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DIFFERENTIAL ANALYZER SOLUTION OF THE DIFFUSION EQUATION WITH A FREE BOUNDARY

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NOMENCLATURE

$(\bar{\ })$	indicates a quantity in the real dimensional space, $(\bar{x},\bar{t})$. [Equations (1)]
$(\tilde{\ })$	indicates a quantity in the non- dimensional space, $(\tilde{x},\tilde{t})$. [Equations (2)]
$(\hat{\ })$	indicates a quantity in the $(\hat{x},\hat{t})$ space for the finite slab. [Equations (38)]
$(\ )^{-1}$	indicates the inverse of the quantity in the parentheses.
$\approx$	indicates approximately equal quantities.
b	steady state boundary speed. [Equation (11)]
B	a matrix of eigenvectors of the matrix A. [Equation (27)]
$\bar{c}$	specific heat of the conducting medium. [Equations (1)]
$\bar{c}_o$	maximum value of the specific heat of the conducting medium. [Equations (2)]
C	constant in the nonlinear power transformation of the space variable. [Equations (45)]
D	length dimension (Appendix A).
dim( )	dimension of the quantity in the parentheses (Appendix A).
e	the base of the natural logarithms. [Equations (12)]
$e^{[\lambda]t}$	a diagonal matrix with elements $e^{\lambda_i t}$. [Equation (27)]
f(x)	functional notation for the nonlinear space variable transformation. [Equations (40)]
$\vec{f}(t)$	a column vector describing the boundary conditions in Equation (26).
F_n	heat flux passing the n^{th} finite difference station (Appendix C).
g_∞	the temperature at $x=\infty$ in the $(\bar{x},\bar{t})$ and $(\tilde{x},\tilde{t})$ spaces. [Equations (1)]
g_o	the temperature at $x=0$ in the (x,t) space. [Equations (9)]
g(x)	initial temperature distribution. [Equations (1)]

$\vec{g}$	a column vector of initial temperatures. [Equation (27)]
G_n	initial temperature distribution in the computer variables. [Equations (33)]
h	maximum amplitude of the heat pulse $H=h(1-\cos\omega t)$ (Page 62)
H(t)	heat flux entering the conducting medium (Figure 1).
$\mathcal{H}(\tau)$	the heating rate in the compute variables. [Equations (33)]
J	Jacobian of a transformation.
k	exponent in the nonlinear space variable transformations. [Equations (45) and (46)]
$\bar{k}$	thermal conductivity of the conducting medium. [Equations (1)]
$\bar{k}_o$	maximum value of the thermal conductivity of the conducting medium. [Equations (2)].
K_1	scaling constant for the space variable. [Equations (2)]
K_2	scaling constant for the time variable. [Equations (2)]
K_3	scaling constant for the heating rate. [Equations (2)]
$\ell(t)$	position of the moving boundary as a function of time (Figure 1).
L	latent heat of fusion of the conducting medium. [Equations (1)]
M	mass dimension (Appendix A).
n	an index designating the n^{th} finite difference station (Appendix C).
q	a point heat source in the conducting medium. [Equations (1)]
Q	The heat added to the finite slab (Page 88).
r	the number of finite difference increments. [Equations (13)]
s	the initial thickness of the finite slab. [Equations (36)]
t	the independent time variable.
t_1	the time at which melting starts. [Equations (1)]

t_2 the time at which melting stops. [Equations (1)]

t_3 the time after which the boundary velocity may be considered to be linear in time (Page 19).

t_m the time at which the finite slab is completely melted. [Equations (36)]

T time dimension (Appendix A).

u the dependent temperature variable (Figure 1).

$\bar{u}_c$ the melting temperature of the conducting medium. [Equations (1)]

$\vec{u}(t)$ a column vector of the temperatures at the finite difference stations. [Equation (26)]

U_n the computer temperature variable. [Equations (33)]

x the independent space variable (Figure 1).

Δx finite difference increment in x. [Equations (13)]

Θ temperature dimension (Appendix A).

$\Theta(x,t)$ an arbitrary function (Appendix C).

λ_i the eigenvalues of the matrix A. [Equation (27)]

$[\lambda]$ the non-singular diagonal matrix of the eigenvalues of the matrix A. [Equation (27)]

$\lambda(\tau)$ the boundary position in the computer variable. [Equations (33)]

π the constant 3.14159... in Equations (29), (30), and (47).

π a dimensionless ratio in Appendix A.

$\bar{\rho}$ the density of the conducting medium. [Equations (1)]

$\bar{\rho}_o$ the maximum value of the density of the conducting medium. [Equations (2)]

τ the independent computer time variable. [Equations (33)]

NOMENCLATURE (CONT'D)

ϕ_c the non-dimensional specific heat function. [Equations (2)]

ϕ_k the non-dimensional conductivity function. [Equations (2)]

ϕ_ρ the non-dimensional density function. [Equations (2)]

CHAPTER I

INTRODUCTION

Background

The problem of the conduction of heat in a solid which is freezing at its boundary was first posed by Fourier in connection with his study of the solidification of the earth.[4] Crank[4] and Evans, Isaakson, and Macdonald[12] list several other problems which are governed by essentially the same equations. Recently, these problems have received renewed interest in such diverse areas as the tarnishing of metals[4] and the ablation of material from bodies entering the earth's atmosphere.

The first solution of the freezing problem was obtained by Neumann about 1840.[2,6] This solution was for a semi-infinite body of liquid which initially existed at the temperature of fusion. In 1891, Stefan[38] solved the problem again for the semi-infinite body, for two particular initial conditions. This freezing or melting problem has since been called the Stefan problem;[6] other similar problems have been called Stefan-like problems;[12] and the published literature is profuse. The article by Brillouin[1] contains an excellent bibliography of the literature up to 1929.

The existence and the uniqueness of the solution to this problem have been shown for very general circumstances by Evans,[11] Douglas,[7] Miranker,[30] Miranker and Frisch,[31] and Kyner.[26] The general solution has not yet been obtained, and it is not in the offing because the problem is nonlinear, as Landau shows.[27]

Several schemes for solving the problem, at least approximately, have been devised. These analytical solutions are for the semi-infinite body and apply only for particular initial and boundary conditions. (All of these solutions provide for the two phases to remain in contact after the change of phase.) Datseff[6] has developed a step-by-step solution which is very laborious and not completely general. Kolodner[24] has reduced the problem by eliminating the diffusion equation and reducing the problem to that of solving a nonlinear, ordinary, integro-differential equation. Friedman[15] has refined a method developed by Rubinstein[36] which reduces the problem to another nonlinear, integral equation that is solved by a method of successive approximations. Douglas and Gallie[8] have developed an iterative process based on a difference analog for a simple case which does not allow for heat flow in the material before the change of phase. Ehrlich[10] has developed an iterative scheme again based on a difference analog which does provide for heat conduction before the change of phase, and Crank[5] has used an iterative process which allows this heat conduction. Crank, in this same article, uses Lagrange's interpolation formula in a second method to allow the variation of finite difference increment size when the boundary passes through a given finite difference cell. Longwell[28] has devised a graphical solution using the Schmidt construction.

The ablative problem, in which the melted material is removed instantaneously, was first studied, according to Landau,[27] by Soodak.[42] Landau[27] has developed a solution for the melting of a semi-infinite solid with perfect ablation, in which none of the input energy is used to heat the liquid. In this solution he considers the case of a prescribed temperature at infinity, a constant heating rate at the finite boundary, and a uniform initial temperature distribution. He solves the warming phase, before melting starts, by means of the classical analytical solution as given in Carslaw and Jaeger.[2] The transient portion of the melting phase is solved numerically. This transient solution is then matched to the steady state solution, which he obtains analytically, by means of an empirical equation.

Goodman[16] and Sutton[40] have studied the semi-infinite body with imperfect ablation in which part of the input energy is used to heat the liquid. Goodman and Shea[17] have applied the heat integral method to the problem for finite slabs, and Citron[3] has developed a method of successive approximations for the solution of the finite slab melting with ablation.

In all of the above methods some form of numerical computation is required. This ranges from the use of large-scale digital computers as required by the iterative schemes through the numerical solution of a Volterra equation of the second type required by Kolodner's solution to the use of a desk calculator in Citron's method.

The analog computer has also been used in recent investigations of this problem. Martini[29] proposed a scheme in which the stations in the finite difference approximation would be switched out of the circuit as the moving boundary passed them. Murray[32] also uses a similar technique in his second method while in his first method he uses a moving network of stations. Aldrich and Paynter[41] designed a special analog computer for the study of frost penetration in the soil. Otis[34] developed a passive network for the solution of the finite thickness slab with perfect ablation. In order to eliminate some of the nonlinear equipment in this computer, Otis employed a nonlinear time transformation which requires an infinite time to melt the slab completely. More recently, Sunderland[39] has used passive networks to solve the melting problem in both finite and semi-infinite slabs for the ablative and non-ablative cases. He attached the coordinate system to the moving boundary in the same manner as Landau.[27] In the past year, Murray[32,33] has used the electronic differential analyzer to solve the melting problem for the finite slab without ablation. His method employs a network of stations which move within the slab as melting progresses. The spaces between the stations change in size as melting progresses but remain equal to each other in each phase. Murray compares his results with solutions obtained by means of approximations suggested by Eyres, *et al.*[13] This article by Eyres, *et al*, is the earliest reference that this author has found to suggest the use of a space network in which the increment size varies with time after melting starts.

The solution of the melting problem presented in this dissertation requires the use of the differential analyzer to solve a set of ordinary, time-varying, nonlinear differential equations. The digital differential analyzer could also be used or the equations could be solved numerically. This method provides for the solution of the melting of the semi-infinite and finite slab problems with perfect ablation. The original contributions contained are believed to be:

1. a new set of non-dimensional parameters which separates the important variables of the problem;
2. the reduction of the semi-infinite interval to a finite interval in order to use the finite difference techniques and the differential analyzer;
3. the development of a method to make certain that the finite difference approximation solution approaches the steady state solution for the continuous case for a constant heating rate in the semi-infinite slab;
4. the use of non-uniform finite difference increments to improve the accuracy of the solution in the finite slab solution.

The solution presented here also provides for arbitrary heating rates (which may be functions of the solution as well as of time) and arbitrary initial temperature distributions. Although only the ablative case is discussed here, the solution may be extended to the non-ablative case. This would require only slight modifications but would approximately double the amount of computing equipment needed.

The semi-infinite medium is considered in detail first; this discussion is followed by a consideration of the finite thickness medium. Diffusion in one dimension is assumed in both cases.

CHAPTER II

THE SEMI-INFINITE SLAB

The Physical Problem

The physical problem is here described in terms of heat conduction in one dimension with melting at one boundary. The conducting medium consists of a slab which extends from negative infinity to positive infinity in two dimensions. In its third dimension, $\bar{x}$, it extends from a boundary which is a positive finite distance, $\bar{\ell}(\bar{t})$, from the origin to positive infinity. Heat enters the slab at $\bar{x} = \bar{\ell}(\bar{t})$. This configuration is shown schematically in Figure 1.

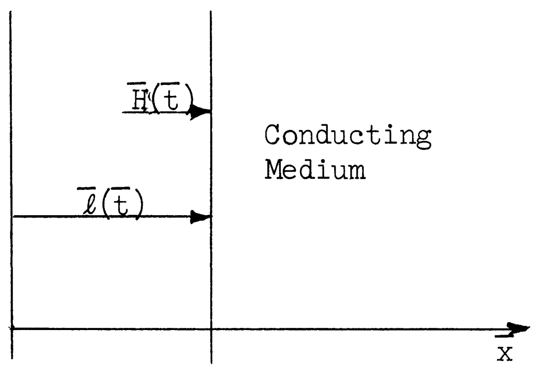


Figure 1. Physical Schematic, Semi-infinite Solid.

Let

$\bar{x}$ = the real space variable.

$\bar{t}$ = the real time variable.

$\bar{u}(\bar{x},\bar{t})$ = the temperature within the conducting medium.

$\bar{H}(\bar{t})$ = the heat flux entering the left boundary.

$\bar{\ell}(\bar{t})$ = the position of the left boundary.

Also let

$$\bar{\ell}(0) = 0$$

The problem starts when heat is first added with the finite boundary at the origin.

If heat is added, i.e. $\bar{H}(\bar{t}) > 0$, and if the medium is at a uniform temperature initially, then the temperature of the slab increases and $\bar{u}(\bar{x},\bar{t})$ becomes a monotonic, non-increasing function of x at any instant of time. The temperature range within the conducting medium is from the temperature on the boundary to that at infinity, which remains at the initial temperature because the adding of heat at the left boundary cannot affect the temperature at a point infinitely removed. This condition is nearly met by bodies of finite thickness in which the heating rate is high and the conductivity is low. Thus physical systems can approximate very closely the conditions set down here and the semi-infinite problem does have practical importance.

If the heat is added in sufficient quantity and at a sufficiently rapid rate, the heated boundary temperature will rise to the melting point and the boundary will move to the right. It is assumed that the liquid produced by the melting is removed immediately and none of the heat flux is used to raise the temperature of the fluid. If the heat flux is removed, melting will stop and the temperature throughout the slab will approach the temperature at infinity. If the heating rate is decreased sufficiently, melting will stop and the problem becomes the usual semi-infinite heating problem with a fixed boundary.

In order to prepare this problem for solution on the electronic differential analyzer the following steps are taken:

1. the mathematical model is established;
2. the problem is made non-dimensional;
3. the semi-infinite interval in space is reduced to a finite interval;
4. the moving boundary is removed;
5. the space derivatives are replaced by finite difference approximations; and
6. the problem is scaled for the computer.

The Mathematical Model

The problem is described by the diffusion Equation (1a) which applies throughout the conducting medium; the boundary condition Equations (1b), (1c), (1d), (1e), and (1f) which apply at the left boundary during the appropriate time intervals; the boundary condition at $\bar{x} = \infty$, Equation (1g); and the initial conditions, Equations (1h) and (1i).

$$
\begin{array}{llll}
\bar{\rho}\bar{c}\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left[\bar{k}\frac{\partial \bar{u}}{\partial \bar{x}}\right] + \bar{q}(\bar{x},\bar{t}) & \bar{\ell}(\bar{t}) \leq \bar{x} \leq \infty;\ \bar{t} \geq 0 & & \text{(a)} \\
\bar{H}(\bar{t}) = \bar{\rho} L \frac{d\bar{\ell}}{d\bar{t}} - \bar{k}\frac{\partial \bar{u}}{\partial \bar{x}} & \bar{x} = \bar{\ell}(\bar{t}) & \bar{t} \geq 0 & \text{(b)} \\
\frac{d\bar{\ell}}{d\bar{t}} = 0 & & 0 \leq \bar{t} \leq \bar{t}_1 & \text{(c)} \\
\bar{u} < \bar{u}_c & \bar{x} = 0 & 0 \leq \bar{t} < \bar{t}_1 & \text{(d)} \\
\frac{d\bar{\ell}}{d\bar{t}} > 0 & & \bar{t}_1 < \bar{t} < \bar{t}_2 & \text{(e)} \\
\bar{u} = \bar{u}_c & \bar{x} = \bar{\ell}(\bar{t}) & \bar{t}_1 \leq \bar{t} \leq \bar{t}_2 & \text{(f)} \\
\bar{q}_\infty = \lim_{\bar{x} \to \infty} \bar{u}(\bar{x},\bar{t}) & & \bar{t} \geq 0 & \text{(g)} \\
\bar{\ell} = 0 & & \bar{t} = 0 & \text{(h)} \\
\bar{u} = \bar{g}(\bar{x}) & 0 \leq \bar{x} \leq \infty & \bar{t} = 0 & \text{(i)}
\end{array}
\quad (1)
$$

In these equations:

$\bar{\rho} = \bar{\rho}(\bar{x},\bar{u})$ = the density of the conducting medium.

$\bar{c} = \bar{c}(\bar{x},\bar{u})$ = the specific heat of the solid.

$\bar{k} = \bar{k}(\bar{x},\bar{u})$ = the conductivity of the solid.

L = the latent heat of fusion of the solid.

$\bar{u}_c$ = the temperature of fusion of the conducting medium.

$\bar{t}_1$ = the time at which melting starts.

$\bar{t}_2$ = the time at which melting stops.

$\bar{g}_\infty$ = the temperature at $\bar{x} = \infty$

$\bar{q}(\bar{x},\bar{t})$ = a point source of heat within the conducting medium.

It is assumed in this thesis that there are no heat sources within the conducting medium. Therefore, $\bar{q}(\bar{x},\bar{t}) = 0$.

The time at which melting stops, $\bar{t}_2$, can be infinite for a heating rate which remains large enough to maintain melting once it has begun to melt. Figure 2 shows the $(\bar{x},\bar{t})$ domain of the problem for such a heating rate.

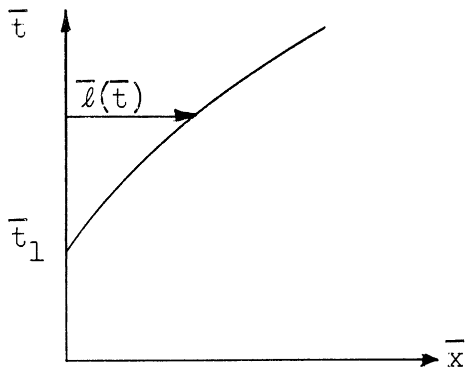


Figure 2. The $(\bar{x},\bar{t})$ Domain.

Equations (1) will now be made non-dimensional to reduce the number of problem parameters and to provide for a more orderly investigation of the problem.

The Equations in Non-dimensional Form

To make Equations (1) non-dimensional the following dimensionless variables are defined.

$$\bar{\rho} = \bar{\rho}_0 \bar{\phi}_\rho(\bar{x}, \bar{u}) = \bar{\rho}_0 \tilde{\phi}(\tilde{x}, \tilde{u}) \qquad \text{(a)}$$

$$\bar{c} = \bar{c}_0 \bar{\phi}_c(\bar{x}, \bar{u}) = \bar{c}_0 \tilde{\phi}(\tilde{x}, \tilde{u}) \qquad \text{(b)}$$

$$\bar{k} = \bar{k}_0 \bar{\phi}_k(\bar{x}, \bar{u}) = \bar{k}_0 \tilde{\phi}_k(\tilde{x}, \tilde{u}) \qquad \text{(c)}$$

$$\tilde{x} = K_1 \frac{\bar{\rho}_0 \bar{c}_0}{\bar{k}_0} \sqrt{L}\, \bar{x} \qquad \text{(d)} \qquad (2)$$

$$\tilde{t} = K_2 \frac{\bar{\rho}_0 \bar{c}_0}{\bar{k}_0} L\, \bar{t} \qquad \text{(e)}$$

$$\tilde{u}(\tilde{x}, \tilde{t}) = \frac{\bar{c}_0}{L}(\bar{u} - \bar{u}_c) \qquad \text{(f)}$$

$$\tilde{H}(\tilde{t}) = K_3 \frac{1}{\bar{\rho} L^{3/2}} \bar{H}(\bar{t}) \qquad \text{(g)}$$

In Equations (2a), (2b), and (2c) the quantities $\bar{\rho}_0$, $\bar{c}_0$, and $\bar{k}_0$ are dimensional constants equal to the maximum value of $\bar{\rho}(\bar{x},\bar{u})$, $\bar{c}(\bar{x},\bar{u})$, and $\bar{k}(\bar{x},\bar{u})$ respectively. The functions $\bar{\phi}_\rho$, $\bar{\phi}_c$, and $\bar{\phi}_k$ are dimensionless functions of temperature and position which are characteristic of the material of which the conducting medium is composed.

In Equations (2d), (2e), and (2f), K_1, K_2, and K_3 are convenient, constant, arbitrary scaling factors.

Equations (2d-2f) are derived in Appendix A. The numerical values of the constants in these equations are included for several common materials.

When Equations (2) are substituted into Equations (1), the non-dimensional Equations (3) are obtained.

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} = \frac{1}{\tilde{\phi}_\rho \tilde{\phi}_c} \frac{K_1^2}{K_2} \frac{\partial}{\partial \tilde{x}}\left[\tilde{\phi}_k \frac{\partial \tilde{u}}{\partial \tilde{x}}\right] \qquad 0 \leq \tilde{x} \leq \infty;\ \tilde{t} \geq 0 \qquad \text{(a)}$$

$$\tilde{H}(\tilde{t}) = \frac{K_2 K_3}{K_1} \tilde{\phi}_\rho \frac{d\tilde{\ell}}{d\tilde{t}} - K_1 K_3 \tilde{\phi}_k \frac{\partial \tilde{u}}{\partial \tilde{x}} \qquad \tilde{x} = \tilde{\ell}(\tilde{t});\ t \geq 0 \qquad \text{(b)}$$

$$\frac{d\tilde{\ell}}{d\tilde{t}} = 0 \qquad 0 \leq \tilde{t} \leq \tilde{t}_1 \qquad \text{(c)}$$

$$\tilde{u} < 0 \qquad \tilde{x} = 0 \quad 0 \leq \tilde{t} \leq \tilde{t}_1 \qquad \text{(d)} \qquad (3)$$

$$\frac{d\tilde{\ell}}{d\tilde{t}} > 0 \qquad \tilde{t}_1 < \tilde{t} < \tilde{t}_2 \qquad \text{(e)}$$

$$\tilde{u} = 0 \qquad \tilde{x} = \tilde{\ell}(\tilde{t}) \quad \tilde{t}_1 \leq \tilde{t} \leq \tilde{t}_2 \qquad \text{(f)}$$

$$\tilde{u} = \tilde{g}_\infty \qquad \tilde{x} = \infty \quad \tilde{t} \geq 0 \qquad \text{(g)}$$

$$\tilde{\ell} = 0 \qquad \tilde{t} = 0 \qquad \text{(h)}$$

$$\tilde{u} = \tilde{g}(\tilde{x}) \qquad 0 \leq \tilde{x} \leq \infty \quad \tilde{t} = 0 \qquad \text{(i)}$$

The $(\tilde{x}, \tilde{t})$ domain for Equations (3) is the same shape as the $(\bar{x}, \bar{t})$ domain of Figure 2 on page 9 .

Two special choices of the scaling factors K_1, K_2, K_3 are of interest:

1. When $\tilde{H}(\tilde{t})$ is not constant, it is desirable to eliminate these arbitrary constants from Equations (3).
2. When $\tilde{H}$ is a constant these factors can be chosen to eliminate $\tilde{H}$.

For the first case define the scaling constants as in Equations (4).

$$K_2 = K_1^2 \qquad \text{(a)}$$
$$K_3 = \frac{1}{K_1} \qquad \text{(b)} \qquad (4)$$

When Equations (4) are substituted into Equations (3) the results are Equations (5).

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{1}{\tilde{\phi}_\rho \tilde{\phi}_c} \frac{\partial}{\partial \tilde{x}}\left[\tilde{\phi}_k \frac{\partial \tilde{u}}{\partial \tilde{x}}\right] \qquad \tilde{\ell}(\tilde{t}) \leq \tilde{x} \leq \infty \; ; \; \tilde{t} \geq 0 \qquad \text{(a)}$$
$$\tilde{H}(\tilde{t}) = \tilde{\phi}_\rho \frac{d\tilde{\ell}}{d\tilde{t}} - \tilde{\phi}_k \frac{\partial \tilde{u}}{\partial \tilde{x}} \qquad \tilde{x} = \tilde{\ell}(\tilde{t}) \; ; \; \tilde{t} \geq 0 \qquad \text{(b)}$$
$$\frac{d\tilde{\ell}}{d\tilde{t}} = 0 \qquad 0 \leq \tilde{t} \leq \tilde{t}_1 \qquad \text{(c)}$$
$$\tilde{u} < 0 \qquad \tilde{x} = 0 \; ; \; 0 \leq \tilde{t} < \tilde{t}_1 \qquad \text{(d)}$$
$$\frac{d\tilde{\ell}}{d\tilde{t}} > 0 \qquad \tilde{t}_1 < \tilde{t} < \tilde{t}_2 \qquad \text{(e)} \qquad (5)$$
$$\tilde{u} = 0 \qquad \tilde{x} = \tilde{\ell}(\tilde{t}) ; \tilde{t}_1 \leq \tilde{t} \leq \tilde{t}_2 \qquad \text{(f)}$$
$$\tilde{u} = g_\infty \qquad \tilde{x} = \infty \; ; \; \tilde{t} \geq 0 \qquad \text{(g)}$$
$$\tilde{\ell} = 0 \qquad \tilde{t} = 0 \qquad \text{(h)}$$
$$\tilde{u} = \tilde{g}(\tilde{x}) \qquad 0 \leq \tilde{x} \leq \infty \; ; \; \tilde{t} = 0 \qquad \text{(i)}$$

In order to eliminate H when the heating rate is constant, the scaling factors are defined as in Equations (6).

$$K_1 = \tilde{H} \quad \text{(a)}$$
$$K_2 = \tilde{H}^2 \quad \text{(b)} \qquad (6)$$
$$K_3 = \frac{1}{1-\tilde{g}_\infty} \quad \text{(c)}$$

This transformation is well defined because $\tilde{g}_\infty < 0$.

When Equations (6) are substituted into Equations (3) Equations (7) result.

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{1}{\tilde{\phi}_k \tilde{\phi}_c} \frac{\partial}{\partial \tilde{x}}\left[\tilde{\phi}_k \frac{\partial \tilde{u}}{\partial \tilde{x}}\right] \qquad 0 \leq \tilde{x} \leq \infty \, ; \tilde{t} \geq 0 \quad \text{(a)}$$

$$1 - \tilde{g}_\infty = \tilde{\phi}_p \frac{d\tilde{\ell}}{d\tilde{t}} - \tilde{\phi}_k \frac{\partial \tilde{u}}{\partial \tilde{x}} \qquad \tilde{x} = \tilde{\ell}(\tilde{t}) \, ; \tilde{t} \geq 0 \quad \text{(b)}$$

$$\frac{d\tilde{\ell}}{d\tilde{t}} = 0 \qquad 0 \leq \tilde{t} \leq \tilde{t}_1 \quad \text{(c)}$$

$$\tilde{u} < 0 \qquad \tilde{x} = 0 \, ; \quad 0 \leq \tilde{t} < \tilde{t}_1 \quad \text{(d)}$$

$$\frac{d\tilde{\ell}}{d\tilde{t}} > 0 \qquad \tilde{t}_1 < \tilde{t} < \tilde{t}_2 \quad \text{(e)} \qquad (7)$$

$$\tilde{u} = 0 \qquad \tilde{x} = \tilde{\ell}(\tilde{t}) \quad \tilde{t}_1 \leq \tilde{t} \leq \tilde{t}_2 \quad \text{(f)}$$

$$\tilde{u} = \tilde{g}_\infty \qquad \tilde{x} = \infty \quad \tilde{t} \geq 0 \quad \text{(g)}$$

$$\tilde{\ell} = 0 \qquad \tilde{t} = 0 \quad \text{(h)}$$

$$\tilde{u} = \tilde{g}(\tilde{x}) \qquad 0 \leq \tilde{x} \leq \infty \, ; \; \tilde{t} = 0 \quad \text{(i)}$$

Equations (5) are the fundamental equations of the problem because, as can be seen by comparing Equations (5b) and (7b), the transformation defined in Equations (6) effectively requires a dimensionless heating rate of $(1-\tilde{g}_\infty)$. This dimensionless heating rate may be related to any real heating rate and temperature at $\tilde{x} = \infty$ by the proper choice of the constant $\tilde{H}$ in Equation (2g) and Equations (6). It will be shown below that this heating rate, $\tilde{H} = 1-\tilde{g}_\infty$, also will assure that the steady state melting rate obtained from the finite difference approximation approaches the steady state melting rate obtained from the solution of the partial differential equations when the density, specific heat, and conductivity are assumed to be constant.

In Equations (5) the problem is described in terms of the following variables:

1. Independent variables, $\tilde{x}$ and $\tilde{t}$;
2. Dependent variables, $\tilde{u}(\tilde{x},\tilde{t})$, $\tilde{\ell}(\tilde{t})$;
3. Problem parameters:
   a. Heating rate $\tilde{H}(\tilde{t})$.
   b. Initial temperature distribution $\tilde{g}(\tilde{x})$.
4. Characteristics of the conducting medium:
   a. Density function, $\tilde{\phi}_\rho(\tilde{x},\tilde{u})$,
   b. Specific heat function, $\tilde{\phi}_c(\tilde{x},\tilde{u})$,
   c. Conductivity function, $\tilde{\phi}_k(\tilde{x},\tilde{u})$.

This is the minimum number of variables to which the problem can be reduced by means of dimensional analysis according to Buckingham's π theorem.[21,9] However, when the heating rate is constant one of the parameters, the dimensionless heating rate, can be eliminated as has been done in Equations (7).

The problem can be further simplified by assuming that

1. the conducting medium is homogeneous, and

2. the density, specific heat, and conductivity are independent of the temperature.

These two assumptions make the functions $\tilde{\phi}_\rho$, $\tilde{\phi}_c$, and $\tilde{\phi}_k$ equal to unity. They also make the problem independent of the material of which the conducting medium is composed. When these assumptions are made the problem is described completely in terms of four variables and two parameters. These assumptions are made at this point and apply to the rest of this dissertation. However, the development of Equations (5) to the finite difference form is carried out in Appendix C for the non-homogeneous, temperature dependent, conducting medium.

When the heating rate is constant and the initial temperature distribution is uniform in $\tilde{x}$ the problem parameters are reduced to one. It is thus only necessary to obtain solutions for the range of initial temperatures which is limited (in the dimensional variable) in minimum value by absolute zero and in maximum value to the melting temperature of the material being considered. The maximum temperature in the non-dimensional form is zero while the minimum value will vary with different materials. For the materials listed in Appendix A the lowest value is -5.01 for stainless steel. Thus the range of initial temperatures over which solutions need be obtained for a complete solution of the problem is limited when the initial temperature distribution is uniform. Of course, if the initial temperature distribution is arbitrary, an infinite number of possibilities exist.

When Equations (5) are considered individually with $\tilde{\phi}_\rho = \tilde{\phi}_c = \tilde{\phi}_k = 1$ the problem seems to be linear. However, the problem is nonlinear as shown by Landau[27] because the position of one space boundary, $\tilde{\ell}(\tilde{t})$, depends upon the temperature distribution which in turn depends upon the position of the boundary, as shown below in the transformed Equations (10).

The space interval for Equation (5a) is semi-infinite in extent and the left boundary moves with time after melting starts. It is necessary to reduce the interval to a finite one and to remove the moving boundary before the finite difference approximations can be applied and the differential analyzer used. The next transformation is designed to accomplish these results.

<u>Reduction of the Problem to a Finite Space Interval and Removal of the Moving Boundary</u>

The transformation defined in Equations (8) will reduce the space interval and remove the moving boundary.

$$x = e^{[\tilde{\ell}(\tilde{t}) - \tilde{x}]} \qquad \tilde{\ell}(\tilde{t}) \leq \tilde{x} \leq \infty \qquad \text{(a)}$$
$$t = \tilde{t} \qquad \tilde{t} \geq 0 \qquad \text{(b)} \qquad (8)$$

Thus x is in the interval $1 \geq x \geq 0$.

Let

$$\ell(t) = \tilde{\ell}(\tilde{t}) \qquad \text{(a)}$$
$$H(t) = \tilde{H}(\tilde{t}) \qquad \text{(b)}$$
$$u(x, t) = \tilde{u}(\tilde{x}, t) \qquad \text{(c)} \qquad (9)$$
$$g(x) = \tilde{g}(\tilde{x}) \qquad \text{(d)}$$
$$g_0 = \tilde{g}_\infty \qquad \text{(e)}$$

When Equations (5) with $\tilde{\phi}_\rho = \tilde{\phi}_c = \tilde{\phi}_k = 1$ are transformed by Equations (8) and the relationships of Equations (9) are used, the results are Equations (10).

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = x\frac{\partial}{\partial x}\left[x\frac{\partial u}{\partial x}\right] - x\frac{d\ell}{dt}\frac{\partial u}{\partial x} && 0 \leq x \leq 1;\ t \geq 0 && \text{(a)}\\
&\frac{d\ell}{dt} = H(t) - \frac{\partial u}{\partial x} && x = 1;\quad t \geq 0 && \text{(b)}\\
&\frac{d\ell}{dt} = 0 && 0 \leq t \leq t_1 && \text{(c)}\\
&u < 0 && x = 1;\quad 0 \leq t < t_1 && \text{(d)}\qquad (10)\\
&\frac{d\ell}{dt} > 0 && t_1 < t < t_2 && \text{(e)}\\
&u = 0 && x = 1;\quad t_1 \leq t \leq t_2 && \text{(f)}\\
&u = g_0 && x = 0;\quad t \geq 0 && \text{(g)}\\
&\ell = 0 && t = 0 && \text{(h)}\\
&u = g(x) && 0 \leq x \leq 1;\ t = 0 && \text{(i)}
\end{aligned}
$$

The Jacobian of the transformation (8) vanishes at $\tilde{x} = \infty$. However, this singularity can be tolerated because only $\tilde{u}(\tilde{x},\tilde{t})$ need be defined at this point. The derivatives of $\tilde{u}(\tilde{x},\tilde{t})$ need be defined only arbitrarily close to $\tilde{x} = \infty$.

Figure 3 shows the (x,t) domain of the problem.

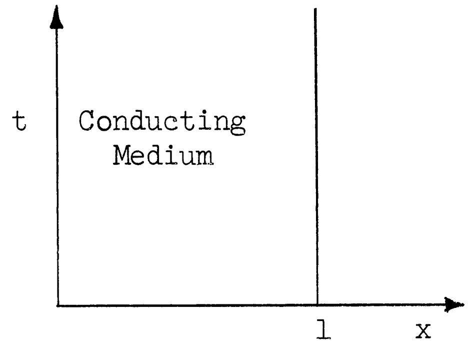


Figure 3. The (x,t) Domain.

Equations (10) describe the problem with boundaries at x = 0 and x = 1. The point $\tilde{x} = \infty$ has been transformed into x = 0 and $\tilde{x} = \tilde{\ell}(\tilde{t})$ into x = 1. With the boundaries fixed, the effect of the boundary motion is now reflected in the term in Equation (10a) which contains $\frac{d\ell}{dt}$. $\frac{d\ell}{dt}$ is a function of the heat flux at x = 1 as shown in Equation (10b). The (x,t) coordinate system is attached to the melting boundary and the heat transfer represented by the term, $- x \frac{d\ell}{dt} \frac{\partial u}{\partial x}$, can be interpreted as the heat transferred by convection due to the relative motion between the conducting medium and the coordinate system.

Equations (10) are now in a form suitable for the application of the finite difference approximations to eliminate the space variable derivatives. Before doing this, however, the steady state melting rate will be discussed.

<u>Steady State Boundary Velocity for a Constant Heating Rate</u>

Equations (10) can be solved, as is done in Appendix B, to obtain the steady state boundary velocity and the temperature distribution for the special case of a constant heating rate. The result of this solution is given in Equations (11).

$$\frac{d\ell}{dt} = \frac{H}{1 - g_o} \qquad \text{(a)}$$
$$u(x) = \left(1 - x^{\frac{d\ell}{dt}}\right) g_o \qquad \text{(b)} \qquad (11)$$

For convenience here the steady state boundary velocity is denoted by $b = \frac{d\ell}{dt}$.

Equations (11) may be transformed to the $(\tilde{x},\tilde{t})$ domain to obtain Equations (12).

$$b = \frac{H}{1 - g_o} \qquad \text{(a)}$$
$$\tilde{u}(\tilde{x}) = \left[1 - e^{b[\tilde{\ell}(\tilde{t}) - \tilde{x}]}\right] \tilde{g}_\infty \qquad \text{(b)} \qquad (12)$$

where $\tilde{g}_\infty = g_o$ as defined in Equation (9e).

The temperature distribution is obtained explicitly as a function of x in Equation (11b). However this does not yield the temperature distribution as a function of $\tilde{x}$, the untransformed distance variable. This is apparent if the boundary position is written as

$$\tilde{\ell}(\tilde{t}) = \tilde{\ell}(\tilde{t}_3) + b(\tilde{t} - \tilde{t}_3)$$

where $\tilde{t} > \tilde{t}_3$ and $\tilde{t}_3$ is the time after which the boundary position may be considered linear in time. Until $\tilde{\ell}(\tilde{t}_3)$ is computed by some other technique, $\tilde{\ell}(\tilde{t})$ cannot be obtained for $\tilde{t} > \tilde{t}_3$.

The steady state boundary velocity is unity when the heating rate is chosen to be $(1-g_o)$, as can be inferred from Equations (7b) and (12a). It will be shown below that the steady state solution for the finite difference approximation is also unity for this heating rate.

The Finite Difference Approximations

The electronic differential analyzer can integrate with respect to one variable only,[19] but Equations (10) contain derivatives with respect to both t and x. It is, therefore, necessary to eliminate either the derivatives with respect to t or to x. The elimination of the latter is more convenient because the resulting set of ordinary differential equations are of first order in t. It is required because it is impossible, in practice, to chose n initial conditions, $u(0,t_n)$, at each station in time such that the n final conditions, $u(1,t_n)$, are satisfied after integrating from $x = 0$ to $x = 1$.

The derivatives with respect to x are eliminated by using finite difference approximations. The approximation used will depend upon the location of the station at which the derivative is to be approximated. Three different types of stations can be identified, (1) the interior stations, (2) the left boundary station, and (3) the right boundary station. Equations (10) are written in such a manner that first order derivatives only need be approximated.

To effect the finite difference approximations let the slab be divided into equal space increments and let

r = the number of finite difference increments or cells in the interval $0 \leq x \leq 1$.

Then

$$\Delta x = \frac{1}{r} \qquad \text{(a)}$$

$$x_n = n\Delta x = \frac{n}{r} \qquad n = 0, 1, 2, \ldots, r \qquad \text{(b)} \qquad (13)$$

$$u_n(t) = u(x_n, t) \qquad n = 0, 1, 2, \ldots, r \qquad \text{(c)}$$

The cell size is constant and is uniform in x. However, in the $(\tilde{x}, \tilde{t})$ domain

$$\tilde{x}_n = \tilde{\ell}(\tilde{t}) - \ln\frac{n}{r} \qquad \text{(a)}$$

while in the $(\bar{x}, \bar{t})$ domain (14)

$$\bar{x}_n = \bar{\ell}(\bar{t}) - \frac{\bar{k}}{K_1 \bar{\rho}\bar{c}\sqrt{L}} \ln\frac{n}{r} \qquad \text{(b)}$$

Thus the station distribution is logarithmic in both the $\tilde{x}$ and $\bar{x}$ coordinates. The values of $\tilde{x}_n$ and $\bar{x}_n$ are functions of time when melting is in progress.

Figure 4 on page 21 shows the station position with respect to the moving boundary for the x coordinate.

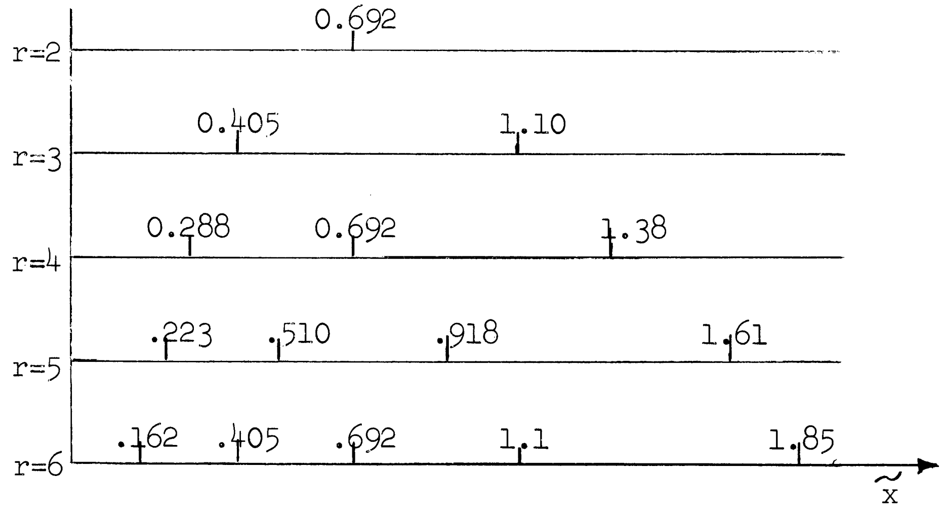


Figure 4. Values of $\tilde{x}_n$ During Warming for Several Values of r.

