \sum_{1 \le \lambda^+ \le n} M_f^{\lambda_+} \cdot \begin{pmatrix} du_1^- \\ \vdots \\ du_n^- \end{pmatrix} \otimes du^+_{\lambda_+} + \sum_{1 \le \lambda^- \le n} du_{\lambda_-} \otimes M_f^{\lambda_-} \cdot \begin{pmatrix} du_1^+ \\ \vdots \\ du_n^+ \end{pmatrix},$$

where $M_f^{\lambda_\pm}$ is the row vector (dependent only on f) defined by

$$\frac{1}{2} \left(df_{1,\lambda_\pm}, df_{2,\lambda_\pm}, \ldots, df_{n,\lambda_\pm} \right) + \left(f_{1,\lambda_\pm}, \ldots, f_{n,\lambda_\pm} \right) \cdot \left(dz^\pm \right) \left(\Xi^\pm \right)$$

Proof. Note that

$$\partial(\rho, C^\infty, 1)(f) = \sum_{1 \le \lambda_-, \lambda_+ \le n} \partial(\rho, C^\infty, 1) \left(f_{\lambda_-,\lambda_+} du^-_{\lambda_-} \otimes du^+_{\lambda_+} \right).$$

We have that

$$\begin{aligned} \partial(\rho, C^\infty, 1) \left(f_{\lambda_-,\lambda_+} du^-_{\lambda_-} \otimes du^+_{\lambda_+} \right) &= \frac{1}{2} df_{\lambda_-,\lambda_+} du^-_{\lambda_-} \otimes du^+_{\lambda_+} + du^-_{\lambda_-} \otimes \frac{1}{2} df du^+ \\ &\quad + f \partial(\rho, C^\infty, 1)(du^-) \otimes du^+ + du^- \otimes f \partial(\rho, C^\infty, 1)(du^+) \end{aligned}$$

So now it suffices to recall $\partial(\rho, C^\infty, 1)(du_{\lambda_\pm})$; applying (3.17) and (3.18), we obtain (8.18).

One can alternatively derive the formula from Proposition VIII.5. ∎

Similarly, we can provide formulae for $\partial(\rho, C^\infty, 1)$ for representations other than ρ_{st}. For now, we conclude with the formulas for $\rho = \mathrm{Sym}^{m_-} \otimes \mathrm{Sym}^{m_+}$ and $\rho = \det^{m_-} \otimes \det^{m_+}$.

Using the above discussion, we see that if

$$f = \tilde{f}(\wedge^n_{du_i^-})^{m_-} \otimes (\wedge^n du_i^+)^{m_+} \tag{8.19}$$

is an automorphic form of weight $\rho = \rho_- \otimes \rho_+$ with $\rho_\pm = \det^{m_\pm}$, then

$$\partial(\rho, C^\infty, 1)(f) = df + \left(m^+ \cdot \det(\Xi^+)^{-1} \cdot (dz)^+ + m^- \cdot \det(\Xi^-)^{-1}(dz)^-\right) f$$

In the following lemma, $\lambda^\pm$ denotes as usual an n-tuple of positive integers $(\lambda_1^\pm, \ldots, \lambda_n^\pm)$.

We have chosen these particular forms ((8.20) and (8.19)) in which to express the action of $\partial(\rho, C^\infty, 1)$ for the determinant and symmetric representations because it helps illustrate the connection with Formula (2.1) for the operators on Siegel modular forms defined in [BSY92].

Lemma VIII.8. *Let $f =$ be an automorphic form of weight $\rho = \rho^- \otimes \rho^+$, with*

$$\rho^\pm = Sym^{m_\pm}$$

for some integers $m_\pm$. Viewing f as an element $\sum_{\lambda_-, \lambda_+} f_{\lambda_-,\lambda_+}(du^-)^{\lambda_-} \otimes (du^+)^{\lambda_+}$ of $Sym^{m_-}(\underline{\omega}^-) \otimes Sym^{m_+}(\underline{\omega}^+)$, we have the following equality

$$\partial(\rho, C^\infty, 1)\left(\sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}\left(du^-\right)^{\lambda_-} \otimes \left(du^+\right)^{\lambda_+}\right) = \tag{8.20}$$

$$\sum_{\lambda_+} M_f^{\lambda_+} \cdot \begin{pmatrix} du_1^- \\ \vdots \\ du_n^- \end{pmatrix} \otimes \left(du^+\right)^{\lambda_+} + \sum_{\lambda_+} \left(du^-\right)^{\lambda_-} \otimes M_f^{\lambda_-} \cdot \begin{pmatrix} du_1^+ \\ \vdots \\ du_n^+ \end{pmatrix}, \tag{8.21}$$

where $M^{\lambda_\pm}$ is the row vector (dependent only on f) defined by

$$\begin{aligned} M_f^{\lambda_\pm} =& \frac{1}{2}\left(\sum_{\lambda_\mp} df_{\lambda_\mp,\lambda\pm} du_1^{\lambda_1^\mp - 1} \prod_{j\neq 1} du_j^{\lambda_j^\mp}, \ldots, \sum_{\lambda_\pm} df_{\lambda_\mp,\lambda_\pm} du_n^{\lambda_n^\mp - 1} \prod_{j \neq n} du_j^{\lambda_j^\mp}\right) \\ &+ \left(\sum_{\lambda_\mp} \lambda_1^\mp f_{\lambda_-,\lambda_+} (du_1^\mp)^{\lambda_1^\mp - 1} \prod_{j\neq 1} du_j^{\lambda_j^\mp}, \ldots, \sum_{\lambda_\mp} \lambda_n^\mp f_{\lambda_-,\lambda_+} (du_1^\mp)^{\lambda_n^\mp - 1} \prod_{j\neq n} du_j^{\lambda_j^\mp}\right) \cdot (dz^\pm)(\Xi^\pm)^{-1} \end{aligned}$$

One sees that in the case $m_- = m+ = 1$, Lemma VIII.8 specializes to Lemma VIII.7.

