ANALYSIS OF THE OPTION PRICES IN JUMP DIFFUSION MODELS

by

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CHAPTER I

Introduction

In mathematical finance, the asset price fluctuations are usually modelled by various stochastic processes. Among them, diffusion processes have long served as the most important class. As a special example, the geometric Brownian motion has become a benchmark process to model the asset price dynamics thanks to the huge success of the option pricing theory starting from Black and Scholes (1973). Decades later, nonlinear diffusions, whose volatility is either a deterministic function of the asset price level or driven by another exogenous process (see e.g. Dupire (1998) and Heston (1993)), were employed to better model the empirical phenomenon observed in the derivative market, for example the "skew/smile" pattern of the implied volatility surface.

During the last decade, stochastic processes with jumps have become increasingly popular tools in modelling the market fluctuations largely due to the following reasons: **(i)** processes with jumps are more natural tools to model the catastrophic events in the market; **(ii)** the increasingly accessible high frequency data indicates the asset price trajectory is not continuous in small time scales; **(iii)** compared to diffusion models, models using jump processes are able to produce rich structures on the distribution of asset returns and option implied volatility surfaces (see Cont

and Tankov (2004) Chapter 1); **(iv)** statistical evidence indicates the existence of "small" jumps along with the diffusion component in the asset price dynamics (see Aït-Sahalia and Jacod (2009)).

Given their advantages, models using stochastic processes with jumps also introduce difficulties. For the option pricing problem, one major difficulty is to handle the non-local integral term in the infinitesimal generator of the jump processes. This non-local integral term causes problems both theoretically and numerically. On the one hand, the option price in jump models solves Partial Integro-Differential Equations (PIDE), instead of Partial Differential Equations (PDE) in diffusion models. The regularity theory for PIDEs is not well established as it is for classical PDEs. Therefore, a priori it is not clear whether the PIDE has classical solutions in order to apply the Itô's Lemma to confirm that the option price function is a solution of a PIDE. On the other hand, because of the non-local integral term, the finite difference method cannot be applied directly to solve the PIDE numerically in order to price options for practical purposes. This thesis attempts to provide some approaches to conquer this difficulty from both probabilistic and PDE perspectives.

1.1   Outline of the thesis

The detailed structure of the thesis is as follows. In Chapter II, the value function of the optimal stopping problem for a process with Lévy jumps is studied. It is known that the value function is a generalized solution of a variational inequality. Assuming the diffusion component of the process is non-degenerate and a mild assumption on the singularity of the Lévy measure (see Assumption (H5)), it is shown that the value function is smooth in the continuation region, no matter whether the jumps of the process have finite or infinite variation. Moreover, the global regularity of the

value function is derived. As a direct corollary of the global regularity, the smooth-fit property of the optimal stopping problem is confirmed. These results confirms the intuition that non-degenerate diffusion component dictates the regularity of the value function in the optimal stopping problem for jump processes. This chapter is based on Bayraktar and Xing (2009). Parts of this work as been presented at Quantitative Products Laboratory, Deutsche Bank, February 23, 2009; Department of Mathematics and Statistics, Boston University, February 19, 2009; Department of Mathematics, Rutgers, February 17, 2009; Department of Mathematical Sciences, University of Cincinnati, January 24, 2009. AMS Annual Meeting, Special Session on Financial Mathematics, Washington D. C., January 8, 2009.

Chapter III investigates the regularity of the optimal exercise boundary/free boundary of the American put option for jump diffusions with compound Poisson jumps. It is proved that his optimal exercise boundary is continuously differentiable before the maturity. This differentiability result has been established by Yang et al. (*European Journal of Applied Mathematics* 17(1): 95 - 127, 2006) in the case where the condition $r \geq q + \lambda \int_{\mathbb{R}_+} (e^z - 1) \, \nu(dz)$ is satisfied. When this condition fails, there is a gap between the strike price and the limit of the optimal exercise boundary close to the maturity. We extend the differentiability result to the case where the condition on parameters fails via an unified approach that treats both cases simultaneously. It is also shown that the boundary is infinitely differentiable under a regularity assumption on the jump distribution (see Corollary 3.5.8). In particular, the optimal exercise boundaries for the American put option in the models of Merton (1976) and Kou (2002) are infinitely differentiable. This chapter is based on Bayraktar and Xing (2008a). Parts of this work has been presented at Quantitative Products Laboratory, Deutsche Bank, February 23, 2009; Department of Mathematics and Statistics,

Boston University, February 19, 2009; Department of Mathematics, Rutgers, February 17, 2009; Department of Mathematics, City University of Hong Kong, February 6, 2009; Department of Systems Engineering & Engineering Management, Chinese University of Hong Kong, February 5, 2009; Department of Industrial Engineering and Logistics Management, Hong Kong University of Science and Technology, February 3, 2009; Department of Mathematical Sciences, University of Cincinnati, January 24, 2009; SIAM Conference on Financial Mathematics & Engineering, New Burnswick, November 22, 2008; 2008 CNA Summer School, Carnegie Mellon, June 2, 2008.

In Chapter IV, the price of the American put option for jump diffusions with compound Poisson jumps is approximated by a sequence of functions. Each of these functions is the value function of an optimal stopping problem for diffusions. It is shown that this approximation sequence converges to the price function uniformly and exponentially fast. This result gives us an efficient numerical algorithm to price American put option in jump diffusion models. Each approximation function is computed iteratively using the classical finite difference methods. Moreover, the convergence and stability of this numerical algorithm are proved. Examples are presented to illustrate the numerical performance of this algorithm. This chapter is based on Bayraktar and Xing (2008b). Parts of this work has been presented at Department of Mathematics, City University of Hong Kong, February 6, 2009; Department of Systems Engineering & Engineering Management, Chinese University of Hong Kong, February 5, 2009; Department of Industrial Engineering and Logistics Management, Hong Kong University of Science and Technology, February 3, 2009.

Along the line of Chapter IV, Chapter V studies the Asian option pricing problem in jump diffusion models. A sequence of functions, which are unique classical

solutions of parabolic differential equations, are constructed to approximated the Asian option value for jump diffusions. It is shown that the convergence is uniform on compact sets and exponential fast. Using this approximation sequence, the price function of Asian option is shown to be the unique solution of an integro-differential equation. This result confirms the assumption in Večeř and Xu (2004). Compared to Chapter IV, some major technical difficulties arise because the pay-off functions may be unbounded and the approximation sequence is not monotonous. As in Chapter IV, an efficient numerical algorithm is proposed to price Asian options in jump diffusions. Numerical convergence and stability are proven and numerical examples are presented. This chapter is based on Bayraktar and Xing (2007). Parts of this work has been presented at 2008 AMS Central Section Meeting, Special Session on Mathematical Finance, Western Michigan University, October 18, 2008; Department of Mathematics, University of Michigan, September 20, 2007.

CHAPTER II

Regularity of the optimal stopping problem for Lévy processes

2.1 Introduction

This chapter studies the finite horizon optimal stopping problem for an n-dimensional jump diffusion process X. In a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such a process $X = \{X_t; t \geq 0\}$ is governed by the following stochastic differential equation:

$$(2.1.1) \qquad dX_t = b(X_{t-}, t)\, dt + \sigma(X_{t-}, t)\, dW_t + d\mathcal{J}_t,$$

in which $W = \{W_t; t \geq 0\}$ is the d-dimensional standard Brownian motion under $\mathbb{P}$ and $\mathcal{J} = \{\mathcal{J}_t; t \geq 0\}$ is a pure jump Lévy process independent of the Brownian motion. This jump process $\mathcal{J}$ can be of finite/infinite activity with finite/infinite variation. We denote the Lévy measure of $\mathcal{J}$ as ν (please refer to Section 2.2 for the definition of $\mathcal{J}$ and its properties).

We investigate the problem of maximizing the discounted terminal reward g by optimally stopping the process X before a fixed time horizon T. The value function of this problem is defined as

$$(2.1.2) \qquad u(x, t) = \sup_{\tau \in \mathcal{T}_{0, T-t}} \mathbb{E}\left[e^{-r\tau} g(X_\tau) \,\middle|\, X_0 = x \right],$$

in which $\mathcal{T}_{0,t}$ is the set of all stopping times (with respect to the filtration $(\mathcal{F})_{0 \leq s \leq t}$) valued between 0 and t. A specific example of such an optimal stopping problem is

the American option pricing problem, where X models the logarithm of the stock price process and g represents the pay-off function.

This value function satisfies, at least intuitively, a variational inequality with a nonlocal integral term (see e.g. Chapter 3 of Bensoussan and Lions (1984)). In general, the value function is not expected to be a smooth solution of the variational inequality. Therefore, notions of generalized solutions are needed to characterize the value function. In the literature, different solution concepts were studied. Pham (1998) showed that the value function of the optimal stopping problem for a controlled jump process is a viscosity solution of a variational inequality using the dynamic programming principle. Lamberton and Mikou (2008) proved that the value function associated to the optimal stopping problem on Lévy processes can be understood as the solution in the distribution sense.

When the jump process X has a nondegenerate diffusion component, intuition tells us that the nondegenerate diffusion component should dominate the jump component, in the sense that the value function can be characterized as a smooth function. This intuition has been confirmed for the partial integro-differential equations associated to the Cauchy problem (e.g. the European option pricing problem) and boundary value problems. For these problems, Sections 1-3 in Chapter 3 of Bensoussan and Lions (1984) and Garroni and Menaldi (1992) proved the existence and uniqueness of second order partial integro-differential equations in both Sobolev and Hölder spaces. These regularity properties ensure that the Cauchy problem and boundary value problems have smooth solutions, as long as the diffusion component is nondegenerate.

On the other hand, for variational inequalities associated to the optimal stopping problems with either finite or infinite activity jumps, Bensoussan and Lions showed

in Theorem 4.4 of Bensoussan and Lions (1984) pp. 250 that the solution of a variational inequality on a bounded domain can be characterized as an element in a certain Sobolev space. These types of variational inequalities were also studied in Chapter 6 of Garroni and Menaldi (2002), where jumps are assumed to be restricted in the bounded domain of the problem. The regularity results for the variational inequality in Bensoussan and Lions (1984) are not enough to ensure the smooth-fit property to hold. Later, these results were extended to variational inequalities on unbounded domains in Jaillet et al. (1990) and Zhang (1994), where processes are assumed to be diffusions or jump diffusions with finite activity jumps. Combining with a probabilistic argument, Jaillet et al. (1990) and Zhang (1994) confirmed the smooth-fit property when there may be finite activity jumps. In addition, assuming jumps have finite activity, Yang et al. (2006) proved that the value function is the unique classical solution of a variational inequality. Following Jaillet et al. (1990) and Pham (1998), Pham (1997) studied the free boundary problem associated to the variational inequality. Bayraktar (2009) also investigated the free boundary problem with alternative techniques. In Pham (1997), Yang et al. (2006) and Bayraktar (2009), the smooth-fit property was proved when the jump has finite activity.

In this chapter, we study the optimal stopping problem (2.1.2) which allows infinite activity jumps. Using the regularity theory for parabolic differential equations, we proved that the value function is the unique solution of a variational inequality, on a unbounded domain, in a certain Sobolev space. The smooth-fit property follows directly from our regularity results. Moreover, based on these regularity result, we further show that the value function is smooth inside the continuation region, under a mild assumption on the Lévy measure.

When the jump has infinite activity, the Lévy measure ν has a singularity. This

singularity introduces difficulties in the analysis of the value function regularity. When ν does not have such a singularity (the jump is of finite activity), after applying the non-local integral operator, which appears in the infinitesimal operator of X, to the value function, the resulting function is expected to have the same regularity with the value function (see Yang et al. (2006)). However, when ν has a singularity, the regularity of the resulting function is reduced compared to the regularity of the value function. This reduction in the regularity gives trouble in defining the resulting function, after applying the integral operator to the value function, in the classical sense. When the jump has finite variation, this resulting function is still well defined in the classical sense, thanks to the a priori regularity of the value function coming from the probabilistic argument in Pham (1998). However, when the jump has infinite variation, the a priori regularity no longer ensures that the resulting function is well defined. We overcome this problem using a fixed point theorem and the verification theorem in Lamberton and Mikou (2008). On the other hand, the unbounded jumps also introduce difficulty in estimating the local regularity of the value function. Because of the unbounded jumps, regularity of the value function inside a bounded domain depends on the value function outside this domain (see Lemmas 2.4.1 and 2.7.1 for more precise explanation). We solve this difficulty via an interior estimate technique in Theorem 2.5.11.

The rest of the chapter is organized as follows. In Section 2.2, we introduce the variational inequality and recall two notions of generalized solutions studied in Pham (1998) and Lamberton and Mikou (2008). In Section 2.3 we discuss the finite variation jump case and analyze the regularity of value function in the continuation region. Section 2.4 is devoted to study the global regularity when jumps may have infinite variation. The the global regularity (Theorem 2.4.5) is proved in Section 2.5.

A key estimate, which is needed to prove Theorem 2.4.5 is showed in Section 2.7. As a corollary of this global regularity result, the smooth-fit property is confirmed. Moreover, based on Theorem 2.4.5, Theorem 2.4.8 shows that the value function is $C^{2,1}$ in the continuation region. At last, proofs of several auxiliary lemmas are listed in Section 2.6.

2.2 The optimal stopping problem and the variational inequality

2.2.1 A priori regularity of the value function

Let us first define the pure jump component $\mathcal{J}$ in (2.1.1). According to the Lévy-Itô decomposition (see e.g. Theorem 19.2 in Sato (1999)), $\mathcal{J}$ can be decomposed as

$$(2.2.1) \qquad \mathcal{J}_t = \mathcal{J}_t^\ell + \lim_{\epsilon \downarrow 0} \mathcal{J}_t^\epsilon,$$

in which

$$(2.2.2) \qquad \mathcal{J}_t^\ell = \int_0^t \int_{|y|>1} y \, \mu(ds, dy), \quad \mathcal{J}_t^\epsilon = \int_0^t \int_{\epsilon \leq |y| \leq 1} y \, \widetilde{\mu}(ds, dy),$$

represent large and small jumps respectively. Here μ is a Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^n \setminus \{0\})$. Its mean measure is the Lévy measure ν, which is a positive Radon measure on $\mathbb{R}^n \setminus \{0\}$ with a possible singularity at 0. Even with this possible singularity at 0, the measure ν still satisfies

$$(2.2.3) \qquad \int_{\mathbb{R}^n} (|y|^2 \wedge 1) \, \nu(dy) < +\infty.$$

Here, the norm $|\cdot|$ is the standard Euclidean norm: $|y| \triangleq \left(\sum_{i=1}^n (y^i)^2\right)^{1/2}$. In (2.2.2), $\widetilde{\mu}(ds, dy) = \mu(ds, dy) - ds \, \nu(dy)$ is the compensated Poisson measure. It is also worth noticing that the convergence in the last term of (2.2.1) is the almost sure convergence. Moreover, the convergence is uniform in t on $[0, T]$.

We assume that the drift and the volatility in (2.1.1) are bounded and Lipschitz continuous, i.e., there exists a positive constant $L_{b,\sigma}$ such that

(H1)
$$|b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq L_{b,\sigma}|x - y|, \quad \forall x, y \in \mathbb{R}^n,$$

moreover, $|b(x,t)|$ and $|\sigma(x,t)|$ are bounded on $\mathbb{R}^n \times [0,T]$.

We name the solution of (2.1.1), with the initial condition $X_0 = x$, as X^x. Thanks to (H1), X^x has the following norm estimates.

Lemma 2.2.1. *Let us assume b and σ satisfy* (H1). *Then there exists a positive constant C such that for any $\tau \in \mathcal{S}_{0,t}$ with $t \leq T$ and $x, y \in \mathbb{R}^n$,*

(2.2.4)
$$\mathbb{E}\,|X_\tau^x - X_\tau^y| \leq C\,|x - y|.$$

Moreover, if the Lévy measure satisfies

(H2)
$$\int_{|y|>1} |y|\,\nu(dy) < +\infty,$$

then we have

(2.2.5)
$$\mathbb{E}\,|X_\tau^x| \leq C,$$

(2.2.6)
$$\mathbb{E}\,|X_\tau^x - x| \leq C\,t^{1/2},$$

(2.2.7)
$$\mathbb{E}\left[\sup_{0 \leq s \leq t} |X_s^x - x|\right] \leq C\,t^{1/2}.$$

Remark 2.2.2. Similar estimates were given in Lemma 3.1 of Pham (1998) under a slightly stronger assumption on the large jumps: $\int_{|y|>1} |y|^2\,\nu(dy) < +\infty$. Using the equivalence between the norm $|y|$ and the norm $\sum_{i=1}^n |y^i|$, one could prove Lemma 2.2.1 under assumption (H2). We give its proof in Section 2.6.

For the optimal stopping problem (2.1.2), let us assume the terminal reward $g : \mathbb{R}^n \to \mathbb{R}$ to be a bounded and Lipschitz continuous function, i.e., there exist positive constants K and L such that

(H3)
$$0 \leq g(x) \leq K \quad \text{and}$$

(H4) $$|g(x) - g(y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}^n.$$

Thanks to (H3), the value function u is uniformly bounded by K. Moreover, the Lipschitz continuity of g in (H4) and norm estimates of X in Lemma 2.2.1 ensure that the value function u has the following regularity properties, which follow from the same proof of Proposition 3.3 in Pham (1998) once its Lemma 3.1 is replaced by our Lemma 2.2.1.

Lemma 2.2.3. *Let us assume that g satisfies* (H3) *and* (H4). *Then there exists a constant $L_x > 0$ such that for any $x_1, x_2 \in \mathbb{R}$, $t \in [0, T]$,*

(2.2.8) $$|u(x_1, t) - u(x_2, t)| \leq L_x |x_1 - x_2|.$$

Moreover, if the Lévy measure satisfies (H2), *then there exists a constant $L_t > 0$ such that for any $t_1, t_2 \in [0, T]$, $x \in \mathbb{R}$,*

(2.2.9) $$|u(x, t_1) - u(x, t_2)| \leq L_t |t_1 - t_2|^{1/2}.$$

The Lipschitz continuity of $u(\cdot, t)$ and semi-Hölder continuity of $u(x, \cdot)$ will be useful to show further regularity properties of u in the next three sections.

For the optimal stopping problem, as usual we define the continuation region $\mathcal{C}$ and the stopping region $\mathcal{D}$ as follows:

$$\mathcal{C} \triangleq \{(x, t) \in \mathbb{R}^n \times [0, T) : u(x, t) > g(x)\} \quad \text{and}$$

$$\mathcal{D} \triangleq \{(x, t) \in \mathbb{R}^n \times [0, T) : u(x, t) = g(x)\}.$$

2.2.2 The variational inequality

Intuitively, one can expect from the Itô's Lemma for Lévy processes (see e.g. Proposition 8.18 in Cont and Tankov (2004) pp. 279) that the value function u,

defined in (2.1.2), satisfies the following variational inequality:

$$\min\left\{(-\partial_t - \mathcal{L} + r)\, u(x,t),\, u(x,t) - g(x)\right\} = 0, \quad (x,t) \in \mathbb{R}^n \times [0,T),$$

(2.2.10)

$$u(x,T) = g(x),$$

in which the integro-differential operator $\mathcal{L}$, the infinitesimal generator of X, is defined via a bounded test function ϕ as

(2.2.11)

$$\mathcal{L}\phi(x,t) \triangleq \mathcal{L}_D\phi(x,t) + I\phi(x,t), \quad \text{with} \quad \mathcal{L}_D\phi(x,t) \triangleq \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2\phi}{\partial x^i \partial x^j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial\phi}{\partial x^i}.$$

Here $A = (a_{ij})_{n \times n} \triangleq \frac{1}{2}\sigma(x,t)\sigma(x,t)^T$ is a $n \times n$ matrix and the integral term

(2.2.12)
$$I\phi(x,t) \triangleq \int_{\mathbb{R}^n} \left[\phi(x+y,t) - \phi(x,t) - \sum_{i=1}^{n} y^i \frac{\partial\phi}{\partial x^i}(x,t)\, 1_{\{|y|\leq 1\}}\right] \nu(dy)$$

$$= \int_{\mathbb{R}^n} \left[\phi(x+y,t) - \phi(x,t) - y \cdot \nabla_x\phi(x,t)\, 1_{\{|y|\leq 1\}}\right] \nu(dy).$$

However, one does not know a priori that the value function u is sufficiently regular (i.e., $u \in C^{2,1}(\mathbb{R}^n \times [0,T))$) to justify applying Itô's Lemma. Moreover, the integral term $I\phi(x,t)$ is only well defined in classical sense when ϕ has certain regularity properties. It is sufficient to require that $\phi(\cdot,t) \in C^1(B_\epsilon(x))$, in which $B_\epsilon(x)$ is an open ball in $\mathbb{R}^n$ centered at x with some radius $\epsilon \in (0,1)$, and that $\nabla_x\phi(\cdot,t)$ to be Lipschitz in $B_\epsilon(x)$ uniformly in t, i.e., for $t \in [0,T)$ there exists a positive constant L_B such that

(2.2.13) $$|\nabla_x\phi(x_1,t) - \nabla_x\phi(x_2,t)| \leq L_B|x_1 - x_2|, \quad \text{for } x_1, x_2 \in B_\epsilon(x).$$

Indeed, using these regularity properties of ϕ we have that

(2.2.14) $$I\phi(x,t) = I_\epsilon\phi(x,t) + I^\epsilon\phi(x,t), \quad \text{where}$$

(2.2.15)

$$I^\epsilon \phi(x,t) \;=\; \int_{|y|>\epsilon} [\phi(x+y,t) - \phi(x,t)]\,\nu(dy) - \nabla_x \phi(x,t) \cdot \int_{\epsilon<|y|\leq 1} y\,\nu(dy),$$

(2.2.16)

$$I_\epsilon \phi(x,t) \;=\; \int_{|y|\leq\epsilon} [\phi(x+y,t) - \phi(x,t) - y\cdot\nabla_x\phi(x,t)]\,\nu(dy)$$

$$=\; \int_{|y|\leq\epsilon} \sum_{i=1}^{n} y^i \left(\partial_{x^i}\phi(z_i,t) - \partial_{x^i}\phi(x,t)\right)\nu(dy) \leq \int_{|y|\leq\epsilon} L_B\,|y|^2\nu(dy).$$

In (2.2.16), z_i are some vectors in $\mathbb{R}^n$ with $|z_i - x| < |y|$ and the second equality follows from the mean value theorem, while the inequality follows from the Cauchy-Schwartz inequality and (2.2.13). Note that $\epsilon \int_{\epsilon<|y|\leq 1} \nu(dy) \leq \int_{\epsilon<|y|\leq 1} |y|\,\nu(dy) < \int_{\epsilon<|y|\leq 1} \nu(dy)$ and $\int_{\epsilon<|y|\leq 1} \nu(dy) \leq \frac{1}{\epsilon^2}\int_{\epsilon<|y|\leq 1}|y|^2\,\nu(dy) < +\infty$ from (2.2.3). These inequalities imply that $\int_{\epsilon<|y|\leq 1}|y|\,\nu(dy) < +\infty$. Hence, we have $I\phi(x,t) < +\infty$.

However, given the regularity of u in Lemma 2.2.3, it is not clear that the value function u has the Lipschitz continuous first derivative to ensure that Iu is well defined in the classical sense in the first place. Yet, the value function u is a solution of (4.3.3) in certain weak senses. In the literature different notions of generalized solutions were explored. For example, Pham analyzed the value function of an optimal stopping problem of controlled jump diffusion processes in Pham (1998) and proved that the value function is a unique viscosity solution of a nonlinear variational inequality. In what follows we will introduce the notions that we will need from Pham (1998). Let us define

$$C_1(\mathbb{R}^n \times [0,T]) \triangleq \left\{ \phi \in C^0(\mathbb{R}^n \times [0,T]) : \sup_{(x,t)\in\mathbb{R}^n\times[0,T]} \frac{|\phi(x,t)|}{1+|x|} < +\infty \right\}.$$

We adapt the notion of viscosity solutions used in Definition 2.1 of Pham (1998) into our context and give the following definition.

Definition 2.2.4. (i) Any $u \in C^0(\mathbb{R}^n \times [0,T])$ is a viscosity supersolution (subsolu-

tion) of (4.3.3) if

(2.2.17)
$$\min\{-\partial_t\phi - \mathcal{L}\,\phi + ru, u(x,t) - g(x)\} \geq 0\,(\leq 0),$$

for any function $\phi \in C^{2,1}(\mathbb{R}^n \times [0,T]) \cap C_1(\mathbb{R}^n \times [0,T])$ such that $u(x,t) = \phi(x,t)$ and $u(\tilde{x}, \tilde{t}) \geq \phi(\tilde{x}, \tilde{t})$ $(u(\tilde{x}, \tilde{t}) \leq \phi(\tilde{x}, \tilde{t}))$ for all $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [0,T]$.

(ii) u is a viscosity solution of (4.3.3) if it is both supersolution and subsolution.

Applying the result of Pham (1998) to our setting, we obtain the following result.

Proposition 2.2.5. *If the Lévy measure ν satisfies* (H2), *the value function $u(x,t)$ is a viscosity solution of* (4.3.3).

Proof. Let us first comment that under the assumption (H2), $I\phi(x,t)$ is well defined for $\phi \in C^{2,1}(\mathbb{R}^n \times [0,T]) \cap C_1(\mathbb{R}^n \times [0,T])$. Indeed, for $\phi \in C_1(\mathbb{R}^n \times [0,T])$, we have $|\phi(x + y,t) - \phi(x,t)| \leq C(1 + |y|)$ for some C independent of y. Therefore, in (2.2.15) $\int_{|y|>\epsilon}[\phi(x + y,t) - \phi(x,t)]\,\nu(dy) < +\infty$ as a result of (H2) and the analysis after (2.2.16).

After replacing Lemma 3.1 of Pham (1998) by Lemma 2.2.1, the statement follows from the same proof of Theorem 3.1 in Pham (1998). □

Remark 2.2.6. As a corollary of Theorem 4.1 in Pham (1998), u is also the unique viscosity solution in the sense of Definition 2.2.4. However, this uniqueness result is not necessary for the later development.

Another notion of generalized solution was studied in Lamberton and Mikou (2008). Lamberton and Mikou showed that u is the unique solution of (4.3.3) in the distribution sense. We will summarize the results of Lamberton and Mikou (2008) that will be used in the sequel. Let Ω be an open subset of $\mathbb{R}^n \times (0,T)$, and let us denote by $\mathcal{S}(\Omega)$ the set of all C^∞ functions with the compact support in Ω,

and by $\mathcal{S}'(\Omega)$ the space of distributions. If $v \in S'(\Omega)$, and it is locally integrable, then the action of the distribution v on the test function ϕ is given by

$$\langle v, \phi \rangle = \int_\Omega v(x,t)\phi(x,t)\,dxdt.$$

Therefore, since the value function u is uniformly bounded, even though it is not clear that u has enough regularity to define $Iu(x,t)$ in classical sense, $Iu(x,t)$ can still be defined as a distribution,

$$(2.2.18) \qquad \langle Iu, \phi \rangle \triangleq \int_{\mathbb{R}^n \times (0,T)} u(x,t)\, I^*\phi(x,t)\,dxdt, \qquad \text{for } \phi \in \mathcal{S}(\Omega),$$

in which the adjoint operator I^* is defined as

$$(2.2.19) \qquad I^*\phi(x,t) = \int_{\mathbb{R}^n} \left[\phi(x-y,t) - \phi(x,t) + y \cdot \nabla_x \phi(x,t) 1_{\{|y| \le 1\}} \right] \nu(dy).$$

Note that since ϕ is infinitely differentiable with compact support, $I^*\phi$ is well defined in the classical sense thanks to the analysis in (2.2.15) and (2.2.16).

Using the theory of the Snell envelope, Lamberton and Mikou proved the following result in Theorem 2.8 of Lamberton and Mikou (2008).

Proposition 2.2.7. *The value function $u(x,t)$ is the only continuous and bounded function on $[0,T] \times \mathbb{R}^n$ that satisfies the following conditions:*

(i) $u(x,T) = g(x)$,

(ii) $u \ge g$,

(iii) the distribution $(\partial_t + \mathcal{L} - r)u$ is a nonpositive measure on $\mathbb{R}^n \times (0,T)$, i.e.,

$(\partial_t + \mathcal{L} - r)u \le 0$ *in the distribution sense,*

(iv) on the open set $\{(x,t) \in \mathbb{R}^n \times (0,T) : u(x,t) > g(x)\}$, $(\partial_t + \mathcal{L} - r)u = 0$.

Remark 2.2.8. In Proposition 2.2.7, the inequality (equality) $(\partial_t + \mathcal{L} - r)\,u \leq 0\,(= 0)$ is understood in the distribution sense, i.e., for any open set $\Omega \subset \mathbb{R}^n \times (0, T)$ and any nonnegative function $\phi(x, t) \in \mathcal{S}(\Omega)$,

(2.2.20)
$$\int_\Omega (\partial_t + \mathcal{L} - r)\,u(x, t)\,\phi(x, t)\,dxdt = \int_\Omega u(x, t)\,(-\partial_t + \mathcal{L}^* - r)\,\phi(x, t)\,dxdt \leq 0\,(= 0),$$

where the adjoint operator $\mathcal{L}^*$ is defined as the adjoint operator of the differential part of $\mathcal{L}$ plus the operator I^* in (2.2.19), i.e.,

$$\mathcal{L}^*\phi(x, t) \triangleq \sum_{i,j=1}^n \frac{\partial^2}{\partial x^i \partial x^j}\,(a_{ij}\phi) - \sum_{i=1}^n \frac{\partial}{\partial x^i}\,(b_i\phi) + I^*\phi(x, t).$$

2.2.3 The classical differentiability

We will apply the regularity theory of parabolic differential equations to analyze the classical differentiability of u in the next three sections. We need foremost make sure that Iu is defined in the classical sense. Throughout this chapter, we assume that the Lévy measure ν has a density, which we will denote by $\rho(y)$. Moreover, There exists a positive constants M such that

(H5) $\qquad \rho(y) \leq \dfrac{M}{|y|^{n+\alpha}}, \quad$ for $|y| \leq 1$ and some constant $\alpha \in [0, 2)$.

Remark 2.2.9. The Lévy measures ν, corresponding to Lévy processes widely used in the financial modelling for the single asset case, satisfy (H5) with $n = 1$. In jump diffusions models where ν is a probability measure, if the density $\rho(y)$ is bounded, (H5) is satisfied with sufficiently large M. Examples of this case are Merton's model and Kou's model. On the other hand, if $\rho(y) \in C^0(B_1(0) \setminus \{0\})$ and $\rho(y) \to C/|y|^\beta$ with $0 < \beta < 1$ as $y \to 0$, (H5) is also fulfilled because $\frac{1}{|y|^{1+\alpha}} > \frac{1}{|y|^\beta}$ for any $\alpha \geq 0$ and $|y| \leq 1$.

Moreover, for Lévy processes that are the Brownian motion subordinated by tempered stable subordinators, it follows from (4.25) in Cont and Tankov (2004) that $\rho(y) \to C/|y|^{1+2\beta}$, with $0 \le \beta < 1$, as $y \to 0$. Therefore (H5) is satisfied by choosing $\alpha = 2\beta$ and sufficiently large M. In particular, this class of Lévy processes contains Variance Gamma and Normal Inverse Gaussian where $\beta = 0$ or $1/2$ respectively.

Furthermore, for the generalized tempered stable processes (see Remark 4.1 in Cont and Tankov (2004)) whose Lévy measure is

$$\rho(y) = \frac{C_-}{|y|^{1+\alpha_-}} e^{-\lambda_-|x|} 1_{\{x<0\}} + \frac{C_+}{|y|^{1+\alpha_+}} e^{-\lambda_+ x} 1_{\{x>0\}},$$

with $\alpha_-, \alpha_+ < 2$, (H5) is satisfied by choosing $\alpha = \max\{\alpha_-, \alpha_+, 0\}$ and $M = \max\{C_-, C_+\}$. In particular, CGMY processes in Carr et al. (2003) are special examples of generalized tempered stable processes. In the similar manner, one can also check that the regular Lévy processes of exponential type (RLPE) in Boyarchenko and Levendorskiǐ (2002) also satisfy (H5).

In order to apply the regularity theory of parabolic differential equations to analyze the regularity of u, let us recall the definition of Sobolev spaces and Hölder spaces on pp. 5 and 7 of Ladyženskaja et al. (1968).

Definition 2.2.10. Let Ω be a domain in $\mathbb{R}^n$, $Q_T = \Omega \times (0, T)$ and $\overline{Q_T}$ be the closure of Q_T. $C^{2,1}(Q_T)$ denotes the class of continuous functions on Q_T with continuous classical derivatives on Q_T of the form $\partial_t v$, $\partial_{x^i} v$ and $\partial^2_{x^i x^j} v$ for $i, j \le n$.

For any positive integer $p \ge 1$, $W_p^{2,1}(Q_T)$ is the Banach space consisting of the elements of $L_p(Q_T)$ having generalized derivatives of the form $\partial_t v$, $\partial_{x^i} v$ and $\partial^2_{x^i x^j} v$ for $i, j \le n$. The norm in it is defined as

$$\|v\|_{W_p^{2,1}(Q_T)} = \|\partial_t v\|_{L_p} + \sum_{i=1}^n \|\partial_{x^i} v\|_{L_p} + \sum_{i,j=1}^n \|\partial^2_{x^i x^j} v\|_{L_p},$$

where $\|v\|_{L_p} = \left(\int_0^T \int_\Omega |v(x,t)|^p \; dxdt \right)^{1/p}$. On the other hand, $W_{p,loc}^{2,1}(Q_T)$ is the Banach space consisting of functions whose $W_p^{2,1}$-norm is finite on any compact subset of Q_T.

For any positive nonintegral real number α, $H^{\alpha,\alpha/2}\left(\overline{Q_T}\right)$ is the Banach space of functions v that are continuous in $\overline{Q_T}$, together with continuous classical derivatives of the form $\partial_t^r \partial_x^s v$ for $2r + s < \alpha$, and have a finite norm

$$\|v\|_{\overline{Q_T}}^{(\alpha)} = |v|_x^{(\alpha)} + |v|_t^{(\alpha/2)} + \sum_{2r+s \leq [\alpha]} \|\partial_t^r \partial_x^s v\|^{(0)}, \quad \text{in which}$$

$$\|v\|^{(0)} = max_{Q_T}|v|, \quad \partial_x^s v = \partial_{x^{i_1}}^{j_1} \cdots \partial_{x^{i_k}}^{j_k} v, \; \text{with } j_1 + \cdots + j_k = s,$$

$$|v|_x^{(\alpha)} = \sum_{2r+s=[\alpha]} < \partial_t^r \partial_x^s v >_x^{(\alpha-[\alpha])}, \quad |v|_t^{(\alpha/2)} = \sum_{\alpha-2<2r+s<\alpha} < \partial_t^r \partial_x^s v >_t^{(\frac{\alpha-2r-s}{2})};$$

$$< v >_x^{(\beta)} = \sup_{\substack{(x,t),(x',t) \in \overline{Q_T} \\ |x-x'| \leq \rho_0}} \frac{|v(x,t) - v(x',t)|}{|x-x'|^\beta}, \quad 0 < \beta < 1,$$

$$< v >_t^{(\beta)} = \sup_{\substack{(x,t),(x,t') \in \overline{Q_T} \\ |t-t'| \leq \rho_0}} \frac{|v(x,t) - v(x,t')|}{|t-t'|^\beta}, \quad 0 < \beta < 1,$$

where ρ_0 is a positive constant.

On the other hand, $H^\alpha\left(\overline{\Omega}\right)$ is the Banach space whose elements are continuous functions $v(x)$ on $\overline{\Omega}$ that have continuous derivatives up to order $[\alpha]$ and the following norm finite

$$\|v\|_{\overline{\Omega}}^{(\alpha)} = \sum_{s \leq [\alpha]} \|\partial_x^s v\|^{(0)} + \left|\partial_x^{[\alpha]} v\right|^{(\alpha-[\alpha])}, \quad \text{in which} \quad |v|^{(\beta)} = \sup_{x,x' \in \overline{\Omega}, |x-x'| \leq \rho_0} \frac{|v(x) - v(x')|}{|x-x'|^\beta}.$$

These Hölder norms depend on ρ_0, but for different $\rho_0 > 0$, the corresponding Hölder norms are equivalent. Hence their dependence on ρ_0 will not be noted in the sequel.

2.3   Finite variation jumps and regularity in the continuation region

In this section, based on Pham's result in Proposition 2.2.5, we will analyze the regularity of the value function u when the jump of X has finite variation, i.e.,

$$(2.3.1) \qquad \int_{\mathbb{R}^n} |y| \wedge 1 \, \nu(dy) < +\infty.$$

It is worth noticing that $\int_{|y| \leq 1} |y| \, \nu(dy) < +\infty$ is satisfied when we assume (H5) with $0 \leq \alpha < 1$. As a result, the infinitesimal generator $\mathcal{L}$ can be rewritten as

$$(2.3.2) \ \ \mathcal{L}\phi(x, t) \ = \ \mathcal{L}_D^f \phi(x, t) + I^f \phi(x, t), \quad \text{where}$$

$$(2.3.3) \mathcal{L}_D^f \phi(x, t) \ = \ \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 \phi}{\partial x^i \partial x^j} + \sum_{i=1}^n \left[b_i(x, t) - \int_{|y| \leq 1} y^i \nu(dy) \right] \frac{\partial \phi}{\partial x^i},$$

$$(2.3.4) \ I^f \phi(t, x) \ \triangleq \ \int_{\mathbb{R}^n} [\phi(x + y, t) - \phi(x, t)] \ \nu(dy).$$

Thanks to this reduced integral form and the Lipschitz continuity of $u(\cdot, t)$ (see Lemma 2.2.3), $I^f u(x, t)$ is well defined in the class sense. Indeed

$$I^f u(x, t) \leq \int_{\mathbb{R}} |u(x + y, t) - u(x, t)| \ \nu(dy) \leq L_x \int_{\mathbb{R}} |y| \nu(dy) < +\infty,$$

as a result of (2.3.1) and (H2). Moreover, assuming (H5) with $0 \leq \alpha < 1$, we will show that $I^f u(x, t)$ is Hölder continuous in both variables in the following lemma.

Lemma 2.3.1. *Let Ω be any compact domain in $\mathbb{R}^n$. If the density $\rho(y)$ of the measure ν satisfies* (H5) *with $0 \leq \alpha < 1$, then $I^f u(x, t)$ is Hölder continuous in both variables on $\Omega \times [0, T]$.*

(i) For any $(x_1, t), (x_2, t) \in \Omega \times [0, T]$, there exist constants $C_{\Omega, \beta}$ and C_Ω independent of x_1, x_2 and t, such that

(2.3.5)

when $\alpha = 0:$ $\left| I^f u(x_1, t) - I^f u(x_2, t) \right| \leq C_{\Omega, \beta} |x_1 - x_2|^{1-\beta}, \quad$ for any $\beta \in (0, 1);$

$$(2.3.6) \qquad \text{when } 0 < \alpha < 1: \quad \left| I^f u(x_1, t) - I^f u(x_2, t) \right| \leq C_\Omega |x_1 - x_2|^{1-\alpha}.$$

(ii) *For any* $(x, t_1), (x, t_2) \in \Omega \times [0, T]$, *there exist constants* $D_{\Omega,\beta}$ *and* D_Ω *independent of* t_1, t_2 *and* x, *such that*

(2.3.7)

$$\text{when } \alpha = 0: \quad \left| I^f u(x, t_1) - I^f u(x, t_2) \right| \leq D_{\Omega,\beta} |t_1 - t_2|^{\frac{1-\beta}{2}}, \quad \forall \beta \in (0, 1);$$

$$(2.3.8) \qquad \text{when } 0 < \alpha < 1: \quad \left| I^f u(x, t_1) - I^f u(x, t_2) \right| \leq D_\Omega |t_1 - t_2|^{\frac{1-\alpha}{2}}.$$

Proof. This proof is motived by Proposition 2.5 in Silvestre (2006). We will show the Hölder continuity in x first. Let us break up the integral into two parts:

(2.3.9)

$$\left| I^f u(x_1, t) - I^f u(x_2, t) \right| \leq \int_{\mathbb{R}} |u(x_1 + y, t) - u(x_1, t) - u(x_2 + y, t) + u(x_2, t)| \, \nu(dy)$$

$$\leq I_1 + I_2,$$

in which

$$(2.3.10) \qquad I_1 = \int_{|y| \leq \epsilon} \left[|u(x_1 + y, t) - u(x_1, t)| + |u(x_2 + y, t) - u(x_2, t)| \right] \nu(dy),$$

$$(2.3.11) \qquad I_2 = \int_{|y| > \epsilon} \left[|u(x_1 + y, t) - u(x_2 + y, t)| + |u(x_1, t) - u(x_2, t)| \right] \nu(dy).$$

Here the constant $\epsilon \in (0, 1]$ will be determined later. Since $x \to u(t, x)$ is globally Lipschitz (see Lemma 2.2.3), we have for $i = 1, 2$

$$|u(x_i + y, t) - u(x_i, t)| \leq L_x |y|, \quad |u(x_1 + y, t) - u(x_2 + y, t)| \leq L_x |x_1 - x_2| \quad \text{and}$$

$$|u(x_1, t) - u(x_2, t)| \leq L_x |x_1 - x_2|.$$

Combining these inequalities with (H5), in which $0 \leq \alpha < 1$, we obtain from (2.3.10)

and (2.3.11) that

(2.3.12)
$$I_1 \leq \int_{|y| \leq \epsilon} 2L_x |y|\, \nu(dy) \leq 2L_x M \int_{|y| \leq \epsilon} |y|^{1-n-\alpha} dy = 2L_x M |S_1(0)| \int_0^\epsilon r^{-\alpha} dr$$
$$= \frac{2L_x M |S_1(0)|}{1-\alpha} \epsilon^{1-\alpha},$$

(2.3.13)
$$I_2 \leq \int_{|y|>\epsilon} 2L_x |x_1 - x_2|\, \nu(dy) \leq 2L_x |x_1 - x_2| \int_{|y|>1} \nu(dy) + 2L_x M |x_1 - x_2| \int_{\epsilon < |y| \leq 1} |y|^{-n-\alpha} dy$$
$$= 2L_x |x_1 - x_2| \int_{|y|>1} \nu(dy) + 2L_x M |S_1(0)|\, |x_1 - x_2| \cdot \begin{cases} \dfrac{\epsilon^{-\alpha}-1}{\alpha} & \text{if } 0 < \alpha < 1 \\[2mm] -\log \epsilon & \text{if } \alpha = 0, \end{cases}$$

where $|S_1(0)|$ is the surface area of a unit ball in $\mathbb{R}^n$. Now picking $\epsilon = |x_1 - x_2| \wedge 1$

and noticing that $0 \leq \alpha < 1$, we have

(2.3.14)
$$\epsilon^{1-\alpha} \leq |x_1 - x_2|^{1-\alpha}, \quad \epsilon^{-\alpha} - 1 \leq |x_1 - x_2|^{-\alpha}.$$

Moreover, when $\epsilon = |x_1 - x_2| < 1$,

(2.3.15)
$$-\log \epsilon = \int_{|x_1-x_2|}^1 \frac{1}{z}\, dz \leq \int_{|x_1-x_2|}^1 \frac{1}{z^{1+\beta}}\, dz = \frac{1}{\beta}\left(|x_1 - x_2|^{-\beta} - 1\right) \leq \frac{1}{\beta}|x_1-x_2|^{-\beta} \quad \forall \beta > 0.$$

Hence choosing $\epsilon = |x_1 - x_2| \wedge 1$, we have $-\log \epsilon \leq \frac{1}{\beta}|x_1 - x_2|^{-\beta}$ for any $\beta > 0$.

Combining (2.3.9) and (2.3.12) - (2.3.15), we conclude that

when $0 < \alpha < 1 :$

$$\left| I^f u(x_1, t) - I^f u(x_2, t) \right| \leq \left[\frac{2L_x M |S_1(0)|}{\alpha(1-\alpha)} + 2L_x d^\alpha \int_{|y|>1} \nu(dy) \right] |x_1 - x_2|^{1-\alpha},$$

when $\alpha = 0 :$

$$\left| I^f u(x_1, t) - I^f u(x_2, t) \right|$$
$$\leq \left[2L_x M |S_1(0)|\, d^\beta + \frac{2L_x M |S_1(0)|}{\beta} + 2L_x d^\beta \int_{|y|>1} \nu(dy) \right] |x_1 - x_2|^{1-\beta},$$

in which $\beta \in (0, 1)$ and $d = \max_{x,y \in \Omega} |x - y|$.

Similarly, in order to show the Hölder continuity in t, we also break up the integral term into two parts:

(2.3.16)

$$\left| I^f u(x, t_1) - I^f u(x, t_2) \right| \leq \int_{\mathbb{R}} |u(x + y, t_1) - u(x, t_1) - u(x + y, t_2) + u(x, t_2)| \, \nu(dy)$$

$$\leq I_1 + I_2,$$

in which

(2.3.17) $\quad I_1 = \int_{|y| \leq \epsilon} \left[|u(x + y, t_1) - u(x, t_1)| + |u(x + y, t_2) - u(x, t_2)| \right] \nu(dy),$

(2.3.18) $\quad I_2 = \int_{|y| > \epsilon} \left[|u(x + y, t_1) - u(x + y, t_2)| + |u(x, t_1) - u(x, t_2)| \right] \nu(dy).$

The constant $\epsilon \in (0, 1]$ will be determined later. We can first bound I_1 in (2.3.17) using (2.3.12). Then it follows from the semi-Hölder continuity of $t \to u(t, x)$ (see Lemma 2.2.3) that

(2.3.19)

$$I_2 \leq \int_{|y| > \epsilon} 2\, L_t |t_1 - t_2|^{\frac{1}{2}} \nu(dy) = 2\, L_t |t_1 - t_2|^{\frac{1}{2}} \int_{\epsilon < |y| \leq 1} \nu(dy) + 2\, L_t |t_1 - t_2|^{\frac{1}{2}} \int_{|y| > 1} \nu(dy)$$

$$\leq 2\, L_t |t_1 - t_2|^{\frac{1}{2}} \int_{|y| > 1} \nu(dy) + 2\, L_t M |S_1(0)| \, |t_1 - t_2|^{\frac{1}{2}} \cdot \begin{cases} \frac{\epsilon^{-\alpha} - 1}{\alpha}, & \text{if } 0 < \alpha < 1 \\ -\log \epsilon, & \text{if } \alpha = 0, \end{cases},$$

in which the second inequality follows from (H5) with $0 \leq \alpha < 1$.

Now picking $\epsilon = |t_1 - t_2|^{\frac{1}{2}} \wedge 1$, we have $\epsilon^{1-\alpha} \leq |t_1 - t_2|^{\frac{1-\alpha}{2}}$ and $\epsilon^{-\alpha} - 1 \leq |t_1 - t_2|^{-\frac{\alpha}{2}}$. A calculation in (2.3.15) gives us that $-\log \epsilon \leq 2|t_1 - t_2|^{-\beta/2}/\beta$ for any $\beta > 0$. Therefore (2.3.7) and (2.3.8) follow from combining (2.3.16), (2.3.12) and (2.3.19). $\qquad \square$

Having shown that the integral term $I^f u$ is well defined in classical sense and is Hölder continuous on compact domains, we will study the variational inequality

(4.3.3) on a given compact domain inside the continuation region $\mathcal{C}$. Let B be an open ball in $\mathbb{R}^n$ with its closure $\overline{B}$ and $B \times (t_1, t_2) \subset \mathcal{C}$ for some $t_1, t_2 \in [0, T)$. Let us consider the following boundary value problem:

$$(-\partial_t - \mathcal{L} + r)\, v(x,t) = 0, \quad (x,t) \in B \times [t_1, t_2],$$

(2.3.20)

$$v(x,t) = u(x,t), \quad (x,t) \in \mathbb{R}^n \times [t_1, t_2] \setminus B \times [t_1, t_2].$$

Due to Lemma 2.2.3, the boundary and terminal value u is continuous in $\overline{B} \times [t_1, t_2]$. The viscosity solution of this boundary value problem can be defined as follows (see e.g. Definition 7.4 in Crandall et al. (1992), Definition 13.1 in Fleming and Soner (2006) or Definition 12.1 in Cont and Tankov (2004)).

Definition 2.3.2. (i) Any $v \in C^0(\overline{B} \times [t_1, t_2])$ is a viscosity subsolution of (2.3.20) if

(2.3.21) $\qquad (-\partial_t - \mathcal{L} + r)\, \phi(x,t) \leq 0, \quad \text{for } (x,t) \in B \times [t_1, t_2),$

(2.3.22)

$$\min\left\{(-\partial_t - \mathcal{L} + r)\, \phi(x,t),\, v(x,t) - u(x,t)\right\} \leq 0, \quad \text{for } (x,t) \in \partial B \times [t_1, t_2) \cup \overline{B} \times t_2,$$

(2.3.23) $\qquad v(x,t) \leq u(x,t), \quad \text{for } (x,t) \in \mathbb{R}^n \times [t_1, t_2] \setminus \overline{B} \times [t_1, t_2],$

for any function $\phi \in C^{2,1}(\mathbb{R}^n \times [t_1, t_2]) \cap C_1(\mathbb{R}^n \times [t_1, t_2])$ such that $\phi(x,t) = v(x,t)$ and $\phi(\tilde{x}, \tilde{t}) \geq v(\tilde{x}, \tilde{t})$ for any $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [t_1, t_2]$. Any $v \in C^0(\overline{B} \times [t_1, t_2])$ is a viscosity supersolution of (2.3.20) if

(2.3.24) $\qquad (-\partial_t - \mathcal{L} + r)\, \phi(x,t) \geq 0, \quad \text{for } (x,t) \in B \times [t_1, t_2),$

(2.3.25)

$$\max\left\{(-\partial_t - \mathcal{L} + r)\, \phi(x,t),\, v(x,t) - u(x,t)\right\} \geq 0, \quad \text{for } (x,t) \in \partial B \times [t_1, t_2) \cup \overline{B} \times t_2,$$

$$(2.3.26) \qquad v(x,t) \geq u(x,t), \quad \text{for } (x,t) \in \mathbb{R}^n \times [t_1, t_2] \setminus \overline{B} \times [t_1, t_2],$$

for any function $\phi \in C^{2,1}(\mathbb{R}^n \times [t_1, t_2]) \cap C_1(\mathbb{R}^n \times [t_1, t_2])$ such that $\phi(x,t) = v(x,t)$ and $\phi(\tilde{x}, \tilde{t}) \leq v(\tilde{x}, \tilde{t})$ for any $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [t_1, t_2]$.

(ii) v is a viscosity solution of (2.3.20) if it is both a subsolution and a supersolution.

Following Definition 2.3.2, it is easy to check the following result.

Lemma 2.3.3. *If the Lévy measure ν satisfies* (H2), *then $u(x,t)$ is a viscosity solution of* (2.3.20).

Proof. We will only show that $u(x,t)$ is a viscosity subsolution. That u is a viscosity supersolution can be checked similarly. For any $(x,t) \in \overline{B} \times [t_1, t_2]$, let ϕ be a test function satisfying conditions in Definition 2.3.2 for subsolutions. Noticing that $u(x,t)$ itself is the boundary and terminal value of (2.3.20), (2.3.22) and (2.3.23) are automatically satisfied. On the other hand, the inequality (2.3.21) follows from (2.2.17) and the fact that $u(t,x) \geq g(x)$. $\qquad \square$

In Definition 2.3.2, it is important to note that the test function ϕ is used in evaluating the integral term $I^f \phi(t,x)$. However, thanks to Lemma 2.3.1, $I^f u$ is well defined in the classical sense. Therefore, we will consider the following parabolic differential equation with an integral driving term

$$(2.3.27) \qquad \begin{aligned} (-\partial_t - \mathcal{L}_D^f + r)\, v(x,t) &= I^f u(x,t), \quad \text{for } (x,t) \in B \times [t_1, t_2), \\ v(x,t) &= u(x,t), \quad \text{for } (x,t) \in \partial B \times [t_1, t_2) \cup \overline{B} \times t_2, \end{aligned}$$

where B is the same as in (2.3.20). The viscosity solution of (2.3.27) is defined as follows.

Definition 2.3.4. Any $v \in C^0(\overline{B} \times [t_1, t_2])$ is a viscosity subsolution of (2.3.27) if

(2.3.28) $\qquad (-\partial_t - \mathcal{L}_D^f + r)\, \phi(x, t) \leq I^f u(x, t), \quad$ for $(x, t) \in B \times [t_1, t_2),$

(2.3.29)
$$\min\left\{(-\partial_t - \mathcal{L}_D^f + r)\, \phi(t, x) - I^f u(x, t),\, v(x, t) - u(x, t)\right\} \leq 0,$$
$$\text{for } (x, t) \in \partial B \times [t_1, t_2)\overline{B} \times t_2$$

for any function $\phi \in C^{2,1}(\mathbb{R}^n \times [t_1, t_2])$ such that $\phi(x, t) = v(x, t)$ and $\phi(\tilde{x}, \tilde{t}) \geq v(\tilde{x}, \tilde{t})$ for any $(\tilde{x}, \tilde{t}) \in \mathbb{R}^n \times [t_1, t_2]$. The supersolution is defined analogously. As usual, v is a viscosity of (2.3.27) if it is both a subsolution and a supersolution.

Actually, it turns out the notion of viscosity solutions for (2.3.20) defined in Definition 2.3.2 is equivalent to the notion of viscosity solutions for (2.3.27) defined in Definition 2.3.4.

Lemma 2.3.5. *The value function u is a viscosity solution of (2.3.20) in the sense of Definition 2.3.2, if and only if u is a viscosity solution of (2.3.27) in the sense of Definition 2.3.4.*

Proof. The proof follows from the argument of Lemma 2.1 in Soner (1986). For the completeness of this chapter, we will repeat this argument in Section 2.6. $\qquad \square$

Now we will apply the regularity theory of parabolic differential equation to analyze the regularity of u in the continuation region $\mathcal{C}$. We assume that there exist a positive constant λ such that

(H6) $\qquad \displaystyle\sum_{i,j=1}^n a_{ij}(x, t)\, \xi^i \xi^j \geq \lambda |\xi|^2, \quad \forall x, \xi \in \mathbb{R}^n, t \geq 0.$

Additionally, for $i, j \leq n$

(H7)

$a_{ij}(x, t), b_i(x, t)$ and $r(x, t)$ are continuously differentiable in both variables on $\mathbb{R}^n \times [0, T]$.

