# How Do You Know that You Know? Making Believe in Mathematics

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Distinguished University Professor Lecture

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**Abstract**

*Knowing in a discipline means not only knowing its products, and their uses, but also about the genesis of its knowledge, how that knowledge is warranted, and how contrary views are reconciled. This shapes the entailments of teaching and learning of that discipline whether at age 8, 18, or 80. This lecture will focus on the discipline of mathematics, illustrating what mathematicians do when they seek to "prove" a claim, taking note of the challenges presented by the advent of computer-based proofs. Proving is a fundamental mathematical practice, learned by few students, and then only late in their education. I will consider how proving could shape the early learning of mathematics, including the use of "generic" proofs. Examples from elementary classrooms will illustrate why the warrants for shared knowledge in a field matters for the curriculum at any point.*

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<u>Introduction</u>

Each discipline (or field of knowledge) presents us with theories, claims, and debates. For example, quantum theories in physics, biology (evolution vs. intelligent design), meteorology (global warming - its existence, causes), history (the causes of the American Civil War), law (legal interpretations of the US Constitution), economics (institutional actions that can best intervene on the collapse of the global economy).

To have a critical understanding of a discipline, it is not enough to know its products and their uses. One needs to understand as well the genesis of its knowledge, how that knowledge is warranted or certified, and how, within the norms of the field, contrary views are reconciled, if at all. Each field – physics, biology, medicine, linguistics, history, economics, law, theology, etc. – has its own versions of this. And, as Joseph Schwab emphasized, this in turn shapes the entailments of the teaching and learning of that discipline, at every level.

The *genesis* of knowledge generally begins with a focused exploration of some field of experience or of ideas, and expressions of creative insight or speculative generalization leading to hypotheses or conjectural claims. The somewhat mysterious creative processes underlying such insights have been the subject of much reflective, but relatively undisciplined, writing, even by some of our greatest creative thinkers.

My focus today is not on exploration and discovery, but on justification. It is about the warrants for well-articulated claims, and about how a professional community agrees to certify the truth of such claims; in particular, about how it reconciles conflicting claims. Each field has its own norms for certifying truth. Mathematics has evolved a unique method – deductive proof – dating back to Greek antiquity. That is the subject of my talk today: proof, as a theoretical concept; proving, as a fundamental practice of the disciplinary community; and finally, what substantial forms proof and proving might appropriately take in mathematics classrooms, even in the early grades.

Proof and truth in mathematics

A mathematical proof is composed of a sequence of assertions of the form, "something is true because something else is true," where the word "because" conveys some well-understood logical rule of inference, for example syllogistic. But how does such a process get started? In mathematics it starts typically from some set of axioms, i.e. statements taken to be assumed without further justification, plus definitions (the precise meanings of terms being used), and eventually also the products of prior proofs, which are not in practice required to be repeated when used to derive further conclusions. (Mathematics is thus hierarchical, building new knowledge on old). Moreover, the system is assumed to be consistent, i.e. free of logical contradiction. The mature notion of formal mathematical proof was finally achieved only in the early 20$^{th}$ century.

Saying that a mathematical claim is true means, for a mathematician, that there exists a proof of it. Notice then that a proof rests only on axioms and precise meanings and rules for the use of language and logical inference. It thus makes no reference to extra-mathematical contexts or experience. In this sense, mathematical truth, being purely analytic, is certain and permanent, thus making mathematical knowledge cumulative – nothing is discarded.

But it is also in a sense "unreal." Einstein once said, "As far as the laws of mathematics refer to reality they are not certain; and as far as they are certain, they do not refer to reality." This contrasts dramatically with the natural sciences, wherein truth is empirical; it is derived from and applies to the experiential world. It is therefore uncertain, and subject to being overthrown by new data. A mathematician once described fundamental particle physics as a field in which the blackboard is periodically erased.

