Equilibrium-preserving deformations of two-dimensional foams

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1 Introduction

1.1 What is a foam?

Fill a glass halfway with chocolate milk. Put one end of a straw in the milk, put the other end of the straw in your mouth, and blow. The structure you have just produced is a *dry liquid foam*: a network of thin liquid films enclosing several gas-filled regions.

To a first approximation, the law governing the evolution of a liquid film is delightfully simple: the surface area of the film is always decreasing as quickly as possible. The film reaches equilibrium when its surface area can no longer be reduced.

The films of a dry liquid foam are prevented from collapsing to zero area by the pressures of the gas bubbles they enclose. In a typical foam, the pressure differences between adjacent bubbles are tiny compared to the ambient pressure, so the volume of each bubble depends only on the number and temperature of the gas molecules inside it.[1] The time it takes for the films to reach mechanical equilibrium is much longer than the time it takes for the bubbles to reach thermal equilibrium, but much shorter than the time it takes for an appreciable number of gas molecules to diffuse across the films. The volumes of the bubbles in a mechanically equilibrated foam are therefore approximately constant.

The facts laid out above suggest the following mathematical model of a dry liquid foam.

Definition 1.1. An *n-dimensional foam* is a collection of bounded, continuously differentiable $(n-1)$-dimensional surfaces, called *films*, that partition $\mathbb{R}^n$ into a finite number of connected regions. These regions, with the exception of the unbounded one, are known as *bubbles*. Films may intersect only at their boundaries, and every boundary must be shared by at least three films.

[1] The surface tension of water is 0.073 J/m^2 so the tension of a thin water film is about 0.146 J/m^2. The pressure difference across a spherical water film 1 cm in diameter is therefore only 58 N/m^2—less than a thousandth of an atmosphere. For water containing a surfactant, the pressure difference is even smaller.

A foam is said to be in *equilibrium* if no continuously differentiable deformation of the foam can reduce its total area while keeping the volumes of the bubbles constant.

This definition can be extended to encompass foams with infinitely many bubbles [3]. It can also be changed to allow films to bump into each other without crossing [1]. These modifications increase the generality of the definition, but they also introduce technical complications that I wish to avoid here.

1.2 Foam dynamics

An *equilibrium-preserving* deformation of a foam is a continuously differentiable deformation that keeps the foam in equilibrium at all times. An equilibrium-preserving deformation does not have to preserve the volumes of bubbles. For simplicity, I will adopt the convention that an equilibrium-preserving deformation may not create or destroy films.

Heuristic arguments suggest that in most cases, the evolution of a two-dimensional foam undergoing an equilibrium-preserving deformation should be determined, up to translation and rotation, by the evolution of the pressures of its bubbles [4, 2]. I will make this statement rigorous under fairly general assumptions, and also describe some cases in which it fails.

2 Preliminaries

2.1 Equilibrium conditions

The variational definition of equilibrium that I have given is conceptually illuminating, but difficult to work with. Fortunately, equilibrium can also be defined in the more practical language of elementary geometry.

Theorem 2.1. *A two-dimensional foam is in equilibrium if and only if it satisfies the following conditions, known as* Plateau's laws*:*

1. *The curvature of each film is constant. In other words, each film is a straight line segment or a circular arc.*

2. *The films meet in threes at* $120°$ *angles.*

3. *If a closed path crosses three intersecting films transversally, as shown in Figure 1, the signed curvatures of the films with respect to the path sum to zero. The sign convention for the curvature is shown in Figure 2.*

Proof. See Lemma 4.1 of [1] and Corollary A.4 of [7]. Although the latter is stated as a result about foams in S^2, the proof goes through in $\mathbb{R}^2$ as well. $\square$

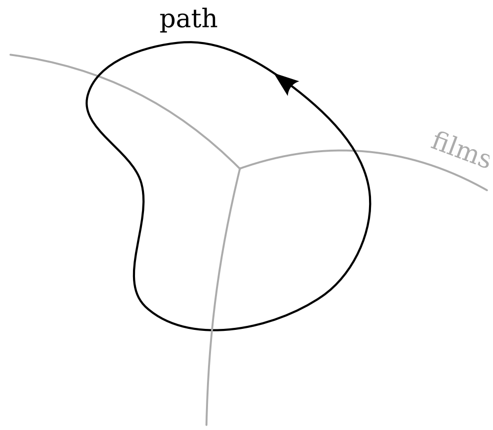


Figure 1

Figure 2

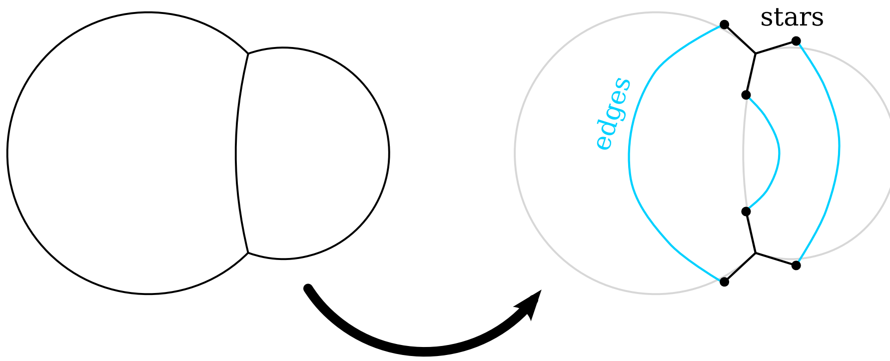


Figure 3

2.2 Constellations

The films of a two-dimensional foam intersect only at isolated points, known as *vertices*. To describe an equilibrated two-dimensional foam in the vicinity of its vertices, I will use a structure called a *constellation*. A constellation is composed of *stars* and *edges*. A star has a *base point*, which is a point in $\mathbb{R}^2$, and three *tangent vectors*, which are unit vectors in $\mathbb{R}^2$ separated by angles of 120°. An edge connects a tangent vector of one star to a tangent vector of another star. Every tangent vector in a constellation belongs to exactly one edge.

The *constellation map* turns equilibrated two-dimensional foams into constellations. It does this by turning vertices into stars and films into edges. The position of a vertex becomes the base point of a star, and the unit tangent vectors of the films that intersect at the vertex become the tangent vectors of the star. Two tangent vectors are connected by an edge if and only if they come from the same film. The effect of the constellation map is demonstrated in Figure 3.

Theorem 2.2. *The constellation map is one-to-one.*

Proof. Let F be a foam, and let G be the constellation of F.

Consider any film of F. The corresponding edge of G is associated with two base points, which are the endpoints of the film, and two tangent vectors, which are the unit tangent vectors of the film at its endpoints. Since the film is a straight line segment or a circular arc, this uniquely determines the film.

Therefore, every film of F is uniquely determined by G. □

The constellation map is not surjective. A constellation is the image of an equilibrated foam if and only if it satisfies the following conditions:

1. For each edge, there exists a constant-curvature arc with the appropriate endpoints and endpoint tangent vectors.

   Equivalently, if the tangent vectors t_1 and t_2, with associated base points x_1 and x_2, are connected by an edge, the signed angle from t_1 to $x_2 - x_1$

is equal to the signed angle from $x_1 - x_2$ to t_2. Counterclockwise angles are positive.

2. The arcs described in Condition 1 satisfy the curvature sum condition from Theorem 2.1.

3. The arcs described in Condition 1 intersect only at their endpoints.

2.3 Constellation dynamics

Consider a foam undergoing an equilibrium-preserving deformation. The deformation is prohibited, by convention, from creating or destroying films, so the constellation of the foam is changing only by the translation and rotation of its stars. Since the motion of the stars is guaranteed to be continuously differentiable, all of the stars have well-defined linear and angular velocities at all times. The linear and angular velocities of the stars determine the evolution of the constellation, which determines, by Theorem 2.2, the evolution of the foam.

A *velocity field* assigns a linear velocity and an angular velocity to each star of a constellation. The velocity fields on a constellation G form a vector space, $V(G)$. The velocity field that assigns linear velocity v and angular velocity ω to star i, leaving all other stars stationary, will be denoted $[v, \omega]_i$. The projection of a velocity field onto the subspace $\{[v, \omega]_i \mid v \in \mathbb{R}^2, \omega \in \mathbb{R}\}$ will be referred to as the projection of the velocity field onto star i.