With these two assumptions, now we are ready to state the main theorem of this section.

Theorem 2.3.6. *Let us assume that the Lévy measure ν satisfies (H2) and (H5) with $0 \leq \alpha < 1$, moreover coefficients of (2.3.20) satisfy (H6) and (H7). Then the value function u is the unique classical solution, i.e., $u \in C^{2,1}$, of the boundary value problem (2.3.20). Moreover, $u \in C^{2,1}(\mathcal{C})$.*

Proof. It follows from Lemmas 2.3.3 and 2.3.5 that the value function $u(x,t)$ is a viscosity solution of (2.3.27) in the sense of Definition 2.3.4. For the boundary value problem (2.3.27), its boundary and terminal values are continuous on $\partial B \times [t_1, t_2) \cup \overline{B} \times t_2$, as a result of the continuity of u (see Lemma 2.2.3). On the other hand, the driving term $I^f u(x,t)$ is uniformly Hölder continuous in both variables in $\overline{B} \times [t_1, t_2]$ (see Lemma 2.3.1). Moreover, thanks to (H7), the coefficients in (2.3.27) are bounded and Hölder continuous in $\overline{B} \times [t_1, t_2]$. Therefore, combining with the nondegenerate assumption (H6), Theorem 9 in Friedman (1964) pp. 69 implies that (2.3.27) has a unique classical solution $u^*(x,t) \in C^{2,1}(B \times (t_1, t_2))$. Since u^* is already a classical solution, u^* is also a viscosity solution of (2.3.27). Therefore, it follows from the Comparison Theorem for viscosity solutions for parabolic differential equations with the driving term (see e.g. Theorem 7.5 in Crandall et al. (1992)) that $u(x,t) = u^*(x,t)$ for $(x,t) \in B \times (t_1, t_2)$. This ensures that the value function u is the unique classical solution of (2.3.20). Since $B \times (t_1, t_2)$ is an arbitrary domain in the continuation region $\mathcal{C}$, we have $u \in C^{2,1}(\mathcal{C})$. $\square$

We have studied the regularity of the value function inside the continuation region when jumps have finite variation. We still want to understand how the value function cross the interface of the continuation region and the stopping region, even when

jumps have finite variation. Moreover, we hope to study problems with infinite variation jumps. These analysis depend on the global regularity of the value function, which we shall study in the following section.

2.4 Infinite variation jumps and the global regularity

2.4.1 The integral term

When the jumps of X have infinite variation, i.e., (2.3.1) is not satisfied, the integral term cannot be reduced to the form in (2.3.4). Therefore, throughout this section we need to work with the integro-differential operator $\mathcal{L}$ and its integral part I in the form of (5.4.54) and (2.2.12). However, given the regularity properties of the value function u in Lemmas 2.2.3, it is not clear that u has Lipschitz continuous first derivative to make sure Iu is well defined in the classical sense (see (2.2.16)). Nevertheless, in the following lemma, we will show that given sufficient regularity properties for the test function ϕ, $I\phi(x,t)$ is Hölder continuous in both variables. Later in this section, we will prove that the value function u does have these regularity properties to guarantee Iu well defined in the classical sense.

Let Ω be a compact domain in $\mathbb{R}^n$, $\Omega^\delta \triangleq \{x \in \mathbb{R}^n : x \in B_\delta(y) \text{ for some } y \in \Omega\}$ for some $\delta > 0$. For $s \in (0, T]$, let us denote $\overline{Q_s} = \overline{\Omega} \times [0, s]$ and $\overline{Q_s^\delta} = \overline{\Omega^\delta} \times [0, s]$. Moreover, we denote $D_s \triangleq \mathbb{R}^n \times [0, s]$.

Lemma 2.4.1. *Let us assume that the Lévy measure satisfies* (H2) *and* (H5) *with* $\alpha \in [1, 2)$.

(i) Let us choose ϕ with finite norms $\max_{\mathbb{R}^n \times [0,s]} |\phi|$ and $\max_{\mathbb{R}^n \times [0,s]} |\nabla_x \phi|$, moreover
$$|\phi(x, t_1) - \phi(x, t_2)| \leq \widetilde{L}_t |t_1 - t_2|^{1/2} \text{ for any } x \in \mathbb{R} \text{ and } t_1, t_2 \in [0, s]. \text{ If } \phi \in$$
$H^{\beta, \frac{\beta}{2}} \left(\overline{Q_s^1} \right)$ *for some* $\beta \in (\alpha, 2)$, *then* $Iu \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}} \left(\overline{Q_s} \right)$. *Additionally, there*

exists a constant $C_\Omega > 0$, depending on Ω, α, β and T, such that

$$(2.4.1) \qquad \|I\phi\|_{\overline{Q_s}}^{\left(\frac{\beta-\alpha}{2}\right)} \le C_\Omega \left(\max_{\mathbb{R}^n \times [0,s]} |\phi| + \max_{\mathbb{R}^n \times [0,s]} |\nabla_x \phi| + \widetilde{L}_t + \|\phi\|_{Q_s^1}^{(\beta)} \right),$$

where the Hölder norm $\| \cdot \|_{\overline{Q_s}}^{(\gamma)}$ is defined in Definition 2.2.10.

(ii) If $\phi \in H^{\beta, \frac{\beta}{2}}(D_s)$ for some $\beta \in (\alpha, 2)$, then $I\phi \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$. Moreover, there exists a constant C, depending on α, β and T, such that

$$(2.4.2) \qquad\qquad \|I\phi\|_{D_s}^{\left(\frac{\beta-\alpha}{2}\right)} \le C \|\phi\|_{D_s}^{(\beta)}.$$

Proof. For the notational simplicity, the constant C denotes a generic constant in different places in the proof.

1. Let us first estimate $\max_{\overline{Q_s}} |I\phi|$. Following (2.2.12), for $(x,t) \in \overline{Q_s}$, we have

$$(2.4.3)$$

$$
\begin{aligned}
|I\phi(x,t)| &\le \int_{|y| \le 1} \left| \phi(x+y,t) - \phi(x,t) - \sum_{i=1}^n y^i \, \partial_{x^i}\phi(x,t) \right| \nu(dy) \\
&\quad + \int_{|y|>1} |\phi(x+y,t) - \phi(x,t)| \, \nu(dy) \\
&\le \int_{|y| \le 1} \sum_{i=1}^n \left| y^i \, \partial_{x^i}\phi(z_i,t) - y^i \, \partial_{x^i}\phi(x,t) \right| \nu(dy) + 2 \max_{\mathbb{R}^n \times [0,s]} |\phi| \int_{|y|>1} \nu(dy) \\
&\le \|\phi\|_{Q_s^1}^{(\beta)} \int_{|y| \le 1} |y|^\beta \nu(dy) + 2 \max_{\mathbb{R}^n \times [0,s]} |\phi| \int_{|y|>1} \nu(dy) \\
&\le C \left(\max_{\mathbb{R}^n \times [0,s]} |\phi| + \|\phi\|_{Q_s^1}^{(\beta)} \right).
\end{aligned}
$$

In the second inequality of (2.4.3), z_i are some vectors in $\mathbb{R}^n$ with $|z_i - x| < |y|$. Therefore, when $x \in \Omega$, we have $x + z_i \in \Omega^1$. The third inequality follows from the Hölder continuity of $\partial_{x^i}\phi$ on $\overline{Q_s^1}$, i.e., $\sum_{i=1}^n |\partial_{x^i}\phi(z_i,t) - \partial_{x^i}\phi(x,t)| \le \|\phi\|_{Q_s^1}^{(\beta)} |y|^{\beta-1}$. TO get the last inequality, we applies (H5). Note that $\beta > \alpha$, hence $\int_{|y| \le 1} |y|^{-n+\beta-\alpha} dy$ is integrable.

The proof of the Hölder continuity of $x \to I\phi(x,t)$ and $t \to I\phi(x,t)$ are similar to the proof in Lemmas 2.3.1. Let us check the Hölder continuity in x first. For any $x_1, x_2 \in \Omega$ and $t \in [0, s]$, breaking up the integral term into three parts, we obtain

$$(2.4.4) \qquad |I\phi(x_1, t) - I\phi(x_2, t)| \leq I_1 + I_2 + I_3,$$

in which

$$I_1(x,t) = \int_{|y| \leq \epsilon} [|\phi(x_1 + y, t) - \phi(x_1, t) - y \cdot \nabla_x \phi(x_1, t)|$$

$$+ |\phi(x_2 + y, t) - \phi(x_2, t) - y \cdot \nabla_x \phi(x_2, t)|] \, \nu(dy),$$

$$I_2(x,t) = \int_{\epsilon < |y| \leq 1} [|\phi(x_1 + y, t) - \phi(x_2 + y, t)| + |\phi(x_1, t) - \phi(x_2, t)|$$

$$+ |y| \, |\nabla_x \phi(x_1, t) - \nabla_x \phi(x_2, t)|] \, \nu(dy),$$

$$I_3(x,t) = \int_{|y| > 1} [|\phi(x_1 + y, t) - \phi(x_2 + y, t)| + |\phi(x_1, t) - \phi(x_2, t)|] \, \nu(dy).$$

Here the constant $\epsilon \leq 1$ will be determined later. Let us estimate each integral term separately. An estimate similar to (2.4.3) shows that

$$(2.4.5) \quad I_1 \leq 2\|\phi\|_{Q_s^1}^{(\beta)} \int_{|y| \leq \epsilon} |y|^\beta \nu(dy) \leq 2M\|\phi\|_{Q_s^1}^{(\beta)} \int_{|y| \leq \epsilon} |y|^{-n+\beta-\alpha} dy = C\|\phi\|_{Q_s^1}^{(\beta)} \epsilon^{\beta-\alpha}.$$

Thanks to the Lipschitz continuity of $x \to \phi(x,t)$ and the Hölder continuity of $x \to \partial_{x_i} \phi(x,t)$, we can estimate I_2 and I_3 as

$$(2.4.6)$$

$$I_2 \leq \int_{\epsilon < |y| \leq 1} \left[2 \max_{\mathbb{R}^n \times [0,s]} |\nabla_x \phi| \, |x_1 - x_2| + \|\phi\|_{Q_s^1}^{(\beta)} |y| \, |x_1 - x_2|^{\beta-1} \right] \nu(dy)$$

$$\leq M \int_{\epsilon < |y| \leq 1} \left[2 \max_{\mathbb{R}^n \times [0,s]} |\nabla_x \phi| \, |x_1 - x_2| + \|\phi\|_{Q_s^1}^{(\beta)} |y| \, |x_1 - x_2|^{\beta-1} \right] |y|^{-n-\alpha} dy$$

$$= C \max_{\mathbb{R}^n \times [0,s]} |\nabla_x \phi| \, |x_1 - x_2|(\epsilon^{-\alpha} - 1) + C \, \|\phi\|_{Q_s^1}^{(\beta)} |x_1 - x_2|^{\beta-1} \cdot \begin{cases} \epsilon^{1-\alpha} - 1 & \text{when } 1 < \alpha < 2, \\ -\log \epsilon & \text{when } \alpha = 1. \end{cases},$$

$$(2.4.7) \qquad I_3 \leq 2 \max_{\mathbb{R}^n \times [0,s]} |\nabla_x \phi| \, |x_1 - x_2| \int_{|y| > 1} \nu(dy).$$

Now pick $\epsilon = |x_1 - x_2|^{1/2} \wedge 1$. Note that $1 \leq \alpha < 2$, we obtain $\epsilon^{\beta-\alpha} \leq |x_1 - x_2|^{\frac{\beta-\alpha}{2}}$,

$\epsilon^{-\alpha} - 1 \leq |x_1 - x_2|^{-\frac{\alpha}{2}}$, $\epsilon^{1-\alpha} - 1 \leq |x_1 - x_2|^{\frac{1-\alpha}{2}}$ and $-\log \epsilon \leq \frac{1}{\delta}|x_1 - x_2|^{-\delta}$ for any $\delta > 0$

(see (2.3.15)). Since $\beta > 1$, we will choose $\delta = \frac{\beta-1}{2}$ in the following. Concluding

from these inequalities and (2.4.4) - (2.4.7), we obtain

$$(2.4.8) \qquad |I\phi(x_1, t) - I\phi(x_2, t)| \leq C_\Omega \left(\max_{\mathbb{R}^n \times [0,s]} |\nabla_x \phi| + \|\phi\|_{Q_s^1}^{(\beta)} \right) |x_1 - x_2|^{\frac{\beta-\alpha}{2}},$$

where C_Ω is a sufficiently large constant independent of x_1, x_2 and t.

For the Hölder continuity of $t \to I\phi(x, t)$, since $\phi \in H^{\beta, \frac{\beta}{2}}(\overline{Q_s^1})$, it follows from

Definition 2.2.10 that

$$\sum_{i=1}^{n} |\partial_{x_i} \phi(x, t_1) - \partial_{x_i} \phi(x, t_2)| \leq \|\phi\|_{Q_s^1}^{(\beta)} |t_1 - t_2|^{\frac{\beta-1}{2}}, \qquad \text{for } x \in \Omega \text{ and } t_1, t_2 \in [0, s].$$

Picking $\epsilon = |x_1 - x_2|^{\frac{1}{4}} \wedge 1$, an estimation similar to Lemma 2.3.1 gives us

$$(2.4.9) \qquad |I\phi(x, t_1) - I\phi(x, t_2)| \leq C_\Omega \left(\widetilde{L}_t + \|\phi\|_{Q_s^1}^{(\beta)} \right) |t_1 - t_2|^{\frac{\beta-\alpha}{4}},$$

where C_Ω is a sufficiently large constant independent of x, t_1 and t_2.

Now the first part of the lemma follows from (2.4.3), (2.4.8) and (2.4.9).

2. Noting that $\max_{D_s} |\phi| \leq \|\phi\|_{D_s}^{(\beta)}$ and $\max_{t_1, t_2 \in [0,s]} \frac{|\phi(x, t_1) - \phi(x, t_2)|}{|t_1 - t_2|^{\frac{1}{2}}} \leq s^{\frac{\beta-1}{2}} \|\phi\|_{D_s}^{(\beta)}$

(see Definition 2.2.10), the second part of the lemma follows from the same argument

which we used in the first part of the proof. □

Remark 2.4.2. When the Lévy measure ν is a finite measure on $\mathbb{R}^n$, the integral form

$\int_{\mathbb{R}^n} \phi(x + y, t) \, \nu(y)$ has the same regularity as $\phi(x, t)$ (see Yang et al. (2006)). When

the Lévy measure has a singularity, as we have seen in Lemma 2.4.1, the regularity

of $I\phi$ decreases compared to the regularity of ϕ. Moreover, as we have seen in

(2.4.1), the Hölder norm of $I\phi$ depends on the Hölder norm of ϕ on a slightly larger

domain. This extension of domains will introduce a technical difficulty in estimating

the Sobolev norm of u. This estimation will be carried out in the following section.

2.4.2 Solutions in the Sobolev sense

As we have seen in Proposition 2.2.5, if the Lévy measure ν satisfies (H2), the value function u is the viscosity solution of the variational inequality (4.3.3). In the following, we will apply the regularity results for partial differential equations to show that u is also a solution of (4.3.3) in the Sobolev sense.

In this subsection, instead of (H7), we assume that

(H7') a_{ij}, b_i and r are constants for $i, j \leq n$, and $r \geq 0$.

Moreover, there exist positive constants λ and Λ such that

(H6') $$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij} \, \xi^i \xi^j \leq \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.$$

Remark 2.4.3. Actually, the following two assumptions

(H6") $$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t) \, \xi^i \xi^j \leq \Lambda |\xi|^2, \quad \forall (x,t) \in \mathbb{R}^n \times [0,T] \text{ and } \xi \in \mathbb{R}^n, \text{ and}$$

(H7")

$$a_{ij}(x,t), b(x,t), r(x,t) \in H^{\ell, \frac{\ell}{2}}(\mathbb{R}^n \times [0,T]), \quad \forall \ell \in (0,1) \text{ and } i, j \leq n, \text{ and } r(x,t) \geq 0$$

are sufficient for all results in this section except Lemma 2.5.7, where the constant coefficient assumption (H7') is necessary.

In order to work with non-smooth payoff functions, we assume that there exists a mollified sequence of g, denoted by $\{g^\epsilon\}_{\epsilon \in (0,\epsilon_0)}$ for some constant $\epsilon_0 < 1$, such that $\lim_{\epsilon \downarrow 0} g^\epsilon(x) = g(x)$ uniformly in compact subsets of $\mathbb{R}^n$ and

(H8)   each $g^\epsilon(x) \in H^{2+\ell}(\mathbb{R}^n) \quad \forall \ell \in (0,1)$.

Moreover, there exist positive constants K, L and J independent of ϵ such that for all $x \in \mathbb{R}^n$

(H3') $0 \leq g^\epsilon(x) \leq K,$

(H4')
$$|\nabla g^\epsilon(x)| \le L, \quad \text{and}$$

(H9)
$$\sum_{i,j=1}^{n} \partial^2_{x^i x^j} g^\epsilon(x)\, \xi^i \xi^j \ge -J\, |\xi|^2, \quad \forall \xi, x \in \mathbb{R}^n.$$

Remark 2.4.4. Actually, for standard put option payoffs on multiple assets: $g(x) = \left[K - \frac{1}{n}\sum_{i=1}^{n} e^{x_i}\right]^+$ (the arithmetic average) and $g(x) = \left[K - \exp\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right]^+$ (the geometric average), mollified sequences can be constructed to satisfy the assumptions (H3'), (H4'), (H8) and (H9). Indeed, we can choose a sequence of functions $H_\epsilon(y) \in C^\infty(\mathbb{R})$ ($\epsilon \in (0, \epsilon_0)$ with ϵ_0 much smaller than K) such that $0 \le H'_\epsilon(y) \le 1$, $H''_\epsilon(y) \ge 0$ and $H_\epsilon(y) = \begin{cases} y, & y \ge \epsilon \\ 0, & y \le -\epsilon \end{cases}$. The mollified sequence $\{g^\epsilon\}_{\epsilon \in (0, \epsilon_0)}$ can be constructed by defining $g^\epsilon(x) = H_\epsilon\left(K - \frac{1}{n}\sum_{i=1}^{n} e^{x_i}\right)$ or $g^\epsilon(x) = H_\epsilon\left(K - \exp\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right)$. It is clear that $\lim_{\epsilon \downarrow 0} g^\epsilon(x) = g(x)$ uniformly in $\mathbb{R}$. Note that $H'(y) > 0$ only when $y > -\epsilon$, one can check that (H3'), (H4'), (H8) and (H9) are satisfied for both cases.

Given these assumptions, we are ready to state main result of this section.

Theorem 2.4.5. *If* (H6'), (H7'), (H3'), (H4'), (H8) *and* (H9) *are satisfied, moreover, the Lévy measure ν satisfies* (H2) *and* (H5) *with $\alpha \in [0, 2)$, then $u \in W_p^{2,1}(B_\rho(x_0) \times (0, T-s))$ for any integer $p \in (1, \infty)$, $\rho, s > 0$ and $x_0 \in \mathbb{R}^n$.*

Before we prove this key estimate in Section 2.5, let us list some corollaries of this result.

Corollary 2.4.6. *If the assumptions in Theorem 2.4.5 are satisfied, then for any $\rho, s > 0$ and $x_0 \in \mathbb{R}^n$*

(i) $u \in H^{\beta, \frac{\beta}{2}}(\overline{B_\rho(x_0)} \times [0, T-s])$ *where $\beta = 2 - \frac{n+2}{p} > 0$. In particular, $\nabla_x u \in C(\mathbb{R}^n \times [0, T))$. Therefore the smooth-fit property holds.*

(ii) if the Lévy measure ν satisfies (H5) *with $\alpha \in [1, 2)$, then Iu is well defined in the classical sense in $B_\rho(x_0) \times [0, T)$. Moreover, $I\,u \in H^{\frac{\beta-\alpha}{2},\frac{\beta-\alpha}{4}}\big(\overline{B_\rho(x_0)} \times [0, T-s]\big)$ for some $\beta \in (\alpha, 2)$.*

Proof. **(i)** Combining the result in Theorem 2.4.5 and the Sobolev Inequality (see e.g. Lemma 3.3 in Ladyženskaja et al. (1968) pp. 80), we have $u \in H^{\beta,\frac{\beta}{2}}\big(\overline{B_\rho(x_0)} \times [0, T-s]\big)$, where $\beta = 2 - \frac{n+2}{p} > 0$. Choosing sufficiently large p such that $\beta > 1$, the continuity of $\nabla_x u$ follows from the definition of Hölder spaces in Definition 2.2.10 and the arbitrary choice of s.

(ii) It follows from the result in (i) for $\rho + 1$ and the estimation (2.4.3) that Iu is well defined in $B_\rho(x_0) \times [0, T-s]$. Then the first statement of (ii) follows, since the choice of s is arbitrary. Choosing sufficiently large p such that $\beta > \alpha$, the second statement of (ii) follows from Lemma 2.4.1. $\qquad\square$

Thanks to Corollary 2.4.6 (ii), we can consider the following boundary value problem with the driving term $I\,u$:

$$
(2.4.10) \qquad
\begin{aligned}
(-\partial_t - \mathcal{L}_D + r)\,v(x, t) &= Iu(x, t), \quad \text{for } (x, t) \in B \times [t_1, t_2), \\
v(x, t) &= u(x, t), \quad \text{for } (x, t) \in \partial B \times [t_1, t_2) \cup \overline{B} \times t_2,
\end{aligned}
$$

where $B \times (t_1, t_2) \subset \mathcal{C}$ is the bounded domain as in (2.3.20). The viscosity solution of (2.4.10) is defined similarly as in Definition 2.3.4, with operators $\mathcal{L}_D^f$ and I^f replaced by $\mathcal{L}_D$ and I respectively.

Rather than extending Lemma 2.3.5 to the infinite variation jump case, the following relation between the solutions in the Sobolev sense and the viscosity sense shows that the value function u is a viscosity solution of the boundary value problem (2.4.10). See Corollary 3 in Lions (1983) or Theorem 9.15 (ii) in Karatzas (1998) for its proof.

Lemma 2.4.7. *If $u \in W_p^{2,1}(B \times (t_1, t_2))$ for $p > n + 1$ satisfies* (2.4.10) *at almost every point in $B \times (t_1, t_2)$, then u is the viscosity solution of* (2.4.10) *in the sense of Definition* 2.3.4.

Thanks to Corollary 2.4.6, Lemmas 2.4.1 and 2.4.7, the argument in Theorem 2.3.6 also works for the infinite variation jump case.

Theorem 2.4.8. *If the Lévy measure ν satisfies* (H2) *and* (H5) *with $1 \le \alpha < 2$, then the value function u is the unique classical solution, i.e., $u \in C^{2,1}$, of the boundary value problem* (2.3.20). *Moreover, $u \in C^{2,1}(\mathcal{C})$.*

Proof. Corollary 2.4.6 (ii) tells us that $Iu(x,t) \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(\overline{B} \times [t_1, t_2])$. As the value function u is shown to be a viscosity solution of (2.4.10) in Lemma 2.4.7, the rest proof follows from the same proof for Theorem 2.3.6. $\qquad\square$

2.5 Proof of Theorem 2.4.5

Because the jump may have infinite variation, the proof of Theorem 2.4.5 needs to conquer several technical difficulties. We will carry the proof of Theorem 2.4.5 in a series of lemmas and point out the difficulties along the way.

Let us first define $v(x,t) = u(x, T - t)$ for $(x,t) \in \mathbb{R}^n \times [0,T]$. It is natural to expect that v solves the following variational inequality

$$
\begin{aligned}
\min\left\{(\partial_t - \mathcal{L}_D - I + r)\, v(x,t),\, v(x,t) - g(x)\right\} &= 0, \quad (x,t) \in \mathbb{R} \times (0,T], \\
v(x,0) &= g(x).
\end{aligned}
$$
(2.5.1)

In this section, we will show that v indeed solves (2.5.1) for almost every point $(x,t) \in \mathbb{R}^n \times [0,T]$. Moreover, its $W_p^{2,1}$-norm is bounded on bounded domains of $\mathbb{R}^n \times [0,T]$. In the following, we will only carry out the proof of Theorem 2.4.5 for the infinite variation jump case, i.e., the Lévy measure ν satisfies (H5) with

$1 \leq \alpha < 2$. Since the integral operator has the reduced form I^f in (2.3.4) for the finite variation jumps, the proof of $0 \leq \alpha < 1$ case in Theorem 2.4.5 will be similar and easier.

Motivated by Lemma 3.1 in Friedman (1982) pp. 24 and Yang et al. (2006), we will study the following penalty problem for each $\epsilon \in (0, \epsilon_0)$, where ϵ_0 is chosen before (H8):

(2.5.2)
$$(\partial_t - \mathcal{L}_D - I + r) v^\epsilon(x, t) + p_\epsilon(v^\epsilon - g^\epsilon) = 0, \quad (x, t) \in \mathbb{R}^n \times (0, T],$$
$$v^\epsilon(x, 0) = g^\epsilon(x),$$

where the mollified sequence $\{g^\epsilon\}_{\epsilon \in (0, \epsilon_0)}$ satisfies (H3'), (H4'), (H8) and (H9). Here the penalty term $p_\epsilon(y) \in C^\infty(\mathbb{R})$ is chosen to satisfy following properties:

(2.5.3)
$$(i)\, p_\epsilon(y) \leq 0, \quad (ii)\, p_\epsilon(y) = 0 \text{ if } y \geq \epsilon,$$

$$(iii)\, p_\epsilon(0) = -n\Lambda J - |b|^{(0)} L - |r|^{(0)} K - J \int_{|y| \leq 1} |y|^2 \nu(dy) - K \int_{|y| > 1} \nu(dy),$$

$$(iv)\, p'_\epsilon(y) \geq 0, \quad (v)\, p''_\epsilon(y) \leq 0 \quad \text{and} \quad (vi)\, \lim_{\epsilon \downarrow 0} p_\epsilon(y) = \begin{cases} 0, & y > 0 \\ -\infty, & y < 0 \end{cases}.$$

Here constants Λ, K, L and J come from (H6") (H3'), (H4') and (H9) respectively. Additionally, $|b|^{(0)} = \max_{\mathbb{R}^n \times [0,T]} |b(x, t)|$ and $|r|^{(0)} = \max_{\mathbb{R}^n \times [0,T]} |r(x, t)|$ are finite due to (H7"). Moreover, $p_\epsilon(0)$ is also finite thanks to (2.2.3). It is also worth noticing that $p_\epsilon(0)$ is independent of ϵ. These properties of p_ϵ will be useful in later development. In particular, (2.5.3) (iii) will be essential for proofs of Lemma 2.5.9 and Corollary 2.5.10.

Let us recall the Schauder Fixed Point Theorem (see e.g. Theorem 2 in Friedman (1964) pp. 189).

Lemma 2.5.1. *Let Θ be a closed convex subset of a Banach space and let $\mathcal{T}$ be a continuous operator on Θ such that $\mathcal{T}\Theta$ is contained in Θ and $\mathcal{T}\Theta$ is precompact.*

Then $\mathcal{T}$ has a fixed point in Θ.

For each $\epsilon \in (0, \epsilon_0)$, we will show that the penalty problem (2.5.2) has a classical solution via the Schauder Fixed Point Theorem. Let us recall $D_s = \mathbb{R}^n \times [0, s]$.

Lemma 2.5.2. *If the Lévy measure ν satisfies* (H2) *and* (H5) *with $1 \le \alpha < 2$, then for any $\epsilon \in (0, \epsilon_0)$ and $\beta \in (\alpha, 2)$, (2.5.2) has a solution $v^\epsilon \in H^{2+\frac{\beta-\alpha}{2}, 1+\frac{\beta-\alpha}{4}}(D_T)$.*

Proof. We will first prove that (2.5.2) has a solution on a sufficiently small time interval $t \in [0, s]$ via the Schauder Fixed Point Theorem. Then we will extend this solution to the interval $[0, T]$.

Let us consider the set $\Theta = \left\{ v \in H^{\beta, \frac{\beta}{2}}(D_s) \text{ with its Hölder norm } \|v\|_{D_s}^{(\beta)} \le U_0 \right\}$, where positive constants s and U_0 will be determined later. It is clear that Θ is a bounded, closed and convex set in the Banach space $H^{\beta, \frac{\beta}{2}}(D_s)$. For any $v \in \Theta$, consider the following Cauchy problem for $u - g^\epsilon$:

$$(\partial_t - \mathcal{L}_D + r)(u - g^\epsilon)(x, t) = Iv(x, t) - p_\epsilon(v - g^\epsilon)(x, t) + (\mathcal{L}_D - r)g^\epsilon(x),$$

(2.5.4) $$(x, t) \in \mathbb{R} \times (0, s],$$

$$u(x, 0) - g^\epsilon(x) = 0.$$

Via the solution u of (2.5.4), the operator $\mathcal{T}$ can be defined as $u = \mathcal{T}v$. Let us check the conditions for the Schauder Fixed Point Theorem in the sequel.

1. Tv is well defined. Note that $v \in H^{\beta, \frac{\beta}{2}}(D_s)$ and $\beta \in (\alpha, 2)$, it follows from Lemma 2.4.1 (ii) that $Iv \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$ with

(2.5.5) $$\|Iv\|_{D_s}^{(\frac{\beta-\alpha}{2})} \le C \|v\|_{D_s}^{(\beta)}, \quad \text{for some constant } C > 0 \text{ independent of } s.$$

On the other hand, we can check that $p_\epsilon(v - g^\epsilon) \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$. Indeed, $p_\epsilon(v - g^\epsilon)$ is bounded in D_s, since both $v, g^\epsilon \in H^{\beta, \frac{\beta}{2}}(D_s)$ (see (H8)) and $p_\epsilon(y) \in C^0(\mathbb{R})$.

Additionally, for any $x_1, x_2 \in \mathbb{R}^n$, $t \in [0, s]$

$$|p_\epsilon(v - g^\epsilon)(x_1, t) - p_\epsilon(v - g^\epsilon)(x_2, t)| \leq \max_{D_s} |p'_\epsilon(v - g^\epsilon)| \, |(v - g^\epsilon)(x_1, t) - (v - g^\epsilon)(x_2, t)|$$

$$\leq \widetilde{C}|x_1 - x_2|.$$

Here $\max_{D_s} |p'_\epsilon(v - g^\epsilon)|$ is finite, which also follows from the boundness of $v - g^\epsilon$ and $p_\epsilon \in C^1(\mathbb{R})$. The positive constant $\widetilde{C}$ depends on $\max_{D_s} |p'_\epsilon(v - g^\epsilon)|$ and the Hölder norms of v and g^ϵ. Meanwhile, the Hölder continuity of $p_\epsilon(v - g^\epsilon)$ in t can be checked similarly. Furthermore, $(\mathcal{L}_D - r) g^\epsilon(x) \in H^{\frac{\beta-\alpha}{2}, \frac{\beta-\alpha}{4}}(D_s)$ as a result of (H8). Therefore, thanks to (H6") and (H7"), it follows from Theorem 5.1 in Ladyženskaja et al. (1968) pp. 320 that (2.5.4) has a uniqueness solution $u - g^\epsilon \in H^{2 + \frac{\beta-\alpha}{2}, 1 + \frac{\beta-\alpha}{4}}(D_s)$. Note that $g^\epsilon \in H^{2 + \frac{\beta-\alpha}{2}, 1 + \frac{\beta-\alpha}{4}}(D_s)$ (see (H8)), we have $u = Tv \in H^{2 + \frac{\beta-\alpha}{2}, 1 + \frac{\beta-\alpha}{4}}(D_s)$.

2. $\mathcal{T}\Theta \subset \Theta$. For $u = Tv$, appealing to Lemma 2 in Friedman (1964) pp. 193, we obtain that there exists a positive constant A_β, depending on β, such that

$$
(2.5.6) \quad
\begin{aligned}
\|u - g^\epsilon\|_{D_s}^{(\beta)} &\leq A_\beta s^\gamma \left[\|Iv\|^{(0)} + \|p_\epsilon(v - g^\epsilon)\|^{(0)} + \|(\mathcal{L}_D - r) g^\epsilon\|^{(0)} \right] \\
&\leq A_\beta C s^\gamma \|v\|_{D_s}^{(\beta)} + \widetilde{A},
\end{aligned}
$$

where $\gamma = \frac{2-\beta}{2}$, C is the constant in (2.5.5) and $\widetilde{A}$ is a sufficiently large constant dependent on $\|g^\epsilon\|_{\mathbb{R}^n}^{(2+\ell)}$ for some $\ell \in (0, 1)$. Let us take sufficiently small s such that $\tau \triangleq A_\beta C s^\gamma < 1/2$. Moreover, let us take $U_0 = \max\{\frac{2\widetilde{A}}{1-2\tau}, 2\|g^\epsilon\|_{D_s}^{(\beta)}\}$. Note that $\|v\|_{D_s}^{(\beta)} \leq U_0$, it follows from (2.5.6) that

$$(2.5.7) \quad \|u\|_{D_s}^{(\beta)} \leq \|u - g^\epsilon\|_{D_s}^{(\beta)} + \|g^\epsilon\|_{D_s}^{(\beta)} \leq \tau U_0 + \widetilde{A} + \frac{U_0}{2} \leq \tau U_0 + \frac{1 - 2\tau}{2} U_0 + \frac{U_0}{2} = U_0.$$

Therefore, $u = \mathcal{T}v \in \Theta$.

3. $\mathcal{T}\Theta$ is a precompact subset of $H^{\beta, \frac{\beta}{2}}(D_s)$. For any $\eta \in (\beta, 2)$, similar estimate as (2.5.6) shows that for any $v \in \Theta$, we have $\|Tv\|_{D_s}^{(\eta)} \leq U_1$ for some constant U_1 depending on U_0 and s. On the other hand, argument similar to Theorem 1 in

Friedman (1964) pp.188 shows that bounded subsets of $H^{\eta,\frac{\eta}{2}}(D_s)$ are precompact subsets of $H^{\beta,\frac{\beta}{2}}(D_s)$. Therefore, $\mathcal{T}\Theta$ is a precompact subset in $H^{\beta,\frac{\beta}{2}}(D_s)$.

4. $\mathcal{T}$ is a continuous operator. Let v_n be a sequence in Θ such that $\lim_{n\to\infty}\|v_n - v\|_{D_s}^{(\beta)} = 0$, we will show $\lim_{n\to\infty}\|\mathcal{T}v_n - \mathcal{T}v\|_{D_s}^{(\beta)} = 0$. From (2.5.4), $w \triangleq \mathcal{T}v_n - \mathcal{T}v$ satisfies the Cauchy problem

$$\left(\partial_t - \mathcal{L}_D + r\right) w(x,t) = I(v_n - v)(x,t) - \left[p_\epsilon(v_n - g^\epsilon) - p_\epsilon(v - g^\epsilon)\right], \quad (x,t) \in \mathbb{R}^n \times (0,s]$$

$$w(x,0) = 0.$$

It follows again from Lemma 2 in Friedman (1964) pp. 193 that

$$\|\mathcal{T}v_n - \mathcal{T}v\|_{D_s}^{(\beta)} = \|w\|_{D_s}^{(\beta)} \leq A_\beta s^\gamma \left[\|I(v_n - v)\|^{(0)} + \|p_\epsilon(v_n - g^\epsilon) - p_\epsilon(v - g^\epsilon)\|^{(0)}\right]$$

$$\leq A_\beta s^\gamma \left[C\|v_n - v\|_{D_s}^{(\beta)} + \max_{D_s,n}\left|p_\epsilon'(v_n - g^\epsilon)\right|\|v_n - v\|^{(0)}\right]$$

$$\to 0 \quad \text{as } n \to \infty.$$

Concluding from **2. - 4.**, we obtain a fixed point of operator $\mathcal{T}$ in $H^{\beta,\frac{\beta}{2}}(D_s)$ as a result of the Schauder Fixed Point Theorem. We denote this fixed point as v^ϵ. Moreover, it follows from the result in **1.** that $v^\epsilon = \mathcal{T}v^\epsilon \in H^{2+\frac{\beta-\alpha}{2},1+\frac{\beta-\alpha}{4}}(D_s)$.

Finally, let us extend v^ϵ to the interval $[0,T]$. Choosing any $\rho \in (0, T-s)$, we replace $g^\epsilon(\cdot)$ by $v^\epsilon(\cdot,\rho)$ in (2.5.4). Note that the choice of s in **2.** only depend on β and C, but not on ρ. If $\|v^\epsilon(\cdot,\rho)\|_{\mathbb{R}^n}^{(2+\frac{\beta-\alpha}{2})}$ is finite, we can choose a sufficiently large U_0, depending on $\|v^\epsilon(\cdot,\rho)\|_{\mathbb{R}^n}^{(2+\frac{\beta-\alpha}{2})}$, such that (2.5.7) holds on $[\rho, \rho+s]$, moreover $\|v^\epsilon(\cdot,\rho+s)\|_{\mathbb{R}^n}^{(2+\frac{\beta-\alpha}{2})}$ is finite thanks to the result after **4.**. Noticing that $\|g^\epsilon\|_{\mathbb{R}^n}^{(2+\ell)}$ is finite for any $\ell \in (0,1)$, one can extend the time interval by s each time, until the time interval contains $[0,T]$. Therefore we have the statement of the lemma. $\qquad\square$

Remark 2.5.3. Because of the regularity decreases after applying the integral operator (see Remark 2.4.2), it is no longer straight forward to use the "bootstraping scheme" which was used in Theorem 2.1 of Yang et al. (2006) to explore the higher regularity

of v^ϵ. Instead, we will use a new technique to study the higher regularity of v^ϵ in the proof of Lemma 2.5.7.

Thanks to the definition of the Hölder spaces, Lemma 2.5.2 also tells us that v^ϵ is bounded in D_T. In order to show that v^ϵ is the unique bounded classical solution of the penalty problem (2.5.2), we need the following Maximum Principle for the parabolic integro-differential operator. The proof of it is provided in Section 2.6. (See Lemma 2.1 of Yang et al. (2006) for a similar Maximum Principle, where ν is assumed to be a finite measure on $\mathbb{R}$.)

Lemma 2.5.4. *Let us assume that $a_{ij}(x,t)$, $b_i(x,t)$ and $c(x,t)$ are bounded in $\mathbb{R}^n \times [0,T]$ with $A = (a_{ij})_{n\times n}$ satisfying $\sum_{i,j=1}^n a_{ij}(x,t)\,\xi^i\xi^j > 0$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$, moreover $c(x,t) \geq 0$ and the Lévy measure satisfies (H2). If $v \in C^0([0,T] \times \mathbb{R}^n) \cap C^{2,1}((0,T] \times \mathbb{R}^n)$ satisfies $(\partial_t - \mathcal{L}_\mathcal{D} - I + c(x,t))\,v(x,t) \geq 0$ in $\mathbb{R} \times (0,T]$ and there exists a sufficiently large positive constant m such that $v(x,t) \geq -m$ for $(x,t) \in \mathbb{R}^n \times [0,T]$. Then $v(x,0) \geq 0$ implies that $v(x,t) \geq 0$ for $(x,t) \in \mathbb{R}^n \times [0,T]$.*

As a corollary of this Maximum Principle, the bounded classical solution of the penalty problem (2.5.2) is unique.

Corollary 2.5.5. *For each $\epsilon \in (0, \epsilon_0)$, the penalty problem (2.5.2) has a unique bounded classical solution.*

Proof. Let us assume v_1 and v_2 are two bounded solutions of (2.5.2). Then $v_1 - v_2$ satisfies

(2.5.8)

$$(\partial_t - \mathcal{L}_D - I + r)\,(v_1 - v_2) + p_\epsilon(v_1 - g^\epsilon) - p_\epsilon(v_2 - g^\epsilon) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T],$$

$$(v_1 - v_2)(x,0) = 0$$

On the other hand, it follows from the mean value theorem that $p_\epsilon(v_1 - g^\epsilon) - p_\epsilon(v_2 - g^\epsilon) = p'_\epsilon(y)(v_1 - v_2)$ for some $y \in \mathbb{R}^n$. Moreover, $p'_\epsilon(y)$ is bounded, say by M, thanks to the fact that $p_\epsilon \in C^1(\mathbb{R})$ and v_1, v_2 and g^ϵ are all bounded. Now applying Lemma 2.5.4 to the equation (2.5.8) and choosing $c = r + M \geq 0$ (see (2.5.3) (iv)), we have $v_1(x,t) \geq v_2(x,t)$ for $(x,t) \in \mathbb{R}^n \times (0,T]$. The other direction of the inequality follows from applying the same argument to $v_2 - v_1$. $\qquad\square$

Applying Lemma 2.5.4, we will analyze some universal properties of v^ϵ for all $\epsilon \in (0, \epsilon_0)$ in the following three lemmas.

Lemma 2.5.6.

$$0 \leq v^\epsilon(x,t) \leq K + 1, \quad \text{for } (x,t) \in \mathbb{R}^n \times [0,T].$$

Proof. Since the proof is similar to the proof of Lemma 2.2 in Yang et al. (2006), we give it in the Section 2.6. $\qquad\square$

Lemma 2.5.7.

$$|\partial_{x^k} v^\epsilon(x,t)| \leq L, \quad \text{for } (x,t) \in \mathbb{R}^n \times [0,T], 1 \leq k \leq n.$$

Proof. Intuitively, thanks to the constant coefficient assumption (H7'), it follows from (2.5.2) that $\partial_{x^k} v^\epsilon$ satisfies

(2.5.9)
$$
\begin{aligned}
&(\partial_t - \mathcal{L}_D - I + r)\, w + p'_\epsilon(v^\epsilon - g^\epsilon)(w - \partial_{x^k} g^\epsilon) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T], \\
&w(x,0) = \partial_{x^k} g^\epsilon(x),
\end{aligned}
$$

where coefficients unchanged compared to (2.5.2). However, given the result in Lemma 2.5.2, it is only known that v^ϵ has continuous derivatives of the form $\partial^2_{x^i x^j} v^\epsilon$, $\partial_{x^i} v^\epsilon$ and $\partial_t v^\epsilon$. While it is necessary for v^ϵ to have derivatives of higher orders to ensure $\partial_{x^k} v^\epsilon$ as the classical solution of (2.5.9). Therefore, we will first prove that $\partial_{x^k} v^\epsilon$ is indeed the classical solution of (2.5.9).

Let us consider the equation

(2.5.10)
$$(\partial_t - \mathcal{L}_D - I + r)\, w = -p'_\epsilon(v^\epsilon - g^\epsilon)(\partial_{x^k} v^\epsilon - \partial_{x^k} g^\epsilon), \quad (x,t) \in \mathbb{R}^n \times (0,T],$$

$$w(x,0) = \partial_{x^k} g^\epsilon(x).$$

Thanks to Lemma 2.5.2 and (H8), $-p'_\epsilon(v^\epsilon - g^\epsilon)(\partial_{x^k} v^\epsilon - \partial_{x^k} g^\epsilon)$ is Hölder continuous. Therefore, it follows from Theorem 3.1 in Garroni and Menaldi (1992) pp. 89 that (2.5.10) has a unique classical solution. Let us call it w.

For any point $(x,t) \in \mathbb{R}^n \times [0,T]$, we will show that $\partial_{x^k} v^\epsilon(x,t) = w(x,t)$. As a vector in $\mathbb{R}^n$, $x = (x^1, \cdots, x^n)$. Let us also denote $x(z) \triangleq (x^1, \cdots, x^{k-1}, z, x^{k+1}, \cdots, x^n)$. One can check that $v(x,t) \triangleq \int_0^{x^k} w(x(z),t)\, dz + v^\epsilon(x(0),t)$ is a classical solution of the following Cauchy problem

(2.5.11)
$$(\partial_t - \mathcal{L}_D - I + r)\, v = -p_\epsilon(v^\epsilon - g^\epsilon), \quad (x,t) \in \mathbb{R}^n \times (0,T],$$

$$v(x,0) = g^\epsilon(x).$$

Moreover, thanks to estimate (3.6) in Theorem 3.1 of Garroni and Menaldi (1992) pp. 89, v is a bounded on $\mathbb{R}^n \times [0,T]$. On the other hand, using Lemma 2.5.4 one can show that (2.5.11) has a unique bounded classical solution. Therefore, it follows from Corollary 2.5.5 that $v(x,t) = v^\epsilon(x,t)$ for $(x,t) \in \mathbb{R}^n \times [0,T]$. As a result $\partial_{x^k} v^\epsilon(x,t) = w(x,t)$ and $\partial_{x^k} v^\epsilon$ is a classical solution of (2.5.9).

The rest of the proof is same as the proof of Lemma 2.4 in Yang et al. (2006). Thanks to Lemma 2.5.2, $|\partial_{x^k} v^\epsilon|$ is already bounded on $\mathbb{R}^n \times [0,T]$. We will show it is bounded uniformly in ϵ in the following. Let $u = L + \partial_{x^k} v^\epsilon$, $u \in C^0([0,T] \times \mathbb{R}^n) \cap C^{2,1}((0,T] \times \mathbb{R}^n)$ and it satisfies

(2.5.12)
$$(\partial_t - \mathcal{L}_D - I + r)\, u + p'_\epsilon(v^\epsilon - g^\epsilon)\, u = p'_\epsilon(v^\epsilon - g^\epsilon)(\partial_{x^k} g^\epsilon + L) + r\, L,$$

$$u(x,0) = L + \partial_{x^k} g(x).$$

Note (H4') and (2.5.3) (iv), $u(x,t) \geq 0$ follows from applying Lemma 2.5.4 to (2.5.12)

by picking $c = r + p'_\epsilon(v^\epsilon - g^\epsilon)$. The proof for the upper bound can be performed similarly by picking $u = L - \partial_{x^k} v^\epsilon$. □

Remark 2.5.8. The constant coefficient assumption (H7') makes sure that the coefficient before u in (2.5.12) is nonnegative in order to apply the Maximal Principle Lemma 2.5.4.

Lemma 2.5.9. *For any $\epsilon \in (0, \epsilon_0)$, $v^\epsilon(x, t) \geq g^\epsilon(x)$ on $\mathbb{R}^n \times [0, T]$.*

Proof. Let us first show that $Ig^\epsilon(x)$ is uniformly bounded from below. Compared to Yang et al. (2006) where ν is a finite measure, we will bound $Ig^\epsilon(x)$ from below in the following way.

(2.5.13)

$$
\begin{aligned}
Ig^\epsilon(x) &= \int_{|y| \leq 1} \left[g^\epsilon(x + y) - g^\epsilon(x) - \sum_{i=1}^n y^i \frac{\partial}{\partial x^i} g^\epsilon(x) \right] \nu(dy) + \int_{|y| > 1} \left[g^\epsilon(x + y) - g^\epsilon(x) \right] \nu(dy) \\
&= \int_{|y| \leq 1} \nu(dy) \int_0^1 dz (1 - z) \sum_{i,j=1}^n y^i y^j \frac{\partial^2}{\partial x^i \partial x^j} g^\epsilon(x + zy) + \int_{|y| > 1} \left[g^\epsilon(x + y) - g^\epsilon(x) \right] \nu(dy) \\
&\geq \int_{|y| \leq 1} \nu(dy) \int_0^1 dz (1 - z) \left(-J|y|^2 \right) - K \int_{|y| > 1} \nu(dy) \\
&\geq -J \int_{|y| \leq 1} |y|^2 \nu(dy) - K \int_{|y| > 1} \nu(dy),
\end{aligned}
$$

where the first inequality follows from (H9) and (H3').

On the other hand, thanks to (H6") and (H9), $\sum_{i,j}^n a_{ij}(x, t) \partial^2_{x^i x^j} g^\epsilon(x)$ is also bounded from below. Note that $\sum_{i,j}^n a_{ij}(x, t) \partial^2_{x^i x^j} g^\epsilon(x) = tr(A H(g^\epsilon))$, where $H(g^\epsilon)$ is the Hessian of g^ϵ, i.e., $H(g^\epsilon)_{ij} = \partial^2_{x^i x^j} g^\epsilon(x)$. It follows from the first inequality in (H6") that A is a positive definite matrix. Then there exists a nonsingular matrix C such that $A = CC'$. Therefore $tr(A H(g^\epsilon)) = tr(CC' H(g^\epsilon)) = tr(C' H(g^\epsilon) C)$. Moreover, (H9) and (H6") give us that

$$
(C\xi)' H(g^\epsilon) (C\xi) \geq -J \left(\xi' C' C \xi \right) = -J \left(\xi' A \xi \right) \geq -J\Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.
$$

Hence $C' H(g^\epsilon) C + J\Lambda I_n$ is a non-negative definite matrix. As a result, we have $tr\left(C' H(g^\epsilon) C\right) + nJ\Lambda = tr\left(C' H(g^\epsilon) C + J\Lambda I_n\right) \geq 0$, which implies

$$(2.5.14) \qquad \sum_{i,j=1}^{n} a_{ij}(x,t)\,\frac{\partial^2}{\partial x^i \partial x^j} g^\epsilon(x) = tr\left(AH(g^\epsilon)\right) \geq -nJ\Lambda.$$

Thanks to (2.5.13) and (2.5.14), we can bound $(\partial_t - \mathcal{L}_D - I + r)\,g^\epsilon(x)$ from above. Indeed,

(2.5.15)

$$(\partial_t - \mathcal{L}_D - I + r)\,g^\epsilon(x)$$

$$= -\sum_{i,j=1}^{n} a_{ij}(x,t)\,\frac{\partial^2}{\partial x^i \partial x^j} g^\epsilon(x) - \sum_{i=1}^{n} b_i(x,t)\,\frac{\partial}{\partial x^i} g^\epsilon(x) + r(x,t)\,g^\epsilon(x) - Ig^\epsilon(x)$$

$$\leq nJ\Lambda + |b|^{(0)} L + |r|^{(0)} K + J \int_{|y| \leq 1} |y|^2 \nu(dy) + K \int_{|y| > 1} \nu(dy)$$

$$= -p_\epsilon(0),$$

where the second equality follows from (2.5.3) (iii).

Now we will show $v^\epsilon \geq g^\epsilon$ via the Maximum Principle in Lemma 2.5.4. It follows from (2.5.15) that

$$(\partial_t - \mathcal{L}_D - I + r)\,(v^\epsilon - g^\epsilon) = -p_\epsilon(v^\epsilon - g^\epsilon) - (\partial_t - \mathcal{L}_D - I + r)\,g^\epsilon$$

$$\geq -p_\epsilon(v^\epsilon - g^\epsilon) + p_\epsilon(0).$$

Combining with the mean value theorem, we obtain

$$(2.5.16) \qquad \left(\partial_t - \mathcal{L}_D - I + r + p_\epsilon'(y)\right)(v^\epsilon - g^\epsilon) \geq 0,$$

where $y \in \mathbb{R}^n$ and $p_\epsilon'(y)$ is bounded. Therefore the statement of the lemma follows applying Lemma 2.5.4 to (2.5.16) and choosing $c = r + p_\epsilon'(y) \geq 0$. $\qquad\square$

As an easy corollary, the penalty terms are uniformly bounded.

Corollary 2.5.10. $p_\epsilon(v^\epsilon - g^\epsilon)$ *is bounded uniformly in* $\epsilon \in (0, \epsilon_0)$.

Proof. Thanks to Lemma 2.5.9 and (2.5.3) (i) and (iv), we have $p_\epsilon(0) \leq p_\epsilon(v^\epsilon - g^\epsilon) \leq$ 0. The statement follows noticing that $p_\epsilon(0)$ (in (2.5.3) (iii)) is independent of ϵ. $\square$

Thanks to Lemmas 2.5.2, 2.5.6, 2.5.7 and Corollary 2.5.10, we can apply the following $W_p^{2,1}$-norm estimate for the parabolic integro-differential equation to each solution v^ϵ of the penalty problem.

Since the proof of the following theorem is technical and independent of the penalty problem, we will perform it in the Section 2.7.

Theorem 2.5.11. *Let us assume the Lévy measure satisfies* (H5) *with* $\alpha \in [0, 2)$, *if* v *is a* $W_{p,loc}^{2,1}$ *solution of the following Cauchy problem for some positive integer* p,

(2.5.17)
$$(\partial_t - \mathcal{L}_D - I + r)\,v = f(x,t), \quad (x,t) \in \mathbb{R}^n \times (0, T],$$
$$v(x, 0) = g(x),$$

where the coefficients satisfy (H6"), (H7") *and* $f \in L_{p,loc}(\mathbb{R}^n \times (0, T))$, *moreover* $|v|$ *is bounded on* $\mathbb{R}^n \times [0, T]$ *and* $|\nabla_x v|$ *is bounded on any compact domain of* $\mathbb{R}^n \times [0, T]$. *Then for any domain* $B_\rho(x_0) \times (s, T)$ *for any* $\rho > 0$, $s \in (0, T)$ *and* $x_0 \in \mathbb{R}^n$

(2.5.18)
$$\|v\|_{W_p^{2,1}(B_\rho(x_0) \times (s,T))} \leq C_\delta \left[\max_{\mathbb{R}^n \times [0,T]} |v| + \max_{B_{\rho+\delta/4+1}(x_0) \times [0,T]} |\nabla_x v| + \|f\|_{L^p(B_{\rho+\delta/4}(x_0) \times (\delta/2,T))} \right],$$

for some positive constant C_δ *and* $\delta < s$.

Remark 2.5.12. The existence of the $W_p^{2,1}$ solution for (2.5.17) was ensured by Theorem 3.2 in Bensoussan and Lions (1984) pp.234. However, the norm estimation was not given there. On the other hand, since the integral operator I is non-local, it is important to study the Cauchy problem (2.5.17) on the entire domain $\mathbb{R}^n \times [0, T]$. Otherwise, for the Cauchy problem on bounded domains of $\mathbb{R}^n \times [0, T]$ with some boundary conditions, $W_p^{2,1}$ solutions are not expected in general, see Gimbert and Lions (1985) for a counterexample.