But despite its "unreality," it must be said that there is still something wonderful and mysterious about proof-based mathematical knowledge. Wonderful, because such purely analytical reasoning can reveal often surprising truths of extraordinary depth, subtlety, and sweep. Mysterious because, as history has demonstrated repeatedly, such curiosity driven mathematical discovery and proof repeatedly produce concepts and structures exquisitely suited to model and understand the world of experience. This is the "unreasonable effectiveness of mathematics" of which the physicist Eugene Wigner spoke. It is what prompts Mario Livio to ask, "Is God a mathematician?" Sidebar illustration of a proof:

| **Claim**: Every whole number > 1 is a product of (one or more) primes | |
|---|---|
| **Steps of the proof** | **Logical justification** |
| 1. If this were not true, then there would be a smallest number N for which it fails. | Every non empty set of whole numbers has a least element. This (intuitively obvious) fact is either part of an axiom system for whole numbers, or else taken here as part of prior established knowledge. |
| 2. Being a counterexample to the claim, N cannot be prime, and so N is a product, A x B, with A and B both whole numbers strictly between 1 and N. | This follows from the definition of prime number, and the fact that a proper factor of a number is smaller than that number (prior knowledge). |
| 3. Since A and B are both > 1 and < N, they cannot be counterexamples to the claim (of which N was, by choice, the smallest), and so A and B are each products of primes. | This uses the hypothesis that N was the smallest counterexample, the defining characteristic of N. |
| 4. It follows then that N = A x B is also a product of primes. | The meaning (definition) and associativity (prior knowledge) of multiplication |
| 5. This contradicts assumption #1, that N is a counterexample to the claim. | |
| 6. This contradiction shows that assumption #1, from which it was derived, cannot be true. I.e. the claim cannot fail to be true, and so it is true. | The system of arithmetic is assumed to be consistent (free of contradiction). |

Certifying truth: What are mathematicians doing when proving?

The truth of a mathematical claim rests on the existence of a proof. Stated this way, such a criterion is absolute, abstract, and independent of human awareness. This criterion is conceptually important, but practically useless.

*The practice of mathematics is concerned with*
*ways to make its practitioners convinced of the existence of proofs*.

This is how mathematical truth is *certified*, in contrast with how it is defined. Thus, while truth is absolute, certification is a social act of human judgment. Of course the standard way for certification to happen is for a mathematician to actually construct a proof, and exhibit it for critical examination by peers. The rules for proof construction are sufficiently exact that the checking of a proof should, in principle, be a straightforward procedure. However, formal mathematical proofs are ponderous and unwieldy constructs. For mathematical claims of any reasonable complexity, mathematicians virtually never produce complete formal proofs. Indeed, requiring that they do so would cause the whole enterprise to grind to a halt. The resulting license in the practices of certifying mathematical knowledge has caused some (non mathematical) observers to conclude that mathematical truth is just another kind of social negotiation, and so, unworthy of its prideful claims of objective certainty.

I think that this is based on a misunderstanding of what mathematicians are doing when they claim to be proving something? Specifically, I suggest that:

*Proving a claim is, for a mathematician,*
*an act of producing, for an audience of peer experts,*
*an argument to convince them that a proof of the claim exists.*

Thus, proving here is an act of producing conviction, i.e. of "making believe." Two things are important to note here. First, that implicit in this description is the understanding that the peer experts possess the conceptual knowledge of the nature and significance of mathematical proof. Second, the notion of "conviction" here is operationalized to mean that:

*the convinced listener feels empowered by the argument,*
*given sufficient time, incentive, and resources, to actually construct a formal proof.*

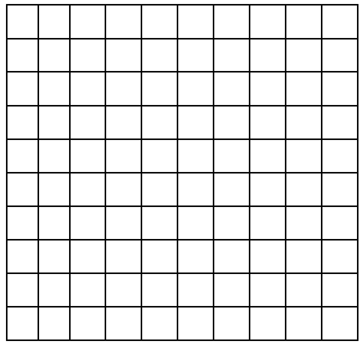
I have not seen this perspective publicly articulated quite this way, so I shall be interested to hear the reactions of my mathematical colleagues to this.