Proof. The lemma follows by expressing the action of $\partial(\rho, C^\infty, 1)$ as follows.

$$\partial(\rho, C^\infty, 1)\left(\sum_{\lambda_-,\lambda} f_{\lambda_-,\lambda_+}(du^-)^{\lambda_-}\otimes(du^+)^{\lambda_+}\right) =$$
$$\sum_{\lambda_-,\lambda_+} df_{\lambda_-,\lambda_+}(du^-)^{\lambda_-}\otimes(du^+)^{\lambda_+} + \sum_{\lambda_-,\lambda_+} f_{\lambda_-,\lambda_+}\partial(\rho, C^\infty, 1)\left((du^-)^{\lambda_-}\otimes(du^+)^{\lambda_+}\right).$$

Now,

$$\begin{aligned}&\partial(\rho, C^\infty, 1)\left((du^-)^{\lambda_-}\otimes(du^+)^{\lambda_+}\right)\\ &\qquad = \partial(\rho, C^\infty, 1)\left((du^-)^{\lambda_-}\right)\otimes(du^+)^{\lambda_+} + \left(du^-\right)^{\lambda_-}\otimes\partial(\rho, C^\infty, 1)\left((du^+)^{\lambda_+}\right),\end{aligned}$$

and

$$\begin{aligned}\partial(\rho, C^\infty, 1)\left((du^\pm)^{\lambda_\pm}\right) &= \partial(\rho, C^\infty, 1)\left(du_1^{\pm\lambda_1^\pm}\cdots du_n^{\pm\lambda_n^\pm}\right)\\ &= \sum_{i=1}^n\left(\partial(\rho, C^\infty, 1)\left((du_i^\pm)^{\lambda_i^\pm}\right)\right)\cdot\prod_{j\neq i} du_j^{\pm\lambda_j^\pm}.\end{aligned}$$

Note that

$$\partial(\rho, C^\infty, 1)\left(du_i^{\pm\lambda_i^\pm}\right) = \lambda_i^\pm\left(du_i^\pm\right)^{\lambda_i^\pm-1}\cdot\left(\partial(\rho, C^\infty, 1)\left(du_i^\pm\right)\right).$$

So applying (3.17) and (3.18) and putting the above equations together, we obtain Equation (8.20). ∎

The above formulas for the action of $\partial(\rho, C^\infty, 1)$ allow one to write down formulas for the action of $\partial(\rho, C^\infty, d)$ for $d > 1$.

One can also similarly work out formulas for other representations, but the main ones of interest to us for our intended applications are powers of the determinant and symmetric product, which we just finished describing.

CHAPTER IX

The p-adic differential operators

In this chapter, we construct p-adic differential operators that act on p-adic automorphic forms, and we discuss basic properties of these operators. We follow the arguments from [Kat78] closely. Rather than the notation from [Kat78], however, we use the notation from [HLS06] and [Hid04], since this is what we will use in our applications.

Throughout this chapter, let R be an $\mathcal{O}_{\mathcal{K}}$-algebra that is separated for the p-adic topology, and let R_0 be the p-adic completion of R.

9.1 Analogue of the differential operators $\partial(\rho, C^\infty, d)$

We define p-adic differential operators

$$\partial(\rho, p\text{-adic}, d) : (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(p\text{-adic}) \to (\underline{\omega}^-)^{\rho_-} \otimes (\underline{\omega}^+)^{\rho_+}(p\text{-adic}) \otimes \Omega(p\text{-adic})$$

in the same way as the C^∞-differential operators $\partial(\rho, C^\infty, d)$, except that we replace $\mathrm{Split}(C^\infty)$ and $\Omega(C^\infty)$ with $\mathrm{Split}(p\text{-adic})$ and $\Omega(p\text{-adic})$, respectively. Replacing all occurrences of C^∞ with p-adic in the proof of Theorem VIII.4, we see that all the properties of the C^∞-operators given in Theorem VIII.4 also hold for the p-adic operators:

Theorem IX.1. *The p-adic differential operators satisfy the following properties.*

1.

$$\partial(\rho, p\text{-}adic, d) = (\partial(\rho, p\text{-}adic, 1))^d$$

for each positive integer d.