An *equilibrium-preserving* velocity field is the instantaneous velocity field of an equilibrium-preserving deformation. A velocity field is equilibrium-preserving if and only if it satisfies the following conditions, obtained by time-differentiating the conditions for a constellation to belong to an equilibrated foam:

1. For each edge, there exists an evolving constant-curvature arc whose endpoints and endpoint tangent vectors have the appropriate linear and angular velocities.

   Equivalently, if the tangent vectors t_1 and t_2, with associated base points x_1 and x_2, are connected by an edge, the signed angle from t_1 to $x_2 - x_1$ is changing at the same rate as the signed angle from $x_1 - x_2$ to t_2.

2. As the arcs described in Condition 1 evolve, they continue to satisfy the curvature sum condition from Theorem 2.1.

The condition requiring arcs to intersect only at their endpoints has no analogue here, because it is always preserved by sufficiently small deformations.

To describe the equilibrium-preserving deformations of a foam, it suffices to describe the equilibrium-preserving velocity fields of the foam's constellation. This task will be the focus of the next section.

3 Equilibrium-preserving velocity fields

3.1 Assumptions and notation

Let F be an equilibrated two-dimensional foam. Since the connected components of a foam are completely independent of one another, I will assume, without loss of generality, that F is connected. For convenience, I will also assume that every bubble of F has at least three sides. This assumption does not lead to any practical loss of generality, because two-sided bubbles can be treated as "decorations" superimposed on the rest of the foam.

Let G be the constellation of F. I will use the following notation in reference to G.

- x_i is the base point of star i.
- $c_{ij} = |x_j - x_i|$.
- $e_{ij} = \frac{1}{c_{ij}}(x_j - x_i)$.
- e^*_{ij} is the vector obtained by rotating e_{ij} $90°$ counterclockwise.
- t_{ij} is the unique tangent vector of star i that is connected to a tangent vector of star j.
- θ_{ij} is the signed angle from t_{ij} to e_{ij}.
- M_{ij} is the unique constant-curvature arc with endpoints x_i and x_j and endpoint tangent vectors t_{ij} and t_{ji}.
- L_{ij} is the unique circle or line containing M_{ij}.
- κ_{ij} is the signed curvature of M_{ij} with respect to a path traveling counterclockwise around x_i.

Suppose G is undergoing a continuously differentiable deformation with instantaneous velocity field u. I will use the following notation in reference to u.

- v_i is the linear velocity of star i.
- ω_i is the angular velocity of star i.

3.2 Symmetry constraints

Suppose stars 0 and 1 are *adjacent*—that is, connected by an edge. The signed angle from t_{01} to $x_1 - x_0$ is changing at the same rate as the signed angle from $x_0 - x_1$ to t_{10} if and only if $\theta'_{01} = -\theta'_{10}$, where $'$ denotes the time derivative.

Let $\cdot$ denote the standard Euclidean inner product. As G evolves, e_{01} and e_{10} both rotate with angular velocity $\phi = \frac{1}{c_{01}} e^*_{01} \cdot (v_1 - v_0)$, as shown in Figure 4. In addition, t_{01} rotates with angular velocity ω_0, and t_{10} rotates with angular

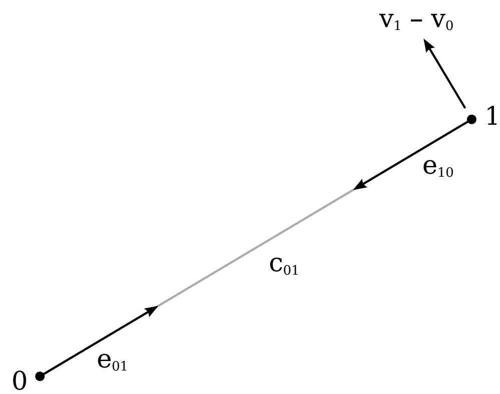


Figure 4

velocity ω_1. Therefore, $\theta'_{01} = \phi - \omega_0$, and $\theta'_{10} = \phi - \omega_1$. As a result, $\theta'_{01} = -\theta'_{10}$ if and only if

$$\begin{aligned} \phi - \omega_0 &= -(\phi - \omega_1) \\ 0 &= -2\phi + \omega_0 + \omega_1 \\ 0 &= -\frac{2}{c_{01}} e^*_{01} \cdot (v_1 - v_0) + \omega_0 + \omega_1 \\ 0 &= \frac{2}{c_{01}} e^*_{01} \cdot v_0 - \frac{2}{c_{01}} e^*_{01} \cdot v_1 + \omega_0 + \omega_1. \end{aligned}$$

Using the fact that $e_{01} = -e_{10}$, this can be rewritten

$$0 = \frac{2}{c_{01}} e^*_{01} \cdot v_0 + \frac{2}{c_{10}} e^*_{10} \cdot v_1 + \omega_0 + \omega_1.$$

In other words, $\theta'_{01} = -\theta'_{10}$ if and only if u is orthogonal to the velocity field

$$S_{01} = \left[\frac{1}{c_{01}} e^*_{01},\ \frac{1}{2}\right]_0 + \left[\frac{1}{c_{10}} e^*_{10},\ \frac{1}{2}\right]_1.$$

There is one *symmetry constraint* S_{ij} for each edge of G. The subspace of $V(G)$ spanned by the symmetry constraints will be denoted $C_S(G)$.

3.3 Balance constraints

In this subsection, I will assume that u is in $C_S(G)^\perp$, the orthogonal complement of $C_S(G)$. The arcs M_{ij} may therefore be taken to be evolving in accordance with u.

Suppose star 0 is adjacent to stars 1, 2, and 3. As G evolves, the arcs M_{01}, M_{02}, and M_{03} continue to satisfy the curvature sum condition from Theorem 2.1 if and only if $\kappa'_{01} + \kappa'_{02} + \kappa'_{03} = 0$.

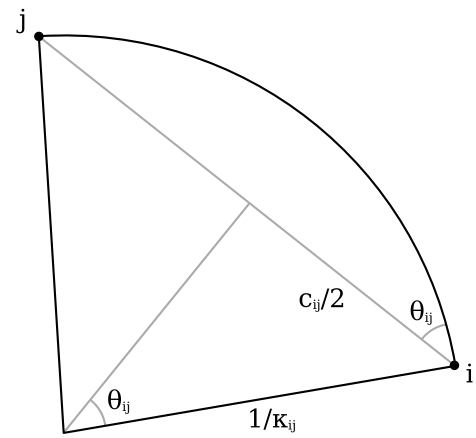


Figure 5

One can deduce from Figure 5 that $c_{ij}\kappa_{ij} = 2\sin\theta_{ij}$. Differentiating with respect to time, one finds that

$$\begin{aligned}
c_{ij}\kappa'_{ij} + c'_{ij}\kappa_{ij} &= (2\cos\theta_{ij})\theta'_{ij} \\
c_{ij}\kappa'_{ij} + [(v_j - v_i)\cdot e_{ij}]\kappa_{ij} &= (2\cos\theta_{ij})\frac{\omega_j - \omega_i}{2} \\
c_{ij}\kappa'_{ij} &= \kappa_{ij}e_{ij}\cdot v_i - \kappa_{ij}e_{ij}\cdot v_j - (\cos\theta_{ij})\omega_i + (\cos\theta_{ij})\omega_j \\
\kappa'_{ij} &= \frac{\kappa_{ij}}{c_{ij}}e_{ij}\cdot v_i - \frac{\kappa_{ij}}{c_{ij}}e_{ij}\cdot v_j - \frac{\cos\theta_{ij}}{c_{ij}}\omega_i + \frac{\cos\theta_{ij}}{c_{ij}}\omega_j \\
\kappa'_{ij} &= \frac{2\sin\theta_{ij}}{c_{ij}^2}e_{ij}\cdot v_i - \frac{2\sin\theta_{ij}}{c_{ij}^2}e_{ij}\cdot v_j - \frac{\cos\theta_{ij}}{c_{ij}}\omega_i + \frac{\cos\theta_{ij}}{c_{ij}}\omega_j.
\end{aligned}$$

Since G is the constellation of an equilibrated foam, $\theta_{ij} = -\theta_{ji}$. Recalling that $e_{ij} = -e_{ji}$, this implies that

$$\kappa'_{ij} = \frac{2\sin\theta_{ij}}{c_{ij}^2}e_{ij}\cdot v_i - \frac{2\sin\theta_{ji}}{c_{ji}^2}e_{ji}\cdot v_j - \frac{\cos\theta_{ij}}{c_{ij}}\omega_i + \frac{\cos\theta_{ji}}{c_{ji}}\omega_j.$$

In other words, $\kappa'_{ij} = K_{ij}\cdot u$, where

$$K_{ij} = \left[\frac{2\sin\theta_{ij}}{c_{ij}^2}e_{ij},\ \ -\frac{\cos\theta_{ij}}{c_{ij}}\right]_i - \left[\frac{2\sin\theta_{ji}}{c_{ji}^2}e_{ji},\ \ -\frac{\cos\theta_{ji}}{c_{ji}}\right]_j.$$

Therefore, $\kappa'_{01} + \kappa'_{02} + \kappa'_{03} = 0$ if and only if u is orthogonal to the velocity field

$$Y_0 = K_{01} + K_{02} + K_{03}.$$

There is one *balance constraint* Y_i for each star of G. The subspace of $V(G)$ spanned by the balance constraints will be denoted $C_Y(G)$.