Generic proof

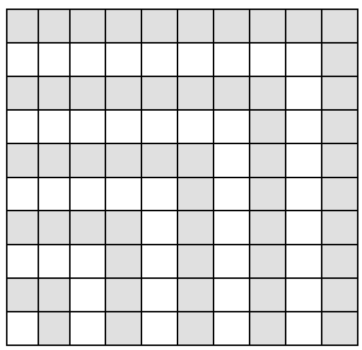
One particular form of proving in the above sense, one commonly used by mathematicians, goes by the name of "*generic proof*." This refers to a situation where an argument is formulated around a representation of a strategically chosen example of a claim, intended to support one's intuition while following the reasoning. But the argument itself appeals only to generic, rather than idiosyncratic, features of the example. To illustrate this, consider the following claim:

*For any whole number N, the sum of the first N odd numbers is $N^2$*

Generic proof: We can represent $N^2$ as the number of cells in an N x N grid square. Here is the case N = 10:



On the other hand, we can build up the square by successively adjoining L-shaped figures, and the sizes of the latter are exactly the first N odd numbers.



$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 \quad = \quad 100$$

Though the picture depicts only the case N = 10, the argument is generic, and many people will find it generally "convincing" in the above sense. In general, it is somewhat vague when such arguments cross the threshold from heuristic evidence to generic proofs. So "generic proof" is not itself a rigorously defined concept.

Generic proofs are especially common in emerging fields where the technical language and notation to facilitate the precise general articulation of ideas is not yet fully developed. This is a condition pertinent to that of elementary classrooms, in which the language and tools of mathematical reasoning are not yet highly developed. And so we can expect that generic proofs, which are decidedly more than checking a sampling of cases, might play a significant role in the development of early reasoning skills.

Size, complexity, and technology

Mathematical knowledge, being permanent and cumulative, has grown in size and complexity. This reflects itself in a corresponding growth in size, depth, and complexity of mathematical claims and their proofs. The published proof by Andrew Wiles and others of Fermat's Last Theorem reached over 200 pages of dense and sophisticated mathematical argument that was reviewed by a team of expert referees.

The classification of finite simple groups[1] is a monument of $20^{th}$ century mathematics. Though relatively simple to state, its proof is decomposed into many major parts, separately executed by dozens of mathematicians around the world, and occupying more than 10,000 published pages. It is safe to say that no single human agent has inspected all of the details of this massive "proof." Some mathematicians have therefore expressed reservations about the level of certification of this result. Efforts to simplify and conceptually unify the proof are now themselves the goal of major research projects.

Some proofs manage to reduce a general claim to the checking of a finite number of cases. However, this number, though finite, and the complexity of the checking, may be so large and demanding as to be accessible only through the use of powerful computers. Such situations raise serious new questions about the criteria for certification, since human agency has been removed from an important part of the process. The first large-scale case of this was the proof by Appel and Haken of the Four Color Theorem: Any planar map can be colored with four colors. A more recent example is the proof, by Tom Hales, while here at Michigan, of the Kepler Conjecture about sphere packings of space. Just as human referees traditionally checked conventional proofs, we now need proof-checking software to certify computer based-proofs. And, in turn, we need rigorous certification of the reliability of the proof checking software. Major efforts by Hale and others, both technical and theoretical, are now underway to provide such reliability.

Another interesting recent development is the notion of "zero-knowledge proofs." These are protocols now used in designing cryptographic security. They permit an agent to reliably convince an interrogator that she is in possession of a certain piece of knowledge, for example a proof of a theorem, but without revealing any of that knowledge to the interrogator.

What kinds of things do mathematicians prove?