2. Associating the space of p-adic automorphic forms of weight $\rho \otimes \tau^f$ with

$$\underline{\omega}(p\text{-}adic)^\rho \otimes \Omega(p\text{-}adic)^{\otimes f}$$

via the natural isomorphism induced by the Kodiara-Spencer isomorphism, we have that $\partial(\rho, p\text{-}adic, d)$ is the same as $\partial(\rho \otimes \tau^f, p\text{-}adic, d)$, i.e.

$$\begin{aligned} &\partial(\rho, p\text{-}adic, d)|_{\underline{\omega}(p\text{-}adic)^\rho \otimes \Omega_{A_{\mathrm{univ}}/\mathcal{M}}(p\text{-}adic)^{\otimes e}} \\ &\qquad = \partial(\rho \otimes \tau^f, p\text{-}adic, d)|_{\underline{\omega}(p\text{-}adic) \otimes \Omega_{A_{\mathrm{univ}}/\mathcal{M}}(p\text{-}adic)^{\otimes e}} \end{aligned} \tag{9.1}$$

for all non-negative integers $f \leq e$.

3.

$$\partial(\rho \otimes \tau^{d-1}, p\text{-}adic, 1) \circ \cdots \circ \partial(\rho \otimes \tau, p\text{-}adic, 1) \circ \partial(\rho, p\text{-}adic, 1) = \partial(\rho, p\text{-}adic, d) \tag{9.2}$$

for all positive integers $d \geq 2$.

One can construct operators $\partial(\rho, p\text{-adic}, d)^Z$ similarly to how one constructs the operators $\partial(\rho, C^\infty, 1)^Z$. The operators $\partial(\rho, p\text{-adic}, d)^Z$ have properties similar to those of $\partial(\rho, C^\infty, 1)^Z$.

9.2 p-adic arithmeticity result

In this section, following [Kat78], we give a p-adic analogue of the algebraicity theorem VIII.2.

Let $x = \underline{X}$ be an R-valued point of $\mathcal{M}_R$ with an element λ of $\mathcal{E}_{x/R}$. Suppose there is a splitting over R

$$\mathrm{Split}(X/R) \oplus \underline{\omega}_{X/R} \xrightarrow{\sim} H^1_{DR}(X/R).$$

Then we have an inclusion

$$\mathrm{Split}(X/R) \hookrightarrow H^1_{DR}(p\text{-adic})(X/R_0) \tag{9.3}$$

coming from the composition of maps

$$\begin{tikzcd}
\mathrm{Split}(X/R) \arrow[r, hook, "\text{ext. of scalars via } R\hookrightarrow R_0"] \arrow[drr, dotted] & \mathrm{Split}(X/R) \otimes R_0 \arrow[r, hook, "(9.3)\otimes \mathrm{Id}"] & H^1_{DR}(X/R) \otimes_R R_0 \arrow[d, "\cong"] \\
 & & H^1_{DR}(p\text{-adic})(X/R_0)
\end{tikzcd}$$

We introduce a p-adic analogue of the condition (†). We say that the the pair $(x, \mathrm{Split}(X/R))$ satisfies the condition (‡)if the following holds:

The image of $\mathrm{Split}(X/R)$ under the inclusion (9.3) is the unit root subspace

$$\underline{U}(X/R_0) \subset H^1_{DR}(p\text{-adic})(X/R_0),$$

$$\text{i.e. } \mathrm{Split}(X/R) \otimes R_0 = \underline{U}(X/R_0). \tag{‡}$$

Let f be an automorphic form of weight ρ over R, and associate it with a corresponding element of $\underline{\omega}_R^{\rho}$. Then by the extension of scalars $R \hookrightarrow R_0$, we can view f as a section $f(p\text{-adic})$ of $\underline{\omega}(p\text{-adic})^{\rho}$.

We now give a p-adic analogue of the algebraicity theorem VIII.2.

Theorem IX.2. *Suppose that $(x, \mathrm{Split}(X/R))$ is an R-valued point of $\mathcal{M}_R$ that satisfies condition (‡), and let λ be an element of $\mathcal{E}_{x/R}$. Let f be an automorphic form in $\mathbf{M}_{\rho\otimes\tau^e}(R)$ with values in the R-module V. Then*

$$(\partial(\rho, p\text{-}adic, d)f(p\text{-}adic))(x, \lambda)_{R_0} = \iota_R(\partial(\rho, e, x, \lambda, \mathrm{Split}(X/R), d)f).$$

Therefore,

$$(\partial(\rho, p\text{-}adic, d) f(p\text{-}adic))(x, \lambda)_{R_0}$$

lies in the R-module $V \otimes (R^n \otimes R^n)^{\otimes d}$.

We note that the only points that matter in our intended applications are certain ordinary CM points. We shall see soon that for each such ordinary CM point there is a splitting that satisfies condition (‡).

The proof of Theorem IX.2 is similar to the proof of Theorem VIII.2: it is obtained by replacing "C^∞" with "p-adic."

We obtain a similar arithmeticity result for the operators $\partial(\rho, p\text{-adic}, d)^Z$.

9.3 Differential operators on p-adic automorphic forms

In this section, we construct a morphism θ that acts on the space of p-adic automorphic forms. The operator θ is a vector-valued analogue of Ramanujan's operator $q\frac{d}{dq}$.