Station number 0 is defined to be the left boundary in x. This corresponds to the point at $\tilde{x}=+\infty$ in the $(\tilde{x},\tilde{t})$ domain as shown in Figure 4 for $\tilde{x}$. Stations 1 through (r-1) are the interior stations, and station r is at the right boundary in x (left boundary in $\tilde{x}$ as shown in Figure 4). The interior station finite difference approximations will be obtained first.

Interior Station Finite Difference Approximations

The central difference approximation is used to obtain the finite difference approximation of $\frac{\partial u}{\partial x}$ at the n^{th} station. This derivative is approximated by the relationship given in Equation (15).

$$\left.\frac{\partial u}{\partial x}\right|_{x_n} \approx \frac{u_{n+1}-u_{n-1}}{2\Delta x} = \frac{r}{2}\left(u_{n+1}-u_{n-1}\right) \tag{15}$$

Figure 5 shows schematically the quantities of Equation (15).

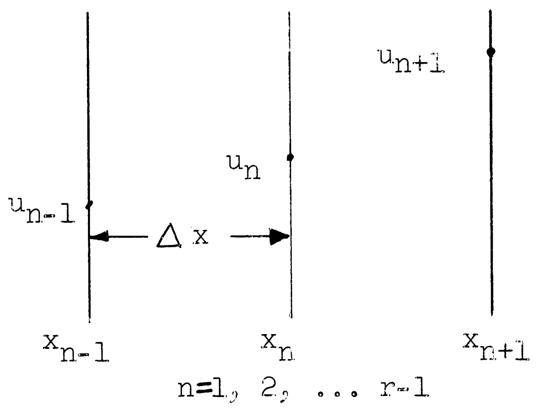


Figure 5. Finite Difference Stations in the Interior of the Slab.

It is necessary to use the double interval in Equation (13) because both the temperature and the first derivative with respect to x must be evaluated at each station. The error introduced by this approximation is of the order of $(\Delta x)^2$ as shown in Appendix C.

It is also necessary to approximate $\frac{\partial}{\partial x}[x \frac{\partial u}{\partial x}]_{x_n}$ and this is done again by using the central difference formula. However, a single interval, Δx, can be used for this approximation which is made in accordance with Equation (16).

$$\frac{\partial}{\partial x}\left[x \frac{\partial u}{\partial x}\right]_{x_n} \approx \frac{x_{n+\frac{1}{2}}\left(\frac{\partial u}{\partial x}\right)_{x_{n+\frac{1}{2}}} - x_{n-\frac{1}{2}}\left(\frac{\partial u}{\partial x}\right)_{x_{n-\frac{1}{2}}}}{\Delta x} \qquad (16)$$

However, since

$$\left(\frac{\partial u}{\partial x}\right)_{x_{n+\frac{1}{2}}} \approx \frac{u_{n+1} - u_n}{\Delta x} = r(u_{n+1} - u_n)$$

and

$$x_{n+\frac{1}{2}} = \left(n+\frac{1}{2}\right)\frac{1}{r}$$

Equation (16) reduces to Equation (17).

$$\frac{\partial}{\partial x}\left[x \frac{\partial u}{\partial x}\right]_{x_n} = \left(n+\frac{1}{2}\right) r\, u_{n+1} - 2nr\, u_n + \left(n-\frac{1}{2}\right) r\, u_{n-1} \qquad (17)$$

Left Boundary Approximation

At the left boundary, x_o, the temperature is fixed. Thus the left boundary equation is Equation (18).

$$u_0(t) = g_0 \tag{18}$$

Right Boundary Finite Difference Approximation

At the heated boundary, x_r, both the gradient and the temperature must be evaluated. The gradient is approximated by a backward difference. As shown in Appendix C, the backward difference approximation introduces an error of the order of Δx when first order differences are used. When second order differences are used with a single interval, the order of magnitude of the error in the first derivative becomes $(\Delta x)^2$. Both these approximations will be investigated to determine the effect on the time domain solutions.

The first backward difference approximation of the first derivative of the temperature at the heated boundary is given by Equation (19).

$$\left(\frac{\partial u}{\partial x}\right)_{x=1} \approx \frac{u_r - u_{r-1}}{\Delta x} \tag{19}$$

When second order differences are used the approximation is that of Equation (20).

$$\left(\frac{\partial u}{\partial x}\right)_{x=1} \approx \frac{1}{\Delta x}\left(\frac{3}{2}u_r - 2u_{r-1} + \frac{1}{2}u_{r-2}\right) \tag{20}$$

The approximations of Equations (19) and (20) are derived in Appendix C.

Determination of the Temperature at the Heated Boundary

The temperature at the heated boundary must be determined as a function of time during the period of warming before melting starts. This may be done by means of Equation (10a) or (10b).

When Equation (10a) is used, the heat capacity lumped at this boundary station is considered in the calculation of u_r, and r differential equations must be solved during the warming period. One difficulty with this method, which will subsequently be termed the implicit method, is that in general $\frac{d\ell}{dt}(t_1)$ will have a non-zero value at the instant melting starts. This introduces an error into the boundary position because $\frac{d\ell}{dt}(t_1)$ must be 0 at t_1 as shown by Citron[3] for the continuous medium.

The backward difference approximations used at the heated boundary for the implicit determination of u_r are:

(a) first order differences

$$x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)_{x=1} \approx 2rH - 2r(r-\tfrac{1}{2})u_r + 2r(r-\tfrac{1}{2})u_{r-1} \tag{21}$$

(b) second order differences

$$x\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)_{x=1} \approx 3rH - \frac{r}{2}(7r-3)u_r + 4r(r-\tfrac{1}{2})u_{r-1} - \frac{r}{2}(r-1)u_{r-2} \tag{22}$$

The derivations of these approximations and the assumptions made in these derivations are contained in Appendix C. Equations (21) and (22) were obtained by using the smallest increments in x possible at each step in the derivations. These formulae are not in as common use as Equations (19) and (20) and other formulae could be developed, but they would require using larger increments in x in some point of their derivations.

In an effort to eliminate the error in $\frac{d\ell}{dt}(t_1)$, the explicit method for determining u_r is introduced. In this case Equation (10b), which becomes

$$H = \frac{\partial u}{\partial x} \qquad x = 1;\ 0 \le t \le t_1$$

during the warming interval because $\frac{d\ell}{dt} = 0$, is used. This equation allows the determination of u_r in terms of the heating rate and the internal temperatures. This approximation forces $\frac{d\ell}{dt}(t_1)$ to be zero. However, it neglects the heat capacity lumped at the boundary station and, if $H(0) \neq 0$, $u_r(0)$ is in error. During the warming period (r-1) differential equations must be solved simultaneously.

When u_r is determined explicitly during warming, the boundary temperature during melting can be calculated and this value can be used to determine when melting stops if the heating rate falls below that required to sustain melting.

It is, of course, possible to calculate u_r implicitly and explicitly simultaneously. Then the implicit calculation can be used to determine the boundary temperature, u_r, when there is no melting; the explicit calculation can be used to determine when melting stops.

The approximations for the explicit determination of u_r are:

(a) first order differences

$$u_r \approx u_{r-1} + \frac{1}{r} H(t) \tag{23a}$$

(b) second order differences

$$u_r \approx \frac{4}{3} u_{r-1} - \frac{1}{3} u_{r-2} + \frac{2}{3r} H(t) \tag{23b}$$

These approximations follow from Equations (10b), (19), and (20) since $\frac{d\ell}{dt} = 0$ in the interval $0 \le t \le t_1$.

Determination of the Boundary Velocity and Position During Melting

During the melting phase the temperature, u_r, is identically zero and Equation (10b) is used to determing the boundary velocity and position. For this case, also, both first and second order backward difference approximations for $\frac{\partial u}{\partial x}$ are to be investigated. The approximations for $\frac{d\ell}{dt}$ are given in Equations (24a) and (24b).

First order differences:

$$\frac{d\ell}{dt} \approx H(t) + r\, u_{r-1} \tag{24a}$$

Second order differences:

$$\frac{d\ell}{dt} \approx H(t) - \frac{r}{2} u_{r-2} + 2r\, u_{r-1} \tag{24b}$$

These approximations follow directly from Equations (19) and (20) with $u_r = 0$ and Equation (10b).

The Finite Difference Equations

The space derivatives of Equations (10) are eliminated by means of the finite difference approximations by introducing Equations (15) through (24) into Equations (10). The results are the following set of ordinary differential equations with boundary and initial conditions given in Equations (25).

$$U_0(t) = g_0 \qquad t \geq 0 \qquad \text{(a)}$$

$$\frac{du_n}{dt} = \frac{n}{2}(2n-1)u_{n-1} + \frac{n}{2}\frac{d\ell}{dt}u_{n-1} - 2n^2 u_n - \frac{n}{2}\frac{d\ell}{dt}u_{n+1} + \frac{n}{2}(2n+1)u_{n+1} \qquad n = 1,2..(r-1),\ t \geq 0 \qquad \text{(b)}$$

$$\left.\begin{array}{ll} \dfrac{du_r}{dt} = 2rH + r(2r-1)u_{r-1} - r(2r-1)u_r & \text{(c1)} \\ \dfrac{du_r}{dt} = 3rH - \frac{r}{2}(r-1)u_{r-2} + 2r(2r-1)u_{r-1} - \frac{r}{2}(7r-3)u_r & \text{(c2)} \\ u_r = \frac{1}{r}H + u_{r-1} & \text{(c3)} \\ u_r = \frac{2}{3r}H + \frac{4}{3}u_{r-1} - \frac{1}{3}u_{r-2} & \text{(c4)} \end{array}\right\} 0 \leq t \leq t_1$$

$$\frac{d\ell}{dt} = 0 \qquad 0 \leq t \leq t_1 \qquad \text{(d)}$$

$$u_r < 0 \qquad 0 \leq t \leq t_1 \qquad \text{(e)} \qquad (25)$$

$$\frac{d\ell}{dt} = H + r u_{r-1} \qquad t_1 < t < t_2 \qquad \text{(f1)}$$

$$\frac{d\ell}{dt} = H - \frac{r}{2}u_{r-2} + 2r u_{r-1} \qquad t_1 < t < t_2 \qquad \text{(f2)}$$

$$\ell = 0 \qquad t = 0 \qquad \text{(g)}$$

$$u_n = g_n \qquad \begin{array}{l} n = 0,1,\ldots r \ \text{for (C1) and (C2)} \\ n = 0,1,\ldots(r-1) \ \text{for (C3) and (C4)} \end{array} \qquad t = 0 \qquad \text{(h)}$$

Equations (25c1) through (25c4) will be used in turn with Equations (25f1) and (25f2) in investigating the effects of first and second order difference approximations. Two combinations, however, are not worthwhile. These are (25c3) with (25f2) and (25c4) with (25f1). Equations (25c3) and (25c4) result from the explicit determination of u_r. One object of this approximation is to force $\frac{d\ell}{dt}(t_1)$ to be zero, and this can be done only if the $\left.\frac{\partial u}{\partial x}\right|_{x_r}$ is approximated by means of the same order differences during the warming and during the melting periods. Thus the combinations in which first order differences are used during warming and second order differences during melting should be eliminated. With these two cases eliminated, there are six cases of interest in studying the effects of the various approximations.

Equations (25) describe the problem in terms of r simultaneous, first order, ordinary differential equations. During warming $\frac{d\ell}{dt}$ is zero and the equations reduce to a set of linear, ordinary, differential equations which can be solved by conventional methods. During melting the equations are nonlinear because $\frac{d\ell}{dt}$ depends upon the temperatures at the interior stations and they in turn depend implicitly upon the boundary velocity. Equations (25) are the equations to be solved upon the differential analyzer during both warming and melting.

It is reiterated here that the stations at which the temperatures are determined move within the solid during melting and remain a fixed distance from the physical boundary in the $(\tilde{x},\tilde{t})$ and $(\bar{x},\bar{t})$ domains but are fixed in the (x,t) domain.

Equations (25) will be analyzed during the warming phase in the next section.

Analysis of the Equations During the Warming Phase

In order to determine the errors introduced by the finite difference approximations and to isolate these errors from those introduced by the computer, Equations (25) are solved analytically for the warming portion of the problem and this solution is compared with the solution obtained from the partial differential Equations (5).

The solution of Equations (25) is facilitated by writing these equations in matrix form and proceeding to the solution in this form. Thus these equations may be written as

$$\frac{d\vec{u}}{dt} = A\vec{u}(t) + \vec{f}(t) \qquad 0 \leq t \leq t_1 \tag{26}$$

where

$\vec{u}(t)$ is a column vector with r elements for the implicit determination of u_r and (r-1) elements for the explicit method;

A is the matrix of Equations (25b) and (25c) for the interval $0 \leq t \leq t_1$. It is an r x r matrix for the implicit method and (r-1)x(r-1) for the explicit method.

$\vec{f}(t)$ is a column vector with the same number of elements as $\vec{u}$. The first element is $1/2\ g_0$ and describes the boundary condition at x = 0. The last element is a constant multiple of H(t) and describes the heated boundary condition. All other elements are zero.

The solution for Equations (26) may be obtained by standard matrix methods for the solution of a set of ordinary, constant coefficient differential equations.[23] This solution is given in Equation (27).

$$\vec{U}(t) = Be^{[\lambda]t}B^{-1}\vec{g} + Be^{[\lambda]t}[\lambda]^{-1}B^{-1}\vec{f}(0) - B[\lambda]^{-1}B^{-1}\vec{f}(t) + Be^{[\lambda]t}\int [\lambda]^{-1}e^{-[\lambda]t}B^{-1}\frac{d\vec{f}}{dt}\,dt \tag{27}$$

where

$[\lambda]$ = the non-singular diagonal matrix of the eigenvalues of A.

$e^{[\lambda]t}$ = the diagonal matrix with elements $e^{\lambda_i t}$

B = a matrix whose columns are eigenvectors of A.

$\vec{g}$ = a column vector whose elements are $g_1, g_2, g_3, \ldots g_r$ for the implicit case and $g_1, g_2, \ldots g_{r-1}$ for the explicit case.

If it is assumed that the heating rate is constant, the initial temperature is zero, and all eigenvalues of A are real, distinct, and non-zero then Equation (27) may be reduced to Equation (28).

$$\vec{U}_r(t) = Be^{[\lambda]t}B^{-1}[\vec{g} + A^{-1}f(0)] - A^{-1}\vec{f}(t) \tag{28}$$

The first term of the right member of Equation (28) is the transient solution and is composed of the sum of exponentials. The eigenvalues of A calculated for r = 2, 3, ... 7 are all negative and are listed in Appendix D. It can be shown that A is negative definite. This can be done by showing that all even order principal minors are positive and all odd order principal minors are negative. Thus the transient solution must approach zero and the complete solution must approach the steady state solution represented by the second term of the right member. The heating rate is constant for this case; therefore $\vec{f}(t)$ is a constant vector and the steady state solution must be a constant.

The solution of Equation (28) can be compared with the solution of the continuous Equations (10) which, for constant heating rate and uniform initial temperature distribution, is given by Carslaw and Jaeger[2] as

$$\tilde{u}(\tilde{x},\tilde{t}) = 2H\left[\sqrt{\frac{\tilde{t}}{\pi}}\, e^{-\frac{\tilde{x}^2}{4\tilde{t}}} - \frac{\tilde{x}}{2}\,\mathrm{erfc}\,\frac{\tilde{x}}{2\sqrt{\tilde{t}}}\right] + \tilde{g}_\infty \tag{29}$$

where

$$\mathrm{erfc}\ (y) = 1 - \mathrm{erf}\ (y)$$

and

$$\mathrm{erf}\ (y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-\xi}\, d\xi$$

This equation is given in the $(\tilde{x},\tilde{t})$ domain so that the heated face is at $\tilde{x} = 0$. Thus $\tilde{x} = 0$ corresponds to $x = 1$ or $x = x_r$. The temperature $\tilde{u}(0,\tilde{t})$ must be compared with $u_r(t)$. The temperatures at each of the stations are calculated from Equations (28) and (29) and are compared graphically in Figures 6 for r = 3 for the several approximations. In these figures the temperature is normalized with respect to the constant heating rate and is plotted as

$$\frac{\tilde{u}_n - \tilde{g}_\infty}{H} = \frac{u_n - g_\infty}{H}.$$

Equations (28) were evaluated manually be obtaining (1) the inverse of the matrix A; (2) the eigen value matrix, $[\lambda]$; (3) the eigen-vector matrix B and (4) its inverse, B^{-1}. The solutions, u_n, were written as explicit functions of time. These functions of time were then evaluated for specific values of time with the aid of tables of exponentials and a desk calculator.

For Equation (29) the temperature at the heated boundary is

$$\tilde{u}(0,\tilde{t}) = \frac{2H}{\sqrt{\pi}}\sqrt{\tilde{t}} + \tilde{g}_\infty \tag{30}$$

This temperature varies as $\sqrt{t}$. However, it was shown above that for the finite difference approximations the solution at the heated boundary approaches a constant temperature which is

$$\vec{u}_{ss} = -A^{-1}\vec{f}.$$

Thus the error, $|\tilde{u}(0,\tilde{t}) - u_r|$, eventually would become unbounded if melting did not occur to stop the rise in surface temperature at $t = t_1$.

The time at which melting starts, t_1, is attained when the boundary temperature u_r reaches zero. This occurs in Figures 6 when $\frac{u_r - g_o}{H} = -\frac{g_o}{H}$. Thus t_1 may be determined from these figures by finding the intersection of the line $-\frac{g_o}{H}$ with the u_r curve. When $H = 1-g_o$, the line $-\frac{g_o}{H}$, becomes $\frac{g_o}{g_o - 1}$. The minimum value of this ratio is zero when $g_o = 0$ while its maximum value is unity because

$$\lim_{g_o \to \infty} \frac{g_o}{g_o - 1} = 1.$$

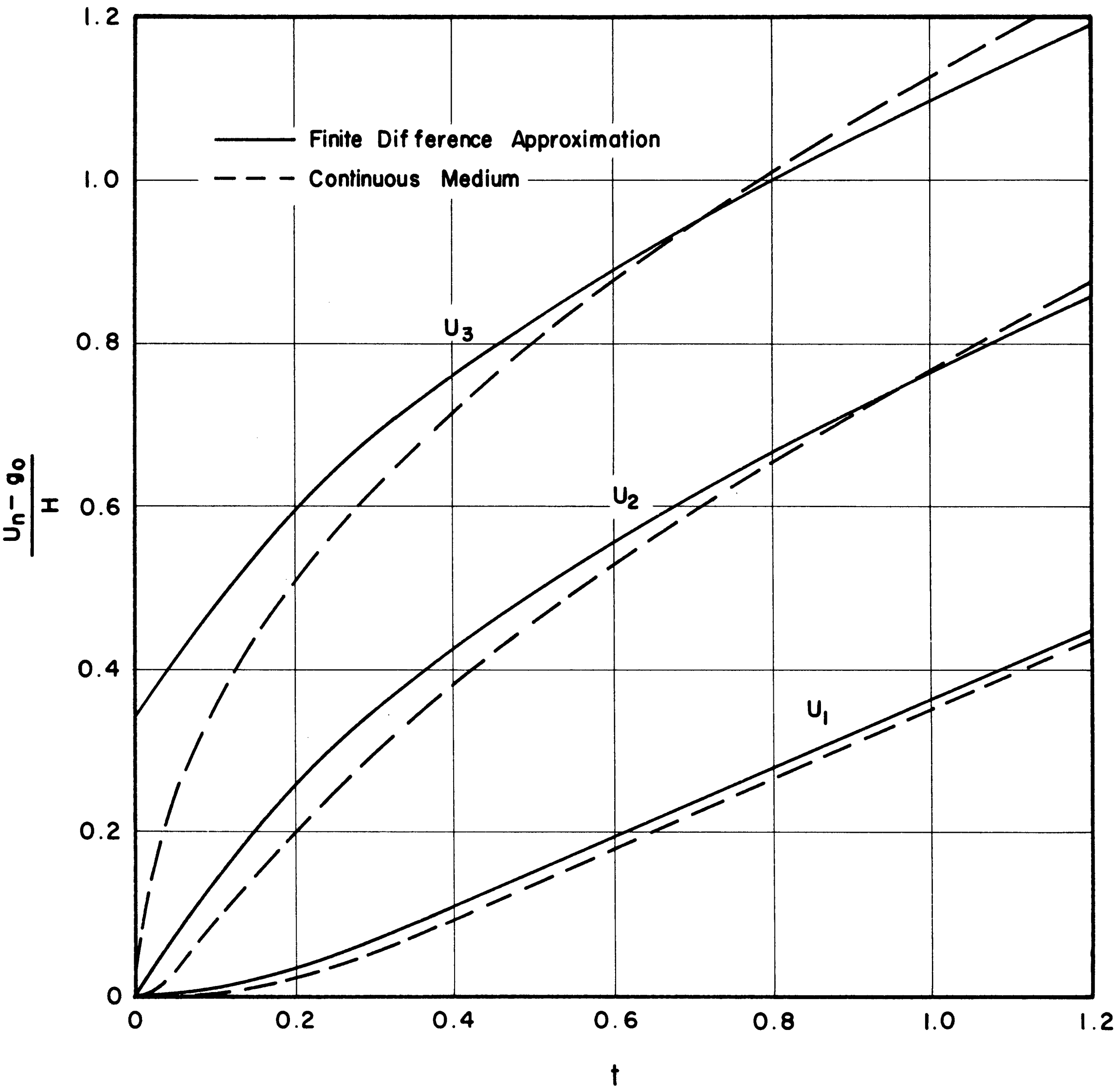


Figure 6a. Temperatures During the Warming Phase vs. Time
u_r Determination Explicit, 1st Differences.

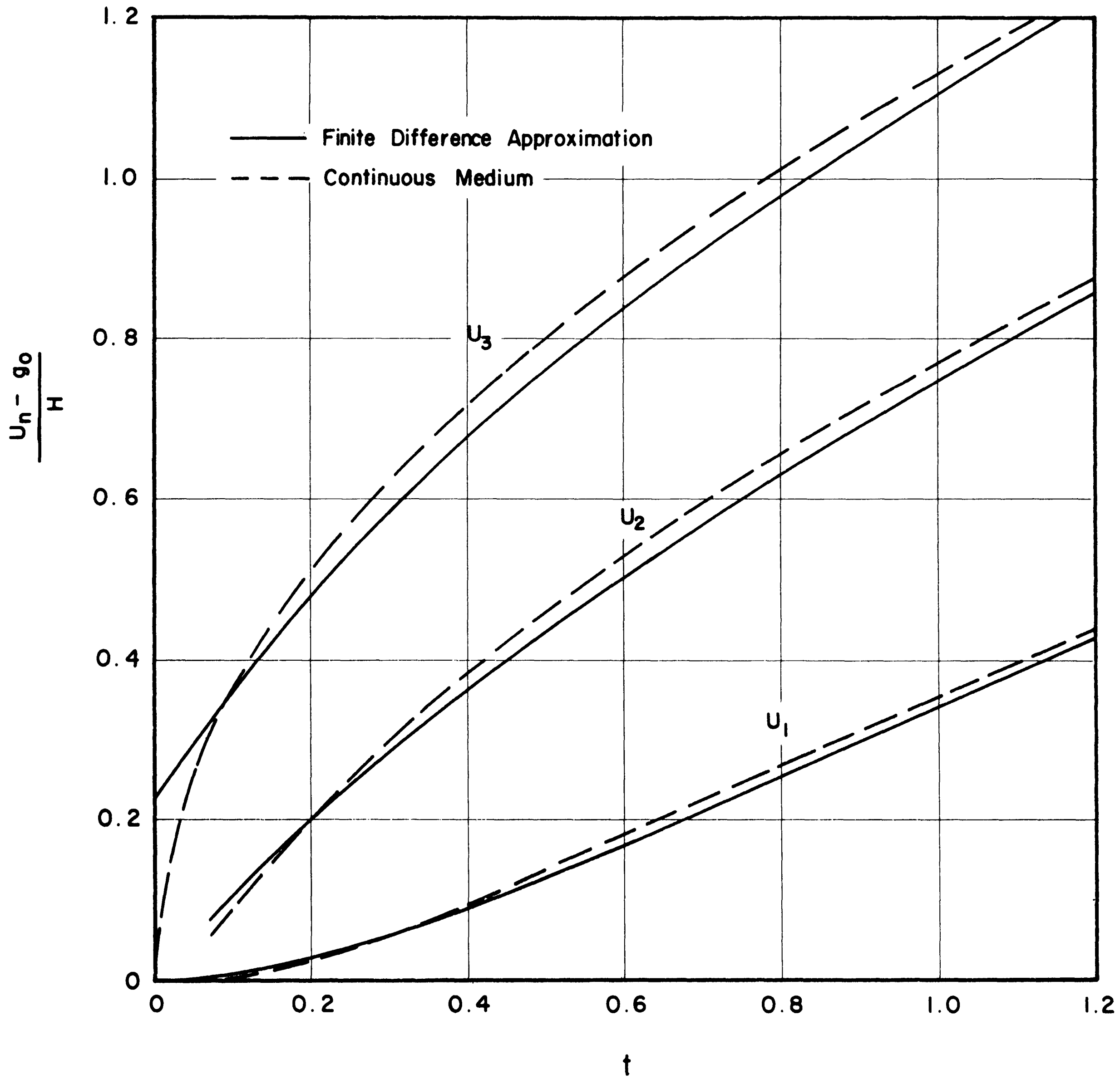


Figure 6b. Temperatures During the Warming Phase vs. Time u_r Determination Explicit, 2nd Differences.

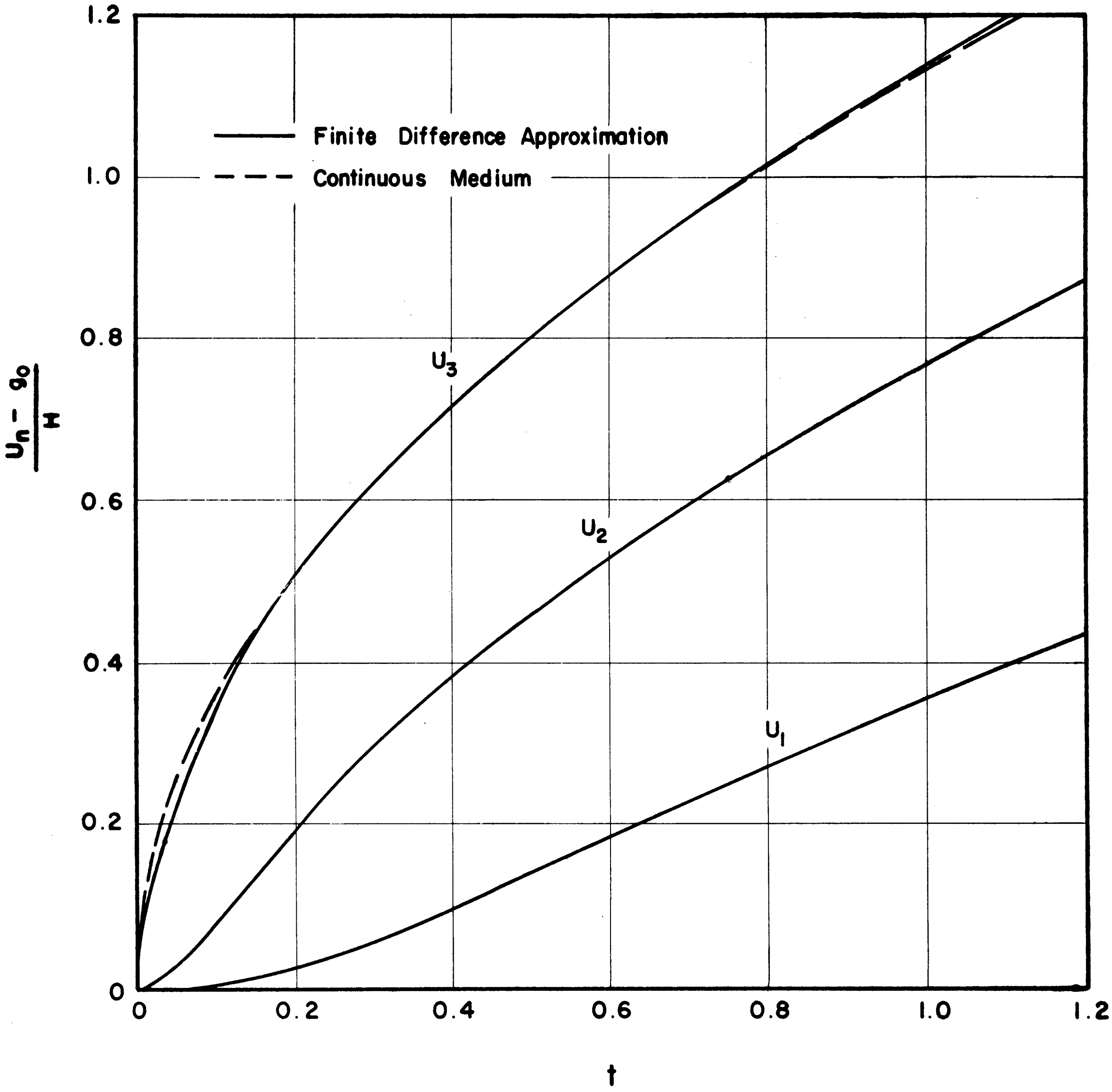


Figure 6c. Temperatures During the Warming Phase vs. Time
u_r Determination Implicit, 1st Differences.

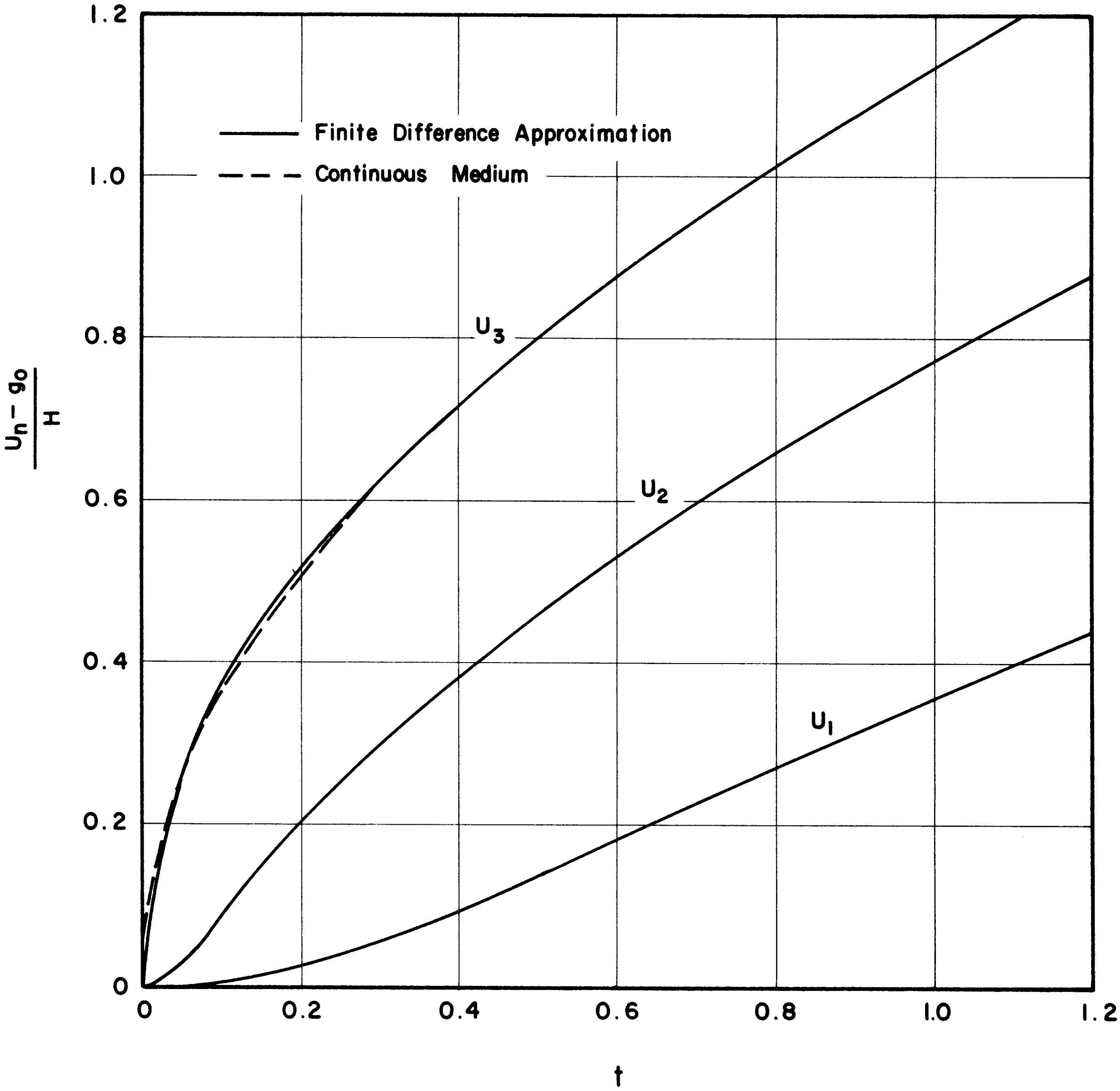


Figure 6d. Temperatures During the Warming Phase vs. Time
u_r Determination Implicit, 2nd Differences.

If the temperature ratio, $\frac{g_o}{g_o-1}$, is in the interval, $0 \leq \frac{g_o}{g_o-1} \leq 1$, the largest error introduced by the finite difference approximations occurs in the temperature, u_r, at the boundary, for each approximation. These results have been obtained with $r = 3$. It is concluded from the small errors that the logarithmic grouping of the finite difference stations in the $(\tilde{x},\tilde{t})$ domain allows a very accurate approximation of the gradient of $\tilde{u}$ by means of the finite difference approximations. It is expected that an increase in the number of finite difference increments will improve the accuracy. This effect will be checked by means of the computer solutions for $r = 6$. First the computer solutions for $r = 3$ will be checked against these analytical results for the finite difference approximations.

The choice of $H = 1-g_o$ as the heating rate allows the solution during warming to be confined to the interval $0 \leq \frac{u_r-g_o}{H} \leq 1$. The errors introduced by the finite difference approximation have been shown to be small in this interval after the first 0.1 units of time, t, even when $r = 3$ when u_r is determined explicitly. When u_r is determined implicitly, the errors are small over the entire interval. Thus this is a good choice of heating rate for the warming phase. It will now be shown that this is a particularly good choice for the steady state melting phase.

Analysis of the Melting Phase

The transient melting phase cannot be solved analytically in either the continuous representation or the finite difference representation. Equations (25) can be solved analytically, however, for the steady state melting rate when the dimensionless heating rate, H, is constant. This solution can be compared with the steady state solution obtained for the continuous case in Equations (11).

Let H(t) be a constant. Then, in steady state

$$\frac{du_n}{dt} = 0 \qquad n = 1, 2, \ldots r$$
$$\frac{d\ell}{dt} = b, \text{ a constant,} \tag{31}$$

and Equations (25) can be written as

$$[1+\tfrac{1}{2}(b-1)]g_0 - 2u_1 + [1+\tfrac{1}{2}(b-1)]u_2 = 0$$
$$[1+\tfrac{1}{2n}(b-1)]u_{n-1} - 2u_n + [1+\tfrac{1}{2n}(b-1)]u_{n+1} = 0 \qquad n = 2, 3, \ldots (r-1)$$
$$b = H + r\,u_{r-1} \qquad (1^{st} \text{ Differences})$$
$$b = H - \tfrac{r}{2}u_{r-2} + 2r\,u_{r-1} \; (2^{nd} \text{ Differences})$$

If b = 1, then the above equations become

$$g_0 - 2u_1 + u_2 = 0$$
$$u_{n-1} - 2u_n + u_{n+1} = 0$$
$$H = 1 - r\,u_{r-1} \qquad (1^{st} \text{ Differences})$$
$$H = 1 + \tfrac{r}{2}u_{r-2} - 2r\,u_{r-1} \qquad (2^{nd} \text{ Differences})$$

Thus

$$u_1 = \tfrac{1}{2}(g_0 + u_2) \qquad \text{(a)}$$
$$u_n = \tfrac{1}{2}(u_{n-1} + u_{n-1}) \qquad \text{(b)} \qquad (32)$$

Thus each temperature is the arithmetic mean of the adjacent temperatures and the temperature distribution is linear in x.

Since $u_r = 0$ the temperature increment between stations can be expressed as

$$u_r - u_{r-1} = u_{r-1}$$

There are r increments so

$$g_0 = r\,u_{r-1} \tag{33}$$

Therefore for first order differences in the $\frac{d\ell}{dt}$ equation

$$H = 1 - g_0$$

The relationship for the case when second order differences are used in the $\frac{d\ell}{dt}$ equation is obtained by using the equation

$$2\,r\,u_{r-1} = 2\,g_0$$

But

$$r\,u_{r-2} = 2\,r\,u_{r-1} = 2\,g_0$$

because

$$u_{r-2} - 2\,u_{r-1} + u_r = 0.$$

Therefore

$$\frac{r}{2}\,u_{r-2} - 2\,u_{r-1} = g_0 - 2g_0$$

and the heating rate of

$$H = 1 - g_0$$

results in a steady state boundary velocity of unity when second order differences are used, also.

From Equations (11), if

$$H = 1 - g_0,$$

then

$$b = 1.$$

Thus when the heating rate is the constant, $H = 1-g_0$, the steady state solution for the finite difference approximation is identical to the steady state solution for the continuous case in the boundary velocity. It is emphasized here that this does not mean that the boundary position is exact for the approximation because the boundary position depends upon the value of the boundary velocity during the transient phase as well as during the steady state portion. Since the transient phase solution for the boundary velocity is not exact, the boundary position will also be in error.

The above solution for the steady state boundary velocity is not affected by the approximations used during the warming phase at the heated boundary nor by the initial temperature distribution.

The problem is now ready to be solved on the differential analyzer. The computer scaling and circuits are the subjects of the next section.

Differential Analyzer Scaling and Circuits

Equations (23) are scaled for the computer by means of the transformation defined in Equations (34).

$$U_n(\tau) = \frac{1}{a_1} u_n(t) \qquad G_0 = \frac{1}{a_1} g_0 \qquad \text{(a)}$$

$$\lambda(\tau) = \frac{1}{a_2} \ell(t) \qquad \text{(b)} \qquad (34)$$

$$\mathcal{H}(\tau) = \frac{1}{a_3} H(t) \qquad \text{(c)}$$

$$\tau = a_4 t \qquad \text{(d)}$$

In this scaling the unit for the dependent variables in the machine unit and 1 mu = 100 volts.[1]

The transformation of Equations (25) by means of Equations (34) yields Equations (35).

$$U(\tau) = G_0 \qquad \tau \geq 0 \qquad \text{(a)}$$

$$\frac{dU_n}{d\tau} = \frac{n}{2a_4}(2n-1)U_{n-1} + \frac{n}{2}a_2 U_{n-1}\frac{d\lambda}{d\tau} - \frac{2n^2}{a_4}U_1 + \frac{n}{2}a_2 U_{n+1}\frac{d\lambda}{d\tau} + \frac{n}{2a_4}(2n+1)U_{n+1} \qquad n = 1, 2, \ldots r-1;\ \tau \geq 0 \qquad \text{(b)}$$

$$\left.\begin{aligned} &\frac{dU_r}{d\tau} = 2r\frac{a_3}{a_1 a_4}\mathcal{H} + \text{[illegible]}\,U_{r-1} - \text{[illegible]}\,U_r && \text{(c1)}\\ &\frac{dU_r}{d\tau} = 3r\frac{a_3}{a_1 a_4}\mathcal{H} \text{[illegible]} + \frac{2}{a_4}\text{[illegible]} - \frac{r}{2a_4}(2r\text{[illegible]})U_r && \text{(c2)}\\ &U_r = \frac{a_3}{r a_1}\mathcal{H} + U_{r-1} && \text{(c3)}\\ &U_r = \frac{2a_3}{3 r a_1}\mathcal{H} + \frac{4}{3}U_{r-1} - \frac{1}{3}U_{r-2} && \text{(c4)} \end{aligned}\right\} 0 \leq \tau \leq \tau_1 \qquad (35)$$

$$\frac{d\lambda}{d\tau} = 0 \qquad 0 \leq \tau \leq \tau_1 \qquad \text{(d)}$$

$$U_r = 0 \qquad \tau_1 \leq \tau \leq \tau_2 \qquad \text{(e)}$$

$$\left.\begin{aligned} &\frac{d\lambda}{d\tau} = \frac{a_3}{a_2 a_4}\mathcal{H} + \frac{r a_1}{a_2 a_4}U_{r-1} && \text{(f1)}\\ &\frac{d\lambda}{d\tau} = \frac{a_3}{a_2 a_4}\mathcal{H} - \frac{r a_1}{2a_2 a_4}U_{r-2} + \frac{2 r a_1}{a_2 a_4}U_{r-1} && \text{(f2)} \end{aligned}\right\} \tau_1 < \tau < \tau_2$$

$$\lambda = 0 \qquad \tau = 0 \qquad \text{(g)}$$

$$U_n = G_n \quad \begin{matrix} n = 0, 1, \ldots r \text{ for (c1) and (c2)} \\ n = 0, 1, \ldots (r-1) \text{ for (c3) and (c4)} \end{matrix} \qquad \tau = 0 \qquad \text{(h)}$$

---

[1] A familiarity with the electronic differential analyzer is assumed. For a general treatment of this computer the books by Johnson[22] and Korn and Korn[25] are recommended. The paper by Howe and Haneman[20] is an excellent reference on the use of finite difference techniques in solving partial differential equations with the differential analyzer. The report by Howe[19] contains an exhaustive study of the solution of the heat equation with fixed boundaries on the electronic differential analyzer.