| **What kinds of mathematical claims do proofs prove?** | |
|---|---|
| **Some Types of Theorems** | **Examples** |
| 1. **[G] Generality**: Something is true for all cases, in some domain beyond what is known. | 1. For any two odd integers, their sum is even.<br>2. For any polygonal decomposition of a sphere, with V vertices, E edges, and F faces, one has V-E+F = 2. |
| 2. **[P] Properties**: A particular mathematical object (or family of objects) has certain properties. Perhaps one can say that these properties actually characterize the object(s) uniquely. | 1. Every positive integer is a sum of four squares of integers. (E.g. 5 = 0+0+1+4, 15 = 1+1+4+9)<br>2. The sum of the angles in a plane triangle is $180^o$.<br>3. Positive numbers $a \le b \le c$ are the side lengths of a right triangle if and only if $a^2 + b^2 = c^2$. |
| 3. **[Eq] Equivalence**: Two apparently distinct mathematical objects (or families of objects) are in fact the "same" is some precise sense. | 1. The sum of the first n odd numbers is $n^2$.<br>2. The powers of 2 are precisely those whole numbers not expressible as the sum of two or more consecutive whole numbers.<br>3. A teacup and a donut are (topologically) the "same."<br>4. The symmetries of an equilateral triangle and the invertible 2 x 2 matrices over $\mathbf{Z}/2\mathbf{Z}$ are the "same" group. |
| 4. **[C] Classification**: This has to do with explicitly describing (by listing or parametrizing) all mathematical objects of a certain species, for example describing <u>all</u> solutions to an equation or problem. Central to such questions is not only the specification of the objects to be classified, but | 1. The set of solutions to the problem,<br>"I have pennies, nickels, and dimes in my pocket. If I pull out two coins, how much money might I have in my hand?"<br>is {2¢, 6¢, 10¢, 11¢, 15¢, 20¢}.<br>2. The set of "Pythagorean triples," i.e. triples (a, b, c) of whole numbers such that $a^2 + b^2 = c^2$, can be rationally parametrized (in terms of rational points on the unit circle). |

[1] Groups are a species of algebraic structure by which mathematicians formalize the concept of symmetry.

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| also making explicit what it means for two of them to be considered the "same" (or equivalent). | 3. There are exactly five regular solids (up to similarity): the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. (In contrast, there are infinitely many regular polygons there exactly one (up to similarity) each integer $n \geq 3$.<br>4. The classification of finite simple groups is given by some discretely parametrized families, plus a finite list of "sporadic" groups. |
| 5. **[Ex] Existence**: Showing that a certain kind of mathematical object exists, for example that an equation has a solution. This has two forms: (C) Constructive (methods to construct, or effectively approximate the solution are given); and (NC) Non-constructive (We are only assured of existence, without knowing how to produce a solution.) | (C1) Every quadratic polynomial has a (complex) root, given by the quadratic formula.<br>(C2) Every non-constant polynomial has a complex root.<br>(C3) Irrational numbers exist: $\sqrt{2}$ is irrational (proved)<br>(NC3) Irrational numbers exist: The cardinal of the set of real numbers is > the cardinal of the set of rational numbers.<br>(NC4) Sometimes equations are shown to have finitely many solutions, but without specifying any bound on their size. This makes it computationally inaccessible to determine the set of all solutions. |
| 6. **[I] Impossibility (Non-existence)**: A certain kind of mathematical object or process does <u>not</u> exist. For example, and equation may have no solution. Or there may be no way to construct a certain object with certain prescribed tools. | 1. There is no solution to $7 \div 0 = x$.<br>2. There is no solution to the system of equations, $4x + 6y = 10, \quad 6x + 9y = 12$<br>3. There is no general method, with straightedge and compass alone, to trisect an angle, in fact an angle of $60^o$.<br>4. A regular n-sided polygon cannot be constructed with straightedge and compass if n is not a power of 2 times a product of distinct "Fermat primes," i.e. primes of the form $2^e + 1$, where e is a power of 2.<br>5. For n an integer $\geq 3$ there is no solution in non-zero integers x, y, z of the equation, $x^n + y^n = z^n$. (Fermat, Wiles) |

<u>The teaching and learning of proof</u>

Deborah Ball and I have argued, along with Schwab, that the learning of a discipline must include some understanding of the genesis and warrants for its knowledge. In the case of mathematics, this means learning about the nature of proof, and the practices of proving. We view this not as a peripheral embellishment of the curriculum, but rather as a basic skill, to be learned by all students. It is a fundamental source of sense making in mathematics. Further, we are persuaded that such learning must be developmental, starting in the early grades. We are fully sensitive to issues of equity in American education, and that concern only strengthens our belief.

Many will hear this as hopelessly romantic, and naively unrealistic. How can mathematical reasoning in a third grade class be anything more than a caricature of substantive and authentic mathematical practice? For example, these children cannot have any robust idea of even the concept of mathematical proof, so how could they be trying to convince classmates that a proof exists.