Recall that p-adic automorphic forms are actually certain sections of modules of the form $\mathcal{O}^n_{T_{\infty,\infty}} \otimes_R V$ with (ρ, V) a representation of $GL_n \times GL_n$. Thus, all p-adic automorphic forms appear as certain sections of the tensor product of total tensor algebras

$$T(\mathcal{O}^n_{T_{\infty,\infty}}) \otimes T(\mathcal{O}^n_{T_{\infty,\infty}}),$$

which we associate with the tensor product of the free algebras on n letters

$$\mathfrak{R} = \mathcal{O}_{T_{\infty,\infty}} \otimes_R R\langle T_1, \ldots, T_n\rangle \otimes_R R\langle T_1, \ldots, T_n\rangle,$$

via

$$e_i \otimes e_j \mapsto T_i \otimes T_j, \tag{9.4}$$

where e_i denotes the i-th standard basis element of $\mathcal{O}^n_{T_{\infty,\infty}}$. This viewpoint allows us to consider p-adic automorphic forms as subsections of the algebra $\mathfrak{R}$. In the p-adic modular forms case, this algebra is simply the ring of p-adic modular forms.

We use this viewpoint in this section, not only because it conveniently allows us to consider modular forms of all different weights at once, but also because this viewpoint is important for applications involving construction of families of p-adic automorphic forms of different weights.

While not in general a derivation of $\mathfrak{R}$ over R, the morphism θ that we construct later in this section extends to an R-derivation of the commutative subalgebra

$$\mathcal{O}_{T_{\infty,\infty}} \otimes_R R[T_1, \ldots, T_n] \otimes R[T_1, \ldots T_n].$$

Note that composition of the canonical isomorphism

$$\underline{\omega}_{can} : \mathcal{O}^n_{T_{\infty,\infty}} \otimes \mathcal{O}^n_{T_{\infty,\infty}} \xrightarrow{\sim} \underline{\omega}^-(p\text{-adic}) \otimes \underline{\omega}^+(p\text{-adic})$$

with (9.4) induces an isomorphism

$$\mathfrak{R} \xrightarrow{\sim} T\left(\underline{\omega}^-(p\text{-adic})\right) \otimes T\left(\underline{\omega}^+(p\text{-adic})\right), \tag{9.5}$$

which we shall also denote by $\underline{\omega}_{can}$.

Before further discussing the map θ, we must recall some facts from Chapter IV on the algebraic theory of q-expansions. Recall the canonical isomorphism

$$\omega_{can} : W'^{\vee} \otimes (\mathcal{O}_{\mathcal{K}})_{(p)}((q, H^{\vee}_{\geq 0})) \xrightarrow{\sim} \underline{\omega}$$

over the Mumford object $\mathrm{Mum}_L(q)$ at a cusp H. We associate $W'^{\vee}$ with W, through the canonical identification

$$W' \xrightarrow{\sim} W^{\vee}.$$

Recall also the Kodaira-Spencer isomorphism

$$W \otimes_{\mathcal{O}_{\mathcal{K}}} W \otimes_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)} ((q, H^{\vee}_{\geq 0})) \xrightarrow{\sim} H \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathcal{K}})_{(p)} ((q, H^{\vee}_{\geq 0})).$$

We shall identify $H \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathcal{K}})_{(p)} ((q, H^{\vee}_{\geq 0}))$ with its image in $(\mathcal{O}_{\mathcal{K}})^n_n \otimes_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)} ((q, H^{\vee}_{\geq 0}))$ via the inclusion

$$\begin{aligned} H \otimes_{\mathbb{Z}} (\mathcal{O}_{\mathcal{K}})_{(p)} ((q, H^{\vee}_{\geq 0})) &\hookrightarrow (\mathcal{O}_{\mathcal{K}})^n_n \otimes_{\mathcal{O}_{\mathcal{K}}} (\mathcal{O}_{\mathcal{K}})_{(p)} ((q, H^{\vee}_{\geq 0})) \\ h \otimes a &\mapsto h \otimes a. \end{aligned}$$

Recall from Chapter IV that over the Mumford object at the cusp H at ∞,

$$\omega^{+}(e_i) \otimes \omega^{-}(e_j) = KS\left(D\left(e_{ij}\right)\right),$$

where e_i denotes the i-th standard basis vector of $W = \mathcal{O}_{\mathcal{K}}^n$ and e_{ij} is the element of $(\mathcal{O}_{\mathcal{K}})^n_n$ with a 1 in the i-th row of the j-th column and zeroes everywhere else.

We are now in a position to define the map θ and state some of its fundamental properties.

Theorem IX.3. *There exists a morphism θ of $\mathfrak{R}$ such that the following hold:*

1. *The diagram*

$$\begin{tikzcd} \mathfrak{R} \arrow[r, dotted, "\theta"] \arrow[d, "\cong \underline{\omega}_{can}"'] & \mathfrak{R} \arrow[d, "\cong \underline{\omega}_{can}"] \\ T(\underline{\omega}(p\text{-}adic)^{-}) \otimes T(\underline{\omega}(p\text{-}adic)^{+}) \arrow[r, "\partial(\rho, p\text{-}adic, 1)"] & T(\underline{\omega}(p\text{-}adic)^{-}) \otimes T(\underline{\omega}(p\text{-}adic)^{+}) \end{tikzcd}$$

   commutes.