The *curvature constraints* K_{ij} will be important later on. There is one curvature constraint for each edge of G.

3.4 The subspace of equilibrium-preserving velocity fields

Let $\Lambda(G) = [C_S(G) + C_Y(G)]^\perp$. A velocity field on G is equilibrium-preserving if and only if it lies in $\Lambda(G)$.

The main result of this paper is an upper bound on the dimension of $\Lambda(G)$. I will also show that if the upper bound is attained, each bubble of F is associated with an equilibrium-preserving velocity field that changes its pressure while keeping the pressures of the other bubbles constant. (Pressure will be defined in Subsection 5.1.)

Before these results can be proven, a few technical details must be worked out. This will be done in the next section.

4 Technical details

4.1 Conjugate vertices

Suppose star 0 is adjacent to stars 1, 2, and 3. Because F is in equilibrium, L_{01}, L_{02}, and L_{03} intersect not only at x_0, but also at the *conjugate vertex* $x_0^\dagger$ [5, 6].

Definition 4.1. A curve *passes through* all of its points except the endpoints.

Lemma 4.1. *None of the arcs M_{01}, M_{02}, and M_{03} pass through $x_0^\dagger$.*

Proof. Let $I\colon \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ be inversion about x_0. (See [6, 8] for discussion of the inversion transformation.) Since I preserves generalized circles and sends x_0 to ∞, it turns M_{01}, M_{02}, and M_{03} into rays and maps L_{01}, L_{02}, and L_{03} to lines intersecting at $I(x_0^\dagger)$, as shown in Figure 6. Since I is conformal, $I(L_{01})$, $I(L_{02})$, and $I(L_{03})$ meet at $120°$ angles.

Inversion preserves Plateau's laws, so $I(F)$ would be an equilibrated foam were it not for the unbounded films $I(M_{01})$, $I(M_{02})$, and $I(M_{03})$. By shortening these films and adding three new films, as shown in Figure 7, $I(F)$ can be turned into an equilibrated foam, F^*.

Suppose M_{01} passes through $x_0^\dagger$. Since I is continuous, $I(M_{01})$ passes through $I(x_0^\dagger)$. This forces the bubbles labeled A and B in Figure 8 to intersect the shaded region.

Cristian Moukarzel has shown that every equilibrated two-dimensional foam is also a Sectional Multiplicative Voronoï Partition (SMVP) [6]. Consider F^* as an SMVP. If you remove all of the sources except the ones corresponding to the exterior region and the bubbles A, B, and C, the resulting SMVP will be the one pictured in Figure 9, because the line or circle on which the interface between two SMVP sources lies depends only on the positions, intensities, and heights of the sources.

In the SMVP in Figure 9, regions A and B do not intersect the shaded region, which is the same as the one in Figure 8. Adding sources to an SMVP can only remove points from the already-existing regions, so it follows that regions A and B of F^* cannot intersect the shaded region. This contradiction proves that $I(M_{01})$ does not pass through $I(x_0^\dagger)$, which proves that M_{01} does

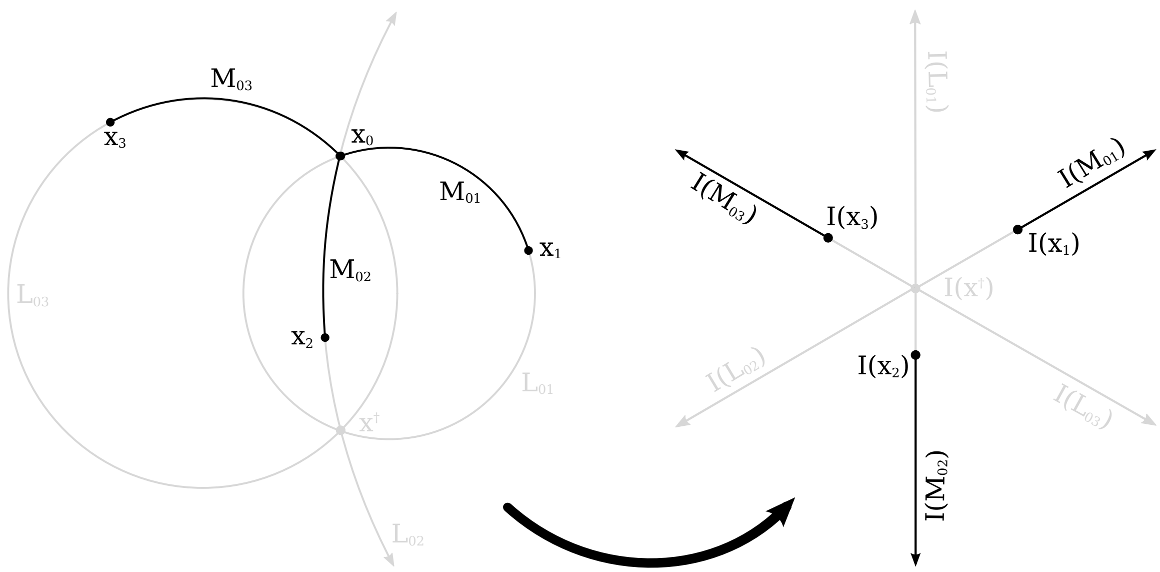


Figure 6

not pass through $x_0^\dagger$. The same argument can be used to show that M_{02} and M_{03} do not pass through $x_0^\dagger$ either. ☐

4.2 Linear independence of the symmetry constraints

Proposition 4.1. *The symmetry constraints are linearly independent.*

Lemma 4.2. *If star* 0 *is adjacent to stars* 1, 2, *and* 3, *the projections of* S_{01}, S_{02}, *and* S_{03} *onto star* 0 *are linearly independent.*

Proof. The projections of S_{01}, S_{02}, and S_{03} onto star 0 are $[\frac{1}{c_{01}}e_{01}^*, \frac{1}{2}]_0$, $[\frac{1}{c_{02}}e_{02}^*, \frac{1}{2}]_0$, and $[\frac{1}{c_{03}}e_{03}^*, \frac{1}{2}]_0$. Since the projections all have the same angular velocity component, they are coplanar in $\mathbb{R}^2 \times \mathbb{R}$ if and only if $\frac{1}{c_{01}}e_{01}^*$, $\frac{1}{c_{02}}e_{02}^*$, and $\frac{1}{c_{03}}e_{03}^*$ are collinear in $\mathbb{R}^2$. This, of course, occurs if and only if $\frac{1}{c_{01}}e_{01}$, $\frac{1}{c_{02}}e_{02}$, and $\frac{1}{c_{03}}e_{03}$ are collinear in $\mathbb{R}^2$.

Notice that $c_{ij}e_{ij} = x_j - x_i$. Therefore, $\frac{1}{c_{ij}}e_{ij}$ is the image of x_j under inversion about x_i. Let $I\colon \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}^2 \cup \{\infty\}$ be inversion about x_0.

Now, suppose $I(x_1)$, $I(x_2)$, and $I(x_3)$ are collinear. Because F does not contain any two-sided bubbles, x_1, x_2, and x_3 must be distinct. Since I is one-to-one, $I(x_1)$, $I(x_2)$, and $I(x_3)$ must be distinct as well. There is therefore a unique line X containing $I(x_1)$, $I(x_2)$, and $I(x_3)$.