Our view of such learning is this: A guiding intellectual imperative in the instruction is embedded in questions that do make sense to children, like:

*"How do you know?"*
*"How can you be sure there is not another solution?"*
*"Does this always work?" In all cases?"*
*"What do other people think about Mary's explanation?"*

These kinds of questions can be applied to anything from the answer to a numerical computation to a conjectural pattern (numerical or geometric, for example) that a student has observed. Now, you may rightly raise doubts, noting that children do not yet have the formalized resources to construct and articulate mathematically adequate responses to such questions. In this sense, the instruction has to be a kind of "bootstrap" undertaking, in which students, while attempting to construct their best approximations of answers to questions like those above, are beginning to gain appreciation of the need and desire for knowing why, and of the need for "making believe," for convincing others. More than establishing claims, they are, at the same time, engaged in constructing, with guidance from the teacher, some of the foundational intellectual architecture of mathematical reasoning itself, so that they become progressively better equipped to provide more rigorous answers to the questions, and to evaluate the rigor of the answers of others.

I can imagine that some of you hear this as impractically theoretical, and divorced from real teaching and learning. On the contrary, these ideas were distilled from our very close study and analysis of actual instruction. To make these ideas more concrete and vivid, I will conclude now by immersing you in the observation of a real classroom, wherein the above imperative, to answer, "How do you know?" prevailed, and wherein the following commitments were integral to the instruction:

- To treat the mathematics with integrity.
- To take student thinking seriously, and to make it an integral part of the instruction.
- To treat the learning and growth of knowledge in the class as the work of an intellectual collective.

<u>The context of the video</u>

- A third grade class (ages 8 – 10).
- Multi-cultural and multi-lingual
- It's January, middle of the academic year
- Norms: The children have already begun to internalize the need to explain why things are true, and to critically – and respectfully – evaluate explanations of their classmates.
- When we see them, they have been studying even and odd numbers, noticing patterns
- "Bernadette's Conjecture:"
  *When you add an odd number to an odd number you get an even number.*
- We enter the classroom when they have been trying to prove this.

Clip 1 (Jan 26): You can't prove Bernadette's Conjecture

Jillian & Shekira:
*- We were trying to prove that . . . you can't prove that Bernadette's conjecture always works*
*- Because . . numbers go on and on forever and that means odd numbers and even numbers . . go on forever and, um, so you couldn't prove that all of them work.*

This stunning assertion is at once insightful and revolutionary. J & S rightly observe that Bernadette's Conjecture is not just one, but rather infinitely many claims. And so, it would be impossible to individually check all of the cases. B's Conjecture is an "infinitely quantified statement," about all of the infinitely many pairs of odd numbers.

Ogechi
*- I think it can always work because I um tried . . . 18 of them, and I also tried a Shea number so I think, I think it can always work.*

Ogechi is still content to believe B's Conjecture, having checked that it holds in the many cases that she checked. Note that this is a type of empirical reasoning that is valid in science, but is not conclusive in mathematics.

Lin
*- I think it could always work because with those conjectures,* (she motions to several previously discussed and widely agreed-upon conjectures, posted above the chalkboard) *we haven't even tried them with all the numbers that there is, so why do you say those work? Well, we haven't even tried all those numbers that there ever could be.*

Lin expresses an interesting, almost legalistic, objection to what she sees as the logical inconsistency of Jillian's & Shekira's argument. Lin does not say directly that the J & S argument is wrong. Rather, she says that the same mode of reasoning would apply equally well to defeat several other conjectures that had previously been agreed to by the class, and in fact "published" as posters in the classroom. So Lin is saying, in effect, if your argument is valid, then you should have objected on the same grounds to those previous conjectures.

This is a pretty sophisticated piece of logical reasoning. It illustrates a step in how the children are progressively constructing the collective norms of mathematical proving.

But all of this leaves the class, and the teacher, with an important mathematical dilemma.

Clip 2 (Jan 31): Bernadette's generic proof

Bernadette reports at the board on a proof of her conjecture developed in her group.