2. *The morphism θ is "homogeneous of degree $1 \otimes 1$", in the sense that if x_1 and x_2 are homogeneous elements of $\mathcal{O}_{T_{\infty,\infty}} \otimes_R R\langle T_1, \ldots, T_n\rangle$ of degrees d_1 and d_2, respectively, then $\theta(x_1 \otimes x_2)$ is homogeneous of degree $(d_1 + 1) \otimes (d_2 + 1)$.*

3. *On the commutative subalgebra* $\mathcal{O}_{T_{\infty,\infty}}[T_1, \ldots, T_n] \otimes_{\mathcal{O}_{T_{\infty,\infty}}} \mathcal{O}_{T_{\infty,\infty}}[T_1, \ldots, T_n]$ *of* $\mathfrak{R}$, θ *is a derivation.*

4. *If* f *is a* p*-adic automorphic form with* q*-expansion at the cusp at infinity* (L, H) *given by*

$$f(q) = \sum_{h=(h_{ij})\in H^\vee_{\geq 0}} \left(c(h)q^h\right),$$

with $c(h) \in R\langle T_1, \ldots, T_n\rangle \otimes R\langle T_1, \ldots, T_n\rangle$, *then*

$$(\theta(f))(q) = \sum_{h=(h_{ij})\in H^\vee_{\geq 0}} \left(d(h)q^h\right), \tag{9.6}$$

where

$$d(h) = \sum_{i,j} h_{ij}c(h) \cdot (T_j \otimes T_i).$$

Proof. We define the morphism θ by

$$\theta = \underline{\omega}_{can}^{-1} \circ \partial(\rho, p\text{-adic}, 1) \circ \underline{\omega}_{can}.$$

Recall that for any positive integers d and e, $\partial(\rho, p\text{-adic}, 1)$ maps each element of $T^d(\underline{\omega}(p\text{-adic})^-)\otimes T^e(\underline{\omega}(p\text{-adic})^+)$ to an element of $T^{d+1}(\underline{\omega}(p\text{-adic})^-)\otimes T^{e+1}(\underline{\omega}(p\text{-adic})^+)$. So θ maps homogenous elements of degree $d \otimes e$ in $\mathfrak{R}$ to homogeneous elements of degree $(d+1) \otimes (e+1)$ in $\mathfrak{R}$. It follows from the definition of $\partial(\rho, p\text{-adic}, 1)$ that θ is a R-derivation of the commutative subalgebra

$$\mathcal{O}_{T_{\infty,\infty}} \otimes_R R[T_1, \ldots, T_n] \otimes_R R[T_1, \ldots T_n]$$

of $\mathfrak{R}$.

Now, we will examine the action of θ over the Mumford object $\mathrm{Mum}_L(q)$. By Lemma V.9, the elements $\nabla(D(\gamma))(\omega(w))$ lie in $\underline{U} \subseteq H^1_{DR}$ for each $\gamma \in H$ and $w \in W$. Since $\nabla(D(\gamma))$ is an R-derivation and $\underline{U}$ an R-module, we in fact have that

$$\nabla\left(D\left(\gamma\right)\right)\left(\omega^{\pm}\left(w\right)\right) \in \underline{U} \tag{9.7}$$

for each $w \in W$. Since ∇ is defined through the chain rule and since $\partial(\rho, p\text{-adic}, 1)$ is defined by

$$\partial(\rho, p\text{-adic}, 1) = \nabla \mod \mathrm{Split}(p\text{-adic}),$$

we have that

$$\partial(\rho, p\text{-adic}, 1)\left(\omega^{\pm}(e_{i_1}) \otimes \cdots \otimes \omega^{\pm}(e_{i_r})\right) = 0$$

for each positive integer r and $1 \leq i_1, \ldots, i_r \leq n$. Let f be a section of $\mathcal{O}_{T_{\infty,\infty}}$. Then

$$\partial(\rho, p\text{-adic}, 1)\left(f \cdot \omega^{\pm}(e_{i_1}) \otimes \cdots \otimes \omega^{\pm}(e_{i_r})\right) = \omega^{\pm}(e_{i_1}) \otimes \cdots \otimes \omega^{\pm}(e_{i_r}) \cdot Df.$$

Suppose the value of f at $\mathrm{Mum}_L(q)$ is

$$f(q) = f(\mathrm{Mum}_L(q)) = \sum_{h=(h_{ij}) \in H^{\vee}_{\geq 0}} a(h) q^h.$$

For the standard basis elements $e_{kl} \in H$,

$$\begin{aligned}(D(e_{kl}))(f(q)) &= \sum_{h=(h_{ij}) \in H^{\vee}_{\geq 0}} a(h) \cdot \mathrm{tr}(e_{kl} h) q^h \\ &= \sum_{h=(h_{ij}) \in H^{\vee}_{\geq 0}} a(h) h_{lk} q^h.\end{aligned}$$

So over the Mumford object, we have

$$\begin{aligned}&\nabla(D(e_{kl}))\left(f(q) \cdot \left(\omega^{\pm}(e_{i_1}) \otimes \cdots \otimes \omega^{\pm}(e_{i_r})\right)\right) \\ &\quad = D(e_{kl})(f(q)) \cdot \left(\omega^{\pm}(e_{i_1}) \otimes \cdots \otimes \omega^{\pm}(e_{i_r})\right) \cdot KS(D(e_{kl}) \mod \mathrm{Split}(p\text{-adic}) \\ &\quad = \sum_{h=(h_{ij}) \in H^{\vee}_{\geq 0}} a(h) h_{lk} q^h \cdot \omega^{+}(e_k) \otimes \omega^{-}(e_l) \mod \mathrm{Split}(p\text{-adic}).\end{aligned}$$