The scaling constants, a_1, a_2, a_3, and a_4, are chosen to assure as large a variation in voltage at each amplifier output as possible without saturating any of the amplifiers during the solution. This procedure is simplified for Equations (35) by the fact that the maximum absolute value of the temperatures occurs at $t = 0$.

The computer circuits for solving Equations (35) are presented in Figures 7 through 9. The heating rate, $\mathcal{H}(\tau)$, shown in Figures 7 and 8 as the input heating rate may be any arbitrary function of time or temperatures or combination of these. This, of course, is one of the prime advantages of the solution by means of the differential analyzer because these functions can be generated and used with little difficulty.

The circuits of Figures 7, 8, and 9 were used to solve Equations (35) for $r = 3$ and $r = 6$. A Reeves Model 101 computer with 0.1% components was used to solve these equations.

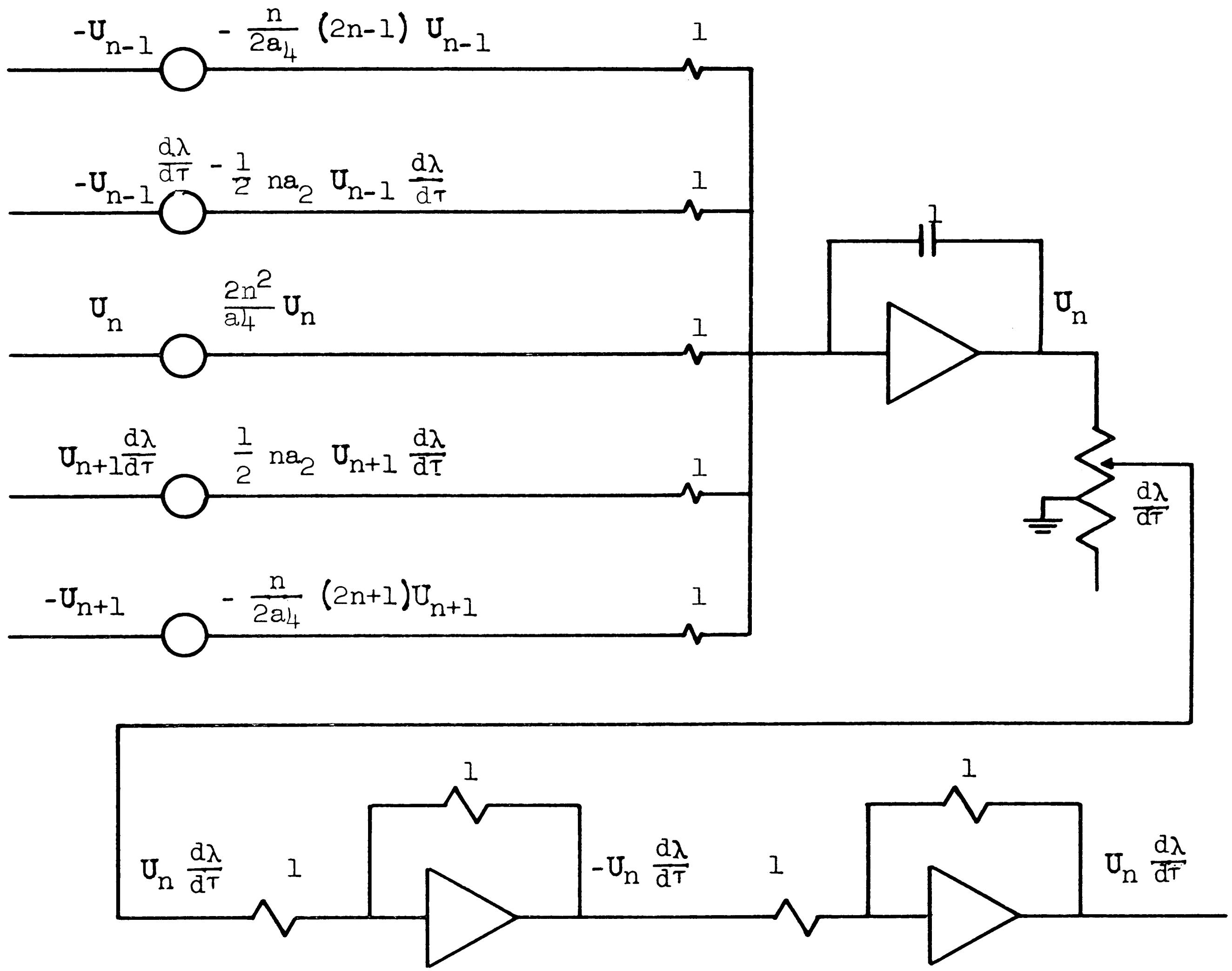
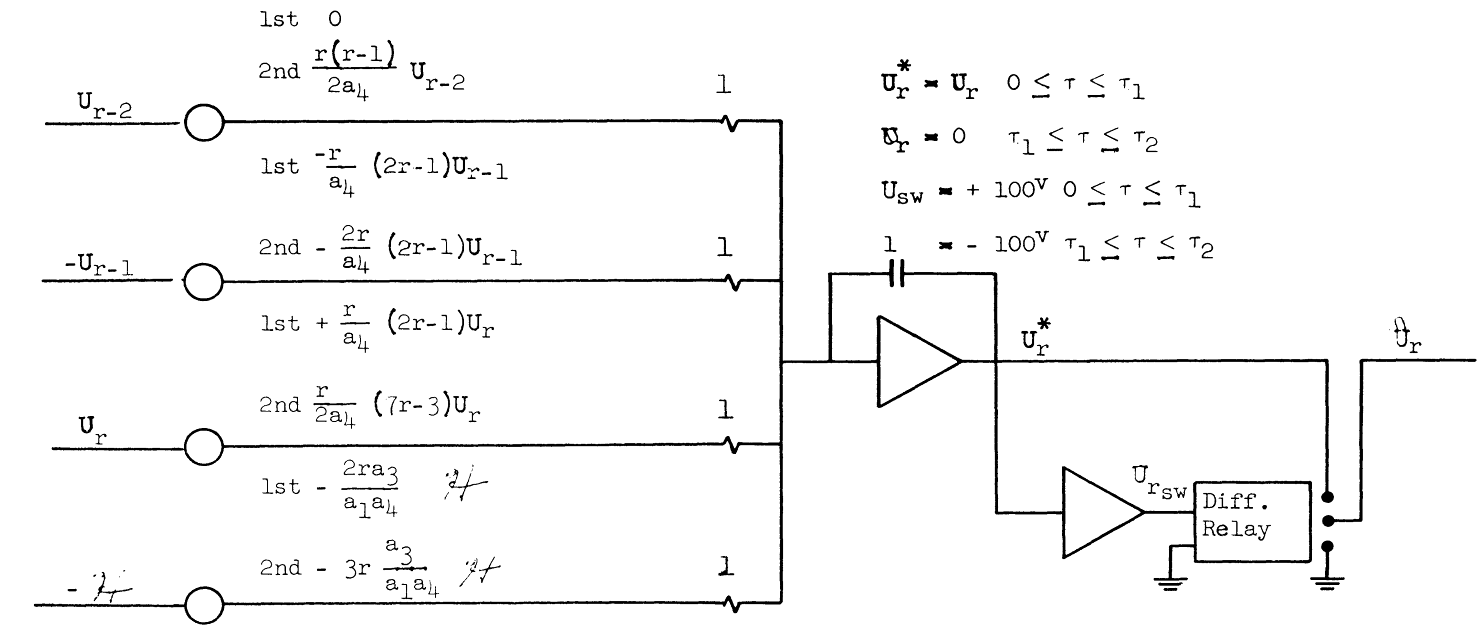
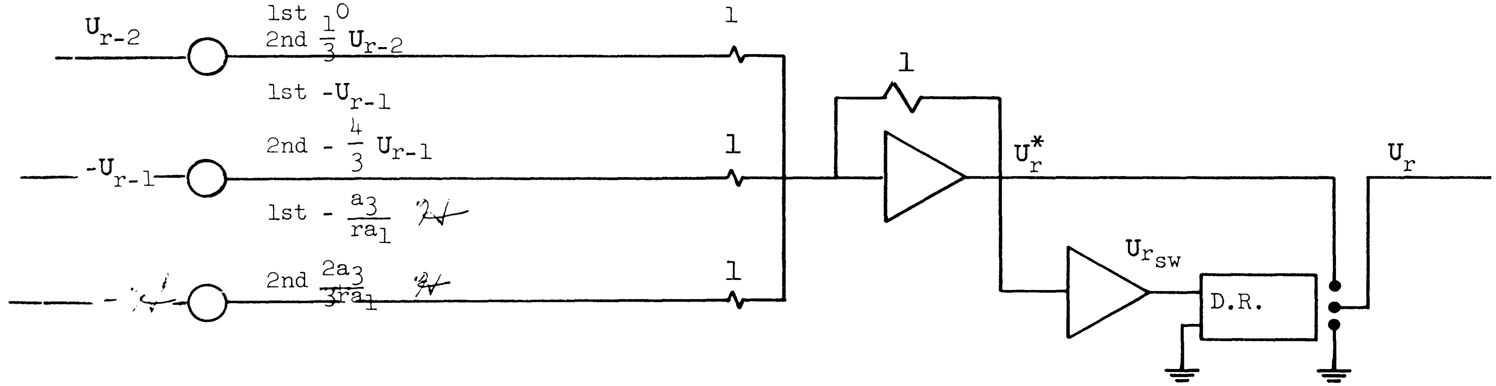


Figure 7. Computer Circuit for Determining the Temperature at the n^{th} Station (Equation 34b).



(a) Implicit Determination of U_r (Equations 34c1 and c2)



(b) Explicit Determination of U_r (Equations 34c3 and c4)

Figure 8. Computer Circuits for the Determination of U_r.

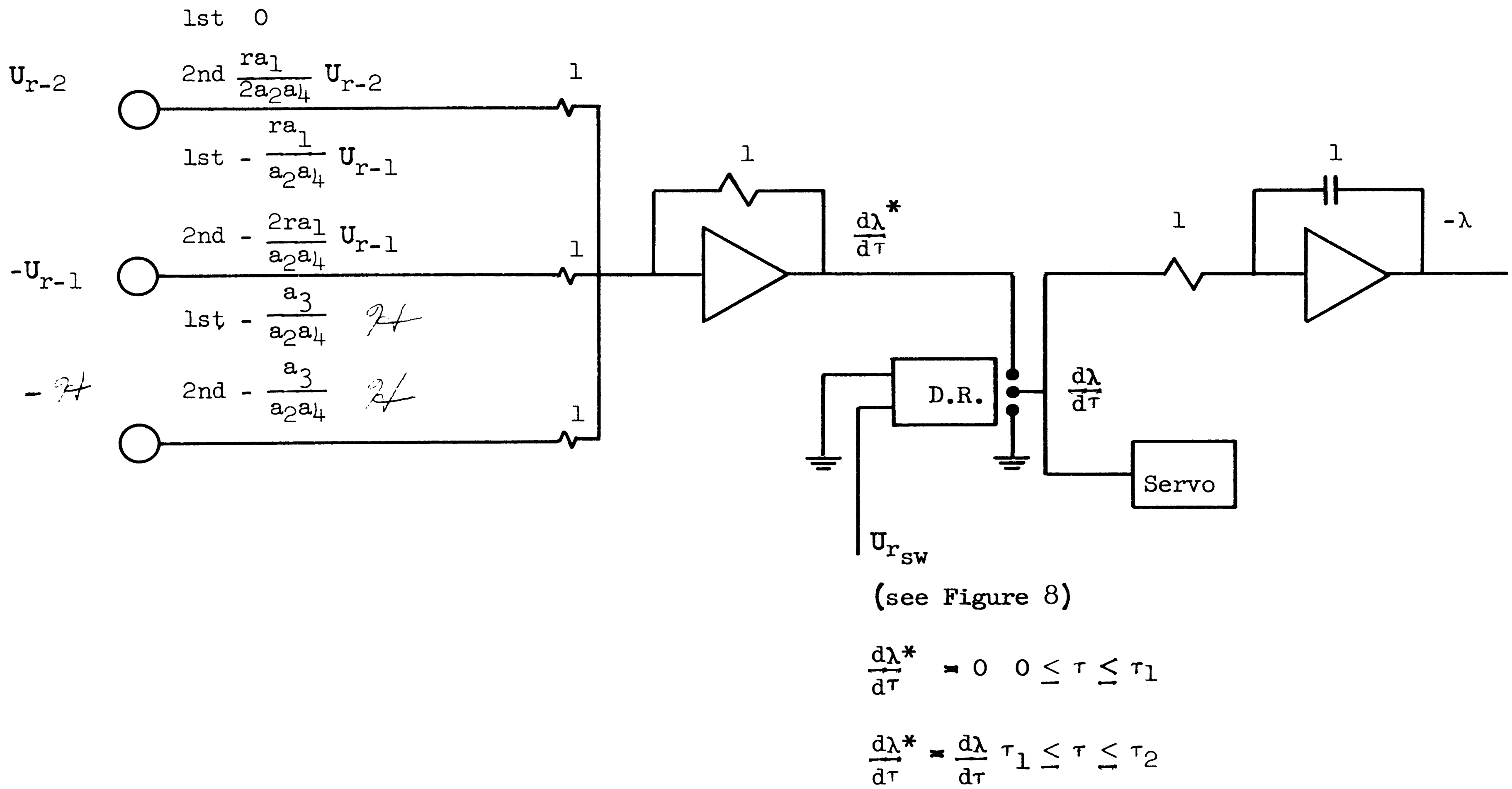


Figure 9. Computer Circuit for the Determination of the Boundary Velocity and Position. (Equations 34f1 and f2)

CHAPTER III

COMPUTER SOLUTION FOR THE SEMI-INFINITE SLAB

Equations (34) were solved on the electronic differential analyzer in order to:

1. Determine the relative accuracy of the finite difference approximations;
2. Determine the effect of increasing the number of finite difference increments;
3. Obtain a set of curves with initial temperature as a parameter which could be used to solve a wide range of physical problems.

To accomplish the first and second goals above, the problem was solved for $r = 3$ and $r = 6$ with $g_o = g_1 = g_2 = g_3 = \dots = g_{r-1} = g_r = -1.128$ (where g_i = the initial temperature at the i^{th} station) and constant heating rates of $H = 1$ and $H = 4$. This value of initial temperature was chosen in order to compare the results obtained in this dissertation with Lundau's solution.(27) First and second difference approximations were used at the heated boundary, and u_r was determined both explicitly and implicitly. The results of these solutions are presented in Tables I, II, III, IV, and V.

Table I contains values of t_1 (time elapsed when melting starts) and b (steady state boundary velocity for constant H) for the continuous

TABLE I

t_1 AND b FOR THE SEVERAL APPROXIMATIONS AND THE CONTINUOUS MEDIUM
$g_o = -1.128$

Approximations				t_1(time melting starts)				b(steady state boundary speed)				
H	Case	u_r Determination	$\frac{d\ell}{dt}$ Determination	Cont.	Diff. Eqns.	Comp.		Cont.	Diff. Eqns.		Comp.	
					r=3	r=3	r=6		r=3	r=6	r=3	r=6
1	A	Explicit, 1st Differences	1st Differences	1.00	1.07	1.076	.99	0.47	.343	.399	.34	.39
1	B	Explicit, 2nd Differences	2nd Differences		1.04	1.032	1.00		.416	.420	.42	.42
1	C	Implicit, 1st Differences	1st Differences		.99	.99	.99		.343	.399	.34	.40
1	D	Implicit, 1st Differences	2nd Differences		.99	.99	1.00		.416	.420	.41	.42
1	E	Implicit, 2nd Differences	1st Differences		1.00	1.00	1.00		.343	.399	.34	.40
1	F	Implicit, 2nd Differences	2nd Differences		1.00	1.00	1.00		.416	.420	.41	.42
4	A	Explicit, 1st Differences	1st Differences	0.0625	0.00	0.00	.034	1.88	2.06	1.96	2.06	1.96
4	B	Explicit, 2nd Differences	2nd Differences		0.042	.04	.061		1.88	1.88	1.88	1.88
4	C	Implicit, 1st Differences	1st Differences		.073	.07	.065		2.06	1.96	2.06	1.96
4	D	Implicit, 1st Differences	2nd Differences		.073	.07	.064		1.88	1.88	1.88	1.88
4	E	Implicit, 2nd Differences	1st Differences		.060	.058	.062		2.06	1.96	2.05	1.96
4	F	Implicit, 2nd Differences	2nd Differences		.060	.059	.062		1.88	1.88	1.88	1.88

medium, Equations (8), the analytical solution of Equations (34) and the computer solution.

When H = 1 the time, t_1, at which melting starts is very close to that for the continuous medium for all cases. However, for the higher heating rate, H = 4, the explicit determination of u_r introduces very large errors into the value of t_1 when only three stations are used. When six stations are used with first order differences the error in t_1 is 45%. When second order differences are used this error drops to 2.5%. The analytic solution of the differential equations indicates that these errors are introduced by the approximations and not by the computer. The errors in t_1 are smaller when the temperature at the r^{th} station is determined implicitly but the error decreases from 12% for Run C, r = 3, to zero for Run F, r = 6, within the accuracy of the solution.

When six finite difference increments are used with second order differences there is little to choose between the implicit and explicit determinations of u_r as far as the time at which melting starts is concerned.

The values of the steady state melting rate, b, were calculated by means of Equation (12a) for the continuous case. The values for the finite difference approximations were obtained by setting all time derivatives of temperatures in Equations (35) equal to zero, letting $\frac{d\ell}{dt} = b$ and solving the resulting set of algebraic equations simultaneously for b. The solutions are polynomials in b of degree r if r is odd and (r-1) if r is even. The coefficients of powers of b are functions of H and g_o.

These polynomials must reduce to b = 1 if $H = (1-g_0)$ as shown in Equations (29-31). These polynomials are presented below for r = 1 through r = 6.

r=1, 1st differences.

$$b-(H+g_0)=0$$

r=2, 1st differences.

$$(2-g_0)b-(2H+g_0)=0$$

r=2, 2nd differences.

$$(1-g_0)b-H=0$$

r=3, 1st differences.

$$b^3-(H+3g_0)b^2+(23-12g_0)b-(23H+9g_0)=0$$

2nd differences.

$$b^3-(H+6g_0)b^2+(23-12g_0)b-(23H+6g_0)=0$$

r=4, 1st differences.

$$(4-g_0)b^3-(4H+9g_0)b^2+(44-23g_0)b-(44H+15g_0)=0$$

2nd differences.

$$(2-g_0)b^3-(2H+6g_0)b^2+(22-11g_0)b-(22H+6g_0)=0$$

r=5, 1st differences.

$$b^5-(H+5g_0)b^4+(230-80g_0)b^3-(230H+430g_0)b^2$$
$$+(1689-880g_0)b-(1689H+525g_0)=0$$

2nd differences.

$$b^5-(H+10g_0)b^4+(230-120g_0)b^3-(230H+500g_0)b^2 \qquad (35)$$
$$+(1689-840g_0)b-(1689H+450g_0)=0$$

r 6, 1st differences.

$$(6-g_0)b^5-(6H+25g_0)b^4+(580-230g_0)b^3-(580H+950g_0)b^2+(3254-1689g_0)b-(3254H+945g_0)=0$$

2nd differences.

$$(3-g_0)b^5-(3H+20g_0)b^4+(290-150g_0)b^3-(290H+520g_0)b^2+(1627-809g_0)b-(1627H+420g_0)=0$$

The values H = 1 and H = 4 were chosen instead of H = 2.128 for the solutions in Table I in order to show the effect of the approximations on b. The results in Table I show this error to be rather large for H = 1 where the minimum error is 10.3% even when second order differences are used with r = 6. For H = 4, when second order differences are used, the value of b is almost exact. These results may be explained by means of Figure 10. In this figure b is plotted as a function of H for $g_o = -1.128$. The values of b as a function of H for the continuous medium are plotted as the dashed line. The curve b vs. H for the finite difference approximation intersects the continuous medium curve at $H = 1-g_o$, $H = 2(1-g_o)$, and $H = 3(1-g_o)$; and in the interval from $H = 1-g_o$ to $H = 3(1-g_o)$ is very close to that for the continuous medium. The two curves diverge outside this interval. Thus the close agreement between the values for b when H = 4 and $g_o = -1.128$ are predicted.

In all these cases the close agreement between the computer results and the analytical solution of the ordinary differential Equations (25) indicates that the computer solution is accurate and establishes a reasonably high confidence in the electronic differential analyzer solution.

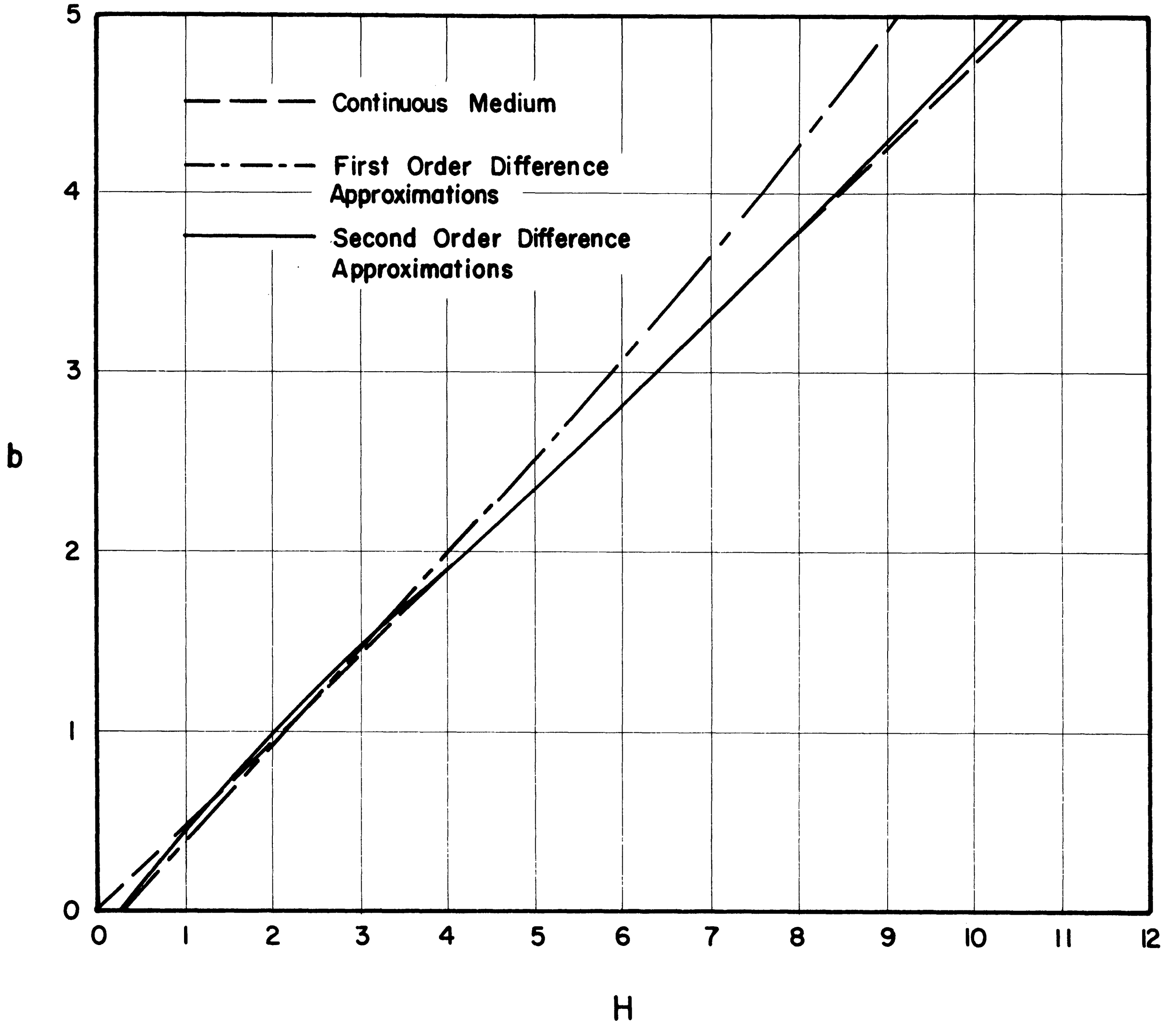


Figure 10. Steady State Boundary Velocity vs. Heating Rate.

Figures 11 through 14 contain sample solutions from the computer results for r = 3 and r = 6 with the second order difference approximations at the boundary. In Figures 11 and 12, 1/2 inch represents one unit of τ; in Figures 13 and 14, one inch represents one unit in τ.

The transient melting solution will now be compared with the solution obtained by Landau for a constant heating rate and uniform initial temperature distribution.

Comparison of the Boundary Velocities

Tables II - V contain the values of the boundary velocity for the several approximations used in this thesis and those obtained by Landau.[27] The agreement of the differential analyzer results with Landau's results is very good for the cases where r = 6 and second order difference approximations were used for $\frac{\partial u}{\partial x}$ at the boundary (cases B and F).

The effect of changing the approximations used during the warming phase is to vary t_1 and, when u_r is determined implicitly, to cause the error described on page 25 in $\frac{d\ell}{dt}$ at t_1. This error in $\frac{d\ell}{dt}$ at t_1 is very apparent for cases C and E, r = 3, in Table II. In these cases the boundary velocity is negative during the first moments of "melting".

The use of second order differences in the velocity equation steepens the slope of the $\frac{d\ell}{dt}$ curve during the first part of melting when compared to the first order difference curves. Increasing the number of increments has the same qualitative effect. The values in Tables II - V

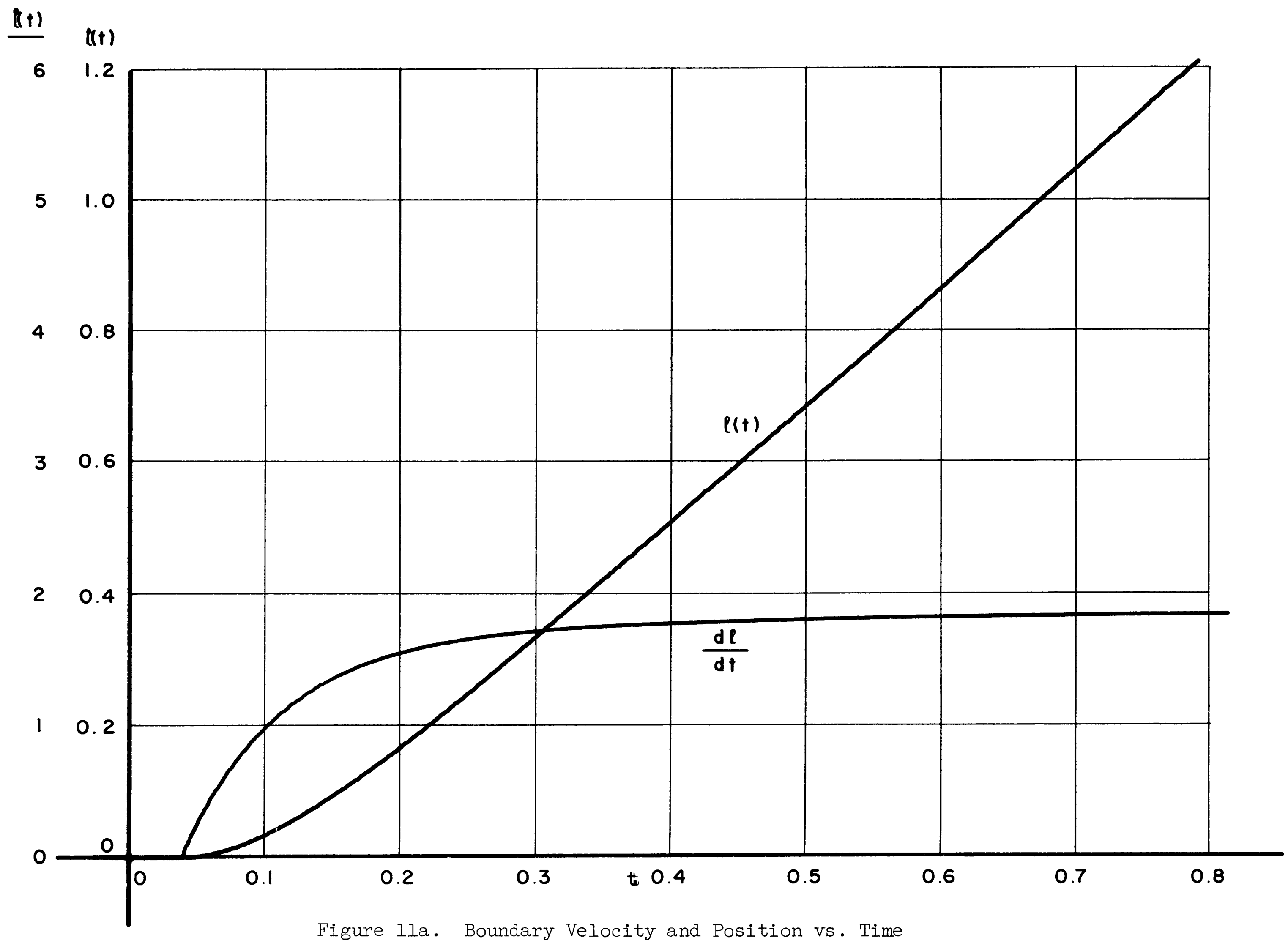


Figure 11a. Boundary Velocity and Position vs. Time
$r=3$; $H=4$; $g_o = -1.128$
Explicit Determination of u_r.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

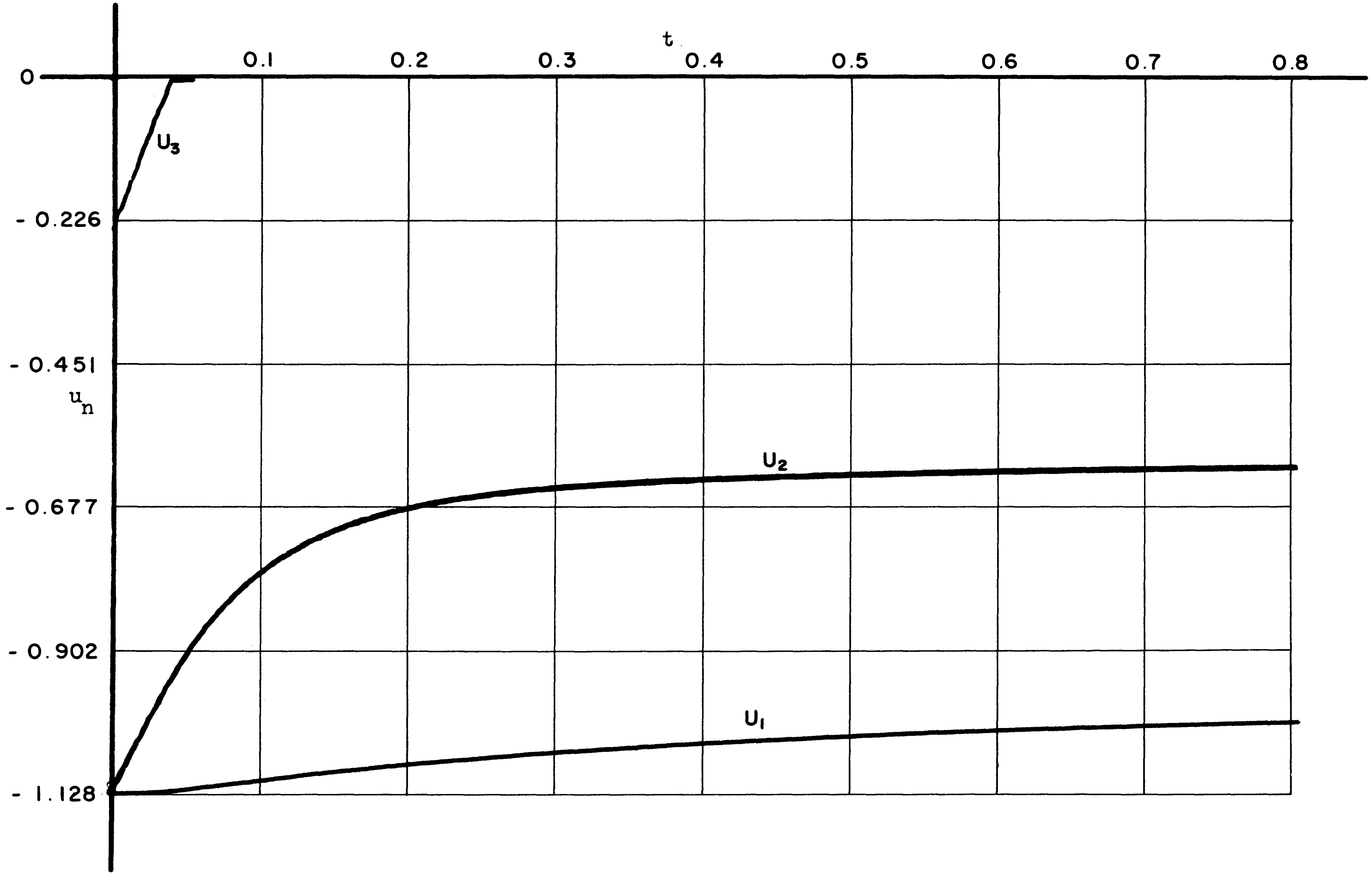


Figure 11b. Temperature at the Finite Difference Stations vs. Time
$r=3$; $H=4$; $g_o = -1.128$
Explicit Determination of u_r.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

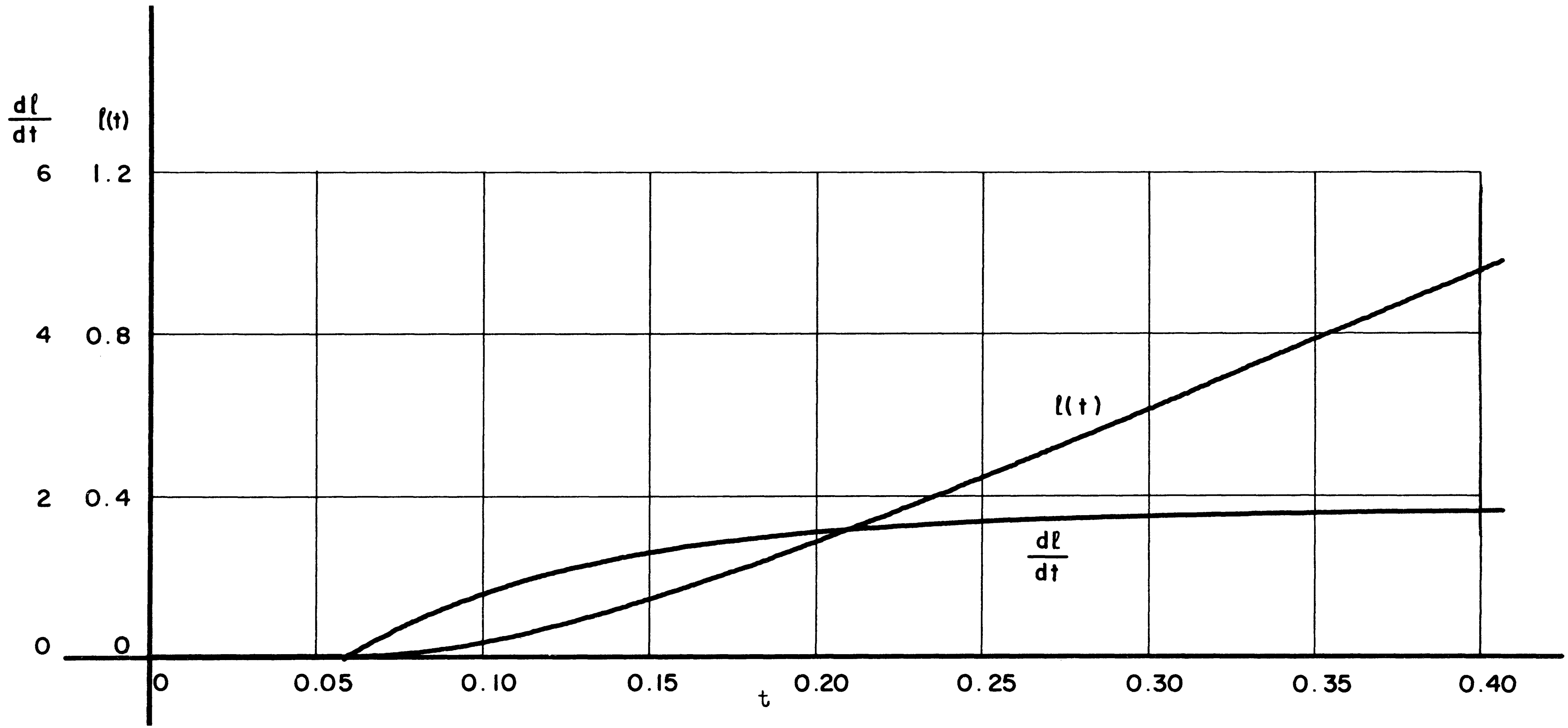


Figure 12a. Boundary Velocity and Position vs. Time
$r=3$; $H=4$; $g_o= -1.128$
Implicit Determination of u_3.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

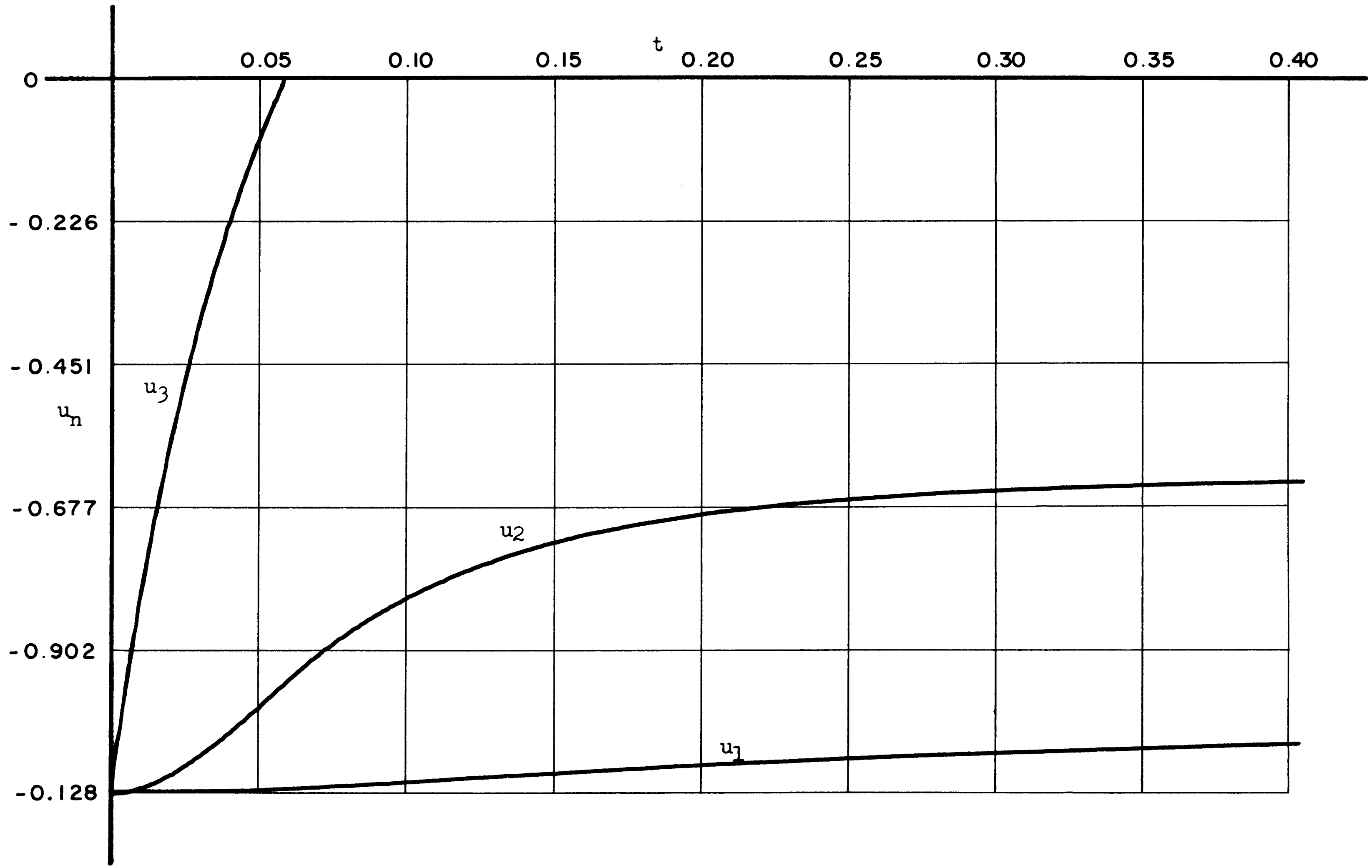


Figure 12b. Temperature at the Finite Difference Stations vs. Time
$r=3$; $H=4$; $g_o= -1.128$
Implicit Determination of u_3.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