A $W_p^{2,1}$-norm estimate, similar to (2.5.18), for the parabolic integro-differential equation was proved in Theorem 3.5 in Garroni and Menaldi (1992) pp. 91. However, the estimation in Garroni and Menaldi (1992) requires the jump restricted in a bounded domain, i.e., if $x \in \Omega$ where Ω is a bounded domain in $\mathbb{R}^n$, the jump size $z(x)$, which is state dependent, can only be chosen such that $x + z(x) \in \Omega$ (see (1.54) in Garroni and Menaldi (1992) pp. 63). However, this restriction is not satisfied in our case, where the jump size is unbounded and independent of the state variable x.

Applying Theorem 2.5.11 to each penalty problem (2.5.2), thanks to Lemmas 2.5.2, 2.5.6, 2.5.7 and Corollary 2.5.10, we have the following corollary.

Corollary 2.5.13. *If the Lévy measure satisfies* (H2) *and* (H5) *with* $\alpha \in [1, 2)$, *moreover* (H6'), (H7'), (H3'), (H4'), (H8) *and* (H9) *are also satisfied, then for any domain* $B_\rho(x_0) \times (s, T)$ *for any* $\rho > 0$, $s \in (0, T)$ *and* $x_0 \in \mathbb{R}^n$, $\|v^\epsilon\|_{W_p^{2,1}(B_\rho(x_0) \times (s,T))}$ *are bounded uniformly in* $\epsilon \in (0, \epsilon_0)$ *for any integer* $p \in (1, \infty)$, *i.e., there is a constant* C *independent of* ϵ *such that*

$$\tag{2.5.19} \|v^\epsilon\|_{W_p^{2,1}(B_\rho(x_0) \times (s,T))} \le C.$$

Proof. It follows from Lemma 2.5.2 that $v^\epsilon \in W_{p,loc}^{2,1}(\mathbb{R}^n \times (0, T))$. Thanks to Lemmas 2.5.6 and 2.5.7, both $\max_{\mathbb{R}^n \times [0,T]} |v^\epsilon|$ and $\max_{\mathbb{R}^n \times [0,T]} |\nabla_x v^\epsilon|$ are also bounded uniformly in ϵ. Moreover, picking $f = -p_\epsilon(v^\epsilon - g^\epsilon)$, it follows from Corollary 2.5.10 that f is also bounded uniformly in ϵ. Concluding from these facts, (2.5.19) follows (2.5.18). $\square$

Remark 2.5.14. The estimate in Theorem 2.5.11 is essential for the proof of Corollary 2.5.13. However, having infinite variation jumps presents two technical difficulties to the proof of Theorem 2.5.11.

First, as we shall see in Lemma 2.7.1, once the Lévy measure has a singularity, the L_p-norm of Iv^ϵ depends on the $W_p^{2,1}$-norm of v^ϵ. Therefore, one could not consider Iv^ϵ as a driving term directly and use the classical $W_p^{2,1}$-norm estimate for parabolic differential equations (without the integral term) to bound the $W_p^{2,1}$-norm of v^ϵ by the L_p-norm of $I\,v^\epsilon$. When the Lévy measure is a finite measure as in Yang et al. (2006), L_p-norm of Iv^ϵ only depends on L^∞-norm of v^ϵ. Therefore, Lemma 2.6 in Yang et al. (2006) follows from the classical $W_p^{2,1}$-norm estimate for parabolic differential equations, i.e., the $W_p^{2,1}$-norm of v^ϵ is bounded by L^∞-norm of v^ϵ.

Second, as we have seen in Remark 2.4.2 and we shall see it again in Lemma 2.7.1, the regularity of Iv^ϵ actually depends on regularity of v^ϵ on a larger domain. This extension of the domain is another technical difficulty we face in the proof of Theorem 2.5.11, because the extension of domains implies that $W_p^{2,1}$-norm of v^ϵ on a bounded domains depends on its $W_p^{2,1}$-norm on a slightly larger domain.

To conclude this section, in the following theorem we will find a limit v^* of the sequence $\{v^\epsilon\}_{\epsilon \in (0,\epsilon_0)}$ such that v^* is indeed the value function v defined at the beginning of this section.

Theorem 2.5.15. *Let us assume that* (H6'), (H7'), (H3'), (H4'), (H8) *and* (H9) *are satisfied, moreover, the Lévy measure ν satisfies* (H2) *and* (H5) *with $\alpha \in [1,2)$. Then for any $s, \rho > 0$ and $x_0 \in \mathbb{R}^n$, there exists a subsequence $\{\epsilon_k\}_{k\geq 0}$ such that v^{ϵ_k} converges uniformly to the limit v^* uniformly in $\overline{B_\rho(x_0)} \times [s, T]$ as $\epsilon_k \to 0$. Moreover, v^* solves the variational inequality* (2.5.1) *for almost every point in $\mathbb{R}^n \times [0,T]$ and $v^* \in W_p^{2,1}(B_\rho(x_0) \times (s,T))$ for any integer $p \in (1,\infty)$.*

Proof. Thanks to Corollary 2.5.13, there exists a subsequence $\{\epsilon_k\}$ with $\epsilon_k \to 0$ and

a function $v^* \in W_p^{2,1}(B_\rho(x_0) \times (s,T))$ such that

$$v^{\epsilon_k} \rightharpoonup v^* \quad \text{in } W_p^{2,1}(B_\rho(x_0) \times (s,T)).$$

Here " $\rightharpoonup$ " represents weak convergence, please refer to Appendix D.4. in Evans (1998) pp. 639 for its definition and properties. The rest of the proof is the same as proof of Theorem 3.2 in Yang et al. (2006). It confirms that v^* solves the variational inequality (2.5.1) for almost every point in $\mathbb{R}^n \times [0,T]$. $\qquad\square$

Finally, thanks to the verification result Proposition 2.2.7, v^* must be the v defined at the beginning of this section. As a result, the $1 \leq \alpha < 2$ case of Theorem 2.4.5 follows from Theorem 2.5.15 after reversing the time.

2.6 Proof of several lemmas in Sections 2.2, 2.3 and 2.4

Proof of Lemma 2.2.1. Throughout this proof, in order to distinguish the Euclidean norm in $\mathbb{R}^n$ from the absolute value in $\mathbb{R}$, we denote the Euclidean norm as $\|\cdot\|$ and the absolute value as $|\cdot|$. Actually, the norm $\|\cdot\|$ is equivalent to the sum of the norms $|\cdot|$ among all components, i.e.,

$$(2.6.1) \qquad \|y\| \leq \sum_{i=1}^{n} |y^i| \leq n \|y\|, \quad \text{for any } y \in \mathbb{R}^n.$$

Thanks to (2.6.1), (2.2.4) - (2.2.7) can be showed under a slighter weaker assumption (H2), compared to the assumption $\int_{|y|>1} |y|^2 \, \nu(dy)$ in Lemma 3.1 of Pham (1998). We will only prove (2.2.6) and (2.2.7) in the following.

Following from (2.1.1) and (2.2.2), we have for any $\tau \in \mathcal{T}_{0,t}$ that

$$(2.6.2) \quad \|X_\tau^x - x\| \leq \left\| \int_0^\tau b\left(X_s^x, s\right) ds \right\| + \left\| \int_0^\tau \sigma\left(X_s^x, s\right) dW_s \right\| + \left\| \mathcal{J}_\tau^\ell \right\| + \left\| \lim_{\epsilon \downarrow 0} \mathcal{J}_\tau^\epsilon \right\|.$$

Comparing to the proof of Lemma 3.1 in Pham (1998), the difference is on the estimation on the large jump term. Therefore, we will focus on $\left\| \mathcal{J}_\tau^\ell \right\|$ in the following.

First it follows from (2.2.2) and the triangle inequality that

(2.6.3)
$$\mathbb{E}\left\|\mathcal{J}_\tau^\ell\right\| = \mathbb{E}\left\|\int_0^\tau \int_{\|y\|>1} y\,\mu(ds,dy)\right\| \leq \mathbb{E}\left\|\int_0^\tau \int_{\|y\|>1} y\,\widetilde{\mu}(ds,dy)\right\| + \mathbb{E}\left\|\int_0^\tau ds \int_{\|y\|>1} y\,\nu(dy)\right\|.$$

Let us estimate the right-hand-side of (2.6.3) separately. On the one hand, $\int_0^t \int_{\|y\|>1} y\,\widetilde{\mu}(ds,dy)$ is a martingale because of (H2). Hence $\left\|\int_0^t \int_{\|y\|>1} y\,\widetilde{\mu}(ds,dy)\right\|$ is a submartingale (see e.g. Problem 3.7 in Karatzas and Shreve (1991) pp. 13). Noticing that $\tau \in \mathcal{T}_{0,t}$, it follows from the Optional Sampling Theorem that

(2.6.4)
$$\mathbb{E}\left\|\int_0^\tau \int_{\|y\|>1} y\,\widetilde{\mu}(ds,dy)\right\| \leq \mathbb{E}\left\|\int_0^t \int_{\|y\|>1} y\,\widetilde{\mu}(ds,dy)\right\|.$$

Thanks to (2.6.1), we can estimate the right-hand-side of (2.6.4) as follows.

(2.6.5)
$$\mathbb{E}\left\|\int_0^t \int_{\|y\|>1} y\,\widetilde{\mu}(ds,dy)\right\| \leq \mathbb{E}\sum_{i=1}^n \left|\int_0^t \int_{\|y\|>1} y^i\,\widetilde{\mu}(ds,dy)\right|$$
$$\leq \mathbb{E}\sum_{i=1}^n \left|\int_0^t \int_{\|y\|>1} y^i\,\mu(ds,dy)\right| + \sum_{i=1}^n \int_0^t ds \int_{\|y\|>1} \left|y^i\right|\,\nu(dy)$$
$$\leq \mathbb{E}\int_0^t \int_{\|y\|\geq 1} \sum_{i=1}^n \left|y^i\right|\,\mu(ds,dy) + \int_0^t ds \int_{\|y\|>1} \sum_{i=1}^n \left|y^i\right|\,\nu(dy)$$
$$= 2\int_0^t ds \int_{\|y\|>1} \sum_{i=1}^n \left|y^i\right|\,\nu(dy) \leq 2\,n \int_{\|y\|>1} \|y\|\,\nu(dy)\cdot t.$$

Here the first and fourth inequalities follow from (2.6.1). Moreover, the third inequality follows since the Poisson random measure μ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^n$ for each $\omega \in \Omega$. On the other hand, the second term on the right-hand-side of (2.6.3) can be estimated similarly using (2.6.1).

Concluding from (2.6.3) - (2.6.5), we can find a positive constant C such that $\mathbb{E}\left\|\mathcal{J}_\tau^\ell\right\| \leq C\,t$ for any $\tau \in \mathcal{T}_{0,t}$. The other three terms on the right-hand-side of (2.6.2) can be estimated in the same way as in Lemma 3.1 of Pham (1998). In particular, the stochastic integral and the small jump terms are bounded by $C\,t^{1/2}$.

Moreover, compared to the estimate (3.3) in Pham (1998), the boundness of b and σ ensures that the constant C in (2.2.6) is independent of x.

For (2.2.7), we will still focus on the large jump term. Instead of applying the Doob's inequality as in Lemma 3.1 in Pham (1998), we will use properties of μ to derive the following estimate.

(2.6.6)

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left\|\mathcal{J}_s^\ell\right\|\right] = \mathbb{E}\left[\sup_{0\leq s\leq t}\left\|\int_0^s\int_{\|y\|>1} y\,\mu(du,dy)\right\|\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}\sum_{i=1}^n\left|\int_0^s\int_{\|y\|>1} y^i\,\mu(du,dy)\right|\right]$$

$$\leq \mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^s\int_{\|y\|>1}\sum_{i=1}^n|y^i|\,\mu(du,dy)\right] \leq \mathbb{E}\left[\int_0^t\int_{\|y\|>1}\sum_{i=1}^n|y^i|\,\mu(du,dy)\right]$$

$$= \int_0^t du\int_{\|y\|>1}\sum_{i=1}^n|y^i|\,\nu(dy) \leq n\int_{\|y\|>1}\|y\|\,\nu(dy)\cdot t.$$

Here the first and fourth inequalities follow from (2.6.1), the second and the third inequalities hold since μ is a non-negative measure for each $\omega \in \Omega$. The rest proof of (2.2.7) follows from the same approach used in Lemma 3.1 of Pham (1998). □

Proof of Lemma 2.3.5. Thanks to Lemma 2.3.1, the driving term $I^f u$ in (2.3.27) is well defined in the classical sense and Hölder continuous in both its variables. We will only prove the statement for the subsolution. The statement for the supersolution can be shown in the similar manner.

Given u as a subsolution of (2.3.27), we will show that u is a viscosity subsolution of (2.3.20). According to Definition 2.3.2, for any $(x_0, t_0) \in \overline{B} \times [t_1, t_2]$, the test function $\phi(x, t)$ is chosen such that

$$u(x_0, t_0) - \phi(x_0, t_0) = \max_{(x,t)\in\mathbb{R}^n\times[t_1,t_2]} \left[u(x,t) - \phi(x,t)\right].$$

Therefore $u(x_0 + y, t_0) - u(x_0, t_0) \leq \phi(x_0 + y, t_0) - \phi(x_0, t_0)$ for any $y \in \mathbb{R}^n$. Since ν is a positive measure, we have from (2.3.4) that

(2.6.7) $$I^f u(x_0, t_0) \leq I^f \phi(x_0, t_0).$$

Here $\phi(x,t)$ is chosen in $C_1(\mathbb{R}^n \times [t_1,t_2])$ so that $I^f \phi(x_0, t_0)$ is finite under the assumption (H2). Thanks to (2.6.7), we obtain from (2.3.28) that

$$(-\partial_t - \mathcal{L}_D + r)\, \phi(x_0, t_0) \le I^f u(x_0, t_0) \le I^f \phi(x_0, t_0), \quad \text{for } (x_0, t_0) \in B \times [t_1, t_2].$$

Moreover, (2.3.22) and (2.3.23) are automatically satisfied because $u(x,t)$ itself is the boundary and terminal value (2.3.27). Therefore according to Definition 2.3.2, $u(x,t)$ is a subsolution of (2.3.20).

Conversely, let us assume that $u(x,t)$ is a subsolution of (2.3.20), for any $(x_0, t_0) \in \overline{B} \times [t_1, t_2]$, given any function $\phi(x,t) \in C^{2,1}(\mathbb{R}^n \times [t_1, t_2])$ such that $\phi(x_0, t_0) = u(x_0, t_0)$ and $\phi(x,t) \ge u(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [t_1, t_2]$, let us construct ϕ^ϵ for $\epsilon \in (0,1)$ as follows.

$$\phi^\epsilon(x,t) \triangleq \phi(x,t)\chi^\epsilon(x) + \widetilde{u}(x,t)\, (1 - \chi^\epsilon(x)),$$

where χ^ϵ is a smooth function satisfying $0 \le \chi^\epsilon \le 1$, $\chi^\epsilon(x) = 1$ when $x \in B_\epsilon(x_0)$ and $\chi^\epsilon(x) = 0$ when $x \in \mathbb{R}^n \setminus B_{2\epsilon}(x_0)$. Moreover, $\widetilde{u} \in C^\infty(\mathbb{R}^n \times [t_1, t_2])$ such that $u \le \widetilde{u} \le u + \epsilon^2$ on $\mathbb{R}^n \times [t_1, t_2]$, for example, the usual mollification $\widetilde{u} = u * \zeta^\delta + \epsilon^2$ for sufficiently small δ (Please see Evans (1998) pp. 629 for the definition of the mollifier ζ^δ).

Observe that $u(x_0, t_0) = \phi(x_0, t_0) = \phi^\epsilon(x_0, t_0)$ and $u(x,t) - \phi^\epsilon(x,t) = (u - \phi)\chi^\epsilon(x) + (u - \widetilde{u})\, (1 - \chi^\epsilon(x)) \le 0$ for $(x,t) \in \mathbb{R}^n \times [t_1, t_2]$. Moreover, $\partial_t \phi^\epsilon(x_0, t_0) = \partial_t \phi(x_0, t_0)$, $\partial_{x_i} \phi^\epsilon(x_0, t_0) = \partial_{x^i} \phi(x_0, t_0)$ and $\partial^2_{x^i x^j} \phi^\epsilon(x_0, t_0) = \partial^2_{x^i x^j} \phi(x_0, t_0)$. Note that $\widetilde{u}$ is uniformly bounded, hence $\phi^\epsilon \in C_1(\mathbb{R}^n \times [t_1, t_2])$, therefore we choose $\phi^\epsilon(x,t)$ as the test function in the Definition 2.3.2 and obtain from (2.3.21) that

(2.6.8) $$(-\partial_t - \mathcal{L}_D + r)\, \phi(x_0, t_0) - I^f \phi^\epsilon(x_0, t_0) \le 0,$$

where $I^f \phi^\epsilon(x_0, t_0)$ is well defined, because one can show $\phi^\epsilon(x, t_0)$ is globally Lipschitz

in x as a result of our choice of χ^ϵ. On the other hand,

(2.6.9)

$$|\phi^\epsilon(x_0 + y, t_0) - u(x_0 + y, t_0)|$$

$$\leq |\phi(x_0 + y, t_0) - u(x_0 + y, t_0)| \chi^\epsilon(x_0 + y) + |\widetilde{u}(x_0 + y, t_0) - u(x_0 + y, t_0)| (1 - \chi^\epsilon(x_0 + y))$$

$$\leq |\phi(x_0 + y, t_0) - u(x_0 + y, t_0)| 1_{\{|y| \leq 2\epsilon\}} + \epsilon^2 1_{\{|y| \geq \epsilon\}}$$

$$\leq [|\phi(x_0 + y, t_0) - \phi(x_0, t_0)| + |u(x_0 + y, t_0) - u(x_0, t_0)|] 1_{\{|y| \leq 2\epsilon\}} + \epsilon^2 1_{\{|y| \geq \epsilon\}}$$

$$\leq (\widetilde{L_x} + L_x) |y| 1_{\{|y| \leq 2\epsilon\}} + \epsilon^2 1_{\{|y| \geq \epsilon\}},$$

where $\widetilde{L_x} = \max_{|x - x_0| \leq 2\epsilon} \partial_x \phi(t_0, x)$ and L_x is the constant in Lemma 2.2.3. Due to

(2.6.9), (2.3.1) and (H2), we have

(2.6.10)

$$\left| I^f \phi^\epsilon(x_0, t_0) - I^f u(x_0, t_0) \right| \leq (\widetilde{L_x} + L_x) \int_{|y| \leq 2\epsilon} |y| \, \nu(dy) + \int_{|y| \geq \epsilon} \epsilon^2 \, \nu(dy)$$

$$\leq (\widetilde{L_x} + L_x) \int_{|y| \leq 2\epsilon} |y| \, \nu(dy) + \epsilon \int_{|y| \geq \epsilon} |y| \, \nu(dy) \to 0 \quad \text{as } \epsilon \downarrow 0.$$

Then the statement that u is a viscosity solution of (2.3.27) follows from combining

(2.6.8) and (2.6.10). □

Proof of Lemma 2.5.4. For any $R_0 > 0$, let us consider the following function

$$w(x, t) = \frac{m}{f(R_0)} [f(|x|) + C_1 t] + v(x, t),$$

where $f(R) = \frac{R^2}{1+R}$ and the positive constant C_1 will be determined later. It is

clear that $f(R)$ is an increasing function on $(0, +\infty)$ and $\lim_{R \to +\infty} f(R) = +\infty$.

On the other hand, $|\partial_{x^i} f(|x|)| \leq \frac{|x|(2+|x|)}{(1+|x|)^2} < 1$ for any $i \leq n$. Moreover, one can

also check that $\lim_{|x| \to +\infty} |\partial^2_{x^i x^j} f(|x|)| = 0$ and $\lim_{|x| \to 0} |\partial^2_{x^i x^j} f(|x|)| = 2 \delta_{ij}$ for any

$i, j \leq n$. Therefore both $\partial_{x^i} f(|x|)$ and $\partial^2_{x^i x^j} f(|x|)$ are bounded on $\mathbb{R}^n$. Thanks to

these properties, we can find an upper bound for $|If(|x|)|$ as follows:

$$|If(|x|)| = \left| \int_{\mathbb{R}^n} \left[f(|x+y|) - f(|x|) - \sum_{i=1}^{n} y^i \, \partial_{x^i} f(|x|) \, 1_{\{|y| \leq 1\}} \right] \nu(dy) \right|$$

$$\leq \int_{|y| \leq 1} \nu(dy) \int_0^1 dz \, (1-z) \sum_{i,j=1}^{n} |y^i y^j| \, |\partial^2_{x^i x^j} f(|x+zy|)|$$

(2.6.11)

$$+ \int_{|y|>1} \nu(dy) \, |f(|x+y|) - f(|x|)|$$

$$\leq C \left(\int_{|y| \leq 1} |y|^2 \nu(dy) + \int_{|y|>1} |y| \, \nu(dy) \right) < +\infty,$$

for some sufficiently large constant $C > 0$. Here the last inequality in (2.6.11) follows from (2.2.3) and (H2).

Now, applying the parabolic integro-differential operator to w, we obtain

$$(\partial_t - \mathcal{L}_D - I + c) \, w(x,t)$$

$$\geq (\partial_t - \mathcal{L}_D - I + c) \left[\frac{m}{f(R_0)} \left(f(|x|) + C_1 \, t \right) \right]$$

$$= \frac{m}{f(R_0)} \left[C_1 - \sum_{i,j=1}^{n} a_{ij} \, \partial^2_{x^i x^j} f(|x|) - \sum_{i=1}^{n} b_i \, \partial_{x^i} f(|x|) + c \, f(|x|) - If(|x|) \right],$$

where the first inequality follows from the assumption that $(\partial_t - \mathcal{L}_D - I + c) \, v(x,t) \geq 0$. We can choose a sufficiently large constant C_1 independent of R_0 such that

(2.6.12) $\qquad (\partial_t - \mathcal{L}_D - I + c) \, w(x,t) > 0, \qquad \text{for } (x,t) \in \mathbb{R}^n \times [0,T].$

This is because $\partial^2_{x^i x^j} f(|x|)$, $\partial_{x^i} f(|x|)$ and coefficients a_{ij}, b_i, c are all bounded, moreover $c \geq 0$ and $|If(|x|)|$ is bounded thanks to (2.6.11).

On the other hand, $w(x,0) = \frac{m}{f(R_0)} f(|x|) + v(x,0) \geq 0$ thanks to the assumption $v(x,0) \geq 0$. Moreover, when $|x| = R_0$, $w(x,t) = \frac{m}{f(R_0)} (f(R_0) + C_1 \, t) + v(x,t) \geq m + v(x,t) \geq 0$ due to the assumption $v(x,t) \geq -m$. Furthermore, when $|x| > R_0$, we also have $w(x,t) \geq m + v(x,t) \geq 0$ since $f(R)$ is an increasing function. Therefore, we claim that $w(x,t) \geq 0$ for $(x,t) \in B_{R_0} \times (0,T_0]$. Indeed, if there are some points

$(x, t) \in B_{R_0} \times (0, T_0]$ such that $w(x, t) < 0$, $w(x, t)$ must take its negative minimum at some point $(x_0, t_0) \in B_{R_0} \times (0, T_0]$. Noticing that $w(x, t) \geq 0$ for $|x| \geq R_0$, we have $w(x_0, t_0) \leq w(x, t)$ for all $(x, t) \in \mathbb{R}^n \times (0, T]$. As a result, we obtain $\partial_t w(x_0, t_0) \leq 0$, $\sum_{i=1}^n b_i \partial_{x^i} w(x_0, t_0) = 0$ and $\sum_{i,j=1}^n a_{ij} \partial^2_{x^i x^j} w(x_0, t_0) \geq 0$ (see e.g. Lemma 1 in Friedman (1964) pp. 34). Moreover, $I w(x_0, t_0) \geq 0$, since w achieves its minimum at (x_0, t_0) and $\nabla_x w(x_0, t_0) = 0$. Therefore, we have

$$(\partial_t - \mathcal{L}_D - I + r) \, w(x_0, t_0) \leq 0,$$

which contradicts with (2.6.12).

Now, for any point $(x, t) \in \mathbb{R}^n \times (0, T]$, taking $R_0 \to +\infty$, we have $v(x, t) \geq 0$ since $\lim_{R_0 \to +\infty} f(R_0) = +\infty$. $\qquad \square$

Proof of Lemma 2.5.6. First, thanks to Lemma 2.5.2, $|v^\epsilon|$ is bounded on $\mathbb{R}^n \times [0, T]$. In the following, we will show it is bounded uniformly in ϵ. It follows from (2.5.3) (i) that $(\partial_t - \mathcal{L}_D - I + r) \, v^\epsilon = -p_\epsilon(v^\epsilon - g^\epsilon) \geq 0$. Note that $v^\epsilon(x, 0) = g^\epsilon(x) \geq 0$ (see (H3')), the first inequality in the statement follows from Lemma 2.5.4 directly. On the other hand, defining $u = K + 1 - v^\epsilon$, u satisfies

$$(2.6.13) \qquad (\partial_t - \mathcal{L}_D - I + r) \, u = r(K + 1) + p_\epsilon(v^\epsilon - g^\epsilon), \quad (x, t) \in \mathbb{R}^n \times (0, T].$$

It follows from (H3') and (2.5.3) (ii) that $p_\epsilon(K + 1 - g^\epsilon) = 0$ with $\epsilon \leq \epsilon_0 \leq 1$. Combining with (2.6.13) and the mean value theorem, we obtain

$(2.6.14)$

$$(\partial_t - \mathcal{L}_D - I + r) \, u + p_\epsilon(K + 1 - g^\epsilon) - p_\epsilon(v^\epsilon - g^\epsilon) = \left[\partial_t - \mathcal{L}_D - I + r + p'_\epsilon(y) \right] u$$

$$= r \, (K + 1) \geq 0,$$

for some $y \in \mathbb{R}$. Note that both $K + 1 - g^\epsilon$ and $v^\epsilon - g^\epsilon$ are bounded, p'_ϵ is bounded in any bounded domain. Therefore, we have that $r + p'_\epsilon(y)$ is bounded and nonnegative

(see (2.5.3) (iv)). Now apply Lemma 2.5.4 to u and pick $c = r + p'_\epsilon(y)$, we obtain

$$u(x,t) = K + 1 - v^\epsilon(x,t) \geq 0 \text{ on } \mathbb{R}^n \times [0,T]. \qquad \square$$

2.7 Proof of Theorem 2.5.11

In this Appendix, for notational simplicity, the constant C denotes a generic constant in different places. Moreover, the center x_0 of the ball $B_\rho(x_0)$ will not be noted in the sequel. For any positive integer p, let us first estimate the L_p-norm of the integral term Iv.

Lemma 2.7.1. *If the assumptions of Theorem 2.5.11 are satisfied, then for any $\eta > 0$, there exists a positive constant C such that*

(2.7.1)

$$\|Iv\|_{L_p(B_\rho(x_0)\times(s,T))}$$

$$\leq C\eta^{2-\alpha}\|v\|_{W_p^{2,1}(B_{\rho+\eta}(x_0)\times(s,T))} + C\left(\max_{\mathbb{R}^n\times[s,T]}|v| + \max_{B_{\rho+1}(x_0)\times[s,T]}|\nabla_x v|\right) \cdot \begin{cases} (1+\eta^{1-\alpha}), & \alpha \neq 1 \\ (1-\log\eta), & \alpha = 1 \end{cases}.$$

Proof. Let us break the integral into three parts.

$$|Iv(x,t)| = \left| \int_{\mathbb{R}^n} \left[v(x+y,t) - v(x,t) - y \cdot \nabla_x v(x,t) 1_{\{|y| \le 1\}} \right] \nu(dy) \right|$$

$$\le \int_{|y| \le \eta} \nu(dy) \int_0^1 dz (1-z) \sum_{i,j=1}^n \left| y^i y^j \frac{\partial^2}{\partial x^i \partial x^j} v(x + zy, t) \right|$$

$$+ \int_{\eta < |y| \le 1} \nu(dy) \left| v(x+y,t) - v(x,t) - y \cdot \nabla_x v(x,t) \right|$$

$$+ \int_{|y| > 1} \nu(dy) \left| v(x+y,t) - v(x,t) \right|$$

$$\le \sum_{i,j=1}^n \int_{|y| \le \eta} |y|^2 \nu(dy) \int_0^1 dz \left| \frac{\partial^2}{\partial x^i \partial x^j} v(x + zy, t) \right|$$

$$+ \int_{\eta < |y| \le 1} \nu(dy) \left| v(x+y,t) - v(x,t) - y \cdot \nabla_x v(x,t) \right|$$

$$+ \int_{|y| > 1} \nu(dy) \left| v(x+y,t) - v(x,t) \right|$$

$$\triangleq \sum_{i,j=1}^n I_{i,j}(x,t) + I_2(x,t) + I_3(x,t).$$

In the following, we will estimate the L_p-norm of each term respectively.

(2.7.2)

$$\|I_{ij}(\cdot,t)\|_{L_p(B_\rho)}^p$$

$$= \int_{B_\rho} dx \left[\int_{|y| \le \eta} |y|^2 \nu(dy) \int_0^1 dz \left| \partial_{x^i x^j}^2 v(x+zy,t) \right| \right]^p$$

$$\le \int_{B_\rho} dx \int_0^1 dz \left[\int_{|y| \le \eta} \nu(dy) |y|^2 \left| \partial_{x^i x^j}^2 v(x+zy,t) \right| \right]^p$$

$$\le M^p \int_{B_\rho} dx \int_0^1 dz \left[\int_{|y| \le \eta} dy \, |y|^{2-n-\alpha} \left| \partial_{x^i x^j}^2 v(x+zy,t) \right| \right]^p$$

$$\le M^p \int_{B_\rho} dx \int_0^1 dz \left(\int_{|y| \le \eta} dy \, |y|^{2-n-\alpha} \right)^{\frac{p}{q}} \cdot \left(\int_{|y| \le \eta} dy \, |y|^{2-n-\alpha} \left| \partial_{x^i x^j}^2 v(x+zy,t) \right|^p \right)$$

$$= M^p \left(|S_1(0)| \frac{\eta^{2-\alpha}}{2-\alpha} \right)^{\frac{p}{q}} \cdot \int_0^1 dz \int_{|y| \le \eta} dy \, |y|^{2-n-\alpha} \int_{B_\rho} dx \left| \partial_{x^i x^j}^2 v(x+zy,t) \right|^p$$

$$\le M^p \left(|S_1(0)| \frac{\eta^{2-\alpha}}{2-\alpha} \right)^{\frac{p}{q}} \cdot \int_0^1 dz \int_{|y| \le \eta} dy \, |y|^{2-n-\alpha} \left\| \partial_{x^i x^j}^2 v(\cdot,t) \right\|_{L_p(B_{\rho+\eta})}^p$$

$$= M^p \left(|S_1(0)| \frac{\eta^{2-\alpha}}{2-\alpha} \right)^p \cdot \left\| \partial_{x^i x^j}^2 v(\cdot,t) \right\|_{L_p(B_{\rho+\eta})}^p.$$

Here the first inequality follows from Fubini's Theorem and Jensen's inequality with respect to the Lebesgue measure dz. The assumption (H5) is used in the second inequality. The third inequality follows from Hölder inequality with $1/p + 1/q = 1$. In the second equality, $|S_1(0)|$ is the surface area of the unit ball in $\mathbb{R}^n$. Note that $x + zy \in B_{\rho+\eta}$ when $x \in B_\rho$, $z \in (0,1)$ and $|y| \leq \eta$, the fourth inequality follows.

For I_2 and I_3, noting that $x + y \in B_{\rho+1}$ when $x \in B_\rho$ and $|y| \leq 1$, we have

$$(2.7.3)\|I_2(\cdot, t)\|_{L_p(B_\rho)} \leq C \cdot \max_{B_{\rho+1} \times [s,T]} |\nabla_x v| \cdot \begin{cases} (1 + \eta^{1-\alpha}), \ \alpha \neq 1 \\ \\ (1 - \log \eta), \ \alpha = 1 \end{cases} \quad \text{and}$$

$$(2.7.4) \qquad \|I_3(\cdot, t)\|_{L_p(B_\rho)} \leq C \cdot \max_{\mathbb{R}^n \times [s,T]} |v| \cdot \int_{|y|>1} \nu(dy).$$

Combining (2.7.2) - (2.7.4), (2.7.1) follows from noting that

$$\|Iv\|_{L_p(B_\rho \times (s,T))} \triangleq \left[\int_s^T \|Iv(\cdot, t)\|_{L_p(B_\rho)} \, dt \right]^{1/p} \text{ and } \|\partial^2_{x^i x^j} v\|_{L_p(B_{\rho+\eta} \times (s,T))} \leq \|v\|_{W_p^{2,1}(B_{\rho+\eta} \times (s,T))}$$

(see Definition 2.2.10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

In (2.7.1), when $\alpha \in [0,1)$ (finite variation jumps), the factors of η in both terms on the right-hand-side converge to 0 as $\eta \to 0$. Therefore, the L_p-norm of Iv on the domain $B_\rho(x_0) \times (s,T)$ essentially only depends on $\max_{\mathbb{R}^n \times [s,T]} |v|$ and $\max_{B_{\rho+1} \times [s,T]} |\nabla_x v|$. This can also be confirmed by working with the reduced integral form $I^f v$ in (2.3.4).

On the contrary, when $\alpha \in [1,2)$ (infinite variation jumps), the factor $1 + \eta^{1-\alpha}$ (or $1 - \log \eta$) in (2.7.1) will blow up as $\eta \to 0$ (a similar phenomenon was also observed in Lemma 1.1 of Bensoussan and Lions (1984) pp.206 for L_p-norm on $\mathbb{R}^n$). Therefore, it is important to note that the L_p-norm of Iv on the domain $B_\rho(x_0) \times (s,T)$ actually depends on $W_p^{2,1}$-norm of v on a larger domain $B_{\rho+\eta}(x_0) \times (s,T)$. Because of the expansion of the domain, instead of using the boundary estimate in Theorem 9.1 in Ladyženskaja et al. (1968) pp. 342, we will use the interior estimate technique in

Figure 2.1: Domains used in the proof of Theorem 2.5.11
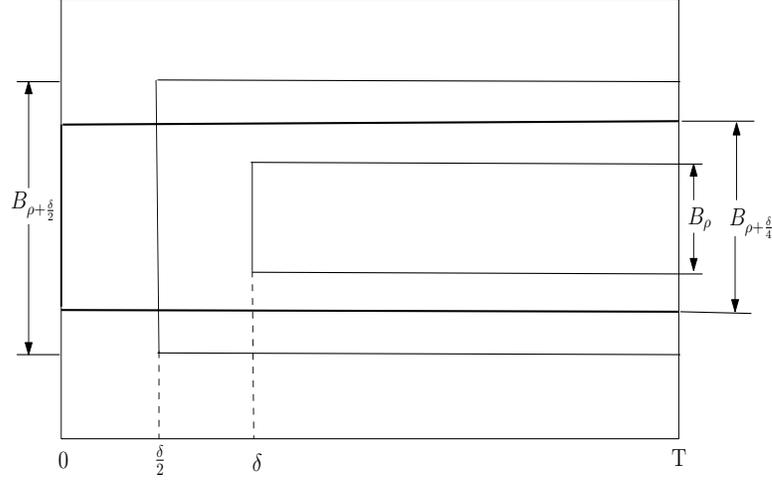
Theorem 10.1 in Ladyženskaja et al. (1968) pp. 351 to prove Theorem 2.5.11 in the following.

Proof of Theorem 2.5.11. Let us choose a cut-off function $\zeta^\delta(x,t)$ such that

$$
\zeta^\delta(x,t) = \begin{cases} 1 & (x,t) \in B_\rho \times (\delta, T) \\ 0 & (x,t) \in \mathbb{R}^n \times (0,T) \setminus B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T) \end{cases}
$$

Here the constant $\delta \in (0, s)$ will be determined later. This cut-off function can be chosen such that

(2.7.5) $\qquad \left|\partial_{x^i}\zeta^\delta\right| \le \frac{C_1}{\delta}, \quad \left|\partial^2_{x^i x^j}\zeta^\delta\right| \le \frac{C_2}{\delta^2} \quad \text{and} \quad \left|\partial_t\zeta^\delta\right| \le \frac{C_3}{\delta},$

for $i, j \le n$ and some constants C_1, C_2 and C_3. Please see Figure 2.1 for the domains used in this proof.

Defining $u(x,t) = \zeta^\delta(x,t)v(x,t)$, it satisfies

$$
(\partial_t - \mathcal{L}_D + r)\, u(x,t) = \zeta^\delta \cdot Iv(x,t) + \zeta^\delta \cdot f(x,t) + h(x,t), \quad (x,t) \in B_{\rho+\frac{\delta}{4}} \times (0,T),
$$

$$
u(x,t) = 0, \quad (x,t) \in \partial B_{\rho+\frac{\delta}{4}} \times (0,T),
$$

$$
u(x,0) = 0, \quad x \in \overline{B_{\rho+\frac{\delta}{4}}},
$$

in which $h(x,t) \triangleq \partial_t \zeta^\delta \cdot v - \sum_{i,j=1}^{n} a_{ij} \left(\partial_{x^i x^j}^2 \zeta^\delta \cdot v + 2 \partial_{x^i} \zeta^\delta \cdot \partial_{x^j} v \right) - \sum_{i=1}^{n} b_i \cdot \partial_{x^i} \zeta^\delta \cdot v$. Appealing to Theorem 9.1 in Ladyženskaja et al. (1968) pp.341, there exists a constant C such that

(2.7.6)

$$\|u\|_{W_p^{2,1}(B_{\rho+\frac{\delta}{4}} \times (0,T))} \leq C \left[\left\| \zeta^\delta \cdot Iv \right\|_{L_p} + \left\| \zeta^\delta \cdot f \right\|_{L_p} + \left\| \partial_t \zeta^\delta \cdot v \right\|_{L_p} + \left\| \sum_{i,j=1}^{n} a_{ij} \, \partial_{x^i x^j}^2 \zeta^\delta \cdot v \right\|_{L_p} \right.$$

$$\left. + \left\| \sum_{i,j=1}^{n} 2 \, a_{ij} \, \partial_{x^i} \zeta^\delta \cdot \partial_{x^j} v \right\|_{L_p} + \left\| \sum_{i=1}^{n} b_i \cdot \partial_{x^i} \zeta^\delta \cdot v \right\|_{L_p} \right],$$

in which all L_p-norms on the right-hand-side represent $L_p \left(B_{\rho+\frac{\delta}{4}} \times (0,T) \right)$.

In the following, we will estimate the terms on the right-hand-side of (2.7.6) respectively.

(2.7.7)

$$\left\| \zeta^\delta \cdot Iv \right\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0,T))}$$

$$\leq \|Iv\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))}$$

$$\leq C \left(\frac{\delta}{4} \right)^{2-\alpha} \|v\|_{W_p^{2,1}(B_{\rho+\frac{\delta}{2}} \times (\frac{\delta}{2}, T))} + C \left(1 + \left(\frac{\delta}{4} \right)^{1-\alpha} \right) \left[\max_{\mathbb{R}^n \times [0,T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1} \times [0,T]} |\nabla_x v| \right].$$

Here the first inequality follows from the choice of the cut-off function ζ^δ, the second inequality follows from Lemma 2.7.1 for $\alpha \neq 1$ case by picking $\eta = \frac{\delta}{4}$ and $s = \frac{\delta}{2}$. When $\alpha = 1$, we also have an estimate similar to (2.7.7). On the other hand, we have

(2.7.8)
$$\left\| \zeta^\delta \cdot f \right\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (0,T))} \leq \|f\|_{L_p(B_{\rho+\frac{\delta}{4}} \times (\frac{\delta}{2}, T))}.$$

Moreover, we obtain from (2.7.5) that

$$\left\|\partial_t \zeta^\delta \cdot v\right\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(0,T))} \leq \max_{\mathbb{R}^n\times[0,T]} |v| \cdot \left\|\partial_t\zeta^\delta\right\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(0,T))}$$

(2.7.9)
$$\leq \max_{\mathbb{R}^n\times[0,T]} |v| \left(\int_{B_{\rho+\frac{\delta}{4}}\times(\frac{\delta}{2},T)\setminus B_\rho\times(\delta,T)} dt\, dx\, \frac{C_3^p}{\delta^p}\right)^{\frac{1}{p}}$$

$$\leq C \max_{\mathbb{R}^n\times[0,T]} |v| \cdot \delta^{\frac{1-p}{p}}.$$

Similarly, thanks to (H7"), we also have

(2.7.10) $\quad \left\|\sum_{i,j=1}^n a_{ij} \partial^2_{x^ix^j}\zeta^\delta \cdot v\right\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(0,T))} \leq C \max_{\mathbb{R}^n\times[0,T]} |v| \cdot \delta^{\frac{1-2p}{p}}$,

(2.7.11)$\left\|\sum_{i,j=1}^n 2\, a_{ij}\, \partial_{x^i}\zeta^\delta \cdot \partial_{x^j}v\right\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(0,T))} \leq C \max_{B_{\rho+\frac{\delta}{4}}\times[0,T]} |\nabla_x v| \cdot \delta^{\frac{1-p}{p}}$ $\quad$ and

(2.7.12) $\quad \left\|\sum_{i=1}^n b_i \cdot \partial_{x^i}\zeta^\delta \cdot v\right\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(0,T))} \leq C \max_{\mathbb{R}^n\times[0,T]} |v| \cdot \delta^{\frac{1-p}{p}}$.

Plugging (2.7.7) - (2.7.12) into (2.7.6) and noticing the choice of the cut-off function ζ^δ, we obtain

$$\|v\|_{w^{2,1}_p(B_\rho\times(\delta,T))}$$

$$\leq \|u\|_{w^{2,1}_p(B_{\rho+\frac{\delta}{4}}\times(0,T))}$$

(2.7.13)
$$\leq C \left(\frac{\delta}{4}\right)^{2-\alpha} \|v\|_{W^{2,1}_p(B_{\rho+\frac{\delta}{2}}\times(\frac{\delta}{2},T))} + C \left[1 + \delta^{1-\alpha} + \delta^{\frac{1-p}{p}} + \delta^{\frac{1-2p}{p}}\right]$$

$$\cdot \left[\max_{\mathbb{R}^n\times[0,T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1}\times[0,T]} |\nabla_x v|\right] + \|f\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(\frac{\delta}{2},T))} \cdot$$

Multiplying δ^2 on both hand side of (2.7.13) and defining

$$K(\delta) = C\left[\delta^2 + \delta^{3-\alpha} + \delta^{\frac{1+p}{p}} + \delta^{\frac{1}{p}}\right] \cdot \left[\max_{\mathbb{R}^n\times[0,T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1}\times[0,T]} |\nabla_x v|\right] + \delta^2 \|f\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(\frac{\delta}{2},T))},$$

we obtain

(2.7.14) $\quad \delta^2 \|v\|_{w^{2,1}_p(B_\rho\times(\delta,T))} \leq 4C \left(\frac{\delta}{4}\right)^{2-\alpha} \cdot \left(\frac{\delta}{2}\right)^2 \|v\|_{w^{2,1}_p(B_{\rho+\frac{\delta}{2}}\times(\frac{\delta}{2},T))} + K(\delta).$

Let $F(\tau) \triangleq \tau^2 \|v\|_{W_p^{2,1}(B_{\rho+\delta-\tau}\times(\tau,T))}$. The inequality (2.7.14) gives us the following recursive inequality

$$(2.7.15) \qquad F(\delta) \leq 4C \left(\frac{\delta}{4}\right)^{2-\alpha} F(\delta/2) + K(\delta).$$

Since $\alpha < 2$, we can choose sufficiently small δ such that $4C\left(\delta/4\right)^{2-\alpha} \leq \frac{1}{2}$. Therefore, we have from (2.7.15) that

$$(2.7.16) \qquad F(\delta) \leq \frac{1}{2}F(\delta/2) + K(\delta).$$

On the other hand, thanks to the assumption $v \in W_{p,loc}^{2,1}(\mathbb{R}^n \times (0,T))$, $F(\delta)$ is finite for any $\delta \in (0,\delta_0)$. Iterating the recursive inequality (2.7.16) gives us

$$F(\delta) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} K\left(\frac{\delta}{2^i}\right) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} K(\delta) = 2\,K(\delta),$$

where the second inequality follows from noticing that $K(\delta)$ is increasing in δ. Therefore, it follows from the definitions of $F(\delta)$ and $K(\delta)$ that

$$\|v\|_{W_p^{2,1}(B_\rho\times(s,T))}$$

$$\leq \|v\|_{W_p^{2,1}(B_\rho\times(\delta,T))}$$

$$\leq 2\,C\left[1 + \delta^{1-\alpha} + \delta^{\frac{1-p}{p}} + \delta^{\frac{1-2p}{p}}\right] \cdot \left[\max_{\mathbb{R}^n\times[0,T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1}\times[0,T]} |\nabla_x v|\right] + \|f\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(\frac{\delta}{2},T))}$$

$$\leq C_\delta\left[\max_{\mathbb{R}^n\times[0,T]} |v| + \max_{B_{\rho+\frac{\delta}{4}+1}\times[0,T]} |\nabla_x v| + \|f\|_{L_p(B_{\rho+\frac{\delta}{4}}\times(\frac{\delta}{2},T))}\right].$$

$\square$

CHAPTER III

Regularity of the optimal exercise boundary of American options

3.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space hosting a Wiener process $W = \{W_t; t \geq 0\}$ and a Poisson random measure N on $\mathbb{R}_+ \times \mathbb{R}$ with the mean measure $\lambda dt \, \nu(dz)$ (in which ν is a probability measure on $\mathbb{R}$) independent of the Wiener process. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ be the (augmented) natural filtration of W and N. We will consider a Markov process $\mathbb{S} = \{S_t; t \geq 0\}$, which follows the dynamics

$$(3.1.1) \qquad dS_t = \mu S_{t-} dt + \widetilde{\sigma}(S_{t-}, t) S_{t-} dW_t + S_{t-} \int_{\mathbb{R}} (e^z - 1) \, N(dt, dz),$$

as the stock price process. We will take $\mu \triangleq r - q + \lambda - \lambda \xi$, in which

$$(3.1.2) \qquad \xi \triangleq \int_{\mathbb{R}} e^z \nu(dz) < \infty,$$

as a standing assumption. We impose this condition on ξ so that the discounted stock prices are martingales. The constant $r \geq 0$ is the interest rate, $q \geq 0$ is the dividend. The volatility $\widetilde{\sigma}(S, t)$ is assumed to be continuously differentiable in both S and t. Moreover, there are positive constants δ and Δ such that

$$(3.1.3) \qquad 0 < \delta \leq \widetilde{\sigma}(S, t) \leq \Delta, \quad \text{for all } S, t \geq 0.$$

We should note that at the time of a jump the stock price moves from S_{t-} to $S_{t-}e^Z$ in which Z is a random variable whose distribution is given by ν. When $Z < 0$ the stock price jumps down, when $Z > 0$ the stock price jumps up. In the classical Merton jump diffusion model, Z is a Gaussian random variable.

In this framework, we will study the American put option pricing problem. The value function of the American put option is defined by

$$(3.1.4) \qquad V(S,t) \triangleq \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}\{e^{-r\tau}(K - S_\tau)^+ \,|\, S_0 = S\},$$

in which $\mathcal{T}_{0,T-t}$ is the set of stopping times (with respect to the filtration $\mathbb{F}$) taking values in $[0, T-t]$. The value function V is the classical solution of a free boundary problem (see Proposition 3.2.1). The main goal of this chapter is to analyze the regularity of the free boundary. We will show that the free boundary is C^1 except at the maturity T, and C^∞ with an appropriate regularity assumption on the jump distribution ν. For notational simplicity we will first change variables and transform the value function V into u and its free boundary s into b (see (3.2.4)) and state our results in terms of u and b.

While the continuity of the free boundary for the American option in jump models has been studied intensively in, for example, Pham (1997), Yang et al. (2006) and Lamberton and Mikou (2008), the differentiability of the free boundary, even when the geometric Brownian motion is the underlying process, is difficult to establish (see the discussion on page 172 of Peskir (2005)) and has only recently been fully analyzed by Chen and Chadam (2007). In the jump diffusion case, Yang et al. (2006) proved that the free boundary is continuously differentiable before the maturity when the parameters satisfy

$$(3.1.5) \qquad r \geq q + \lambda \int_{\mathbb{R}_+} (e^z - 1)\,\nu(dz).$$

When the condition (3.1.5) is violated, the free boundary of the American option for jump diffusions exhibits a discontinuity at the maturity (see Theorem 5.3 in Yang et al. (2006) and equation (3.3.20) in this chapter). This behavior of the free boundary was also observed by Levendorskiĭ (2004) and Lamberton and Mikou (2008) in the exponential Lévy models. The purpose of this chapter is to extend the regularity results of the free boundary to the case where (3.1.5) is not satisfied. We will see that the same differentiability result of the free boundary still holds without the condition (3.1.5).

There are two critical points in showing the differentiability properties without the condition (3.1.5): 1) to show the Hölder continuity of the free boundary, 2) to show that $\partial_S^2 V(S, t)$ is strictly larger than 0 when the point (S, t) is close to the free boundary in the continuation region. We achieve these two results in Theorem 3.3.7 and Corollary 3.3.5 respectively. Combining these two properties and a generalization of the result in Cannon et al. (1974) (see Lemma 3.4.2), we upgrade the regularity of the free boundary from Hölder continuity to continuous differentiability in Theorem 3.4.3. Then we analyze the higher order regularity of the free boundary making use of a technique Schaeffer (1976) used for the free boundary of a one dimensional Stefan problem on a bounded domain.

In our method, it is essential to have the value function $V(S, t)$ as the classical solution of the free boundary problem. In the jump diffusion models, this has been shown by Pham (1997) under the condition that

$$(3.1.6) \qquad r > q + \lambda \int_{\mathbb{R}_+} (e^z - 1)\, \nu(dz).$$

This condition was removed in Yang et al. (2006) and also in Bayraktar (2009). In the Lévy models with infinite activity jumps, the value function is not expected to be a classical solution in general. Yet in the literature different notions of generalized

solutions were explored. For example, Pham (1998) used the viscosity solution, Achdou (2008) showed that the value function is the solution in the Sobolev sense and Lamberton and Mikou (2008) proved that the value function is the solution in the distribution sense. Moreover, the smooth-fit property (see (3.2.2)) is also necessary in our analysis (see Theorem 3.4.3 and equation (3.5.1)). While this property may not hold for general pay-off functions (see Peskir (2007)), it has been shown to hold for the put option pay-off in Zhang (1994), Pham (1997) and Bayraktar (2009) in the jump diffusion models. The analysis in this chapter also applies to the pay-off functions which are continuously differentiable, bounded, convex on $[0, +\infty)$ and equal to zero in $[K, +\infty)$ for some $K > 0$. However, the singularity at the strike of the put option pay-off introduces technical difficulties. Therefore, we will focus on the put option pay-off in this chapter and leave the investigation of general pay-off functions to the future work.

The rest of the chapter is organized as follows: In Section 3.2, after changing variables we will collect several useful properties of the value function, which will be crucial in establishing our main results in the next three sections about the regularity of its free boundary. In Section 3.3, we will introduce an auxiliary function and use it to show that the the free boundary is Hölder continuous. In Section 3.4, we will prove the continuous differentiability of the free boundary. In Section 3.5, we will upgrade the regularity of the boundary curve and show that it is infinitely differentiable under an appropriate regularity assumption on the jump distribution. Finally, in Section 3.6, we will show that the approximation free boundaries, constructed in Bayraktar (2009), have the similar regular properties with the original free boundary. Proofs of some auxiliary results are presented in the Section 3.7.

Our main results are Theorems 3.3.7, 3.4.3 and 3.5.6. In Figure 3.1 we show the

Figure 3.1: Our results and the relationships among them.
$A \to B$ means that statement A is used in the proof of statement B.
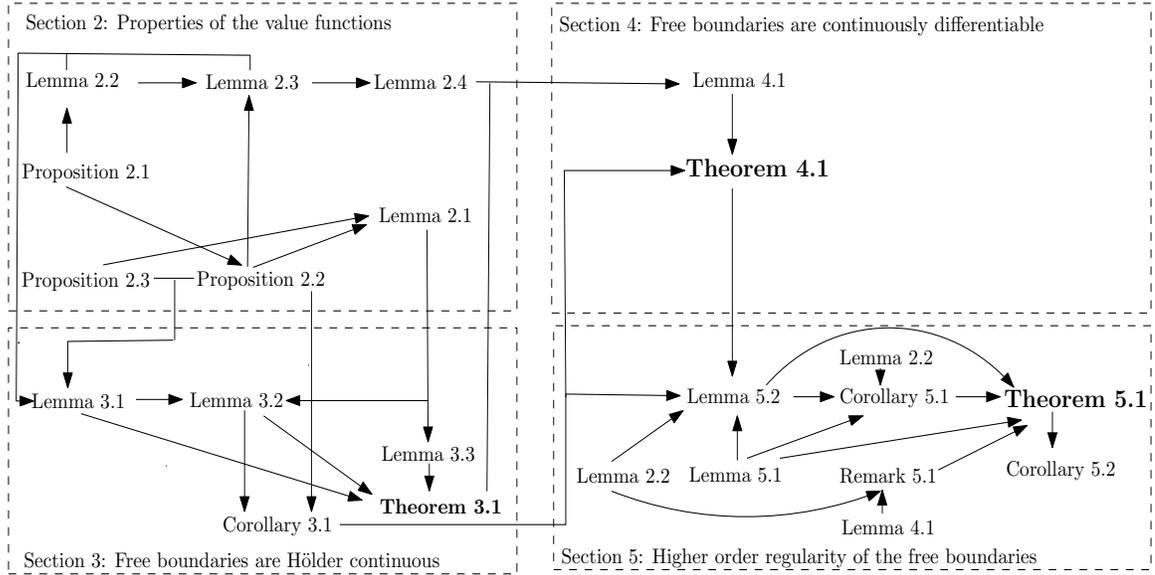
logical flow of the chapter, i.e. we show how several results proved in the chapter are related to each other.