Therefore,

$$\nabla f(q) = \sum_{h=(h_{ij}) \in H^{\vee}_{\geq 0}} \sum_{1 \leq l, k, \leq n} a(h) h_{lk} q^h \cdot \omega^{\pm}(e_k) \otimes \omega^{\pm}(e_l) \mod \mathrm{Split}(p\text{-adic}),$$

so

$$\partial(\rho, p\text{-adic}, 1) f(q) = \sum_{h=(h_{ij}) \in H^{\vee}_{\geq 0}} \sum_{1 \leq l, k, \leq n} a(h) h_{lk} q^h \cdot \omega^{\pm}(e_k) \otimes \omega^{\pm}(e_l).$$

Thus, θ acts on q-expansions of automorphic forms as in Equation (9.6). $\blacksquare$

Remark IX.4. The operators θ can be thought of as a vector-valued generalization of Ramanujan's operator $\theta = q\frac{d}{dq}$ and Katz's analogous operator for Hilbert modular forms in [Kat78]. Indeed, in the one-variable situation, our operator takes the form $\theta = q\frac{d}{dq}$ (or, more precisely, $T_1^2 q\frac{d}{dq}$).

Remark IX.5. The proof of Theorem IX.3 shows that in the case of $U(1,1)$, the morphism θ can be viewed as a derivation

$$\mathcal{O}_{T_{\infty,\infty}} \to \mathcal{O}_{T_{\infty,\infty}}.$$

Definition IX.6. For each subrepresentation Z of $\rho \otimes \tau^d$ we also define an operator

$$\theta^Z : (\mathcal{O}_{T_{\infty,\infty}})^\rho \to (\mathcal{O}_{T_{\infty,\infty}})^Z$$

by

$$\theta^Z := \phi_Z \circ \theta^d|_{(\mathcal{O}_{T_{\infty,\infty}})^\rho}.$$

From the definition of θ, we see that

$$\theta^Z = \underline{\omega}_{can}^{-1} \circ \partial(\rho, p\text{-adic}, d)^Z \circ \underline{\omega}_{can}.$$

We note that, in practice, the above discussion of θ can often be simplified according to the properties of the particular representation with which one works. For example, in our intended applications, we will only be interested in representations of the form $\rho_- \otimes \rho_+$ with $\rho_\pm = \det^{k_\pm} \otimes Sym^{l_\pm}$. In this case, we will be able to restrict our discussion to the commutative subalgebra

$$\mathcal{O}_{T_{\infty,\infty}} \otimes_R R[T_1, \ldots, T_n] \otimes_R R[T_1, \ldots, T_n]$$

of $\mathfrak{R}$, on which θ will be a derivation.

Now we compare the values of θ to the values of $\partial(\rho, p\text{-adic}, d)$.

Lemma IX.7. *Let $\underline{A}$ be an R-valued point of $\mathcal{M}(p\text{-}adic)$ that satisfies condition $(\ddagger)$, and let $\omega+$ and ω^- be elements of $\mathcal{E}^+_{\underline{A}/R}$ and $\mathcal{E}^-_{\underline{A}/R}$ respectively. Let $c = (c_{ij}) \in (GL_n \times GL_n)(R_0)$ satisfy*

$$\omega^{\pm} = c^{\pm} \cdot \omega^{\pm}_{can}(\underline{A}). \tag{9.8}$$

Let f be an automorphic form of weight (ρ, V) over R, and let

$$\tilde{f} = \omega^{-1}_{can}(f) \in \underline{\omega}.$$

Then

$$\partial(\rho, p\text{-}adic, d)(\tilde{f})(\underline{A}, \omega) = ((\rho \otimes \tau^d)(c^{-1}))(\theta^d f)(\underline{A}).$$

Proof. We have

$$\partial(\rho, p\text{-adic}, d)(\tilde{f}) = \omega_{can} \circ \theta^d(f).$$

So by (9.8),

$$\begin{aligned}\partial(\rho, p\text{-adic}, d)(\tilde{f})(\underline{A}, \omega) &= (\rho \otimes \tau^d)(c^{-1})\partial(\rho, p\text{-adic}, d)(\tilde{f})(\underline{A}, \omega_{can}(\underline{A})) \\ &= ((\rho \otimes \tau^d)(c^{-1}))(\theta^d f)(\underline{A})\end{aligned}$$

∎

The same method also gives a similar result when we restrict to subrepresentations Z of $\rho \otimes \tau^d$:

Corollary IX.8. *With hypotheses as in Lemma IX.7,*

$$\partial(\rho, p\text{-}adic, d)^Z(f)(\underline{A}, \omega) = ((\rho \otimes \tau^d)|_Z(c^{-1}))(\theta^Z \tilde{f})(\underline{A})$$

As a corollary of Theorem IX.2 and Lemma IX.7, we obtain the following theorem.

Theorem IX.9. *Let $\underline{A}$ be an R-valued point of the moduli scheme $\mathcal{M}$(p-adic), and let f be an automorphic form over R of weight (ρ, V), and let Z be a subrepresentation of $\rho \otimes \tau^d$. Then*

$$((\rho \otimes \tau^d)|_Z(c^{-1}))(\theta^Z \tilde{f})(\underline{A}) = \iota_R(\partial(\rho, e, \underline{A}, \omega, \mathrm{Split}(A/R), d)f).$$

Therefore,

$$((\rho \otimes \tau^d)|_Z(c^{-1}))(\theta^Z \tilde{f})(\underline{A})$$

lies in the R-submodule $Z = R \otimes_R Z$ of $R_0 \otimes_R Z$.