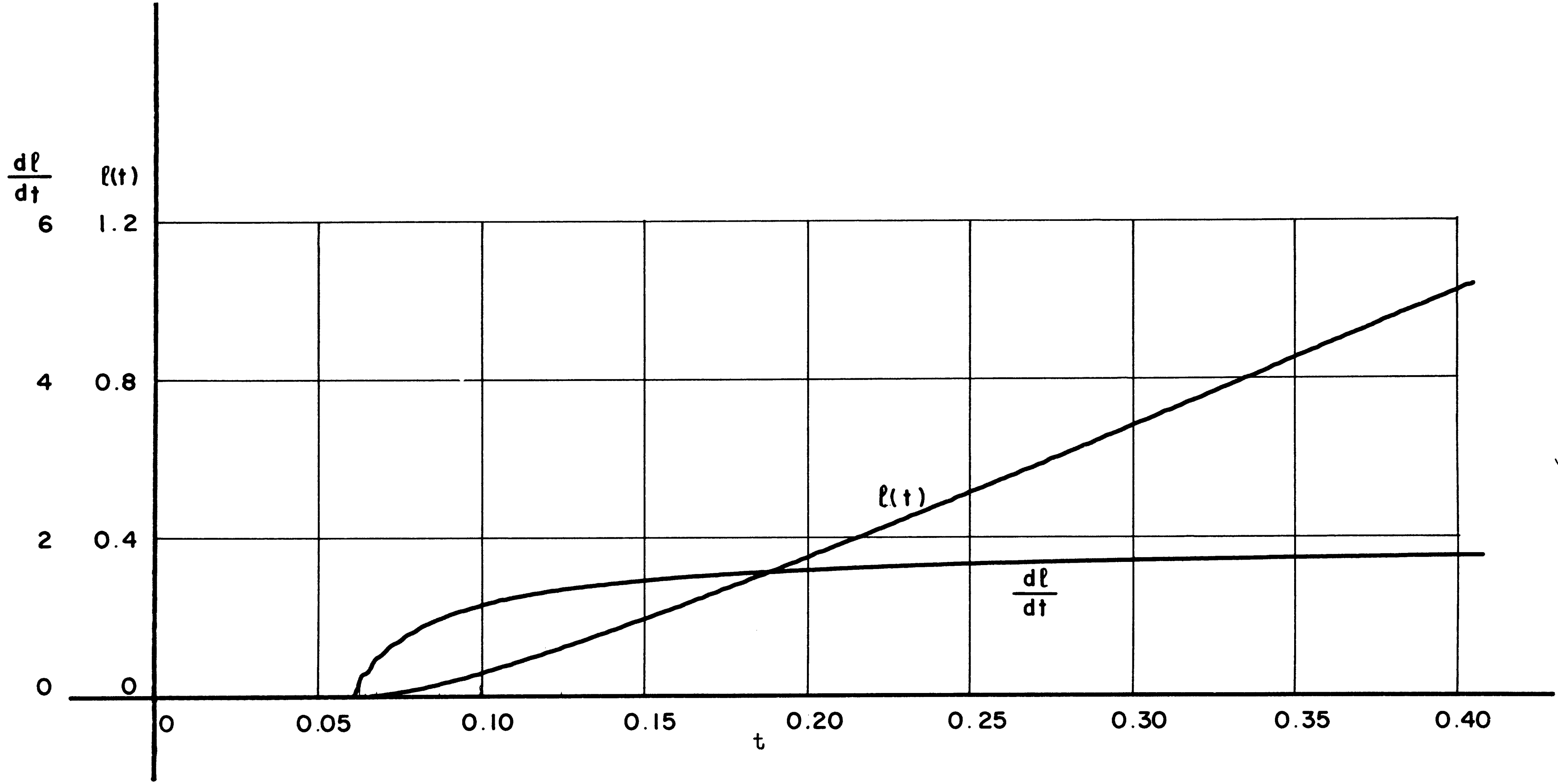


Figure 13a. Boundary Velocity and Position vs. Time
$r=6$; $H=4$; $g_o= -1.128$
Explicit Determination of u_6.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

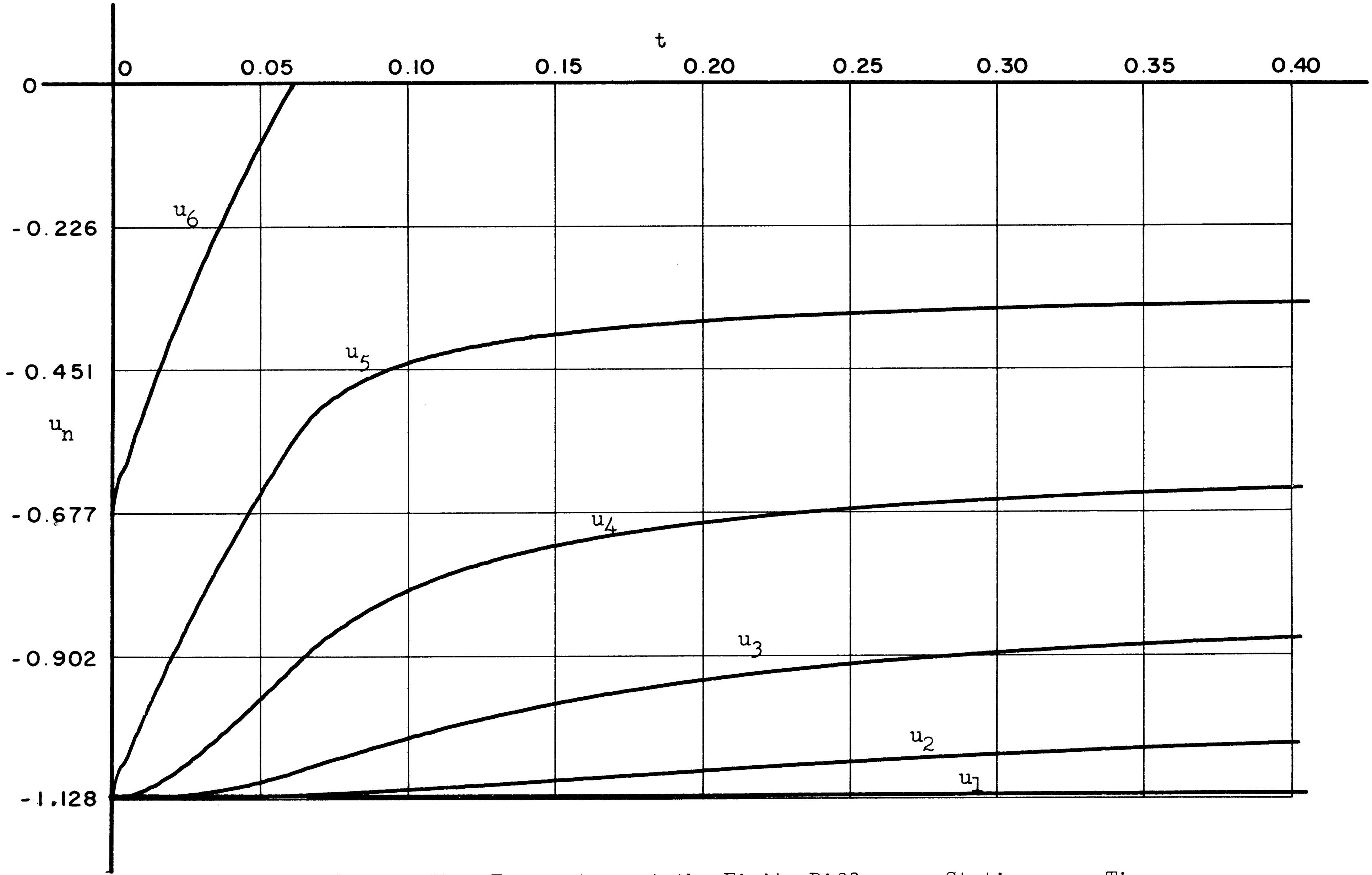


Figure 13b. Temperature at the Finite Difference Stations vs. Time
$r=6$; $H=4$; $g_o = -1.128$
Explicit Determination of u_6.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

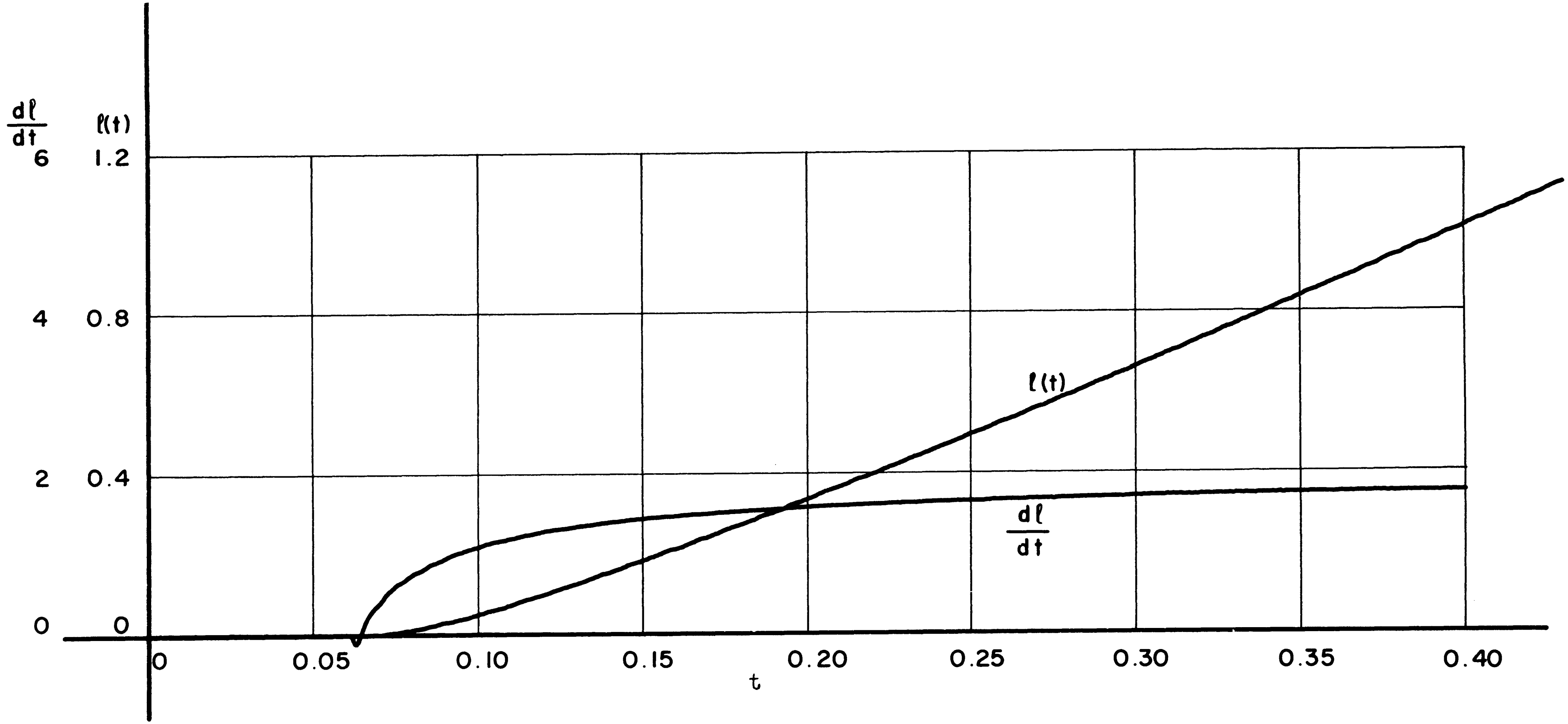


Figure 14a. Boundary Velocity and Position vs. Time
$r=6$; $H=4$; $g_o = -1.128$
Implicit Determination of u_6.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

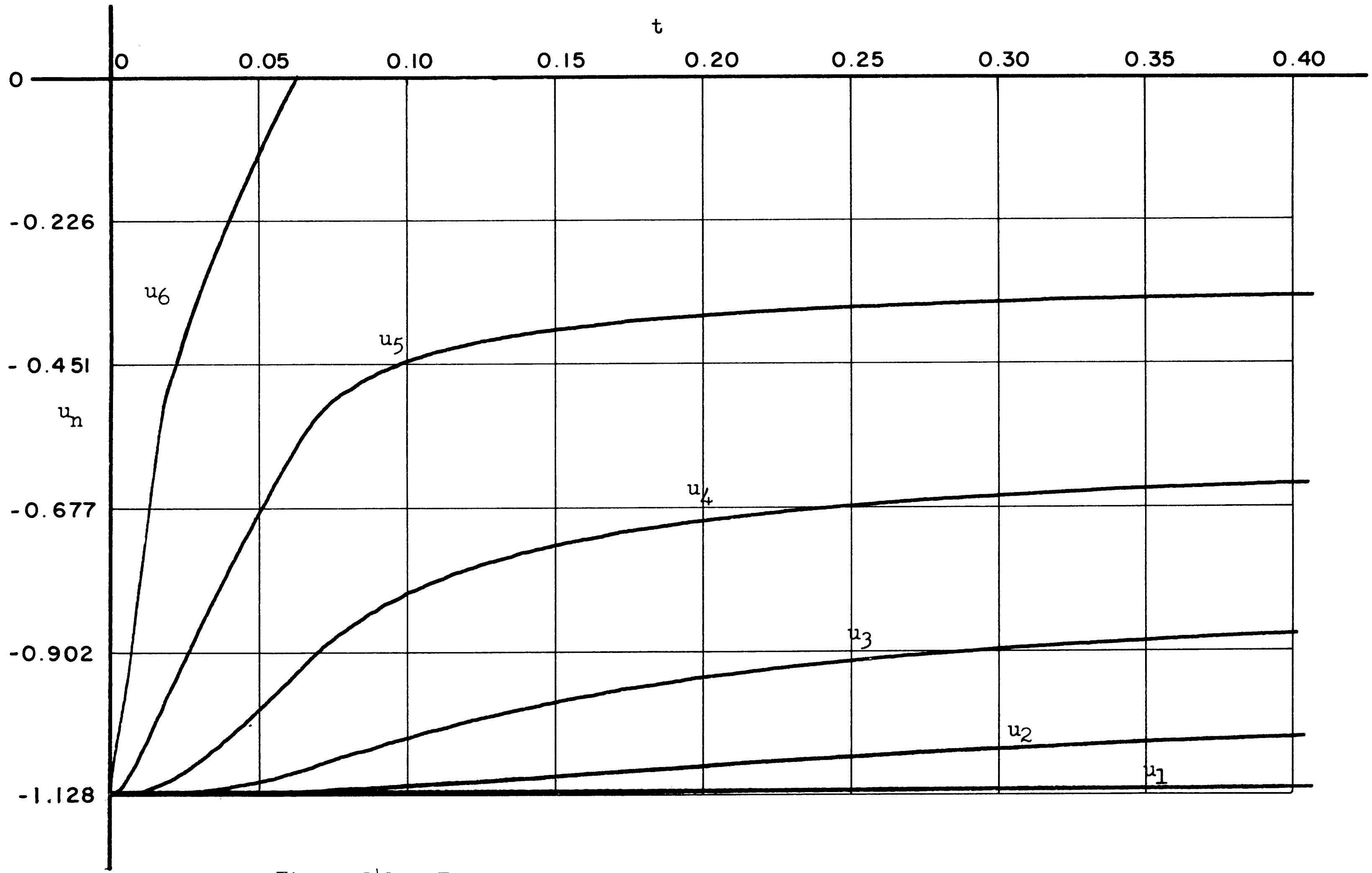


Figure 14b. Temperature at the Finite Difference Stations vs. Time
$r=6$; $H=4$; $g_o = -1.128$
Implicit Determination of u_6.
Second Order Difference Approximation of $\left.\frac{\partial u}{\partial x}\right|_{x_r}$.

indicate that after approximately 0.1 time units of melting only the approximation for $\left.\frac{\partial u}{\partial x}\right|_{x_r}$ in the $\frac{d\ell}{dt}$ equation affects $\frac{d\ell}{dt}$. This can be seen by comparing the values of $\frac{d\ell}{dt}$ for cases A, C, and E with each other and cases B, D, and F with each other.

General Solution Curves for Constant Heating Rate

In Appendix E solution curves are presented for initial uniform temperatures of g_o = -0.5, -1.0, -2.0, and -3.0. In each of these solutions $H = 1-g_o$, and u_r was determined implicitly. Thus these curves can be used to obtain values of the temperature at any of the six stations and the position and velocity of the boundary at any given time. Equations (2), (6), and (8) are used in conjunction with the curves of Appendix E.

The temperature curves of Appendix E indicate that the temperature distribution is linear in x in steady state as predicted above. The steady state boundary velocity in each solution is $(1-g_o)$ as predicted.

The relatively few values of g_o for which solutions are obtained in Appendix E do not allow direct solution of the problem for all possible g_∞. These curves could be used for values of $\tilde{g}_\infty$ which do not correspond to any value of g_o contained in Appendix E by graphical interpolation. A better solution could be obtained by solving the problem for small increments of g_o.

Solutions for a Time Varying Heating Rate

Solutions for a heat pulse of the form $H = h(1- \cos 2\pi t)$, shown in Figure F1, are presented in Appendix F. In these solutions the initial

TABLE II

BOUNDARY VELOCITY FOR THE SEVERAL APPROXIMATIONS AND LANDAU'S SOLUTION
$g_o = -1.128$, H=1

		Approximations		$\frac{d\ell}{dt}(t)$									
r	Case	u_r Calculation	$\frac{d\ell}{dt}$ Calculation	1.00	1.02	1.06	1.10	1.20	1.60	2.00	2.40	2.80	3.20
	A	Explicit, 1st Differences	1st Differences	0	0	0	.03	.115	.24	.28	.30	.32	.32
	B	Explicit, 2nd Differences	2nd Differences	0	0	.065	.13	.22	.33	.37	.40	.41	.41
	C	Implicit, 1st Differences	1st Differences	0	-.05	.00	.04	.11	.24	.28	.31	.32	.33
3	D	Implicit, 1st Differences	2nd Differences	0	.065	.125	.17	.24	.33	.37	.39	.40	.41
	E	Implicit, 2nd Differences	1st Differences	0	-.03	.01	.05	.12	.24	.28	.31	.32	.32
	F	Implicit, 2nd Differences	2nd Differences	0	.05	.11	.16	.23	.32	.37	.39	.40	.41
	A	Explicit, 1st Differences	1st Differences	.025	.055	.10	.125	.18	.26	.31	.34	.36	.37
	B	Explicit, 2nd Differences	2nd Differences	0	.07	.15	.19	.26	.34	.37	.39	.40	.41
	C	Implicit, 1st Differences	1st Differences	0	.04	.08	.105	.17	.26	.32	.34	.36	.37
6	D	Implicit, 1st Differences	2nd Differences	0	.09	.135	.165	.21	.30	.34	.37	.39	.40
	E	Implicit, 2nd Differences	1st Differences	-.04	.03	.07	.105	.16	.26	.31	.34	.36	.37
	F	Implicit, 2nd Differences	2nd Differences	0	.085	.12	.16	.21	.30	.34	.37	.38	.40
		LANDAU'S SOLUTION		.00	.0712	.1289	.1600	.2103	.292	.336	.362	.381	.395

TABLE III

BOUNDARY POSITION FOR THE SEVERAL APPROXIMATIONS AND LANDAU'S SOLUTION
$g_o = -1.128$, H=1

Approximations				$\ell(t)$									
r	Case	u_r Calculation	$\frac{d\ell}{dt}$ Calculation	1.00	1.02	1.06	1.10	1.20	1.60	2.00	2.40	2.80	3.20
	A	Explicit, 1st Differences	1st Differences	0	0	0	0	.008	.085	.190	.307	.431	.554
	B	Explicit, 2nd Differences	2nd Differences	0	0	0	.005	.023	.138	.278	.433	.595	--
	C	Implicit, 1st Differences	1st Differences	0	-.002	-.002	-.001	.006	.080	.184	.305	.430	.558
3	D	Implicit, 1st Differences	2nd Differences	0	.001	.002	.012	.032	.147	.287	.439	.597	--
	E	Implicit, 2nd Differences	1st Differences	0	-.001	-.002	0	.009	.087	.190	.308	.434	.558
	F	Implicit, 2nd Differences	2nd Differences	0	0	0	.011	.030	.146	.283	.436	.594	--
	A	Explicit, 1st Differences	1st Differences	0	.002	.003	.008	.023	.113	.229	.368	.500	--
	B	Explicit, 2nd Differences	2nd Differences	0	.001	.005	.012	.035	.158	.299	.451	.612	--
	C	Implicit, 1st Differences	1st Differences	0	0	0	.005	.020	.108	.220	.348	.438	--
6	D	Implicit, 1st Differences	2nd Differences	0	.001	.006	.011	.030	.032	.256	.400	.547	--
	E	Implicit, 2nd Differences	1st Differences	0	0	.001	.005	.018	.104	.218	.346	.484	--
	F	Implicit, 2nd Differences	2nd Differences	0	0	.004	.010	.030	.132	.258	.397	.543	--
		LANDAU'S SOLUTION		--	--	.00517	.0106	.0296	.1318	.263	.400	.555	.705

TABLE IV

BOUNDARY VELOCITY FOR THE SEVERAL APPROXIMATIONS AND LANDAU'S SOLUTION
$g_o = -1.128$, H=4

	Approximations			$\frac{d\ell}{dt}(t)$									
r	Case	u_r Calculation	$\frac{d\ell}{dt}$ Calculation	0.00	.05	.0625	.075	.10	.15	.20	.30	.40	.60
	A	Explicit,1st Differences	1st Differences	.60	1.17	1.28	1.35	1.50	1.68	1.81	1.91	1.97	2.02
	B	Explicit,2nd Differences	2nd Differences	0	.24	.47	.67	.97	1.34	1.53	1.70	1.75	1.80
	C	Implicit, 1st Differences	1st Differences	0	0	0	1.00	1.40	1.54	1.78	1.92	1.96	--
3	D	Implicit, 1st Differences	2nd Differences	0	0	0	.16	.61	1.19	1.47	1.69	1.76	--
	E	Implicit, 2nd Differences	1st Differences	0	0	1.18	1.40	1.46	1.69	1.81	1.94	2.00	--
	F	Implicit, 2nd Differences	2nd Differences	0	0	.08	.35	.78	1.28	1.52	1.72	1.79	--
	A	Explicit, 1st Differences	1st Differences	0	.52	.98	1.16	1.36	1.58	1.66	1.80	1.88	--
	B	Explicit, 2nd Differences	2nd Differences	0	0	.20	.72	1.16	1.44	1.60	1.72	1.76	--
	C	Implicit, 1st Differences	1st Differences	0	0	0	.66	1.24	1.54	1.68	1.82	1.86	--
6	D	Implicit, 1st Differences	2nd Differences	0	0	0	.48	1.06	1.42	1.56	1.68	1.76	--
	E	Implicit, 2nd Differences	1st Differences	0	0	.20	.96	1.28	1.54	1.68	1.80	1.86	--
	F	Implicit, 2nd Differences	2nd Differences	0	0	-.10	.62	1.10	1.43	1.57	1.69	1.75	--
		LANDAU'S SOLUTION		0	0	0	.84	1.17	1.45	1.57	1.69	1.76	--

TABLE V

BOUNDARY POSITION FOR THE SEVERAL APPROXIMATIONS AND LANDAU'S SOLUTION

$g_o = -1.128$, H=4

Approximations				$\ell(t)$								
r	Case	u_r Calculation	$\frac{d\ell}{dt}$ Calculation	0.00	.05	.0625	.075	.10	.15	.20	.30	.40
	A	Explicit, 1st Differences	1st Differences	0	.046	.060	.074	.110	.192	.280	.470	.670
	B	Explicit, 2nd Differences	2nd Differences	0	0	.004	.010	.030	.100	.162	.330	.504
	C	Implicit, 1st Differences	1st Differences	0	0	0	0	.045	.130	.200	.390	.586
3	D	Implicit, 1st Differences	2nd Differences	0	0	0	0	.008	.056	.124	.286	.456
	E	Implicit, 2nd Differences	1st Differences	0	0	.006	.020	.056	.136	.244	.410	.600
	F	Implicit, 2nd Differences	2nd Differences	0	0	0	.004	.020	.072	.140	.304	.476
	A	Explicit, 1st Differences	1st Differences	0	.006	.014	.030	.062	.140	.220	.394	.572
	B	Explicit, 2nd Differences	2nd Differences	0	0	0	.005	.030	.096	.174	.342	.512
	C	Implicit, 1st Differences	1st Differences	0	0	0	.006	.034	.104	.184	.354	.532
6	D	Implicit, 1st Differences	2nd Differences	0	0	0	0	.020	.038	.162	.324	.498
	E	Implicit, 2nd Differences	1st Differences	0	0	0	.008	.038	.110	.192	.370	.552
	F	Implicit, 2nd Differences	2nd Differences	0	0	0	.002	.026	.086	.166	.328	.502
		LANDAU'S SOLUTION		0	0	0	.0074	.0328	.0998	.176	.340	.506

temperature distribution was uniform in x. The heated boundary temperature was obtained explicitly in each case.

In Figure F2 the heating rate, $H = 0.5(1-\cos 2\pi t)$, was not sufficiently high to cause the boundary to reach the melting temperature. This corresponds to a heat sink. After the heat input returns to zero the temperatures within the body approach g_o, the temperature at x = 0 or $\tilde{x} = \infty$, because there is an infinite heat capacity in the slab.

Solutions are also presented for peak values of h = 1, and 1.5 in Figures F3 and F4. In both these cases melting starts at t_1 and eventually stops when the heating rate drops to a sufficiently low value. After the heating stops the temperatures again approach the initial temperature.

CHAPTER IV

THE FINITE SLAB

The finite thickness slab with one dimensional heat flow can be treated by finite difference techniques also, as has been done by Eyres, et al.,[13] Otis,[34] and Sunderland,[39] and Murray.[32] In each of these references the finite difference increments used were of uniform size. A non-uniform grouping of stations in which the finite difference increments are smaller where the temperature gradient is larger allows a more accurate approximation of the gradient. Such a method for regrouping the finite difference stations for the heating problem with fixed boundaries was developed by Howe.[19] This method is applied to the melting problem in this chapter after the problem has been made dimensionless by means of Equations (2) above and the moving boundary removed by means of another transformation below.

The Physical Problem

The physical problem is presented schematically in Figure 15.

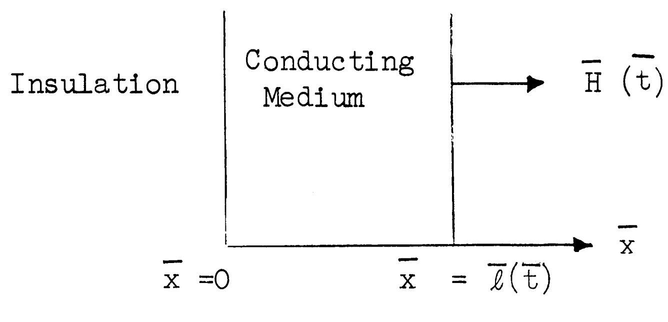


Figure 15. Physical Schematic, Finite Slab.

The slab extends to $\pm\infty$ in both directions perpendicular to $\bar{x}$ thereby giving one dimensional heat flow. The heat flux, $\bar{H}$, is defined to be positive when heat flow is in the direction of positive $\bar{x}$.

The Mathematical Model

The physical problem is described by Equations (36) in the $(\bar{x},\bar{t})$ domain of Figure 16.

$$\bar{\rho}\bar{c}\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left[\bar{k}\frac{\partial \bar{u}}{\partial \bar{x}}\right] \qquad 0 \le \bar{x} \le \bar{\ell}(\bar{t});\ 0 \le \bar{t} \le \bar{t}_m \quad \text{(a)}$$

$$\bar{H}(\bar{t}) = \bar{\rho}L\frac{d\bar{\ell}}{d\bar{t}} - \bar{k}\frac{\partial \bar{u}}{\partial \bar{x}} \qquad \bar{x} = \bar{\ell}(\bar{t}),\ 0 \le \bar{t} \le \bar{t}_m \quad \text{(b)}$$

$$\frac{d\bar{\ell}}{d\bar{t}} = 0 \qquad 0 \le \bar{t} \le \bar{t}_1 \quad \text{(c)}$$

$$\bar{u} < \bar{u}_c \qquad \bar{x} = \bar{\ell}(\bar{t});\ 0 \le \bar{t} < \bar{t}_1 \quad \text{(d)}$$

$$\frac{d\bar{\ell}}{d\bar{t}} < 0 \qquad \bar{t}_1 < \bar{t} < \bar{t}_2 \quad \text{(e)} \qquad (36)$$

$$\bar{u} = \bar{u}_c \qquad \bar{x} = \bar{\ell}(\bar{t});\ \bar{t}_1 \le \bar{t} \le \bar{t}_2 \quad \text{(f)}$$

$$\bar{u} = \bar{g}(\bar{x}) \qquad 0 \le \bar{x} \le \bar{\ell}(\bar{t});\ \bar{t} = 0 \quad \text{(g)}$$

$$\bar{\ell} = \bar{s} \qquad \bar{t} = 0 \quad \text{(h)}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} = 0 \qquad \bar{x} = 0;\ 0 \le \bar{t} \le \bar{t}_m \quad \text{(i)}$$

Here

$\bar{t}_m$ = the time at which the slab becomes completely melted.

All other symbols are as defined on page 9 . It is assumed in Equations (36) that there are no heat sources in the conducting medium.

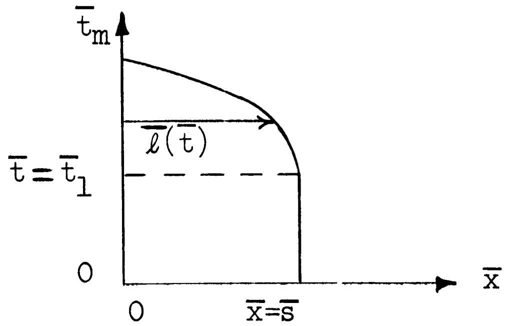


Figure 16. The $(\bar{x},\bar{t})$ Domain, Finite Slab.

Figure 16 is drawn for a heating rate which remains negative sufficiently long to cause complete melting of the slab. If the heating rate becomes zero after t_1, melting will stop and the slab will approach an equilibrium temperature. If the heating flux becomes positive the temperature will decrease. Freezing cannot occur because perfect ablation has been assumed and thus the liquid is removed immediately upon formation. Thus $d\ell/dt \leq 0$ for all time.

Equations (36) are made non-dimensional by means of Equations (2). Equations (37) result from this transformation.

$$
\begin{array}{lll}
\dfrac{\partial \tilde{u}}{\partial \tilde{t}} = \dfrac{1}{\tilde{\phi}_\rho \tilde{\phi}_c} \dfrac{K_1^2}{K_2} \dfrac{\partial}{\partial \tilde{x}}\left[\tilde{\phi}_k \dfrac{\partial \tilde{u}}{\partial \tilde{x}}\right] & 0 \leq \tilde{x} \leq \tilde{\ell}(\tilde{t});\ 0 \leq \tilde{t} \leq \tilde{t}_m & \text{(a)} \\
\tilde{H}(\tilde{t}) = \dfrac{K_3 K_2}{K_1} \tilde{\phi}_\rho \dfrac{d\tilde{\ell}}{d\tilde{t}} - K_1 K_3 \tilde{\phi}_k \dfrac{\partial \tilde{u}}{\partial \tilde{x}} & \tilde{x} = \tilde{\ell}(\tilde{t});\ 0 \leq \tilde{t} \leq \tilde{t}_m & \text{(b)} \\
\dfrac{d\tilde{\ell}}{d\tilde{t}} = 0 & 0 \leq \tilde{t} \leq \tilde{t}_1 & \text{(c)} \\
\tilde{u} < 0 & \tilde{x} = \tilde{s};\ 0 \leq \tilde{t} \leq \tilde{t}_1 & \text{(d)} \\
\dfrac{d\tilde{\ell}}{d\tilde{t}} < 0 & \tilde{t}_1 < \tilde{t} < \tilde{t}_2 & \text{(e)} \\
\tilde{u} = 0 & \tilde{x} = \tilde{\ell}(\tilde{t});\ \tilde{t}_1 \leq \tilde{t} \leq \tilde{t}_2 & \text{(f)} \\
\tilde{u} = \tilde{g}(\tilde{x}) & 0 \leq \tilde{x} \leq \tilde{s};\ \tilde{t} = 0 & \text{(g)} \\
\tilde{\ell} = \tilde{s} & \tilde{t} = 0 & \text{(h)} \\
\dfrac{\partial \tilde{u}}{\partial \tilde{x}} = 0 & \tilde{x} = 0;\ 0 \leq \tilde{t} \leq \tilde{t}_m & \text{(i)}
\end{array}
\qquad (37)
$$

The $(\tilde{x},\tilde{t})$ domain is the same shape as the $(\bar{x},\bar{t})$ domain of Figure 16.

Choice of Scaling Constants K_1, K_2, K_3

The scaling constants, K_1, K_2, and K_3, can be eliminated from Equations (37) by means of Equations (4) as was done in Chapter II for the semi-infinite slab.

When the heating rate is constant, Equations (6) can be used to eliminate $\tilde{H}$ in a manner analogous to that used to obtain Equations (7) from Equations (4).

Problem Variables

Equations (37) describe the problem in terms of the following variables:

1. independent variables, $\tilde{x}$ and $\tilde{t}$;
2. dependent variables, $\tilde{u}(\tilde{x},\tilde{t})$ and $\tilde{\ell}(\tilde{t})$;
3. problem parameters:
    a. heating rate, $\tilde{H}(\tilde{t})$;
    b. initial temperature distribution, $\tilde{g}(\tilde{x})$;
    c. initial thickness, $\tilde{s}$;
4. characteristics of the conducting medium;
    a. density function, $\tilde{\phi}_\rho(\tilde{x},\tilde{u})$;
    b. specific heat function, $\tilde{\phi}_c(\tilde{x},\tilde{u})$;
    c. conductivity function, $\tilde{\phi}_k(\tilde{x},\tilde{u})$.

The finite slab problem has one more parameter, namely the initial thickness, $\tilde{s}$, than the semi-infinite slab.

It will now be assumed that the thermal characteristics of the conducting medium are constant both in temperature and in $\tilde{x}$. Thus $\tilde{\phi}_\rho = \tilde{\phi}_c = \tilde{\phi}_k = 1$.

The moving boundary will be eliminated in the next section.

Removal of the Moving Boundary

The moving boundary is removed by applying the transformation defined in Equations (38) to Equations (37) to obtain Equations (39).

The transformation is

$$\hat{x} = \frac{\tilde{x}}{\tilde{\ell}(\tilde{t})} \qquad (38)$$
$$\hat{t} = \tilde{t}$$

Also let

$$\hat{u}(\hat{x}, \hat{t}) = \tilde{u}(\tilde{x}, \tilde{t})$$
$$\hat{\ell}(\hat{t}) = \tilde{\ell}(\tilde{t})$$
$$\hat{H}(\hat{t}) = \tilde{H}(\tilde{t})$$

The transformed equations are:

$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{1}{\hat{\ell}^2(\hat{t})} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \hat{x} \frac{1}{\hat{\ell}(\hat{t})} \frac{d\hat{\ell}}{d\hat{t}} \frac{\partial \hat{u}}{\partial \hat{x}}$	$0 \le \hat{x} \le 1;\ 0 \le \hat{t} \le \hat{t}_m$	(a)	
$\hat{H}(\hat{t}) = \frac{d\hat{\ell}}{d\hat{t}} - \frac{1}{\hat{\ell}(\hat{t})} \frac{\partial \hat{u}}{\partial \hat{x}}$	$\hat{x} = 1;\ 0 \le \hat{t} \le \hat{t}_m$	(b)	
$\frac{d\hat{\ell}}{d\hat{t}} = 0$	$0 \le \hat{t} \le \hat{t}_1$	(c)	
$\hat{u} < 0$	$\hat{x} = 1;\ 0 \le \hat{t} < \hat{t}_1$	(d)	
$\frac{d\hat{\ell}}{d\hat{t}} < 0$	$\hat{t}_1 < \hat{t} < \hat{t}_2$	(e)	(39)
$\hat{u} = 0$	$\hat{x} = \hat{\ell}(\hat{t});\ \hat{t}_1 \le \hat{t} \le \hat{t}_2$	(f)	
$\hat{u} = \hat{g}(\hat{x})$	$0 \le \hat{x} \le 1;\ \hat{t} = 0$	(g)	
$\hat{\ell} = \hat{s}$	$\hat{t} = 0$	(h)	
$\frac{\partial \hat{u}}{\partial \hat{x}} = 0$	$\hat{x} = 0;\ 0 \le \hat{t} < \hat{t}_m$	(i)	

The $(\hat{x},\hat{t})$ domain is shown in Figure 17. This figure emphasizes the singularity in the problem at $\hat{t} = \hat{t}_m$ when the conducting medium disappears.

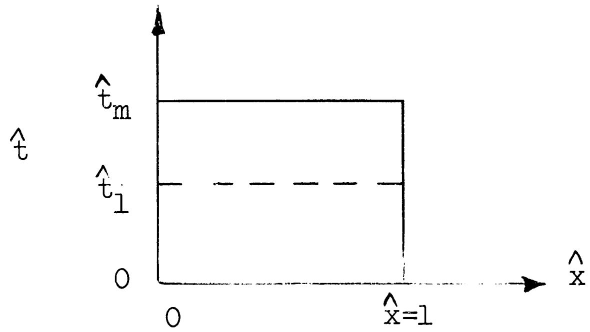


Figure 17. The $(\hat{x},\hat{t})$ Domain, Finite Slab.

If the finite difference approximations of Appendix C were applied at this point, the increments of $\hat{x}$ would be uniform. In order to improve the accuracy of the finite difference approximations the stations will now be grouped closer together near the heated boundary. This grouping will be obtained by transforming the $\hat{x}$ variable nonlinearly.

The Nonlinear Space Variable Transformation

The transformation used to group the finite difference stations near the heated boundary must satisfy the following two criteria:

1. The region near $\hat{x} = 1$ must be expanded.
2. The Jacobian of the transformation must not vanish in the interval $0 \leq \hat{x} \leq 1$.

For convenience it is also required that, if x is the transformed variable, its range must be $0 \leq x \leq 1$.

The nonlinear transformation is defined in functional form to be:

$$\hat{x} = f(x) \qquad 0 \le \hat{x} \le 1; \; 0 \le x \le 1 \qquad \text{(a)}$$
$$\hat{t} = t \qquad 0 \le \hat{t} = t \le t_m \qquad \text{(b)} \qquad (40)$$

The Jacobian of this transformation is

$$J = \frac{d\hat{x}}{dx} = f'(x)$$

This Jacobian must be positive for all values of x in the interval $0 \le x \le 1$. This requirement is justified by a consideration of Figure 18 and the two criteria above.