Since $I(x_i)$ must lie on $I(L_i)$ for all $i \in \{1, 2, 3\}$, X can contain $I(x_0^\dagger)$ only if two of the $I(x_i)$ lie on $I(x_0^\dagger)$, contradicting the fact that all of the $I(x_i)$ are distinct (Figure 10). Therefore, X does not contain $I(x_0^\dagger)$. As a result, one of

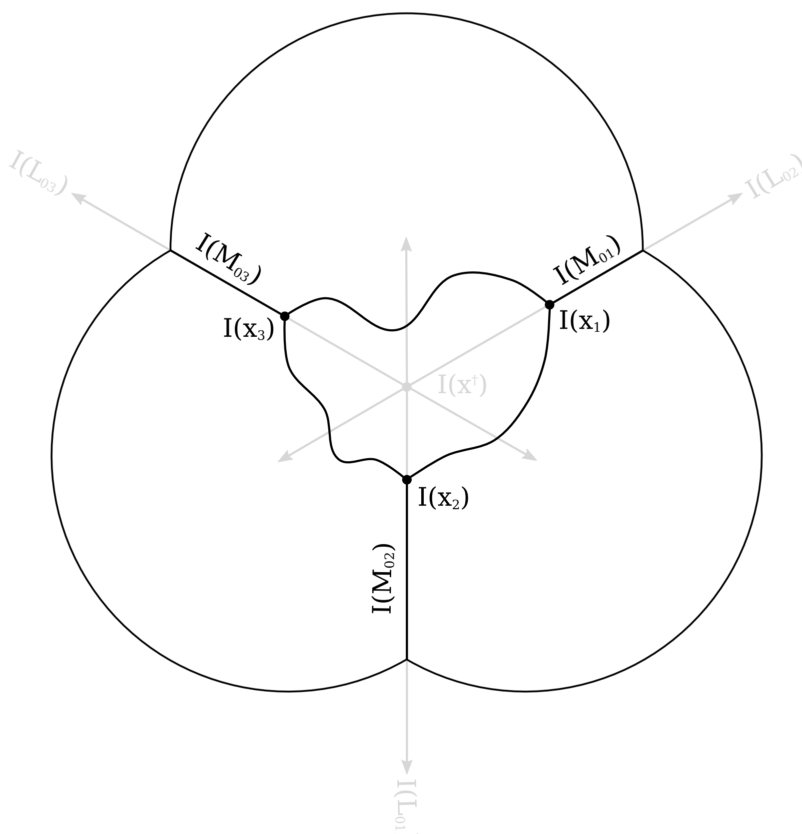


Figure 7

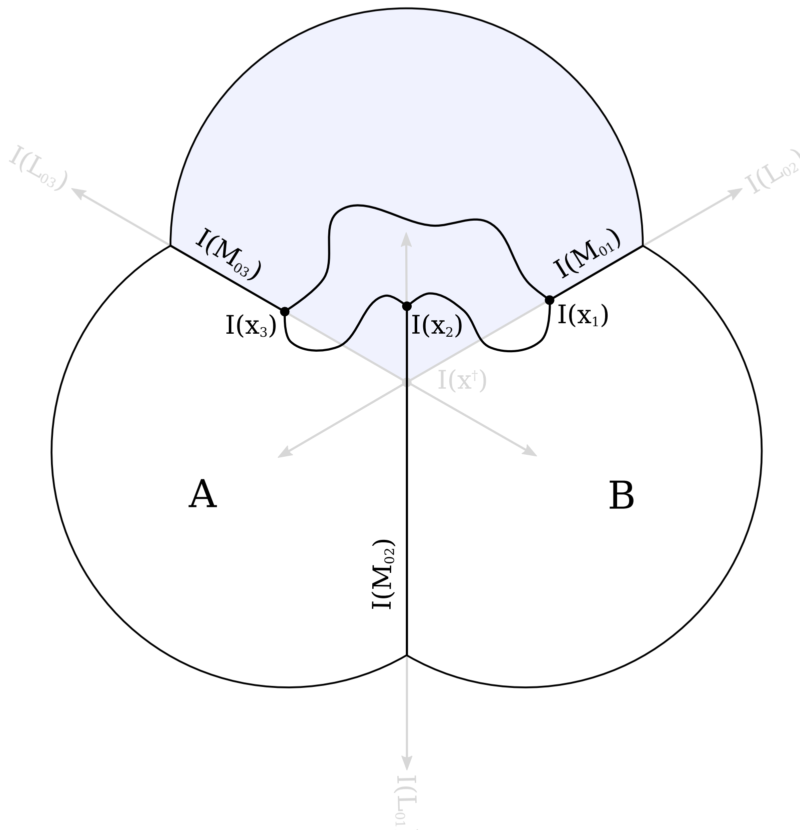


Figure 8

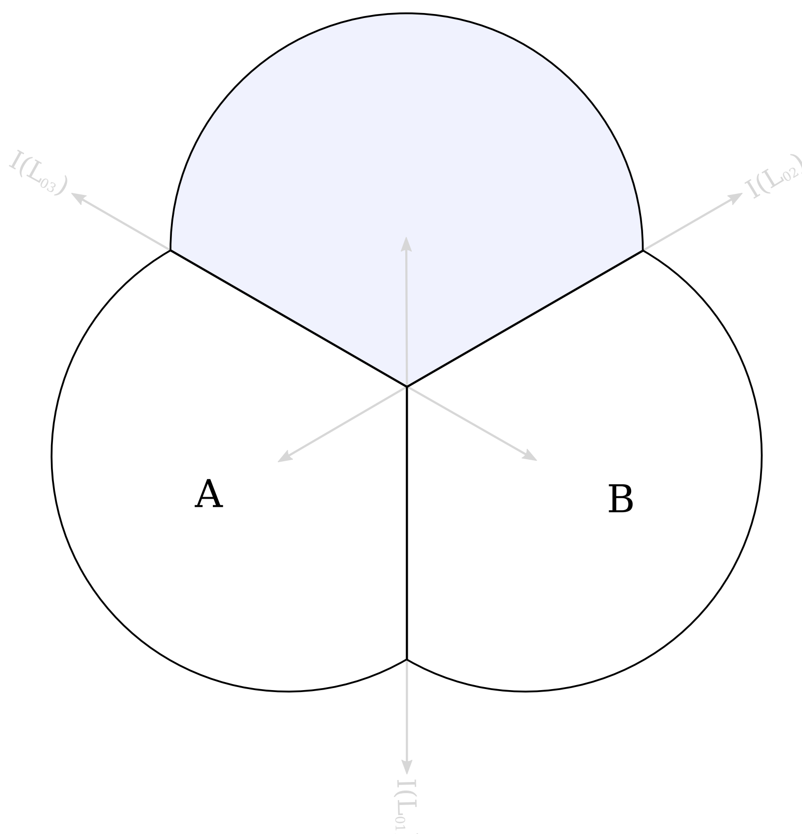


Figure 9

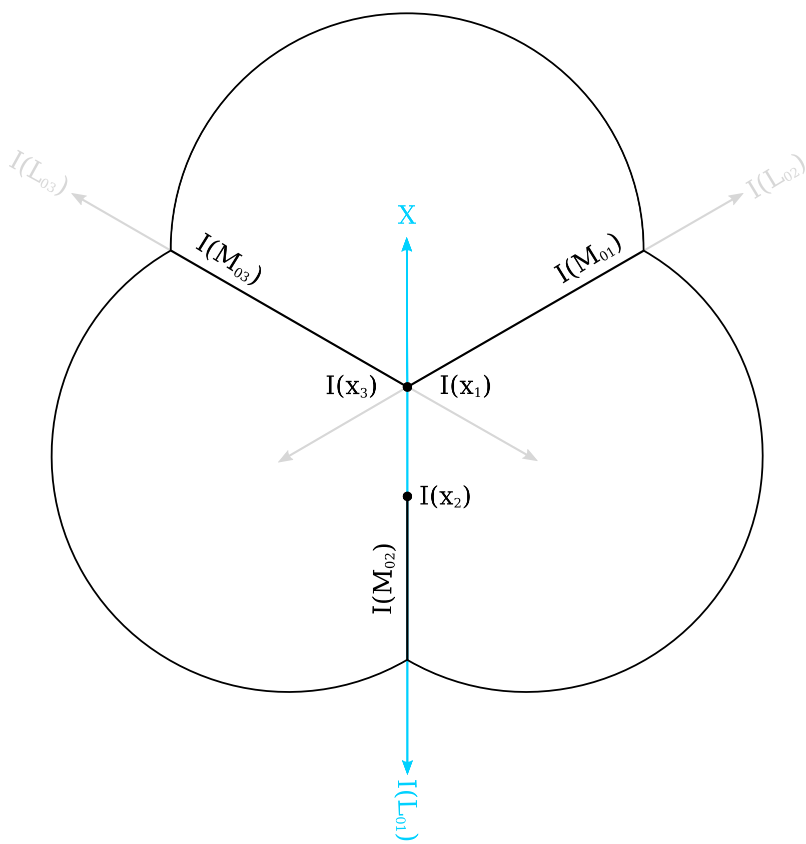


Figure 10

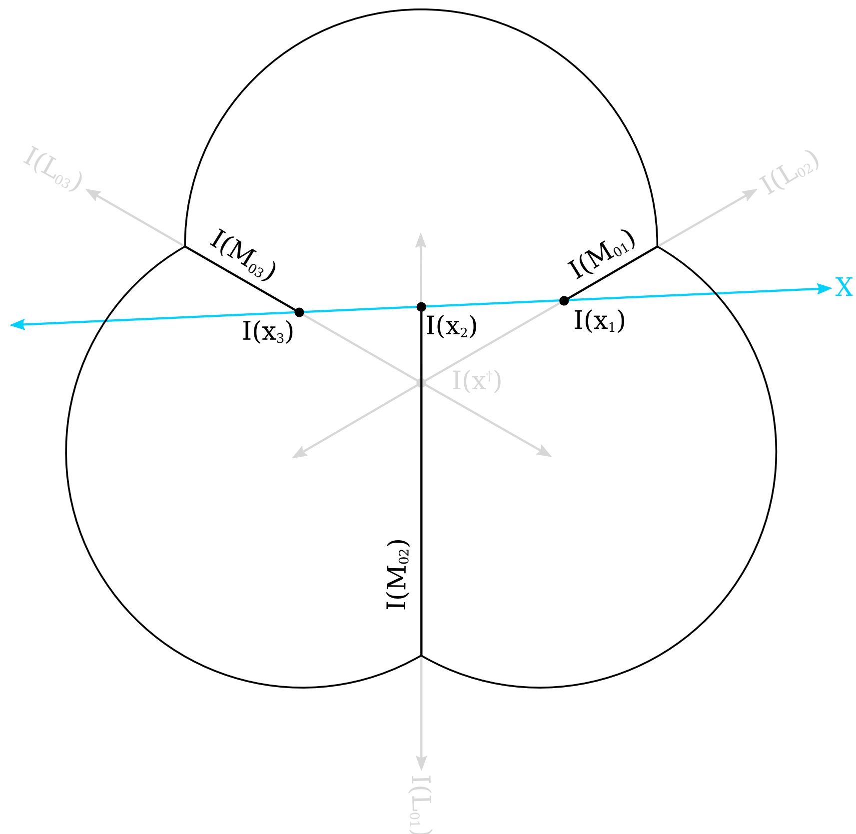


Figure 11

the $I(M_i)$ must pass through $I(x_0^\dagger)$, as shown in Figure 11. Since I is a homeomorphism, one of the M_i must pass through $x_0^\dagger$, contradicting Lemma 4.1. This contradiction shows that $I(x_1)$, $I(x_2)$, and $I(x_3)$ are not collinear. Therefore, the projections of S_{01}, S_{02}, and S_{03} onto star 0 are linearly independent. □

Proof of Proposition 4.1. Consider a linear combination of the symmetry constraints. If the coefficient of S_{ij} is nonzero, the projection of the linear combination onto stars i and j will also be nonzero, by Lemma 4.2. Therefore, a linear combination of the symmetry constraints can only be zero if all of the coefficients are zero. □

4.3 Modified balance constraints

The dimension of $C_Y(G)$ is not obvious. I will therefore introduce a set of *modified balance constraints* $\widetilde{Y}_i$, which will turn out to be linearly independent.

If star 0 is adjacent to stars 1, 2, and 3,

$$\widetilde{Y}_0 = \widetilde{K}_{01} + \widetilde{K}_{02} + \widetilde{K}_{03}, \text{ where}$$
$$\widetilde{K}_{ij} = K_{ij} - \frac{2\cos\theta_{ij}}{c_{ij}} S_{ij}.$$

The subspace of $V(G)$ spanned by the modified balance constraints will be denoted $\widetilde{C}_Y(G)$. Since $C_Y(G) \subset C_S(G) + \widetilde{C}_Y(G)$, it follows that $C_S(G) + C_Y(G) = C_S(G) + \widetilde{C}_Y(G)$. Therefore, $\Lambda(G) = [C_S(G) + \widetilde{C}_Y(G)]^\perp$.

Knowing that

$$\begin{aligned}\frac{2\cos\theta_{ij}}{c_{ij}} S_{ij} &= \left[\frac{2\cos\theta_{ij}}{c_{ij}} \frac{1}{c_{ij}} e^*_{ij},\ \ \frac{2\cos\theta_{ij}}{c_{ij}} \frac{1}{2}\right]_i + \left[\frac{2\cos\theta_{ij}}{c_{ij}} \frac{1}{c_{ji}} e^*_{ji},\ \ \frac{2\cos\theta_{ij}}{c_{ij}} \frac{1}{2}\right]_j \\ &= \left[\frac{2\cos\theta_{ij}}{c_{ij}^2} e^*_{ij},\ \ \frac{\cos\theta_{ij}}{c_{ij}}\right]_i + \left[\frac{2\cos\theta_{ji}}{c_{ji}^2} e^*_{ji},\ \ \frac{\cos\theta_{ji}}{c_{ji}}\right]_j,\end{aligned}$$