One also obtains similar theorems for subrepresentations Z of $\rho \otimes \tau^d$.

CHAPTER X

Splitting of H^1_{DR} for CM abelian varieties

In this chapter, we discuss conditions under which an abelian variety A/R has a splitting over R

$$H^1_{DR}(A/R) = \underline{\omega} \oplus M$$

that simultaneously satisfies both condition (†)and (‡). Such abelian varieties are important for applications to construction of L-functions via the doubling method.

Let $E \times E'$ be a product of CM algebras E and E', with

$$E = L_1 \times \cdots \times L_{m_E}$$

$$E' = L'_1 \times \cdots \times L'_{m_{E'}}$$

products of CM fields L_i, L'_i such that each field L_i, L'_i is a totally real extension of the CM field $\mathcal{K}$ fixed in Section 1.2. Let

$$\mathfrak{S} = \mathfrak{S}_E \times \mathfrak{S}_{E'}$$

be a CM type for $E \times E'$, with

$$\mathfrak{S}_E = \mathfrak{S}_{L_1} \times \cdots \times \mathfrak{S}_{L_{m_E}}$$

$$\mathfrak{S}'_E = \mathfrak{S}_{L'_1} \times \cdots \times \mathfrak{S}_{L'{}_{m_{E'}}}$$

CM types for E and E'.

Definition X.1. We shall say the CM type $\mathfrak{S}$ **is compatible with** Σ if the following two conditions are both met:

1. For each σ in $\mathfrak{S}_E$, $\sigma|_{\mathcal{K}}$ is in Σ.
2. For each σ in $\mathfrak{S}_{E'}$, $\sigma|_{\mathcal{K}}$ is in $\bar{\Sigma}$.

Suppose $(E \times E', \mathfrak{S})$ is a CM type compatible with $(\mathcal{K}, \Sigma)$. So

$$E = L_1 \times \cdots \times L_{m_E}$$
$$E' = L'_1 \times \cdots \times L'_{m_{E'}}$$

with each L_i and each L'_i a totally real extension of K, i.e. L_i (resp. L'_i) is of the form $F_i \otimes K$ (resp. $F'_i \otimes K$) for some totally real fields F_i. We use the following notation:

$$\mathcal{O}_E = \mathcal{O}_{L_1} \times \cdots \times \mathcal{O}_{L_{m_E}}$$
$$\mathcal{O}_{E'} = \mathcal{O}_{L'_1} \times \cdots \times \mathcal{O}_{L_{m_{E'}}}$$
$$\mathcal{O}_{E\times E'} = \mathcal{O}_E \times \mathcal{O}_{E'}$$
$$\mathcal{O} = \mathcal{O}_{F_1} \times \cdots \times \mathcal{O}_{F_{m_E}} \times \mathcal{O}_{F'_1} \times \cdots \times \mathcal{O}_{F_{m_{E'}}}.$$

Let R be a $\mathbb{Z}_{(p)}$-subalgebra of $\bar{\mathbb{Q}}$ containing each $\mathcal{O}_{L_i}$ and each $\mathcal{O}_{L'_i}$. For each CM type $(L_i, \mathfrak{S}_{L_i})$, there is a natural ring isomorphism

$$\mathcal{O}_{L_i} \otimes R \xrightarrow{\sim} \mathcal{O}_{F_i} \otimes R \times \mathcal{O}_{F_i} \otimes R$$
$$a \otimes r \mapsto \phi_{\mathfrak{S}_i}(a \otimes r) \times \phi_{\mathfrak{S}_i}(\bar{a} \otimes r),$$

where

$$\phi_{\mathfrak{S}_i} : \mathcal{O}_{L_i} \otimes R \to \mathcal{O}_{F_i} \otimes R \cong R^{\mathfrak{S}_i}$$
$$a \otimes r \mapsto \prod_{\sigma \in \mathfrak{S}_i} \sigma(a) r.$$

is the projection A similar isomorphism holds for each $(L'_i, \mathfrak{S}_{L'_i})$. So there is a corresponding ring homomorphism

$$\mathcal{O}_{E\times E'} \otimes R \to \mathcal{O} \otimes R \times \mathcal{O} \times R$$
$$a \otimes r \mapsto (\phi_{\mathfrak{S}}(a)r, \phi_{\mathfrak{S}}(\bar{a})),$$

and for any $\mathcal{O}_{E\times E'} \otimes R$-module M, there is a corresponding $\mathcal{O} \otimes R$-decomposition

$$M \cong M(\mathfrak{S}) \oplus M(\mathfrak{S}),$$

where

$$M(\mathfrak{S}) = \{m \in M | a \cdot m = \phi_{\mathfrak{S}}(a)m \text{ for all } a \in \mathcal{O}_{E\times E'}\}$$
$$M(\bar{\mathfrak{S}}) = \{m \in M | a \cdot m = \phi_{\mathfrak{S}}(\bar{a})m \text{ for all } a \in \mathcal{O}_{E\times E'}\}$$

If M is an invertible $\mathcal{O}_{E\times E'} \otimes R$-module, then so are $M(\mathfrak{S})$ and $M(\bar{\mathfrak{S}})$ as $\mathcal{O} \otimes R$-modules.