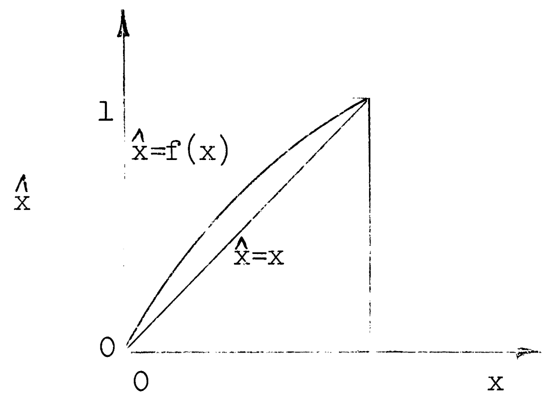


Figure 18. The Nonlinear Transformation.

In order to expand $\hat{x}$ near $\hat{x} = 1$, the function $\hat{x} = f(x)$ must pass through the origin and through the point (1,1). Therefore f'(x) must be positive somewhere in the interval if f(x) is continuous. Since $f'(x) \neq 0$ in the interval $0 \le x \le 1$, then $f'(x) > 0$ throughout this interval.

Equations (39) are now transformed by means of Equations (40) to obtain Equations (41) in which

$$u(x,t) = \hat{u}(\hat{x},\hat{t})$$
$$\ell(t) = \hat{\ell}(\hat{t})$$
$$H(t) = \hat{H}(\hat{t})$$

$$\frac{\partial u}{\partial t} = \frac{f(x)}{f'(x)}\frac{1}{\ell(t)}\frac{d\ell}{dt}\frac{\partial u}{\partial x} + \frac{1}{f'(x)}\frac{1}{\ell^2(t)}\frac{\partial}{\partial x}\left[\frac{1}{f'(x)}\frac{\partial u}{\partial x}\right] \qquad 0 \le x \le 1;\ 0 \le t \le t_m \quad \text{(a)}$$

$$H(t) = \frac{d\ell}{dt} - \frac{1}{f'(x)}\frac{1}{\ell(t)}\frac{\partial u}{\partial x} \qquad x = 1;\ 0 \le t \le t_m \quad \text{(b)}$$

$$\frac{d\ell}{dt} = 0 \qquad 0 \le t \le t_1 \quad \text{(c)}$$

$$u < 0 \qquad x = 1;\ 0 \le t < t_1 \quad \text{(d)}$$

$$\frac{d\ell}{dt} < 0 \qquad t_1 < t < t_2 \quad \text{(e)} \qquad (41)$$

$$u = 0 \qquad x = 1;\ t_1 \le t \le t_2 \quad \text{(f)}$$

$$u = g(x) \qquad 0 \le x \le 1;\ t = 0 \quad \text{(g)}$$

$$\ell = s \qquad t = 0 \quad \text{(h)}$$

$$\frac{\partial u}{\partial x} = 0 \qquad x = 0;\ 0 \le t \le t_m \quad \text{(i)}$$

The Equations in Finite Difference Form

Equations (41) are in a form suitable for applying the finite difference approximations of Appendix C. When these approximations are introduced into Equations (41), the results are Equations (42).

$$\frac{du_1}{dt} = -\frac{r^2}{f'_0}\left(\frac{1}{f'_{\frac{1}{2}}} + \frac{1}{f'_{-\frac{1}{2}}}\right)\frac{u_0}{\ell^2(t)} + \frac{r^2}{f'_0}\left(\frac{1}{f'_{\frac{1}{2}}} + \frac{1}{f'_{-\frac{1}{2}}}\right)\frac{u}{\ell^2(t)} \qquad 0 \le t \le t_m \qquad \text{(a)}$$

$$\frac{du_n}{dt} = \frac{r^2}{f'_n f'_{n-\frac{1}{2}}}\frac{u_{n-1}}{\ell^2(t)} - \frac{r}{2}\frac{f_n}{f'_n}\frac{d\ell}{dt}\frac{u_{n-1}}{\ell(t)} - \frac{r^2}{f'_n}\left(\frac{1}{f'_{n-\frac{1}{2}}} + \frac{1}{f'_{n+\frac{1}{2}}}\right)\frac{u_n}{\ell^2(t)}$$
$$+\frac{r}{2}\frac{f_n}{f'_n}\frac{d\ell}{dt}\frac{u_{n+1}}{\ell(t)} + \frac{r^2}{f'_n}\frac{1}{f'_{n+\frac{1}{2}}}\frac{u_{n+1}}{\ell^2(t)} \qquad n = 1, 2, \ldots, (r-1) \quad 0 \le t \le t_m \qquad \text{(b)}$$

$$u_r = \frac{4}{3}u_{r-1} - \frac{1}{3}u_{r-2} - \frac{2}{3r}f'_r\,\ell(t)\,H(t) \qquad 0 \le t < t_1 \qquad \text{(c1)}$$

$$\frac{du_r}{dt} = -\frac{3r}{f'_r\,\ell(t)}H(t) - \frac{r^2}{f'_r f'_{r-1}}\frac{u_{r-2}}{\ell^2(t)} + \frac{4r^2}{f'_r f'_{r-\frac{1}{2}}}\frac{u_{r-1}}{\ell^2(t)} - \frac{r^2}{f'_r}\left(\frac{4}{f'_{r-\frac{1}{2}}} - \frac{1}{f'_{r-1}}\right)\frac{u_r}{\ell^2(t)} \qquad 0 \le t < t_1 \qquad \text{(c2)}$$

$$\frac{d\ell}{dt} = 0 \qquad 0 \le t < t_1 \qquad \text{(d)}$$

$$\frac{d\ell}{dt} = H(t) + \frac{r}{f'_r}\,\frac{1}{2}\,\frac{u_{r-2}}{\ell(t)} - \frac{2r}{f'_r}\frac{u_{r-1}}{\ell(t)} \qquad t_1 < t < t_2 \qquad \text{(e)}$$

$$u_r < 0 \qquad 0 \le t < t_1 \qquad \text{(f)}$$

(42)

$$u_r = 0 \qquad t_1 \le t \le t_2 \qquad \text{(g)}$$

$$u_n = g_n \qquad n = 0, 1, \ldots, (r-1) \text{ for (C1)}, \quad n = 0, 1, \ldots, r \text{ for (C2)} \qquad t = 0 \qquad \text{(h)}$$

$$\ell = S \qquad t = 0 \qquad \text{(i)}$$

Second order differences have been used in Equations (42) at the heated boundary because the results of the semi-infinite slab clearly demonstrate the improved accuracy of the second order over the first order approximations. In Equation (42-C1) the boundary temperature u_r, is determined explicitly; in Equation (42-C2) it is determined implicitly.

Scaling for the Differential Analyzer

Equations (42) are scaled for the differential analyzer by means of Equations (43).

$$U_n(\tau) = \frac{1}{a_1} u_n(t) \qquad n = 0, 1, \ldots r. \qquad \text{(a)}$$

$$\lambda(\tau) = \frac{1}{a_2} \ell(t) \qquad \text{(b)}$$

$$\mathcal{H}(\tau) = \frac{1}{a_3} H(t) \qquad \text{(c)} \qquad (43)$$

$$\tau = a_4 r^2 t \qquad \text{(d)}$$

Equations (43a, b, and c) are identical with Equations (34a, b, and c). In Equation (43d) r^2 is written explicitly in order to eliminate r^2 from Equations (42a and b).

When Equations (43) are introduced into Equations (42), the results are Equations (44) which are the computer equations.

$$\frac{dU_0}{dT} = -\frac{1}{f'_0}\left(\frac{1}{f'_{\frac{1}{2}}}+\frac{1}{f'_{-\frac{1}{2}}}\right)\frac{1}{a_2^2 a_4}\,\frac{U_0}{\lambda^2(T)} + \frac{1}{f'_0}\left(\frac{1}{f'_{\frac{1}{2}}}+\frac{1}{f'_{-\frac{1}{2}}}\right)\frac{1}{a_2^2 a_4}\,\frac{U_1}{\lambda^2(T)} \qquad 0 \le T \le T_m \qquad \text{(a)}$$

$$\begin{aligned} \frac{dU_n}{dT} = {} & \frac{1}{f'_n f'_{n-\frac{1}{2}}}\,\frac{1}{a_2^2 a_4}\,\frac{U_{n-1}}{\lambda^2(T)} - \frac{r f_n}{2 f'_n}\,\frac{d\lambda}{dT}\,\frac{U_{n-1}}{\lambda(T)} \\ & - \frac{1}{f'_n}\left(\frac{1}{f'_{n-\frac{1}{2}}}+\frac{1}{f'_{n+\frac{1}{2}}}\right)\frac{1}{a_2^2 a_4}\,\frac{U_n}{\lambda^2(T)} \\ & + \frac{r f_n}{2 f'_n}\,\frac{d\lambda}{dT}\,\frac{U_{n+1}}{\lambda(T)} + \frac{1}{f'_n f'_{n+\frac{1}{2}}}\,\frac{1}{a_2^2 a_4}\,\frac{U_n}{\lambda^2(T)} \end{aligned} \qquad \begin{aligned} & n = 1, 2, \ldots, r-1 \\ & 0 \le T \le T_m \end{aligned} \qquad \text{(b)}$$

$$U_r = \frac{4}{3} U_{r-1} - \frac{1}{3} U_{r-2} - \frac{2 f'_r}{3 r}\,\frac{a_2 a_3}{a_1}\,\lambda(T)\mathcal{H}(T) \qquad 0 \le T < T_1 \qquad \text{(c1)}$$

$$\begin{aligned} \frac{dU_r}{dT} = {} & -\frac{3 a_3}{r f'_r \lambda a_1 a_2 a_4}\,\mathcal{H}(T) - \frac{1}{f'_r f'_{r-1}}\,\frac{1}{a_2^2 a_4}\,\frac{U_{r-2}}{\lambda^2(T)} \\ & + \frac{4}{f'_r f'_{r-\frac{1}{2}}}\,\frac{1}{a_2^2 a_4}\,\frac{U_{r-1}}{\lambda^2(T)} - \frac{1}{f'_r}\left(\frac{4}{f'_{r-\frac{1}{2}}} - \frac{1}{f'_r}\right)\frac{1}{a_2^2 a_4}\,\frac{U_r}{\lambda^2(T)} \end{aligned} \qquad 0 \le T < T_1 \qquad \text{(c2)}$$

(44)

$$\begin{aligned} \frac{d\lambda}{dT} = {} & \frac{1}{r^2}\,\frac{a_3}{a_2 a_4}\,\mathcal{H}(T) + \frac{1}{2 r f'_r}\,\frac{a_1}{a_2^2 a_4}\,\frac{U_{r-2}}{\lambda(T)} \\ & - \frac{2}{r f'_r}\,\frac{a_1}{a_2^2 a_4}\,\frac{U_{r-1}}{\lambda(T)} \end{aligned} \qquad T_1 < T < T_2 \qquad \text{(d)}$$

$$\frac{d\lambda}{dT} = 0 \qquad 0 \le T \le T_1 \qquad \text{(e)}$$

$$U_r < 0 \qquad 0 \le T < T_1 \qquad \text{(f)}$$

$$U_r = 0 \qquad T_1 \le T \le T_2 \qquad \text{(g)}$$

$$U_n = G_n \qquad \begin{aligned} & n = 0, 1, \ldots, r-1 \ \text{ for (C1)} \\ & n = 0, 1, \ldots, r \ \text{ for (C2)} \end{aligned} \qquad T = 0 \qquad \text{(h)}$$

$$\lambda = S \qquad T = 0 \qquad \text{(i)}$$

The computer circuits for solving Equations (44) are given in Figures (19a-d) on the following pages.

Specific nonlinear $\hat{x}$ transformations must be selected before solutions can be obtained on the differential analyzer. Two possible choices of this transformation are presented in the next section.

Nonlinear Space Variable Transformations

Two nonlinear space variable transformations are considered here. The first is a power transformation; the second an exponential transformation.

The power transformation is given in Equations (45).

$$\hat{x} = \frac{[1-(1-Cx)^k]}{1-(1-C)^k} \qquad \text{(a)}$$

$$\hat{t} = t \qquad \text{(b)} \qquad (45)$$

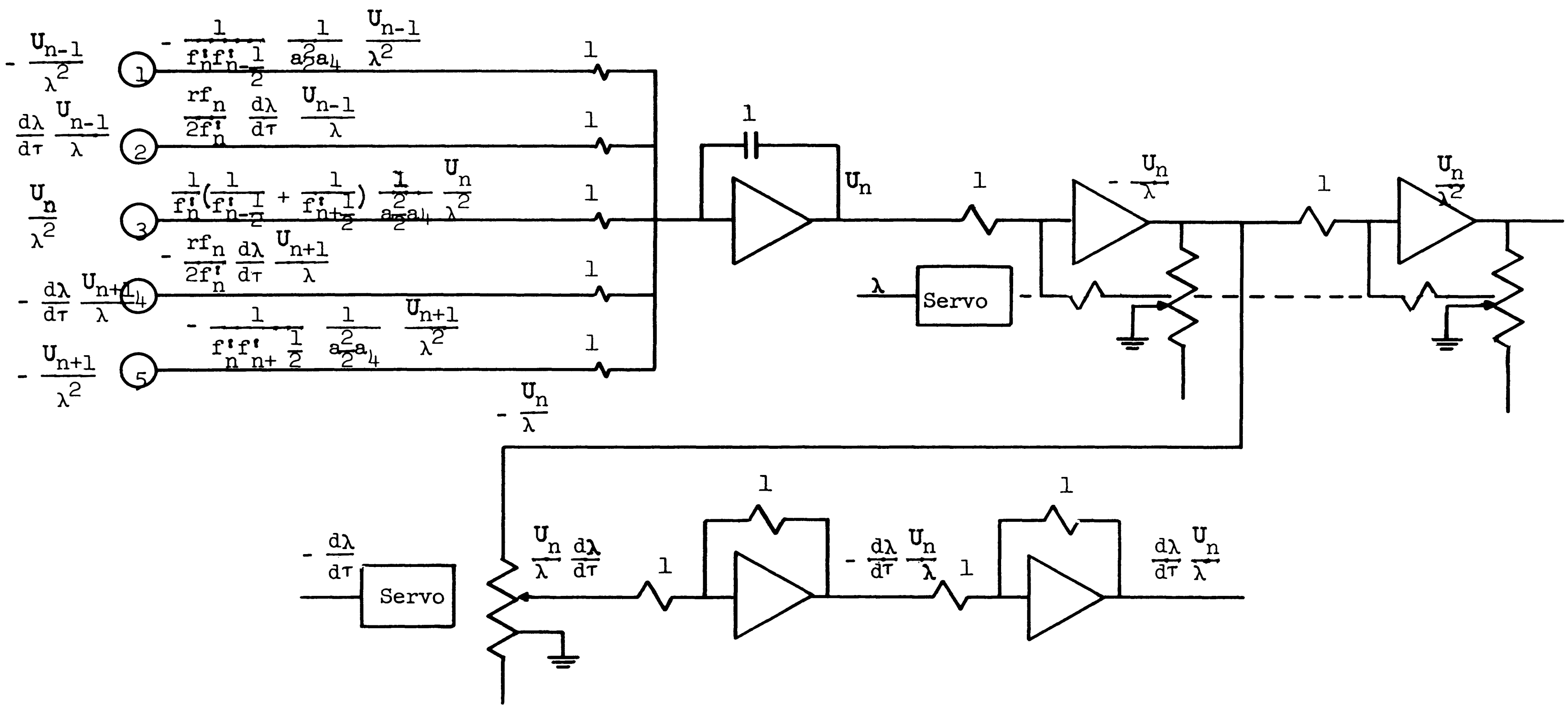
C and k are arbitrary parameters.

In Equations (45) the interval of x is $0 \leq x \leq 1$ as can be verified by direct substitution. The Jacobian, $J = d\hat{x}/dx$ of the transformation is

$$J = \frac{kC}{1-(1-C)^k}(1-Cx)^{k-1} \qquad 0 \leq x \leq 1$$

and

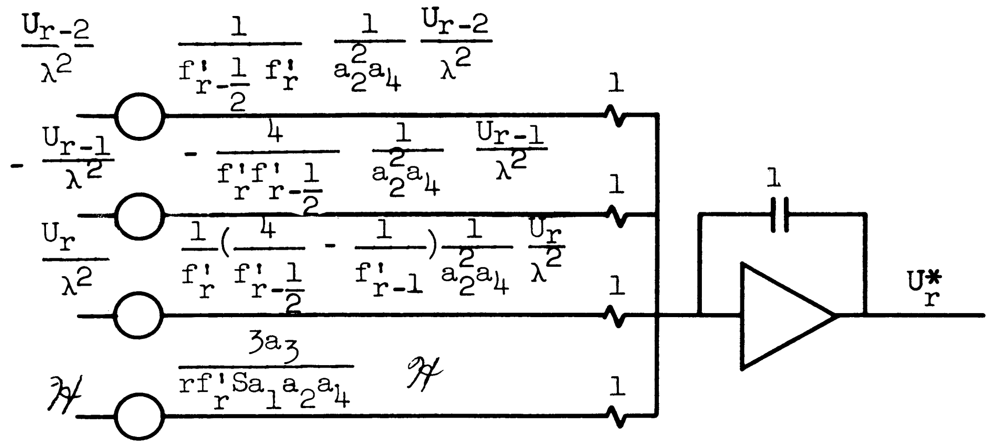
$$\frac{d^2\hat{x}}{dx^2} = -\frac{kC^2(k-1)}{1-(1-C)^k}(1-Cx)^{k-2}.$$



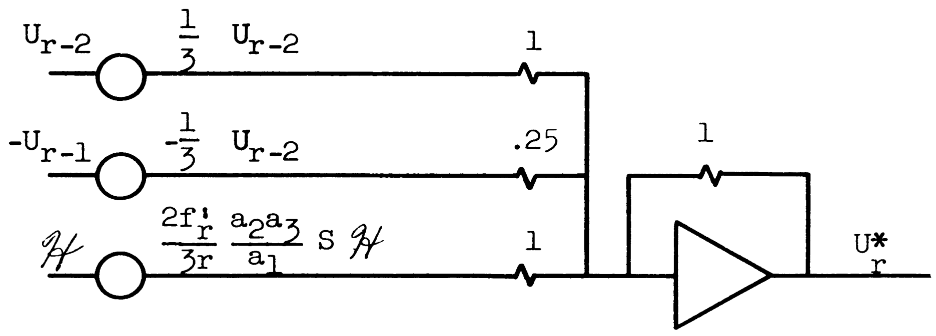
NOTE:

1. Potentiometer settings are given for n 1,2,...,r-1.
2. For n=0: Pots 1, 2, and 4 are set at 0.
   Pots 3 and 5 are set at $1/f'_0(1/f'_{1/2} + 1/f'_{1/2})$.
3. 100^V 1 machine unit.

Figure 19a. Computer Schematic for the n^{th} Station.



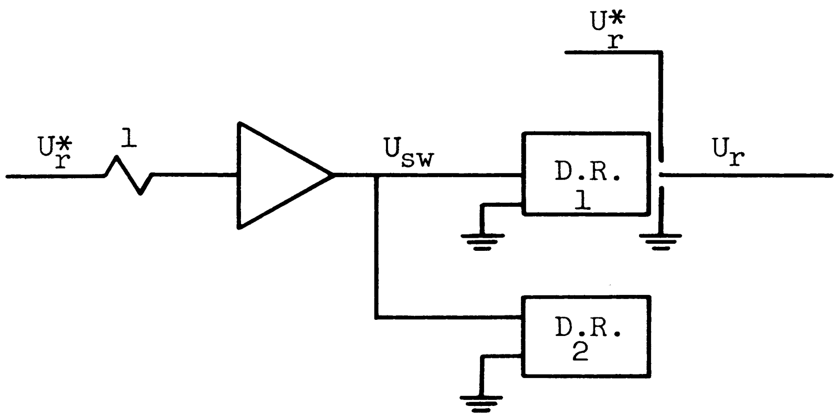
Implicit Determination of U_r.



Explicit Determination of U_r

NOTE: $U_r = U^*_r \quad 0 \leq \tau < \tau_1$

Figure 19b. Computer Schematic for Forming U_r.



NOTE: $U_{sw} = +100^V$ when $U^*_r < 0$

$= -100^V$ when $U^*_r > 0$

D.R. = Differential Relay.

Figure 19c. Switching Circuit for U_r.

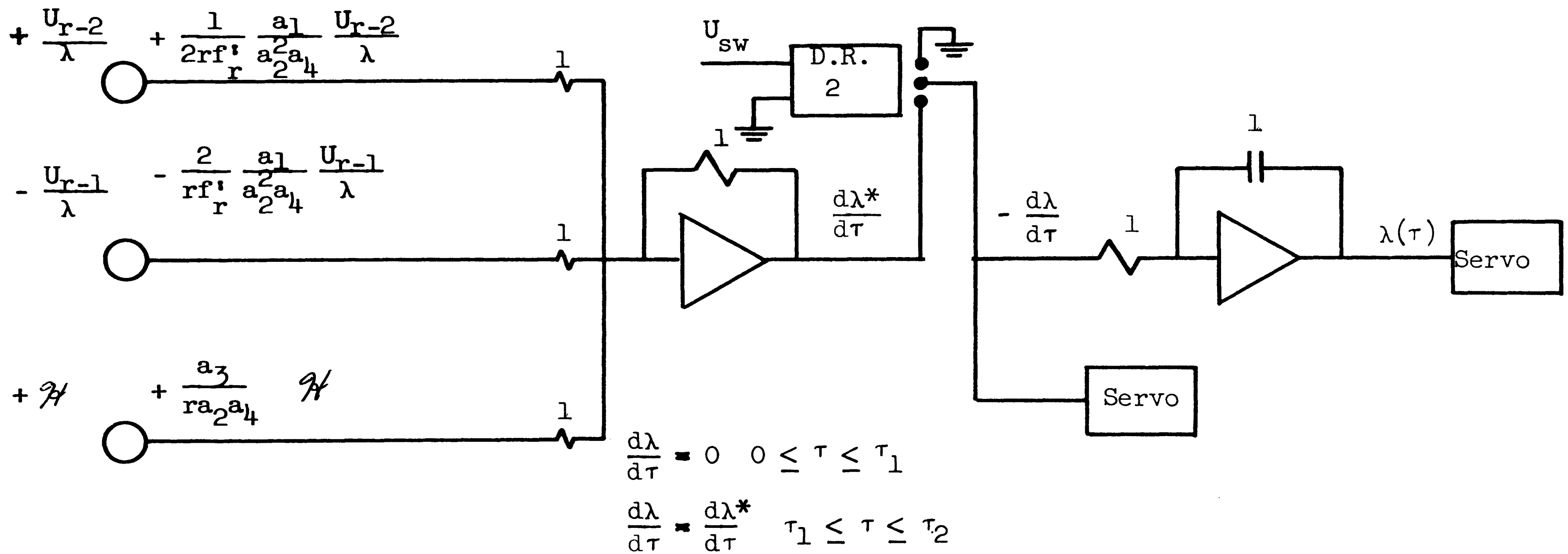


Figure 19d. Computer Schematic for the Formation of $d\lambda/d\tau$ and $\lambda(\tau)$.

Thus the Jacobian is positive for C in the interval $0 < C < 1$ and requirement 2 above is satisfied. Also, if $k > 1$ and $0 < C < 1$, then $d^2\hat{x}/dx^2 = dJ/dx < 0$ and requirement 1 is satisfied also because the curve $x = [1/1-(1-C)^k]/[1-(1-Cx)^k]$ is convex upward.

When Equations (45) are used with $k = 2$, a uniform finite difference station distribution in x causes the finite difference stations in $\hat{x}$ to be distributed parabolically. This is shown in Table VI.

A second nonlinear transformation is the exponential transformation given in Equations (46).

$$\hat{x} = \frac{1-e^{-kx}}{1-e^{-k}} \qquad \text{(a)}$$

$$\hat{t} = t \qquad \text{(b)} \qquad (46)$$

Again the interval in x is $0 \leq x \leq 1$ for $\hat{x}$ in the interval $0 \leq \hat{x} \leq 1$.

The Jacobian of this transformation is

$$J = \frac{d\hat{x}}{dx} = \frac{ke^{-kx}}{1-e^{-k}}$$

Thus $J > 0$ for all $k > 0$.

Since $d^2x/dx^2 = -k^2e^{-kx}/1-e^{-k}$, the curve $x = 1-e^{-kx}/1-e^{-k}$ is convex upward in the interval $0 \leq x \leq 1$ and must lie above the line $\hat{x} = x$ as required.

The station locations for a six station problem are shown in Table VI for the power transformation, Equations (45), and the exponential transformation, Equations (46). The values of $f'_n = d\hat{x}/dx$ are also shown at each station.

The stations can be grouped more closely near the heated boundary by increasing C and k in Equations (45) or (46). While the approximation of the temperature gradients improves in these areas the increased accuracy is not without its price. As the stations are grouped closer together, the temperature difference between stations decreases. In the computing circuits the differences are taken in the summing amplifiers and depend upon the coefficient potentiometer settings on the inputs. Thus as the differences become smaller the errors in the coefficient potentiometer settings become larger by comparison and eventually limit the accuracy of the solution. The same inaccuracies become predominant also when very large numbers of finite difference increments are used to improve the accuracy of the solution.

TABLE VI

STATION POSITIONS AND $d\hat{x}/dx$ FOR SIX STATIONS
LINEAR, PARABOLIC, AND EXPONENTIAL STATION DISTRIBUTIONS

	n						
n	0	1	2	3	4	5	6
x_n	0	0.1667	0.3333	.5000	.6667	.8333	1.0000
$\hat{x}_n = \frac{1-(1-0.7x)^2}{0.91}$	0	0.2414	0.4530	.6346	.7863	.9081	1.0000
$f'_n = \frac{1.4(1-0.7x)}{0.91}$	1.538	1.346	1.179	1.0000	.821	.641	.462
$\hat{x}_n = \frac{1-e^{-x}}{0.37}$	0	0.2428	.4483	.6225	.7698	.8945	1.0000
$f'_n = \frac{e^{-x}}{0.37}$	1.582	1.3392	1.134	.9595	.8122	.6875	0.5820

Computer Solution Results

The finite slab problem was solved by means of the same electronic differential analyzer used for the semi-infinite solid solution above with six finite difference increments. A linear distribution of stations in $\hat{x}$

and a parabolic distribution, $\hat{x} = \frac{1-(1-Cx)^2}{1-(1-C)^2}$, with C = 0.7 were used. Samples of these solutions are presented in Appendix G. In each case u_6 was determined explicitly.

The time t_1, at which melting starts, and the time, t_m, at which melting is completed when the heating rate is a constant are shown for the linear and parabolic station groupings in Table VII for three thickness, s = 0.5, 1.0, and 2.0. In all these cases the initial temperature was g_n = -1.0 and the heating rate was $H = -(1-g_o) = -2.0$.

The time, t_m, obtained from the computer agrees very closely with that obtained from the continuous case. The value to t_m for the continuous case was obtained from the equation $t_m = [(g-1)/H]\, \ell(o)$ which is derived in Appendix H. The errors in this value are about one per cent.

The value for the time at which melting starts is obtained for the continuous medium by solving Equations (39) during the interval $0 \leq \hat{t} \leq \hat{t}_1$ in terms of a Fourier series. The results of this solution are given here for a constant heating rate and a uniform initial temperature.

$$\hat{u}(\hat{x},\hat{t}) = \hat{g} + \frac{1}{6}\hat{H}\hat{\ell}(0) - \frac{\hat{H}}{2}\hat{\ell}^2(0)\,\hat{x} - \frac{1}{\ell(0)}\hat{H}\,\hat{t} + \frac{2\hat{H}\hat{\ell}(0)}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}e^{-\frac{n^2\pi^2}{\ell^2(0)}t}\cos n\pi x \qquad (47)$$

$$0 \leq \hat{t} \leq \hat{t}_1, \quad 0 \leq \hat{x} < 1$$

The value of $\hat{t}_1$ is obtained from this equation by solving it numerically.

The values of t_1 obtained from the computer and from Equation (47) are compared in Table VII. When $\ell(0) = 0.5$, these values agree

TABLE VII

t_1 AND t_m FOR SEVERAL THICKNESSES AND STATION GRUPINGS

$H = -2;\ g_n = -1.0$		t_1			t_m		
$\ell(0)$	$\hat{x}_n$	Continuous	Computer	% Error	Continuous	Computer	% Error
0.5	$\hat{x}_n = n/r$	0.167	0.164	-1.8	0.500	0.494	-1.2
0.5	$\frac{1-(1-.7x_n)^2}{.91}$	0.167	0.164	-1.8	0.500	0.494	-1.2
1.0	$\hat{x}_n = n/r$	0.19598	0.178	-9.2	1.000	0.986	-1.14
1.0	$\frac{1-(1-.7x_n)^2}{.91}$	0.19598	0.199	+1.5	1.000	0.983	-1.7
2.0	$\hat{x}_n = n/r$	0.1963	0.160	-18.4	2.000	1.95	-2.5
2.0	$\frac{1-(1-.7x_n)^2}{.91}$	0.1963	0.232	+18.4	2.000	1.98	-1.0
∞	$\ell n\ n/r$	0.1963	0.196	0			

within less than two per cent. The regrouping of stations does not affect appreciably this value. When $\ell(0) = 1.0$, the percentage error in t_1 increases to -9.2%. Here regrouping the stations lowers this error to 1.5%.

When $\ell(0) = 2.0$, the error in t_1 becomes -18.4%. Regrouping the stations makes the error +18.4% which is still large. Grouping the stations more closely together near the heated boundary might decrease this error still further. However, when the value of t_1 from the continuous case (0.1963) is compared to t_1 for the continuous semi-infinite case (0.1963), from Equation (29), it is apparent that the slab with $\ell(0) = 2.0$ behaves as a semi-infinite body during warm-up. This is further corroborated by Figure G5a and G6a in which u_o does not vary until after melting starts. Thus the semi-infinite formulation could be used with good accuracy for this portion of the problem at least for initial temperatures $g_o \leq -1$.

The decrease in accuracy with an increase in initial thickness was expected because the same number of stations was used in each case. As the initial thickness increases the accuracy of the gradient approximations must decrease because the error in the finite difference approximations varies as the square of the increment.

The value of u_6 for the continuous case is plotted on Figures G3a and G4a. These values were obtained from Equation (47). The improvement in accuracy with the regrouping of stations is very apparent when these two figures are compared. The maximum percentage error when the stations are linearly distributed in $\hat{x}$ is 22%; for the parabolic distribution it is 10.3%.

The effect of the station regrouping is to make the differences between adjacent stations more nearly uniform compared to the linear station grouping in $\hat{x}$. This can be seen by comparing Figure G1a with G2a, G3a with G4a, and G5a with G6a. A possible desideratum for station grouping could be that the temperatures be linearly distributed over the stations in the part of the solution of most interest.

Figures G7 and G8 are solutions of the melting problem for a heat flux of the form $H = 1 - \cos 4\pi t$. It can be seen that with this heat input melting starts at t_1 and ends at t_2. The accuracy of the solution can be checked by considering the energy exchange during the heating. The total heat input is

$$Q = \int_0^{\frac{1}{2}} (1 - \cos 4\pi t)\, dt = \frac{1}{2}$$

From Figure G7a the equilibrium temperature is -0.048. The rise in temperature was thus 0.952 and the heat required to raise the remaining part of the slab to this temperature was

$$Q_1 = 0.952\left(\frac{1}{2} - 0.036\right) = 0.440$$

From Figure G7b, 0.036 of the slab was melted. The heat required to raise this much of the slab to 0 was

$$Q_2 = 0.036$$

The heat required to melt 0.036 of the slab is

$$Q_3 = 0.036$$

The total heat required is then

$$Q = Q_1 + Q_2 + Q_3 = 0.512$$

The error is

$$\frac{Q - (Q_1 + Q_2 + Q_3)}{Q} = 0.06$$

This error could be caused by inaccuracies in reading the curves.

When Figure G8 is compared with Figure G7, the effect of the station regrouping is seen to be to spread out the temperature curves and make the temperature increment between stations more nearly equal as noted above.

In this chapter the finite slab has been analyzed and solutions obtained on the electronic differential analyzer. The results are not as striking as those obtained with the semi-infinite solid, but it has been shown that the accuracy of the solutions can be improved appreciably by regrouping the finite difference stations and that, for slabs of such thickness that the finite thickness formulation gives unacceptable errors, the semi-infinite formulation can be used to reduce the errors during the warming phase and the first part of melting.

CHAPTER V

CONCLUSION

The problem of the diffusion equation with a free boundary has been analyzed in this dissertation for both semi-infinite and finite thickness conducting media in which diffusion proceeds in one dimension. The object of the analysis was to render the problem into a form which could be readily solved on the differential analyzer and which would yield accurate solutions with a modest amount of computing equipment. It is believed that these objectives have been attained.

Description of the Solution

The problem equations were first made non-dimensional. The particular dimensionless ratios for the time variable, the space variable and the heating rate which are derived in this dissertation are believed to be original. In the process of eliminating the dimensions, scale factors were introduced which allow physical temperature, time and space variables, and physical heating rates of any range to be related to any convenient non-dimensional range. It has also been shown that, for a constant heating rate, these scale factors can be chosen to eliminate the heating rate. This reduces the semi-infinite problem to one parameter and the finite slab problem to two. The effect of varying these scale factors is to expand or contract the space and time variables in the non-dimensional variable domain.

For the semi-infinite solid the infinite space variable was reduced to the unit interval. This permitted the approximation of the derivatives of the temperature with respect to the space variable by means of the finite differences over the entire interval.

The finite difference approximation reduced the problem to a set of ordinary differential equations which could be solved on the differential analyzer. It was shown for a uniform initial temperature distribution and a constant heating rate that during warming the transient solution of these differential equations is a sum of decaying exponentials plus a constant at each station, while the solution of the continuous case proceeds as the square root of time at the heated surface. Thus the error between the continuous and finite difference solutions at the heated boundary could eventually become unbounded. However, it was also shown how the dimensionless heating rate could be chosen to keep this error very small until melting started. Further it was shown that this particular choice of constant heating rate caused the steady state melting rate obtained from the finite difference equations to be exactly the same as that for the continuous case.

Two methods for determining the heated boundary temperature during the warming phase were developed. The implicit method has the advantages of (1) allowing the initial boundary temperature to be arbitrary and (2) being more accurate than the explicit method. The explicit method (1) forces the boundary velocity to be zero when melting starts and (2) allows the heat balance at the boundary to be determined during melting in order to determine when the heating rate decreases sufficiently to stop melting.

The semi-infinite slab problem was solved on an electronic differential analyzer for three and for six finite difference increments. The results show that reasonable accuracy can be obtained with six increments. The one-parameter problem for constant heating rate and uniform initial conditions was solved for several initial temperatures. These solutions are presented in Appendix E. The problem was also solved for a heat pulse of the form $(1-\cos 2\pi t)$ to demonstrate the ability of the electronic differential analyzer to solve the problem with arbitrary heating rates. Samples of these solutions are included in Appendix F.

The finite thickness slab solution was made more accurate by grouping the stations closer together near the heated boundary. It was shown further that when the slab became too thick to obtain accurate results with only six finite difference increments, the semi-infinite slab formulation could be used during the warming phase to obtain accurate results for a constant heating rate. This problem also was solved for a heat pulse of the form $(1-\cos 4\pi t)$ and sample solution curves are presented in Appendix G.

Extensions of the Solution

The problem considered in this dissertation has been described in terms of heat flow and the melting of a solid. The solution obtained is directly applicable, in its non-dimensional form, to any problem described by the diffusion equation with a free boundary of this type. Thus it is directly extended to problems of mass diffusion, crystal growth, sublimation, and neutron diffusion and the control of nuclear reactors. In each case, of course, one dimensional diffusion must occur and the equivalent of perfect ablation must occur. Also, suitable non-dimensional ratios would have to be derived to relate the non-dimensional solution to the dimensional problem.

The solution obtained in this dissertation is limited to the case of perfect ablation. It can be extended to the two phase problem in which the liquid and solid phase remain in contact at the free boundary by writing the equations for each phase. The space variable must be normalized with respect to the liquid phase thickness in the equations for this phase and with respect to the solid phase thickness in the solid phase equations. The finite difference approximations could then be applied. Care would have to be exercised in approximating the gradient at the free boundary, where it is discontinuous. This difficulty could be overcome by using forward and backward differences at this point. Care would also have to be exercised during the warming period and just after melting starts, because the liquid thickness would be zero during these times and the equations have a singularity when the thickness is zero. This could be overcome by starting melting with a small, non-zero liquid thickness and assuming that the melting rate would be linear just after melting starts.

In order to solve the two-phase problem it must be assumed that there are no convection currents in the liquid. This is a very restrictive assumption and is unrealistic after the liquid thickness becomes large enough to allow free motion of the fluid. This restricts the validity of the solution.

The one dimensional heat flow problem has been described in this dissertation in a Cartesian reference system. It can also be described as radial heat flow in cylindrical or spherical coordinates. The same methods could be used in the latter two coordinate systems.

The extension of the results of this dissertation to two and three dimensional heat flow is complicated by having to determine what occurs at the "corners". No attempt has been made in this dissertation to solve this "corner" problem.

The solution obtained here does provide a set of useful dimensionless ratios for the melting problem; provides for the selection of an optimum heating rate for the semi-infinite slab when the heating rate is constant; demonstrates inaccuracies inherent in the finite difference approach and developes means for avoiding large errors due to these inaccuracies; provides a means for solving the melting problem which yields very good accuracies with only a nominal amount of computing equipment; refines the solution of the finite slab by regrouping the stations and shows that the semi-infinite solution may be employed for slabs sufficiently thick to cause large errors when using only a few finite difference increments.

APPENDIX A

DIMENSIONAL ANALYSIS

Equations (1a-i) are expressed in terms of $\bar{\rho}$, $\bar{c}$, $\bar{k}$, L, $\bar{x}$, $\bar{t}$, $\bar{H}$, and $\bar{u}$. These physical quantities must have units of mass (M), length (D), time (T), and temperature (Θ) or combinations of these four basic quantities. The eight physical quantities and the four basic dimensions then allow, according to Buckingham's π theorem of dimensional analysis,[1] the formation of four dimensionless ratios. The problem can thus be expressed in terms of four dimensionless variables.

The problem will be analyzed in terms of $\bar{x}$, $\bar{t}$, $\bar{u}$, and $\bar{H}$. That these form a dimensionally independent set may be shown by forming the product

$$\dim(\bar{x}^a \bar{t}^b \bar{H}^d \bar{u}^e) = M^0 D^0 T^0 \theta^0$$

Now

$$\dim(\bar{x}) = D$$
$$\dim(\bar{t}) = T$$
$$\dim(\bar{H}) = MT^{-3}$$
$$\dim(\bar{u}) = \theta$$

Thus

$$D^a T^b M^d T^{-3d} \theta^e = M^0 D^0 T^0 \theta^0$$

1 A complete discussion of Buckingham's theorem can be found in the books by Durand(9) and by Huntley.(21)

The following set of linear equations is obtained by setting the exponents of like factors equal.

$$a = 0$$
$$b - 3d = 0$$
$$c = 0$$
$$d = 0$$

Clearly the only solution is the trivial one thereby proving the dimensional independence of the chosen set.

The dimensionless ratios are determined by forming the product of $\bar{\rho}$, $\bar{c}$, $\bar{k}$, and L, each raised to arbitrary power, with the first power of $\bar{x}$, $\bar{u}$, $\bar{t}$, and $\bar{H}$. Thus four dimensionless ratios π_1, π_2, π_3, and π_4 are formed. The first of these is derived here.

Let

$$\pi_1 = \bar{\rho}^a \bar{c}^b \bar{k}^d L^e \bar{x}$$

Then

$$\dim(\pi_1) = M^0 D^0 T^0 \theta^0$$

$$\dim(\bar{\rho}^a \bar{c}^b \bar{k}^d L^e \bar{x}) = M^a D^{-3a} D^{2b} T^{-2b} \theta^{-b} M^d D^d T^{-3d} \theta^{-d} D^{2e} T^{-2e}$$

Equating exponents of like factors yields:

$$a + d = 0 \qquad (M)$$
$$-3a + 2b + d + 2e + 1 = 0 \qquad (D)$$
$$-2b - 3d - 2e = 0 \qquad (T)$$
$$-b - d = 0 \qquad (\theta)$$

The solution of these equations yields

$$\pi_1 = \frac{\bar{\rho}\,\bar{c}}{\bar{k}} \sqrt{L}\; \bar{x} \; .$$

Similarly

$$\Pi_2 = \frac{\bar{\rho}\,\bar{c}\,L}{\bar{k}}\,\bar{t}$$

$$\Pi_3 = \frac{1}{\bar{\rho}\,L^{3/2}}\,\bar{H}$$

$$\Pi_4 = \frac{\bar{c}}{L}\,\bar{u}$$

In order to make Equations (1a-i) dimensionless, the following dimensionless variables are defined.

$$\tilde{x} = K_1\,\Pi_1$$

$$\tilde{t} = K_2\,\Pi_2\,.$$

$$\tilde{H} = K_3\,\Pi_3$$

$$\tilde{u} = \Pi_4 - \frac{\bar{c}}{L}\,\bar{u}_c$$

In these definitions K_i is a non-dimensional constant of arbitrary value. Its introduction here allows the non-dimensional variables to be chosen in any convenient range and still be related to any given set of dimensional variables.