one finds that

$$\widetilde{K}_{ij} = \left[\frac{2\sin\theta_{ij}}{c_{ij}^2} e_{ij} - \frac{2\cos\theta_{ij}}{c_{ij}^2} e^*_{ij},\ \ -\frac{2\cos\theta_{ij}}{c_{ij}}\right]_i - \left[\frac{2\sin\theta_{ji}}{c_{ji}^2} e_{ji} + \frac{2\cos\theta_{ji}}{c_{ji}^2} e^*_{ji},\ \ 0\right]_j.$$

This expression will be useful in the next subsection.

4.4 Linear independence of the modified balance constraints

Proposition 4.2. *The modified balance constraints are linearly independent.*

Lemma 4.3. *If star* 0 *is adjacent to stars* 1, 2, *and* 3,

$$\frac{\cos\theta_{01}}{c_{01}} + \frac{\cos\theta_{02}}{c_{02}} + \frac{\cos\theta_{03}}{c_{03}} > 0.$$

Proof. I will start by proving the lemma in the case that κ_{01}, κ_{02}, and κ_{03} are all nonzero.

Consider the vertices x_0, x_1, x_2, and x_3 independently of the rest of F. If x_0 is fixed, and the curvatures of M_{01}, M_{02}, and M_{03} are held constant, x_1, x_2, and x_3 can only move by sliding back and forth along L_{01}, L_{02}, and L_{03}. For $i \in \{1, 2, 3\}$, the position of x_i is uniquely determined by θ_{0i}. We can therefore consider c_{0i} to be a function of θ_{0i}. Let

$$\Omega(\theta_{01}, \theta_{02}, \theta_{03}) = \frac{\cos\theta_{01}}{c_{01}(\theta_{01})} + \frac{\cos\theta_{02}}{c_{02}(\theta_{02})} + \frac{\cos\theta_{03}}{c_{03}(\theta_{03})}.$$

For all $i \in \{1, 2, 3\}$, the angle swept out by the arc M_{0i} is $2\theta_{0i}$. Therefore, $|\theta_{0i}| \in (0, \pi]$, with $M_{0i} = L_{0i}$ when $|\theta_{0i}| = \pi$.

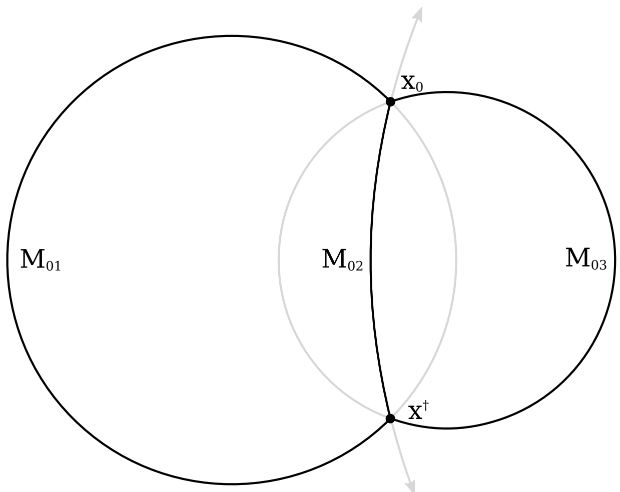


Figure 12

Special case. Let Θ_{01}, Θ_{02}, and Θ_{03} be the unique angles such that $x_1(\Theta_{01}) = x_2(\Theta_{02}) = x_3(\Theta_{03}) = x_0^\dagger$. Suppose that $\theta_{0i} = \Theta_{0i}$ for all $i \in \{1, 2, 3\}$. In this case, M_{01}, M_{02}, and M_{03} join up at $x_0^\dagger$, creating a standard double bubble (Figure 12) [6]. Since $\Omega(\theta_{01}, \theta_{02}, \theta_{03})$ is a symmetric function, we can assume, without loss of generality, that the arrangement of t_{01}, t_{02}, and t_{03} around star 0 is as shown in Figure 12.