3.2 Properties of the value function

The value function $V(S,t)$ of the American put option for jump diffusions solves a free boundary problem with the free boundary $s(t)$. In particular, Theorem 4.2 of Yang et al. (2006) and Theorem 3.1 of Bayraktar (2009) state the following:

Proposition 3.2.1. $V(S,t)$ *is the unique classical solution of the following boundary value problem:*

(3.2.1)
$$\frac{\partial V}{\partial t} + \frac{1}{2}\widetilde{\sigma}(S,t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - (r+\lambda)V + \lambda \int_{\mathbb{R}} V(Se^z,t)\nu(dz) = 0, \ S > s(t),$$

$$V(s(t),t) = K - s(t), \quad t \in [0,T),$$

$$V(S,T) = (K-S)^+, \quad S \geq s(T).$$

Moreover, the smooth fit property is satisfied, i.e.

$$(3.2.2) \qquad \frac{\partial}{\partial S} V(s(t), t) = -1, \quad t \in [0, T).$$

In the region $\{(S, t) : S < s(t), t \in [0, T)\}$, $V(S, t)$ also satisfies the following inequality:

$$(3.2.3) \qquad \frac{\partial V}{\partial t} + \frac{1}{2} \widetilde{\sigma}(S, t)^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - (r + \lambda)V + \lambda \int_{\mathbb{R}} V(Se^z, t)\nu(dz) \leq 0.$$

In the following, let us first change the variables to state (3.2.1)-(3.2.3) in a more convenient form:

(3.2.4)

$$x = \log(S), \quad u(x, t) = V(S, T - t), \quad b(t) = \log(s(T - t)) \quad \text{and} \quad \sigma(x, t) = \widetilde{\sigma}(S, t).$$

It is clear from the assumptions of $\widetilde{\sigma}(S, t)$ that

(3.2.5)

σ is continuously differentiable in both variables and

there are positive constants δ and Δ such that $0 < \delta < \sigma(x, t) < \Delta$ for all $(x, t) \in \mathbb{R} \times [0, T]$.

While the first part of the assumption will be used in equation (5.4.6) and Lemma 3.4.2,

the second part will be necessary for Lemma 3.2.8, Corollary 3.5.5 and Theorem 3.5.6.

For the simplicity of the notation, we will omit the variables for σ in the sequel. In

terms of the new variables, (3.2.1) - (3.2.3) reduce to

(3.2.6)

$$\mathcal{L}u \triangleq \frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - \left(\mu - \frac{1}{2}\sigma^2\right) \frac{\partial u}{\partial x} + (r + \lambda)u - \lambda \int_{\mathbb{R}} u(x + z, t)\nu(dz) = 0, \quad x > b(t),$$

$$u(b(t), t) = K - e^{b(t)}, \quad t \in (0, T],$$

$$u(x, 0) = (K - e^x)^+, \quad x \geq b(0),$$

$$(3.2.7) \qquad \frac{\partial}{\partial x} u(b(t), t) = -e^{b(t)},$$

(3.2.8) $$\mathcal{L}u(x,t) \geq 0, \quad x < b(t),\ t \in (0,T].$$

Let us define the continuation region $\mathcal{C}$ and the stopping region $\mathcal{D}$ as follows

$$\mathcal{C} \triangleq \{(x,t) \,|\, b(t) < x < +\infty, 0 < t \leq T\}, \quad \mathcal{D} \triangleq \{(x,t) \,|\, -\infty < x \leq b(t), 0 < t \leq T\}.$$

From Proposition 3.2.1, it is clear that the boundary value problem (3.2.6) has a unique classical solution $u(x,t)$ in $\mathcal{C}$.

Remark 3.2.2. The integral term in (3.2.6) can also be considered as a driving term, then the integro-differential equation (3.2.6) can be viewed as the following parabolic differential equation with a driving term $f(x,t) = \lambda \int_{\mathbb{R}} u(x+z,t)\nu(dz)$:

(3.2.9) $$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} - \left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial x} + (r+\lambda)u = f(x,t).$$

This point of view will be useful in the proof of some results in later sections.

In this section, we will study the properties of u in both the continuation and the stopping regions. Let us start from the following proposition from Yang et al. (2006). It shows that the time derivative of u is continuously differentiable across the free boundary.

Proposition 3.2.3. *$\partial_t u(x,t)$ is a continuous function in $\mathbb{R} \times (0,T]$. In particular, for any $t \in (0,T]$,*

(3.2.10) $$\lim_{x \downarrow b(t)} \frac{\partial}{\partial t} u(x,t) = 0.$$

Proof. The proof is given in Theorem 5.1 in Yang et al. (2006), which summarized Lemmas 2.8 and 2.11 in the same chapter and used a special case of Lemma 4.1 in page 239 of Friedman (1976). $\square$

Moreover, we will show in the following that $t \to u(x,t)$ is strictly increasing function in the continuation region.

Proposition 3.2.4.

$$(3.2.11) \qquad \frac{\partial u}{\partial t}(x,t) > 0, \quad (x,t) \in \mathcal{C}.$$

Proof. The inequality (3.2.11) is proved in Proposition 4.1 in Yang et al. (2006) using the Maximum Principle for the integro-differential equations, which can be found in Theorem 2.7 in Chapter 2 of Garroni and Menaldi (1992). However, it can be proved using the ordinary Maximum Principle for parabolic differential equations (see Corollary 7.4 in Lieberman (1996)). We know that $w = \partial_t u$ satisfies the following equation in $\mathcal{C}$,

$$(3.2.12) \qquad \mathcal{L}_{\mathcal{D}} w = \lambda \int_{\mathbb{R}} w(x+z,t)\nu(dz),$$

Since $w = \partial_t u \geq 0$ in $\mathbb{R} \times (0,T)$, (3.2.12) implies that $\mathcal{L}_{\mathcal{D}} w \geq 0$. If there is a point $(x_0, t_0) \in \mathcal{C}$ such that $w(x_0, t_0) = 0$ (i.e. w achieves its non-positive minimum at (x_0, t_0)), it follows from the strong Maximum Principle that $w(x,t) = 0$ in $\mathcal{C} \cap \mathbb{R} \times \{0 < t \leq t_0\}$. Together with the fact that $w(x,t) = 0$ in $\mathcal{D}$, we have that $w(x,t) = 0$ in $\mathbb{R} \times \{0 < t \leq t_0\}$. As a result, from

$$u(x_0, t_0) - u(x_0, 0) = \int_0^{t_0} w(x_0, s)ds = 0,$$

we obtain $u(x_0, t_0) = (K - e^{x_0})^+$. This contradicts with the definition of the free boundary $b(t)$, because $b(t_0) = \max\{x \in \mathbb{R} : u(x, t_0) = (K - e^x)^+\}$ and $x_0 > b(t_0)$. $\square$

Combining Propositions 3.2.3 and 3.2.4 with the Hopf's Lemma for parabolic integro-differential equations (see Theorem 2.8 in page 78 of Garroni and Menaldi (1992)), we obtain that the free boundary is strictly decreasing.

Lemma 3.2.5. *The function $t \to b(t)$ is strictly decreasing for $t \in (0,T]$.*

Proof. The proof is given in Theorem 5.4 in Yang et al. (2006). $\square$

In order to investigate the regularity of the free boundary in the later sections, we need more properties of u, which we will develop in the following three lemmas. Since the results of these lemmas are intuitive but proofs are technical, we will list the proofs of these lemmas in the Section 3.7.1.

It has been proven in Bayraktar (2009) that $V(S, \cdot)$ is uniformly Lipschitz in $\mathbb{R}_+$ and $V(\cdot, t)$ is uniformly semi-Hölder continuous in $[0, T]$. The following lemma shows the same properties also holds for $u(x, t)$, the function that we obtained after the change of variables in (3.2.4). (The globally Lipschitz continuity with respect to x is not a priori clear and one needs to check whether $\partial_x u(x, t)$ is bounded.)

Lemma 3.2.6. *Let $u(x, t)$ be the solution of equation (3.2.6), then we have*

$$(3.2.13) \qquad |u(x, t) - u(y, t)| \leq C|x - y|, \quad x, y \in \mathbb{R}, t \in [0, T],$$

$$(3.2.14) \qquad |u(x, t) - u(x, s)| \leq D|t - s|^{\frac{1}{2}}, \quad x \in \mathbb{R}, 0 \leq t, s \leq T,$$

where C and D are positive constants independent of x and t.

Proof. See Section 3.7.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In the rest of this section, we will investigate the boundness of $\partial_t u(x, t)$ and its behavior when $x \to +\infty$. These two properties will be useful to show several results in Sections 3.4 and 3.5 (see e.g. (3.4.2), proof of Lemma 3.4.2 and Remark 3.5.1). Let us first recall the definition of the Hölder spaces on page 7 of Ladyženskaja et al. (1968).

Definition 3.2.7. Let Ω be a domain in $\mathbb{R}$, $Q_T = \Omega \times (0, T)$. We denote $\overline{Q_T}$ the closure of Q_T. For any positive nonintegral real number α, $H^{\alpha, \alpha/2}\left(\overline{Q_T}\right)$ is the Banach space of functions $v(x, t)$ that are continuous in $\overline{Q_T}$, together with continuous

derivatives of the form $\partial_t^r \partial_x^s v$ for $2r + s < \alpha$, and have a finite norm

$$||v||^{(\alpha)} = |v|_x^{(\alpha)} + |v|_t^{(\alpha/2)} + \sum_{2r+s\leq[\alpha]} ||\partial_t^r \partial_x^s v||^{(0)},$$

in which

$$||v||^{(0)} = max_{Q_T}|v|,$$

$$|v|_x^{(\alpha)} = \sum_{2r+s=[\alpha]} < \partial_t^r \partial_x^s v >_x^{(\alpha-[\alpha])}, \quad |v|_t^{(\alpha/2)} = \sum_{\alpha-2<2r+s<\alpha} < \partial_t^r \partial_x^s v >_t^{(\frac{\alpha-2r-s}{2})};$$

$$< v >_x^{(\beta)} = \sup_{\substack{(x,t),(x',t) \in \overline{Q_T} \\ |x-x'| \leq \rho_0}} \frac{|v(x,t) - v(x',t)|}{|x-x'|^\beta}, \quad 0 < \beta < 1,$$

$$< v >_t^{(\beta)} = \sup_{\substack{(x,t),(x,t') \in \overline{Q_T} \\ |t-t'| \leq \rho_0}} \frac{|v(x,t) - v(x,t')|}{|t-t'|^\beta}, \quad 0 < \beta < 1,$$

where ρ_0 is a positive constant.

On the other hand, $H^\alpha\left(\overline{\Omega}\right)$ is the Banach space whose elements are continuous functions $f(y)$ on $\overline{\Omega}$ that have continuous derivatives up to order $[\alpha]$ and the following norm finite

$$||f||^{(\alpha)} = \sum_{j\leq[\alpha]} ||d_y^j f||^{(0)} + |d_y^{[\alpha]} f|^{(\alpha-[\alpha])},$$

in which

$$|f|^{(\beta)} = \sup_{y,y'\in\overline{\Omega},|y-y'|\leq\rho_0} \frac{|f(y) - f(y')|}{|y-y'|^\beta}.$$

Here $d_y^j f$ is the jth derivative of f. These Hölder norms depend on ρ_0, but for different $\rho_0 > 0$, the corresponding Hölder norms are equivalent hence their dependence on ρ_0 will not be noted in the sequel.

Using the Hölder spaces and regularity results for parabolic equations, we have the following result.

Lemma 3.2.8. *For any $\epsilon > 0$, $\partial_t u(x,t)$ is uniformly bounded in $\mathbb{R} \times [\epsilon, T]$.*

Proof. See Section 3.7.1. □

Remark 3.2.9. **(i)** In the statement of Lemma 3.2.8, $t = 0$ cannot be included, i.e., $\lim_{t \to 0} \partial_t u(x,t)$ is not uniformly bounded in $x \in \mathbb{R}$, because $\partial_t u = \frac{1}{2}\sigma^2 \partial_x^2 u + \left(\mu - \frac{1}{2}\sigma^2\right)\partial_x u - (r + \lambda)u + \lambda \int_{\mathbb{R}} u(x + z, t)\nu(dz)$ and $\lim_{t \to 0} \partial_x^2 u(x,t)$ is not bounded as a result of non-smoothness of the initial value at $x = \log K$.

In the following, we will use the previous lemma to analyze the behavior of $\partial_t u(x,t)$ as $x \to +\infty$.

Lemma 3.2.10.

$$\lim_{x \to +\infty} \partial_t u(x,t) = 0, \quad t \in (0,T].$$

Proof. See Section 3.7.1. □

Remark 3.2.11. Given the result in Lemma 3.2.10, it is clear from the differential equation (3.2.9) that $\partial_x^2 u$ is uniformly bounded in $\mathbb{R} \times [\epsilon, T]$, since $\partial_x u$ is uniformly bounded (see Lemma 3.2.6). Combining with semi-Hölder continuity of $u(x, \cdot)$ in Lemma 3.2.6, Lemma 3.1 in page 78 of Ladyženskaja et al. (1968) now tells us that $\partial_x u(x, \cdot) \in H^{1/2}([\epsilon, T])$. Therefore, combining with the smooth fit property and Proposition 3.2.3, we have

$$u \in C^1\left(\mathbb{R} \times (0,T]\right).$$

In the following three sections we will use the properties of the value function we have shown in this section to investigate the regularity of the free boundary $b(t)$.

3.3 The free boundary is Hölder continuous

3.3.1 An auxiliary function

Before we begin to analyze the regularity of the free boundary, let us introduce the following important auxiliary function, which was also used in Lamberton and Mikou (2008) to prove the continuity of the free boundary in an exponential Lévy model:

$$(3.3.1) \quad J(x,t) \triangleq qe^x - rK + \lambda \int_{\mathbb{R}} \left[u(x+z,t) + e^{x+z} - K \right] \nu(dz), \quad x \in \mathbb{R},\ t \in [0,T].$$

As a result of the assumption (3.1.2), $J < \infty$. Moreover, J is closely related to the behavior of the value function u in the stopping region, since one can check that

$$(3.3.2) \qquad\qquad \mathcal{L}u(x,t) = -J(x,t), \quad \text{for } x < b(t),\ t \in (0,T],$$

(3.3.3)

$$\mathcal{L}g(x) = \mathcal{L}u(x,0) = -\left[qe^x - rK + \lambda \int_{\mathbb{R}} \left(e^{x+z} - K \right)^+ \nu(dz) \right] = -J(x,0), \quad \text{for } x < \log K,$$

in which $g(x) \triangleq (K - e^x)^+$. As we shall see in the rest of this section, the function $J(x,0)$ is of special importance. We rename it as $J_0(x)$, i.e.,

$$(3.3.4) \qquad\qquad J_0(x) \triangleq qe^x - rK + \lambda \int_{\mathbb{R}} \left(e^{x+z} - K \right)^+ \nu(dz).$$

Let us analyze the properties of J.

Lemma 3.3.1. *(i)* $J(x,t) \geq -rK$, $\lim_{x\downarrow-\infty} J(x,t) = -rK$ *and* $\lim_{x\uparrow+\infty} J(x,t) = +\infty$,

(ii) $J(x,t) \in C^1 \left(\mathbb{R} \times (0,T] \right) \cap C \left(\mathbb{R} \times [0,T] \right)$,

(iii) The functions $x \to J(x,t)$ *and* $t \to J(x,t)$ *are non-decreasing. If either either* $q > 0$ *or*

$$(3.3.5) \qquad\qquad \nu \left((M, +\infty) \right) > 0, \quad \text{for any } M > 0;$$

then $x \to J(x,t)$ is a strictly increasing function. On the other hand, if

(3.3.6)
$$v((0, \infty)) > 0$$

(3.3.7)
$$\partial_t J(x,t) > 0, \quad x \geq b(t), t \in (0, T].$$

Proof. **(i)** The first statement follows from $u(x + z, t) \geq (K - e^{x+z})^+ \geq K - e^{x+z}$. The two limit statements follow from the Bounded Convergence Theorem.

(ii) The continuity of $u(x, t)$ on $\mathbb{R} \times [0, T]$ implies that J is continuous on the same region. For the differentiability, since $\partial_x u$ and $\partial_t u$ are uniformly bounded in $\mathbb{R} \times [\epsilon, T]$ for any $\epsilon > 0$ (see Lemmas 3.2.6 and 3.2.8), the Bounded Convergence Theorem gives us

(3.3.8)
$$\frac{\partial}{\partial x} J(x,t) = qe^x + \lambda \int_{\mathbb{R}} \left[\frac{\partial}{\partial x} u(x + z, t) + e^{x+z} \right] \nu(dz) < +\infty,$$
$$\frac{\partial}{\partial t} J(x,t) = \lambda \int_{\mathbb{R}} \frac{\partial}{\partial t} u(x + z, t) \nu(dz) < +\infty.$$

These partial derivatives are also continuous in $\mathbb{R} \times [\epsilon, T]$ as a result of Remark 3.2.11. Then the statement in (ii) follows since the choice of ϵ is arbitrary.

(iii) It is clear that the functions $x \to J(x,t)$ and $t \to J(x,t)$ are nondecreasing functions since $x \to u(x,t) + e^x$ and $t \to u(x,t)$ are nondecreasing.

The condition (3.3.5) means that the support of the measure ν is not bounded from above. As a result we have that the set $A = \{z : x+z \in \mathcal{C}\}$ has positive measure, i.e., $\nu(A) > 0$ for any $x \in \mathbb{R}$. For any $z \in A$ we have that $\partial_x u(x + z, t) + e^{x+z} > 0$, which is equivalent to $\partial_S V(Se^z, t) + 1 > 0$. The latter follows from the convexity of the function V and (3.2.2). If $z \notin A$, then clearly $\partial_x u(x + z, t) + e^{x+z} = 0$. Using these facts in the first equation in (3.3.8), we see that 3.3.5 yields $\partial_x J(x,t) > 0$ in $\mathbb{R} \times [0, T]$. On the other hand, when $q > 0$ the condition assumed on ν can be dropped.

Moreover, when $x \geq b(t)$ (3.3.6) ensures that $\nu(A) > 0$. Then (3.3.7) follows from Proposition 3.2.4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the rest of the chapter, we will assume either (3.3.5) or $q > 0$ and (3.3.6) are satisfied. Indeed, in the two well-known examples of jump diffusions, Kou's model and Merton's model (see Cont and Tankov (2004) p.111), in which ν is the double exponential and normal distribution respectively, condition (3.3.5) is fulfilled.

As the consequence of Lemma 3.3.1, the level curve

$$(3.3.9) \qquad\qquad B(t) \triangleq \{x : J(x,t) = 0, t \in [0,T]\}.$$

is well defined. $B(0)$, which is the unique solution of the integral equation,

$$(3.3.10) \qquad J_0(x) = qe^x - rK + \lambda \int_{\mathbb{R}} \left(e^{x+z} - K\right)^+ \nu(dz) = 0.$$

will be crucial in describing the behavior of $b(t)$ close to 0 (see Section 3.3.2).

Remark 3.3.2. When $r = 0$, Lemma 3.3.1 (i) implies that $B(t) = -\infty$. On the other hand, the proof in the following lemma tell us that $B(t) \geq b(t)$. Therefore $b(t) = -\infty$ in this case. We will assume $r > 0$ in the rest of the chapter to exclude this trivial case.

This level curve $B(t)$ will be crucial in analyzing the regularity properties of the free boundaries in the rest of this section. Let us analyze its properties first.

Lemma 3.3.3. *(i) $B(t)$ is non-increasing,*

(ii) $B(t) \in C^1(0,T] \cap C[0,T]$,

(iii) $B(t) > b(t)$ for $t \in (0,T]$. Here $b(t)$ is the free boundary in (3.2.6).

Proof. **(i)** The proof follows from Lemma 3.3.1 (iii).

(ii) We have the continuity of B because $J(x,t)$ is continuous and strictly increasing

in x (see Lemma 3.3.1 (ii) and (iii)). Let us focus on the differentiability in the following. It follows from Lemma 3.3.1 (ii) that $J(x,t)$ is a C^1 function in $\mathbb{R} \times (0,T]$. Moreover, it follows from (3.3.7) and $B(t) \geq b(t)$ (which we will prove in the Step 1 in (iii)) that

$$\partial_t J(x,t_0)|_{x=B(t_0)} > 0, \quad t_0 \in (0,T_0].$$

Therefore, the Implicit Function Theorem implies that there exists an open set U containing t_0 such that

$$B(t) \in C^1(U).$$

Then the statement in (ii) follows after pasting different neighborhoods for all points $t \in (0,T]$ together.

(iii) The proof consists of two steps:

Step 1: First we show that $B(t) \geq b(t)$. If these is a $t_0 \in (0,T]$ such that $B(t_0) < b(t_0)$, from the definition of $B(t)$ and the fact that $x \to J(x,t)$ is strictly increasing, we obtain $J(x,t_0) > 0$ for all $x \in (B(t_0), b(t_0))$. Combining with (3.3.2), we have

$$\mathcal{L}u(x,t_0) < 0, \quad \text{for any } x \in (B(t_0), b(t_0)),$$

which contradicts with (3.2.8).

Step 2: Second, we show that $B(t) \neq b(t), t \in (0,T]$. Since $b(t) < \log K$ (thanks to Lemma 3.2.5) and $t \to B(t)$ is non-increasing, it is clear that $B(t) > b(t)$ for any $t \in (0,t^*)$ where $t^* = T \wedge \sup\{t \in \mathbb{R}_+ : B(t) = \log K\}$. Hence we only need to focus on the region where $B(t) < \log K$. If there is a $t_0 \in (0,T]$ such that $B(t_0) = b(t_0)$, we will derive a contradiction in the following.

First, let us define the region $\Omega \triangleq \{(x,t) \mid B(t) < x < \log K, t \in (0,T]\}$. Because

of the result in Step 1, $\Omega \subset \mathcal{C}$. Hence $u(x,t)$ satisfies

$$\mathcal{L}_{\mathcal{D}} u(x,t) = \lambda \int_{\mathbb{R}} u(x+z,t)\nu(dz), \quad (x,t) \in \Omega.$$

Let us define $\xi \triangleq x - B(t)$, $\tilde{u}(\xi,t) \triangleq u(x,t)$ and $\tilde{g}(\xi,t) \triangleq (K - e^{\xi+B(t)})^+ = g(x)$. In the region $\widetilde{\Omega} \triangleq \{(\xi,t)| 0 < \xi < \log K - B(t), t \in (0,T]\}$ we have

(3.3.11)
$$\tilde{\mathcal{L}}_{\mathcal{D}}\tilde{u} \triangleq \frac{\partial \tilde{u}}{\partial t} - \frac{1}{2}\sigma^2\frac{\partial^2 \tilde{u}}{\partial \xi^2} - \left(\mu + B'(t) - \frac{1}{2}\sigma^2\right)\frac{\partial \tilde{u}}{\partial \xi} + (r+\lambda)\tilde{u} = \lambda \int_{\mathbb{R}} \tilde{u}(\xi+z,t)\nu(dz).$$

since $B(t) \in C^1(0,T]$. On the other hand,

(3.3.12)
$$\tilde{\mathcal{L}}_{\mathcal{D}}\tilde{g} = -e^{\xi+B(t)}B'(t) + \frac{1}{2}\sigma^2 e^{\xi+B(t)} + \left(\mu + B'(t) - \frac{1}{2}\sigma^2\right)e^{\xi+B(t)} + (r+\lambda)\left(K - e^{\xi+B(t)}\right)$$
$$= -\left[q e^{\xi+B(t)} - rK + \lambda \int_{\mathbb{R}} \left(e^{\xi+B(t)+z} - K\right)\nu(dz)\right].$$

Therefore, we obtain from (3.3.11) and (3.3.12) that

(3.3.13)
$$\tilde{\mathcal{L}}_{\mathcal{D}}\left(\tilde{u} - \tilde{g}\right)(\xi,t) = q e^{\xi+B(t)} - rK + \lambda \int_{\mathbb{R}} \left[\tilde{u}(\xi+z,t) + e^{\xi+B(t)+z} - K\right]\nu(dz) = J(\xi+B(t),t),$$

for $(\xi,t) \in \widetilde{\Omega}$. Note that $J(x,t) > 0$ when $x > B(t)$. Therefore (3.3.13) yields

(3.3.14)
$$\tilde{\mathcal{L}}_{\mathcal{D}}\left(\tilde{u} - \tilde{g}\right) > 0, \quad (\xi,t) \in \widetilde{\Omega}.$$

On the other hand, from our assumption $\xi_0 \triangleq b(t_0) - B(t_0) = 0$. Moreover, there clearly exists a ball $\mathcal{B} \subset \widetilde{\Omega}$ such that 1) $\overline{\mathcal{B}} \cap \{\xi = 0\} = (\xi_0, t_0)$; 2) $(\tilde{u} - \tilde{g})(\xi,t) > (\tilde{u} - \tilde{g})(\xi_0, t_0) = 0$ for all $(\xi,t) \in \mathcal{B}$, since $(\tilde{u} - \tilde{g})(\xi,t) = (u-g)(x,t) > 0$ when $x > \mathcal{B}(t) \geq b(t)$. Now apply Hopf Lemma (see Theorem 17 in page 49 of Friedman (1964)) to $\tilde{u} - \tilde{g}$ in $\mathcal{B}$, we obtain

(3.3.15)
$$\frac{\partial}{\partial \xi}\left(\tilde{u} - \tilde{g}\right)(\xi_0, t_0) > 0,$$

which contradicts with the smooth fit property at (ξ_0, t_0), i.e., $\partial_\xi(\tilde{u} - \tilde{g})(\xi_0, t_0) = \partial_x(u-g)(b(t_0), t_0) = 0$. $\qquad \square$

Remark 3.3.4. In the proof of Lemma 3.3.3 (iii), the reason we work with the domain $\widetilde{\Omega}$ instead of the domain Ω is that Ω may not satisfy the interior ball condition (see Theorem 17 in page 49 of Friedman (1964)), which is a crucial assumption of the Hopf Lemma. If one can show $B(t) \in C^2$, the interior ball condition automatically holds for Ω (see Remark in page 330 of Evans (1998)). However, $B(t) \in C^2$ does not follow directly from the Implicit Function Theorem, because $J(x,t)$ is not expected to be a C^2 function in a neighborhood of the point $(b(t_0), t_0)$, for any t_0, as a result of the discontinuity of $\partial_x^2 u(x,t)$ across the free boundary $b(t)$ (see the following corollary).

As a corollary of Lemma 3.3.3 (iii), $\partial_x^2 u(x,t)$ does not cross the free boundary continuously.

Corollary 3.3.5.

$$(3.3.16) \qquad \frac{\partial^2}{\partial x^2} u\left(b(t)+, t\right) \triangleq \lim_{x \downarrow b(t)} \frac{\partial^2}{\partial x^2} u(x,t) > -e^{b(t)}, \quad t \in (0, T].$$

(This is equivalent to $\lim_{S \downarrow s(t)} \partial_S^2 V(S,t) > 0, t \in [0, T).$)

Proof. On the one hand, since $B(t) > b(t)$ and $x \to J(x,t)$ is strictly increasing, we have

$$(3.3.17) \qquad J(b(t), t) < 0, \quad t \in (0, T],$$

On the other hand, from the continuity of u, (3.2.7), (3.2.6) and Proposition 3.2.3, it follows that

$$
\begin{aligned}
(3.3.18) \quad 0 &= \lim_{x \downarrow b(t)} \mathcal{L}u(x,t) \\
&= -\frac{1}{2}\sigma^2 \lim_{x \downarrow b(t)} \frac{\partial^2}{\partial x^2} u(x,t) - \frac{1}{2}\sigma^2 e^{b(t)} \\
&\quad - \left\{ qe^{b(t)} - rK + \lambda \int_{\mathbb{R}} \left[u(b(t) + z, t) + e^{b(t)+z} - K \right] \nu(dz) \right\} \\
&= -\frac{1}{2}\sigma^2 \lim_{x \downarrow b(t)} \frac{\partial^2}{\partial x^2} u(x,t) - \frac{1}{2}\sigma^2 e^{b(t)} - J(b(t), t).
\end{aligned}
$$

The inequality (3.3.16) now follows from combining (3.3.17) and (3.3.18). $\qquad\square$

3.3.2 The behavior of the free boundary close to maturity

We are ready to analyze the regularity of the free boundaries. The continuity of the free boundaries for differential equations with or without integral terms have been studied intensively, see e.g. Friedman (1975), Pham (1997), Yang et al. (2006) and Lamberton and Mikou (2008). For the American option in jump diffusions, Pham (1997) showed the continuity of the free boundary under the technical condition

$$(3.3.19) \qquad r > q + \lambda \int_{\mathbb{R}_+} (e^z - 1)\,\nu(dz).$$

In Yang et al. (2006), this condition was removed in the proof of the continuity. Moreover, in their Theorem 5.3, they showed that

(3.3.20)

$$b(0+) \triangleq \lim_{t \to 0^+} b(t) = \min\{\log K, B(0)\} = \begin{cases} \log K, & r \geq q + \lambda \int_{\mathbb{R}_+} (e^z - 1)\nu(dz) \\ B(0), & r < q + \lambda \int_{\mathbb{R}_+} (e^z - 1)\nu(dz) \end{cases},$$

in which $B(0)$ is the unique solution of (3.3.10). The same result has been shown for the exponential Lévy models in Lamberton and Mikou (2008).

3.3.3 Hölder continuity of the free boundary

In the following, the function $J_0(x)$ in (3.3.4) and the Maximum Principle will play a crucial role in showing that $t \to b(t)$ is Hölder continuous.

Lemma 3.3.6. *Let $b(t)$ be the free boundary in Lemma 3.2.5. For any $\epsilon > 0$, if there exists $\delta > 0$ such that for any t_1 and t_2 satisfying $\epsilon \leq t_1 < t_2 \leq T$ and $t_2 - t_1 \leq \delta$ one has*

$$(3.3.21) \qquad u(b(t_1), t) - u(b(t_1), t_1) \leq C_\epsilon (t_2 - t_1)^\alpha, \quad t_1 \leq t \leq t_2,$$

in which $0 < \alpha \leq 1$ and C_ϵ is a constant that does not depend on t_1 and t_2, then

there exists $\delta' \in (0, \delta]$ such that

(3.3.22)
$$b(t_1) - b(t_2) \leq C'_\epsilon (t_2 - t_1)^{\frac{\alpha}{2}}, \quad 0 \leq t_2 - t_1 \leq \delta',$$

in which C'_ϵ is another positive constant that is independent of t_1 and t_2.

Proof. This proof is motivated by Lemma 5.1 in Friedman and Shen (2002). For any t_1 and t_2 such that $\epsilon \leq t_1 < t_2 \leq T$ and $t_2 - t_1 \leq \delta$, let us consider the domain $D \triangleq \{(x,t) : b(t) < x < b(t_1), t_1 < t < t_2\}$. (In what follows, we will choose t_1 and t_2 close to each other, i.e. we will find an appropriate δ' such that $t_2 - t_1 \leq \delta'$.) Let $\overline{D}$ be the closure of the domain D.

In the following, we will show that the function

(3.3.23)
$$\chi(x) = \left\{ \left[\sqrt{C_\epsilon}(t_2 - t_1)^{\frac{\alpha}{2}} + \beta(x - b(t_1)) \right]^+ \right\}^2, \quad b(t_2) \leq x \leq b(t_1)$$

satisfies $\chi(x) \geq (u - g)(x, t)$ on the domain D for suitably chosen positive constant β.

It is clear that $\chi(x) = 0$, when $x \leq b(t_1) - \frac{\sqrt{C_\epsilon}}{\beta}(t_2 - t_1)^{\frac{\alpha}{2}} \triangleq \xi$. We also have $\chi(b(t_1)) = C_\epsilon(t_2 - t_1)^\alpha \geq u(b(t_1), t) - g(b(t_1))$ for $t_1 \leq t \leq t_2$ because of the assumption (3.3.21). On the other hand, $\chi(b(t)) \geq 0 = u(b(t), t) - g(b(t))$. Therefore on the parabolic boundary of the domain D, we have that $\chi \geq u - g$. We will show that this holds for all $(x, t) \in D$. To this end, we will compare $\mathcal{L}\chi$ with $\mathcal{L}(u - g)$ using the Maximum Principle. Note that χ is carefully chosen so that it has a continuous first derivative and a bounded second derivative. These properties of χ makes the application of the Maximum Principle for weak solutions (see e.g. Corollary 7.4 in Lieberman (1996)) possible.

First, for $(x, t) \in D$ let us estimate the integral term:

$$
\begin{aligned}
\lambda \int_{\mathbb{R}} \chi(x+z) \nu(dz) = \lambda \int_{z \geq \xi - x} & \left\{ \sqrt{C_\epsilon}(t_2 - t_1)^{\frac{\alpha}{2}} + \beta(x + z - b(t_1)) \right\}^2 \nu(dz) \\
& \leq \lambda \int_{z \geq \xi - x} \left\{ \sqrt{C_\epsilon}(t_2 - t_1)^{\frac{\alpha}{2}} + \beta z \right\}^2 \nu(dz) \\
& \leq 2\lambda \int_{z \geq \xi - x} \left[C_\epsilon(t_2 - t_1)^\alpha + \beta^2 z^2 \right] \nu(dz) \\
& \leq 2\lambda \left[C_\epsilon(t_2 - t_1)^\alpha + \beta^2 M \right].
\end{aligned}
$$

(3.3.24)

for a sufficiently large positive constant M independent of t_1 and t_2. To obtain the first inequality, we used $x < b(t_1)$ for $(x, t) \in D$. The third inequality follows, because $\int_{\mathbb{R}} e^z \nu(dz) < +\infty$ in (3.1.2) and z is bounded from below.

With the estimate (3.3.24), we can calculate $\mathcal{L}\chi$ inside the domain D.

(3.3.25)

$$
\begin{aligned}
\mathcal{L}\chi(x) = & \left[-\sigma^2 \beta^2 - \left(\mu - \frac{1}{2}\sigma^2 \right) 2\beta \chi^{\frac{1}{2}} + (r+\lambda)\chi \right] 1_{\{x > \xi\}} - \lambda \int_{\mathbb{R}} \chi(x+z, t) \nu(dz) \\
\geq & - \left[\frac{(\mu - \sigma^2/2)^2}{r+\lambda} + \sigma^2 \right] \beta^2 1_{\{x > \xi\}} - 2\lambda \left[C_\epsilon(t_2 - t_1)^\alpha + \beta^2 M \right] \\
\geq & -E\beta^2 - F(t_2 - t_1)^\alpha,
\end{aligned}
$$

in which $E \triangleq \frac{(\mu - \sigma^2/2)^2}{r+\lambda} + \sigma^2 + 2\lambda M$ and $F \triangleq 2\lambda C_\epsilon$ are positive constants.

Recall that for any $\varepsilon > 0$, $b(\epsilon) < \min\{\log K, B(0)\}$ and that the strictly increasing function J_0 defined in (3.3.4) satisfies $J_0(x) < 0$ for $x < B(0)$. Using these observations and (3.3.3) it can be seen that for any $x \leq b(\epsilon)$ we have

(3.3.26)
$$
\mathcal{L}g(x) = -J_0(x) \geq -J_0(b(\epsilon)) > 0.
$$

Now choosing

(3.3.27)
$$
c = -J_0(b(\epsilon)) > 0
$$

and $\delta' = \min\{ \left(\frac{c}{2F} \right)^{1/\alpha}, \delta \}$ and $\beta \leq \sqrt{\frac{c}{2E}}$, we have that

$$
\mathcal{L}\chi(x)(x) \geq -c \geq \mathcal{L}(u - g)(x, t), \quad (x, t) \in D.
$$

Considering $\Psi = \chi - u + g$, we have $\mathcal{L}\Psi \geq 0$ in D and $\Psi \geq 0$ on the parabolic boundary of D. It follows from the Maximum Principle for weak solutions that $\Psi \geq 0$ in D, i.e.,

$$(3.3.28) \qquad \chi(x) \geq (u - g)(x, t), \quad (x, t) \in D.$$

Observe that $(u - g)(x, t) = 0$ if $x \leq \xi$. For any $(x, t) \in D$, since $(u - g)(x, t) > 0$, we can see that $x > \xi$. This gives us

$$(3.3.29) \qquad \inf_{t_1 \leq t \leq t_2} b(t) \geq b(t_1) - \frac{\sqrt{C_\epsilon}}{\beta}(t_2 - t_1)^{\frac{\alpha}{2}}, \quad 0 < t_2 - t_1 \leq \delta'.$$

We have shown the free boundary $b(t)$ is continuous and strictly decreasing in Lemma 3.2.5. Along with this fact, the inequality (3.3.29) gives us (3.3.22) with $C'_\epsilon = \sqrt{C_\epsilon}/\beta$. $\qquad \square$

Now we are ready to state the main result of this section.

Theorem 3.3.7. *Let $b(t)$ be the free boundary in problem (3.2.6), then for any $\epsilon > 0$ if $\epsilon \leq t_1 < t_2 \leq T$, and $t_2 - t_1$ is sufficiently small, then*

$$(3.3.30) \qquad b(t_1) - b(t_2) \leq C_\epsilon(t_2 - t_1)^{\frac{5}{8}},$$

in which C_ϵ is a positive constant independent of t_1 and t_2.

Proof. The proof will follow by applying Lemma 3.3.6 twice. The first application will show that $b(t)$ is Hölder continuous with exponent $\frac{1}{2}$. Applying Lemma 3.3.6 for the second time we will upgrade the Hölder exponent to $\frac{5}{8}$.

As a result of Propositions 3.2.3 and 3.2.4 for any $\epsilon > 0$, t_1 and t_2 satisfying $\epsilon \leq t_1 < t_2 \leq T$ we have that

$$(3.3.31) \qquad u(b(t_1), t) - u(b(t_1), t_1) \leq \max_{t_1 \leq s \leq t} \frac{\partial u}{\partial t}(b(t_1), s)(t - t_1) \leq C_1(t_2 - t_1),$$

where $C_1 = \max_{\epsilon \le s \le T} \partial_t u(b(t_1), s)$ is a positive constant. Now as a result of Lemma 3.3.6, we know that there exists a sufficiently small constant $\delta_1 \in (0, T - \epsilon]$ such that

$$(3.3.32) \qquad b(t_1) - b(t_2) \le C'_1 (t_2 - t_1)^{\frac{1}{2}}, \quad 0 \le t_2 - t_1 \le \delta_1,$$

in which C'_1 is a positive constant that does not depend on t_1, t_2 and δ_1.

It follows from Lemmas 2.8 and 2.11 in Yang et al. (2006) and the Sobolev Embedding Theorem (see also (3.7.27) in Appendix A.3) that for any $a < b < \log K$ and $t \in [t_1, t_2]$,

$$(3.3.33) \qquad \left| \frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial t}(\overline{x}, t) \right| \le \tilde{C} \, |x - \overline{x}|^{\frac{1}{2}}, \quad x, \overline{x} \in (a, b),$$

in which $\tilde{C}$ is a positive constant that does not depend on t. Taking $x = b(t_1)$ and $\overline{x} = b(t)$ in (3.3.33) and using Proposition 3.2.3, we obtain

$$(3.3.34) \qquad 0 \le \frac{\partial u}{\partial t}(b(t_1), t) \le \tilde{C} \, |b(t_1) - b(t)|^{\frac{1}{2}} \le \tilde{C} |b(t_1) - b(t_2)|^{\frac{1}{2}}, \quad t_1 \le t \le t_2,$$

where the third inequality follows from $b(t)$ being strictly decreasing in Lemma 3.2.5. Combining (3.3.32) and (3.3.34), we get

$$(3.3.35) \qquad 0 \le \frac{\partial u}{\partial t}(b(t_1), t) \le C_2 (t_2 - t_1)^{\frac{1}{4}}, \quad t_1 \le t \le t_2, 0 \le t_2 - t_1 \le \delta_1.$$

As a result

$$(3.3.36) \qquad u(b(t_1), t) - u(b(t_1), t_1) \le \max_{t_1 \le s \le t_2} \frac{\partial u}{\partial t}(b(t_1), s)(t_2 - t_1) \le C_2 (t_2 - t_1)^{\frac{5}{4}}.$$

Applying Lemma 3.3.6 for the second time, we know that there exists $\delta_2 \in (0, \delta_1]$ such that

$$(3.3.37) \qquad b(t_1) - b(t_2) \le C_\epsilon (t_2 - t_1)^{\frac{5}{8}}, \quad 0 \le t_2 - t_1 \le \delta_2,$$

where C_ϵ is a positive constant that does not depend on t_1, t_2 and δ_2. $\qquad \square$

3.4 The free boundary is continuously differentiable

In this section, we will investigate the continuous differentiability of the free boundary. In Theorem 5.6 in Yang et al. (2006), the authors have shown $b(t) \in C^1(0, T]$, with the extra condition

$$(3.4.1) \qquad r \geq q + \lambda \int_{\mathbb{R}_+} (e^z - 1) \nu(dz).$$

Thanks to Corollary 3.3.5 and Theorem 3.3.7, we can show the continuous differentiability of the free boundary without imposing this extra condition.

Remark 3.4.1. If condition (3.4.1) is not satisfied, we can see from (3.3.20) that there is a gap between $\lim_{t \to 0^+} b(t)$ and $b(0) = \log K$. Therefore it is impossible to have $b(t)$ to be even continuous at $t = 0$. But we shall see that it is continuously differentiable for all $t \in (0, T]$.

Let us consider the time derivative $\partial_t u(x, t)$. Recall that $u(x, t)$ is the solution of (3.2.6). Using the assumption (3.2.5), the time derivative $w = \partial_t u(x, t)$ satisfies the following partial differential equation

$$\mathcal{L}_{\mathcal{D}} w = h(x, t), \quad x > b(t),\ t \in (0, T],$$

$$(3.4.2) \qquad w(b(t), t) = 0, \quad \lim_{x \to +\infty} w(x, t) = 0, \quad t \in (0, T],$$

$$w(x, 0) = \lim_{t \to 0} \partial_t u(x, t), \quad x \geq b(0),$$

in which

$$(3.4.3) \qquad h(x, t) \triangleq \lambda \int_{\mathbb{R}} \frac{\partial}{\partial t} u(x + z, t) \nu(dz) + \sigma \partial_t \sigma \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \right).$$

When $x < b(t)$, we also have $w(x, t) = 0$. Given $u(x, t)$ and $b(t)$, (3.4.2) is a parabolic differential equation for $w(x, t)$. In this equation, the boundary conditions for $w(x, t)$ along $b(t)$ and at the infinity follow from Proposition 3.2.3 and Lemma 3.2.10.

In order to show the differentiability of the free boundary, we need to study the behavior of $\frac{\partial^2}{\partial x \partial t} u$ at the free boundary (by first making sure that the cross derivatives exist in the classical sense), which is carried out in the following lemma.

Lemma 3.4.2. *(i) As a function of t, $\frac{\partial^2}{\partial x \partial t} u(b(t)+, t) \triangleq \lim_{x \downarrow b(t)} \frac{\partial^2}{\partial x \partial t} u(x, t)$ is continuous on $(0, T]$.*

(ii) Moreover, the function $\frac{\partial^2}{\partial x \partial t} u(x, t)$ is continuous for $x > b(t)$, $t \in (0, T]$.

This lemma is a slight generalization of the result in Cannon et al. (1974) to the parabolic integro-differential equation (3.4.2). Considering the integral term h in (3.4.2) as the driving term, this lemma follows from using the same technique presented in Section 1 of Chapter 8 in Friedman (1964). We will postpone this proof to the Appendix 3.7.2. We are now ready to state and prove the main theorem of this section.

Theorem 3.4.3. *Let $b(t)$ be the free boundary in the boundary value problem (3.2.6), then $b(t) \in C^1(0, T]$.*

Proof. First, we will show $b(t)$ is differentiable at $t_0 \in (0, T]$. Let us define $\rho = \partial_x^2 u(b(t_0)+, t_0) + e^{b(t_0)}$. Corollary 3.3.5 implies that $\rho > 0$.

For sufficiently small $\epsilon > 0$, it follows from (3.2.7) that

$$\frac{1}{\epsilon} \left[\frac{\partial}{\partial x} u(b(t_0), t_0) - \frac{\partial}{\partial x} u(b(t_0 - \epsilon), t_0 - \epsilon) + e^{b(t_0)} - e^{b(t_0-\epsilon)} \right] = 0.$$

Applying the Mean Value Theorem yields

(3.4.4)
$$\left(\frac{\partial^2}{\partial x^2} u(b(t_0) + y, t_0) + e^{b(t_0)+y} \right) \frac{b(t_0) - b(t_0 - \epsilon)}{\epsilon} = -\frac{\partial^2}{\partial x \partial t} u(b(t_0 - \epsilon), t_0 - \tau),$$

for some $y \in (0, b(t_0 - \epsilon) - b(t_0))$ and $\tau \in (0, \epsilon)$. Letting $\epsilon \to 0$ in (3.4.4) and using

Lemma 3.4.2 (ii), we obtain

$$(3.4.5) \qquad \lim_{\epsilon \to 0} \frac{b(t_0) - b(t_0 - \epsilon)}{\epsilon} = -\frac{\frac{\partial^2}{\partial x \partial t} u(b(t_0)+, t_0)}{\frac{\partial^2}{\partial x^2} u(b(t_0)+, t_0) + e^{b(t_0)}},$$

which implies that $b(t)$ is differentiable since $\rho > 0$. Moreover, from (3.2.9) and Proposition 3.2.3, we have

$$\frac{\partial^2}{\partial x^2} u(b(t)+, t) = \frac{2(r + \lambda)}{\sigma(b(t), t)^2} K + \left(\frac{2(\mu - r - \lambda)}{\sigma(b(t), t)^2} - 1 \right) e^{b(t)} - \frac{2}{\sigma(b(t), t)^2} f(b(t), t),$$

which is clearly a continuous function of t on $t \in (0, T]$, since $b(t)$ is a continuous function and $\sigma(x, t)$ is continuous from our assumption (3.2.5). Along with Lemma 3.4.2 (i), we can see from (3.4.5) that $b(t) \in C^1(0, T]$. $\qquad \square$

3.5 Higher order regularity of the free boundary

In the previous section, we have proved that the free boundary $b(t)$ is continuously differentiable. In this section, we will upgrade their regularity. Throughout this section, for the simplicity of the notation, we will assume that σ is a positive constant. In this case, $h(x, t) = \lambda \int_{\mathbb{R}} \frac{\partial}{\partial t} u(x + z, t) \nu(dz)$, which is bounded thanks to Lemma 3.2.8. More generally, if $\sigma = \sigma(x, t)$, $h(x, t)$ is given in (5.4.6). If we assume $\sigma(x, t) \in C^\infty(\mathbb{R} \times [0, T])$ with all its derivatives bounded and $\delta \leq \sigma \leq \Delta$ for some positive constants δ and Δ, the same arguments in this section can still be carried through. Because of Lemmas 3.2.6 and 3.2.8, we can see from the equation (3.2.6) that $\partial_x^2 u(x, t)$ is also bounded in $\mathbb{R} \times [\epsilon, T]$ for any $\epsilon > 0$. Hence, $h(x, t)$ is also bounded in this general case.

First, let us derive an identity on $b'(t)$. Since $b(t)$ is differentiable, taking derivative with respect to t on both sides of (3.2.7), we have

$$(3.5.1) \qquad \frac{\partial^2}{\partial x^2} u(b(t)+, t) b'(t) + \frac{\partial^2}{\partial x \partial t} u(b(t)+, t) = -e^{b(t)} b'(t).$$

The term $\partial_x^2 u(b(t)+, t)$ can be represented as

(3.5.2) $\quad \dfrac{\partial^2}{\partial x^2} u(b(t)+, t) = \left(\dfrac{2(\mu - r - \lambda)}{\sigma^2} - 1 \right) e^{b(t)} + \dfrac{2(r + \lambda)}{\sigma^2} K - \dfrac{2}{\sigma^2} f(b(t), t).$

Plugging (3.5.2) back into (3.5.1) and recalling $w = \partial_t u$, we obtain

(3.5.3) $\quad b'(t) = - \dfrac{\frac{\sigma^2}{2} \frac{\partial}{\partial x} w(b(t)+, t)}{(\mu - r - \lambda) e^{b(t)} + (r + \lambda) K - f(b(t), t)}, \quad t \in (0, T].$

We can see from equations (3.4.2) that $w(x, t)$ is the solution of a formal Stefan problem in the unbounded continuation regions $\mathcal{C}$. Schaeffer (1976) gave a proof of the infinite differentiability of the free boundary of a one dimensional Stefan problem in a bounded domain. By introducing the new variable $\xi = \frac{x}{b(t)}$, he reduced the problem into a fixed boundary problem on a bounded domain. However, if we apply the same change of variables we will have unbounded coefficients in the corresponding fixed boundary problem. Instead, similar to the change of variables in the proof of Lemma 3.3.3 (iii), we will define

$$\xi \triangleq x - b(t), \quad v(\xi, t) \triangleq w(x, t),$$

in which $b(t)$ is the free boundary in (3.2.6). The function $v(\xi, t)$ satisfies the following fixed boundary equation,

(3.5.4)
$$\dfrac{\partial v}{\partial t} - \dfrac{1}{2} \sigma^2 \dfrac{\partial^2 v}{\partial \xi^2} - \left(\mu + b'(t) - \dfrac{1}{2} \sigma^2 \right) \dfrac{\partial v}{\partial \xi} + (r + \lambda) v = h(\xi + b(t), t), \quad (\xi, t) \in (0, +\infty) \times (0, T],$$

$$v(0, t) = 0, \quad t \in (0, T],$$

$$v(\xi, 0) = w(\xi + b(0), 0), \quad \xi \geq 0.$$

Moreover, we have the following identity

(3.5.5) $\quad b'(t) = - \dfrac{\frac{\sigma^2}{2} \frac{\partial}{\partial \xi} v(0, t)}{(\mu - r - \lambda) e^{b(t)} + (r + \lambda) K - f(b(t), t)}, \quad t \in (0, T].$

Remark 3.5.1. Since $b(t) \in C^1(0, T]$, so for any $\epsilon > 0$, $b'(t)$ is continuous and bounded in $[\epsilon, T]$. On the other hand, since $\partial_t u$ is bounded by Lemma 3.2.8, so $h(\xi + b(t), t) = \lambda \int_{\mathbb{R}} \partial_t u(\xi + b(t) + z, t)\nu(dz)$ is also bounded when $(\xi, t) \in [0, +\infty) \times [\epsilon, T]$. As a result, it follows from Theorem 2.6 in page 19 of Ladyženskaja et al. (1968) that the parabolic differential equation (3.5.4) with the initial condition $v(\xi, \epsilon) = w(\xi + b(\epsilon), \epsilon)$ has at most one bounded classical solution. It follows from the proof of Lemma 3.4.2 (i) that $\partial_t u(x, t)$ is a bounded classical solution, so it is the unique bounded solution of (3.5.4).

The following result for parabolic differential equations will be an essential tool in the proof of the main result in this section.

Lemma 3.5.2. *Let us assume* $w(\xi, t) \in H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T])$ *(for some α and $\delta > 0$) satisfies the following equation*

(3.5.6)
$$\frac{\partial w}{\partial t} - a\frac{\partial^2 w}{\partial \xi^2} + \ell\frac{\partial w}{\partial \xi} + cw = d\int_{\mathbb{R}} \phi(\xi + z, t)\nu(dz), \quad (\xi, t) \in ((0, +\infty) \times (\delta, T))$$

$$w(0, t) = g(t), \quad t \in [\delta, T].$$

We assume that $d\int_{\mathbb{R}} \phi(\xi + z, t)\nu(dz) \in H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T])$ *and that coefficients* a, ℓ, c *also belong to* $H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T])$ *with* $\delta \leq a \leq \Delta$ *for some positive constants δ and Δ, moreover $g(t) \in H^{1+\alpha}([\delta, T])$. Then $w(\xi, t) \in H^{2+2\alpha, 1+\alpha}([0, +\infty) \times [\delta', T]$, for any $\delta' > \delta$.*

Proof. Consider a cut-off function $\eta(t) \in C_0^\infty((0, T])$, such that $\eta(t) = 0$ when $t \in$

$(0, \delta]$ and $\eta(t) = 1$ for $t \in [\delta', T]$. The function $\widetilde{w}(\xi, t) = \eta(t) w(\xi, t)$ satisfies

$$\frac{\partial \widetilde{w}}{\partial t} - a \frac{\partial^2 \widetilde{w}}{\partial \xi^2} + \ell \frac{\partial \widetilde{w}}{\partial \xi} + c \widetilde{w} = d \int_{\mathbb{R}} \eta(t) \phi(\xi + z, t) \nu(dz) + \frac{\partial \eta}{\partial t} w(\xi, t),$$

$$(\xi, t) \in (0, +\infty) \times (\delta, T],$$

$$\widetilde{w}(0, t) = \eta(t) g(t), \quad t \in [\delta, T],$$

$$\widetilde{w}(\xi, \delta) = 0, \quad \xi \geq 0.$$

From our assumptions we have that

$$d \int_{\mathbb{R}} \eta(t) \phi(\xi + z, t) \nu(dz) + \frac{\partial \eta}{\partial t} w(\xi, t) \in H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T]),$$

$$\eta(t) g(t) \in H^{1+\alpha}([\delta, T]).$$

Moreover, the coefficients of the above differential equation are all inside space $H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T])$. In addition, this equation is uniformly parabolic as the result of $0 < \delta \leq a \leq \Delta$. It follows from regularity estimation for parabolic differential equations (see Theorem 5.2 in page 320 of Ladyženskaja et al. (1968)) that $\widetilde{w}(\xi, t) \in H^{2+2\alpha, 1+\alpha}([0, +\infty) \times [\delta, T])$, which implies $w(\xi, t) \in H^{2+2\alpha, 1+\alpha}([0, +\infty) \times [\delta', T])$ by the choice of $\eta(t)$. $\square$

Remark 3.5.3. We will apply the previous lemma to $w(x, t) = \partial_t u(x, t)$. Because the initial condition for $w(x, t)$, $\lim_{t \to 0} \partial_t u(x, t)$, is not smooth, therefore we can not apply Theorem 5.2 in page 320 of Ladyženskaja et al. (1968) to upgrade the regularity of w directly. This is the reason we work with $\widetilde{w}$ in the proof of the previous lemma.

In order to apply Lemma 3.5.2 to (3.5.4), we need Hölder continuous coefficients and value functions. Let us first show that the coefficients in equation (3.5.4) are Hölder continuous.