The numerical values values of the non-dimensional variables are presented in Table A-II for several common materials whose physical properties are listed in Table A-I. In all cases the physical properties are assumed to be independent of temperature.

In many ablation problems heating rates on the order of 10^2 to 10^3 are of interest. For these values of H, a convenient value for K would be 10^{-4}.

TABLE A1

PHYSICAL CONSTANTS FOR SEVERAL COMMON MATERIALS[1]

Material	$\bar{\rho}$ $\frac{gm}{cm^3}$	$\bar{k}$ $\frac{cal}{gm\ sec\ ^\circ C}$	$\bar{c}$ $\frac{cal}{gm\ ^\circ C}$	L $\frac{cal}{gm}$	$\bar{u}_c$ °C
Aluminum	2.7	1.01	0.25	76.8	658
Copper	8.9	0.858	0.096	42.0	1083
Ice	0.92	0.004	0.50	79.7	0
Magnesium	1.8	0.376	0.28	70.1	615
Silver	10.3	0.992	0.07	21.1	961
Stainless Steel	7.8	0.107	0.12	48(iron)	1475
Water	1.0	0.0014	1.0	79.7	0

[1]All data are from the Handbook of Chemistry and Physics, 36th Edition, Chemical Rubber Publishing Co., Cleveland, Ohio, (1954).

TABLE A2

DIMENSIONLESS VARIABLES FOR SEVERAL COMMON MATERIALS

Material	$\tilde{x} \times 10^{-4}$	$\tilde{t} \times 10^{-9}$	$\tilde{u} \times 10^{3}$	$\tilde{H} \times 10^{8}$
Aluminum	$3.78\ \bar{x}$	$2.14\ \bar{t}$	$3.26(\bar{u}-\bar{u}_c)$	$8.50\ \bar{H}$
Copper	$4.17\ \bar{x}$	$1.75\ \bar{t}$	$2.28(\bar{u}-\bar{u}_c)$	$6.37\ \bar{H}$
Ice	$0.663\ \bar{x}$	$382\ \bar{t}$	$6.27(\bar{u}-\bar{u}_c)$	$23.7\ \bar{H}$
Magnesium	$7.25\ \bar{x}$	$3.94\ \bar{t}$	$3.99(\bar{u}-\bar{u}_c)$	$14.6\ \bar{H}$
Silver	$21.6\ \bar{x}$	$6.41\ \bar{t}$	$3.32(\bar{u}-\bar{u}_c)$	$15.5\ \bar{H}$
Stainless Steel	$39.2\ \bar{x}$	$17.6\ \bar{t}$	$2.87(\bar{u}-\bar{u}_c)$	$5.97\ \bar{H}$

In this table

$\bar{x}$ is measured in centimeters

$\bar{t}$ is measured in seconds

$\bar{u}$ is measured in ° Centigrade

$\bar{H}$ is measured in gram calories/square centimeter-second

$K_1 = K_2 = K_3 = 1$

APPENDIX B

STEADY STATE SOLUTION FOR THE SEMI-INFINITE SLAB WITH A CONSTANT HEATING RATE

In steady state the temperature distribution in the (x,t) domain becomes a function of the space variable, x, only when the heating rate, H, is constant. This provides a means for solving Equations (8) for the steady state boundary velocity and the temperature distribution.

Since the temperatures are functions of x only

$$\frac{\partial u}{\partial t} = 0. \tag{B1}$$

The boundary velocity, $d\ell/dt$, must be a constant, also, because

$$\frac{d\ell}{dt} = H - \frac{\partial u}{\partial x} \qquad x = 1 \tag{B2}$$

Here H is constant by hypothesis and $\left.\frac{\partial u}{\partial x}\right|_{x=1}$ is constant because it is evaluated at x=1. Therefore

$d\ell/dt = b$, a constant.

The steady state problem now may be stated:

$$x\frac{d}{dx}\left[x\frac{du}{dx}\right] - b\,x\frac{du}{dx} = 0 \qquad 0 \leq x \leq 1 \qquad \text{(a)}$$

$$u = 0 \qquad x = 1 \qquad \text{(b)} \quad \text{(B3)}$$

$$u = g_0 \qquad x = 0 \qquad \text{(c)}$$

Expand (B3a) to obtain

$$x\frac{du}{dx} + x^2\frac{d^2u}{dx^2} - b\,x\frac{du}{dx} = 0$$

This equation may be written

$$x\left[x\frac{d^2u}{dx^2} + (1-b)\frac{du}{dx}\right] = 0$$

The second factor is now equated to zero to obtain

$$x\frac{d^2u}{dx^2} + (1-b)\frac{du}{dx} = 0 \tag{B4}$$

Equation (B4) may be integrated once to obtain a first order differential equation. Thus

$$x\frac{du}{dx} - u + (1-b)u = C_1$$

or

$$x\frac{du}{dx} - bu = C_1 \tag{B5}$$

where C_1 is a constant of integration.

The solution of (B5) is

$$u = C_2 e^{\int \frac{b}{x}dx} + e^{\int \frac{b}{x}dx}\int \frac{C_1}{x} e^{-\int \frac{b}{x}dx} dx$$

or

$$u = C_2 e^{\ln x^b} + C_1 e^{\ln x^b}\int \frac{1}{x} e^{\ln x^b} dx$$

But

$$e^{\ln x^b} = x^b$$

Thus

$$u = C_2 x^b + C_1 x^b \int \frac{x^{-b}}{x} dx$$

$$u = C_2 x^b + \frac{C_1}{b} \tag{B6}$$

The constants are evaluated by means of the boundary conditions (B3b) and (B3c). Thus

$$C_1 = b g_0$$

and

$$C_2 = -g_0$$

The solution, with constants evaluated, is

$$u = (1 - x^b) g_0 \tag{B7}$$

Equation (B2) provides a means for computing b:

$$\left.\frac{du}{dx}\right|_1 = -b x^{b-1} g_0 = -b g_0$$

$$b = \frac{H}{1 - g_0} \tag{B8}$$

Equations (B7) and (B8) indicate that if $H = 1 - g_0$ then the temperature will be linearly distributed in the x coordinate. This property will be used to demonstrate that the finite difference approximation solution in steady state is identical to the continuous medium steady state solution for this heating rate.

APPENDIX C

FINITE DIFFERENCE APPROXIMATIONS AND ERRORS

When Equations (5) of Chapter II are transformed by means of Equations (8), the results are Equations (C1).

$$\begin{aligned}
&\phi_\rho(x,u)\phi_c(x,u)\frac{\partial u}{\partial t} = x\frac{\partial}{\partial x}\left[x\,\phi_k(x,u)\frac{\partial u}{\partial x}\right] - \phi_\rho(x,u)\,\phi_c(x,u)\,x\frac{d\ell}{dt}\frac{\partial u}{\partial x} && 0 \le x \le 1;\ 0 \le t && \text{(a)}\\
&H(t) = \phi(1,u)\frac{d\ell}{dt} + \phi_k(1,u)\frac{\partial u}{\partial x} && x=1;\ t \geq 0 && \text{(b)}\\
&\frac{d\ell}{dt} = 0 && 0 \le t \le t_1 && \text{(c)}\\
&u < 0 && x=1;\ 0 \le t < t_1 && \text{(d)}\\
&\frac{d\ell}{dt} > 0 && t_1 < t < t_2 && \text{(e)}\\
&u = 0 && x = 1;\ t_1 \le t \le t_2 && \text{(f)}\\
&u = g_0 && x=0,\ t \geq 0 && \text{(g)}\\
&u = g(x) && 0 \le x \le 1;\ t = 0 && \text{(h)}\\
&\ell = 0 && t = 0 && \text{(i)}
\end{aligned} \qquad \text{(C1)}$$

In these equations the derivatives with respect to x are to be replaced by finite difference approximations. The equations are written in such a manner that only the finite difference approximation of the first derivative is required. These approximation formulae and the errors introduced by their use will be derived for a general function, $\theta(x,t)$, and then the formulae will be applied to Equations (C1).

Central Difference Approximations

To obtain the finite difference approximations the conducting medium is divided into r increments as in Figure C1.

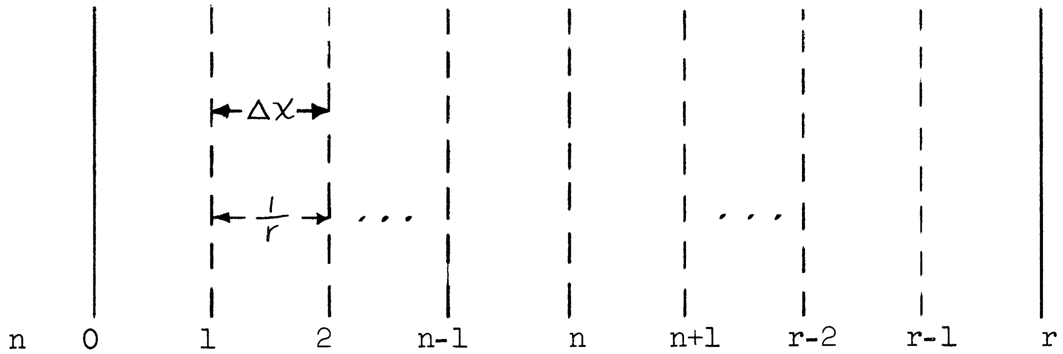


Figure C1. Finite Difference Increments.

The function $\theta(x,t)$ is first expanded in a Taylor series in x about the n^{th} station. Thus

$$\theta(x+\Delta x,t) = \theta_n(t) + \frac{\partial \theta}{\partial x}\Big|_{x_n}\Delta x + \frac{\partial^2\theta}{\partial x^2}\Big|_{x_n}\frac{(\Delta x)^2}{2!} + \ldots + \frac{\partial^m\theta}{\partial x^m}\Big|_{x_n}\frac{(\Delta x)^m}{m!} + \ldots \qquad \text{(C2)}$$

By means of Equation (C2), $\theta(x_{n+1},t)$ can be evaluated with

$$x_n = n\,\Delta x = \frac{n}{r}.$$

Thus

$$\theta_{n+1} = \theta(x_{n+1},t) = \theta_n + \frac{\partial\theta}{\partial x}\Big|_{x_n}\Delta x + \frac{\partial^2\theta}{\partial x^2}\Big|_{x_n}\frac{(\Delta x)^2}{2!} + \frac{\partial^3\theta}{\partial x^3}\Big|_{x_n}\frac{(\Delta x)^3}{3!} + \ldots$$

and

$$\theta_{n-1} = \theta_n - \frac{\partial\theta}{\partial x}\Big|_{x_n}\Delta x + \frac{\partial^2\theta}{\partial x}\Big|_{x_n}\frac{\Delta x}{2!} - \frac{\partial^3\theta}{\partial x^3}\Big|_{x_n}\frac{(\Delta x)^3}{3!} + \ldots$$

θ_n and $\left.\frac{\partial^2\theta}{\partial x^2}\right|_{x_n}\frac{(\Delta x)^2}{2!}$ and all even order derivative terms are eliminated from these two equations by subtracting the second from the first. Thus

$$\theta_{n+1}-\theta_{n-1} = 2\left.\frac{\partial\theta}{\partial x}\right|_{x_n}\Delta x + 2\frac{\partial^3\theta}{\partial x^3}\frac{(\Delta x)^3}{3!}+\ldots$$

The approximation for $\left.\frac{\partial\theta}{\partial x}\right|_{x_n}$ is obtained by solving for $\left.\frac{\partial\theta}{\partial x}\right|_{x_n}$ from this equation and neglecting all terms of higher order than the first in (Δx). The required approximation is

$$\left.\frac{\partial\theta}{\partial x}\right|_{x_n} \approx \frac{\theta_{n+1}-\theta_{n-1}}{2\Delta x} \tag{C3}$$

and the error introduced by neglecting the higher order terms is

$$\epsilon = -\left.\frac{\partial^3\theta}{\partial x^3}\right|_{x_n}\frac{(\Delta x)^2}{3!} - \left.\frac{\partial^5\theta}{\partial x^5}\right|_{x_n}\frac{(\Delta x)^4}{5!} - \ldots$$

If Δx is small and all derivatives exist as required by the Taylor series expansion, the error will then vary as $(\Delta x)^2$.

Equation (C3) is the central difference approximation of the first derivative of θ with respect to x evaluated at the n^{th} finite difference station. It requires that the values of θ at each adjacent station be known.

At the right hand boundary, the r^{th} station, it is impossible to evaluate θ at a station to the right. Thus backward differences must be employed to evaluate $\left.\frac{\partial\theta}{\partial x}\right|_{x_r}$.

Backward Difference Approximations

The function, $\theta(x,t)$, is now expanded about the r^{th} station (refer to Figure C1). Thus

$$\theta(x_r+\Delta x, t) = \theta_r + \left.\frac{\partial\theta}{\partial x}\right|_{x_r}\Delta x + \frac{\partial^2\theta}{\partial x^2}\frac{(\Delta x)^2}{2!}+\ldots$$

Then $\left.\frac{\partial \theta}{\partial x}\right|_{x_r}$ may be approximated by means of the values of θ at the r^{th} and $(r-1)^{st}$ stations.

$$\theta_r = \theta_r$$

$$\theta_{r-1} = \theta_r - \left.\frac{\partial \theta}{\partial x}\right|_{x_r} \Delta x + \left.\frac{\partial^2 \theta}{\partial x^2}\right|_{x_r} \frac{(\Delta x)^2}{2!} - \left.\frac{\partial^3 \theta}{\partial x^3}\right|_{x_r} \frac{(\Delta x)^3}{3!} + \ldots$$

Solving for $\left.\frac{\partial \theta}{\partial x}\right|_{x_r}$ from these two equations yields

$$\left.\frac{\partial \theta}{\partial x}\right|_{x_r} = \frac{\theta_r - \theta_{r-1}}{\Delta x} + \left.\frac{\partial^2 \theta}{\partial x^2}\right|_{x_r} \frac{\Delta x}{2!} - \left.\frac{\partial^3 \theta}{\partial x^3}\right|_{x_r} \frac{(\Delta x)^2}{3!} + \ldots \qquad (C4)$$

Equation (C4) is the backward difference approximation of when all terms of degree greater than unity if (Δx) are neglected. The error introduced by this approximation is

$$\epsilon = \left.\frac{\partial^2 \theta}{\partial x^2}\right|_{x_r} \frac{\Delta x}{2!} - \left.\frac{\partial^3 \theta}{\partial x^3}\right|_{x_r} \frac{\Delta x^2}{3!} + \ldots$$

In this case the error varies as (Δx) for small Δx.

In Equation (C4) it is required that the value of the function θ be known at two points. This equation employs first order differences only. A more accurate approximation is obtained by using second order differences in which the values of θ at the r^{th}, $r-1^{st}$, and $r-2^{nd}$ stations are required. These values are

$$\theta_r = \theta_r$$

$$\theta_{r-1} = \theta_r - \left.\frac{\partial \theta}{\partial x}\right|_{x_r} \Delta x + \left.\frac{\partial^2 \theta}{\partial x^2}\right|_{x_r} \frac{(\Delta x)^2}{2!} - \left.\frac{\partial^3 \theta}{\partial x^3}\right|_{x_r} \frac{(\Delta x)^3}{3!} + \left.\frac{\partial^4 \theta}{\partial x^4}\right|_{x_r} \frac{(\Delta x)^4}{4!} - \ldots$$

$$\theta_{r-2} = \theta_r - \left.\frac{\partial \theta}{\partial x}\right|_{x_r} (2\Delta x) + \left.\frac{\partial^2 \theta}{\partial x^2}\right|_{x_r} \frac{(2\Delta x)^2}{2!} - \left.\frac{\partial^3 \theta}{\partial x^3}\right|_{x_r} \frac{(2\Delta x)^3}{3!} + \left.\frac{\partial^4 \theta}{\partial x^4}\right|_{x_r} \frac{(2\Delta x)^4}{4!} - \ldots$$

When θ_r and $\left.\frac{\partial^2\theta}{\partial x^2}\right|_{x_r}$ are eliminated from the right members of these three equations the result is

$$\frac{3}{2}\theta_r - 2\theta_{r-1} + \frac{1}{2}\theta_{r-2} = \left.\frac{\partial\theta}{\partial x}\right|_{x_r} - \frac{1}{3}\left.\frac{\partial^3\theta}{\partial x^3}\right|_{x_r}(\Delta x)^3 + \ldots$$

This equation is solved for $\left.\frac{\partial\theta}{\partial x}\right|_{x_r}$ to obtain

$$\left.\frac{\partial\theta}{\partial x}\right|_{x_r} = \frac{1}{\Delta x}\left[\frac{3}{2}\theta_r - 2\theta_{r-1} + \frac{1}{2}\theta_{r-2}\right] + \frac{1}{3}\left.\frac{\partial^3\theta}{\partial x^3}\right|_{x_r}(\Delta x)^2 + \ldots$$

The approximation of $\left.\frac{\partial\theta}{\partial x}\right|_{x_r}$ is thus

$$\left.\frac{\partial\theta}{\partial x}\right|_{x_r} \approx \frac{1}{\Delta x}\left[\frac{3}{2}\theta_r - 2\theta_{r-1} + \frac{1}{2}\theta_{r-2}\right] \tag{C5}$$

The error introduced by this approximation is

$$\epsilon = \frac{1}{3}\left.\frac{\partial^3\theta}{\partial x^3}\right|_{x_r}(\Delta x)^2 + \ldots$$

This error varies as $(\Delta x)^2$ and thus is of the same order of magnitude as the central difference error of Equation (C3).

Equations (C3), (C4), and (C5) will now be used to eliminate the x variable from Equations (C1).

The Finite Difference Approximations of the Problem Equations

The space variable, x, is eliminated from Equation (C1a) for the stations n=1 through n=r-1 by means of the central difference approximation of Equation (C3). Equation (C1a) is

$$\phi_\rho(x,u)\phi_c(x,u)\frac{\partial u}{\partial t} = x\frac{\partial}{\partial x}\left[x\,\phi_k(x,u)\frac{\partial u}{\partial x}\right] - \phi_\rho(x,u)\phi_c(x,u)\,x\frac{d\ell}{dt}\frac{\partial u}{\partial x}$$

The first term of the right member is considered first.

Let

$$\theta = x\,\phi_k(x,u)\frac{\partial u}{\partial x}$$

$$\phi_{k_n} = \phi_k[x_n,\ u(x_n,t)]$$

Then, by means of Equation (C3)

$$\frac{\partial}{\partial x}\left[x\,\phi_k(x,u)\frac{\partial u}{\partial x}\right] \approx \frac{1}{\Delta x}\left[x_{n+\frac{1}{2}}\ \phi_{k_{n+\frac{1}{2}}}\frac{\partial u}{\partial x}\Big|_{x_{n+\frac{1}{2}}} - x_{n-\frac{1}{2}}\,\phi_{k_{n-\frac{1}{2}}}\frac{\partial u}{\partial x}\Big|_{x_{n-\frac{1}{2}}}\right]$$

But

$$\Delta x = \frac{1}{r}\ ;\ \ x_n = \frac{n}{r}$$

$$\frac{\partial u}{\partial x}\Big|_{x_{n+\frac{1}{2}}} \approx \frac{1}{\Delta x}(u_{n+1} - u_n)$$

$$\frac{\partial u}{\partial x}\Big|_{x_{n-\frac{1}{2}}} \approx \frac{1}{\Delta x}(u_n - u_{n-1})$$

Thus

$$x\frac{\partial}{\partial x}\left[x\,\phi_k\frac{\partial u}{\partial x}\right] \approx n\Big\{(n-\tfrac{1}{2})\phi_{k_{n-\frac{1}{2}}}u_{n-1} - \left[(n+\tfrac{1}{2})\phi_{k_{n+\frac{1}{2}}} + (n-\tfrac{1}{2})\phi_{k_{n-\frac{1}{2}}}\right]u_n + (n+\tfrac{1}{2})\,\phi_{k_{n+\frac{1}{2}}}\,u_{n+1}\Big\}$$

The second term is approximated by

$$\phi_\rho\,\phi_c\,x\frac{d\ell}{dt}\frac{\partial u}{\partial x} \approx \phi_{\rho_n}\phi_{c_n}\frac{d\ell}{dt}\,\frac{n}{2}(u_{n+1} - u_{n-1})$$

since

$$\frac{\partial u}{\partial x}\Big|_{x_r} \approx \frac{r}{2}(u_{n+1} - u_{n-1})$$

Thus Equation (Cla) is approximated by Equation (C6).

$$\begin{aligned}\frac{du_n}{dt} = {} & \frac{\phi_{k_{n-\frac{1}{2}}}}{\phi_{\rho_n}\phi_{c_n}}\,n(n-\tfrac{1}{2})\,u_{n-1} + \frac{n}{2}\frac{d\ell}{dt}\,u_{n-1} \\ & - \left[\frac{\phi_{k_{n-\frac{1}{2}}}}{\phi_{\rho_n}\phi_{c_n}}\,n(n-\tfrac{1}{2}) + \frac{\phi_{k_{n+\frac{1}{2}}}}{\phi_{\rho_n}\phi_{c_n}}\,n(n+\tfrac{1}{2})\right]u_n \\ & - \frac{n}{2}\frac{d\ell}{dt}\,u_{n+1} + \frac{\phi_{k_{n+\frac{1}{2}}}}{\phi_{\rho_n}\phi_{c_n}}\,n(n+\tfrac{1}{2})\,u_{n+1} \qquad \text{(C6)}\end{aligned}$$

This is Equation (25b) of the text when $\phi_\rho = \phi_c = \phi_k = 1$.

When the heated boundary temperature is determined implicitly during warming, Equation (C1a) is used to obtain du_r/dt. During warming $d\ell/dt = 0$ and the second term of the right member of Equation (C1a) is zero. Also, from Equation (C1b)

$$H(t) = \phi_k(1,u)\frac{\partial u}{\partial x}\Big|_{x_r}$$

Let

$$\theta = x\,\phi_k \frac{\partial u}{\partial x}$$

Then

$$\frac{\partial}{\partial x}\left[x\,\phi_k(x,u)\frac{\partial u}{\partial x}\right]_{x_r} = 2r\left[\phi_{k_r}\frac{\partial u}{\partial x}\Big|_{x_r} - \phi_{k_{r-\frac{1}{2}}}\frac{\partial u}{\partial x}\Big|_{x_{r-\frac{1}{2}}}\frac{1}{r}(r-\tfrac{1}{2})\right]$$

Here $\Delta x = 1/2r$ and it is possible to use the half interval to improve the accuracy of the approximation. Thus

$$\frac{du_r}{dt} = \frac{2r}{\phi_{\rho_r}\phi_{c_r}}H - 2\frac{\phi_{k_{r-\frac{1}{2}}}}{\phi_{\rho_r}\phi_{c_r}}r(r-\tfrac{1}{2})u_r + 2\frac{\phi_{k_{r-\frac{1}{2}}}}{\phi_{\rho_r}\phi_{c_r}}r(r-\tfrac{1}{2})u_{r-1}\,.$$

Equation (C7) reduces to Equation (25c1) of the text when $\phi_\rho = \phi_c = \phi_k = 1$.

When second order backward differences are used to determine the boundary temperature implicitly let

$$\theta = x\,\phi_k(x,u)\frac{\partial u}{\partial x}$$

Again

$$H = \phi_k(1,u)\frac{\partial u}{\partial x}\Big|_{x_r}$$

In this case the following approximations are used

$$\frac{\partial u}{\partial x}\Big|_{r-\frac{1}{2}} \approx \frac{u_r - u_{r-1}}{1/r}$$

and

$$\frac{\partial u}{\partial x}\Big|_{r-1} \approx \frac{u_r - u_{r-2}}{2/r}$$

When these four relationships are substituted into Equation (C1a) with Equation (C5) in which $\Delta x = 1/2r$ the result is Equation (C8).

$$\frac{du_r}{dt} = \frac{3r}{\phi_{\rho_r}\phi_{c_r}} H - \frac{r}{2}(r-1)\frac{\phi_{k_{r-1}}}{\phi_{\rho_r}\phi_{c_r}} u_{r-2} + 2r(2r-1)\frac{\phi_{k_{r-\frac{1}{2}}}}{\phi_{\rho_r}\phi_{c_r}} u_{r-1} + \frac{r}{2}\left[(r-1)\frac{\phi_{k_{r-1}}}{\phi_{\rho_r}\phi_{c_r}} - 4(2r-1)\frac{\phi_{k_{r-\frac{1}{2}}}}{\phi_{\rho_r}\phi_{c_r}}\right] u_r \quad \text{(C8)}$$

Equation (C8) reduces to Equation (25C2) of the text.

Equation (C1b) is approximated with first order differences by means of Equation (C4) to obtain u_r explicitly. Thus

$$H(t) = \phi_{\rho_r}\frac{d\ell}{dt} + \phi_{k_r} r(u_r - u_{r-1}) \quad \text{(C9)}$$

Equation (C9) reduces to Equation (25C3) during warming when $\frac{d\ell}{dt} = 0$ if $\phi_\rho = \phi_k = 1$. It reduces to Equation (25f1) during melting when $u_r = 0$.

When second order backward difference approximations are used in Equation (C1b) the result is Equation (C10).

$$H(t) = \phi_{\rho_r}\frac{d\ell}{dt} + \phi_{k_r} r\left(\frac{3}{2}u_r - 2u_{r-1} + \frac{1}{2}u_{r-2}\right) \quad \text{(C10)}$$

Equation (C10) reduces to Equation (25C4) during warming and to Equation (25f2) during melting if $\phi_\rho = \phi_k = 1$.

<u>Additional Approximations for the Finite Slab</u>

To approximate the gradient at the insulated boundary, $x = 0$, a virtual station is placed at $x = -\Delta x$. The temperature at this virtual station is required to be such that

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \frac{u_1 - u_{-1}}{2\Delta x} = 0$$

Thus

$$u_1 = u_{-1} \tag{C11}$$

This boundary condition approximation could have been obtained also by ending the conducting medium at x = 1/2Δx and then forcing $u_o = u_1$.

The approximation of du_r/dt during warming to obtain the implicit formulation of u_r requires that Equation (40a) be written as

$$\frac{\partial u}{\partial t} = \frac{1}{f'(x)} \frac{1}{\ell(t)} \frac{\partial}{\partial x}\left[\frac{1}{f'(x)} \frac{1}{\ell(t)} \frac{\partial u}{\partial x}\right]$$

During warming, of course, $d\ell/dt$ is zero. The quantity, $\frac{1}{f'(x)}\frac{1}{\ell(t)}\frac{\partial u}{\partial x}$, is the heat flux at any point in the slab. Thus if F represents the flux and if a second order backward difference is used to approximate $\partial F/\partial x$ at the heated boundary with a finite difference interval of $\Delta x/2$, the result is

$$\frac{\partial F}{\partial x} = \frac{1}{\Delta x/2}\left[-\frac{3}{2}F_r + 2F_{r-\frac{1}{2}} - \frac{1}{2}F_{r-1}\right]$$

But

$$F_r = H$$

$$F_{r-\frac{1}{2}} = -\frac{1}{f'_{r-\frac{1}{2}}} \frac{1}{\ell(t)} \frac{u_r - u_{r-1}}{\Delta x}$$

$$F_{r-1} = -\frac{1}{f'_{r-1}} \frac{1}{\ell(t)} \frac{u_r - u_{r-2}}{2\Delta x}$$

Thus

$$\frac{\partial}{\partial x}\left[\frac{1}{f'(x)}\frac{1}{\ell(t)}\frac{\partial u}{\partial x}\right] \approx -3rH - \frac{r^2}{f'_{r-1}}\frac{u_{r-2}}{\ell(t)} + \frac{4r^2}{f'_{r-\frac{1}{2}}}\frac{u_{r-1}}{\ell(t)} - \left[\frac{4r^2}{f'_{r-\frac{1}{2}}} - \frac{r^2}{f'_{r-1}}\right]\frac{u_r}{\ell(t)} \tag{C12}$$

Only second order differences at the heated boundary are used for the finite slab.

APPENDIX D

EIGEN VALUES OF THE MATRIX A

The eigen values for the matrix, A, of Equation (26) are listed below for r = 3 through r = 7 for the explicit determination of u_r and for r = 2 through r = 6 for the implicit determination of u_r.

1. Explicit method, First differences

$$u_r = u_{r-1} + (1/r)H$$

	λ_n				
n	r = 3	r = 4	r = 5	r = 6	r = 7
1	-0.32056	- 0.25155	- 0.21413	- 0.19008	- 0.17310
2	-4.67945	- 3.17196	- 2.42348	- 2.15232	- 1.90842
3	-	-14.07558	- 9.91283	- 7.97425	- 6.82072
4	-	-	-29.34803	-21.31184	-17.34784
5	-	-	-	-50.87021	-37.86498
6	-	-	-	-	-78.88755

2. Explicit method, Second differences

$$u_r = 4/3\ u_{r-1} - 1/3\ u_{r-2} + 2/3r\ H$$

	λ_n				
n	r = 3	r = 4	r = 5	r = 6	r = 7
1	-0.21370	- 0.20967	- 0.19190	- 0.176217	- 0.16326
2	-3.11963	- 2.54390	- 2.19925	- 1.95634	- 1.77689
3	-	-11.12466	- 8.48754	- 7.13172	- 6.26940
4	-	-	-25.12152	-18.90817	-15.80432
5	-	-	-	-45.16184	-34.35316
6	-	-	-	-	-71.63703

3. Implicit method, First differences

$$du_r/dt = r(2r-1)u_{r-1} - r(2r-1)u_r + 2rH$$

	λ_n				
n	r = 2	r = 3	r = 4	r = 5	r = 6
1	-0.39445	- 0.28273	- 0.23174	- 0.20160	- 0.18132
2	-7.60555	- 3.80059	- 2.81687	- 2.32581	- 2.02428
3	-	-20.91580	-11.67256	- 8.85285	- 7.35941
4	-	-	-41.27602	-24.47488	-19.15244
5	-	-	-	-68.86556	-43.47948
6	-	-	-	-	-103.79791

4. Implicit method, Second differences

$$du_r/dt = -r/2(r-1)u_{r-2} + 2r(2r-1)u_{r-1} - r/2(7r-3)u_r + 3rH$$

	λ_n				
n	r = 2	r = 3	r = 4	r = 5	r = 6
1	-0.31534	- 0.27128	- 0.22794	- 0.19985	- 0.17792
2	-6.68466	- 3.67350	- 2.76862	- 2.30338	- 1.97764
3	-	-33.05158	-11.69403	- 8.80312	- 7.16368
4	-	-	-63.30421	-25.16050	-18.64387
5	-	-	-	-103.52709	-42.78379
6	-	-	-	-	-119.24925

All the eigen values that have been calculated for the matrix A, negative.