Imagine sliding a unit tangent vector along M_{0i} from x_0 to x_i. The vector will rotate through the angle $2\Theta_{0i}$. Now, imagine sliding a unit tangent vector along M_{01} from x_0 to $x_0^\dagger$, rotating it $\frac{\pi}{3}$ radians clockwise to line up with M_{02}, sliding it along M_{02} from $x_0^\dagger$ to x_0, and rotating it another $\frac{\pi}{3}$ radians clockwise to line up with M_{01}. The vector is now back in its starting position, and it has rotated through one complete clockwise circle. Therefore,

$$2\Theta_{01} + \frac{\pi}{3} - 2\Theta_{02} + \frac{\pi}{3} = 2\pi.$$

A similar argument shows that

$$2\Theta_{02} + \frac{\pi}{3} - 2\Theta_{03} + \frac{\pi}{3} = 2\pi.$$

In other words,

$$\Theta_{01} = \Theta_{02} + \frac{2\pi}{3}$$
$$\Theta_{03} = \Theta_{02} - \frac{2\pi}{3}$$

Since $x_1(\Theta_{01}) = x_2(\Theta_{02}) = x_3(\Theta_{03})$, it follows that $c_{01}(\Theta_{01}) = c_{02}(\Theta_{02}) =$

$c_{03}(\Theta_{03})$. Therefore,

$$\Omega(\Theta_{01},\Theta_{02},\Theta_{03}) = \frac{\cos(\Theta_{02}+\frac{2\pi}{3})}{c_{01}(\Theta_{01})} + \frac{\cos\Theta_{02}}{c_{01}(\Theta_{01})} + \frac{\cos(\Theta_{02}-\frac{2\pi}{3})}{c_{01}(\Theta_{01})}.$$

A little algebra reveals that

$$\cos(\Theta_{02}+\tfrac{2\pi}{3}) + \cos(\Theta_{02}-\tfrac{2\pi}{3}) = 2\cos(\Theta_{02})\cos(\tfrac{2\pi}{3}).$$

Consequently,

$$\begin{aligned}\Omega(\Theta_{01},\Theta_{02},\Theta_{03}) &= \tfrac{1}{c_{01}(\Theta_{01})}[\cos(\Theta_{02}) + 2\cos(\Theta_{02})\cos(\tfrac{2\pi}{3})] \\ &= \tfrac{1}{c_{01}(\Theta_{01})}[\cos(\Theta_{02}) + 2(-\tfrac{1}{2})\cos(\Theta_{02})] \\ &= 0.\end{aligned}$$

General case. Recall that $c_{0i}\kappa_{0i} = 2\sin\theta_{0i}$. Therefore,

$$\frac{\cos\theta_{0i}}{c_{0i}(\theta_{0i})} = \frac{\kappa_{0i}}{2}\cot\theta_{0i}.$$

Since cot is an odd function, and κ_{0i} and θ_{0i} always have the same sign,

$$\frac{\cos\theta_{0i}}{c_{0i}(\theta_{0i})} = \frac{|\kappa_{0i}|}{2}\cot|\theta_{0i}|.$$

As a result,

$$\Omega(\theta_{01},\theta_{02},\theta_{03}) = \frac{|\kappa_{01}|}{2}\cot|\theta_{01}| + \frac{|\kappa_{02}|}{2}\cot|\theta_{02}| + \frac{|\kappa_{03}|}{2}\cot|\theta_{03}|.$$

Because $|\theta_{0i}| \in (0,\pi]$, it follows that $\Omega(\theta_{01},\theta_{02},\theta_{03})$ decreases whenever $|\theta_{01}|$, $|\theta_{03}|$, or $|\theta_{03}|$ decreases.

Now, go back to considering x_0, x_1, x_2, and x_3 as part of F. This fixes the values of θ_{01}, θ_{02}, and θ_{03}. Lemma 4.1 guarantees that $|\theta_{0i}| \le |\Theta_{0i}|$ for all $i \in \{1,2,3\}$. In addition, the fact that F has no two-sided bubbles means that $|\theta_{0i}| < |\Theta_{0i}|$ for some $i \in \{1,2,3\}$. Therefore, $\Omega(\theta_{01},\theta_{02},\theta_{03}) > \Omega(\Theta_{01},\Theta_{02},\Theta_{03})$. In other words, $\Omega(\theta_{01},\theta_{02},\theta_{03}) > 0$.

I will now consider the possibility that some of the κ_{0i} may be zero. If all three of the κ_{0i} are zero, the lemma is obviously true, because all of the c_{0i} are positive. If two of the κ_{0i} are zero, Plateau's laws force the third to be zero as well. If only one of the κ_{0i} is zero, the argument for the nonzero case can be re-used with some minor modifications. I leave this as an exercise for the reader. □

Proof of Proposition 4.2. Notice that $\widetilde{K}_{ij}\cdot[0,1]_k$ can only be nonzero if $i=k$. Therefore, $\widetilde{Y}_0$ is the only modified balance constraint whose projection onto $[0,1]_0$ can be nonzero. Furthermore,

$$\begin{aligned}\widetilde{Y}_0\cdot[0,1]_0 &= \widetilde{K}_{01}\cdot[0,1]_0 + \widetilde{K}_{02}\cdot[0,1]_0 + \widetilde{K}_{03}\cdot[0,1]_0 \\ &= \frac{\cos\theta_{01}}{c_{01}} + \frac{\cos\theta_{02}}{c_{02}} + \frac{\cos\theta_{03}}{c_{03}}\end{aligned}$$

Therefore, by Lemma 4.3, $\widetilde{Y}_0 \cdot [0,1]_0$ is never zero.

Now, consider a linear combination of the modified balance constraints. If the coefficient of $\widetilde{Y}_i$ is nonzero, the projection of the linear combination onto $[0,1]_i$ will also be nonzero. Therefore, a linear combination of the modified balance constraints can only be zero if all of the coefficients are zero. □

5 The dimension of $\Lambda(G)$

5.1 Pressure

In a physical foam, the pressure difference between two adjacent bubbles is proportional to the curvature of the film between them. This relationship, which can be deduced from basic mechanical principles, is known as the law of Young and Laplace [10].

The physical notion of pressure turns out to have a purely mathematical analogue, first described by Cox *et al.* [1]. The *pressure* of a bubble in an equilibrated two-dimensional foam can be found by drawing a path that begins in the exterior region and ends inside the bubble, crossing films transversally and never crossing vertices. The pressure of the bubble is the sum of the signed curvatures of the films the path crosses. Plateau's laws guarantee that the pressure will not depend on the choice of path.

5.2 Isobaric velocity fields

Suppose u is in $\Lambda(G)$. If the pressure of a certain bubble in F is given by $\kappa_{12} + \kappa_{34} + \ldots + \kappa_{mn}$, the time derivative of the pressure is given by $(K_{12} + K_{34} + \ldots + K_{mn}) \cdot u$.

An *isobaric* velocity field is an equilibrium-preserving velocity field that keeps the pressures of all of the bubbles constant. To identify the isobaric velocity fields on G, one must first obtain a set of "special" films with the property that any bubble of F can be connected to the exterior by a path that crosses only special films. Such a set can be constructed in the following way:

1. Pick a bubble adjacent to the exterior. Add the film between the bubble and the exterior to the set of special films. Put a "mark" on the bubble to show that you have dealt with it.

2. Pick a bubble adjacent to a marked bubble (or to the exterior). Add the film between the bubble and the marked bubble (or the exterior) to the set of special films. Mark the bubble.

3. Repeat Step 2 until all of the bubbles are marked.

When this procedure is complete, there will be one special film associated with each bubble of F.

Suppose F is undergoing an equilibrium-preserving deformation with instantaneous velocity field u. Since any bubble of F can be connected to the exterior by a path that crosses only special films, the pressures of all of the bubbles are constant if and only if the curvatures of all of the special films are constant. In other words, u is isobaric if and only if it is orthogonal to all of the curvature constraints corresponding to the special films.