Lemma 3.5.4. *Let $b(t)$ be the free boundary in (3.2.6). Then $b(t) \in H^{1+\alpha}([\delta, T])$ with $0 < \alpha < \frac{1}{2}$ for any $\delta > 0$.*

Proof. For any $\delta > 0$, since $b(t) \in C^1(0, T]$ by Theorem 3.4.3, the coefficients in equation (3.5.4) are bounded and continuous in $[\delta, T]$. On the other hand, because $\partial_t u(x, t)$ is bounded in $\mathbb{R} \times [\delta, T]$ by Lemma 3.2.8, the function $h(\xi + b(t), t) = \lambda \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \nu(dz)$ is also bounded when $(\xi, t) \in [0, +\infty) \times [\delta, T]$. It follows from Theorem 9.1 in page 341 of Ladyženskaja et al. (1968) that equation (3.5.4) has a unique solution $v(\xi, t) \in W_q^{2,1}([0, M] \times [\delta, T])$ for any $q > 1$ and $M > 0$.

By the Sobolev Embedding Theorem (see, for example, Theorem 2.1 in page 61 of Ladyženskaja et al. (1968)), for $q > 3$, we have $v(\xi, t) \in H^{\beta, \beta/2}([0, M] \times [\delta, T])$ with $\beta = 2 - \frac{3}{q}$ $(1 < \beta < 2)$. as a result, we have

$$(3.5.7) \qquad \frac{\partial}{\partial \xi} v(0, t) \in H^{\frac{\beta-1}{2}}([\delta, T]), \quad \text{with } 0 < \frac{\beta - 1}{2} < \frac{1}{2}.$$

Let us analyze the terms in the denominator on the right hand side of (3.5.5). We have that $b(t) \in C^1([\delta, T])$ and that

$$f(b(t), t) = \lambda \int_{\mathbb{R}} u(b(t) + z, t) \nu(dz) \in C^1([\delta, T]),$$

since $u(x, t) \in C^1(\mathbb{R} \times [\delta, T])$ (see Remark 3.2.11). Moreover, this denominator is also bounded away from 0, because

$$(\mu - r - \lambda)\sigma^2 e^{b(t)} + (r + \lambda)K - f(b(t), t) = \frac{\sigma^2}{2} \left(\frac{\partial^2}{\partial x^2} u(b(t), t) + e^{b(t)} \right) > 0, \quad t \in [\delta, T],$$

where the last inequality follows from Corollary 3.3.5. With (3.5.5) and (3.5.7), it is clear that

$$b'(t) \in H^{\frac{\beta-1}{2}}([\delta, T]).$$

$\square$

As a corollary of Lemmas 3.5.2 and 3.5.4, we can improve the regularity of the functions $u(x, t)$.

Corollary 3.5.5. *Let $u(x,t)$ be the classical solution of the boundary value problem*
(3.2.6). Then $u(\xi + b(t), t) \in H^{2+2\alpha, 1+\alpha}([0, +\infty) \times [\delta', T])$ for any $\delta' > 0$, with
$\alpha \in (0, 1/2)$.

Proof. Let $\xi = x - b(t)$, $\kappa(\xi, t) = u(x, t)$ and $\phi(\xi + z, t) = u(\xi + b(t) + z, t)$. Then $\kappa(\xi, t)$
satisfies a differential equation of the form (3.5.6) in Lemma 3.5.2 with $g(t) = K - e^{b(t)}$
(in fact κ satisfies (3.5.4) when h in the driving term is replaced by f). Moreover, by
Lemma 3.5.4, the coefficients in this equation (3.5.6) are inside space $H^\alpha([\delta, T])$ for
any $\delta > 0$, and $g(t) \in H^{1+\alpha}([\delta, T])$. In addition, thanks to the assumption (3.2.5),
the equation (3.5.6) is uniformly parabolic.

On the other hand, since $u(x, t)$ is uniformly Lipschitz in $x \in \mathbb{R}$ and uniformly
semi-Hölder continuous in $t \in [0, T]$ (see Lemma 3.2.6), and $b(t)$ is continuously
differentiable, it is not hard to see that $\int_{\mathbb{R}} u(\xi + b(t) + z, t)\nu(dz) \in H^{2\alpha, \alpha}([0, +\infty) \times$
$[\delta, T])$. Moreover, $u(\xi + b(t), t) \in H^{2\alpha, \alpha}([0, +\infty) \times [\delta, T])$ again because of Lemma
3.2.6. Now, the statement follows directly from Lemma 3.5.2. $\qquad\square$

Armed with Lemmas 3.5.2, 3.5.4 and Corollary 3.5.5, we can state and prove the
main theorem of this section.

Theorem 3.5.6. *Let $b(t)$ be the free boundary in (3.2.6). Assume that ν has a*
density, i.e. $\nu(dz) = \rho(z)dz$. Let $\alpha \in (0, 1/2)$. If $\rho(z)$ satisfies $\int_{-\infty}^{u} \rho(z)dz \in$
$H^{2\alpha}(\mathbb{R}_-)$, then $b(t) \in H^{\frac{3}{2}+\alpha}([\epsilon, T])$. On the other hand, if $\rho(z) \in H^{\ell-1+2\alpha}(\mathbb{R}_-)$ for
$\ell \geq 1$, then $b(t) \in H^{\frac{3}{2}+\frac{\ell}{2}+\alpha}([\epsilon, T])$, for any $\epsilon > 0$.

Proof. The proof consists of four steps.

Step 1. From Lemma 3.5.4 and Corollary 3.5.5, we have that $b(t) \in H^{1+\alpha}([\delta, T])$
and that $u(\xi + b(t), t) \in H^{2+2\alpha, 1+\alpha}([0, +\infty) \times [\delta', T])$ for any $\delta' > \delta > 0$ with $\alpha \in$
$(0, 1/2)$, which implies that $\partial_t u(\xi + b(t), t) \in H^{2\alpha, \alpha}([0, +\infty) \times [\delta', T])$ (see Definition

3.2.7).

Step 2. Assume that there is a positive nonintegral real number β with $2\beta \leq 2\alpha + \ell$, such that

$$(3.5.8) \qquad\qquad b(t) \in H^{1+\beta}([\delta, T]),$$

$$(3.5.9) \qquad\qquad \frac{\partial}{\partial t} u(\xi + b(t), t) \in H^{2\beta, \beta}([0, +\infty) \times [\delta', T]),$$

$$(3.5.10) \qquad\qquad u(\xi + b(t), t) \in H^{2+2\beta, 1+\beta}([0, +\infty) \times [\delta', T]),$$

for $\delta' > \delta > 0$. We will upgrade the regularity exponent from β to $1/2 + \beta$, in steps 2 and 3.

Let us analyze $\partial_t u(\xi + b(t), t)$. For any integers $r, s \geq 0$, $2r + s < 2\beta$, since $\partial_t u(\xi + b(t) + z, t) = 0$ when $z \leq -\xi$, we have

(3.5.11)

$$\frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \nu(dz) = \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{-\xi}^{+\infty} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \rho(z) dz$$

$$= 1_{\{s \geq 1\}} \sum_{i=0}^{s-1} \frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \Big|_{z \downarrow -\xi} \frac{d^{s-1-i}}{d\xi^{s-1-i}} \rho(-\xi)$$

$$+ \int_{-\xi}^{+\infty} \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \rho(z) dz,$$

for any $\xi \geq 0$.

When t is fixed, in the following, we will show

$$(3.5.12) \quad \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \nu(dz) \in H^{2\beta - [2\beta]}([0, +\infty)), \quad \text{for } 2r + s = [2\beta].$$

For any $\xi_1 > \xi_2 \geq 0$ such that $\xi_1 - \xi_2 \leq \rho_0$, we have

(3.5.13)

$$\left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi_1 + b(t) + z, t) \nu(dz) - \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi_2 + b(t) + z, t) \nu(dz) \right|$$

$$\leq 1_{\{s \geq 1\}} \sum_{i=0}^{s-1} \left| \frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \right|_{z \downarrow -\xi} \left| \left| \frac{d^{s-1-i}}{d\xi^{s-1-i}} \left(\rho(-\xi_1) - \rho(-\xi_2) \right) \right| \right.$$

$$+ \int_{-\xi_2}^{+\infty} \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} \left(u(\xi_1 + b(t) + z, t) - u(\xi_2 + b(t) + z, t) \right) \right| \rho(z) dz$$

$$+ \int_{-\xi_1}^{-\xi_2} \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} u(\xi_1 + b(t) + z, t) \right| \rho(z) dz.$$

Let us analyze the right hand side of (3.5.13) term by term. When $s > 1$, since $s - 1 < 2\beta - 1 \leq 2\alpha + \ell - 1$, we have $\rho(z) \in H^{2\beta-1}(\mathbb{R}_-)$, which implies

(3.5.14)
$$1_{\{s \geq 1\}} \sum_{i=0}^{s-1} \left| \frac{\partial^i}{\partial \xi^i} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \right|_{z \downarrow -\xi} \left| \left| \frac{d^{s-1-i}}{d\xi^{s-1-i}} \left(\rho(-\xi_1) - \rho(-\xi_2) \right) \right| \right.$$

$$\leq C \|\partial_t u\|^{(2\beta)} |\xi_1 - \xi_2|^{2\beta - [2\beta]},$$

in which C is a positive constant and $\| \cdot \|^{(2\beta)}$ is the Hölder norm (see Definition 3.2.7). On the other hand, it follows from (3.5.9) that

(3.5.15)
$$\int_{-\xi_2}^{+\infty} \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} \left(u(\xi_1 + b(t) + z, t) - u(\xi_2 + b(t) + z, t) \right) \right| \rho(z) dz$$

$$\leq \|\partial_t u\|^{(2\beta)} |\xi_1 - \xi_2|^{2\beta - [2\beta]} \int_{-\xi_2}^{+\infty} \rho(z) dz \leq \|\partial_t u\|^{(2\beta)} |\xi_1 - \xi_2|^{2\beta - [2\beta]}.$$

Moreover, because $\rho(z) \in H^{\ell-1+2\alpha}(\mathbb{R}_-)$ for $\ell \geq 1$ or $\int_{-\infty}^u \rho(z) dz \in H^{2\alpha}(\mathbb{R}_-)$, we have $\int_{-\infty}^u \rho(z) dz \in H^{\ell+2\alpha}(\mathbb{R}_-)$ for $\ell \geq 0$. In particular, using $2\beta \leq 2\alpha + \ell$, we can see $\int_{-\infty}^u \rho(z) dz \in H^{2\beta-[2\beta]}(\mathbb{R}_-)$. As a result,

(3.5.16)
$$\int_{-\xi_1}^{-\xi_2} \left| \frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \frac{\partial}{\partial t} u(\xi_1 + b(t) + z, t) \right| \rho(z) dz$$

$$\leq \|\partial_t u\|^{(2\beta)} \left(\int_{-\infty}^{-\xi_2} \rho(z) dz - \int_{-\infty}^{-\xi_1} \rho(z) dz \right) \leq \tilde{C} \|\partial_t u\|^{(2\beta)} |\xi_1 - \xi_2|^{2\beta - [2\beta]},$$

where $\tilde{C}$ is also a positive constant. Plugging the estimates (3.5.14) - (3.5.16) into (3.5.13), we observe that (3.5.12) holds.

When ξ is fixed, using (3.5.11), it directly follows from (3.5.8) and (3.5.9) that

(3.5.17)
$$\frac{\partial^s}{\partial \xi^s} \frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \nu(dz) \in H^{\beta - \frac{2r+s}{2}}([\delta', T]), \quad \text{for } 2\beta - 2 < 2r + s < 2\beta.$$

Now, (3.5.12) and (3.5.17) imply that

(3.5.18)
$$\int_{\mathbb{R}} \frac{\partial}{\partial t} u(\xi + b(t) + z, t) \nu(dz) \in H^{2\beta, \beta}([0, +\infty) \times [\delta', T]).$$

Let $v(\xi, t)$ be a bounded solution of the boundary value problem (3.5.4) with the initial condition $v(\xi, \delta') = \partial_t u(\xi + b(\delta'), \delta')$. The uniqueness in Remark 3.5.1 implies that

(3.5.19)
$$v(\xi, t) = \frac{\partial}{\partial t} u(\xi + b(t), t), \quad (\xi, t) \in [0, +\infty) \times [\delta', T].$$

As a result, the assumption (3.5.9) implies that

(3.5.20)
$$v(\xi, t) \in H^{2\beta, \beta}([0, +\infty) \times [\delta', T]).$$

We will apply Lemma 3.5.2 to (3.5.4) with $\phi(\xi + z, t) = \partial_t u(\xi + b(t) + z, t)$, $a = \sigma^2/2$, $\ell = -(\mu + b'(t) - \sigma^2/2)$, $c = r + \lambda$ and $d = \lambda$. Thanks to (3.5.8), the coefficient l belongs to $H^\beta([\delta, T])$. The other coefficients already happen to reside there since they are constants. Along with (3.5.18) and (3.5.20), Lemma 3.5.2 yields

(3.5.21)
$$v(\xi, t) \in H^{2+2\beta, 1+\beta}([0, +\infty) \times [\delta'', T]) \quad \text{for any } \delta'' > \delta' > \delta,$$

which implies that

(3.5.22)
$$\frac{\partial}{\partial \xi} v(0, t) \in H^{\frac{1}{2}+\beta}([\delta'', T]),$$

and

(3.5.23)
$$\frac{\partial}{\partial t} u(\xi + b(t), t) \in H^{2+2\beta, 1+\beta}([0, +\infty) \times [\delta'', T]),$$

by (3.5.19).

Using (3.5.5) and (3.5.22), we will improve the regularity of $b(t)$ in the following.

From (3.7.1) we have

$$
\begin{aligned}
f(b(t),t) &= \lambda \int_{\mathbb{R}} u(b(t)+z,t)\nu(dz) \\
&= \lambda \int_0^{+\infty} u(b(t)+z,t)\nu(dz) + \lambda \int_{-\infty}^0 (K - e^{b(t)+z})\nu(dz).
\end{aligned}
$$

(3.5.24)

Along with (3.5.8) and (3.5.10), we can see from (3.5.24) that

$$
f(b(t),t) \in H^{1+\beta}([\delta'',T]). \tag{3.5.25}
$$

Together with (3.5.8), (3.5.22) and (3.5.25), we can see from the identity (3.5.5) that $b'(t) \in H^{\frac{1}{2}+\beta}([\delta'',T])$ for any $\delta'' > \delta'$. It in turn implies that

$$
b(t) \in H^{\frac{3}{2}+\beta}([\delta'',T]). \tag{3.5.26}
$$

Step 3. Let us investigate $u(\xi + b(t),t)$. For any $r,s \geq 0$, $2r+s < 2+2\beta$, we have

$$
\begin{aligned}
&\frac{\partial^s}{\partial \xi^s}\frac{\partial^r}{\partial t^r} \int_{\mathbb{R}} u(\xi+b(t)+z,t)\nu(dz) \\
&= \frac{\partial^s}{\partial \xi^s}\frac{\partial^r}{\partial t^r} \int_{-\xi}^{+\infty} u(\xi+b(t)+z,t)\rho(z)dz + \frac{\partial^s}{\partial \xi^s}\frac{\partial^r}{\partial t^r} \int_{-\infty}^{-\xi} u(\xi+b(t)+z,t)\rho(z)dz \\
&= 1_{\{s\geq 1\}} \sum_{i=0}^{s-1} \left[\frac{\partial^i}{\partial \xi^i}\frac{\partial^r}{\partial t^r} u(\xi+b(t)+z,t)\Big|_{z\downarrow-\xi} - \frac{\partial^i}{\partial \xi^i}\frac{\partial^r}{\partial t^r} u(\xi+b(t)+z,t)\Big|_{z\uparrow-\xi} \right] \frac{d^{s-1-i}}{d\xi^{s-1-i}}\rho(-\xi) \\
&\quad + \int_{-\xi}^{+\infty} \frac{\partial^s}{\partial \xi^s}\frac{\partial^r}{\partial t^r} u(\xi+b(t)+z,t)\rho(z)dz + \int_{-\infty}^{-\xi} \frac{\partial^s}{\partial \xi^s}\frac{\partial^r}{\partial t^r} u(\xi+b(t)+z,t)\rho(z)dz,
\end{aligned}
$$

for any $\xi \geq 0$. It is worth noticing that $\partial_\xi^i \partial_t^r u(\xi+b(t)+z,t)|_{z\downarrow-\xi} \neq \partial_\xi^i \partial_t^r u(\xi+b(t)+z,t)|_{z\uparrow-\xi}$ for some i and r. Following the same arguments that lead up to (3.5.18), we can show

$$
\int_{\mathbb{R}} u(\xi+b(t)+z,t)\nu(dz) \in H^{2+2\beta,1+\beta}([0,+\infty)\times[\delta',T]), \tag{3.5.27}
$$

given $1 + 2\beta \leq 2\alpha + \ell - 1$.

Now, we can apply Lemma 3.5.2 to the differential equation $u(\xi + b(t), t)$ satisfies, taking (3.5.10) and (3.5.26) into account. This results in

$$(3.5.28) \qquad u(\xi + b(t), t) \in H^{3+2\beta, \frac{3}{2}+\beta}([0, +\infty) \times [\delta''', T]),$$

for any $\delta''' > \delta''$. As a result, we have improved the regularities from (3.5.8), (3.5.9) and (3.5.10) to (3.5.26), (3.5.23) and (3.5.28), respectively.

Step 4. For any $\epsilon > 0$, we apply Steps 2 and 3 inductively starting from $\beta = \alpha$ in Step 1. Let n be the number of time we apply Steps 2 and 3. Let $\delta'_1 = \delta'$, in which $\delta' > 0$ is as in Step 1. Running Step 2 and 3 once, we obtain two constants δ''_1 and δ'''_1 such that (3.5.26), (3.5.28) hold with $\beta = \alpha$. In the n-th time, $n \geq 2$, we choose $\delta'_n = \delta'''_{n-1}$ and $\delta'''_n > \delta''_n > \delta'_n$, such that $\delta'''_n < \epsilon$ for any n so that $[\epsilon, T] \subset [\delta'''_n, T]$.

The application of Step 2 for the n-th time will give us that $b(t) \in H^{1+\alpha+\frac{n}{2}}([\epsilon, T])$. Applying Step 2 for $\ell + 1$ and Step 3 for ℓ times the result follows. $\qquad \square$

Remark 3.5.7. (i) The previous proof has also shown the higher order regularity of $u(x, t)$, i.e. $u(\xi + b(t), t) \in H^{2+2\alpha+\ell, 1+\alpha+\frac{\ell}{2}}([0, +\infty) \times [\epsilon, T])$, for any $\epsilon > 0$, under the assumptions of Theorem 3.5.6, .

(ii) Note that $b(t) \in C^1((0, T])$ without any assumption on the density $\rho(z)$. If $\rho(z) \in H^{2m-1+2\alpha}(\mathbb{R}_-)$ for some $m \geq 1$, then $b(t) \in H^{\frac{3}{2}+m+\alpha}([\epsilon, T])$. From Definition 3.2.7 and the arbitrary choice of ϵ, we have that $b(t) \in C^{m+1}((0, T])$ under this assumption.

As a corollary of Theorem 3.5.6, we have the following sufficient condition for the infinitely differentiability of $b(t)$.

Corollary 3.5.8. *Let $b(t)$ be the free boundary in (3.2.6). Assume that ν has a*

density, i.e. $\nu(dz) = \rho(z)dz$. *If* $\rho(z) \in C^\infty(\mathbb{R}_-)$ *with* $\frac{d^\ell}{dz^\ell}\rho(z)$ *bounded for each* $\ell \geq 1$, *but not necessarily uniformly, then* $b(t) \in C^\infty((0,T])$.

Proof. For any $m \geq 1$ with $\rho(z) \in C^{2m+1}(\mathbb{R}_-)$ and derivatives of $\rho(z)$ up to order $2m+1$ are bounded, it follows from Definition 3.2.7 that $\rho(z) \in H^{2m-1+2\alpha}(\mathbb{R}_-)$. As a result of Remark 3.5.7 (ii), we have $b(t) \in C^{m+1}((0,T])$. $\qquad\qquad\square$

Remark 3.5.9. There are two well-known examples of jump diffusion models in the literature, Kou's model and Merton's model (see Cont and Tankov (2004), p.111), in which the density $\rho(z)$ is double exponential and normal, respectively. For both of these densities, it is easy to see that the conditions for Corollary 3.5.8 are satisfied. Therefore, the free boundaries in both models are infinitely differentiable.

3.6   The approximation problems

This section is completely independent of the regularity properties of the free boundary $b(t)$. We want to show that the approximating free boundaries $b_n(t)$, constructed in Bayraktar (2009), have the similar regularity properties with the free boundary $b(t)$.

Bayraktar (2009) constructed a monotone increasing sequence $\{u_n\}_{n\geq 0}$ that converges to the unique solution $u(x,t)$ of the parabolic integro-differential equation (3.2.6), uniformly. In this sequence, $u_0(x,t) = (K - e^x)^+$, and each $u_n(x,t)$ $(n \geq 1)$ is the unique classical solution of the following parabolic differential equation:

(3.6.1)
$$\mathcal{L}_{\mathcal{D}}u_n \triangleq \frac{\partial u_n}{\partial t} - \frac{1}{2}\sigma^2\frac{\partial^2 u_n}{\partial x^2} - \left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial u_n}{\partial x} + (r+\lambda)u_n = f_n(x,t), \quad x > b_n(t),$$

$$u_n(b_n(t),t) = K - e^{b_n(t)}, \quad t \in (0,T],$$

$$u_n(x,0) = (K - e^x)^+, \quad x \geq b_n(0),$$

in which

$$(3.6.2) \qquad f_n(x,t) \triangleq \lambda \int_{\mathbb{R}} u_{n-1}(x+z,t)\nu(dz),$$

and the free boundary $b_n(t) \triangleq \log\left(s_n\left(T-t\right)\right)$ is defined in terms of $s_n(\cdot)$, which is the approximating free boundary in Bayraktar (2009). Moreover, the smooth fit property is also satisfied for each u_n, i.e.

$$(3.6.3) \qquad \frac{\partial}{\partial x} u_n(b_n(t),t) = -e^{b_n(t)}, \quad t \in (0,T].$$

In the region $\{(x,t)\,|\,x < b_n(t), t \in (0,T]\}$, one also has that

$$(3.6.4) \qquad \mathcal{L}_{\mathcal{D}}u_n(x,t) - f_n(x,t) \geq 0.$$

For all $n \geq 1$, we can define the approximating continuation regions $\mathcal{C}_n$ and the stopping regions $\mathcal{D}_n$ as follows

$$\mathcal{C}_n \triangleq \{(x,t)\,|\,b_n(t) < x < +\infty, 0 < t \leq T\}, \quad \mathcal{D}_n \triangleq \{(x,t)|-\infty < x \leq b_n(t), 0 < t \leq T\}.$$

Since $\{u_n\}_{n\geq0}$ is a monotone increasing sequence, the approximating free boundary $\{b_n\}_{n\geq1}$ is a monotone decreasing sequence. As a result, we have $\cup_{n\geq1}\mathcal{C}_n = \mathcal{C}$ and $\cap_{n\geq1}\mathcal{D}_n = \mathcal{D}$.

The approximating sequences $\{u_n\}_{n\geq1}$ and $\{b_n\}_{n\geq1}$ have the similar properties with the value function u and its free boundary b. Proposition 3.2.4, Lemmas 3.2.5, 3.2.6 and 3.2.10 have their analogous versions for u_n and b_n via the same proofs only replacing the integral term f in (3.2.9) by f_n in (3.6.2). While Proposition 3.2.3 and Lemma 3.2.8 have slight modifications as follows:

Proposition 3.6.1. *For all $n \geq 1$,*

(i) *If $\partial_t u_{n-1}(x,t)$ is bounded in $\mathbb{R} \times [\epsilon, T]$ for any $\epsilon > 0$, then $\partial_t u_n(x,t)$ is continuous*

in $\mathbb{R} \times (0, T]$ *and*

$$(3.6.5) \qquad \lim_{x \downarrow b_n(t)} \frac{\partial}{\partial t} u_n(x, t) = 0.$$

(ii) *If* $\lim_{x \downarrow b_n(t)} \partial_t u_n(x, t) = 0$ *for* $t \in (0, T]$ *then* $\partial_t u_n(x, t)$ *is uniformly bounded in*

$\mathbb{R} \times [\epsilon, T]$, *for any* $\epsilon > 0$.

Proof. See Appendix 3.7.3 for the proof of (i). Under the assumption that $\lim_{x \downarrow b_n(t)} u(x, t) = 0$ for $t \in (0, T]$, we have $\partial_t u(x, t)$ is bounded in the domain $\{(x, t) \mid b_n(t) \le x \le X_0, \epsilon \le t \le T\}$ for any $\epsilon \ge 0$. Then the rest proof of (ii) is similar with the proof of Lemma 3.2.8. $\qquad \square$

Remark 3.6.2. To show that assumptions in both (i) and (ii) are satisfied for all u_n, $n \ge 1$, we need to walk through (i) and (ii) successively. Starting from $\partial_t u_0(x, t) = 0$ (since $u_0(x, t) = (K - e^x)^+$), (i) tells us that $\lim_{x \downarrow b_1(t)} \partial_t u_1(x, t) = 0$. Then it follows from (ii) that $\partial_t u_1(x, t)$ is bounded in $\mathbb{R} \times [\epsilon, T]$ for any $\epsilon > 0$. This result feeds back to (i) which shows the assumptions in both (i) and (ii) are fulfilled by induction.

Results similar to Lemmas 3.3.1, 3.3.3 and Corollary 3.3.5 can also be shown for each u_n, $n \ge 1$. Defining

$$J_n(x, t) \triangleq q e^x - rK + \lambda \int_{\mathbb{R}} \left[u_{n-1}(x + z, t) + e^{x+z} - K \right] \nu(dz), \quad x \in \mathbb{R}, t \in [0, T],$$

$$B_n(t) \triangleq \{x : J_n(x, t) = 0, t \in [0, T]\}.$$

we obtain the following:

$$(3.6.6) \quad \mathcal{L}_{\mathcal{D}} u_n(x, t) - \lambda \int_{\mathbb{R}} u_{n-1}(x + z, t) \nu(dz) = -J_n(x, t), \quad x < b_n(t), t \in [0, T],$$

$(3.6.7)$

$x \to J_n(x, t)$ is strictly increasing and $t \to J_n(x, t)$ is non-decreasing for $(x, t) \in \mathbb{R} \times [0, T]$,

$$(3.6.8) \qquad\qquad B_n(t) > b_n(t), \quad t \in (0, T],$$

$$(3.6.9) \qquad \lim_{x \downarrow b_n(t)} \frac{\partial^2}{\partial x^2} u_n(x,t) > -e^{b_n(t)}, \quad t \in (0,T].$$

Moreover, as we can see in the following Proposition, the approximating free boundaries b_n have the same critical value with b when the time is close to maturity.

Proposition 3.6.3. *For the approximating sequence $b_n(t)$, we have*

$(3.6.10)$

$$b_n(0+) \triangleq \lim_{t \to 0^+} b_n(t) = \min\{\log K, B(0)\} = \begin{cases} \log K, & r \geq q + \lambda \int_{\mathbb{R}_+} (e^z - 1)\nu(dz) \\ B(0), & r < q + \lambda \int_{\mathbb{R}_+} (e^z - 1)\nu(dz) \end{cases},$$

in which $B(0)$ the unique solution of $(3.3.10)$.

Proof. When $x < b_n(t)(t > 0)$, it follows from $(3.6.4)$, $(3.6.6)$ and $(3.6.7)$ that

$$0 \leq \mathcal{L}_D u_n(x,t) - \lambda \int_{\mathbb{R}} u_{n-1}(x+z,t)\nu(dz) = -J_n(x,t) \leq -J_n(x,0) = -J_0(x).$$

The fact that $J_0(B(0)) = 0$ and $x \to J_0(x)$ is strictly increasing tells us $x \leq B(0)$, hence $b_n(t) \leq B(0)$ by the choice of x. It is also clear that $b_n(t) \leq \log K$. Then we obtain

$$(3.6.11) \qquad b_n(0+) \leq \min\{\log K, B(0)\}.$$

Now, the corollary results from combining $(3.3.20)$ and $(3.6.11)$, since $\{b_n\}_{n \geq 1}$ is an monotone decreasing sequence. $\qquad \square$

Furthermore, the Hölder continuity in Theorem $3.3.7$ also holds for b_n, $n \geq 1$. In the proof of Lemma $3.3.6$, we only need to replace c in $(3.3.27)$ by $\min\{-2/\sigma^2 J_n(x,t)|\, b_n(t) < x < B_n(t), \epsilon \leq t \leq T\} > 0$. On the other hand, results in Lemma $3.4.2$ also hold for $\partial_{xt} u_n$, $n \geq 1$. Therefore, combining with $(3.6.9)$, we have from $(3.6.3)$ that

Proposition 3.6.4. $b_n(t) \in C^1(0,T]$, $n \geq 1$.

Finally, using the following representation

$$(3.6.12) \qquad b_n'(t) = -\frac{\frac{\sigma^2}{2}\frac{\partial^2}{\partial x \partial t} u_n(b_n(t)+, t)}{(\mu - r - \lambda)e^{b_n(t)} + (r + \lambda)K - f_n(b_n(t), t)}, \qquad t \in (0, T],$$

one can follow the proof of Lemma 3.5.4 to show that there is $\alpha \in (0, 1/2)$ such that

$$b_n(t) \in H^{1+\alpha}([\delta, T]), \qquad \text{for any } \delta > 0.$$

3.7 Proof of some auxiliary results

3.7.1 Proof of Lemmas 3.2.6, 3.2.8 and 3.2.10

Proof of Lemma 3.2.6. The inequality (3.2.14) is clear, because we have

$$|u(x, t) - u(x, s)| = |V(e^x, T - t) - V(e^x, T - s)| \leq D|t - s|^{\frac{1}{2}}.$$

In order to prove (3.2.13), it suffices to check that $\partial_x u(x, t)$ is uniformly bounded in the domain $\mathbb{R} \times [0, T]$. Choose a constant $X > \log K + 1$, we will first prove $\partial_x u(x, t)$ is uniformly bounded in $[X, +\infty) \times [0, T]$. Let us consider a cut-off function $\eta(x) \in C^\infty(\mathbb{R})$, such that $\eta(x) = 0$ when $x \leq X - 1$ and $\eta(x) = 1$ when $x \geq X$. Using (3.2.9) we see that $v(x, t) = \eta(x)u(x, t)$ satisfies

$$\mathcal{L}_{\mathcal{D}} v = \eta(x)f(x, t) + \tilde{f}(x, t),$$

$$v(x, 0) = \eta(x)(K - e^x)^+,$$

where

(3.7.1)

$$f(x, t) = \lambda \int_{\mathbb{R}} u(x + z, t)\nu(dz), \quad \tilde{f}(x, t) = -\frac{1}{2}\sigma^2\left(\eta'' u + 2\eta'\frac{\partial u}{\partial x}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\eta' u.$$

It is worth noticing that the term $\eta'\partial_x u$ in the expression for $\tilde{f}$ vanishes outside a compact domain. Since we also have that $u(x, t) \leq K$, both $f(x, t)$ and $\tilde{f}(x, t)$ are bounded in $\mathbb{R} \times [0, T]$.

Let $G(x, t; y, s)$ be the Green function corresponding to the differential operator $\mathcal{L}_{\mathcal{D}}$. We can represent $v(x, t)$ in terms of G as

(3.7.2)
$$v(x, t) = \int_{\mathbb{R}} dy\, G(x, t; y, 0)\eta(y)(K - e^y)^+ + \int_0^t ds \int_{\mathbb{R}} dy\, G(x, t; y, s)\left(f(y, s)\eta(y) + \tilde{f}(y, s)\right).$$

The first term on the right-hand-side of (3.7.2) will vanish by the choice of $\eta(y)$. On the other hand, Green function $G(x, t; y, s)$ satisfies

$$|\partial_x G(x, t; y, s)| \le c(t - s)^{-1} \exp\left(-c\frac{|x - y|^2}{t - s}\right),$$

for some positive constant c, (see Theorem 16.3 in page 413 of Ladyženskaja et al. (1968)). Since $\int_{\mathbb{R}} dy\, \exp(-c\frac{(x-y)^2}{t-s}) \le d\,(t - s)^{\frac{1}{2}}$ for some other positive constant d, we have that

$$\int_0^t ds \int_{\mathbb{R}} dy\, |\partial_x G(x, t; y, s)| \le \int_0^t ds\, \tilde{c}(t - s)^{-\frac{1}{2}} = 2\tilde{c}\, t^{\frac{1}{2}},$$

Using this estimate and the boundness of f and $\tilde{f}$, the Dominated Convergence Theorem implies that

$$\partial_x v(x, t) = \int_0^t ds \int_{\mathbb{R}} dy\, \partial_x G(x, t; y, s)(f(y, s)\eta(y) + \tilde{f}(y, s)),$$

which is uniformly bounded. On the other hand, $\partial_x v = \eta' u + \eta \partial_x u$. By our choice of $\eta(x)$, we have that $\partial_x u(x, t)$ is uniformly bounded on $[X, +\infty) \times [0, T]$.

Moreover, in the stopping region $\mathcal{D}$, we have $\partial_x u(x, t) = -e^x$. This implies that $0 > \partial_x u(x, t) \ge -e^{b(t)} \ge -K$. On the other hand, since it is continuous $\partial_x u$ is also bounded in the compact closed domain $\{(x, t)|b(t) \le x \le X, 0 \le t \le T\}$. As a result we have that $\partial_x u(x, t)$ is uniformly bounded in $\mathbb{R} \times [0, T]$. $\qquad\square$

Proof of Lemma 3.2.8. Let us choose X_0 such that $X_0 > \log K$. We will first prove that $\partial_t u(x, t)$ is uniformly bounded in the domain $[X_0, +\infty) \times [0, T]$. Let

$k(x, t) \in C_0^\infty(\mathbb{R} \times [0, T])$ be such that

$$\partial_x k(x, t)|_{x=X_0} = \partial_x u(x, t)|_{x=X_0}, \quad t \in [0, T],$$

and that $k(x, 0) = 0$, $x \in \mathbb{R}$. These two conditions on k are consistent since $\partial_x u(x, 0)|_{x=X_0} = 0$. The function $v(x, t) \triangleq u(x, t) - k(x, t)$ satisfies

(3.7.3)
$$\partial_x v(x, t)|_{x=X_0} = 0,$$

and

(3.7.4)
$$\mathcal{L}_{\mathcal{D}} v(x, t) = f(x, t) + g(x, t), \quad x > b(t), t \in (0, T],$$

in which $g(x, t) = -\mathcal{L}_{\mathcal{D}} k(x, t)$ and f is given by (3.7.1). Let us define the even extension of $v(x, t)$ with respect to the line $x = X_0$ as

(3.7.5)
$$\hat{v}(x, t) \triangleq \begin{cases} v(x, t) & x \geq X_0, \\ v(2X_0 - x, t) & x < X_0. \end{cases}$$

We similarly define $\hat{f}(x, t)$ and $\hat{g}(x)$. From (3.7.3) and (3.7.5), we have $\hat{v}(x, t) \in C^{2,1}(\mathbb{R} \times (0, T])$ and that it satisfies the equation

$$\mathcal{L}_{\mathcal{D}} \hat{v} = \hat{f}(x, t) + \hat{g}(x, t), \quad (x, t) \in \mathbb{R} \times (0, T],$$

$$\hat{v}(x, 0) = 0, \quad x \in \mathbb{R}.$$

Here the initial condition follows from (3.2.6) and the choice of X_0 and $k(x, t)$.

It follows from (3.2.13) and (3.2.14) that $f(x, t)$ is uniformly Lipschitz in x and semi-Hölder coninuous in t. So for any $x_1 < x_2$, if we have either $x_2 \leq X_0$ or $X_0 \leq x_1$, then

$$\left| \hat{f}(x_1, t) - \hat{f}(x_2, t) \right| \leq \lambda C(x_2 - x_1),$$

for the same constant C as in (3.2.13). On the other hand, if $x_1 < X_0 < x_2$, then

$$
\begin{aligned}
|\hat{f}(x_1,t) - \hat{f}(x_2,t)| &\leq |\hat{f}(x_1,t) - \hat{f}(X_0,t)| + |\hat{f}(X_0,t) - \hat{f}(x_2,t)| \\
&\leq \lambda C(X_0 - x_1) + \lambda C(x_2 - X_0) = \lambda C(x_2 - x_1).
\end{aligned}
$$

As a result of the last two equations we observe that $\hat{f}(x,t)$ is uniformly Lipschitz in its first variable. It is also clear that $\hat{f}(x,t)$ is semi-Hölder continuous in its second variable. Thus, it follows from Definition 3.2.7 that

$$
\hat{f}(x,t) \in H^{\alpha,\frac{\alpha}{2}}(\mathbb{R} \times [0,T]), \quad \text{for some } 0 < \alpha < 1.
$$

On the other hand, $\hat{g}(x,t) \in H^{\alpha,\alpha/2}(\mathbb{R} \times [0,T])$, because $k(x,t) \in C_0^\infty(\mathbb{R} \times [0,T])$. Combining with the assumption (3.2.5) on σ, the regularity property of parabolic differential equation (see Theorem 5.1 in page 320 of Ladyženskaja et al. (1968)) implies that

$$
\hat{v}(x,t) \in H^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R} \times [0,T]).
$$

In particular, $u(x,t) \in H^{2+\alpha,1+\alpha/2}([X_0,+\infty) \times [0,T])$. As a result, in $[X_0,+\infty) \times [0,T]$, $\partial_t u(x,t)$ is uniformly bounded by the Hölder norm of $u(x,t)$. Now, the result follows from the continuity of $\partial_t u(x,t)$ inside domain $\{(x,t) \,|\, b(t) \leq x \leq X_0, \epsilon \leq t \leq T\}$ for any $\epsilon > 0$ (see Proposition 3.2.3). $\qquad\square$

Proof of Lemma 3.2.10. Let $X_0 > \log K$ be the same as in the proof of Lemma 3.2.8, again choose a cut-off function $\eta(x) \in C^\infty(\mathbb{R})$, such that $\eta(x) = 1$ when $x \geq 2X_0$ and $\eta(x) = 0$ when $x \leq X_0$. Then formally the function $\eta(x)\partial_t u(x,t)$ satisfies the following Cauchy problem

$$
\mathcal{L}_{\mathcal{D}} w = \eta(x)h(x,t) + \tilde{h}(x,t), \quad (x,t) \in \mathbb{R} \times [t_0, T],
$$

where

$$
h(x,t) = \lambda \int_{\mathbb{R}} \partial_t u(x+z,t)\nu(dz), \quad \tilde{h}(x,t) = -\frac{1}{2}\sigma^2 \left(2\eta'\partial_x\partial_t u + \eta''\partial_t u\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\eta'\partial_t u,
$$

and we choose $\eta(x)\partial_t u(x, t_0)$, for some $t_0 \in (0, T)$, as the initial condition. It follows from Theorem 3.1 in page 346 of Garroni and Menaldi (1992) that this Cauchy problem has an unique classical solution, we call it w. On the other hand, we have $w(x, t) = \eta(x)\partial_t u(x, t)$. Indeed, it is easy to check that $\int_{t_0}^t w(x, s)ds$ is the unique classical solution of the Cauchy problem

$$\mathcal{L}_D v = \int_{t_0}^t ds \left(\eta(x)h(x, s) + \tilde{h}(x, s) \right) + \eta(x)\partial_t u(x, t_0), \quad v(x, t_0) = 0.$$

Note that $\eta(x)\left[u(x, t) - u(x, t_0)\right]$ is another classical solution. Therefore $w(x, t) = \eta(x)\partial_t u(x, t)$ by the uniqueness.

Using the Green function $G(x, t; y, s)$ corresponding to the differential operator $\mathcal{L}_\mathcal{D}$, the solution $w(x, t)$ can be represented as

(3.7.6)
$$w(x, t) = \int_{\mathbb{R}} dy\, G(x, t; y, t_0)w(y, t_0) + \int_{t_0}^t ds \int_{\mathbb{R}} dy\, G(x, t; y, s)(\eta(y)h(y, s) + \tilde{h}(y, s)),$$

for all $(x, t) \in \mathbb{R} \times (t_0, T]$. Since the Green function satisfies

$$|G(x, t; y, s)| \le C(t - s)^{-\frac{1}{2}} \exp\left(-\frac{c(x - y)^2}{t - s} \right), \quad (y, s) \in \mathbb{R} \times [0, t).$$

The first term in (3.7.6) is bounded, as long as $w(y, t_0)$ is uniformly bounded. The contribution of $\eta'\partial_x\partial_t u$ (in the expression for $\tilde{h}$) to w is given by,

$$-\int_{\mathbb{R}} dy\, G(x, t; y, s)\eta'(y)\frac{\partial^2}{\partial y \partial s}u(y, s) = \int_{\mathbb{R}} dy\frac{\partial}{\partial y}[G(x, t; y, s)\eta'(y)]\frac{\partial}{\partial s}u(y, s).$$

Now it follows from Lemma 3.2.8 that both $w(x, t_0)$ and $h(x, t)$ are uniformly bounded for $x \in \mathbb{R}, t \in [t_0, T]$. We also have that η' and η'' vanish outside $[X_0, 2X_0]$. Since $\lim_{x\to+\infty} G(x, t; y, s) = 0$ and it can easily be shown that $\lim_{x\to+\infty} \partial_y G(s, t; y, s) = 0$, the Dominated Convergence Theorem implies that

$$\lim_{x\to+\infty} w(x, t) = 0, \quad t \in (t_0, T].$$

Then the statement follows from the choice of η. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.7.2   Proof of Lemma 3.4.2

We will first establish a one to one correspondence between solutions of (3.4.2) and solutions of an integral equation of Volterra type.

Lemma 3.7.1. *(i) Let $G(x,t;y,s)$ be the Green function associated to the differential operator $\mathcal{L}_\mathcal{D}$ and let us consider the following nonlinear integral equation of Volterra type,*

(3.7.7)
$$\left(1 + \frac{1}{4}\sigma^2(b(t),t)\right)v(t) = -\int_{t_0}^t ds\, v(s)\frac{1}{2}\sigma^2(b(s),s)\,\partial_x G(b(t),t;b(s),s) + \sum_{i=1}^2 N_i(t),$$

where $t_0 \le t \le T$,

$$N_1(t) = \int_{b(t)}^{+\infty} dy\, \partial_x G(b(t),t;y,t_0)w(y,t_0) \quad and$$

$$N_2(t) = \int_{t_0}^t ds \int_{b(s)}^{+\infty} dy\, \partial_x G(b(t),t;y,s)h(y,s).$$

There exists a unique solution v to (3.7.7). The function $v(t)$ is continuous.

(ii) Let $w(x,t)$ be a classical solution of the equation (3.4.2) on $[t_0,T]$ with the initial condition $w(x,t_0) = \partial_t u(x,t_0)$, such that $t \to \partial_x w(b(t)+,t)$ is continuous. Then there is a one to one correspondence between $w(x,t)$ and $v(t)$. Moreover $\partial_x w(b(t)+,t) = v(t)$, $t_0 \le t \le T$.

The initial value of equation (3.4.2) may not be smooth. This is the reason we take $w(x,t_0) = \partial_t u(x,t_0)$, $0 < t_0 < T$, as the initial condition of (3.4.2) and consider the differential equation on $t \in [t_0,T]$.

Remark 3.7.2. The correspondence in Lemma 3.7.1 is well known for the Stefan problem on heat equation with Lipschitz continuous free boundary (see Section 1 Chapter 8 of Friedman (1964)). Along Friedman's line of proof, we will extend the correspondence to our parabolic differential equation with Hölder continuous free boundary.

Proof of Lemma 3.7.1.

Proof of (i). First, because $G(b(t), t; b(s), s)$ and $\sigma(b(s), s)$ are continuous for $s \in (0, t)$ (see (3.2.5)), it follows from the classical result on Volterra equations (see Rust (1934)) that the integral equation (3.7.7) has a unique solution $v(t)$ and it is continuous with respect to $t \in [t_0, T]$, as long as $N_i(t)$, $i = 1, 2$, are continuous with respect to t. It is not hard to show these functions are indeed continuous, using the continuity of $b(t)$ and the following estimates on the Green function G and its derivatives:

$$|\partial_x^\ell G(x, t; y, s)| \leq C(t - s)^{-\frac{1+\ell}{2}} \exp\left(-c\frac{|x - y|^2}{t - s}\right),$$

$$|\partial_x G(x, t; y, s) - \partial_x G(x, \tilde{t}; y, s)| \leq C(t - \tilde{t})^{\frac{\alpha}{2}}(\tilde{t} - s)^{-\frac{2+\alpha}{2}} \exp\left(-c\frac{|x - y|^2}{t - s}\right),$$

$$|\partial_x G(x, t; y, s) - \partial_{\tilde{x}} G(\tilde{x}, t; y, s)| \leq C|x - \tilde{x}|^\alpha (t - s)^{-\frac{2+\alpha}{2}} \exp\left(-c\frac{|x'' - y|^2}{t - s}\right),$$

where $\ell = 0, 1$, $s < \tilde{t} < t$, $|x'' - y| = |x - y| \wedge |\tilde{x} - y|$, $0 < \alpha < 1$, C and c are positive constants. These estimates are from Theorem 16.3 in page 413 of Ladyženskaja et al. (1968).

Proof of (ii) Let us assume that $w(x, t)$ is a classical solution of (3.4.2). As a result, the following Green's identity (see page 27 of Friedman (1964)) is satisfied

$$\frac{\partial}{\partial y}\left(\frac{1}{2}\sigma^2(y, s)G(x, t; y, s)\frac{\partial}{\partial y}w(y, s) - \frac{1}{2}\sigma^2(y, s)w(y, s)\frac{\partial}{\partial y}G(x, t; y, s)\right.$$

(3.7.8) $$\left. -w(y, s)G(x, t; y, s)\sigma\sigma_y(y, s)\right) - \frac{\partial}{\partial s}\left(G(x, t; y, s)w(y, s)\right)$$

$$+ \frac{\partial}{\partial y}\left(\left(\mu - \frac{1}{2}\sigma^2(y, s)\right)G(x, t; y, s)w(y, s)\right) = -G(x, t; y, s)h(y, s),$$

where $t_0 \leq s < t \leq T$, $x > b(t)$ and $y > b(s)$. Integrating both hand side of (3.7.8)

over the domain $b(s) < y < +\infty$, $t_0 < s < t - \epsilon$, we obtain

(3.7.9)

$$\int_{t_0}^{t-\epsilon} ds \lim_{y \to +\infty} \frac{1}{2}\sigma^2(y,s)\,\partial_y w(y,s)\,G(x,t;y,s) - \int_{t_0}^{t-\epsilon} ds \frac{1}{2}\sigma^2(b(s),s)\,\partial_y w(b(s)+,s)\,G(x,t;b(s),s)$$

$$- \int_{t_0}^{t-\epsilon} ds \lim_{y \to +\infty} \frac{1}{2}\sigma^2(y,s)\,w(y,s)\,\partial_y G(x,t;y,s) + \int_{t_0}^{t-\epsilon} ds \frac{1}{2}\sigma^2(b(s),s)\,w(b(s),s)\,\partial_y G(x,t;b(s),s)$$

$$- \int_{t_0}^{t-\epsilon} ds \lim_{y \to +\infty} w(y,s)\,G(x,t;y,s)\,\sigma\sigma_y(y,s) + \int_{t_0}^{t-\epsilon} ds\, w(b(s),s)\,G(x,t;b(s),s)\,\sigma\sigma_y(b(s),s)$$

$$- \int_{b(t-\epsilon)}^{+\infty} dy\,[G(x,t;y,t-\epsilon)w(y,t-\epsilon) - G(x,t;y,t_0)w(y,t_0)]$$

$$+ \int_{t_0}^{t-\epsilon} ds \left[\lim_{y \to +\infty} \left(\mu - \frac{1}{2}\sigma^2(y,s) \right) w(y,s)\,G(x,t;y,s) \right.$$

$$\left. - \left(\mu - \frac{1}{2}\sigma^2(b(s),s) \right) w(b(s),s)\,G(x,t;b(s),s) \right]$$

$$= - \int_{t_0}^{t-\epsilon} ds \int_{b(s)}^{+\infty} dy\,G(x,t;y,s)h(y,s).$$

In the seventh term on the left of (3.7.9), we used $w(x,t) = 0$ when $x < b(t)$. Using the boundary and initial conditions for $w(x,t)$ and the facts that $\lim_{y \to +\infty} G(x,t;y,s) = 0$ and $\lim_{y \to +\infty} \partial_y G(x,t;y,s) = 0$, letting $\epsilon \to 0$, we can write

(3.7.10)

$$w(x,t) = - \int_{t_0}^{t} ds\,\partial_x w(b(s)+,s)\,\frac{1}{2}\sigma^2(b(s),s)\,G(x,t;b(s),s) + \int_{b(t)}^{+\infty} dy\,G(x,t;y,t_0)w(y,t_0)$$

$$+ \int_{t_0}^{t} ds \int_{b(s)}^{+\infty} dy\,G(x,t;y,s)h(y,s)$$

$$\triangleq -M_0(x,t) + M_1(x,t) + M_2(x,t).$$

Before differentiating both sides of (3.7.10) with respect to x, let us recall the jump identity: if $\rho(t)$, $t_0 \le t \le T$, is a continuous function and $b(t)$ is the Hölder continuous with Hölder exponent $\alpha > \frac{1}{2}$, then for every $t_0 \le t \le T$,

(3.7.11)

$$\lim_{x \downarrow b(t)} \frac{\partial}{\partial x} \int_{t_0}^{t} ds\,\rho(s)G(x,t;b(s),s) = \frac{1}{2}\rho(t) + \int_{t_0}^{t} ds\,\rho(s)\,\partial_x G(x,t;b(s),s)|_{x=b(t)}.$$

This identity can be proved in the similar way as in Lemma 1 in Chapter 8 of Friedman (1964). As commented in the paragraph after Lemma 4.5 in Friedman (1975), the proof of Lemma 1 can go through when we replace Lipschitz free boundary with Hölder continuous free boundary with the Hölder exponent $\alpha > \frac{1}{2}$.

Now we will take the derivative of (3.7.10) with respect to x to obtain

$$(3.7.12) \qquad \frac{\partial}{\partial x} w(x,t) = \sum_{i=0}^{2} \frac{\partial}{\partial x} M_i(x,t)$$

and let $x \downarrow b(t)$. Since $\partial_x w(b(s)+, s)$ and $\sigma(b(s), s)$, $t_0 \leq s < t$, are continuous and $b(t)$ is Hölder continuous with exponent $\alpha > \frac{1}{2}$ (see Theorem 3.3.7), taking $\rho(s) = \frac{1}{2}\sigma^2(b(s), s) \, \partial_x w(b(s)+, s)$ in (3.7.11), we obtain

(3.7.13)

$$\lim_{x \downarrow b(t)} \frac{\partial}{\partial x} M_0(t) = \lim_{x \downarrow b(t)} \frac{\partial}{\partial x} \int_{t_0}^{t} ds \, \frac{1}{2}\sigma^2(b(s), s) \, \partial_x w(b(s)+, s) G(x, t; b(s), s)$$

$$= \frac{1}{4}\sigma^2(b(t), t) \, \partial_x w(b(t)+, t) + \int_{t_0}^{t} ds \, \frac{1}{2}\sigma^2(b(s), s) \, \partial_x w(b(s)+, s) \, \partial_x G(b(t), t; b(s), s).$$

On the other hand, by Lemmas 3.2.6 and 3.2.8, $w(y, t_0)$ and $h(y, s)$ are bounded in $\mathbb{R} \times [t_0, T]$. Using the Dominated Convergence Theorem we get

$$(3.7.14) \quad \lim_{x \downarrow b(t)} \frac{\partial}{\partial x} M_1(x,t) = \int_{b(t)}^{+\infty} dy \, \partial_x G(b(t), t; y, t_0) w(y, t_0) \triangleq N_1(t),$$

$$(3.7.15) \quad \lim_{x \downarrow b(t)} \frac{\partial}{\partial x} M_2(x,t) = \int_{t_0}^{t} ds \int_{b(s)}^{+\infty} dy \, \partial_x G(b(t), t; y, s) h(y, s) \triangleq N_2(t),$$

It follows from (3.7.12) - (3.7.15) that $\partial_x w(b(t)+, t)$ satisfies (3.7.7).

Let us prove the converse. For any solution $v(t)$ of the integral equation (3.7.7), we can define $w(x, t)$ as follows

(3.7.16)

$$w(x,t) := -\int_{t_0}^{t} ds \, v(s) \frac{1}{2}\sigma^2(b(s), s) \, G(x, t; b(s), s) + \int_{b(t)}^{+\infty} dy \, G(x, t; y, t_0) w(y, t_0)$$

$$+ \int_{t_0}^{t} ds \int_{b(s)}^{+\infty} dy \, G(x, t; y, s) h(y, s), \quad t_0 \leq t \leq T, x \geq b(t),$$

and $w(x, t_0) := \partial_t u(x, t_0)$. We will show in the following that $w(x, t)$ is a classical solution of (3.4.2) and that $t \rightarrow \partial_x w(b(t)+, t)$ is continuous.

Now we will show that $w(x, t)$ defined in (3.7.16) is a classical solution of (3.4.2) on $[t_0, T]$ with initial condition $\partial_t u(x, t_0)$. By definition $w(x, t_0) = \partial_t u(x, t_0)$. On the other hand we have that $\lim_{x \to +\infty} w(x, t) = 0$, which follows from the facts that $\lim_{x \to +\infty} G(x, t; y, t_0) = 0$ and σ, $v(s)$, $w(y, t_0)$ and $h(y, s)$ are all bounded. Furthermore, using the properties of the Green function and the definition of w (see 3.7.16), we also have that $\mathcal{L}_{\mathcal{D}} w(x, t) = h(x, t)$ for $x > b(t)$, $t \in [t_0, T]$. Observe that $\partial_t w$, $\partial_x w$ and $\partial_x^2 w$ all exist and are all continuous in this domain.