- She begins by drawing two strings of seven hash marks, separated by a + sign.
- Then, working from left to right, she circles the hash marks by twos, until the seventh one circled alone, and likewise for the first hash mark to the right of the plus sign. She continues to circle the remaining six hash marks by twos.
- Then she draws an arc to connect the two individually circled hash marks.
- This achieves a representation of the combined collection as grouped by twos, with none left over.

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= (||) (||) (||) (|+|) (||) (||) (||)

This drawing, which appears to show that 7 + 7 is even, is done without explanatory narrative. When completed, Bernadette turns to the class, and never again looks or points at the board. Moreover, her discourse to the class is entirely generic, with no specific reference to the numbers (7 and 14) depicted on the board.

After fumbling a bit to get the wording straight, she says,

*"if you added two odd numbers together you can add the ones left over and it would always equal an even number."*

Notice that she is talking generally about <u>any</u> odd numbers, not about the two sevens on the board. Her verbal argument, unlike her drawing, is generic.

What is it that permitted Bernadette to surmount Jillian & Shekira's dilemma? Bernadette's is using implicit <u>definitions</u> of odd and even numbers: A (whole) number is odd if you can group it by twos with one left over, and it is even if there is none left over. These definitions are themselves infinitely quantified, they apply to all numbers, and therefore they embody the capacity to support conclusions of similar purview.

It is interesting to consider how older students, with algebraic resources, might have constructed a proof of Bernadette's Conjecture. It might for example rest on expressions of the form

$$(2x + 1) + (2y + 1) = 2x + 2y + 1 + 1$$

$$= 2(x + y + 1)$$

where x and y stand for arbitrary integers. It is instructive to appreciate how much is compressed in such very efficient and compact reasoning. It is more than the use of symbolic expressions, and their algebraic manipulation. The very use of the letters x and y here embodies the passage to the infinite, from the consideration of individual numbers to whole (infinite) systems of numbers, which are the range of possible values of x and y. So infinite quantification is almost imperceptibly embedded already in the use of symbolic variables, and so in particular in these algebraic expressions of evenness and oddness.

Further, the multiplication, $2x = 2 \cdot x$ stands for $x + x$. This would correspond to a "fair share" definition of even number (two equal groups). The "pair definition," as a sum of groups of two, $2 + 2 + 2 + \ldots + 2$ ($x$ twos), as used by Bernadette, would correspond algebraically to $x \cdot 2$. But, by algebraic convention, we always put the scalar factor on the left. Thus our algebraic notation already shrouds a distinction that remains mathematically significant for the third graders.

It is because Bernadette did not have such algebraic notation at her disposal that her illustration of the argument was necessarily confined to a particular case. In the absence of such notation, and of formalized definitions of even and odd, she was nonetheless able to compose a generic formulation of her thinking, and so construct what we can see as a generic proof of her conjecture.

Clip 3 (Jan 31, later):            Does Bernadette's proof make you a believer?

Teacher

*- Do people think that does prove that an odd plus an odd would always be even?*

*- Lin thinks it doesn't because you wouldn't know about big, very big numbers.*

Lin does not accept the genericity of Bernadette's argument, seeing it as only the proof of one case. Jillian, on the other hand, seems to grasp the spirit of the generic argument:

Jillian

*- I don't think so* (disagreeing with Lin)*, because she* (Jillian) *didn't say it had to be those two numbers, those two odd numbers. It could be any two odd numbers because, um, there's always one left.*

Lin still insists that one can't really know what happens for very large numbers, so large that they don't even have names, and we don't know how to think of them. For these children, the infinite remains something still mysterious and beyond comprehension.

Lin

*- I'm trying to say that that's only one example. You can't really say that it will work for every odd number, even-- I don't think it would work for numbers that we can't say or figure out what they are.*

Rania

*- How does . . . Jillian know that it would always work? She never tried all the numbers.*

Bernadette

*- ...Mathematicians can't even do that. You would die before you counted every number.*

Bernadette agrees that one could not check all cases. But in her reasoning that is not an obstacle to her proof. But Rania still disagrees:

In summary, the generic proof offered by Bernadette has persuaded Jillian, but not yet Lin or Rania. But the lines of disagreement have been clearly articulated. The children are negotiating their norms for establishing conviction.