APPENDIX E

GENERAL SOLUTION CURVES FOR CONSTANT HEATING RATE

AND

UNIFORM INITIAL TEMPERATURE

The Semi-Infinite Solid

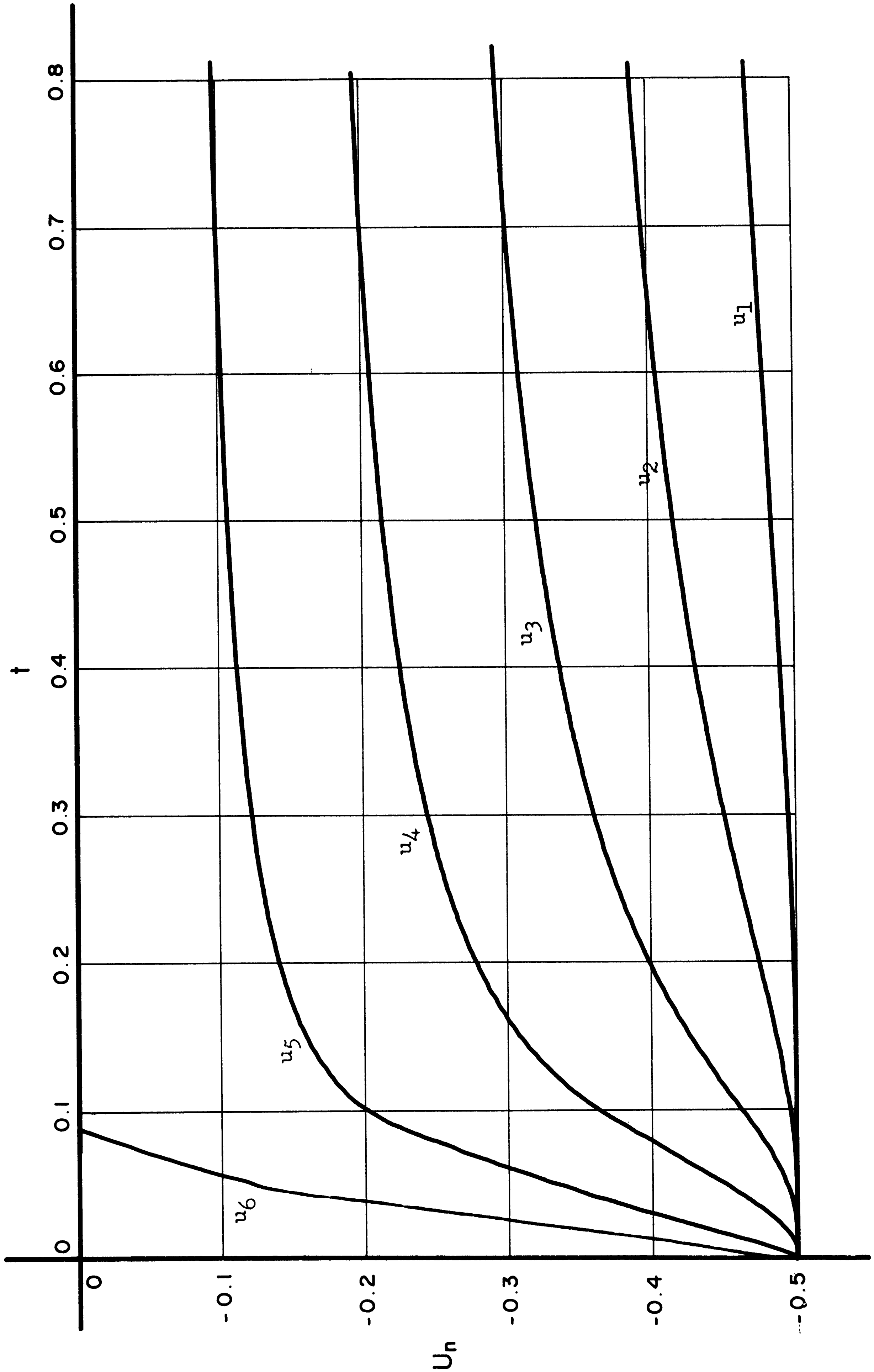


Figure E1. Temperature vs. Time
$g_n = -0.5$; $r = 6$

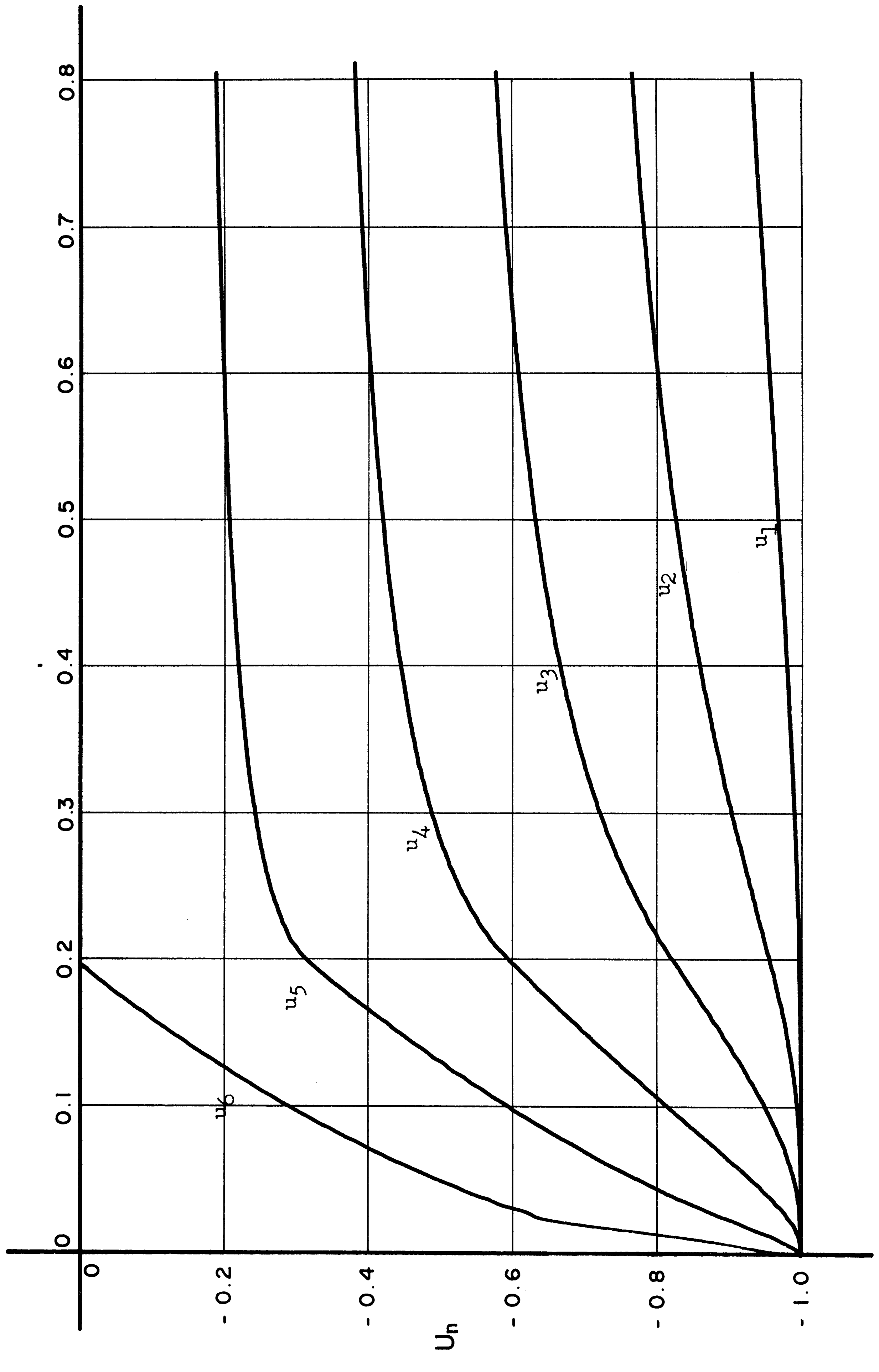


Figure E2. Temperature vs. Time
$g_n = -1.0$; $r = 6$

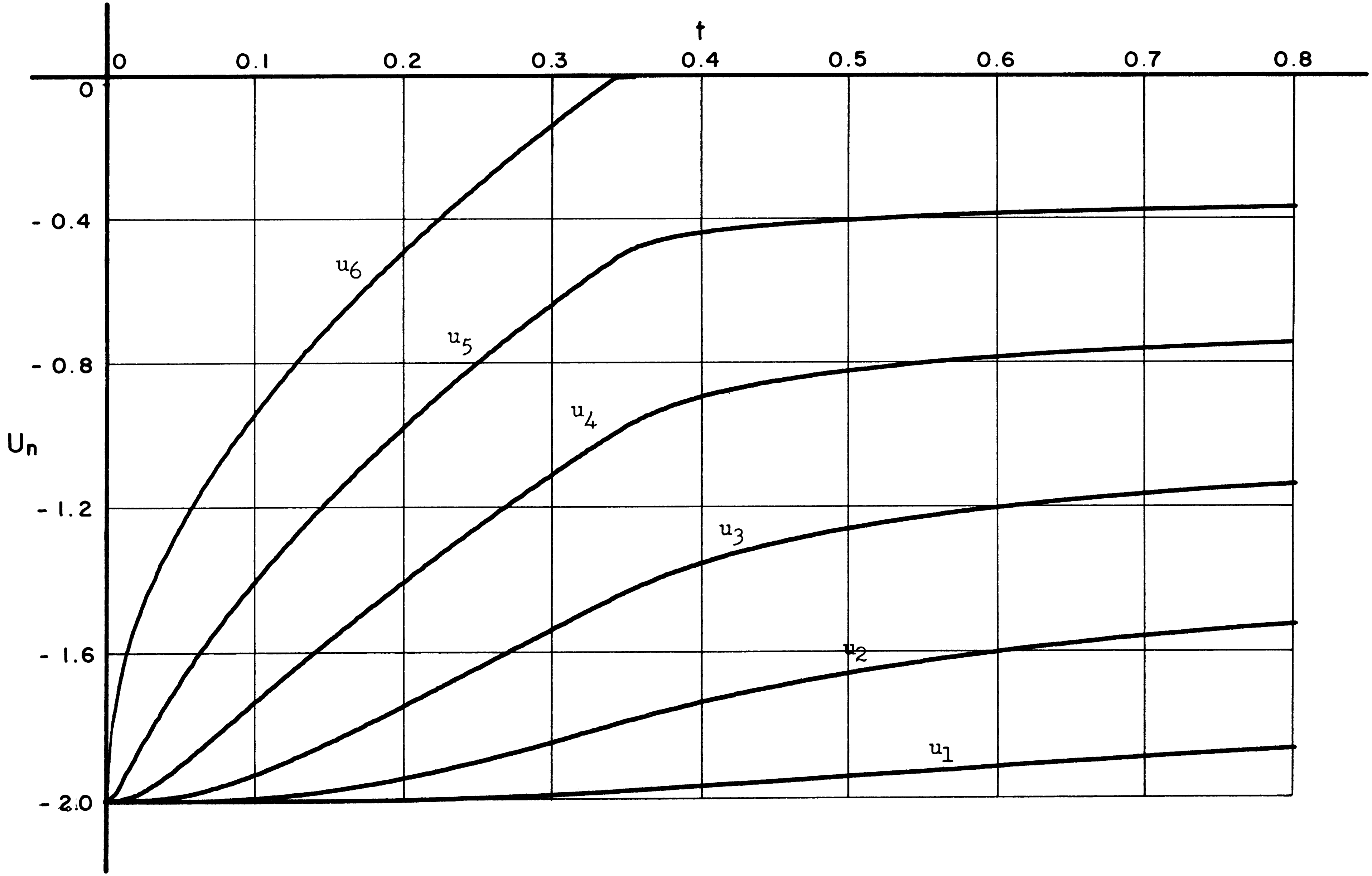


Figure E3. Temperature vs. Time.
$g_n = -2.0$; $r = 6$

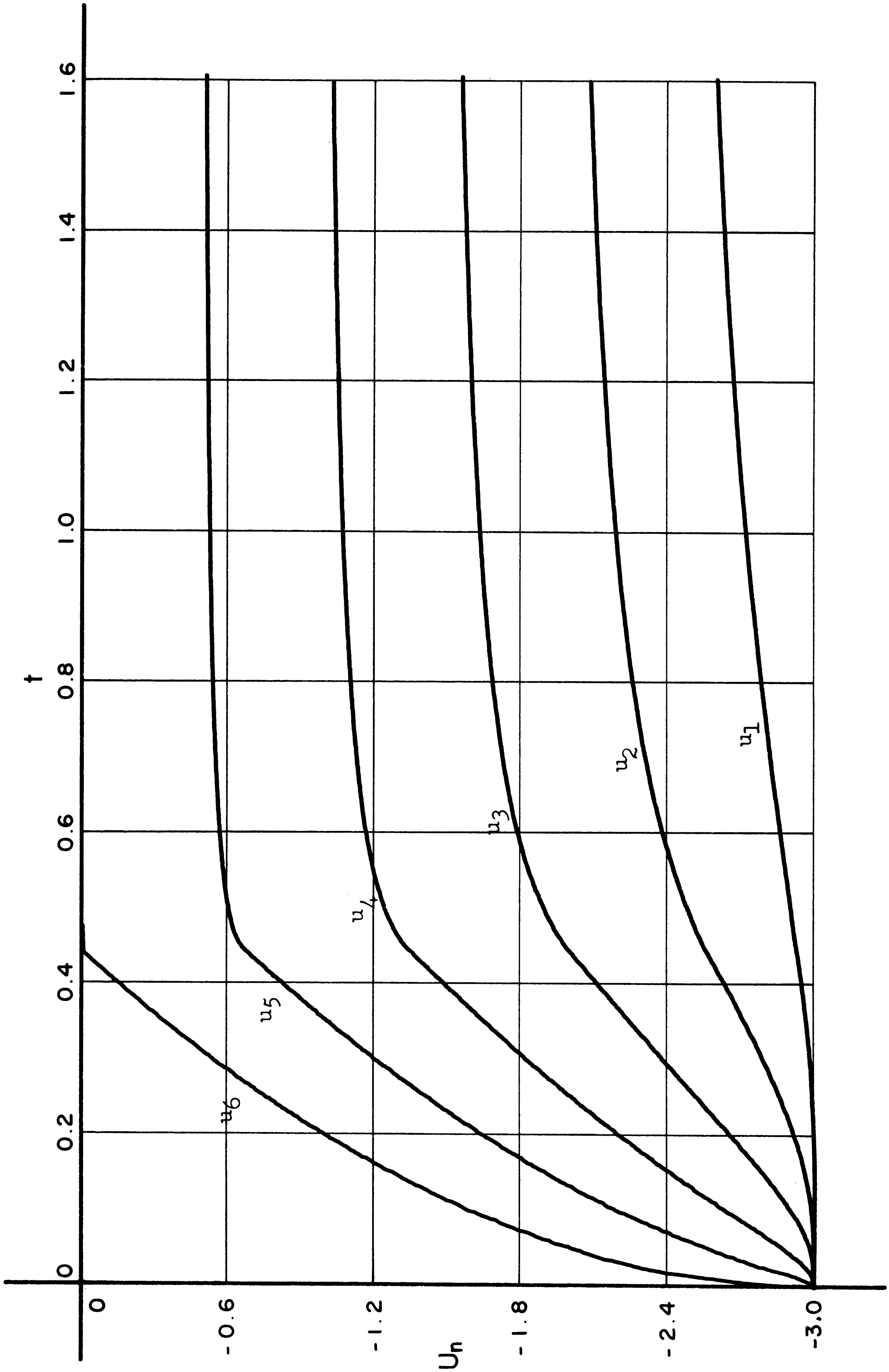


Figure E4. Temperature vs. Time
$g_n = -3.0$; $r = 6$

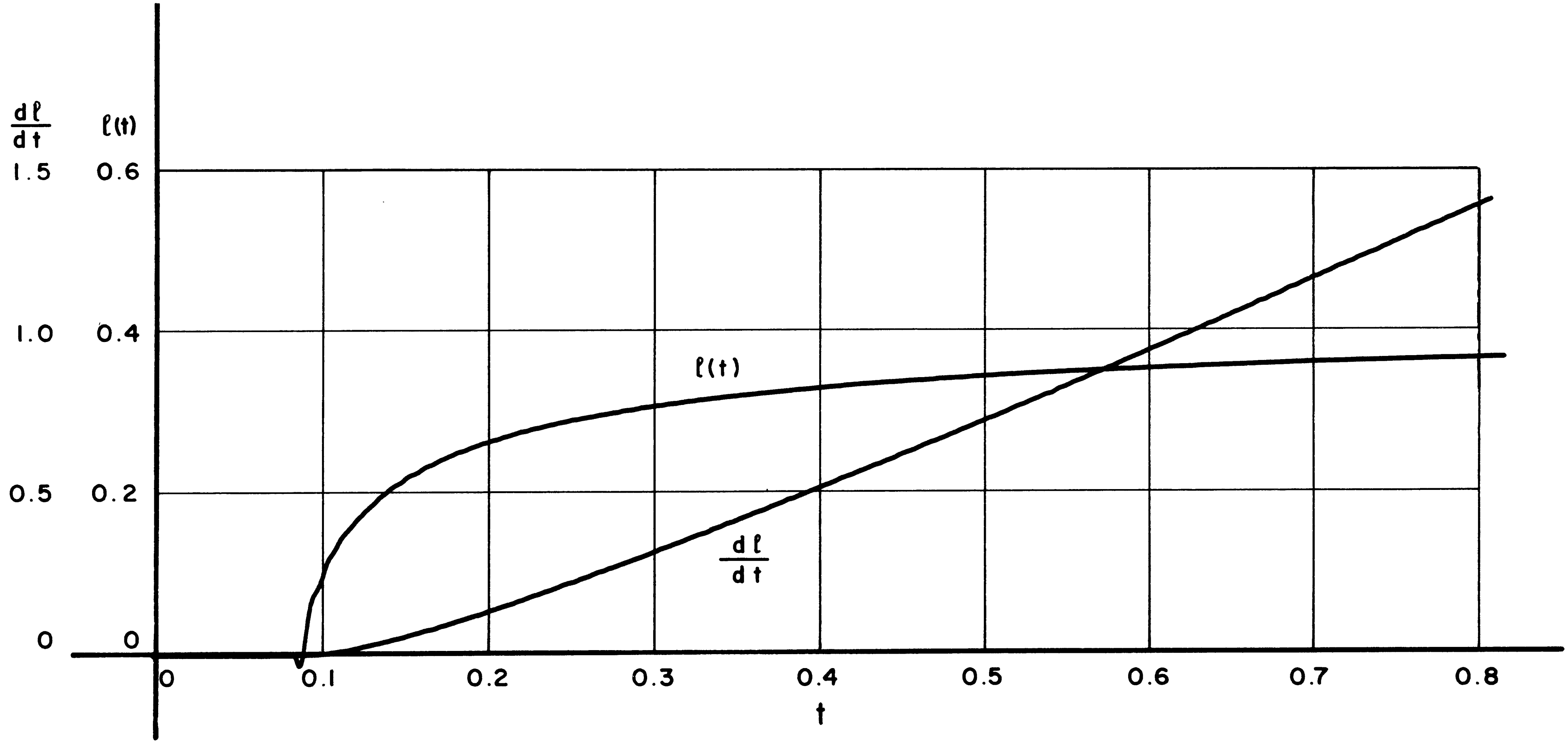


Figure E5. Boundary Velocity and Position vs. Time
$g_n = -0.5$; $r = 6$

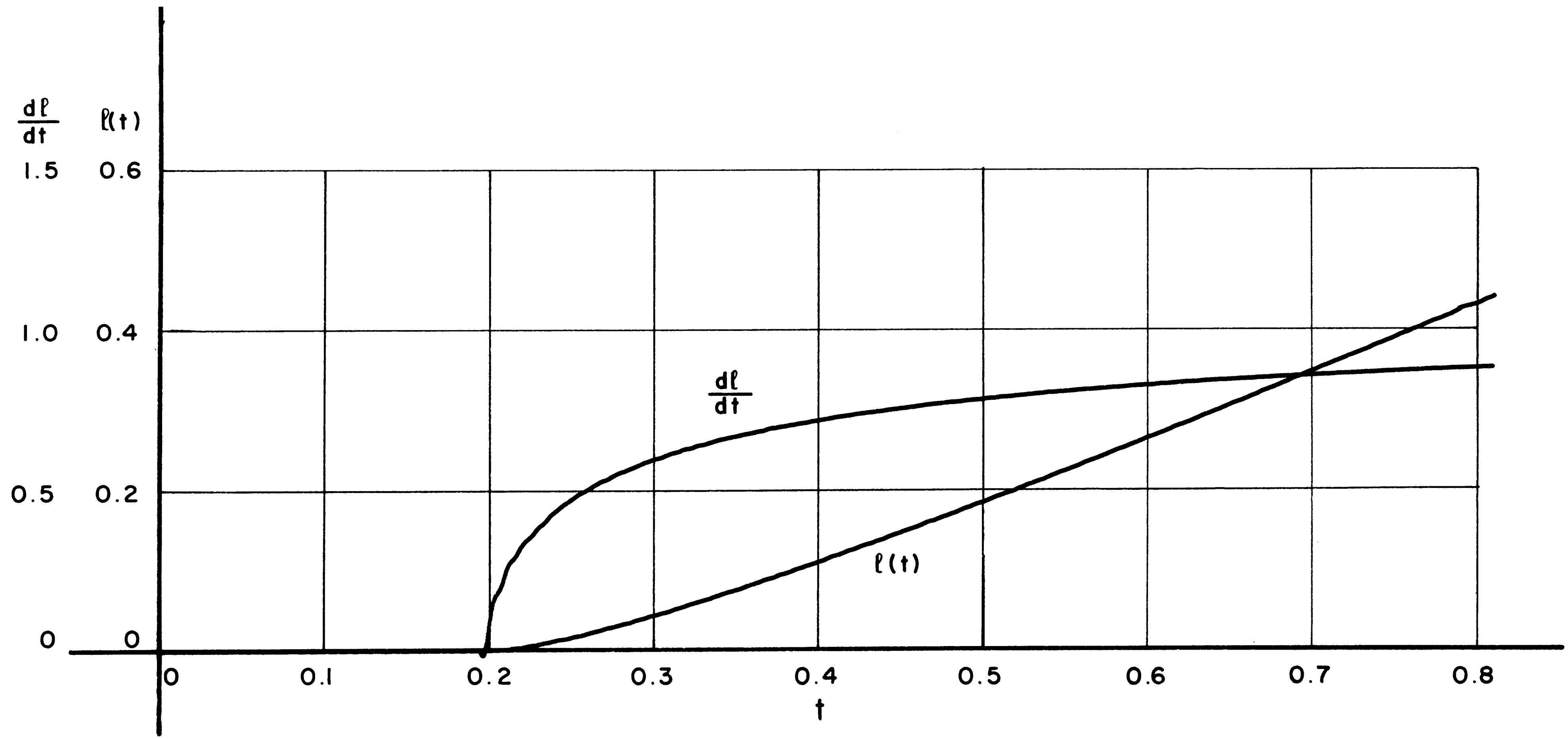


Figure E6. Boundary Velocity and Position vs. Time
$g_n = -1.0$; $r = 6$

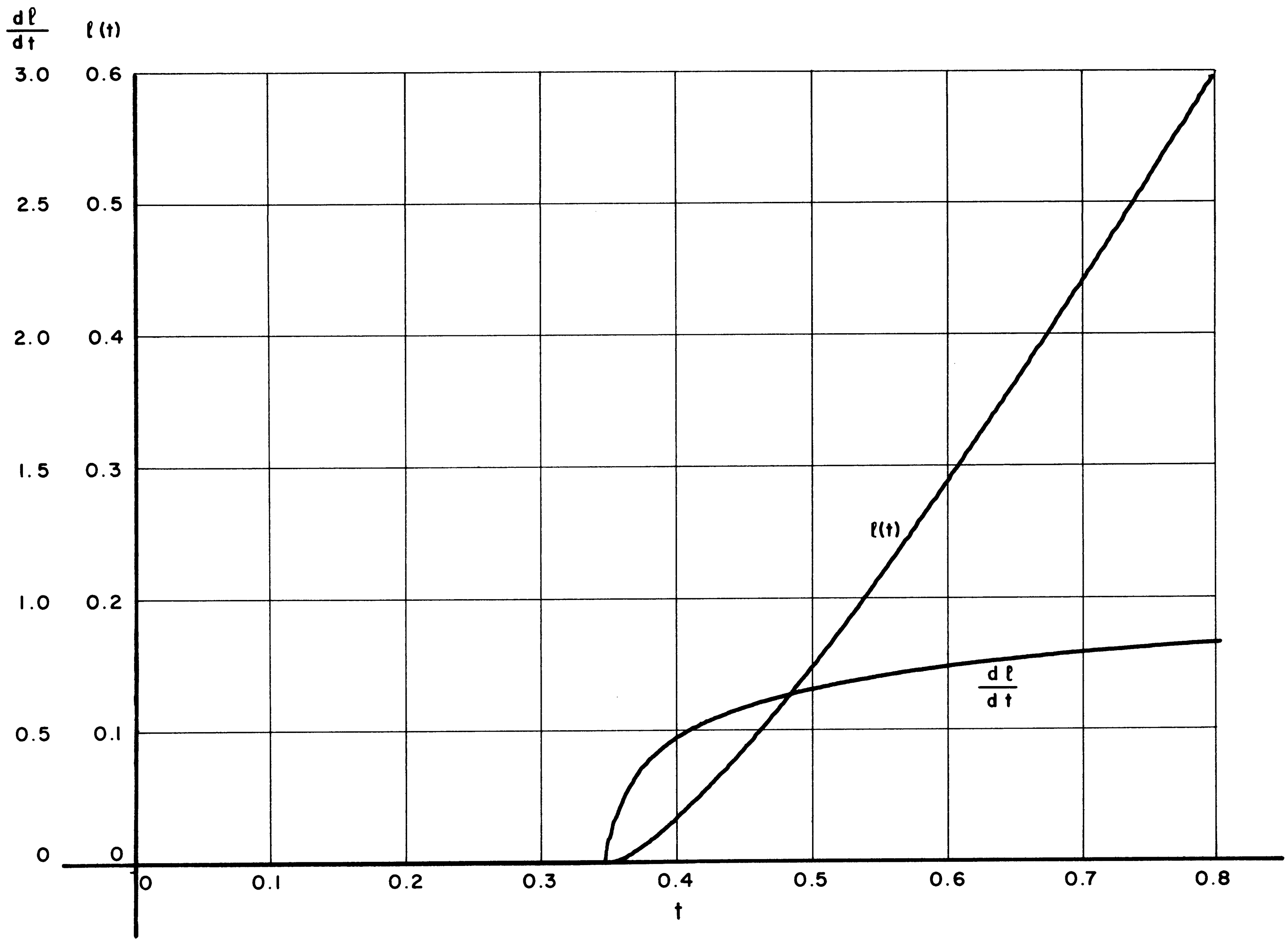


Figure E 7. Boundary Position and Velocity vs. Time.
$g_n = -2.0$; $r = 6$

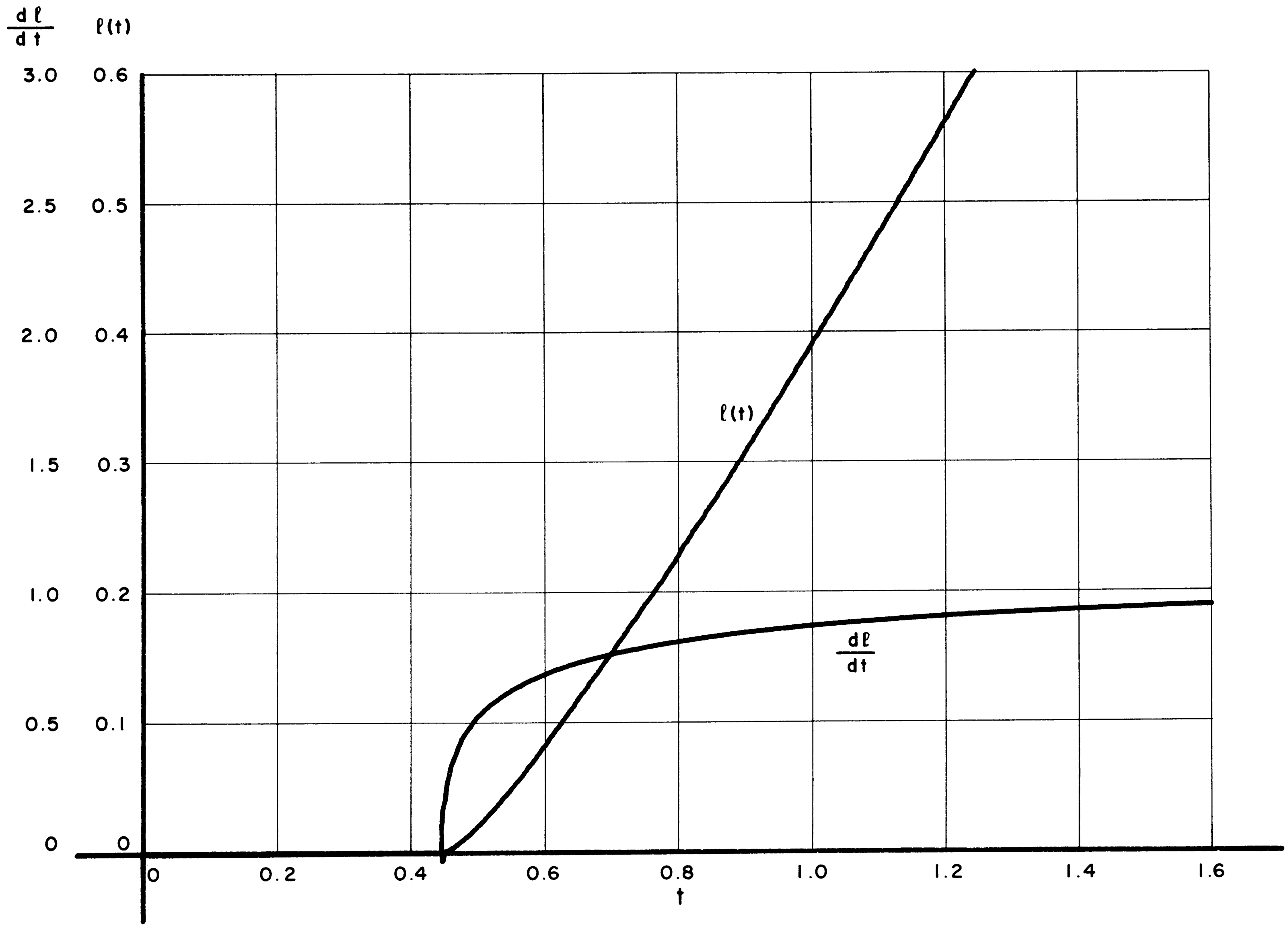


Figure E8. Boundary Velocity and Position vs. Time.
$g_n = -3.0$; $r = 6$

APPENDIX F

SOLUTIONS FOR HEATING RATES OF THE FORM

$$H = h(1-\cos 2\pi t)$$

Semi-Infinite Slab

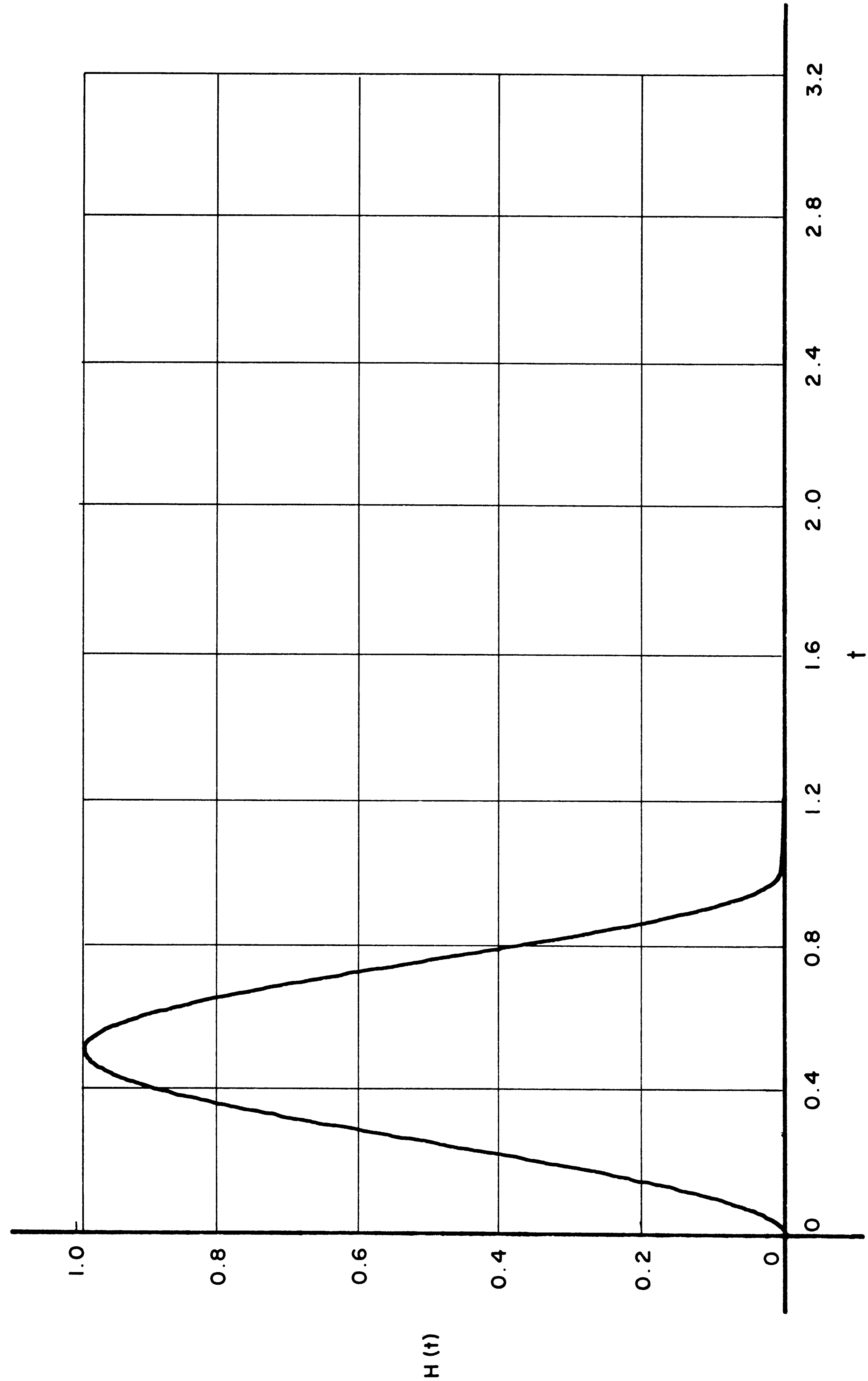


Figure F1. The Heating Rate Pulse.
$H(t) = 1 - \cos 2\pi t$

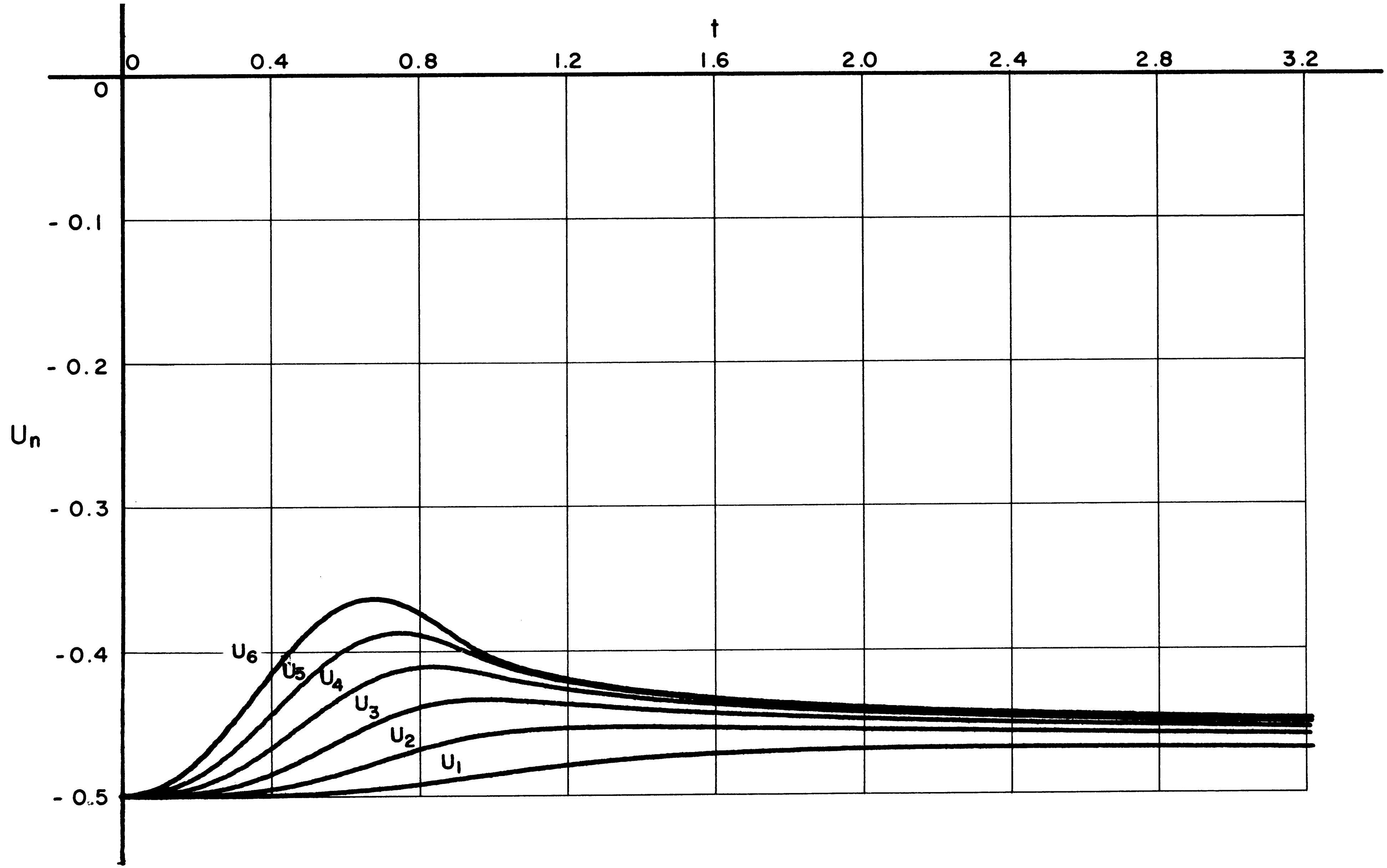


Figure F2. Temperature vs. Time
$H = 0.1(1-\cos 2\pi t)$; $g_n = -0.5$; $r = 6$

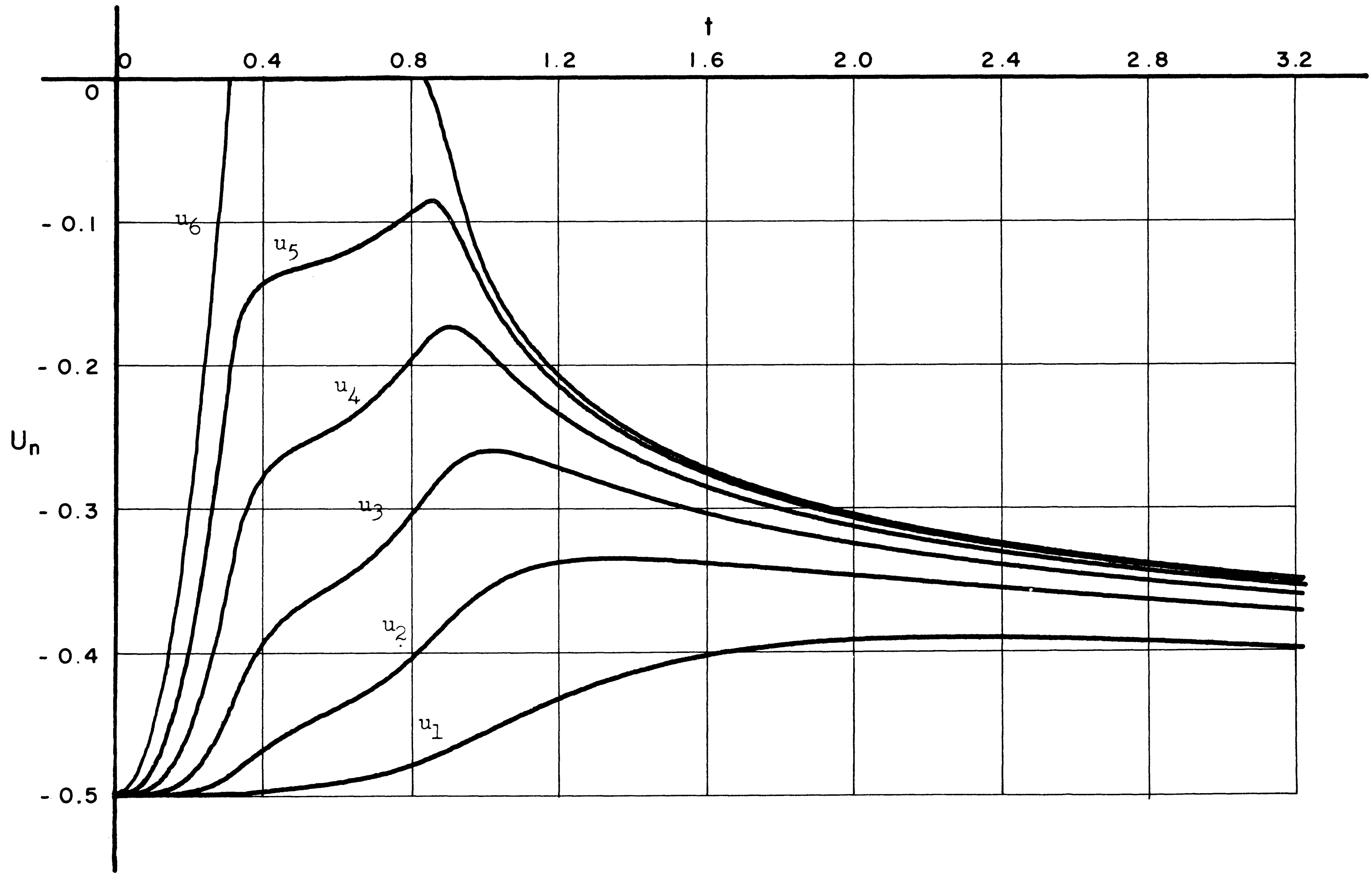


Figure F3. Temperature vs. Time
$H = 1.0(1-\cos 2\pi t)$; $g_n = -0.5$; $r = 6$

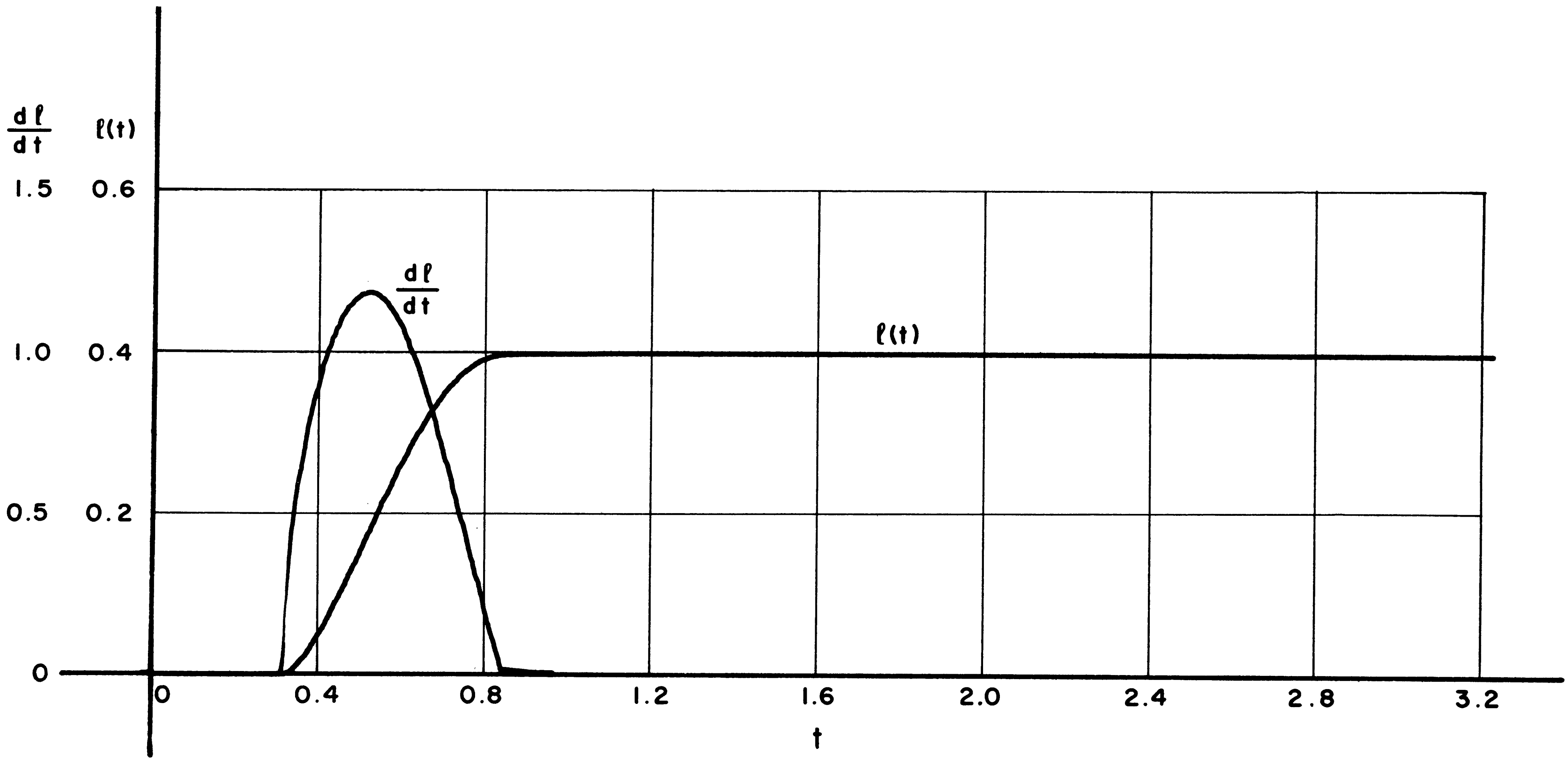


Figure F4. Boundary Velocity and Position vs. Time
$H = 1.0(1-\cos 2\pi t)$; $g_n = -0.5$; $r = 6$