In the following we will show that $\partial_x w(b(t)+, t) = v(t)$, which implies the continuity of $\partial_x w(b(t)+, t)$. We differentiate $w(x, t)$ with respect to x and let $x \downarrow b(t)$. Since $v(t)$ and σ are continuous and $b(t)$ is Hölder continuous with exponent $\alpha > \frac{1}{2}$, we can apply the jump identity (3.7.11) with $\rho(s) = \frac{1}{2}\sigma^2(b(s), s)v(s)$. Following the steps that lead to (3.7.7) in the first part of the proof, we obtain

(3.7.17)
$$\partial_x w(b(t)+, t) = -\frac{1}{4}\sigma^2(b(t), t)\, v(t) - \int_{t_0}^t ds\, v(s)\frac{1}{2}\sigma^2(b(s), s)\, \partial_x G(b(t), t; b(s), s) + \sum_{i=1}^2 N_i(t).$$

Comparing (3.7.17) to (3.7.7), we see that $\partial_x w(b(t)+, t) = v(t)$, $t_0 \leq t \leq T$.

Then it remains to show that $w(b(t), t) = 0$, $t_0 \leq t \leq T$. To this end, since we have already shown $\mathcal{L}_{\mathcal{D}} w = h$, w satisfies the Green's identity given by (3.7.8). Integrating the identity (3.7.8) and using (3.7.16) and the fact that $\lim_{x \to +\infty} w(x, t) = 0$ we can write

(3.7.18)
$$\int_{t_0}^t ds\, w(b(s), s) \left[\left(\frac{1}{2}\sigma^2(b(s), s) + \sigma\sigma_x(b(s), s) \right) \partial_y G(x, t; b(s), s) \right.$$
$$\left. - \left(\mu - \frac{1}{2}\sigma^2(b(s), s) \right) G(x, t; b(s), s) \right] = 0, \quad x > b(t), t_0 \leq t \leq T.$$

Let $x > b(t)$. Integrating both sides of (3.7.18) on $[x, +\infty)$ and using the fact that $\partial_x G = -\partial_y G$, we obtain

$$
\begin{aligned}
0 &= \int_{t_0}^t ds\, w(b(s), s) \left[-\left(\frac{1}{2}\sigma^2(b(s), s) + \sigma\sigma_x \right) \int_x^{+\infty} du\, \partial_x G(u, t; b(s), s) \right. \\
&\qquad\qquad \left. -\left(\mu - \frac{1}{2}\sigma^2(b(s), s) \right) \int_x^{+\infty} du\, G(u, t; b(s), s) \right] \\
&= \int_{t_0}^t ds\, w(b(s), s) \left[\left(\frac{1}{2}\sigma^2(b(s), s) + \sigma\sigma_x \right) G(x, t; b(s), s) \right. \\
&\qquad\qquad \left. -\left(\mu - \frac{1}{2}\sigma^2(b(s), s) \right) \int_x^{+\infty} du\, G(u, t; b(s), s) \right].
\end{aligned}
$$

Taking the derivative with respect to x, letting $x \downarrow b(t)$ and using the jump identity (3.7.11) with

$\rho(s) = \left(\frac{1}{2}\sigma^2(b(s), s) + \sigma\sigma_x \right) w(b(s), s)$, we arrive at

$$
\begin{aligned}
&\frac{1}{2} \left(\frac{1}{2}\sigma^2(b(s), s) + \sigma\sigma_x(b(s), s) \right) w(b(t), t) \\
(3.7.19) \qquad &= \int_{t_0}^t ds\, w(b(s), s) \left[\left(\frac{1}{2}\sigma^2(b(s), s) + \sigma\sigma_x(b(s), s) \right) \partial_y G(b(t), t; b(s), s) \right. \\
&\qquad\qquad \left. -\left(\mu - \frac{1}{2}\sigma^2(b(s), s) \right) G(b(t), t; b(s), s) \right].
\end{aligned}
$$

Since $b(t)$ is Hölder continuous with exponent $\alpha > 1/2$, we have

$$
|\partial_y G(b(t), t; b(s), s)| \leq \frac{C}{(t-s)^{\frac{3}{2}-\alpha}}.
$$

Therefore both $\partial_y G(b(t), t; b(s), s)$ and $G(b(t), t; b(s), s)$ are integrable. Consequently, it follows from (3.7.18), (3.7.19) and the Dominated Convergence Theorem that $w(b(t), t) = 0$, $t_0 \leq t \leq T$. $\qquad\square$

Proof of Lemma 3.4.2. Proof of (i). Let $v(t)$ be the unique continuous solution of the Volterra equation (3.7.7). Define $w(x, t)$ as in (3.7.16). The Lemma 3.7.1 shows that $w(x, t)$ is a classical solution to equation (3.4.2). Let us define

$$
\tilde{u}(x, t) = u(x, t_0) + \int_{t_0}^t w(x, s)ds, \quad x \geq b(t), t_0 \leq t \leq T.
$$

It is easy to check that $\tilde{u}(x,t)$ is a classical solution of the equation (3.2.6) with initial condition $u(x,t_0)$. Since (3.2.6) has a unique solution, we conclude that $u(x,t) = \tilde{u}(x,t)$, $x \geq b(t)$ and $t_0 \leq t \leq T$. Lemma 3.7.1 also implies that

$$\partial_x \partial_t u(b(t)+,t) = \partial_x w(b(t)+,t) = v(t), \quad t_0 \leq t \leq T,$$

which implies that $\partial_x \partial_t u(b(t)+,t)$, $t_0 \leq t \leq T$, is continuous. The statement follows since $t_0 > 0$ is arbitrary.

Proof of (ii). Let (x,t) be such that $x > b(t)$. Choosing $t_0 < t$ such that $b(t_0) < x$, we can see that $\int_{t_0}^{t} ds \partial_x G(x,t;b(s),s) < +\infty$. As a result, we have

$$\frac{\partial}{\partial x} M_0(x,t) = \int_{t_0}^{t} ds \, \frac{1}{2} \sigma^2(b(s),s) \, \partial_x w(b(s)+,s) \partial_x G(x,t;b(s),s).$$

We have shown in part (i) that $\partial_x w(b(s)+,s)$ is continuous with respect to s. It is easy to show $\partial_x M_0(x,t)$ is continuous around a sufficiently small neighborhood of (x,t). One can also show that the functions $\partial_x M_i(x,t)$, $i \in \{1,2\}$ are also continuous by similar means. Thus, it is clear from (3.7.12) that $\partial_x \partial_t u(x,t)$ is continuous in this small neighborhood around (x,t). Therefore, the part (ii) of Lemma 3.4.2 follows, because of the arbitrary choice of x and t. $\square$

3.7.3 Proof of Proposition 3.6.1 (i)

We will use the following result in Lemma 4.1 in page 239 of Friedman (1976):

Lemma 3.7.3. *For any $a < b < \log K$, $0 < t_1 < t_2 < T$, if both $u(x,t)$ and $\partial_t u(x,t)$ belong to $L^2((t_1,t_2); L^2(a,b))$, then $u(t)$ belongs to $C((t_1,t_2); L^2(a,b))$.*

In this lemma, $L^2((t_1,t_2); L^2(a,b))$ is the class of L^2 maps which map $t \in (t_1,t_2)$ to the Hilbert space $L^2(a,b)$. On the other hand $C((t_1,t_2); L^2(a,b))$ is the class of continuous maps which map $t \in (t_1,t_2)$ to $L^2(a,b)$.

The proof of (3.6.5) is similar to that of (3.2.10): First, we will study the penalty problem associated to the free boundary problem (3.6.1) - (3.6.3). Then, we will list some key estimates for the solution of the penalty problem. And finally using Lemma 3.7.3 we will conclude. We will give a sketch of this proof below.

Let us consider the following penalty problem

(3.7.20)
$$\mathcal{L}_{\mathcal{D}} u_n^\epsilon + \beta_\epsilon(u_n^\epsilon - g_\epsilon) = f_n^\epsilon(x,t), \quad x \in \mathbb{R}, \, 0 < t < T,$$

$$u_n^\epsilon(x,0) = g_\epsilon(x), \quad x \in \mathbb{R},$$

in which $0 < \epsilon < 1$, $g_\epsilon(x) \in C^\infty(\mathbb{R})$ such that $g_\epsilon(x) = (K - e^x)^+$ when x satisfies $|K - e^x| \geq \epsilon$. We define $f_n^\epsilon(x,t) = \zeta_\epsilon * f_n(x,t)$, where ζ_ϵ is the standard mollifier in x and t (see Evans (1998) Appendix C4 in page 629). As a result, we have $f_n^\epsilon(x,t) \in C^\infty(\mathbb{R} \times (0,T))$. Moreover, because $f_n(x,t)$ is continuous, $f_n^\epsilon(x,t)$ uniformly converge to $f_n(x,t)$ on any compact domains as $\epsilon \to 0$. On the other hand, from our assumption that $\partial_t u_{n-1}(x,t)$ is bounded for any $\epsilon > 0$ and ν is a probability measure on R, we obtain that

(3.7.21) $\partial_t f_n(x,t)$ is bounded in $\mathbb{R} \times [\epsilon, T]$,   for any $\epsilon > 0$.

Thanks to (3.7.21), it is easy to see that $\partial_t f_n^\epsilon(x,t)$ are uniformly bounded for any $\epsilon > 0$. The penalty functions $\beta_\epsilon(x)$ is a sequence of infinitely differentiable, negative, increasing and concave functions such that $\beta_\epsilon(0) = -C_\varepsilon \leq -(r + \lambda)K - r\epsilon$. The limit of the sequence is

$$\lim_{\epsilon \to 0} \beta_\epsilon(x) = \begin{cases} 0, & x \geq 0, \\ -\infty, & x < 0. \end{cases}$$

It is well known that the penalty problem has a classical solution (see page 1009 of Friedman and Kinderlehrer (7475)). Moreover, a proof similar to that of the proof

of Theorem 2.1 of Yang et al. (2006) shows that $u_n^\epsilon(x,t) \in C^\infty(\mathbb{R} \times (0,T)) \cap L^\infty(\mathbb{R} \times (0,T))$.

On the other hand, $u_n^\epsilon(x,t)$ satisfy the following estimates for any $a < b < \log K$, $0 < t_1 < t_2 \le T$,

$$(3.7.22) \qquad \int_a^b \left(\frac{\partial u_n^\epsilon}{\partial t}\right)^2 (x,t)dx \le C, \quad t \in [t_1, t_2],$$

$$(3.7.23) \qquad \int_{t_1}^{t_2} \int_a^b \left(\frac{\partial^2 u_n^\epsilon}{\partial x \partial t}\right)^2 dxdt \le C,$$

$$(3.7.24) \qquad \int_{t_1}^{t_2} \int_a^b \left(\frac{\partial^2 u_n^\epsilon}{\partial t^2}\right)^2 dxdt + \int_a^b \left(\frac{\partial^2 u_n^\epsilon}{\partial x \partial t}\right)^2 (x,t)dx \le C, \quad t \in [t_1, t_2],$$

in which C is a constant independent of ϵ. These estimates use similar techniques to the ones used in the proofs of Lemmas 2.8, 2.10 and 2.11 in Yang et al. (2006), since $f_n(x,t)$ satisfies (3.7.21). (Similar estimates can also be found in Friedman and Kinderlehrer (7475)). We will give the proof for the inequality (3.7.24) below. The other inequalities can be similarly obtained.

Proof of inequality (3.7.24). Let us consider $w_n(x,t) = \partial_t u_n^\epsilon(x,t)$. Since $u_n^\epsilon(x,t) \in C^\infty(\mathbb{R} \times (0,T))$, it follows from (3.7.20) that $w_n(x,t)$ satisfies

$$(3.7.25) \qquad \mathcal{L}_\mathcal{D} w_n + \beta_\epsilon'(u_n^\epsilon - g_\epsilon)w_n = \frac{\partial}{\partial t} f_n^\epsilon(x,t).$$

Let $\eta(x,t) \in C_0^\infty(\mathbb{R} \times (0,T))$, such that $\eta(x,t) = 1$ for $(x,t) \in [a,b] \times [t_1, t_2]$, and $\eta(x,t) = 0$ outside a small neighborhood of $[a,b] \times [t_1, t_2]$. Multiplying both sides of (3.7.25) by $\eta^2 \partial_t w_n$ and integrating over the domain $\Omega_t = \mathbb{R} \times (0,t)$ in which

$t_1 \leq t \leq t_2$, we obtain

$$
\begin{aligned}
0 &= \int\int_{\Omega_t} \eta^2 \left(\frac{\partial w_n}{\partial t}\right)^2 dxds - \int\int_{\Omega_t} \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 w_n}{\partial x^2}\frac{\partial w_n}{\partial t} dxds \\
&\quad - \int\int_{\Omega_t} \left(\mu - \frac{1}{2}\sigma^2\right)\eta^2 \frac{\partial w_n}{\partial x}\frac{\partial w_n}{\partial t} dxds + (r+\lambda)\int\int_{\Omega_t} \eta^2 w_n \frac{\partial w_n}{\partial t} dxds \\
&\quad + \int\int_{\Omega_t} \eta^2 \beta_\epsilon'(u_n^\epsilon - g_\epsilon) w_n \frac{\partial w_n}{\partial t} dxds - \int\int_{\Omega_t} \eta^2 \frac{\partial w_n}{\partial t}\frac{\partial}{\partial t} f_n^\epsilon(x,s) dxds \\
&\triangleq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned}
$$

where I_j is the j-th term on the left and $\sigma = \sigma(x,t)$ satisfying the assumption (3.2.5). In the following, we will estimate each I_j separately. In deriving these estimates we will make use of the inequality

$$
(3.7.26) \qquad\qquad \frac{1}{6}A^2 + AB + \frac{9}{6}B^2 \geq 0,
$$

for any $A, B \in \mathbb{R}$. In the following estimations, C will represent different constants independent of ϵ.

$$
\begin{aligned}
I_2 &= -\frac{1}{2}\int\int_{\Omega_t} \sigma^2\eta^2 \frac{\partial^2 w_n}{\partial x^2}\frac{\partial w_n}{\partial t} dxds \\
&= \frac{1}{2}\int\int_{\Omega_t} \sigma^2\eta^2 \frac{\partial w_n}{\partial x}\frac{\partial^2 w_n}{\partial x\partial t} dxds + \int\int_{\Omega_t} \sigma\eta \frac{\partial\sigma\eta}{\partial x}\frac{\partial w_n}{\partial x}\frac{\partial w_n}{\partial t} dxds \\
&= \frac{1}{4}\int\int_{\Omega_t} \sigma^2\eta^2 \frac{\partial}{\partial t}\left(\frac{\partial w_n}{\partial x}\right)^2 dxds + \int\int_{\Omega_t} \sigma\eta \frac{\partial\sigma\eta}{\partial x}\frac{\partial w_n}{\partial x}\frac{\partial w_n}{\partial t} dxds \\
&= \frac{1}{4}\int_{\mathbb{R}} \sigma^2\eta^2 \left(\frac{\partial w_n}{\partial x}\right)^2 (x,t)\, dx - \frac{1}{2}\int\int_{\Omega_t} \sigma\eta \frac{\partial\sigma\eta}{\partial t}\left(\frac{\partial w_n}{\partial x}\right)^2 dxds \\
&\quad + \int\int_{\Omega_t} \sigma\eta \frac{\partial\sigma\eta}{\partial x}\frac{\partial w_n}{\partial x}\frac{\partial w_n}{\partial t} dxds \\
&\geq \frac{\delta^2}{4}\int_{\mathbb{R}} \eta^2 \left(\frac{\partial w_n}{\partial x}\right)^2 (x,t)dx - \frac{1}{2}\left|\sigma\eta \frac{\partial\sigma\eta}{\partial t}\right|_{L^\infty} \int\int_{\Omega_t}\left(\frac{\partial^2 u_n^\epsilon}{\partial x\partial t}\right)^2 dxds \\
&\quad - \frac{9}{6}\int\int_{\Omega_t}\left(\sigma\frac{\partial\sigma\eta}{\partial x}\right)^2\left(\frac{\partial^2 u_n^\epsilon}{\partial x\partial t}\right)^2 dxds - \frac{1}{6}\int\int_{\Omega_t}\eta^2\left(\frac{\partial w_n}{\partial t}\right)^2 dxds \\
&\geq \frac{\delta^2}{4}\int_{\mathbb{R}} \eta^2\left(\frac{\partial w_n}{\partial x}\right)^2 (x,t)dx - C - \frac{1}{6}\int\int_{\Omega_t}\eta^2\left(\frac{\partial w_n}{\partial t}\right)^2 dxds.
\end{aligned}
$$

The first four equalities follow from integration by part. The first inequality follows

from the assumption (3.2.5) and the inequality (3.7.26) with $A = \eta \frac{\partial w_n}{\partial t}$ and $B = \sigma \frac{\partial \sigma \eta}{\partial x} \frac{\partial w_n}{\partial x}$. The last inequality follows from estimation (3.7.23).

For I_i (i=3, 4, 5), a similar procedure yields

$$I_3 \geq -C - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left(\frac{\partial w_n}{\partial t} \right)^2 dx ds,$$

$$I_4 \geq -C - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left(\frac{\partial w_n}{\partial t} \right)^2 dx ds,$$

$$I_5 \geq -C - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left(\frac{\partial w_n}{\partial t} \right)^2 dx ds.$$

For I_6, we have

$$I_6 = -\int \int_{\Omega_t} \eta^2 \frac{\partial w_n}{\partial t} \frac{\partial}{\partial t} f_n^\epsilon dx ds \geq -\frac{9}{6} \int \int_{\Omega_t} \eta^2 \left(\frac{\partial}{\partial t} f_n^\epsilon \right)^2 dx ds - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left(\frac{\partial w_n}{\partial t} \right)^2 dx ds$$

$$\geq -C - \frac{1}{6} \int \int_{\Omega_t} \eta^2 \left(\frac{\partial w_n}{\partial t} \right)^2 dx ds.$$

The first inequality can be obtained using (3.7.26), whereas to obtain the last inequality, we use the fact that $\partial_t f_n^\epsilon(x, t)$ is uniformly bounded. Combining all these estimates for I_j, we obtain

$$\frac{1}{6} \int \int_{\Omega_t} \eta^2 \left(\frac{\partial w_n}{\partial t} \right)^2 dx ds + \frac{\delta^2}{4} \int_{\mathbb{R}} \eta^2 \left(\frac{\partial w_n}{\partial x} \right)^2 (x, t) dx \leq C.$$

This completes the proof of (3.7.24). □

Using a similar proof to that of Lemma 2.2 of Yang et al. (2006), we can show that $u_n^\epsilon(x, t)$ is uniformly bounded. Thus there is a subsequence that $\{u_n^{\epsilon_k}\}$ converges weakly to u_n in $L^2((a, b); L^2(t_1, t_2))$ for any $a < b < \log K$, $0 < t_1 < t < t_2 < T$ (see Appendix D in Evans (1998) for an account of the concept of weak convergence). On the other hand, it follows from the estimates in (3.7.22) - (3.7.24) that $\frac{\partial u_n^\epsilon}{\partial t}$ and $\frac{\partial^2 u_n^\epsilon}{\partial x \partial t}$ are uniformly bounded in $L^2(a, b)$, $\frac{\partial^2 u_n^\epsilon}{\partial x \partial t}$ and $\frac{\partial^2 u_n^\epsilon}{\partial t^2}$ are uniformly bounded in $L^2((t_1, t_2); L^2(a, b))$. Therefore there exists a further subsequence satisfying

$$\frac{\partial u_n^{\epsilon_{k_j}}}{\partial t} \rightharpoonup \frac{\partial u_n}{\partial t}, \quad \frac{\partial^2 u_n^{\epsilon_{k_j}}}{\partial x \partial t} \rightharpoonup \frac{\partial^2 u_n}{\partial x \partial t}, \quad \frac{\partial^2 u_n^{\epsilon_{k_j}}}{\partial t^2} \rightharpoonup \frac{\partial^2 u_n}{\partial t^2},$$

where derivatives of u_n are defined in weak sense (see Appendix D in Evans (1998)). Here, the convergences are weak convergences. Since $||u|| \leq \liminf_j ||u_n^{\epsilon_{k_j}}||$ (see Appendix D in Evans (1998)) (3.7.22) - (3.7.24) imply that

$$\frac{\partial u_n}{\partial t} \in L^\infty((t_1, t_2); L^2(a, b)), \quad \frac{\partial^2 u_n}{\partial t^2} \in L^2((t_1, t_2); L^2(a, b)).$$

Then it follows from Lemma 3.7.3 that the derivative $\partial_t u_n$ exists and is inside the space $C((t_1, t_2); L^2(a, b))$. On the other hand, for fixed $t \in [t_1, t_2]$, it also follows from (3.7.22) and (3.7.24) and the Sobolev Embedding Theorem (see, for example, Theorem 4 in page 266 of Evans (1998)) that

$$(3.7.27) \qquad \left| \frac{\partial u_n}{\partial t}(x, t) - \frac{\partial u_n}{\partial t}(\bar{x}, t) \right| \leq C|x - \bar{x}|^{1/2}, \quad x, \bar{x} \in (a, b),$$

in which C is a positive constant that does not depend on t. We already know that $\partial_t u_n(\cdot, t)$ is a continuous map with respect to t, therefore (3.7.27) implies that

$$\frac{\partial u_n}{\partial t} \in C((a, b) \times (t_1, t_2)).$$

Therefore $\partial_t u_n \in C(\mathbb{R} \times (0, T])$ because the choice of a, b, t_1 and t_2 are arbitrary and $\partial_t u_n \in C([\log K, +\infty) \times (0, T])$ since $[\log K, +\infty) \times (0, T] \in \mathcal{C}_n$. Moreover, we have

$$(3.7.28) \qquad \lim_{x \downarrow b(t_0)} \frac{\partial u_n}{\partial t}(x, t_0) = \lim_{t \to t_0^-} \frac{\partial u_n}{\partial t}(b(t_0), t) = 0,$$

because $(b_n(t_0), t)$ is inside the stopping region for $t < t_0$ as $b_n(t)$ is decreasing.

$\square$

CHAPTER IV

Pricing American options for jump diffusions

4.1 Introduction

Jump diffusion models are heavily used in modelling stock prices since they can capture the excess kurtosis and skewness of the stock price returns, and they can produce the smile in the implied volatility curve (see Cont and Tankov (2004)). Two well-known examples of these models are i) the model of Merton (1976), in which the jump sizes are log-normally distributed, and ii) the model of Kou and Wang (2004), in which the logarithm of jump sizes have the so called double exponential distribution. Based on the results of Bayraktar (2009) we propose a numerical algorithm to calculate the American option prices for jump diffusion models and analyze the convergence behavior of this algorithm.

As observed by Bayraktar (2009), we can construct an increasing sequence of functions, which are value functions of optimal stopping problems (see (4.2.8) and also (4.2.11)), that converge to the price function of the American put option uniformly and exponentially fast. Because each element of this sequence solves an optimal stopping problem it shares the same regularity properties, such as convexity and smoothness, with the original price function. Even the corresponding free boundaries have the same smoothness properties (when they have a discontinuity,

which can only happen at maturity, the magnitude of the discontinuity is the same).
Therefore, the elements in this approximating sequence provide a good imitation to
the value function besides being close to it numerically (see Remark 4.2.1). On the
other hand, each of these functions can be represented as classical solutions of free
boundary problems (see (4.2.9)) for geometric Brownian motion, and therefore can
be implemented using classical finite difference methods. We build an iterative nu-
merical algorithm based on discretizing these free boundary problems (see (4.3.10)).
When the mesh sizes are fixed, we show that the iterative sequence we constructed is
monotonous and converges uniformly and exponentially fast (see Proposition 4.3.4).
We also show, in a rather direct way, that when the mesh sizes go to zero our algo-
rithm converges to the true price function (see Proposition 4.3.6).

The pricing in the context of jump models is difficult since the prices of options
satisfy integro-partial differential equations (integro-pdes), i.e. they have non-local
integral terms, and the usual finite-difference methods are not directly applicable
because the integral term leads to full matrices. Recently there has been a lot
of interest in developing numerical algorithms for pricing in jump models, see e.g.
Aitsahlia and Runnemo (2007), Almendral and Oosterlee (2007), Andersen and An-
dreasen (2000), Cont and Voltchkova (2005), d'Halluin et al. (2004), Hirsa and Madan
(2004), Jackson et al. (2008), Kou et al. (2005), Kou and Wang (2004), Metwally and
Atiya (2003), Zhang (1994), among them Almendral and Oosterlee (2007), Cont and
Voltchkova (2005), Hirsa and Madan (2004) and Jackson et al. (2008) treated specific
or general jump models with infinite activity jumps. These algorithms have been ex-
tensively discussed in Section 12 of Cont and Tankov (2004). In this chapter, relying
on the results of Bayraktar (2009) as described above, we give an efficient numerical
algorithm (and analyze its error versus accuracy characteristics) to efficiently com-

pute American option prices for jump diffusion models with finite activity. One can handle infinite activity models by increasing the volatility coefficient appropriately as suggested on p. 417 of Cont and Tankov (2004).

An ideal numerical algorithm, which is most often an iterative scheme, *should monotonically converge to the true price uniformly (across time and space) and exponentially fast*, that is, the error bounds should be very tight. This is the only way one can be sure that the price output of the algorithm is close to the true price after a reasonable amount of runtime and without having to compare the price obtained from the algorithm to other algorithms' output. It is also desirable to obtain a scheme **that does not deviate from the numerical pricing schemes, such as finite difference methods, that were developed for models that do not account for jumps**. Financial engineers working in the industry are already familiar with finite difference schemes such as projected successive over relaxation, PSOR, (see e.g. Wilmott et al. (1995)) and Brennan-Schwartz algorithm (see Brennan and Schwartz (1977) and Jaillet et al. (1990)) to solve the partial differential equations associated with free boundary problems, but may not be familiar with the intricacies involved in solving integro-partial differential equations developed in the literature. It would be ideal for them if they could use what they already know with only a slight modification to solve for the prices in a jump diffusion model. In this chapter, we develop an algorithm which establishes both * and **. We will name this algorithm, depending on which classical method we use to solve the sparse linear systems in (4.3.10), as either "Iterated PSOR" or "Iterated Brennan-Schwartz" in Section 4.4.

In the Section 4.2, we introduce a sequence of optimal stopping problems that approximate the price function of the American options, and discuss their properties. In Section 4.3, we introduce a numerical algorithm and analyze its convergence

properties. In the Section 4.4, we give numerical examples to illustrate the competitiveness of our algorithm and price American, Barrier and European options for the models of Kou and Wang (2004) and Merton (1976).

4.2 A sequence of optimal stopping problems for geometric Brownian motion approximating the American option price for jump diffusions

We will consider a jump diffusion model for the stock price S_t with $S_0 = S$, and assume that return process $X_t := \log(S_t/S)$, under the risk neutral measure, is given by

$$(4.2.1) \qquad dX_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t + \sum_{i=1}^{N_t} Z_i, \quad X_0 = 0.$$

In (5.4.35), $\mu = r + \lambda - \lambda\xi$, r is the risk-free rate, W_t is a Brownian motion, N_t is a Poisson process with rate λ independent of the Brownian motion, Z_i are independent and identically distributed, and come from a common distribution F on $\mathbb{R}$, that satisfies $\xi := \int_{\mathbb{R}} e^z F(dz) < \infty$. The last condition guarantees that the stock prices have finite expectation. We will assume that the volatility σ is strictly positive. The price function of the American put with strike price K is

$$(4.2.2) \qquad V(S,t) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\{e^{-r(\tau-t)}(K - S_\tau)^+ \big| S_t = S\},$$

in which $\mathcal{T}_{t,T}$ is the set of stopping times with respect to the filtration generated by X that belong to the interval $[t, T]$ (t it the current time, T is the maturity of the option). Instead of working with the pricing function V directly, which is the unique classical solution of the following integro-differential free boundary problem

(see Theorem 3.1 of Bayraktar (2009))

(4.2.3)

$$\frac{\partial}{\partial t}V(S,t) + \mathcal{A}V(S,t) + \lambda \cdot \int_{\mathbb{R}} V(e^z \cdot S, t)F(dz) - (r+\lambda) \cdot V(S,t) = 0 \quad S > s(t),$$

$$V(S,t) = K - S, \quad S \le s(t),$$

$$V(S,T) = (K - S)^+,$$

in which, $\mathcal{A}$ is the differential operator

(4.2.4)
$$\mathcal{A} := \frac{1}{2}\sigma^2 S^2 \frac{d^2}{dS^2} + \mu S \frac{d}{dS},$$

and $t \to s(t)$, $t \in [0, T]$, is the exercise boundary that needs to be determined along with the pricing function V; we will construct a sequence of pricing problems for the geometric Brownian motion

(4.2.5)
$$dS_t^0 = \mu S_t^0 dt + \sigma S_t^0 dW_t, \quad S_0^0 = S.$$

To this end, let us introduce a functional operator J, whose action on a test function $f : \mathbb{R}_+ \times [0, T] \to \mathbb{R}_+$ is the solution of the following pricing problem for the geometric Brownian motion: $(S_t^0)_{t \ge 0}$

(4.2.6)
$$Jf(S,t) = \sup_{\tau \in \tilde{\mathcal{T}}_{t,T}} \mathbb{E}\left\{ \int_t^\tau e^{-(r+\lambda)(u-t)}\lambda \cdot Pf(S_u^0, u)du + e^{-(r+\lambda)(\tau-t)}(K - S_\tau^0)^+ \big| S_t^0 = S \right\},$$

in which

(4.2.7)
$$Pf(S, u) = \int_{\mathbb{R}} f(e^z \cdot S, u)F(dz) = \mathbb{E}[f(e^Z S, u)], \quad S \ge 0,$$

for a random variable Z whose distribution is F, and $\tilde{\mathcal{T}}_{t,T}$ is the set of stopping times with respect to the filtration generated by W that take values in $[t, T]$. Let us define a sequence of pricing functions by

(4.2.8)
$$v_0(S,t) = (K - S)^+, \quad v_{n+1}(S,t) = Jv_n(S,t), \ n \ge 0, \quad \text{for all } (S,t) \in \mathbb{R}_+ \times [0, T].$$

For each $n \geq 1$, the pricing function v_n is the unique solution of the classical free-boundary problem (instead of a free boundary problem with an integro-diffential equation)

$$\frac{\partial}{\partial t} v_n(S, t) + \mathcal{A} v_n(S, t) - (r + \lambda) \cdot v_n(S, t) = -\lambda \cdot (P v_{n-1})(S, t), \quad S > s_n(t),$$

(4.2.9) $\quad v_n(S, t) = K - S, \quad S \leq s_n(t),$

$$v_n(S, T) = (K - S)^+,$$

in which $t \to s_n(t)$ is the free-boundary (the optimal exercise boundary) which needs to be determined (see Lemma 3.5 of Bayraktar (2009)). Now starting from v_0, we can calculate $\{v_n\}_{n \geq 0}$ sequentially. For v_n, the solution of (4.2.9) can be determined using a classical finite difference method (we use the Crank-Nicolson discretization along with Bernnan-Schwartz algorithm or PSOR in the the following sections) given that the function v_{n-1} is available. The term on the right-hand-side of (4.2.9) can be computed either using Monte-Carlo or a numerical integrator (we use the numerical integration with the Fast Fourier Transformation (FFT) in our examples). Iterating the solution for (4.2.9) a few times we are able to obtain the American option price V accurately since the sequence of functions $\{v_n\}_{n \geq 0}$ converges to V uniformly and exponentially fast:

(4.2.10)

$$v_n(S, t) \leq V(S, t) \leq v_n(S, t) + K \left(1 - e^{-(r + \lambda)(T - t)}\right)^n \left(\frac{\lambda}{\lambda + r}\right)^n, \quad S \in \mathbb{R}_+, \ t \in (0, T),$$

see Remark 3.3 of Bayraktar (2009). Note that the usual values of T for the traded options is 0.25, 0.5, 0.75, 1 year.

Remark 4.2.1. The approximating sequence $\{v_n\}_{n \geq 0}$ goes beyond approximating the value function V. Each v_n and its corresponding free boundary have the same regularity properties which V and its corresponding free boundary have. In a sense, for

large enough n, v_n provides a good imitation of V. Below we list these properties:

1) The function v_n can be written as the value function of an optimal stopping problem:

$$(4.2.11) \qquad v_n(S,t) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\{e^{-r(\tau \wedge \sigma_n - t)}(K - S_{\tau \wedge \sigma_n})^+ \big| S_t = S\},$$

in which σ_n is the n-th jump time of the Poisson process N_t.

2) Each v_n is a convex function in the S-variable, which is a property that is also shared by V. Moreover, the sequence $\{v_n\}_{n \geq 0}$ is a monotone increasing sequence converging to the value function V (see (4.2.10)).

3) The free boundaries $s(t)$ and $s_n(t)$ have the same regularity properties (see Chapter III):

 a) They are strictly decreasing.

 b) They may exhibit discontinuity at T: If the parameters satisfy

$$(4.2.12) \qquad\qquad r < \lambda \int_{\mathbb{R}_+} (e^z - 1)\, F(dz),$$

 we have

$$(4.2.13) \qquad\qquad \lim_{t \to T} s(t) = \lim_{t \to T} s_n(t) = S^* < K, \quad n \geq 1,$$

 where S^* is the unique solution of the following integral equation

$$(4.2.14) \qquad -rK + \lambda \int_{\mathbb{R}} \big[(K - Se^z)^+ - (K - Se^z)\big] F(dz) = 0.$$

 We will see such an example in Section 4.4, where the equation (4.2.14) can be solved analytically for some jump distribution F.

 c) Both $s(t)$ and $s_n(t)$ are continuously differentiable on $[0, T)$.

4.3 A numerical algorithm and its convergence analysis

4.3.1 The numerical algorithm

In this section, we will discretize the algorithm introduced in the last section and give more details. For the convenience of the numerical calculation, we will first change the variable: $x \triangleq \log S$, $x(t) \triangleq \log s(t)$ and $u(x,t) \triangleq V(S,t)$. u satisfies the following integro-differential free boundary problem

(4.3.1)
$$\frac{\partial}{\partial t}u + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x}u - (r+\lambda)u + \lambda \cdot (Iu)(x,t) = 0, \quad x > x(t)$$

$$u(x,t) = K - e^x, \quad x \leq x(t)$$

$$u(x,T) = (K - e^x)^+,$$

in which

(4.3.2)
$$(Iu)(x,t) = \int_{\mathbb{R}} u(x+z,t)\rho(z)dz,$$

with $\rho(z)$ as the density of the distribution F. Similarly, $u_n(x,t) \triangleq v_n(S,t)$ satisfies the similar free boundary problem where u in (4.3.1) is replaced by u_n in differential parts and by u_{n-1} in the integral part. In addition, it was shown in Theorem 4.2 of Yang et al. (2006) that the free boundary problem (4.3.1) is equivalent to the following variational inequality

$$\mathcal{L}_D u(x,t) + \lambda \cdot (Iu)(x,t) \leq 0$$

(4.3.3) $u(x,t) \geq g(x)$

$$[\mathcal{L}_D u(x,t) + \lambda \cdot (Iu)(x,t)] \cdot [u(x,t) - g(x)] = 0, \quad (x,t) \in \mathbb{R} \times [0,T],$$

in which

$$\mathcal{L}_D u \triangleq \frac{\partial}{\partial t}u + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x}u - (r+\lambda)u$$

$$g(x) = (K - e^x)^+.$$

Since the second spacial derivative of u does not exist along the free boundary $x(t)$, the variational inequality (4.3.3) does not have a classical solution. However, Theorem 3.2 of Yang et al. (2006) showed that u is the solution of (4.3.3) in the Sobolev sense. In the same sense, $u_n(x,t)$ satisfies a similar variational inequality

$$\mathcal{L}_D u_n(x,t) + \lambda \cdot (I u_{n-1})(x,t) \leq 0$$

(4.3.4) $u_n(x,t) \geq g(x)$

$$[\mathcal{L}_D u_n(x,t) + \lambda \cdot (I u_{n-1})(x,t)] \cdot [u_n(x,t) - g(x)] = 0, \quad (x,t) \in \mathbb{R} \times [0,T].$$

Let us discretize (4.3.3) using Crank-Nicolson scheme. For fixed Δt, Δx, x_{min} and x_{max}, let $M\Delta t = T$ and $L\Delta x = x_{max} - x_{min}$. Let us denote $x_l = x_{min} + l\Delta x$, $l = 0, \cdots, L$. By $\tilde{u}^{l,m}$ we will denote the solution of the following difference equation

$$-\theta p_- \tilde{u}^{l-1,m} + (1 + \theta p_0)\tilde{u}^{l,m} - \theta p_+ \tilde{u}^{l+1,m} - \tilde{b}^{l,m} \geq 0$$

(4.3.5) $\tilde{u}^{l,m} \geq g^l$

$$\left[-\theta p_- \tilde{u}^{l-1,m} + (1 + \theta p_0)\tilde{u}^{l,m} - \theta p_+ \tilde{u}^{l+1,m} - \tilde{b}^{l,m}\right] \cdot \left[\tilde{u}^{l,m} - g^l\right] = 0,$$

for $m = M - 1, \cdots, 0$, $l = 0, \cdots, L$, satisfying the terminal condition $\tilde{u}^{l,M} = g^l = (K - e^{x_l})^+$ and Dirichlet boundary conditions. θ is the weight factor. When $\theta = 1$, the scheme (4.3.5) is the completely implicit Euler scheme; when $\theta = 1/2$, it is the classical Crank-Nicolson scheme. The coefficients p_-, p_+ and p_0 are given by

$$p_- = \frac{1}{2}\sigma^2 \frac{\Delta t}{(\Delta x)^2} - \frac{1}{2}\left(\mu - \frac{1}{2}\sigma^2\right)\frac{\Delta t}{\Delta x},$$

(4.3.6) $p_+ = \frac{1}{2}\sigma^2 \frac{\Delta t}{(\Delta x)^2} + \frac{1}{2}\left(\mu - \frac{1}{2}\sigma^2\right)\frac{\Delta t}{\Delta x},$

$$p_0 = p_- + p_+ + (r + \lambda)\Delta t.$$

The term $\tilde{b}$ is defined by

$$\tilde{b}^{l,m} = (1 - \theta)p_- \tilde{u}^{l-1,m+1} + (1 - (1 - \theta)p_0)\tilde{u}^{l,m+1} + (1 - \theta)p_+ \tilde{u}^{l+1,m+1}$$

(4.3.7)
$$+ \lambda\Delta t \cdot \left[(1 - \theta)(\tilde{I}\tilde{u})^{l,m+1} + \theta(\tilde{I}\tilde{u})^{l,m}\right].$$

$\tilde{I}$ in (5.4.8) is the discrete version of the convolution operator I in (4.3.2). It will be convenient to approximate this convolution integral using Fast Fourier Transformation (FFT). Discretizing a sufficiently large interval $[z_{min}, z_{max}]$ into J sub-intervals. For the convenience of the FFT, we will choose these J sub-intervals equally spaced, such that $J\Delta z = z_{max} - z_{min}$. We also choose $\Delta x = \alpha \Delta z$, where α is a positive integer, so that the numerical integral may have finer grid than the grid in x. Let $z_j = z_{min} + j\Delta z$, $j = 0, \cdots, J$. $\tilde{I}$ is defined by

(4.3.8)
$$\left(\tilde{I}\tilde{u}\right)^{l,m} = \sum_{j=0}^{J-1} \tilde{u}_{interp}\left(x_l + z_j, m\Delta t\right) \rho(z_j)\Delta z,$$

in which the value of $\tilde{u}_{interp}$ is determined by the linear interpolation $\tilde{u}$. That is if there is some l' satisfying

$$x_{l'} \le x_l + z_j \le x_{l'+1},$$

then

$$\tilde{u}_{interp}\left(x_l + z_j, m\Delta t\right) = (1-w)\tilde{u}^{l',m} + w\tilde{u}^{l'+1,m},$$

for some $w \in [0, 1]$. On the other hand, if $x_l + z_j$ is outside the interval $[x_{min}, x_{max}]$, the value of $\tilde{u}_{interp}$ is determined by the boundary conditions. Moreover, in (4.3.8) we also assume

(4.3.9)
$$\rho(z_j) \ge 0, \quad \text{for all } j, \quad \text{and} \quad \sum_{j=0}^{J-1}\rho(z_j) \le 1.$$

Now (4.3.8) can be calculated using FFT. See Section 6.1 in Almendral and Oosterlee (2007) for implementation details.

Note that numerically solving the system (4.3.5) is difficult due to the contribution of the integral term $\tilde{I}\tilde{u}$. Therefore, following the results in Section 4.2, we will discretize (4.3.4) recursively (using the Crank-Nicoslon scheme) to obtain the sequence $\{\tilde{u}_n\}_{n\ge 0}$ recursively. Let $\tilde{u}_0^{l,m} = g^l$. For $n \ge 1$, $\tilde{u}_n$ is defined recursively by

$$- \theta p_- \tilde{u}_n^{l-1,m} + (1 + \theta p_0)\tilde{u}_n^{l,m} - \theta p_+ \tilde{u}_n^{l+1,m} - \tilde{b}_n^{l,m} \geq 0$$

(4.3.10)
$$\tilde{u}_n^{l,m} \geq g^l$$

$$\left[-\theta p_- \tilde{u}_n^{l-1,m} + (1 + \theta p_0)\tilde{u}_n^{l,m} - \theta p_+ \tilde{u}_n^{l+1,m} - \tilde{b}_n^{l,m} \right] \cdot \left[\tilde{u}_n^{l,m} - g^l \right] = 0,$$

with the terminal condition $\tilde{u}_n^{l,M} = g^l$ and Dirichlet boundary conditions. Similar to (5.4.8), $\tilde{b}_n$ is defined by

(4.3.11)
$$\tilde{b}_n^{l,m} = (1 - \theta)p_- \tilde{u}_n^{l-1,m+1} + (1 - (1 - \theta)p_0)\tilde{u}_n^{l,m+1} + (1 - \theta)p_+ \tilde{u}_n^{l+1,m+1}$$
$$+ \lambda \Delta t \cdot \left[(1 - \theta)(\tilde{I}\tilde{u}_{n-1})^{l,m+1} + \theta(\tilde{I}\tilde{u}_{n-1})^{l,m} \right].$$

For each n, we will solve the sparse linear system of equations (4.3.10) using the projected PSOR method (see eg. Wilmott et al. (1995)).

Remark 4.3.1. We will iterate (4.3.10) to approximate the solution of (4.3.5), which can be seen as a global fixed point iteration algorithm. This global fixed point algorithm is different from the local fixed point algorithm in d'Halluin et al. (2004), where d'Halluin et al. implemented the Crank Nicolson time stepping of a non-linear integro-partial differential equation coming from an alternative representation (due to the penalty method) of the American option price function. Also see d'Halluin et al. (2005) for the case of European options. Note that discretizing the non-linear PDE that arises from the penalized formulation introduces an extra error. We work with the variational formulation directly.

Each $\tilde{u}_n$ approximates u_n, which itself is the value function of an optimal stopping problem, and as we have discussed in Remark 4.2.1 provides a good imitation of the American option price function. Each of these iterations provide strictly decreasing free boundary curves with the same regularity and jump properties as the free boundary curve for the American option price function, see Remarks 4.2.1 and

4.4.2. The approximating sequence in d'Halluin et al. (2004) does not carry the same meaning, it is a technical step to carry out the Crank Nicolson time stepping of their non-linear integro-PDE.

4.3.2 Convergence of the numerical algorithm

In the following, we will show the convergence of the numerical algorithm for the completely implicit Euler scheme ($\theta = 1$). We first show that $\{\tilde{u}_n\}_{n \geq 0}$ is a monotone increasing sequence. Extra care has to be given to make the approximating sequence monotone in the penalty formulation of d'Halluin et al. (2004) (see Remark 4.3 on page 341), but the monotonicity comes out naturally in our formulation. Next, we prove that the sequence $\{\tilde{u}_n\}_{n \geq 0}$ is uniformly bounded above by the strike price K and converges to $\tilde{u}$ at an exponential rate. At last, we will argue that as the mesh sizes Δx and Δt go to zero $\tilde{u}$ converges to the American option value function u. In the following four propositions, we let Δt and Δx to be sufficiently small so that constants p_- and p_+ defined in (4.3.6) are positive.

Proposition 4.3.2. *The sequence $\{\tilde{u}_n\}_{n \geq 0}$ is a monotone increasing sequence.*

Proof. When $\theta = 1$, subtracting the third equality for n-th iteration in (4.3.10) from the equality for $(n + 1)$-th iteration, we obtain

(4.3.12)

$$
\left[-p_- \tilde{u}_n^{l-1,m} + (1 + p_0)\tilde{u}_n^{l,m} - p_+ \tilde{u}_n^{l+1,m} - \tilde{b}_n^{l,m} \right] \left[\tilde{u}_{n+1}^{l,m} - \tilde{u}_n^{l,m} \right]
$$
$$
+ \left\{ -p_- \left(\tilde{u}_{n+1}^{l-1,m} - \tilde{u}_n^{l-1,m} \right) + (1 + p_0) \left(\tilde{u}_{n+1}^{l,m} - \tilde{u}_n^{l,m} \right) - p_+ \left(\tilde{u}_{n+1}^{l+1,m} - \tilde{u}_n^{l+1,m} \right) \right.
$$
$$
\left. - \left(\tilde{u}_{n+1}^{l,m+1} - \tilde{u}_n^{l,m+1} \right) - \lambda \Delta t \cdot \left(\tilde{I} \left(\tilde{u}_n - \tilde{u}_{n-1} \right) \right)^{l,m} \right\} \left[\tilde{u}_{n+1}^{l,m} - g^l \right] = 0.
$$

in which we used the linearity of the operator $\tilde{I}$. Let us define the vectors

$$e_{n+1}^m = \left(\tilde{u}_{n+1}^{0,m} - \tilde{u}_n^{0,m}, \cdots, \tilde{u}_{n+1}^{L,m} - \tilde{u}_n^{L,m}\right)^T,$$

$$f_{n+1}^m = \left(\left[(\tilde{u}_{n+1}^{0,m+1} - \tilde{u}_n^{0,m+1}) + \lambda\Delta t \cdot \left(\tilde{I}(\tilde{u}_n - \tilde{u}_{n-1})\right)^{0,m}\right]\left[\tilde{u}_{n+1}^{0,m} - g^0\right], \cdots,\right.$$

$$\left.\left[(\tilde{u}_{n+1}^{L,m+1} - \tilde{u}_n^{L,m+1}) + \lambda\Delta t \cdot \left(\tilde{I}(\tilde{u}_n - \tilde{u}_{n-1})\right)^{L,m}\right]\left[\tilde{u}_{n+1}^{L,m} - g^L\right]\right)^T.$$

Equation (4.3.12) can be represented as

$$(4.3.13) \qquad\qquad A\, e_{n+1}^m = f_{n+1}^m,$$

in which the matrix A's entries are

$$a_{l,j} = \begin{cases} -p_-\left(\tilde{u}_{n+1}^{l,m} - g^l\right) & j = l-1 \\ (1+p_0)\left(\tilde{u}_{n+1}^{l,m} - g^l\right) + \left(-p_-\tilde{u}_n^{l-1,m} + (1+p_0)\tilde{u}_n^{l,m} - p_+\tilde{u}_n^{l+1,m} - \tilde{b}_n^{l,m}\right) & j = l \\ -p_+\left(\tilde{u}_{n+1}^{l,m} - g^l\right) & j = l+1 \\ 0 & \text{others.} \end{cases}$$

On the other hand, using the first and second inequalities in (4.3.10) and the fact that p_- and p_+ are positive, we see that A is an M-matrix, i.e. A has positive diagonals, non-positive off-diagonals and the row sums are positive. As a result all entries of A^{-1} are nonnegative.

Now we can prove the proposition by induction. Note that $\tilde{u}_1 \geq \tilde{u}_0 = g$, as a result of the second inequality in (4.3.10) and the definition of $\tilde{u}_0$. Assuming $\tilde{u}_n \geq \tilde{u}_{n-1}$, we will show that $\tilde{u}_{n+1} \geq \tilde{u}_n$, i.e. $\tilde{u}_{n+1}^{l,m} - \tilde{u}_n^{l,m} \geq 0$ for all l and m, in the following.

First, the terminal condition of $\tilde{u}_n$ gives us $\tilde{u}_{n+1}^{l,M} - \tilde{u}_n^{l,M} = 0$. Second, $\left(\tilde{I}(\tilde{u}_n - \tilde{u}_{n-1})\right)^{l,m}$ is nonnegative from the assumption (4.3.9). Assuming $\tilde{u}_{n+1}^{l,m+1} - \tilde{u}_n^{l,m+1}$ nonnegative, we have f_{n+1}^m in (4.3.13) as a nonnegative vector. Combining with the fact that all entries of A^{-1} are nonnegative, the nonnegativity of $\tilde{u}_{n+1}^{l,m} - \tilde{u}_n^{l,m}$ follows from multiplying A^{-1} on both sides of (4.3.13). Then the result follows from an induction m. $\qquad\square$

Proposition 4.3.3. $\{\tilde{u}_n\}_{n\geq 0}$ *are uniformly bounded above by the strike price K.*

Proof. When $\theta = 1$, in the third equality of (4.3.10), there are some (l, m) such that $\tilde{u}_n^{l,m} = g^l$. Otherwise we have

$$(1 + p_0)\tilde{u}_n^{l,m} = p_-\tilde{u}_n^{l-1,m} + p_+\tilde{u}_n^{l+1,m} + \tilde{u}_n^{l,m+1} + \lambda\Delta t \left(\tilde{I}\tilde{u}_{n-1}\right)^{l,m}.$$

However, in both cases, we obtain the following inequality

(4.3.14)

$$(1+p_0)\left|\tilde{u}_n^{l,m}\right| \leq p_-B_n^m + p_+B_n^m + B_n^{m+1} + \lambda\Delta t B_{n-1} + r\Delta t K, \quad 0 \leq l \leq L, 0 \leq m \leq M-1,$$

in which we define

$$B_n^m = \left(\max_l \left|\tilde{u}_n^{l,m}\right|\right) \bigvee K, \quad B_n = \max_m B_n^m.$$

Note that the right hand side of (4.3.14) is independent of l. Moreover, $(1 + p_0)K$ is also less than or equal to the right hand side of (4.3.14). Therefore, (4.3.14) gives us

(4.3.15) $$(1 + (r + \lambda)\Delta t) B_n^m \leq B_n^{m+1} + \lambda\Delta t B_{n-1} + r\Delta t K.$$

Given $B_n^{m+1} \leq K$ and $B_{n-1} \leq K$, it clear from (4.3.15) that $B_n^m \leq K$. Now the proposition follows from double induction on m and n with initial steps $\tilde{u}_n^M = g \leq K$ and $\tilde{u}_0 = g \leq K$. $\square$

As a result of Propositions 4.3.2, we can define

(4.3.16) $$\tilde{u}_\infty^{l,m} = \lim_{n\to+\infty} \tilde{u}_n^{l,m}, \quad 0 \leq l \leq L, 0 \leq m \leq M.$$

It follows from Proposition 4.3.3 that $\tilde{u}_\infty^{l,m} \leq K$. Letting n go to $+\infty$, we can see from (4.3.10) that $\tilde{u}_\infty$ satisfies the difference equation (4.3.5). Therefore,

(4.3.17) $$\tilde{u}_\infty = \tilde{u}.$$

In the following, we will study the convergence rate of $\{\tilde{u}_n\}_{n\geq 0}$.

Proposition 4.3.4. $\tilde{u}_n$ *converges to* $\tilde{u}$ *uniformly and*

(4.3.18)
$$\max_{l,m} \left(\tilde{u}^{l,m} - \tilde{u}_n^{l,m} \right) \leq \left(1 - \eta^M \right)^n \left(\frac{\lambda}{\lambda + r} \right)^n \tilde{K},$$

where $\eta = \frac{1}{1+(\lambda+r)\Delta t} \in (0,1)$, $\tilde{K}$ *is a positive constant.*

Proof. Let us define

$$e_n^{l,m} = \tilde{u}^{l,m} - \tilde{u}_n^{l,m}, \quad E_n^m = \max_l e_n^{l,m}, \quad E_n = \max_m E_n^m.$$

Proposition 4.3.2 and (4.3.17) ensure that $e_n^{l,m}$ is nonnegative. Moreover e_n satisfies

(4.3.19)

$$\left[-p_- \tilde{u}_n^{l-1,m} + (1+p_0)\tilde{u}_n^{l,m} - p_+ \tilde{u}_n^{l+1,m} - \tilde{b}_n^{l,m} \right] e_n^{l,m}$$
$$+ \left\{ -p_- e_n^{l-1,m} + (1+p_0)\, e_n^{l,m} - p_+ e_n^{l+1,m} - e_n^{l,m+1} - \lambda \Delta t \cdot \left(\tilde{I} e_{n-1}^{l,m} \right)^{l,m} \right\} \left[\tilde{u}^{l,m} - g^l \right] = 0.$$

We can drop the first term on the left-hand-side of (4.3.19) because of the first inequality in (4.3.10) and $e_n^{l,m}$ being nonnegative. It gives us the inequality

(4.3.20)

$$(1+p_0)e_n^{l,m} \left[\tilde{u}^{l,m} - g^l \right] \leq \left[p_- e_n^{l-1,m} + p_+ e_n^{l+1,m} + e_n^{l,m+1} + \lambda \Delta t E_{n-1} \right] \left[\tilde{u}^{l,m} - g^l \right],$$

in which we also used the assumption (4.3.9) to derive the upper bound for the integral term.