Proposition X.2. *[Analogue of Key Lemma 5.1.27 in [Kat78]] Let $(\mathfrak{S}, E \times E')$ be a CM type compatible with $(\Sigma, \mathcal{K})$, and let R be as above. Suppose A is an ordinary CM abelian variety of PEL type over R with complex multiplication by $(\mathcal{O}_{E\times E'}, \mathfrak{S})$. Then*

$$\underline{\omega}_{A/R} = H^1_{DR}(\mathfrak{S}),$$

and the splitting

$$H^1_{DR}(A/R) = H^1_{DR}(\mathfrak{S}) \oplus H^1_{DR}(\bar{\mathfrak{S}})$$

simultaneously satisfies both conditions (†)and (‡).

Proof. Let $H = H^1_{DR}(A/R)$. Since $(A, \mathfrak{S})$ is a CM abelian variety over R,

$$H = H(\mathfrak{S}) \oplus H(\bar{\mathfrak{S}})$$

is an invertible $\mathcal{O}_{E\times E'}\otimes R$-module. So $H(\mathfrak{S})$ and $H(\bar{\mathfrak{S}})$ are invertible $\mathcal{O}\otimes R$-modules. Note that the action of $\mathcal{O}_{E\times E'}$ on $Lie(A^\vee/R)$ is through $a\mapsto\phi_{\bar{\mathfrak{S}}}$. Therefore, in the exact sequence

$$0\to\underline{\omega}\to H(\mathfrak{S})\oplus H(\bar{\mathfrak{S}})\to Lie(A^\vee/R)\to 0,$$

$H(\mathfrak{S})$ maps to 0 in $Lie(A/R)$. So $H(\mathfrak{S})$ is contained in $\underline{\omega}$. Since A is ordinary, $\underline{\omega}$ is an invertible $\mathcal{O}\otimes R$-module. So $H(\mathfrak{S})=\underline{\omega}$. The rest of the proof now follows exactly as in [Kat78] Key Lemma 5.1.27. ■

Let $U(1)^n$ denote the subgroup

$$\underbrace{U(1)\times\cdots\times U(1)}_{n\text{ times}}.$$

of $U(n)$. Consider the natural embedding

$$Sh\left(U(1)^n\times U(1)^n\right)\hookrightarrow Sh(U(n)\times U(n))\hookrightarrow Sh(U(n,n)).$$

The points of $Sh\left(U(1)^n\times U(1)^n\right)$ parametrize abelian varieties isogenous to a CM abelian variety of the form

$$\underbrace{A\times\cdots\times A}_{2n\text{ copies of }A}$$

(where each copy of A is one-dimensional) with CM type

$$\left(\underbrace{(\mathcal{K},\Sigma)\times\cdots\times(\mathcal{K},\Sigma)}_{n\text{ times}}\times\underbrace{\left(\mathcal{K},\bar{\Sigma}\right)\times\cdots\times\left(\mathcal{K},\bar{\Sigma}\right)}_{n\text{ times}}\right).$$

Each abelian variety parametrized by a point of $Sh(U(n)\times U(n))$ is isogenous to an abelian variety parametrized by $Sh\left(U(1)^n\times U(1)^n\right)$. Thus points of $Sh(U(n)\times U(n))$ parametrize CM abelian varieties compatible with Σ. Since each of the abelian varieties in $Sh(U(n)\times U(n))$ is a CM abelian variety of CM type compatible with Σ, we arrive at the following corollary.

Corollary X.3. *Each of the abelian varieties in $Sh(U(n) \times U(n))$ has a splitting simultaneously satisfying conditions* (†) *and* (‡).

Corollary X.3 is crucial to our applications involving the pullback method to construct L-functions. The pullback method only requires evaluating functions at points of $U(n) \times U(n)$. So we have the algebraicity result at all points relevant to construction of L-functions in the pullback method. (Functions are pulled back from $Sh(GU(2V))$ to $Sh(U(V) \times U(V))$.

Remark X.4. Note that the proof of Proposition X.2 shows that there are also other abelian varieties which have a splitting simultaneously satisfying both conditions (†) and (‡). For example, suppose A is a CM abelian variety with CM by a CM field $L = F \otimes K$ in which p splits completely in F, where $\deg F = 2n = \dim A$. Let

$$\mathfrak{S} = \{\sigma_1, \ldots, \sigma_{2n}\}$$

be a CM type for L such that

$$\sigma_i|_{\mathcal{K}} \in \Sigma$$

for $1 \leq i \leq n$ and

$$\sigma_i|_{\mathcal{K}} \in \bar{\Sigma}$$

for $n+1 \leq i \leq 2n$. Then the proof of Proposition X.2 shows that $H^1_{DR}(\mathfrak{S}) \oplus H^1_{DR}(\bar{\mathfrak{S}})$ gives a splitting of H^1_{DR} satisfying conditions (†)and (‡) simultaneously. Thus, there are also abelian varieties (over R) in $Sh(U(2V))$ not in $Sh(U(n) \times U(n))$ that have a splitting (over R) simultaneously satisfying (†) and (†). For our applications involving the pullback method, however, only abelian varieties of the type in Proposition X.2 will be relevant.

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