Let $C_\Pi(G)$ be the subspace of $V(G)$ spanned by the curvature constraints corresponding to the special films. Let $\Pi(G) = [C_S(G) + \widetilde{C}_Y(G) + C_\Pi(G)]^\perp$. A velocity field is isobaric if and only if it lies in $\Pi(G)$.

5.3 The dimension of $\Lambda(G)$

Theorem 5.1. *The dimension of $\Lambda(G)$ is no greater than $B + \dim \Pi(G)$, where B is the number of bubbles in F.*

Proof. In this proof, I will write V for $V(G)$, Λ for $\Lambda(G)$, and so forth.

Notice that $\Pi^\perp = C_S + \widetilde{C}_Y + C_\Pi$. Taking the dimension of both sides,

$$\begin{aligned}\dim V - \dim \Pi &= \dim(C_S + \widetilde{C}_Y + C_\Pi) \\ &= \dim C_S + \dim(\widetilde{C}_Y + C_\Pi) - \dim[C_S \cap (\widetilde{C}_Y + C_\Pi)].\end{aligned}$$

Let N be the number of stars in G. Since every velocity field assigns a two-dimensional linear velocity and a one-dimensional angular velocity to each star of G, $\dim V = 3N$.

Let E be the number of edges in G. Every edge touches two stars, and every star touches three edges, so $E = \frac{3}{2}N$. There is one symmetry constraint for each edge of G, so $\dim C_S = \frac{3}{2}N$ by Lemma 4.1. Therefore,

$$\begin{aligned}3N - \dim \Pi &= \tfrac{3}{2}N + \dim(\widetilde{C}_Y + C_\Pi) - \dim[C_S \cap (\widetilde{C}_Y + C_\Pi)] \\ \dim[C_S \cap (\widetilde{C}_Y + C_\Pi)] &= \dim(\widetilde{C}_Y + C_\Pi) + \dim \Pi - \tfrac{3}{2}N.\end{aligned}$$

There is one modified balance constraint for each star of G, so $\dim \widetilde{C}_Y = N$ by Lemma 4.2. Since C_Π is spanned by B velocity fields, its dimension can be no greater than B. Therefore, $\dim(\widetilde{C}_Y + C_\Pi) \leq N + B$. As a result,

$$\begin{aligned}\dim[C_S \cap (\widetilde{C}_Y + C_\Pi)] &\leq N + B + \dim \Pi - \tfrac{3}{2}N \\ \dim[C_S \cap (\widetilde{C}_Y + C_\Pi)] &\leq B + \dim \Pi - \tfrac{1}{2}N.\end{aligned}$$

Recall that $\Lambda = (C_S + \widetilde{C}_Y)^\perp$. Therefore,

$$\begin{aligned}\dim \Lambda &= \dim V - \dim(C_S + \widetilde{C}_Y) \\ &= 3N - [\dim C_S + \dim \widetilde{C}_Y - \dim(C_S \cap \widetilde{C}_Y)] \\ &= 3N - [\tfrac{3}{2}N + N - \dim(C_S \cap \widetilde{C}_Y)] \\ &= \tfrac{1}{2}N + \dim(C_S \cap \widetilde{C}_Y).\end{aligned}$$

Since $\widetilde{C}_Y \subset \widetilde{C}_Y + C_\Pi$, it follows that $\dim(C_S \cap \widetilde{C}_Y) \leq \dim[C_S \cap (\widetilde{C}_Y + C_\Pi)]$. Thus,

$$\begin{aligned}
\dim \Lambda &\leq \tfrac{1}{2}N + B + \dim \Pi - \tfrac{1}{2}N \\
\dim \Lambda &\leq B + \dim \Pi.
\end{aligned}$$

□

5.4 Compressions

A *compression* is an equilibrium-preserving velocity field that changes the pressure of one bubble while keeping all of the other pressures constant.

Theorem 5.2. *If $\dim \Lambda(G) = B + \dim \Pi(G)$, then $\Lambda(G)$ has a basis of the form $p_1, \dots, p_B, \pi_1, \dots, \pi_n$, where p_i is a compression of bubble i, and $\pi_1, \dots, \pi_n$ are a basis for $\Pi(G)$.*

Proof. Once again, I will write V for $V(G)$, Λ for $\Lambda(G)$, and so forth.

Pick any bubble of F and call it the "target bubble." Construct a set of special films using the procedure in Subsection 5.2, making sure that the target bubble is the last bubble marked. Let C_Π be as it was before, and let C_Π^* be the subspace of V spanned by the curvature constraints corresponding to all of the special films except the one associated with the target bubble.

Let $\Pi^* = (C_S + \widetilde{C}_Y + C_\Pi^*)^\perp$. A velocity field in Π^* keeps constant the pressures of all of the bubbles except the target bubble.

Recall that $\Pi = (C_S + \widetilde{C}_Y + C_\Pi)^\perp$. Therefore,

$$\begin{aligned}
\dim \Pi &= \dim V - \dim(C_S + \widetilde{C}_Y + C_\Pi) \\
&= \dim V - \dim(C_S + \widetilde{C}_Y) - \dim C_\Pi + \dim[(C_S + \widetilde{C}_Y) \cap C_\Pi].
\end{aligned}$$

Notice that $C_S + \widetilde{C}_Y = \Lambda^\perp$. Therefore, $\dim(C_S + \widetilde{C}_Y) = \dim V - \dim \Lambda$. If $\dim \Lambda = B + \dim \Pi$, then $\dim(C_S + \widetilde{C}_Y) = \dim V - B - \dim \Pi$. Hence,

$$\begin{aligned}
\dim \Pi &= \dim V - (\dim V - B - \dim \Pi) - \dim C_\Pi + \dim[(C_S + \widetilde{C}_Y) \cap C_\Pi] \\
&= B + \dim \Pi - \dim C_\Pi + \dim[(C_S + \widetilde{C}_Y) \cap C_\Pi] \\
\dim C_\Pi &= B + \dim[(C_S + \widetilde{C}_Y) \cap C_\Pi].
\end{aligned}$$

Since C_Π is spanned by B velocity fields, its dimension can be no greater than B. Therefore, $\dim[(C_S + \widetilde{C}_Y) \cap C_\Pi] = 0$.

Finally,

$$\begin{aligned}
\dim \Pi^* &= \dim V - \dim(C_S + \widetilde{C}_Y + C_\Pi *) \\
&= \dim V - \dim(C_S + \widetilde{C}_Y) - \dim C_\Pi^* + \dim[(C_S + \widetilde{C}_Y) \cap C_\Pi^*] \\
&= B + \dim \Pi - \dim C_\Pi^* + \dim[(C_S + \widetilde{C}_Y) \cap C_\Pi^*].
\end{aligned}$$

Since $C_\Pi^* \subset C_\Pi$, it follows that $\dim[(C_S + \widetilde{C}_Y) \cap C_\Pi^*] \le \dim[(C_S + \widetilde{C}_Y) \cap C_\Pi]$. Recall, however, that $\dim[(C_S + \widetilde{C}_Y) \cap C_\Pi] = 0$. Therefore,

$$\dim \Pi^* = B + \dim \Pi - \dim C_\Pi^*.$$

Since C_Π^* is spanned by $B - 1$ velocity fields, its dimension can be no greater than $B - 1$. Therefore,

$$\dim \Pi^* \ge \dim \Pi + 1.$$

All of the isobaric velocity fields are in Π, so there must be at least one velocity field in Π^* that changes the pressure of some bubble. Recall, however, that a velocity field in Π^* cannot change the pressure of any bubble except the target bubble. It follows that Π^* contains a compression of the target bubble.