If there are some (l,m) such that $\tilde{u}^{l,m} = g^l$, since $\{\tilde{u}_n\}_{n\geq0}$ is an increasing sequence from Proposition 4.3.2, we have $\tilde{u}^{l,m} = \tilde{u}_n^{l,m}$ for all n. Therefore, $e_n^{l,m} = 0$ for these (l,m). On the other hand, if $\tilde{u}^{l,m} > g^l$ for some (l,m), we can divide $\tilde{u}^{l,m} - g^l$ on both sides of (4.3.20) to get

(4.3.21)
$$(1+p_0)e_n^{l,m} \leq p_- e_n^{l-1,m} + p_+ e_n^{l+1,m} + e_n^{l,m+1} + \lambda \Delta t E_{n-1}$$
$$\leq p_- E_n^m + p^+ E_n^m + E_n^{m+1} + \lambda \Delta t E_{n-1}.$$

Since the right-hand-side of (4.3.21) does not depend on l, we can write

$$(4.3.22) \qquad E_n^m \leq \eta E_n^{m+1} + (1-\eta)\frac{\lambda}{\lambda+r}E_{n-1},$$

in which $\eta = \frac{1}{1+(\lambda+r)\Delta t} \in (0,1)$. Note that (4.3.22) is also satisfied for all m, because even if $\tilde{u}^{l,m} = g^l$ for some (l,m), $e_n^{l,m} = 0$ as we proved above. If follows from (4.3.22) that

$$(4.3.23) \qquad E_n^m \leq \eta^{M-m}E_m^M + (1-\eta)(1+\eta+\cdots+\eta^{M-m-1})\frac{\lambda}{\lambda+r}E_{n-1}.$$

Since the terminal condition of $\tilde{u}_n$, we have $E_n^M = 0$. Now maximizing the right-hand-side of (4.3.23) over m, we obtain that

$$E_n \leq (1-\eta^M)\frac{\lambda}{\lambda+r}E_{n-1}.$$

As a result,

$$(4.3.24) \qquad E_n \leq \left(1-\eta^M\right)^n \left(\frac{\lambda}{\lambda+r}\right)^n E_0 \to 0, \quad \text{as } n \to +\infty.$$

$\square$

Remark 4.3.5. As $M \to +\infty$

$$1-\eta^M = 1 - \left(\frac{1}{1+(\lambda+r)T/M}\right)^M \to 1 - e^{-(r+\lambda)T},$$

which agree with the convergent rate (4.2.10) in the continuous case.

Proposition 4.3.6.

$$(4.3.25) \qquad |u(x_k, m\Delta t) - \tilde{u}(x_k.m\Delta t)| \to 0,$$

as Δx, Δt, $\Delta z \to 0$.

Proof. Using the triangle inequality, let us write

$$|u(x_k, m\Delta t) - \tilde{u}(x_k, m\Delta t)|$$

$$\leq |u(x_k, m\Delta t) - u_n(x_k, m\Delta t)| + |u_n(x_k, m\Delta t) - \tilde{u}_n(x_k, m\Delta t)|$$

(4.3.26) $$+ |\tilde{u}_n(x_k, m\Delta t) - \tilde{u}(x_k, m\Delta t)|$$

$$\leq K \left(1 - e^{-(r+\lambda)(T-m\Delta t)}\right)^n \left(\frac{\lambda}{\lambda+r}\right)^n + n \cdot O\left((\Delta t) + (\Delta x)^2 + (\Delta z)^2\right)$$

$$+ \tilde{K} \left(1 - \eta^M\right)^n \left(\frac{\lambda}{\lambda+r}\right)^n,$$

for some positive constants K and $\tilde{K}$. The first and third terms on the right-hand-side of the second inequality are due to (4.2.10) and (4.3.18). The second term arises since the order of error from discretizing a PDE using implicit Euler scheme is $O((\Delta t) + (\Delta x)^2)$, the interpolation and discretization error from numerical integral are of order $(\Delta x)^2$ and $(\Delta z)^2$ and the total error made at each step propagates at most linearly in n when we sequentially discretize (4.3.4).

Letting $\Delta t, \Delta x, \Delta z \to 0$ in (4.3.26), we obtain that

$$\lim_{\Delta t, \Delta x, \Delta z \to 0} |u(x_k, m\Delta t) - \tilde{u}(x_k, m\Delta t)| \leq \left(K + \tilde{K}\right) \left(\frac{\lambda}{\lambda+r}\right)^n \left(1 - e^{-(r+\lambda)T}\right)^n,$$

in which we used (5.5.6). Since n is arbitrary the result follows. $\qquad\square$

Remark 4.3.7. In Propositions 4.3.2 - 4.3.6, we have shown the convergence of the algorithm for completely implicit Euler scheme ($\theta = 1$). In order to have the time discretization error as $O((\Delta t)^2)$, we will choose Crank-Nicolson scheme with $\theta = 1/2$ in the numerical experiments in the next section. From numerical results in Table 4, we shall see that Crank-Nicolson Scheme is also stable and the convergence is fast.

4.4 The numerical performance of the proposed numerical algorithm

In this section, we present the numerical performance of the algorithm proposed in the previous section. First, we compare the prices we obtain to the prices obtained

in the literature. To demonstrate our competitiveness we also list the time it takes to obtain the prices for certain accuracy. We will use either the PSOR or the Brennan-Schwartz algorithm to solve the sparse linear system in (4.3.10); see Remark 4.4.1. All our computations are performed with C++ on a Pentium IV, 3.0 GHz machine.

In Table 1, we take the jump distribution F to be the double exponential distribution

$$(4.4.1) \qquad F(dz) = \left(p\eta_1 e^{-\eta_1 z} 1_{\{z \geq 0\}} + (1-p)\eta_2 e^{\eta_2 z} 1_{\{z < 0\}} \right) dz.$$

We compare our performance with that of Kou and Wang (2004) and Kou et al. (2005). Kou and Wang (2004) obtain an approximate American option price formula, for by reducing the integro-pde equation V satisfied to a integro-ode following Barone-Adesi and Whaley (1987). This approximation is accurate for small and large maturities. Also, they do not provide error bounds, the magnitude of which might depend on the parameters of the problem, therefore one might not be able to use this price approximation without the guidance of another numerical scheme. A more accurate numerical scheme using an approximation to the exercise boundary and Laplace transform was later developed by Kou et al. (2005). Our performance has the same order of magnitude as theirs. Our method's advantage is that it works for a more general jump distribution and we do not have to assume a double exponential distribution for jumps as Kou and Wang (2004) and Kou et al. (2005) do.

In Table 2 we compute the prices of American and European options in a Merton jump diffusion model, in which the jump distribution F is specified to be the Gaussian distribution

$$(4.4.2) \qquad F(dz) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left(\frac{-(z - \tilde{\mu})^2}{\tilde{\sigma}^2} \right) dz.$$

We list the accuracy and time characteristics of the proposed numerical algorithm

algorithm. We compare our prices to the ones obtained by d'Halluin et al. (2004, 2005). d'Halluin et al. (2004) used a penalty method to approximate the American option price, while we analyze the variational inequalities directly (see (4.3.5) and (4.3.10)). Moreover, our approximating sequence is monotone (see Proposition 4.3.2).

In Table 3, We also list the approximated prices of Barrier options. We compare the prices we obtain with Metwally and Atiya (2003) where a Monte Carlo method is used. We do not list the time it takes for the alternative algorithms in Tables 2 and 3 either because they are not listed in the original papers or they take unreasonably long time.

In Table 4, we list the numerical convergence of the proposed algorithm algorithm with respect to grid sizes. We choose Crank-Nicolson scheme with $\theta = 1/2$ in (4.3.10) and solve the sparse linear system by either the Bernnan-Schwartz algorithm or the PSOR.

Remark 4.4.1. Here we will analyze the complexity of our algorithm. Let us fix $\Delta x/\Delta t$ as a constant and choose the number of grid point in x to be N. For each time step, using the FFT to calculate the integral term in (4.3.10) costs $O(N \log N)$ computations. On the other hand, the Brennan-Schwartz algorithm, which uses the LU decomposition to solve the sparse linear system in (4.3.10) (see Jaillet et al. (1990) pp. 283), needs $2N$ computations for each time step. However, PSOR needs $C \cdot N$ computations for each time step to solve (4.3.10) at each time step. Here, C is the number of iterations PSOR requires to converge to a fixed small error tolerance ϵ. We will see in the following that PSOR is numerically more expensive than the Brennan-Schwartz algorithm.

For PSOR, the number of iterations C increases with respect to N. To see this,

we start from the tri-diagonal matrix on the left-hand-side of (4.3.10)

$$A = \left\{ \begin{array}{ccccc} 1 + \theta p_0 & -\theta p_+ & & & \\ -\theta p_- & 1 + \theta p_0 & \ddots & & \\ & \ddots & \ddots & -\theta p_+ & \\ & & -\theta p_- & 1 + \theta p_0 \end{array} \right\}.$$

For the SOR (without projection), the optimal relaxation parameter ω is given by (see Young (1971))

$$\omega = \frac{2}{1 + \sqrt{1 - \rho_J^2}},$$

where ρ_J is the spectral radius of the Jacobi iteration matrix $J = D^{-1}(A - D)$ with D as the diagonal matrix of A. Since $\rho_J \leq \|J\|_\infty = \theta(p_+ + p_-)/(1 + \theta p_0)$, we have

$$(4.4.3) \qquad \omega \leq \omega_0 = \frac{2}{1 + \sqrt{1 - \|J\|_{+\infty}^2}}.$$

We will use ω_0 as the optimal relaxation parameter in our numerical experiments. On the other hand, since the largest eigenvalue λ_{max} of the SOR iteration matrix is bounded above by $\omega - 1$, using (4.3.6) and (4.4.3) we obtain that

$$(4.4.4) \qquad C = \min\{c \geq 0 | (\lambda_{max})^c \leq \epsilon\} = O(\sqrt{N}).$$

Since $O(N^{3/2})$ dominates $O(N \log N)$, the complexity of the Iterated PSOR algorithm at each time step will be $O(N^{3/2})$. Therefore, with $O(N)$ time steps, the complexity for Iterated PSOR algorithm is $O(N^{5/2})$. On the other hand, for the Iterated Brennan-Schwartz algorithm, since $O(N \log N)$ dominates $O(N)$, the complexity at each time step will be $O(N \log N)$. Therefore, the complexity of the Iterated Brennan-Schwartz algorithm is $O(N^2 \log N)$ since we have $O(N)$ time steps.

Please refer to Tables 1, 2, 3 and 4 for numerical performance of both algorithms.

Next, we illustrate the behavior of the sequence of functions $\{v_n(S,t)\}_{n\geq 0}$ and its limit V in Figures 1, 2 and 3. All the figures are obtained for an American put option in the case of the double exponential jump with $K = 100$, $S_0 = 100$, $T = 0.25$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$ (the same parameters are used in the 8th row of Table 1) at a single run.

Remark 4.4.2.   (i) In Figure 1, we show, how $V(S,0)$ depends on the time to maturity, and that it fits smoothly to the put-pay-off function at $s(0)$ (the exercise boundary). The y-axis is the difference between the option price and the pay-off function. As the time to maturity increases, the option price $V(S,0)$ increases while the exercise boundary $s(0)$ decreases. Even though the stock price process has jumps, the option price smoothly fits the pay-off function at $s(0)$, as in the classical Black-Scholes case without the jumps.

(ii) In Figure 2, we illustrate the convergence of the exercise boundaries $t \to s_n(t)$, $n \geq 1$. We can see from the figure that all $s_n(t)$ are convex functions. Also, the sequence $\{s_n\}_{n\geq 1}$ is a monotone decreasing sequence, which implies that the continuation region is getting larger, and that the convergence of the free boundary sequence is fast.

Moreover, we notice that, when the parameters are chosen such that (4.2.12) is satisfied, the free boundaries are discontinuous at the maturity time. In addition, we have $s(T-) = s_n(T-) = S^* < K$, where S^* is the unique solution of (4.2.14). Furthermore, if F is the double exponential distribution as in (4.4.1), the integral equation (4.2.14) can be solved analytically to obtain

$$(4.4.5) \qquad\qquad S^* = \left(\frac{(\eta_1 - 1)r}{\lambda p} \right)^{1/\eta_1} \cdot K.$$

With the parameters we choose, we get from (4.4.5) that $S^* = 98.39$. It is close

to our numerical result as one can see from Figure 2.

(iii) In Figure 3, we illustrate the convergence of the sequence of prices $\{v_n(S,0)\}_{n\geq 0}$. Observe that this is a monotonically increasing sequence and it converges to its limit $V(S,0)$ very fast.

Table 4.1:
     Comparison between the proposed algorithm with the methods in Kou and Wang (2004)
and Kou et al. (2005)

The parameters are chosen as $r = 0.05$, $S(0) = 100$ and $p = 0.6$ as in Kou and Wang (2004) and
Kou et al. (2005). Amin's price is calculated in Kou and Wang (2004) using the enhanced binomial
tree method as in Amin (1993). The accuracy of Amin's price is up to about a penny. The KPW
5EXP price from Kou et al. (2005) is calculated on a Pentium IV, 1.8 GHz, while the iterated price
is calculated on Pentium IV, 3.0GHz, both using C++ implementation. Run times are in seconds.
For numerical algorithm we propose, the number of grid points in x is chosen as 2^6 and $\Delta t = \Delta x$.
The option prices from both Iterated Brennan-Schwartz and Iterated PSOR are the same. Below
"B-S" stands for the Brennan-Schwartz.

American Put Double Exponential Jump Diffusion Model														
Parameter Values						Amin's	KW		KPW 5EXP			Proposed Algorithm			
K	T	σ	λ	η_1	η_2	Price	Value	Error	Value	Error	Time	Value	Error	B-S Time	PSOR Time
90	0.25	0.2	3	25	25	0.75	0.76	0.01	0.74	-0.01	3.21	0.75	0	0.08	0.12
90	0.25	0.2	3	25	50	0.65	0.66	0	0.65	0	3.25	0.66	0.01	0.08	0.12
90	0.25	0.2	3	50	25	0.68	0.69	0.01	0.68	0	2.97	0.69	0.01	0.08	0.12
90	0.25	0.2	3	50	50	0.59	0.60	0.01	0.59	0	2.89	0.59	0	0.12	0.12
90	0.25	0.3	3	25	25	1.92	1.93	0.01	1.92	0	2.40	1.93	0.01	0.09	0.13
90	0.25	0.2	7	25	25	1.03	1.04	0.01	1.02	-0.01	3.18	1.03	0	0.12	0.17
90	0.25	0.3	7	25	25	2.19	2.20	0.01	2.18	-0.01	2.97	2.20	0.01	0.12	0.20
100	0.25	0.2	3	25	25	3.78	3.78	0	3.77	-0.01	3.08	3.78	0	0.12	0.12
100	0.25	0.2	3	25	50	3.66	3.66	0	3.65	-0.01	3.29	3.66	0	0.10	0.12
100	0.25	0.2	3	50	25	3.62	3.62	0	3.62	0	2.88	3.63	0.01	0.09	0.12
100	0.25	0.2	3	50	50	3.50	3.50	0	3.50	0	3.00	3.50	0	0.13	0.12
100	0.25	0.3	3	25	25	5.63	5.62	-0.01	5.63	0	2.44	5.63	0	0.13	0.15
100	0.25	0.2	7	25	25	4.26	4.27	0.01	4.26	0	3.48	4.27	0.01	0.17	0.17
100	0.25	0.3	7	25	25	5.99	5.99	0	5.99	0	2.95	6.00	0.01	0.17	0.18
90	1	0.2	3	25	25	2.91	2.96	0.05	2.90	-0.01	2.43	2.92	-0.01	0.63	0.78
90	1	0.2	3	25	50	2.70	2.75	0.05	2.69	-0.01	2.38	2.70	0	0.69	0.81
90	1	0.2	3	50	25	2.66	2.72	0.06	2.67	0.01	2.55	2.68	0.02	0.64	0.82
90	1	0.2	3	50	50	2.46	2.51	0.05	2.45	-0.01	2.30	2.45	-0.01	0.68	0.82
90	1	0.3	3	25	25	5.79	5.85	0.06	5.79	0	2.48	5.77	-0.02	0.70	0.94

Table 4.2: Option price in Merton jump-diffusion model

K=100, T=0.25, r=0.05, $\sigma = 0.15$, $\lambda = 0.1$. Stock price has lognormal jump distribution with $\tilde{\mu} = -0.9$ and $\tilde{\sigma} = 0.45$. For the iterated jump schemes, the number of grid points in x is chosen as 2^7 and $\Delta t = \Delta x$. Below "B-S" stands for the Brennan-Schwartz.

Option Type [a]	S(0)	dFLV[b]	Proposed Algorithm			
			Value	Error	LU(B-S) Time	PSOR Time
American Put	90	10.004	10.004[c]	0	0.18	0.24
	100	3.241	3.242	0.001		
	110	1.420	1.420	0		
European Put	100	3.149	3.150	0.001	0.21	0.18
European Call	90	0.528	0.528	0	0.18	0.18
	100	4.391	4.392	0.001		
	110	12.643	12.643	0		

[a]The option prices (for the same kind of option) for different $S(0)$ are obtained from a single run.
[b]The dFLV price comes from d'Halluin et al. (2004, 2005).
[c]the option price is 10.001 using the iterated Brennan-Schwartz scheme.

Table 4.3: European down-and-out barrier call option with Merton jump-diffusion model

K=110, S(0)=100, T=1, r=0.05, $\sigma = 0.25$, $\lambda = 2$, rebate R=1, the Stock price has lognormal jump distribution with $\tilde{\mu} = 0$ and $\tilde{\sigma} = 0.1$. For the algorithm we propose the number of grid points in x is chosen as 2^6 and $\Delta t = \Delta x$. Below we use the acronyms LU or SOR to tell wheher we use the LU factorization or the SOR to solve for the sparse linear systems at each time step.

Barrier H	MA Price [a]	Proposed Algorithm			
		Value	Error	LU Time	SOR Time
85	9.013	8.988	-0.025	0.52	0.71
95	5.303	5.290	-0.013	0.64	0.86

[a]The MA price comes from Metwally and Atiya (2003)

Table 4.4: Convergence of the numerical algorithm with respect to grid sizes

K=100, T=0.25, r=0.05, $\sigma = 0.15$, $\lambda = 0.1$, stock price has lognormal jump distribution with $\tilde{\mu} = -0.9$ and $\tilde{\sigma} = 0.45$ (the same parameters that are used in d'Halluin et al. (2004)). The differential equation is discretized by the Crank-Nicolson scheme as (4.3.10) with $\theta = 1/2$. The logarithmic variable $x = \log S$ is equally spaced discretized on an interval $[x_{min}, x_{max}]$ with $\Delta x = \Delta t$. The numerical integral is truncated on the smallest interval $[z_{min}, z_{max}]$, such that $[x+\tilde{\mu}-4\tilde{\sigma}, x+\tilde{u}+4\tilde{\sigma}]$ will be inside $[z_{min}, z_{max}]$ for any $x \in [x_{min}, x_{max}]$. The step length for the numerical integral is chosen the same as the step length in x, i.e. $\Delta z = \Delta x$. The number of grid points for to implement the FFT is chosen as an integral power of 2. The error tolerance for PSOR method is 10^{-8} and for the global iteration is 10^{-6}. Run times are in seconds. Each row in the "Difference" column of the following table is $v_{PSOR}(L, M) - v_{PSOR}(L/2, M/2)$. "B-S" stands for the Brennan-Schwartz algorithm. The number of global iteration is 3 for all the following numerical experiments.

S(0)	No. of grid points in x (L)	No. of time steps (M)	B-S Value v_{B-S}	B-S Time	PSOR Value v_{PSOR}	Difference	PSOR Time	Max. No. of PSOR iterations
	64	30	10.00230	0.06	10.00573	n.a.	0.06	16
90	128	58	10.00142	0.21	10.00429	-0.00144	0.24	21
	256	115	10.00192	0.84	10.00396	-0.00033	0.99	28
	512	230	10.00218	3.51	10.00387	-0.00009	4.50	39
	64	30	3.24074	0.06	3.24465	n.a.	0.06	16
100	128	58	3.24008	0.21	3.24180	-0.00285	0.24	21
	256	115	3.24046	0.84	3.24115	-0.00065	0.99	28
	512	230	3.24058	3.51	3.24103	-0.00012	4.50	39
	64	30	1.42048	0.06	1.42146	n.a.	0.06	16
110	128	58	1.41941	0.21	1.41991	-0.00155	0.24	21
	256	115	1.41958	0.84	1.41966	-0.00025	0.99	28
	512	230	1.41962	3.51	1.41960	-0.00006	4.50	39

Using (4.4.4), the number of SOR iterations can be calculated. The calculation gives $11, 16, 22$ and 31. Comparing with the last column of above table, the maximum numbers of PSOR iteration are slightly larger than these theoretical predicted SOR iteration times. Moreover, when $L = 512$ the the ratio between the maximum number of PSOR iteration and $\sqrt{L}$ is 1.72. This confirms the analysis in Remark 4.4.1 that the maximal PSOR iteration time grows as the order of $\sqrt{L}$.

The parameters for the following three figures are $K = 100$, $S_0 = 100$, $T = 0.25$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 3$, the stock price has double exponential jump with $p = 0.6$, $\eta_1 = 25$ and $\eta_2 = 25$ (the same parameters used in the 8th row of Table 1).

Figure 4.1:   Smooth-fit

The option price function $S \to V(S, 0)$ smoothly fits the pay-off function $(K - S)^+$ at $s(0)$. $V(S, 0)$ increases and $s(0)$ $(V(S, 0) - (K - S)^+ = 0$ at s(0)) decreases as time to maturity T increases.

Figure 4.2: Iteration of the exercise boundaries

$s_n(t) \downarrow s(t)$, $t \in [0, T)$. Both $s_n(t)$ and $s(t)$ will converge to $S^* < K$ as $t \to T$.
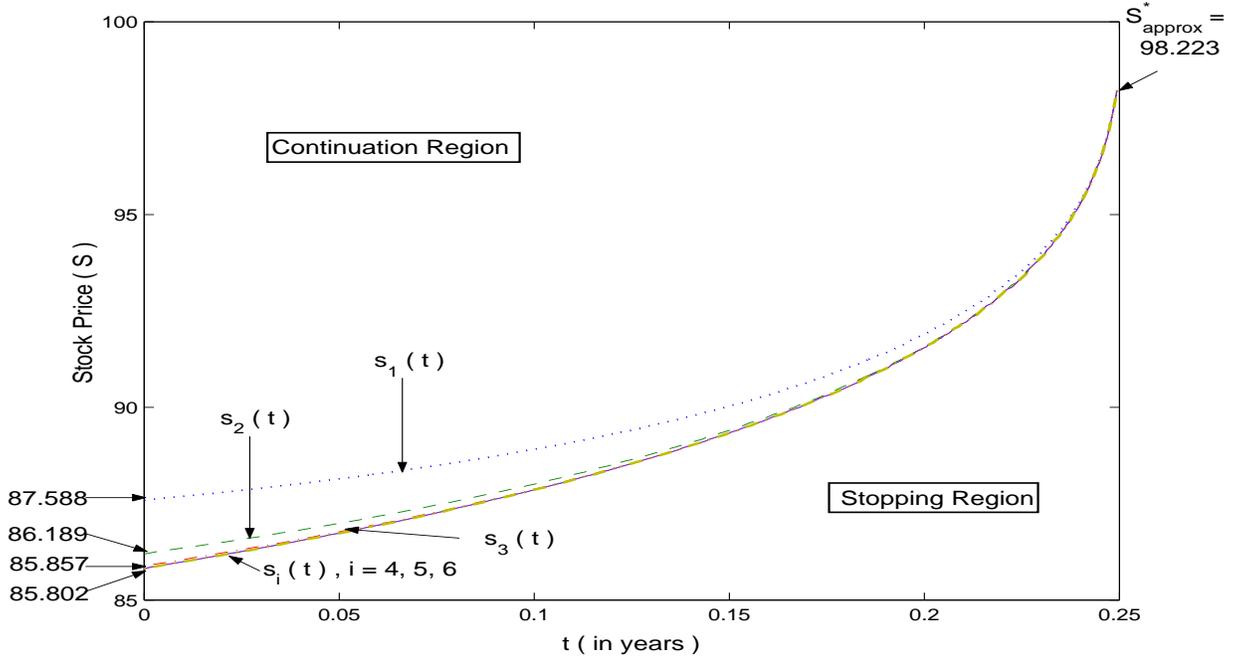
Figure 4.3: Iteration of the price functions

$v_n(S, 0) \uparrow V(S, 0)$, $S \geq 0$.
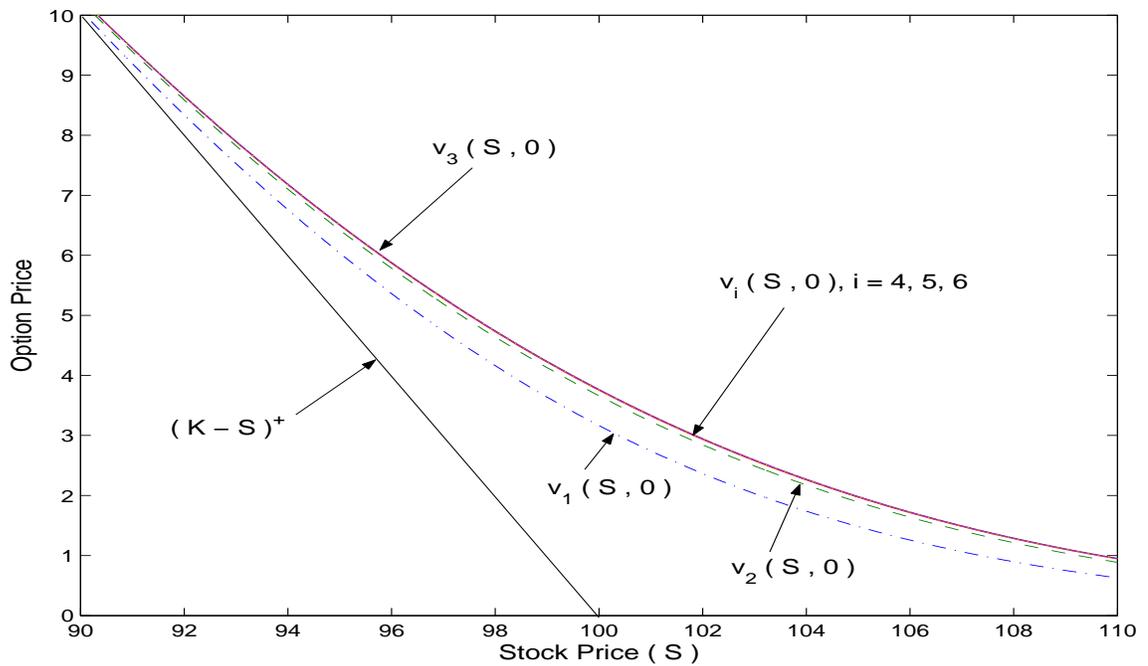
CHAPTER V

Pricing Asian options for jump diffusions

5.1 Introduction

We develop an efficient numerical algorithm to price Asian options, which are derivatives whose pay-off depends on the average of the stock price, for jump diffusions. The jump diffusion models are heavily used in the option pricing context since these models can capture the excess kurtosis of the stock price returns along with the skew in the implied volatility surface (see Cont and Tankov (2004)). Two well-known examples of these models are i) the model of Merton (1976), in which the jump sizes are log-normally distributed, and ii) the model of Kou (2002), in which the logarithm of jump sizes have the so called double exponential distribution. Here we consider a large class of jump diffusion models including these two.

The pricing of Asian options is complicated because it involves solving a partial differential equation (PDE) with two space dimensions, one variable accounting for the average stock price, the other for the stock price itself. However, Večeř was able to reduce the dimension of the problem in Večeř (2001) by using a change of measure argument (also see Section 2.1). When the stock price is a geometric Brownian motion, Večeř (2001) showed that the price of the Asian option at time $t = 0$, which we will denote by $S_0 \to V(S_0)$, satisfies $V(S_0) = S_0 \cdot v(z = z^*, t = 0)$

for a suitable constant z^*, in which the function v solves a one dimensional parabolic PDE. When the stock price is a jump diffusion, then under the assumptions that v_t, v_z and v_{zz} are continuous, Večeř and Xu (see their Theorem 3.3 and Corollary 3.4 in Večeř and Xu (2004)) showed that the function v solves a integro partial differential equation using Itô's lemma. However, a priori it is not clear that these assumptions are satisfied. In this chapter, we show that for the jump diffusion models these assumptions are indeed satisfied (see Theorem 5.2.1), i.e., we directly show that v is the unique classical solution of the partial integro-differential equation in Večeř and Xu (2004) (This integro-PDE is given in (5.2.17) and (5.2.17) in this chapter). We do this by first showing that v is the limit of a sequence of functions constructed by iterating a suitable functional operator, which we will denote by J. This functional operator J takes functions with certain regularity properties into the unique classical solutions of parabolic differential equations and gives them more regularity. We show that v is the fixed point of the functional operator. Finally, we show that v satisfies the certain regularity properties, which ensures that it is the classical solution of the partial integro-differential equation in Večeř and Xu (2004). This proof technique is similar to that of Bayraktar (2009), in which the regularity of the American put option prices are analyzed. In this chapter, some major technical difficulties arise because the pay-off functions we consider are not bounded and also because the sequence of functions constructed is not monotonous (Bayraktar (2009) was able to construct a monotonous sequence because of the early exercise feature of the American options).

The iterative construction of the sequence of functions which converge to the Asian option price naturally leads to an efficient numerical method for computing the price of Asian options. We prove that the constructed sequence of functions converges to

the function v uniformly (on compact sets) and exponentially fast. Therefore, after a few iterations one can obtain an approximation of v within the desired level of accuracy, i.e., the accuracy versus speed characteristics of our numerical method can be controlled. On the other hand, since each element of the approximating sequence solves a parabolic PDE (not an integro-differential equation), we can use one of the classical finite difference schemes to determine it. We propose a numerical scheme in Section 5.3 and analyze the performance of it in the same section.

Numerical methods for pricing Asian options for diffusion models were studied extensively in the literature: various PDE methods were proposed in Večeř (2001), Rogers and Shi (1995), Zhang (2001) and Zhang (2003), a single Laplace inversion method was developed in Geman and Yor (1993), a spectral expansion approach was investigated in Linetsky (2004), a double Laplace inversion method was discovered in Cai and Kou (2007). Meanwhile, tight bounds for the Asian option prices were obtained in Rogers and Shi (1995) and Thompson (1998). However, methods for pricing Asian options in jump models are still under development. We should mention that Cai and Kou (2007) also considered pricing Asian option for the double exponential jump diffusion model of Kou.

The rest of the chapter is organized as follows: In Section 5.2.1, we summarize the findings of Večeř and Xu (2004) in the context of jump-diffusion models. In Section 5.2.2, we present our main theoretical results: Theorem 5.2.1 and Corollary 5.2.2. We propose a numerical algorithm and analyze its convergence properties in Section 5.3.1. Then we perform numerical experiments for two particular jump diffusion models and analyze their performance in Section 5.3.2. Section 5.4 is devoted to development of the proof of Theorem 5.2.1. In Sections 5.4.1 and 5.4.2 we develop the properties of the functional operator J and the properties of the sequence ob-

tained by iterating J, respectively. These results are used to prove Theorem 5.2.1 in Section 5.4.3. A brief summary of our proof technique can be found in Section 5.2.2. right after the definition of the operator J in (5.2.11). The proof of the numerical convergence of our algorithm is given in Section 5.5.

5.2 A sequential approximation to price of an Asian option

5.2.1 Dimension reduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space hosting a Wiener process $\{B_t; t \geq 0\}$ and a Poisson random measure N, whose mean measure is $\lambda\nu(dy)dt$, independent of the Wiener process. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the natural filtration of B and N. In this filtered probability space, let us define a Markov process $S = \{S_t; t \geq 0\}$ via its dynamics as

$$(5.2.1) \qquad dS_t = (r - \mu)S_{t-}\, dt + \sigma S_{t-}\, dB_t + S_{t-}\int_{\mathbb{R}_+} (y-1)N(dt, dy),$$

in which r is the risk free rate, $\mu \triangleq \lambda(\xi - 1)$, where $\xi \triangleq \int_{\mathbb{R}_+} y\nu(dy) < \infty$. The process S is the price of a traded stock, and under the measure $\mathbb{P}$, the discounted stock price $(e^{-rt}S_t)_{t\geq 0}$ is a martingale. In this framework the stock price jumps at time t from S_{t-} to $S_{t-}Y$, in which Y's distribution is given by ν. Y is a positive random variable and note that when $Y < 1$ then the stock price S jumps down, when $Y > 1$ the stock price jumps up. In Merton's jump diffusion model (see Merton (1976)) $Y = \exp(\mathcal{X})$ where $\mathcal{X}$ is a Gaussian random variable. In Kou's model (see Kou (2002)) $\mathcal{X}$ has a double exponential distribution.

To reduce the dimension of the Asian option pricing problem, Večeř and Xu in Večeř and Xu (2004) introduced a new measure $\mathbb{Q}$ by

$$(5.2.2) \qquad \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = e^{-rt}\frac{S_t}{S_0}, \quad t \in [0, T].$$

Here, T is the maturity of the Asian option. Večeř and Xu also introduced the following process

$$(5.2.3) \qquad Z_t^J \triangleq \frac{X_t}{S_t}, \quad t \in [0, T],$$

where $X = \{X_t; t \in [0, T]\}$ is a self-financing portfolio that replicates the pay-off of the Asian forward contract with fixed strike K_2. The dynamics of X are given by

$$(5.2.4) \qquad dX_t = q_t\, dS_t + r(X_{t-} - q_t\, S_{t-})\, dt, \quad X_0 = x,$$

in which q_t defined as

$$(5.2.5) \qquad q_t \triangleq \frac{1}{rT}(1 - e^{-r(T-t)}), \quad t \in [0, T],$$

is the number of shares invested in the stock. The initial value of the portfolio process X is given given in terms of q_0 and K_2 as

$$(5.2.6) \qquad x = q_0 S_0 - e^{-rT} K_2.$$

Večeř and Xu showed in Večeř and Xu (2004) that the price of the continuously averaged Asian option with floating strike K_1 and fixed strike K_2 defined by

$$(5.2.7) \qquad V(S_0) \triangleq \mathbb{E}^{\mathbb{P}}\left\{ e^{-rT}\left(\zeta \cdot \left(\frac{1}{T}\int_0^T S_t dt - K_1 S_T - K_2 \right) \right)^+ \right\}$$

can also be represented as

$$(5.2.8) \qquad V(S_0) = S_0 \cdot \mathbb{E}^{\mathbb{Q}}_{0, Z_0^J}[(\zeta \cdot (Z_T^J - K_1))^+],$$

in which $\zeta \in \{-1, 1\}$ indicates whether the option is a put or a call. Throughout this chapter, the short hand notation $\mathbb{E}^{\mathbb{Q}}_{t,z}$ represents the conditional expectation under $\mathbb{Q}$, given the value of the underlying process at time t is z. Under the measure $\mathbb{Q}$, the mean measure of the Poisson random measure is $\lambda \tilde{\nu}(dy)dt$, in which $\tilde{\nu}(dy) = y\nu(dy)$.

5.2.2 Main theoretical results

In this section we show that

$$V(S_0) = S_0 \cdot v(Z_0^J, 0), \tag{5.2.9}$$

for some v that is the classical solution, i.e., $v \in C^{2,1}$, of an integro-partial differential equation (see Theorem 5.2.1 and Corollary 5.2.2). We will also prove that v is the limit of a sequence of functions constructed by iterating a functional operator, which is defined in (5.2.11). We will show that each of the functions in this sequence are classical solutions of partial differential equations (not integro-differential equations) and that they converge to v locally uniformly and exponentially fast (see Theorem 5.2.1). The analytical properties of the functional operator (listed in the lemmas of Appendix 5.4.1) used in the construction of the approximating sequence plays an important role in proving our main mathematical result. We will summarize the role of the functional operator below after we introduce it.

Let us introduce the following sequence of functions

(5.2.10)

$$v_0(z,t) \triangleq (\zeta \cdot (z - K_1))^+, \quad v_{n+1}(z,t) \triangleq J v_n(z,t) \quad n \geq 0, \quad \text{for all } (z,t) \in \mathbb{R} \times [0,T],$$

in which the functional operator J is defined, through its action on a test function $f : \mathbb{R} \times [0,T] \to \mathbb{R}_+$, as follows:

(5.2.11)

$$J f(z,t) = \mathbb{E}_{t,z}^{\mathbb{Q}} \left\{ e^{-\lambda \xi (T-t)} \left(\zeta \cdot (Z_T - K_1) \right)^+ + \int_t^T e^{-\lambda \xi (s-t)} \lambda \cdot P f(Z_s, s) \, ds \right\},$$

in which $Z = \{Z_t; t \geq 0\}$ is a diffusion process with the dynamics

$$dZ_t = -\mu (q_t - Z_t) \, dt + \sigma (q_t - Z_t) \, dW_t,$$

and

(5.2.12)

$$Pf(Z_t, t) = \int_{\mathbb{R}_+} f\left(Z_t + (q_t - Z_t)\frac{y-1}{y}, t\right) y\nu(dy) \int_{\mathbb{R}_+} f\left(\frac{Z_t}{y} + q_t\frac{y-1}{y}, t\right) y\nu(dy).$$

We will show that the sequence of functions defined in (5.2.10) by iterating J are classical solutions of PDEs thanks to the following analytical properties of the operator J (which are developed in Appendix 5.4.1) : 1) J maps functions that are Lipschitz continuous with respect to the z-variable (uniformly in the t-variable) and Hölder continuous with respect to the t-variable into classical solutions of PDEs (see Proposition 5.4.6), 2) J preserves Lipschitz continuity with respect to the z-variable (see Lemma 5.4.1), 3) J transforms Lipschitz continuous functions, with respect to the z-variable, that satisfy a linear (in the z-variable) growth condition (uniformly in the t-variable) into Hölder continuous functions of the t-variable (see Lemma 5.4.5), 4) J preserves the linear growth condition in the $z-$variable (see Lemma 5.4.3 and Remark 5.4.2). Therefore, the analytical properties of J can be summarized as "J maps nice functions (set of functions with a few regularity properties), to nicer functions (set of functions that are the classical solutions of partial differential equations, and have the same regularity properties as functions before applying J).

It is a priori not clear that the sequence of functions defined in (5.2.10) has a limit. Using the properties of the operator J we show that this sequence is Cauchy (see Lemma 5.4.10) and therefore has a limit (in fact the sequence converges locally uniformly and exponentially fast). We show that, the limit of this sequence, which we denote by v_∞, is a classical solution of an integro-PDE using 1) the fact that it is a fixed point of the operator J (see Lemma 5.4.12), 2) the facts that it is Lipschitz continuous with respect to the z-variable (uniformly in the t-variable) (see Lemma 5.4.13) and Hölder continuous with respect to the t-variable (see Corollary

5.4.14).

Finally, using a verification argument we will show that the limit v_∞ is indeed the function v in (5.2.9) (see Corollary 5.2.2).

The main theoretical results that are summarized above will be stated in the next theorem and its corollary. The proof of Theorem 5.2.1 is given in Appendix 5.4.3 which uses the results of Appendix 5.4.1 and 5.4.2 as summarized above.

Theorem 5.2.1. *(i) The sequence of functions defined in (5.2.10) has a pointwise limit. Let us denote this limit by $v_\infty(z, t)$.*

(ii)For any compact domain $\mathcal{D} \subset \mathbb{R}$, $v_n(z, t)$ converges uniformly to $v_\infty(z, t)$ for $(z, t) \in \mathcal{D} \times [0, T]$. Moreover,

$$(5.2.13) \qquad |v_\infty(z, t) - v_n(z, t)| \leq M_\mathcal{D} \left(1 - e^{-\lambda \eta (T - t)} \right)^n,$$

where $M_\mathcal{D}$ is a constant depending on $\mathcal{D}$ and $\eta = \max\{\xi, 1\}$.

(iii) For $n \geq 0$, the function v_{n+1} is the unique classical solution, i.e., $v_{n+1} \in C^{2,1}$, of

$$(5.2.14) \qquad \mathcal{A}(t) v_{n+1}(z, t) - \lambda \xi v_{n+1}(z, t) + \lambda \cdot (P v_n)(z, t) + \frac{\partial}{\partial t} v_{n+1}(z, t) = 0$$

$$(5.2.15) \qquad v_{n+1}(z, T) = (\zeta \cdot (z - K_1))^+,$$

for $(z, t) \in \mathbb{R} \times [0, T]$. The operator $\mathcal{A}(t)$ is defined as

$$(5.2.16) \qquad \mathcal{A}(t) \triangleq -\mu(q_t - z) \frac{\partial}{\partial z} + \frac{1}{2} \sigma^2 (q_t - z)^2 \frac{\partial^2}{\partial z^2}.$$

(iv) The function v_∞ is the unique classical solution, i.e., $v_\infty \in C^{2,1}$, of the following partial integro-differential equation in Večeř and Xu (2004)

$$(5.2.17) \qquad \mathcal{A}(t) v_\infty(z, t) - \lambda \xi v_\infty(z, t) + \lambda \cdot (P v_\infty)(z, t) + \frac{\partial}{\partial t} v_\infty(z, t) = 0$$

$$(5.2.18) \qquad v_\infty(z, T) = (\zeta \cdot (z - K_1))^+.$$

Proof. See Appendix 5.4.3. □

The iterative procedure in (5.2.14) simply collapses to Vecer's PDE (see Večeř (2001)) when $\lambda = 0$, i.e., when the underlying asset is a geometric Brownian motion. Therefore, the iteration in (5.2.14) is designed for the models in which the asset price jumps.

Corollary 5.2.2. *Let $V(S_0)$ be as in (5.2.7), i.e., $V(S_0)$ is the value of the Asian option for jump diffusion S whose dynamics is given in (5.2.1). Then we have*

$$(5.2.19) \qquad V(S_0) = S_0 \cdot v_\infty(z, 0),$$

in which $v_\infty(\cdot, \cdot)$ is the unique solution of the integro-partial differential equation (5.2.17) with terminal condition (5.2.18), and

$$(5.2.20) \qquad z = \frac{X_0}{S_0} = \frac{1}{rT}\left(1 - e^{-rT}\right) - e^{-rT}\frac{K_2}{S_0}.$$

Proof. Let us define

$$(5.2.21) \qquad M_t = v_\infty(Z_t^J, t), \quad t \in [0, T]$$

where Z^J, defined in (5.2.3), has the initial value $Z_0^J = z$. It follows from (5.2.17) and the Itô's lemma that M_t is a $\mathbb{Q}$-martigale, i.e., $M_t = \mathbb{E}^{\mathbb{Q}}\{M_T | \mathcal{F}_t\}$. As a result

$$(5.2.22)$$
$$v_\infty(z, 0) = M_0 = \mathbb{E}^{\mathbb{Q}}_{0,z}\{M_T\} = \mathbb{E}^{\mathbb{Q}}_{0,z}\{v_\infty(Z_T^J, T)\} = \mathbb{E}^{\mathbb{Q}}_{0,z}\{(\zeta \cdot (Z_T^J - K_1))^+\} = \frac{V(S_0)}{S_0},$$

where the last identity follows from the representation (5.2.8). □

5.3 Computing the prices of Asian options numerically

Note that numerically solving (5.2.17) and (5.2.18) is quite difficult due to the contribution from the non-local integral term (i.e., the Pv_∞ term). However, It follows from Theorem 5.2.1 that the sequence of functions $(v_n)_{n \geq 0}$ converges uniformly

and exponentially fast to v_∞ on any compact domain. Therefore, starting from v_0, a few iterations of (5.2.14) and (5.2.15) will produce an accurate approximation to v_∞. To perform the iterations we make use of the finite difference methods for PDEs to solve (5.2.14) and (5.2.15) numerically. We describe this numerical method and show its convergence in Section 5.3.1. In Section 5.3.2 we determine the performance (the speed and accuracy characteristics) of our numerical method for the jump diffusion models of Kou (2002) and Merton (1976). In the same section we take the Monte-Carlo simulation results as a benchmark.

5.3.1   A numerical algorithm and its convergence

Our numerical algorithm is given by the following iterative scheme:

Step 1: Discretizing the PDEs. For any $n \geq 1$, we discretize (5.2.14) using the Crank-Nicolson method (see e.g. Wilmott et al. (1995) pp. 155). For fixed Δt, Δz, $z_{\max}$ and $z_{\min}$, such that $M\Delta t = T$ and $K\Delta z = z_{\max} - z_{\min}$, let us denote $z_k \triangleq z_{min} + k\Delta z$, $k = 0, 1, \cdots, K$. By $\tilde{v}_{n+1}$ we will denote the solution of the difference equation

(5.3.1)

$$(1 + p_{k,m}^0)\tilde{v}_{n+1}(k, m) - p_{k,m}^+\tilde{v}_{n+1}(k+1, m) - p_{k,m}^-\tilde{v}_{n+1}(k-1, m)$$

$$= p_{k,m+1}^+\tilde{v}_{n+1}(k+1, m+1) + p_{k,m+1}^-\tilde{v}_{n+1}(k-1, m+1)$$

$$+ (1 - p_{k,m+1}^0)\tilde{v}_{n+1}(k, m+1) + \frac{1}{2}\lambda\Delta t\left[\left(\widetilde{P}\tilde{v}_n\right)(k, m+1) + \left(\widetilde{P}\tilde{v}_n\right)(k, m)\right].$$

for $m = M-1, M-2, \cdots, 0$, $k = 0, 1, \cdots, K$, satisfying the terminal condition $\tilde{v}_{n+1}(k, M) = (\zeta \cdot (z_k - K_1))^+$ and boundary conditions $\tilde{v}_{n+1}(0, m) = (\zeta \cdot (z_{min} - K_1))^+$ and $\tilde{v}_{n+1}(K, m) = (\zeta \cdot (z_{max} - K_1))^+$. The coefficients $p_{k,m}^+$, $p_{k,m}^-$ and $p_{k,m}^0$

in (5.3.1) are given by

$$p_{k,m}^+ = \frac{1}{4}\left[\sigma^2\left(\frac{q_{m\Delta t} - k\Delta z}{\Delta z}\right)^2 - \mu\frac{q_{m\Delta t} - k\Delta z}{\Delta z}\right]\Delta t,$$

(5.3.2)
$$p_{k,m}^- = \frac{1}{4}\left[\sigma^2\left(\frac{q_{m\Delta t} - k\Delta z}{\Delta z}\right)^2 + \mu\frac{q_{m\Delta t} - k\Delta z}{\Delta z}\right]\Delta t,$$

$$p_{k,m}^0 = p_{k,m}^+ + p_{k,m}^- + \frac{1}{2}\lambda\xi\Delta t.$$

The numerical integral term $\widetilde{P}\tilde{v}_n$ in (5.3.1) will be described in Step 2.

Step 2: Numerical Integration. Let $x = \log y$, the integral term Pv_n (choosing $f = v_n$ in (5.2.12)) can be rewritten as

(5.3.3)
$$Pv_n(z,t) = \int_{\mathbb{R}} v_n\left(\frac{z}{e^x} + q_t\frac{e^x - 1}{e^x}, t\right)e^x F(dx),$$

in which $F(dx)$ is the distribution of a random variable X, such that the distribution of e^X is given by the jump measure ν. We approximate the integral in (5.3.3) by trapezoidal rule as follows.

Discretizing a sufficiently large interval $[x_{min}, x_{max}]$ into L subintervals, we obtain the grid $x_{min} = x_0 < x_1 < \cdots < x_L = x_{max}$. We choose unequally spaced grids, so that the grid is finer where density of the distribution F is large. Starting from $\tilde{v}_0(k,m) = (\zeta \cdot (z_k - K_1))^+$, the integral in (5.3.3) evaluated on the grid point $(z_k, m\Delta t)$ is approximated by its discrete version

(5.3.4)

$$\widetilde{P}\tilde{v}_n(z_k, m\Delta t)$$

$$= \sum_{\ell=0}^{L-1}\frac{1}{2}\left[\tilde{v}_n\left(\frac{z_k}{e^{x_\ell}} + q_{m\Delta t}\frac{e^{x_\ell} - 1}{e^{x_\ell}}, m\Delta t\right)e^{x_\ell}g(x_\ell)\right.$$

$$\left. + \tilde{v}_n\left(\frac{z_k}{e^{x_{\ell+1}}} + q_{m\Delta t}\frac{e^{x_{\ell+1}} - 1}{e^{x_{\ell+1}}}, m\Delta t\right)e^{x_{\ell+1}}g(x_{\ell+1})\right]\cdot(x_{\ell+1} - x_\ell) + O((\Delta x)^2),$$

where $\Delta x = \max_\ell (x_{\ell+1} - x_\ell)$ and g is the density of F. On the right-hand-side of (5.3.4), the value of $\tilde{v}_n$ not on grid points are determined by the value of $\tilde{v}_n$

on grid points via the linear interpolation, i.e.,

(5.3.5)
$$\tilde{v}_n\left(\frac{z_k}{e^{x_\ell}} + q_{m\Delta t}\frac{e^{x_\ell}-1}{e^{x_\ell}}, m\Delta t\right)$$

$$= \begin{cases} (1-w)\,\tilde{v}_n(z_{k'}, m\Delta t) + w\,\tilde{v}_n(z_{k'+1}, m\Delta t) + O((\Delta z)^2), \\ \qquad\qquad \text{if } z_{k'} \le \frac{z_k}{e^{x_\ell}} + q_{m\Delta t}\frac{e^{x_\ell}-1}{e^{x_\ell}} \le z_{k'+1} \text{ for some } k', \\ \left(\zeta \cdot \left(\frac{z_k}{e^{x_\ell}} + q_{m\Delta t}\frac{e^{x_\ell}-1}{e^{x_\ell}} - K_1\right)\right)^+, \quad \text{if } \frac{z_k}{e^{x_\ell}} + q_{m\Delta t}\frac{e^{x_\ell}-1}{e^{x_\ell}} \notin [z_{min}, z_{max}]. \end{cases}$$

Here w is the linear interpolation weight.

Step 3: Solving the sparse system of linear equations. After evaluating the integral term $\widetilde{P}\tilde{v}_n$ on the grid points using (5.3.4), we solve the sparse system of linear equations in (5.3.1) by using the SOR algorithm (see e.g. Wilmott et al. (1995) pp. 150) to obtain $\tilde{v}_{n+1}$.

Step 4: Updating n. Unless $\max_k |\tilde{v}_{n+1}(k,0) - \tilde{v}_n(k,0)| \le \epsilon$ increase the value of n by 1 and go to Step 1. Here ϵ is the iteration error tolerance. In the numerical experiments in Section 5.3.2, ϵ is chosen as 10^{-5}.

The convergence of the above numerical algorithm is ensured by the following proposition. We denote $\tilde{v}$ as the solution of the difference equation obtained by discretizing (5.2.17) via the Crank-Nicolson scheme, i.e., $\tilde{v}$ satisfies (5.3.1) with both $\tilde{v}_{n+1}$ and $\tilde{v}_n$ replaced by $\tilde{v}$. The following proposition shows that $\tilde{v}_n$ converges to $\tilde{v}$ as $n \to \infty$ and $\tilde{v}$ converges to v_∞ as the mesh sizes go to zero. In this proposition, we choose Δt and Δz sufficiently small such that $p_{k,m}^+$, $p_{k,m}^-$ and $1 - p_{k,m}^0$ are positive for all (k,m). Moreover, for the simplicity of the presentation, in what follows we assume that $(\widetilde{P}1)(k,m) \le \int_{R_+} y\nu(dy) = \xi$ (otherwise the order of error of the discretization of the integral will have to be sufficiently small in the following proposition).

Proposition 5.3.1. *(i) Let $E_n = \max_{m,k} |\tilde{v}(k, m) - \tilde{v}_n(k, m)|$. Then*

$$(5.3.6) \qquad E_n \leq \left(1 - \theta^M\right)^n E_0 \to 0 \qquad as \ n \to \infty,$$

where $\theta \triangleq \frac{1 - 1/2 \cdot \lambda \xi \Delta t}{1 + 1/2 \cdot \lambda \xi \Delta t} \in (0, 1]$ with $1 - \theta^M \to 1 - e^{-\lambda \xi T}$ as $M \to \infty$.

(ii)

$$(5.3.7) \qquad |v_\infty(z_k, m\Delta t) - \tilde{v}(k, m)| \to 0, \qquad as \ \Delta z, \Delta t, \Delta x \to 0.$$

Moreover, the convergence rate is given by

$$(5.3.8)$$

$$|v_\infty(z_k, m\Delta t) - \tilde{v}(k, m)| \sim O\left[\Delta^2 \log 1/\Delta^2\right], \qquad where \ \Delta \triangleq \sqrt{(\Delta t)^2 + (\Delta z)^2 + (\Delta x)^2}.$$

Proof. See Appendix 5.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5.3.2 Numerical results for Kou's and Merton's models

There are two well-known examples of jump diffusion in the literature, the double exponential model as in Kou (2002) and the normal model as in Merton (1976). In this section, we demonstrate our algorithm in Section 5.3.1 in pricing Asian options for these two models. We will introduce the jump distributions chosen by Kou (2002) and Merton (1976) next. Let X be a random variable whose probability distribution function is equal to a given distribution F and let the jump measure ν be equal to the distribution of the random variable e^X. In Kou's model, F is the double exponential distribution whose density is

$$(5.3.9) \qquad F(dx) = \left(p \, \eta_1 e^{-\eta_1 x} 1_{\{x \geq 0\}} + (1 - p) \, \eta_2 e^{\eta_2 x} 1_{\{x < 0\}}\right) dx.$$

In Merton's model, F is the normal distribution whose density is

$$(5.3.10) \qquad F(dx) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left(\frac{-(x - \mu)^2}{\tilde{\sigma}^2}\right) dx.$$

In the following, we demonstrate the numerical performance of the algorithm developed in Section 5.3.1. Since we could not find any numerical results on European Asian options for jump diffusion models in the literature, we use the Put-Call parity for Asian options as a consistency check for our results. For the European Asian option with floating strike K_1 and fixed strike K_2, the put - call parity gives the following identity between call and put option price

$$(5.3.11) \qquad C(S_0, 0) - P(S_0, 0) = \frac{1}{rT}(1 - e^{-rT})S_0 - K_1 S_0 - e^{-rT}K_2.$$

This identity does not depend on dynamics of the underlying process S. Using our algorithm, we calculate the call and put option price independently. Then we compare the difference between our call and put price with the difference predicted by (5.3.11). In addition, we also compare our numerical results with the results from Monte-Carlo simulations.

In Tables 1 and 2, we list the numerical results for the prices of European Asian options for both Kou's model and Merton's model. Run times are in seconds. All our computations are performed on a Pentium IV 3.0 GHz machine with C++ implementation. In Tables 3, 4 and 5, we list the convergence results for the double exponential jump model. We consider the effects of the truncation error in evaluating (5.3.3) in Table 3. On the other hand, Table 4 demonstrates the convergence of the price as the grid size used in the numerical integration becomes smaller. In table 5, our focus is on the grid size used in the finite difference scheme. The parameters for both call and put options in Tables 3, 4, and 5 are the same as the 7th row in Table 1, i.e., $r = 0.15$, $S_0 = 100$, $K_1 = 0$, $K_2 = 90$, $T = 1$, $\sigma = 0.2$, $\lambda = 1$ and $\eta_1 = \eta_2 = 25$.

As we can see from these tables, our algorithm is stable with respect to all parameters and the convergence is fast. Moreover, our difference between call and put option prices are within ± 0.01 comparing to the difference predicted by put-call par-

ity in (5.3.11). The call option prices are almost within the standard error of the Monte Carlo results.

5.4 Mathematical analysis towards proving Theorem 5.2.2

The purpose of this section is to provide the necessary background to prove Theorem 5.2.1. First, in Section 5.4.1 we analyze the properties of the functional operator J: We study how J increases the regularity of certain class of functions. In Lemmas 5.4.1 - 5.4.5 and Proposition 5.4.6, we will see that J takes functions with certain regularity properties into the unique solutions of parabolic differential equations and gives them more regularity properties. Next, in Section 5.4.2, we develop the properties of the functions defined in (5.2.10) in a sequence of lemmas and corollaries using the results developed in Section 5.4.1. These properties will then be used to prove Theorem 5.2.1 in Section 5.4.3.

5.4.1 Properties of operator J

First, we will develop a representation of the functional operator J that is amenable to regularity analysis, which is carried out in this section. Using the notation in page 8 of Pham (1998), we can rewrite J as

(5.4.1)
$$Jf(z,t) = \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\lambda\xi(T-t)} \left(\zeta \cdot \left(Z_{T-t}^{t,z} - K_1 \right) \right)^+ + \int_0^{T-t} e^{-\lambda\xi s} \lambda \cdot Pf \left(Z_s^{t,z}, t+s \right) ds \right\},$$

in which

(5.4.2)
$$Pf \left(Z_s^{t,z}, t+s \right) = \int_{\mathbb{R}^+} f \left(\frac{Z_s^{t,z}}{y} + q_{t+s} \frac{y-1}{y}, t+s \right) y\nu(dy).$$

The process $Z^{t,z} = \{Z_s^{t,z}; s \geq 0\}$ has the dynamics

(5.4.3)
$$dZ_s^{t,z} = -\mu(q_{t+s} - Z_s^{t,z})ds + \sigma(q_{t+s} - Z_s^{t,z})dW_s, \quad Z_0^{t,z} = z,$$

where $\{W_s\}_{s\geq 0}$ is a Wiener process under the measure $\mathbb{Q}$. It is possible to determine the solution to (5.4.3) explicitly. For this purpose it will be convenient to work with the process $\tilde{Z}_s \triangleq q_{t+s} - Z_s^{t,z}$. It follows from (5.4.3) that the dynamics of $\tilde{Z}$ are given by

$$(5.4.4) \qquad d\tilde{Z}_s = \mu\tilde{Z}_s ds - \sigma\tilde{Z}_s dW_s + g(t+s)ds, \quad \tilde{Z}_0 = \tilde{z} = q_t - z,$$

in which $g(t) = \frac{d}{dt}q_t$. Now it is easy to obtain the solution of stochastic differential equation (5.4.4) as

$$(5.4.5) \qquad \tilde{Z}_s = \tilde{z}H_s^0 + \int_0^s H_{s-v}^0\, g(t+v)dv \quad \text{for } s \geq 0,$$

in which

$$(5.4.6) \qquad H_s^0 \triangleq \exp((\mu - \frac{1}{2}\sigma^2)s - \sigma W_s).$$

As a result we have that the solution of (5.4.3) is given by

$$(5.4.7) \qquad Z_s^{t,z} = zH_s^0 + b_s, \quad s \geq 0,$$

in which

$$(5.4.8) \qquad b_s \triangleq q_{t+s} - q_t H_s^0 - \int_0^s H_{s-v}^0\, g(t+v)dv.$$

It follows from (5.4.7) that the solution of the stochastic differential equation (5.4.3) is linear with respect to its initial value z. Inserting its solution (5.4.7) back into the definition of the operator J in (5.4.1), we obtain

(5.4.9)

$$Jf(z,t) = \mathbb{E}^{\mathbb{Q}}\left\{ e^{-\lambda\xi(T-t)}\left(\zeta\cdot\left(zH_{T-t}^0 + b_{T-t} - K_1\right)\right)^+ + \int_0^{T-t} e^{-\lambda\xi s}\lambda\cdot Pf(zH_s^0 + b_s, t+s)\,ds\right\}.$$

In the following, we will study the regularity properties of the operator J with respect to both space and time. When the function f is Lipschitz continuous with

respect to its first variable, the following lemmas show Jf is not only Lipschitz with respect to its first variable, but also Hölder continuous with respect to the second variable.

Lemma 5.4.1. *For any $t \in [0, T]$, let us assume the function f satisfies*

$$(5.4.10) \qquad |f(z, t) - f(\tilde{z}, t)| \leq D\, |z - \tilde{z}|, \quad z, \tilde{z} \in \mathbb{R},$$

for a positive constant D that only depends on T. Then Jf satisfies

$$(5.4.11) \qquad |Jf(z, t) - Jf(\tilde{z}, t)| \leq E\, |z - \tilde{z}|, \quad z, \tilde{z} \in \mathbb{R},$$

with $E = \max\{1, D\}$.

Proof. From the definition of operator J in (5.4.9), we have

$$(5.4.12)$$

$$|Jf(z, t) - Jf(\tilde{z}, t)|$$

$$\leq \mathbb{E}^{\mathbb{Q}}\bigg\{ e^{-\lambda \xi(T-t)} \left| \left(\zeta \cdot \left(z H^0_{T-t} + b_{T-t} - K_1 \right) \right)^+ - \left(\zeta \cdot \left(\tilde{z} H^0_{T-t} + b_{T-t} - K_1 \right) \right)^+ \right| $$
$$+ \int_0^{T-t} ds\, e^{-\lambda \xi s} \lambda \cdot \left| Pf(z H^0_s + b_s, t + s) - Pf(\tilde{z} H^0_s + b_s, t + s) \right| \bigg\}.$$

Let us obtain a bound on the right-hand-side of (5.4.12). First observe that

$$(5.4.13) \quad \left| \left(\zeta \cdot \left(z H^0_{T-t} + b_{T-t} - K_1 \right) \right)^+ - \left(\zeta \cdot \left(\tilde{z} H^0_{T-t} + b_{T-t} - K_1 \right) \right)^+ \right| \leq |z - \tilde{z}| H^0_{T-t},$$

and

$$(5.4.14)$$

$$\left| Pf(z H^0_s + b_s, t + s) - Pf(\tilde{z} H^0_s + b_s, t + s) \right|$$

$$\leq \int_{\mathbb{R}_+} \left| f\left(\frac{z H^0_s}{y} + \frac{b_s}{y} + q_{t+s}\frac{y-1}{y}, t + s \right) - f\left(\frac{\tilde{z} H^0_s}{y} + \frac{b_s}{y} + q_{t+s}\frac{y-1}{y}, t + s \right) \right| y\nu(dy)$$

$$\leq \int_{\mathbb{R}_+} D\, |z H^0_s - \tilde{z} H^0_s|\, \nu(dy)$$

$$= D\, |z - \tilde{z}|\, H^0_s.$$

On the other hand, from the definition of H_s^0 in (5.4.6), we have that

$$(5.4.15) \qquad\qquad \mathbb{E}^{\mathbb{Q}}\{H_s^0\} = e^{\mu s}.$$

Inserting (5.4.13), (5.4.14) and (5.4.15) back into the equation (5.4.12), we have

$$
\begin{aligned}
|Jf(z,t) - Jf(\tilde{z},t)| \;&\leq\; e^{-\lambda\xi(T-t)}\,|z-\tilde{z}|\,\mathbb{E}^{\mathbb{Q}}\{H_{T-t}^0\} + \int_0^{T-t} ds\, e^{-\lambda\xi s}\,\lambda\,D\,|z-\tilde{z}|\,\mathbb{E}^{\mathbb{Q}}\{H_s^0\} \\
&=\; |z-\tilde{z}|\left(e^{(\mu-\lambda\xi)(T-t)} + D\int_0^{T-t} ds\,\lambda\,e^{(\mu-\lambda\xi)s}\right) \\
&\leq\; \left(D + (1-D)e^{-\lambda(T-t)}\right)|z-\tilde{z}| \\
(5.4.16)\qquad &\leq\; \max\{1,D\}\,|z-\tilde{z}|.
\end{aligned}
$$

$\square$

Remark 5.4.2. Let us define

$$(5.4.17) \qquad\qquad M_f \;\triangleq\; \sup_{t\in[0,T]} f(0,t),$$

$$(5.4.18) \qquad\qquad M_{Jf} \;\triangleq\; \sup_{t\in[0,T]} Jf(0,t).$$

It follows from the Lipschitz conditions (5.4.10) and (5.4.11) that both f and Jf satisfy linear growth conditions, if M_f and M_{Jf} are finite, since for $(z,t) \in \mathbb{R} \times [0,T]$

$$(5.4.19) \qquad\qquad f(z,t) \leq f(0,t) + D\,|z|,$$

$$(5.4.20) \qquad\qquad Jf(z,t) \leq Jf(0,t) + E\,|z|.$$

In the next two lemmas we will need the following moment estimates of $Z_s^{t,z}$.

$$(5.4.21) \qquad\qquad \mathbb{E}^{\mathbb{Q}}\left\{|Z_s^{t,z}|\right\} \;\leq\; C(1+|z|),$$

$$(5.4.22) \qquad\qquad \mathbb{E}^{\mathbb{Q}}\left\{|Z_s^{t,z} - z|\right\} \;\leq\; C(1+|z|)s^{\frac{1}{2}},$$

in which $0 \leq s \leq T$ and C is a constant depending on T. These estimates can be found in Pham (1998) (Lemma 3.1).

Lemma 5.4.3. *We have that*

$$(5.4.23) \qquad M_{Jf} \le U + \alpha \left(M_f + \frac{B}{\xi} \right),$$

in which $\alpha = 1 - e^{-\lambda \xi T} < 1$, and U, B are positive constants depending on T.

Proof. We will estimate M_{Jf} using the definition of the operator J in (5.4.1). First, we have that

$$\mathbb{E}^{\mathbb{Q}} \left\{ e^{-\lambda \xi (T-t)} \left(\zeta \cdot \left(Z_{T-t}^{t,0} - K_1 \right) \right)^+ \right\} \le \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\lambda \xi (T-t)} \left(\left| Z_{T-t}^{t,0} \right| + K_1 \right) \right\}$$

$$= \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\lambda \xi (T-t)} \left(|b_{T-t}| + K_1 \right) \right\},$$

in which we obtain the last inequality using the expression of $Z^{t,z}$ in (5.4.7) with $z = 0$. First, it follows from (5.4.21) with $z = 0$ that

$$(5.4.24) \qquad \mathbb{E}^{\mathbb{Q}}\{|b_{T-t}|\} = \mathbb{E}^{\mathbb{Q}}\{|Z_{T-t}^{t,0}|\} \le C.$$

Letting $U \triangleq C + K_1$, which is a finite positive constant depending on T, we have that

$$(5.4.25) \qquad \mathbb{E}^{\mathbb{Q}} \left\{ e^{-\lambda \xi (T-t)} \left(|b_{T-t}| + K_1 \right) \right\} \le U.$$

Second, we will estimate the second term in the definition of J in (5.4.1). From the definition of Pf in (5.4.2), we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}} \left\{ Pf(Z_s^{t,0}, t+s) \right\} &= \mathbb{E}^{\mathbb{Q}} \left\{ \int_{\mathbb{R}_+} f\left(\frac{Z_s^{t,0}}{y} + q_{t+s} \frac{y-1}{y}, t+s \right) y \nu(dy) \right\} \\
&\le \mathbb{E}^{\mathbb{Q}} \left\{ \int_{\mathbb{R}_+} \left(f(0, t+s) + D \frac{|Z_s^{t,0}|}{y} + D\, q_{t+s} \frac{|y-1|}{y} \right) y \nu(dy) \right\} \\
&\le \xi f(0, t+s) + D\, q_{t+s}(\xi + 1) + D\, \mathbb{E}^{\mathbb{Q}} \left\{ |Z_s^{t,0}| \right\} \\
(5.4.26) \qquad &\le \xi f(0, t+s) + D\, q_{t+s}(\xi + 1) + C \cdot D.
\end{aligned}$$

To obtain the first inequality we use the inequality (5.4.19), whereas the second inequality follows from $|y - 1| \le y + 1$. To obtain the last inequality, we use the

inequality (5.4.21) with $z = 0$. Now, using (5.4.26) we obtain

$$\mathbb{E}^{\mathbb{Q}} \left\{ \int_0^{T-t} e^{-\lambda \xi s} \lambda \cdot Pf\left(Z_s^{t,0}, t+s\right) \, ds \right\}$$

(5.4.27)
$$\leq \int_0^{T-t} e^{-\lambda \xi s} \lambda \cdot [\xi f(0, t+s) + D\, q_{t+s}(\xi+1) + C \cdot D] \, ds.$$

Since $0 \leq s \leq T - t$, we have $q_{t+s} \leq \frac{1}{rT}$. Let us define

(5.4.28)
$$B \triangleq \left[\frac{1}{rT}(\xi+1) + C \right] \cdot D,$$

which is a finite positive constant depending on T. Now, we have the following estimation on the left-hand-side of (5.4.27)

$$\mathbb{E}^{\mathbb{Q}} \left\{ \int_0^{T-t} e^{-\lambda \xi s} \lambda \cdot Pf(Z_s^{t,0}, t+s) \, ds \right\} \leq \int_0^{T-t} e^{-\lambda \xi s} \lambda \cdot [\xi f(0, t+s) + B] \, ds$$

$$\leq \left(1 - e^{-\lambda \xi (T-t)}\right) \left(M_f + \frac{B}{\xi} \right)$$

(5.4.29)
$$\leq \left(1 - e^{-\lambda \xi T}\right) \left(M_f + \frac{B}{\xi} \right), \quad \text{for } t \in [0, T].$$

From inequalities (5.4.25) and (5.4.29), we conclude that

(5.4.30)
$$Jf(0, t) \leq U + \left(1 - e^{-\lambda \xi T}\right) \left(M_f + \frac{B}{\xi} \right).$$

$\square$

Remark 5.4.4. Lemma 5.4.3 and Remark 5.4.2 indicate that

(5.4.31)
$$f(z, t) \leq M_f + D\,|z| \leq \tilde{D}(1 + |z|),$$

(5.4.32)
$$Jf(z, t) \leq U + \alpha \left(M_f + \frac{B}{\xi} \right) + E\,|z| \leq \tilde{E}(1 + |z|),$$

in which $\tilde{D} = \max\{M_f, D\}$ and $\tilde{E} = \max\{U + \alpha(M_f + B/\xi), E\}$. We will use these linear growth properties to show a regularity property of the operator J with respect to time in the next lemma.

Lemma 5.4.5. *Assume the function* $z \mapsto f(z,t)$ *satisfies*

$$(5.4.33) \qquad\qquad |f(z,t) - f(\tilde{z},t)| \leq D\,|z - \tilde{z}|,$$

for $z, \tilde{z} \in \mathbb{R}$ *as in Lemma 5.4.1 and* $M_f < \infty$. *Then* $t \mapsto Jf(z,t)$ *satisfies*

$$(5.4.34) \qquad |Jf(z,t) - Jf(z,s)| \leq F\,(1 + |z|)\,(s - t)^{\frac{1}{2}}, \quad 0 \leq t < s \leq T,$$

in which F *is a positive constant that only depends on* λ, ξ, T *and* M_f.

Proof. For any $h \in [t, T]$, it follows from the definition of operator J in (5.4.1) and the Markov property of $Z_s^{t,z}$ that

$$(5.4.35) \quad Jf(z,t) = \mathbb{E}^{\mathbb{Q}} \left\{ \int_0^{h-t} e^{-\lambda\xi v}\,\lambda \cdot Pf(Z_v^{t,z}, t+v)\,dv + e^{-\lambda\xi(h-t)} Jf(Z_{h-t}^{t,z}, h) \right\}.$$

With $h = s$,

(5.4.36)

$$|Jf(z,t) - Jf(z,s)|$$

$$\leq \mathbb{E}^{\mathbb{Q}} \left\{ \int_0^{s-t} e^{-\lambda\xi v}\,\lambda \cdot Pf(Z_v^{t,z}, t+v)\,dv + |e^{-\lambda\xi(s-t)} Jf(Z_{s-t}^{t,z}, s) - Jf(z,s)| \right\}$$

$$\leq \mathbb{E}^{\mathbb{Q}} \left\{ \int_0^{s-t} e^{-\lambda\xi v}\,\lambda \cdot Pf(Z_v^{t,z}, t+v)\,dv + e^{-\lambda\xi(s-t)} \left| Jf(Z_{s-t}^{t,z}, s) - Jf(z,s) \right| \right.$$

$$\left. + \left| e^{-\lambda\xi(s-t)} - 1 \right| Jf(z,s) \right\}.$$

In what follows we will bound the terms on the right-hand-side of this inequality. Since the condition (5.4.33) holds, Lemma 5.4.1 applies. As a result it follows from (5.4.11) that

$$(5.4.37) \qquad \mathbb{E}^{\mathbb{Q}} \left\{ |Jf(Z_{s-t}^{t,z}, s) - Jf(z,s)| \right\} \leq E\,\mathbb{E}^{\mathbb{Q}} \left\{ |Z_{s-t}^{t,z} - z| \right\},$$

Using the estimate in (5.4.31) we have that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}} \left\{ Pf(Z_v^{t,z}, t+v) \right\} &= \int_{\mathbb{R}_+} y\nu(dy)\, \mathbb{E}^{\mathbb{Q}} \left\{ f\left(\frac{Z_v^{t,z}}{y} + q_{t+v}\frac{y-1}{y}, t+v \right) \right\} \\
&\leq \int_{\mathbb{R}_+} y\nu(dy)\, \tilde{D}\left(1 + \frac{1}{y}\mathbb{E}^{\mathbb{Q}}\left\{ |Z_v^{t,z}| \right\} + q_{t+v}\frac{|y-1|}{y} \right) \\
&\leq \tilde{D}\left(\xi + \mathbb{E}^{\mathbb{Q}}\left\{ |Z_v^{t,z}| \right\} + (\xi+1)q_{t+v} \right) \\
(5.4.38) \qquad\qquad &\leq \tilde{D}\left(\xi + \frac{1}{rT}(\xi+1) + C(1+|z|) \right).
\end{aligned}
$$

To obtain the last inequality we use the estimation (5.4.21) and the fact that $q_{t+v} \leq \frac{1}{rT}$ for $v \in [0, s-t]$. On the other hand, from (5.4.32), we have that

$$
(5.4.39) \qquad\qquad |Jf(z,s)| \leq \tilde{E}(1+|z|).
$$

In the inequalities above, the constants $E, \tilde{D}$ and $\tilde{E}$ are as in Lemma 5.4.1 and Remark 5.4.4.

Now, using (5.4.37), (5.4.38), (5.4.39) and the inequalities

$$
(5.4.40) \qquad\qquad e^{-\lambda\xi v} < 1, \quad \text{and} \quad 1 - e^{-\lambda\xi(s-t)} \leq \lambda\xi(s-t),
$$

we can bound (5.4.36) as follows:

(5.4.41)

$$
|Jf(z,t) - Jf(z,s)|
$$

$$
\leq \tilde{D}\lambda\left(\xi + \frac{1}{rT}(\xi+1) + C(1+|z|) \right)(s-t) + E\,\mathbb{E}^{\mathbb{Q}}\left\{ |Z_{s-t}^{t,z} - z| \right\} + \lambda\xi\tilde{E}\,(1+|z|)\,(s-t)
$$

$$
\leq \tilde{D}\lambda\left(\xi + \frac{1}{rT}(\xi+1) + C(1+|z|) \right)(s-t) + E \cdot C\,(1+|z|)\,(s-t)^{\frac{1}{2}}
$$

$$
+ \lambda\xi\tilde{E}\,(1+|z|)\,(s-t)
$$

$$
\leq F\,(1+|z|)\,(s-t)^{\frac{1}{2}},
$$

where F is a positive constant only depending on λ, ξ, T and M_f. In (5.4.41), to obtain the second inequality, we use the moment estimates (5.4.22); to obtain the third inequality, we use the fact that $s - t \leq T$. $\qquad\square$

In the following proposition we show that Jf satisfies a parabolic partial differential equation.

Proposition 5.4.6. *Assume function* $f : \mathbb{R} \times [0, T] \to \mathbb{R}_+$ *satisfies the following condition*

$$(5.4.42) \quad |f(z,t) - f(\tilde{z}, s)| \leq D|z - \tilde{z}| + F(1 + |z|)\,|s - t|^{\frac{1}{2}}, \quad (z,t), (\tilde{z}, s) \in \mathbb{R} \times [0, T],$$

in which D *and* F *are constants, then the function* $Jf : \mathbb{R} \times [0, T] \to \mathbb{R}_+$ *is the unique classical solution, i.e.,* $Jf \in C^{2,1}$, *of*

$$(5.4.43) \qquad \mathcal{A}(t)Jf(z,t) - \lambda \xi Jf(z,t) + \lambda \cdot Pf(z,t) + \frac{\partial}{\partial t}Jf(z,t) = 0$$

$$(5.4.44) \qquad Jf(z, T) = (\zeta \cdot (z - K_1))^+.$$

Proof. It is clear from (5.4.1) that Jf satisfies the terminal condition. For any point $(z, t) \in \mathbb{R} \times [0, T]$, let us take a rectangle $R = [z_1, z_2] \times [0, T]$, so that $(z, t) \in R$. Denote the parabolic boundary of R by $\partial_0 R := \partial R - [z_1, z_2] \times \{0\}$. Consider the following parabolic partial differential equation

$$(5.4.45) \qquad \mathcal{A}(t)u(z,t) - \lambda \xi u(z,t) + \lambda \cdot Pf(z,t) + \frac{\partial}{\partial t}u(z,t) = 0$$

$$(5.4.46) \qquad u(z,t) = Jf(z,t), \quad \text{on } \partial_0 R.$$

Because of the condition (5.4.42), $z \to f(z,t)$ is Lipschitz in its first variable uniformly in the second variable, it follows from Lemmas 5.4.1 and 5.4.5 that $z \to Jf(z,t)$ is Lipschitz and $t \to Jf(z,t)$ is Hölder continuous. As a result $Jf(\cdot, \cdot)$ is a continuous function on $\mathbb{R} \times \mathbb{R}_+$, in particular, continuous on $\partial_0 R$.

On the other hand, for $(z,t),(\tilde{z},s) \in R$, it follows from the condition (5.4.42) that

(5.4.47)

$$
\begin{aligned}
&|Pf(z,t) - Pf(\tilde{z},s)| \\
&\leq \int_{\mathbb{R}_+} \left| f\left(\frac{z}{y} + q_t\frac{y-1}{y},t\right) - f\left(\frac{\tilde{z}}{y} + q_s\frac{y-1}{y},s\right) \right| y\nu(dy) \\
&\leq \int_{\mathbb{R}_+} \left[D|z - \tilde{z}| + D|q_t - q_s|\,|y - 1| + F\left(y + |z| + q_t|y-1|\right)|s - t|^{\frac{1}{2}} \right] \nu(dy) \\
&\leq D|z - \tilde{z}| + D(\xi + 1)\frac{e^{-rT}}{T}\left| \int_t^s e^{ru}du \right| + F\left(\xi + q_t(\xi + 1) + |z|\right)|s - t|^{\frac{1}{2}} \\
&\leq D|z - \tilde{z}| + \tilde{F}(1 + |z|)|s - t|^{\frac{1}{2}},
\end{aligned}
$$

in which $\tilde{F}$ only depends on T and ξ. Since R is a bounded domain, the factor $1+|z|$

in (5.4.47) is bounded in R, so $z \to Pf(z,t)$ is Lipschitz and $t \to Pf(z,t)$ is Hölder,

uniformly with respect to the other variable, in R. Now by Theorem 5.2 in Chapter

6 of Friedman (1964), the parabolic partial differential equation (5.4.45) and (5.4.46)

has a unique classical solution in the bounded domain R. Moreover this solution can

be represented by

$$
\begin{aligned}
u(z,t) &= \mathbb{E}^{\mathbb{Q}}\left\{ e^{-\lambda\xi\tau}Jf(Z_\tau^{t,z}, \tau + t) + \int_0^\tau e^{-\lambda\xi s}\lambda \cdot Pf(Z_s^{t,z}, t + s)\,ds \right\} \\
&= Jf(z,t),
\end{aligned}
$$

in which the exit time $\tau \triangleq \inf_{s \in [0,T-t]}\{Z_s^{t,z} = z_1 \text{ or } z_2\} \wedge (T - t)$, while the second

equality follows from the definition of the operator J in (5.4.1) and the strong Markov

property of $Z^{t,z}$.

So far we have shown that Jf agrees with the unique classical solution of (5.4.45)

and (5.4.46) in the bounded domain R. Since this statement holds for arbitrary R,

it is clear that Jf is a solution of the parabolic partial differential equation (5.4.43)

and (5.4.44) for all $(z,t) \in \mathbb{R} \times [0,T]$.

The uniqueness of the solution for (5.4.43) and (5.4.44) follows from Corollary 4.4

in Chapter 6 in Friedman (1964), since the coefficients of the derivative operators in (5.2.16) satisfy linear and quadratic growth conditions respectively. $\square$

5.4.2 Properties of the sequence of functions defined in (5.2.10)

Our first goal is to prove $z \to v_n(z,t)$ is Lipschitz and $t \to v_n(z,t)$ is Hölder continuous for all n. To this end we will apply Lemmas 5.4.1 and 5.4.5. To be able to apply the latter lemma we need to show that

$$(5.4.48) \qquad M_n \triangleq \sup_{t \in [0,T]} \{v_n(0,t)\} < \infty, \quad \text{for } n \geq 0.$$

In the next lemma, we will dominate the sequence of constants $(M_n)_{n \geq 0}$ by a universal constant M_∞, which depends only on T.

Lemma 5.4.7. *Let us define the sequence of constants $(M_n)_{n \geq 0}$ as in (5.4.48), then*

$$M_n < M_\infty \triangleq \frac{U}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{B}{\xi} + K_1 < \infty,$$

in which the constants U, B and α are defined in Lemma 5.4.3.

Proof. When $n = 0$, by the definition of $v_0(\cdot, \cdot)$ in (5.2.10), we have

$$M_0 = \sup_{t \in [0,T]} v_0(0,t) = (\zeta \cdot (0 - K_1))^+ \leq K_1,$$

in which the last inequality is saturated when $\zeta = -1$. It follows from Lemma 5.4.3 that

$$(5.4.49) \qquad M_{n+1} \leq U + \alpha \left(M_n + \frac{B}{\xi} \right), \quad \text{for } n \geq 0,$$

in which $\alpha < 1$. It can be proven by induction that

$$(5.4.50) \qquad M_n \leq U \left(\sum_{i=0}^{n} \alpha^i - \alpha^n \right) + \alpha \left(\sum_{i=0}^{n} \alpha^i - \alpha^n \right) \frac{B}{\xi} + \alpha^n K_1, \quad \text{for } n \geq 0.$$

Since U, B and ξ are positive constants and $0 < \alpha < 1$, it is clear from (5.4.50) that

$$M_n \leq \frac{U}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{B}{\xi} + K_1 = M_\infty < \infty.$$

$\square$

Lemma 5.4.8. *Let $(v_n(\cdot, \cdot))_{n \geq 0}$ be as in (5.2.10). We have that*

$$(5.4.51) \qquad |v_n(z, t) - v_n(\tilde{z}, t)| \leq |z - \tilde{z}|, \quad z, \tilde{z} \in \mathbb{R}$$

and

$$(5.4.52) \qquad |v_n(z, t) - v_n(z, s)| \leq F_n (1 + |z|) (s - t)^{\frac{1}{2}}, \quad 0 \leq t < s \leq T,$$

in which F_n are all finite constants depending on T.

Proof. From the definition of $v_0(\cdot, \cdot)$ in (5.2.10), we have

$$(5.4.53) \qquad |v_0(z, t) - v_0(\tilde{z}, t)| = \left| (\zeta \cdot (z - K_1))^+ - (\zeta \cdot (\tilde{z} - K_1))^+ \right| \leq |z - \tilde{z}|.$$

Now, the inequality (5.4.51) follows from induction and Lemma 5.4.1. On the other hand, (5.4.52) holds thanks to Lemma 5.4.5 which we can apply to each v_n as a result of Lemma 5.4.7. $\qquad \square$

As a corollary of Remark 5.4.4 and Lemma 5.4.7, we can show that $(v_n(z, t))_{n \geq 0}$ satisfies a linear growth condition in the z-variable, uniformly in the t-variable. This will be used to show that this sequence has a limit.

Corollary 5.4.9. *For any $n \geq 0$,*

$$(5.4.54) \qquad v_n(z, t) \leq M_\infty + |z| \triangleq L(z), \quad (z, t) \in \mathbb{R} \times [0, T].$$

Proof. Combining (5.4.48) and (5.4.51), we have

$$v_n(z, t) \leq M_n + |z|, \quad \text{for } n \geq 0.$$

Now, the result follows from Lemma 5.4.7. $\qquad \square$

As a result of Corollary 5.4.9, next we show that, for a fixed $(z, t) \in \mathbb{R} \times [0, T]$, the sequence $\{v_n(z, t)\}_{n \geq 0}$ is a Cauchy sequence.

Lemma 5.4.10. *For any $(z, t) \in \mathbb{R} \times [0, T]$ and $n, m \geq 0$.*

(5.4.55)

$$|v_{n+m}(z,t) - v_m(z,t)| \leq 2M_\infty A^m + 2\left(\frac{1}{rT}\frac{\xi+1}{\xi} + \frac{C}{\xi}\right)\left[\sum_{i=0}^m A^{m-i}B^i - B^m\right] + 2|z|B^m,$$

where $A = 1 - e^{-\lambda\xi(T-t)}$, $B = 1 - e^{-\lambda(T-t)}$ and C is the same constant as in (5.4.21).

Proof. We will prove the estimation (5.4.55) by induction on m. When $m = 0$, it follows from Corollary 5.4.9 that

$$|v_n(z,t) - v_0(z,t)| \leq 2M_\infty + 2|z|.$$

It is clear that (5.4.55) is satisfied in this case. Assuming (5.4.55) holds for the m case, we will show that it holds when we replace m by $m + 1$. From the definition of $\{v_n(\cdot, \cdot)\}_{n \geq 0}$, we have

$$|v_{n+m+1}(z,t) - v_{m+1}(z,t)| \leq \mathbb{E}^{\mathbb{Q}}\left\{\int_0^{T-t} ds\, e^{-\lambda\xi s}\lambda \cdot \left|Pv_{n+m}(Z_s^{t,z}, t+s) - Pv_m(Z_s^{t,z}, t+s)\right|\right\}.$$

In the right hand side of above inequality, the induction assumption gives us

(5.4.56)

$$\left|Pv_{n+m}(Z_s^{t,z}, t+s) - Pv_m(Z_s^{t,z}, t+s)\right|$$

$$\leq \int_{\mathbb{R}_+}\left|v_{n+m}\left(\frac{Z_s^{t,z}}{y} + q_{t+s}\frac{y-1}{y}, t+s\right) - v_m\left(\frac{Z_s^{t,z}}{y} + q_{t+s}\frac{y-1}{y}, t+s\right)\right| y\nu(dy)$$

$$\leq 2\xi M_\infty \left(1 - e^{-\lambda\xi(T-t-s)}\right)^m$$

$$+ 2\xi\left(\frac{1}{rT}\frac{\xi+1}{\xi} + \frac{C}{\xi}\right)\left[\sum_{i=0}^m \left(1 - e^{-\lambda(T-t-s)}\right)^i\left(1 - e^{-\lambda\xi(T-t-s)}\right)^{m-i} - \left(1 - e^{-\lambda(T-t-s)}\right)^m\right]$$

$$+ 2\int_{\mathbb{R}_+}\left(\frac{|Z_s^{t,z}|}{y} + q_{t+s}\frac{|y-1|}{y}\right)\left(1 - e^{-\lambda(T-t-s)}\right)^m y\nu(dy)$$

$$\leq 2\xi M_\infty(1 - e^{-\lambda\xi(T-t)})^m$$

$$+ 2\xi\left(\frac{1}{rT}\frac{\xi+1}{\xi} + \frac{C}{\xi}\right)\left[\sum_{i=0}^m \left(1 - e^{-\lambda(T-t)}\right)^i\left(1 - e^{-\lambda\xi(T-t)}\right)^{m-i} - \left(1 - e^{-\lambda(T-t)}\right)^m\right]$$

$$+ 2|Z_s^{t,z}|\left(1 - e^{-\lambda(T-t)}\right)^m + \frac{2}{rT}(\xi+1)\left(1 - e^{-\lambda(T-t)}\right)^m.$$

In (5.4.56), the third inequality follows, because $q_{t+s} \leq \frac{1}{rT}$, and for $m \geq 1$ and $s \geq 0$

$$\sum_{i=0}^{m-1} \left(1 - e^{-\lambda(T-t-s)}\right)^i \left(1 - e^{-\lambda\xi(T-t-s)}\right)^{m-i} \leq \sum_{i=0}^{m-1} \left(1 - e^{-\lambda(T-t)}\right)^i \left(1 - e^{-\lambda\xi(T-t)}\right)^{m-i}.$$

On the other hand, from (5.4.7), we have

$$|Z_s^{t,z}| \leq |z|H_s^0 + |b_s|,$$

where $\mathbb{E}^{\mathbb{Q}}\{|b_s|\} = \mathbb{E}^{\mathbb{Q}}\{|Z_s^{t,0}|\} \leq C$ from (5.4.21). Therefore we have

(5.4.57) $$\mathbb{E}^{\mathbb{Q}}\{|Z_s^{t,z}|\} \leq |z|e^{\mu s} + C.$$

Taking expectation on both side of (5.4.56) and plugging (5.4.57) back into (5.4.56), we have

(5.4.58)
$$\mathbb{E}^{\mathbb{Q}}\left|Pv_{n+m}(Z_s^{t,z}, t+s) - Pv_m(Z_s^{t,z}, t+s)\right|$$
$$\leq 2\xi M_\infty A^m + 2\xi \left(\frac{1}{rT}\frac{\xi+1}{\xi} + \frac{C}{\xi}\right)\left[\sum_{i=0}^m A^{m-i}B^i - B^m\right]$$
$$+ 2|z|e^{\mu s}B^m + 2\left(\frac{1}{rT}(\xi+1) + C\right)B^m.$$

Multiplying both sides of (5.4.58) with $e^{-\lambda\xi s}\lambda$ and integrating with respect to s over $[0, T-t]$, and using the identity $\mu - \lambda\xi = -\lambda$, we obtain the inequality (5.4.55) with m replaced by $m+1$. $\qquad\qquad\square$

As a result of the previous lemma we can define the pointwise limit for the sequence $(v_n(\cdot,\cdot))_{n\geq 0}$:

(5.4.59) $$v_\infty(z,t) \triangleq \lim_{n\geq 0} v_n(z,t), \quad (z,t) \in \mathbb{R} \times [0,T].$$

Moreover, as a corollary of Lemma 5.4.10, we have

Corollary 5.4.11. *For any compact domain $\mathcal{D} \subset \mathbb{R}$, $v_n(z,t)$ converges uniformly to $v_\infty(z,t)$ for $(z,t) \in \mathcal{D} \times [0,T]$. Moreover,*

(5.4.60) $$|v_\infty(z,t) - v_m(z,t)| \leq M_\mathcal{D}\left(1 - e^{-\lambda\eta(T-t)}\right)^m,$$

where $M_{\mathcal{D}}$ is a constant depending on $\mathcal{D}$ and $\eta = \max\{\xi, 1\}$.

Proof. Observing that the right hand side of (5.4.55) is independent of n and $|z|$ is uniformly bounded in $\mathcal{D}$, the result follows from Lemma 5.4.10. □

In the following, we will begin to study properties of $v_\infty(\cdot, \cdot)$.

Lemma 5.4.12. *The function v_∞ is a fixed point of the operator J.*

Proof. For any $s \in [0, T-t]$,

$$\mathbb{E}^{\mathbb{Q}}\left\{PL(Z_s^{t,z})\right\} = \mathbb{E}^{\mathbb{Q}}\left\{\int_{\mathbb{R}_+} L\left(\frac{Z_s^{t,z}}{y} + q_{t+s}\frac{y-1}{y}\right)y\nu(dy)\right\}$$

$$(5.4.61) \qquad \leq \mathbb{E}^{\mathbb{Q}}\left\{\int_{\mathbb{R}_+}\left[M_\infty + \frac{|Z_s^{t,z}|}{y} + q_{t+s}\frac{|y-1|}{y}\right]y\nu(dy)\right\}$$

$$\leq \xi M_\infty + \frac{1}{rT}(\xi+1) + C(1+|z|).$$

As a result, we have

(5.4.62)

$$v_\infty(z,t) = \lim_{n\geq 0} v_{n+1}(z,t)$$

$$= \lim_{n\geq 0}\mathbb{E}^{\mathbb{Q}}\left\{e^{-\lambda\xi(T-t)}\left(\zeta \cdot \left(Z_{T-t}^{t,z} - K_1\right)\right)^+ + \int_0^{T-t} e^{-\lambda\xi s}\lambda \cdot (Pv_n)(Z_s^{t,z}, t+s)\, ds\right\}$$

$$= \mathbb{E}^{\mathbb{Q}}\left\{e^{-\lambda\xi(T-t)}\left(\zeta \cdot \left(Z_{T-t}^{t,z} - K_1\right)\right)^+ + \int_0^{T-t} e^{-\lambda\xi s}\lambda \cdot (P\lim_{n\geq 0} v_n)(Z_s^{t,z}, t+s)\, ds\right\}$$

$$= Jv_\infty(z,t),$$

where the third equality follows by applying dominated convergence theorem three times. We can use the dominated convergence theorem due to Corollary 5.4.9 and (5.4.61). □

Using Lemma 5.4.8 and Corollary 5.4.11, we can show that $z \to v_\infty(z,t)$ is Lipschitz continuous and that $t \to v_\infty(z,t)$ is Hölder continuous.

Lemma 5.4.13. *$v_\infty(\cdot, \cdot)$ satisfies*

$$(5.4.63) \qquad |v_\infty(z,t) - v_\infty(\tilde{z},t)| \leq |z - \tilde{z}|, \quad \text{for } (z,t), (\tilde{z},t) \in \mathbb{R} \times [0,T].$$

Proof. For fixed z and $\tilde{z}$, let us choose a compact domain $\mathcal{D}_{z,\tilde{z}} \subseteq \mathbb{R}$, so that $z, \tilde{z} \in \mathcal{D}_{z,\tilde{z}}$. Then we have

$$|v_\infty(z,t) - v_\infty(\tilde{z},t)| \leq |v_\infty(z,t) - v_n(z,t)| + |v_n(z,t) - v_n(\tilde{z},t)| + |v_n(\tilde{z},t) - v_\infty(\tilde{z},t)|$$

$$(5.4.64) \qquad \leq 2\left(1 - e^{-\lambda\eta(T-t)}\right)^n M_{\mathcal{D}_{z,\tilde{z}}} + |z - \tilde{z}|.$$

In order to obtain the last inequality, we use Lemmas 5.4.8 and Corollary 5.4.11. Since n in the second inequality in (5.4.64) is arbitrary, the result follows. $\qquad\square$

Corollary 5.4.14. $v_\infty(\cdot,\cdot)$ *satisfies*

$$(5.4.65) \qquad\qquad |v_\infty(z,t) - v_\infty(z,s)| \leq F_\infty (1 + |z|) |t - s|^{\frac{1}{2}},$$

in which constant $F_\infty < \infty$.

Proof. This is a direct application of Lemmas 5.4.5, 5.4.7 and 5.4.13. Note that Lemma 5.4.7 is needed to show that $\sup_{t\in[0,T]}\{v_\infty(0,t)\} < \infty$, which is required by Lemma 5.4.5. $\qquad\square$

5.4.3 Proof of Theorem 5.2.1

Proof of (i). This is a direct consequence of Lemma 5.4.10, which shows that the sequence $\{v_n(z,t)\}_{n\geq 0}$ is a Cauchy sequence.

Proof of (ii). See Corollary 5.4.11.

Proof of (iii). Using the inequalities (5.4.51) and (5.4.52) in Lemma 5.4.8, we can apply Proposition 5.4.6 to the function $f = v_n$. It indicates $Jv_n(\cdot,\cdot)$ is the unique classical solution of the following equation

$$\mathcal{A}(t)Jv_n(z,t) - \lambda\xi Jv_n(z,t) + \lambda \cdot (Pv_n)(z,t) + \frac{\partial}{\partial t}Jv_n(z,t) = 0$$

$$(5.4.66)$$

$$Jv_n(z,T) = (\zeta \cdot (z - K_1))^+,$$

for $(z,t) \in \mathbb{R} \times [0,T]$. By the definition of the sequence $(v_n(\cdot,\cdot))_{n\geq 0}$ in (5.2.10), we have $Jv_n(\cdot,\cdot) = v_{n+1}(\cdot,\cdot)$. So v_{n+1} is the unique solution of (5.2.14) and (5.2.15).

Proof of (iv). Because of Lemma 5.4.13 and Corollary 5.4.14, we can apply Proposition 5.4.6 to the function $f = v_\infty$. It shows $Jv_\infty(\cdot, \cdot)$ is the unique classical solution of the following parabolic partial differential equation

$$(5.4.67) \qquad \mathcal{A}(t)Jv_\infty(z,t) - \lambda\xi Jv_\infty(z,t) + \lambda \cdot (Pv_\infty)(z,t) + \frac{\partial}{\partial t}Jv_\infty(z,t) = 0$$

$$(5.4.68) \qquad Jv_\infty(z,T) = (\zeta \cdot (z - K_1))^+,$$

However, $Jv_\infty = v_\infty$ by Lemma 5.4.12. Therefore, $v_\infty(\cdot, \cdot)$ is the unique classical solution of the integro-partial differential equation (5.2.17) and (5.2.18). □

5.5 Proof of Proposition 5.3.1

Proof of (i). We will first prove the statements in (i). Let us define $e_n(k,m) \triangleq \tilde{v}(k,m) - \tilde{v}_n(k,m)$. Since $\widetilde{P}$ is a linear operator, e_n will satisfy (5.3.1) when we replace $\tilde{v}_n$ by e_n and $\tilde{v}_{n+1}$ by e_{n+1}. Now let us define $E_n^m \triangleq \max_k |e_n(k,m)|$ and recall $E_n = \max_{m,k}|e_n(k,m)|$. Since $p_{k,m}^+$, $p_{k,m}^-$ and $1 - p_{k,m}^0$ are positive for all (k,m), it follows from the difference equation of e_n that

$$(1 + p_{k,m}^0)\,|e_{n+1}(k,m)|$$

$$\begin{aligned}
(5.5.1) \qquad &\le (p_{k,m}^+ + p_{k,m}^-)E_{n+1}^m + (p_{k,m+1}^+ + p_{k,m+1}^- + (1 - p_{k,m+1}^0))E_{n+1}^{m+1} + \lambda\Delta t\,\xi E_n \\
&= (p_{k,m}^+ + p_{k,m}^-)E_{n+1}^m + \left(1 - \frac{1}{2}\lambda\xi\Delta t\right)E_{n+1}^{m+1} + \lambda\Delta t\,\xi E_n,
\end{aligned}$$

in which we used the assumption that $(\widetilde{P}1)(k,m) \le \xi$. It follows from (5.5.1) that

$$(5.5.2)$$

$$(1 + p_{k,m}^0)\,|e_{n+1}(k,m)| - \left(p_{k,m}^0 - \frac{1}{2}\lambda\xi\Delta t\right)E_{n+1}^m \le \left(1 - \frac{1}{2}\lambda\xi\Delta t\right)E_{n+1}^{m+1} + \lambda\Delta t\,\xi E_n.$$

Let k^* be such that $|e_{n+1}(k^*,m)| = E_{n+1}^m$. Since the right-hand-side of (5.5.2) does not depend on k, we can take $k = k^*$ on the left-hand-side to write

$$(5.5.3) \qquad\qquad\qquad E_{n+1}^m \le \theta E_{n+1}^{m+1} + (1 - \theta)E_n,$$

in which $\theta = \frac{1-1/2\cdot\lambda\xi\Delta t}{1+1/2\cdot\lambda\xi\Delta t} \in (0,1]$, as a result of the assumption that $p_{k,m}^+$, $p_{k,m}^-$ and $1 - p_{k,m}^0$ are positive for all (k,m). It follows from (5.5.3) that

$$(5.5.4) \qquad E_{n+1}^m \le \theta^{M-m} E_{n+1}^M + (1-\theta)(1+\theta+\cdots+\theta^{M-m-1})E_n.$$

Because of the terminal condition of $\tilde{v}_n$, we have $E_{n+1}^M = 0$. In addition, (5.5.4) is satisfied for all m we get that

$$(5.5.5) \qquad\qquad\qquad E_{n+1} \le (1-\theta^M)E_n.$$

As a result (5.3.6) follows from iterating (5.5.5) on n.

On the other hand, as $M \to \infty$

$$(5.5.6) \qquad 1-\theta^M = 1 - \left(\frac{1-1/2\cdot\lambda\xi\cdot T/M}{1+1/2\cdot\lambda\xi\cdot T/M}\right)^M \to 1 - e^{-\lambda\xi T},$$

which shows that the convergence rate in (5.3.6) agrees with the convergence rate in (5.2.13).

Proof of (ii). Using the triangle inequality let us write

$$(5.5.7)$$

$$|v_\infty(z_k, m\Delta t) - \tilde{v}(k,m)|$$

$$\le |v_\infty(z_k, m\Delta t) - v_n(z_k, m\Delta t)| + |v_n(z_k, m\Delta t) - \tilde{v}_n(k,m)| + |\tilde{v}_n(k,m) - \tilde{v}(k,m)|$$

$$\le C\left(1 - e^{-\lambda\eta(T-m\Delta t)}\right)^n + n\cdot O((\Delta t)^2 + (\Delta z)^2 + (\Delta x)^2) + \widetilde{C}(1-\theta^M)^n,$$

for some positive constants C and $\widetilde{C}$. The first and the third terms in the right-hand-side of the second inequality are due to (5.2.13) and (5.3.6). The second term arises since the order of error from discretizing a PDE using a Crank-Nicolson scheme is $O((\Delta z)^2 + (\Delta t)^2)$, the interpolation and the discretization error from the numerical integration are of order $(\Delta z)^2$ and $(\Delta x)^2$ and that the total error made at each step propagates at most linearly in n when we sequentially discretize the PDEs in Step 1 of our numerical algorithm.

Letting $\Delta t, \Delta z \to 0$ in (5.5.7) we obtain that

$$(5.5.8) \qquad \lim_{\Delta t, \Delta z \to 0} |v_\infty(z_k, m\Delta t) - \tilde{v}(k, m)| \leq C \left(1 - e^{-\lambda \eta T}\right)^n + \widetilde{C} \left(1 - e^{-\lambda \xi T}\right)^n,$$

in which we used (5.5.6). Since n is arbitrary the result follows.

On the other hand, in order to get the convergence rate in (5.3.8), we can choose $n = O(\log(1/\Delta^2))$ in (5.5.7), which would guarantee that the first and the third terms on the right-hand-side of (5.5.7) are of order $O(\Delta^2)$. This choice of n makes the right-hand-side of (5.5.7) be on the order of $\Delta^2 \log(1/\Delta^2)$. Note that this order of convergence is better than that of $O((\Delta t)^{2-\gamma} + (\Delta z)^{2-\gamma} + (\Delta x)^{2-\gamma})$ for any $\gamma > 0$.

$\square$

Table 5.1:
The approximated price for a continuously averaged European type Asian options in a double exponential jump model.

$r = 0.15$, $S_0 = 100$, $T = 1$, $p = 0.6$ and $\eta_1 = \eta_2 = 25$. Monte Carlo method uses 10^6 simulations and 10^3 time steps. "C - P" is the difference between our approximated call and put option prices. "Parity" is the difference predicted by the put-call parity (see (5.2.7)). Run times are in seconds.

European Asian option prices for a double exponential jump diffusion model										
			The algorithm of Section 5.3.1						Monte Carlo (Call Option)		
σ	K_2	λ	Call Option (C)		Put Option (P)		C - P	Parity			
			Value	Time	Value	Time			Value	Std. Err.	Time
0.1	90	1	15.419	1.0	0.012	1.0	15.407	15.398	15.410	0.006	913
		3	15.457	1.5	0.045	1.5	15.412		15.442	0.007	976
	100	1	7.170	1.0	0.376	1.0	6.794	6.791	7.170	0.006	919
		3	7.456	1.5	0.656	1.6	6.800		7.439	0.007	987
	110	1	1.702	1.0	3.520	1.0	-1.818	-1.817	1.697	0.004	906
		3	2.220	1.5	4.040	1.6	-1.820		2.207	0.004	981
0.2	90	1	15.699	1.0	0.292	1.0	15.407	15.398	15.686	0.012	908
		3	15.802	1.5	0.390	1.6	15.412		15.806	0.012	983
	100	1	8.540	1.0	1.745	1.0	6.795	6.791	8.540	0.010	935
		3	8.790	1.5	1.994	1.6	6.796		8.784	0.010	996
	110	1	3.723	1.0	5.541	1.0	-1.818	-1.817	3.721	0.007	921
		3	4.045	1.6	5.864	1.6	-1.819		4.038	0.007	983

Table 5.2:
The approximated price for a continuously averaged European type Asian options in a normal jump diffusion model.

$r = 0.15$, $S_0 = 100$, $T = 1$, $\lambda = 1$, $\tilde{\mu} = -0.1$ and $\tilde{\sigma} = 0.3$. Monte Carlo method uses 10^6 simulations and 10^3 time steps. "C - P" is the difference between our approximated call and put option prices. "Parity" is the difference predicted by the put-call parity. Run times are in seconds.

Prices of European Asian option with normal jumps										
		The algorithm of Section 5.3.1						Monte Carlo (Call Option)		
σ	K_2	Call Option (C)		Put Option (P)		C - P	Parity			
		Value	Time	Value	Time			Value	Std. Err.	Time
0.1	90	16.997	0.5	1.601	0.5	15.396	15.398	16.991	0.014	913
	100	10.062	0.5	3.272	0.5	6.789	6.791	10.046	0.013	910
	110	4.836	0.5	6.653	0.5	-1.817	-1.816	4.834	0.011	915
0.2	90	17.346	0.5	1.950	0.5	15.396	15.398	17.339	0.017	919
	100	10.959	0.5	4.170	0.5	6.789	6.791	10.968	0.015	917
	110	6.303	0.5	8.120	0.5	-1.817	-1.816	6.310	0.012	913

Table 5.3: The convergence of the option prices with respect to the truncation length of the numerical integral.

The parameters are the same as the parameters used in the 7th row in Table 1, i.e., $r = 0.15$, $S_0 = 100$, $K_1 = 0$, $K_2 = 90$, $T = 1$, $\sigma = 0.2$, $\lambda = 1$, $p = 0.6$ and $\eta_1 = \eta_2 = 25$.

As we introduced in (5.3.4), the integral term in (5.2.12) is approximated by the trapezoidal rule on an interval $[x_{min}, x_{max}]$ with $x_{min} = x_0 < x_1 < \cdots < x_L = x_{max}$. In this table, fixing Δx, we study the convergence with respect to the length of the truncation interval $[x_{min}, x_{max}]$. We choose $x_{min} = -N/\eta_2$ and $x_{max} = N/\eta_1$. In (5.3.3), if the distribution F is the double exponential, when N is large, the probability that the random variable X be outside the interval $[-N/\eta_2, N/\eta_1]$ is very small (for example, when $N = 15$ the probability is less than 10^{-6}).

Convergence with respect to truncation length used in the numerical integration					
N	Call Option (C)	Time	Put Option (P)	Time	(C - P) - Parity
5	15.5832	0.500	0.2858	0.516	-0.1002
8	15.6953	0.765	0.2916	0.797	0.0061
10	15.6994	0.969	0.2921	1.000	0.0097
12	15.6995	1.141	0.2921	1.187	0.0098
15	15.6995	1.391	0.2921	1.500	0.0098[a]

[a]Because we fix Δx, the difference between the calculated value and predicted value in the last column does not seem to converge to 0. But as $\Delta x \to 0$, the difference will converge to 0 as we will see in the next Table.

Table 5.4: The convergence of the option prices with respect to the grid size of the numerical integral.

In this table, we fix the truncation length of the numerical integral as $x_{min} = -10/\eta_2$ and $x_{max} = 10/\eta_1$, we will show the convergence with respect to the number of grids L in the discretization of numerical integral in (5.3.4). Since the density of double exponential distribution has a cusp at zero, we choose an unequaly spaced grid around zero. (The closer x is to zero, the finer the grid.) Since the truncation interval $[x_{min}, x_{max}]$ is fixed, choosing larger L gives a finer grid. We use the same parameters as in Table 3.

Convergence with respect grid sizes used in the numerical integration					
L	Call Option (C)	Time	Put Option (P)	Time	(C - P) - Parity
200	15.7295	0.422	0.2926	0.422	0.0393
300	15.7103	0.578	0.2923	0.609	0.0204
400	15.7034	0.766	0.2924	0.797	0.0134
500	15.6944	0.969	0.2921	1.000	0.0097
600	15.6968	1.141	0.2920	1.172	0.0072
700	15.6954	1.344	0.2920	1.360	0.0058
800	15.6943	1.516	0.2920	1.562	0.0047

Table 5.5: The convergence of the option prices with respect to the grid sizes in the finite difference scheme.

In this table we fix $x_{min} = -10/\eta_2$ and $x_{max} = 10/\eta_1$, $L = 1000$. Moreover, we fix $z_{min} = z - 0.5$ and $z_{max} = z + 0.5$ with $z = \left(1 - e^{-rT}\right)/(rT) - e^{-rT}K_2/S_0$ defined in (5.2.20). We will show the convergence with respect to time and space grid sizes that are used in implementing the finite difference scheme. We use the same parameters as in Table 3.

Convergence with respect to grid sizes used in the finite difference scheme				
Number of Time Steps	Number of Space Steps	Call Option Price	Changes	Time
10	40	15.7093	n.a.	0.438
25	100	15.6929	0.0164	1.890
50	200	15.688	0.0049	7.500
100	400	15.6864	0.0016	29.406

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