Since any bubble could have been chosen as the target bubble, every bubble has at least one compression. For each $i \in \{1, \ldots, B\}$, let p_i be a compression of bubble i, and let $P_i \colon \Lambda \to \mathbb{R}$ be the map that sends each equilibrium-preserving velocity field to rate at which it changes the pressure of bubble i. Let $P \colon \Lambda \to \mathbb{R}^B$ be the map $P(u) = (P_1(u), \ldots, P_B(u))$. Notice that P is linear, and the images of $p_1, \ldots, p_B$ under P are linearly independent. It follows that $p_1, \ldots, p_B$ are linearly independent, and their span intersects the kernel of P only at zero. The kernel of P, however, is precisely Π, so if $\pi_1, \ldots, \pi_n$ is a basis for Π, then $p_1, \ldots, p_B, \pi_1, \ldots, \pi_n$ are linearly independent. If $\dim \Lambda = B + \dim \Pi$, any set of $B + \dim \Pi$ linearly independent velocity fields in Λ is a basis for Λ. Thus, $p_1, \ldots, p_B, \pi_1, \ldots, \pi_n$ is a basis for Λ. $\square$

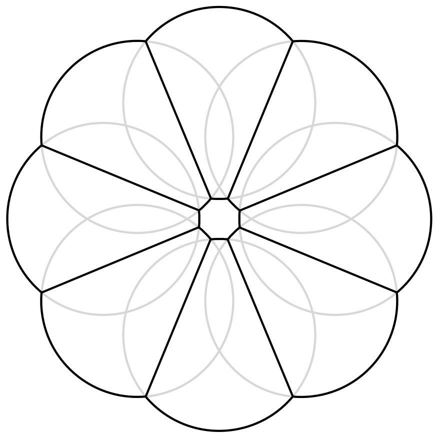
6 Singular foams

6.1 Extra isobaric dimensions

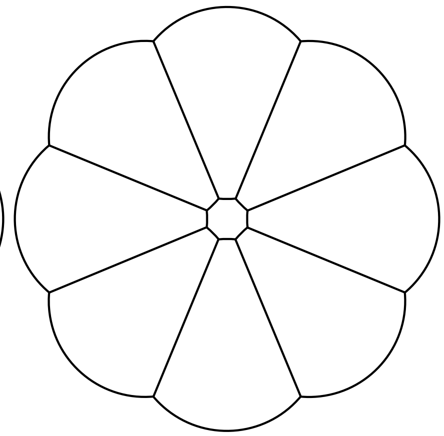
The subspace of isobaric velocity fields must always contain a three-dimensional subspace comprising the velocity fields of rigid rotations and translations. Intuition and numerical experiments suggest that for most foams, the subspace of isobaric velocity fields contains *only* the velocity fields of rigid rotations and translations.

One type of foam with additional isobaric velocity fields is a "necklace": a ring of bubbles surrounding a zero-pressure central bubble. A necklace can be constructed by arranging a sequence of circles $L_1, \ldots, L_n$, with radii $R_1, \ldots, R_n$, so that the distance between the centers of L_i and L_{i+1} is $\sqrt{R_i^2 + R_{i+1}^2 - R_1 R_2}$. This guarantees that L_i intersects L_{i+1} at two points, and the angle of intersection is 120° [8]. The intersection points become the vertices of the foam, and the films are added in the obvious way.

An example of a necklace is shown in Figure 13. In this example, all of the circles have the same radius, and their centers are located at the vertices of a regular polygon. This kind of necklace will be called "perfect." A perfect necklace with n outer bubbles exists if and only if $n \ge 7$.



(a) Circles shown



(b) Circles not shown

Figure 13

If $R_1, \ldots, R_n$ are fixed, the configuration of the necklace is completely determined by the positions of the centers of $L_1, \ldots, L_n$. From this fact, it is not hard to deduce that if G is the constellation of a necklace with n outer bubbles, $\dim \Pi(G) = n = B - 1$.

6.2 Missing compressions

Let F be a necklace whose n circles all have the same radius, and let G be the constellation of F. As I have said, $\dim \Pi(G) = n$. It turns out that if F is perfect, $\dim \Lambda(G) = B + n$. Thus, by Theorem 5.1, there is a compression for each bubble of F. If F is not perfect, however, there is no compression of the central bubble.

To see why, let F^* be a foam obtained from F by compressing the central bubble—that is, by applying an equilibrium-preserving deformation whose velocity field is a compression of the central bubble at all times. Because all of the outer bubbles of F have the same pressure, all of the outer bubbles of F^* have the same pressure. Therefore, the films between the outer bubbles of F^* are straight lines. As a result, each outer bubble of F^* looks like the one in Figure 14. Because all of the outer bubbles of F^* have the same pressure, the values of R and r in Figure 14 are the same for all outer bubbles. Therefore, Δ depends only on γ, and it is fairly easy to see that the relationship between Δ and γ is one-to-one. Since adjacent bubbles share a film, adjacent bubbles must have the same value of Δ. Therefore, all of the outer bubbles have the same Δ, so all of the outer bubbles have the same γ. It quickly follows that F is perfect.

In short, if F^* is a foam obtained from F by compressing the central bubble, then F is perfect. Thus, if F is not perfect, there is no compression of the central bubble. In this case, by Theorem 5.1 and the converse of Theorem 5.2,

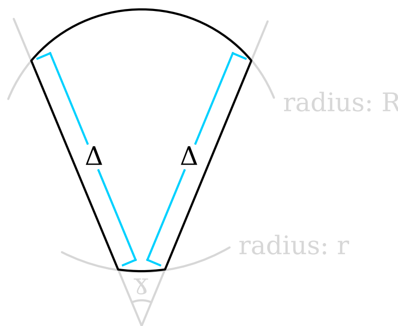


Figure 14

$\dim \Lambda(G) < B + \dim \Pi(G)$.

6.3 Pressure-related instabilities

A *flower* is a foam obtained by compressing the central bubble of a perfect necklace. Let G be the constellation of a flower with n bubbles, and let P_c be the pressure of the central bubble. The dimension of $\Lambda(G)$ is $B+n$ when $P_c = 0$, and numerical experiments suggest that $\dim \Lambda(G)$ is $B+3$ when $P_c \neq 0$. If you start with $P_c > 0$ and steadily decrease it, you will see the dimension of $\Lambda(G)$ jump from $B+3$ to $B+n$ when P_c hits zero. This is precisely the buckling instability described by Weaire *et al.* in [9]. Weaire and his coauthors remark that "it is not immediately clear why the instability should only show itself for $n > 6$." In light of the discussion above, the answer is obvious: there is no buckling instability for $n < 7$ because there is no perfect necklace for $n < 7$. If you steadily decrease P_c with $n < 7$, the central bubble will shrink to a point before P_c reaches zero.

7 Conclusions

7.1 Summary of results

The main results of this paper are summarized below.

Let F be an equilibrated two-dimensional foam. Assume that F is connected, and that every bubble of F has at least three sides. (These assumptions do not lead to any practical loss of generality, as discussed in Subsection 3.1.) Let G be the constellation of F.

1. The dimension of $\Lambda(G)$ is no greater than $B + \dim \Pi(G)$, where B is the number of bubbles in F (Theorem 5.1).

2. If $\dim \Lambda(G) = B + \dim \Pi(G)$, then $\Lambda(G)$ has a basis of the form $p_1, \ldots, p_B, \pi_1, \ldots, \pi_n$, where p_i is a compression of bubble i, and $\pi_1, \ldots, \pi_n$ are a basis for $\Pi(G)$ (Theorem 5.2).

3. There exists F for which $\Pi(G)$ contains more than just the velocity fields of rigid rotations and translations (Subsection 6.1).
4. There exists F for which the dimension of $\Lambda(G)$ is less than $B + \dim \Pi(G)$ (Subsection 6.2).

7.2 The space of equilibrated foams

Two constellations are *combinatorically equivalent* if they have the same edges and the same number of stars. A *constellation space* is an equivalence class of constellations. A constellation in a given constellation space is uniquely determined by the positions and angular orientations of its stars. A constellation space of constellations with n stars may therefore be thought of as $(\mathbb{R}^2 \times S^1)^n$ plus edge information.

Let $\mathcal{C}$ be a constellation space, and let $\mathcal{E}$ be the set of constellations in $\mathcal{C}$ that are the constellations of equilibrated foams. For each $E \in \mathcal{E}$, $\Lambda(E)$ is the set of time derivatives of continuously differentiable curves in $\mathcal{E}$ passing through E. If $\mathcal{E}$ were a differentiable manifold, $\Lambda(E)$ would be the tangent space of $\mathcal{E}$ at E. It seems likely that $\mathcal{E}$ is diffeomorphic to $\mathbb{R}^{B+3}$ at most points, but the examples in Section 6 demonstrate that there are constellation spaces for which $\mathcal{E}$ is not diffeomorphic to $\mathbb{R}^{B+3}$ at every point.

Many different classes of generalized manifolds have been studied, and I suspect that the set of all possible $\mathcal{E}$ falls into one of them. Learning more about the possible geometries of $\mathcal{E}$ may shed light on questions about the geometry, existence, and uniqueness of equilibrated foams.

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