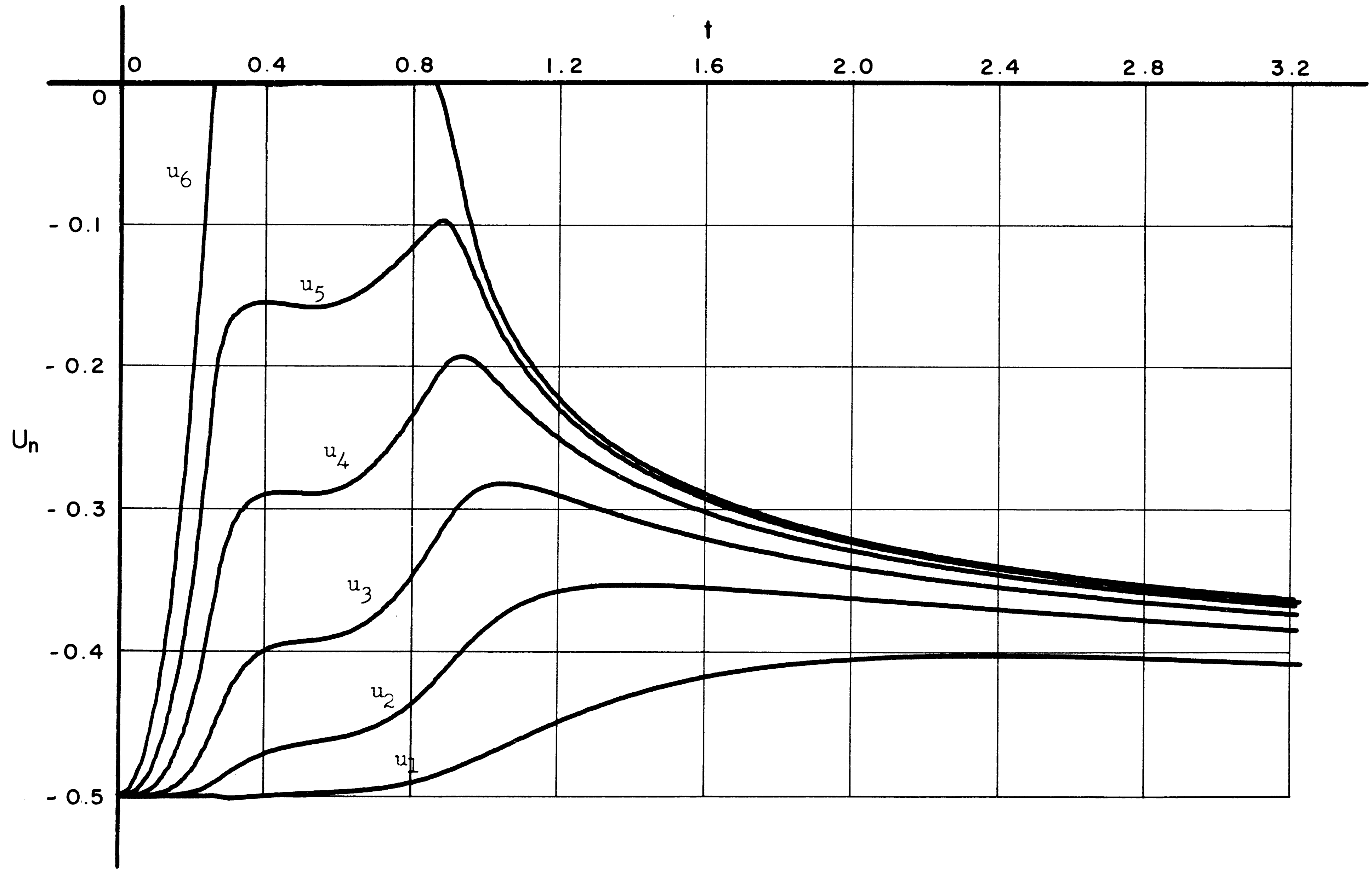


Figure F5. Temperature vs. Time
$H = 1.5(1-\cos 2\pi t)$; $g_n = -0.5$; $r = 6$

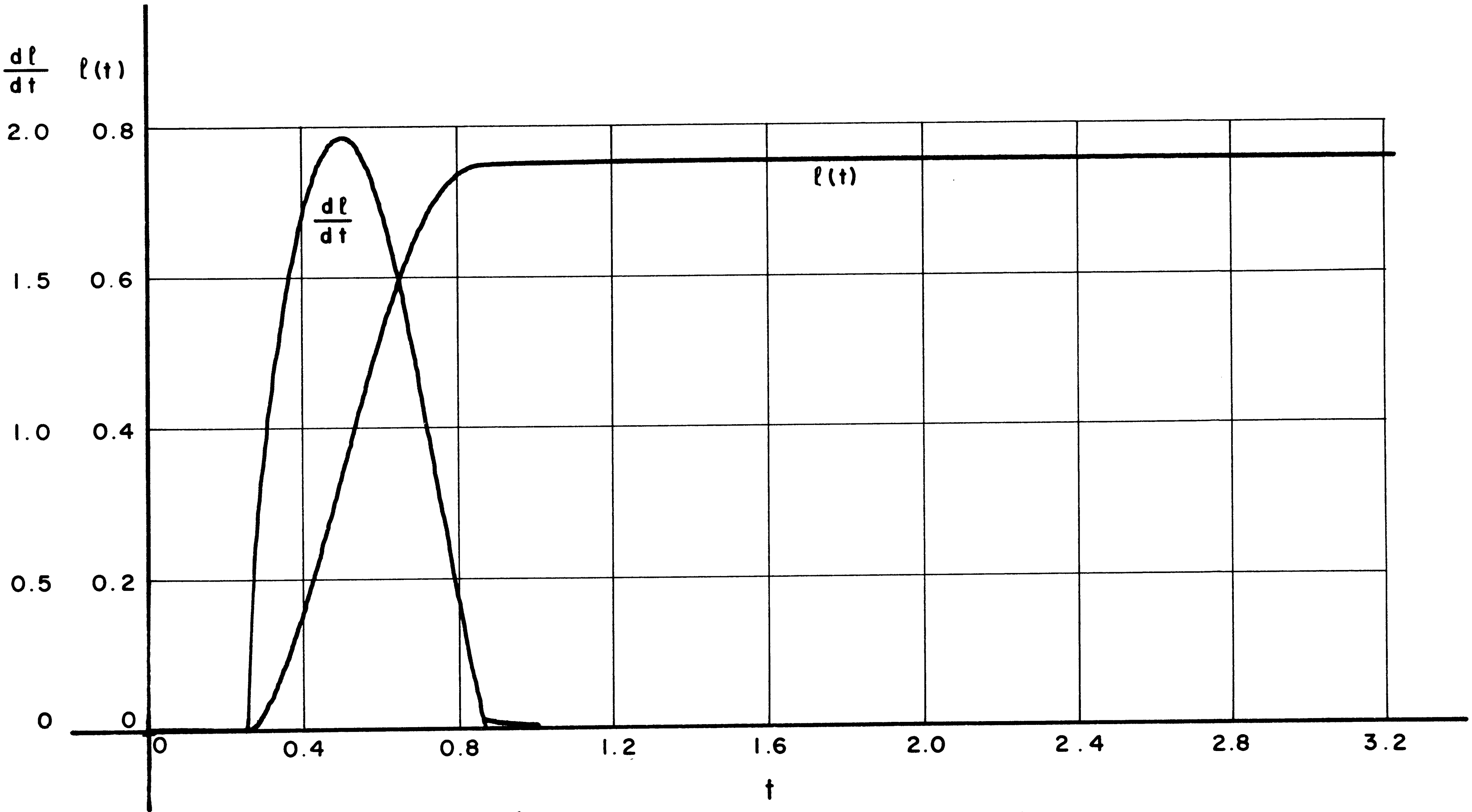


Figure F6. Boundary Velocity and Position vs. Time
$d = 1.5(1-\cos 2\pi t)$; $g_n = -0.5$; $r = 6$

APPENDIX G

SOLUTION CURVES FOR THE FINITE SLAB

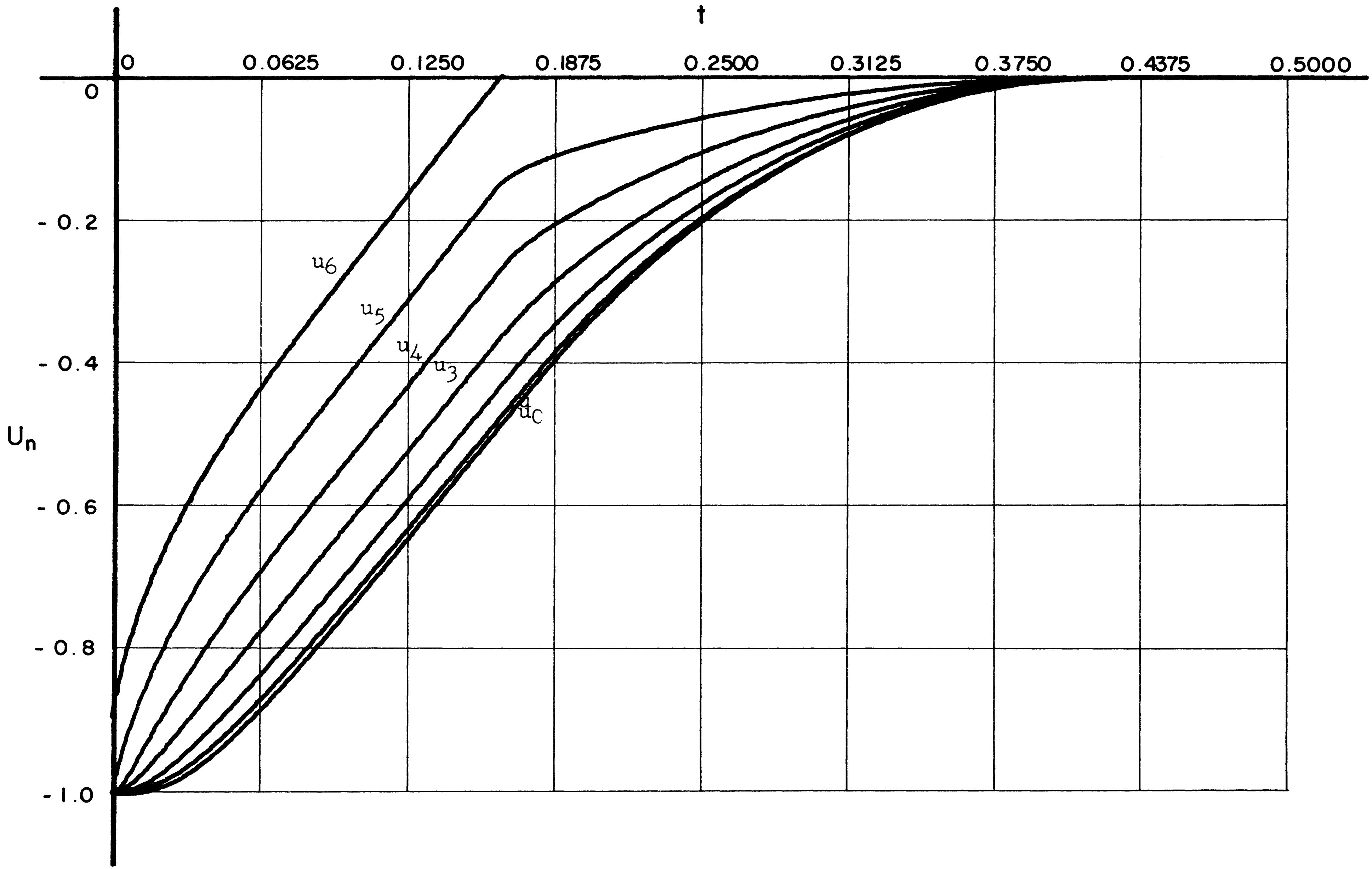


Figure G1a. Temperature vs. Time
$\ell(0) = 0.5$; $g_n = -1.0$; $H = 2.0$; $r = 6$
Linear Station Distribution in $\hat{x}$.

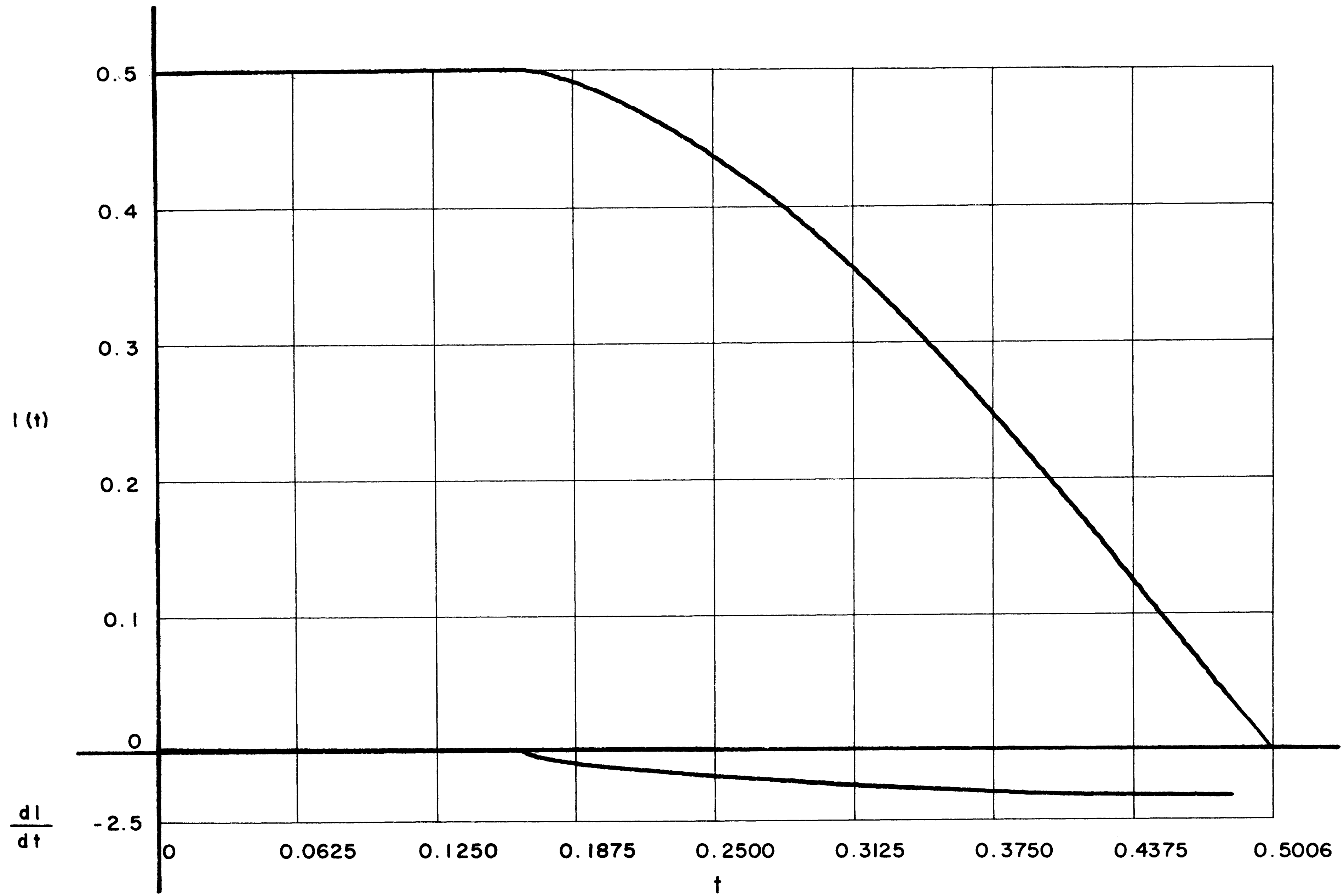


Figure G1b. Boundary Velocity and Position vs. Time
$\ell(0) = 0.5$; $g_n = -1.0$; $H = 2.0$; $r = 6$
Linear Station Distribution in $\hat{x}$.

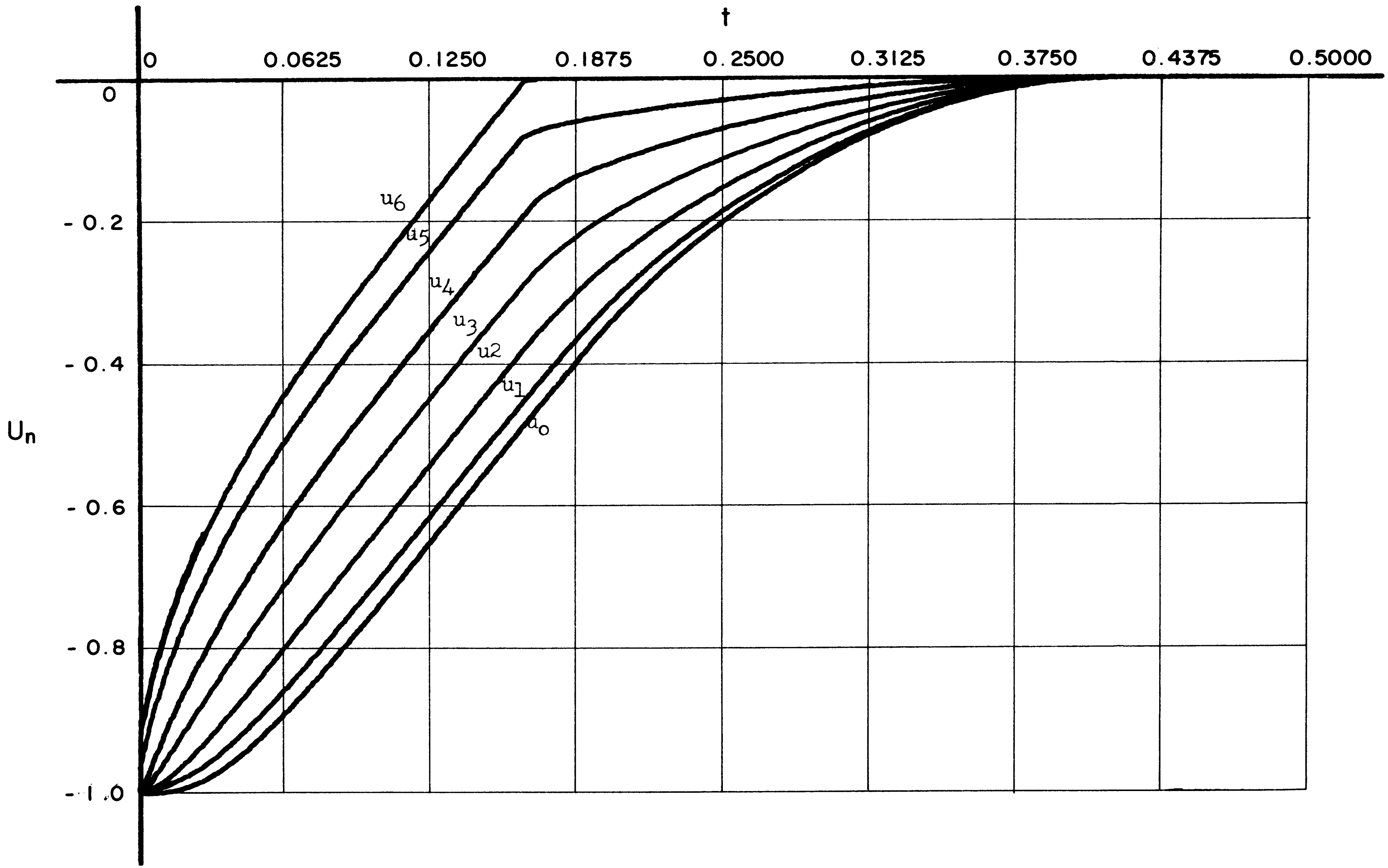


Figure G2a. Temperature vs. Time
$\ell(0) = 0.5$; $g_n = -1.0$; $H = 2.0$; $r = 6$
Parabolic Station Distribution in $\hat{x}$.

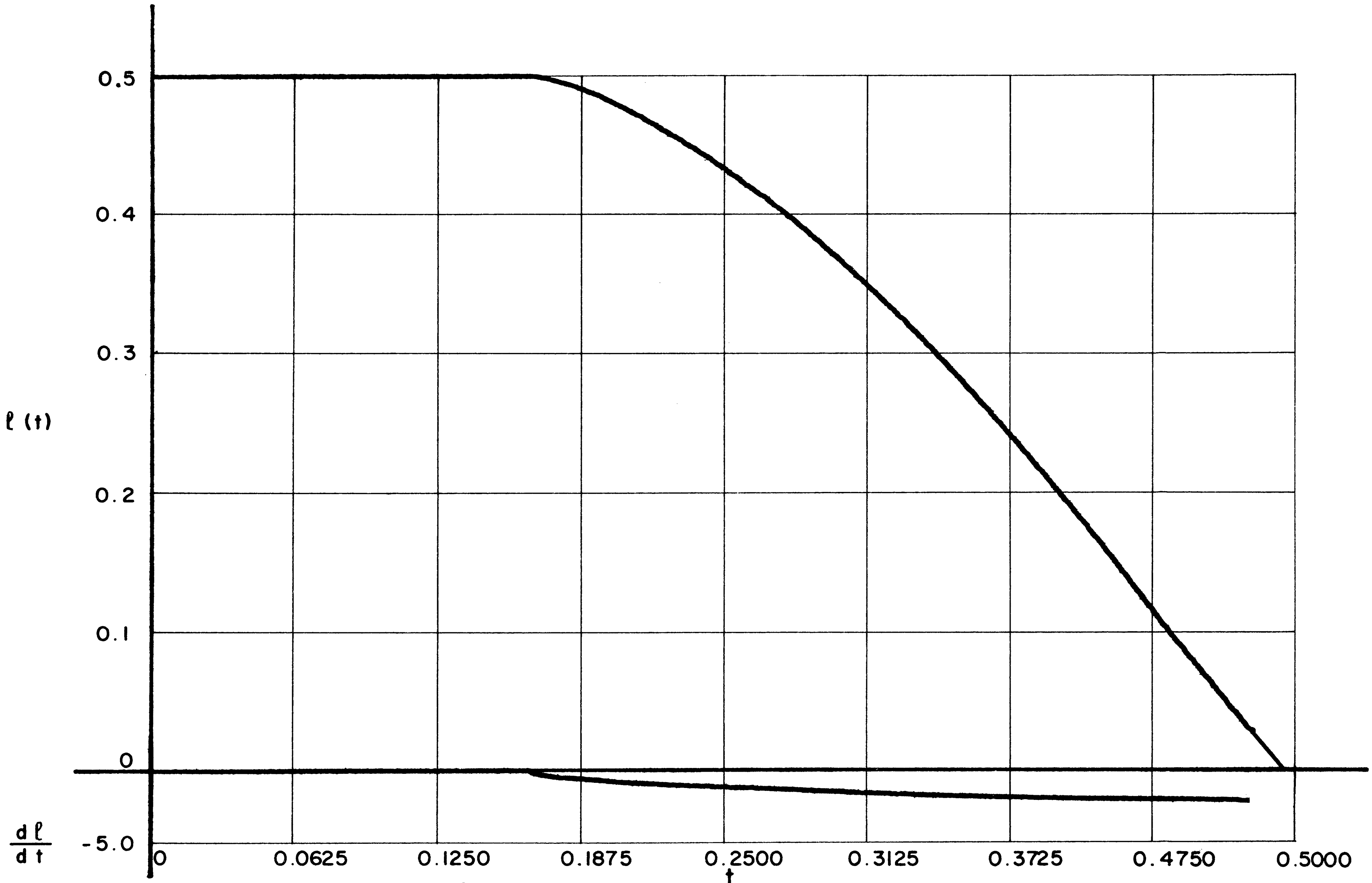


Figure G2b. Boundary Velocity and Position vs. Time
$\ell(0)=0.5$; $g_n=-1.0$; $H=2.0$; $r=6$
Parabolic Station Distribution in $\hat{x}$.

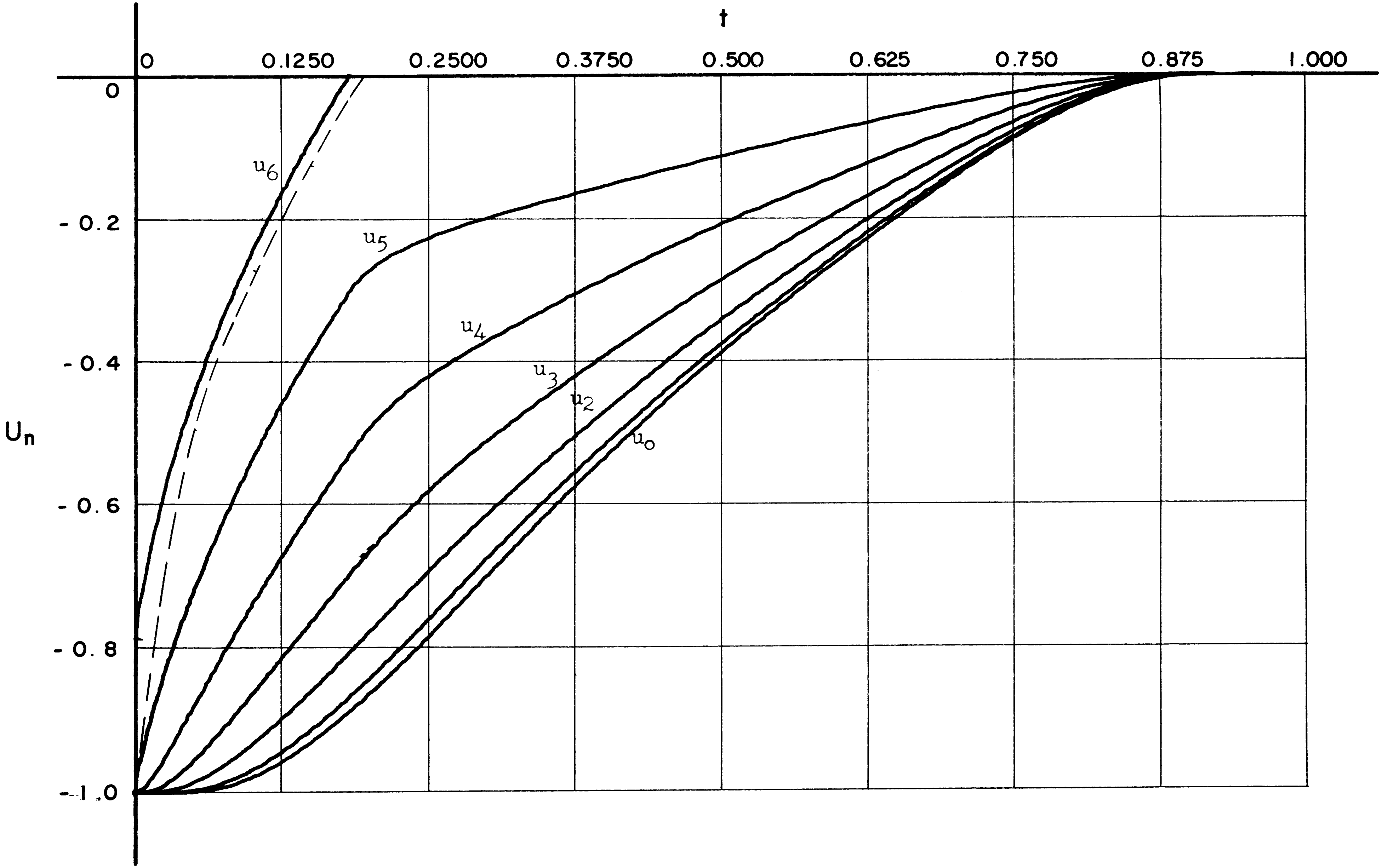


Figure G3a. Temperature vs. Time.
$\ell(0) = 1.0$; $g_n = -10.0$; $H = 2$; $r = 6$
Linear Station Distribution in $\hat{x}$.

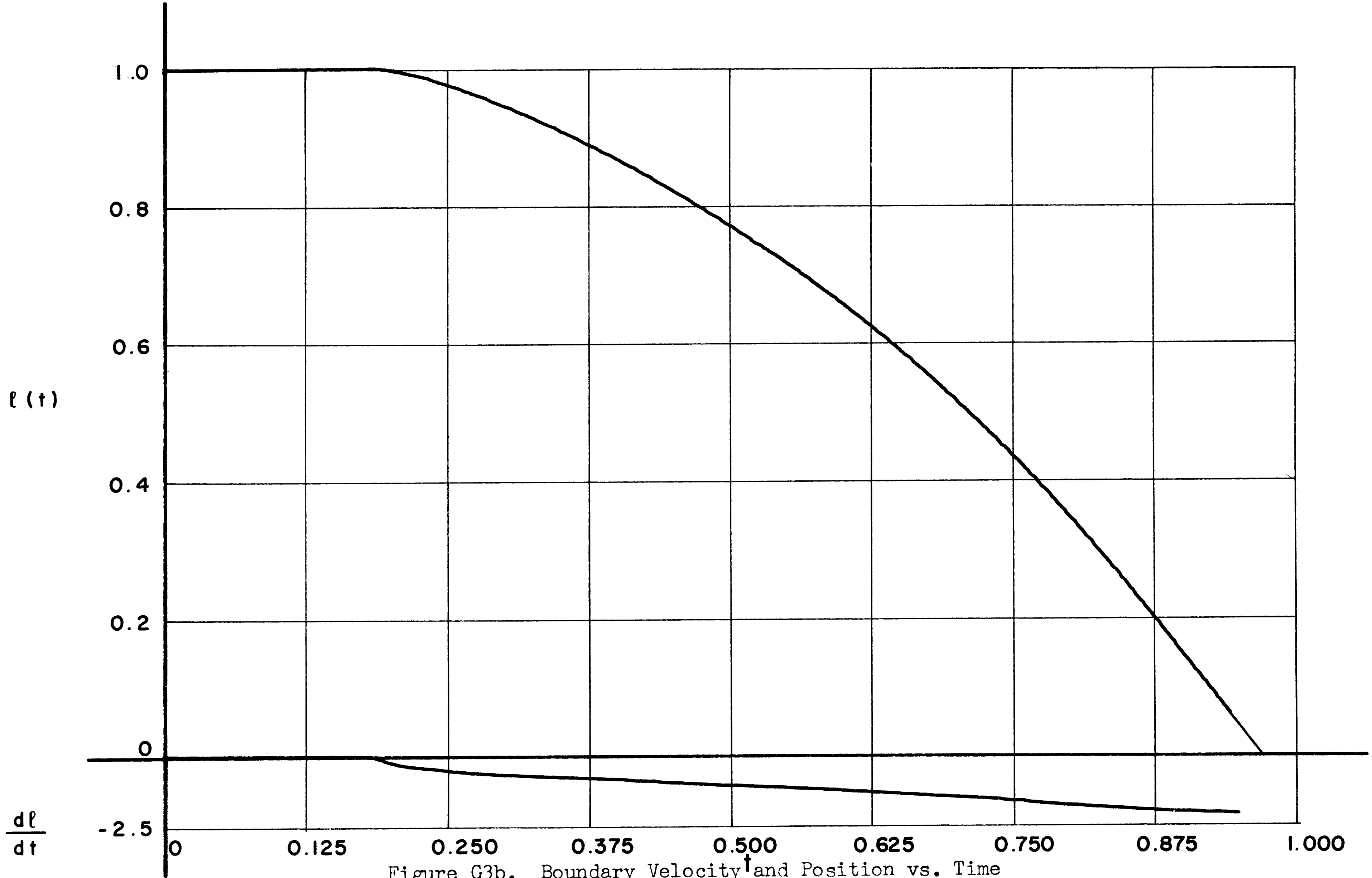


Figure G3b. Boundary Velocity and Position vs. Time
$\ell(C) = 1.0$; $g_n = -1.0$; $H = 2$; $r = 6$
Linear Station Distribution in $\hat{x}$.

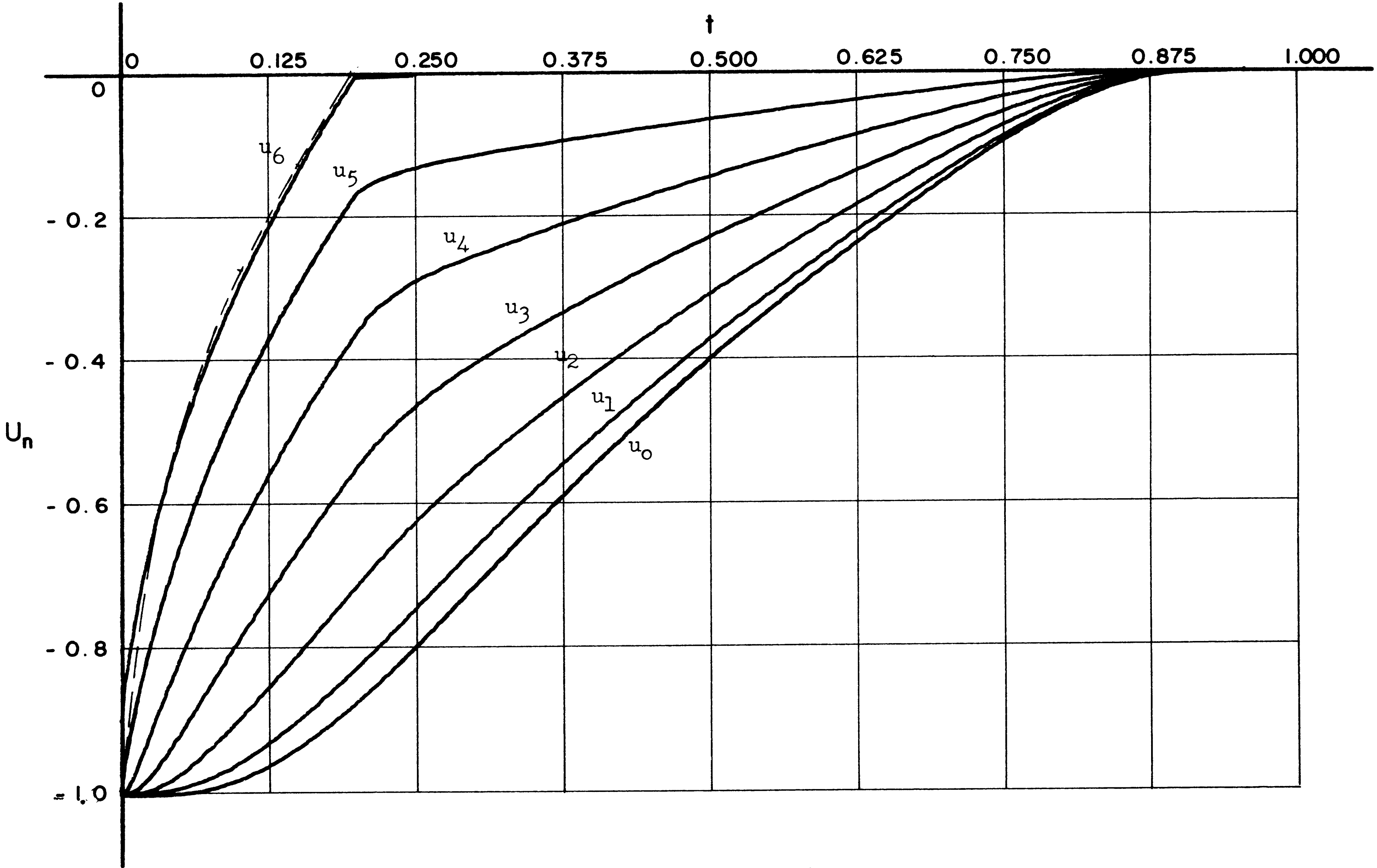


Figure G4a. Temperature vs. Time
$\ell(C) = 1.0$; $g_n = -1.C$, $H = 2.C$; $r = 6$
Parabolic Station Distribution in $\hat{x}$.

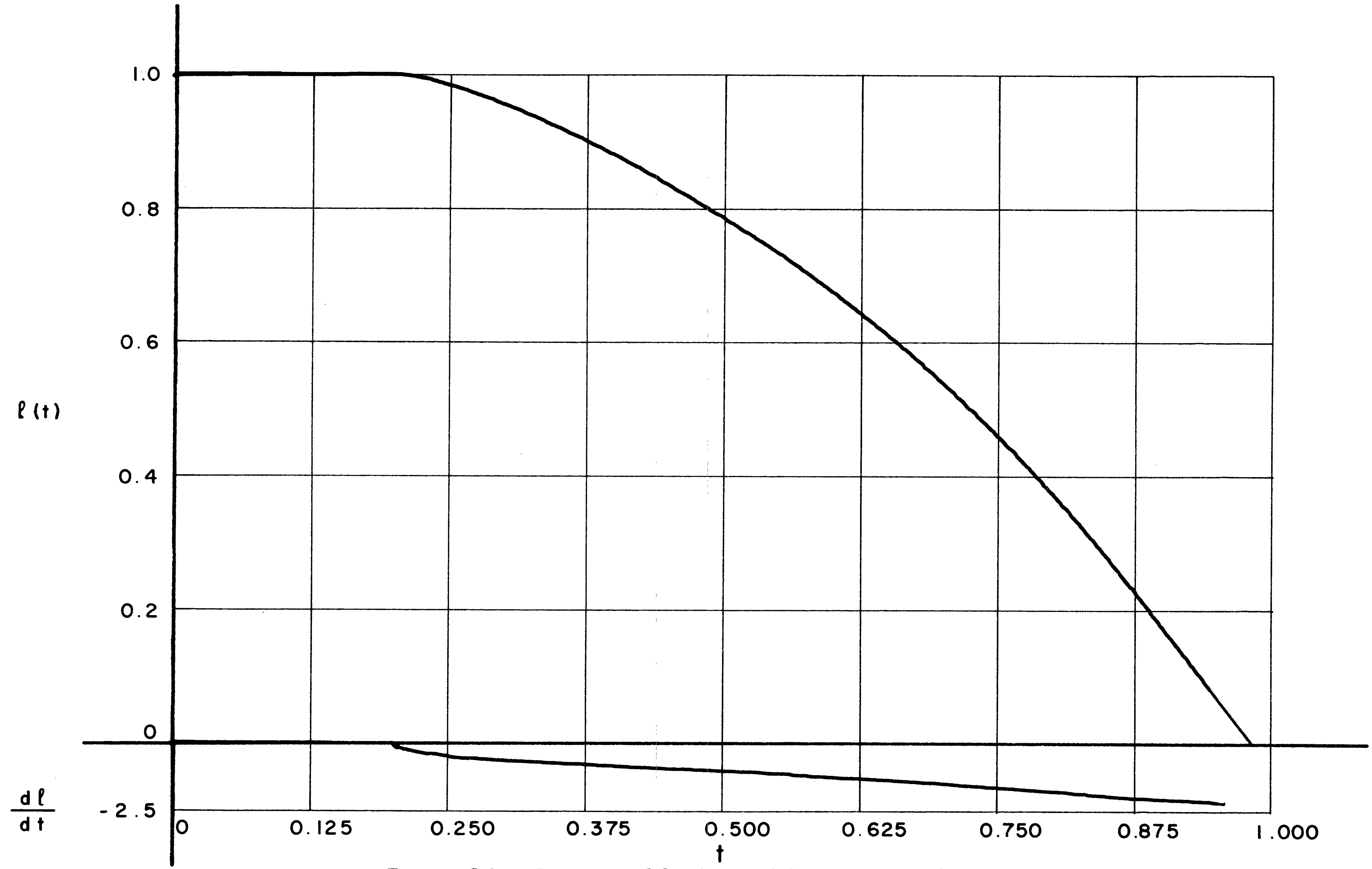


Figure G4b. Boundary Velocity and Position vs. Time.
$\ell(0) = 1.0$; $g_n = -1.0$; $H = 2.0$; $r = 6$
Parabolic Station Distribution in $\hat{x}$.

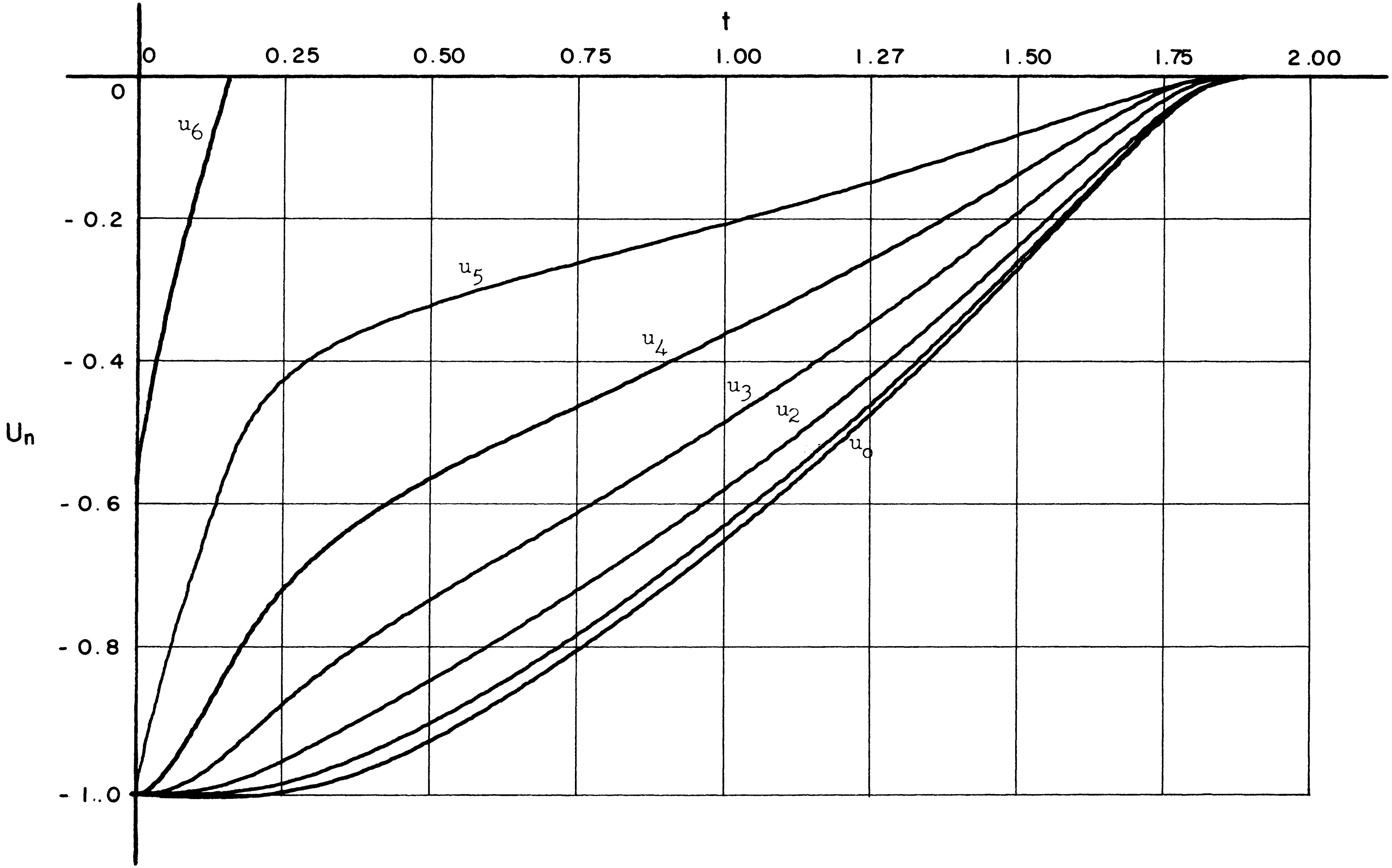


Figure G5a. Temperature vs. Time
$\ell(C) = 2.0$; $g_n = -1.0$; $H = 2.0$; $r = 6$
Linear Station Distribution in $\hat{x}$

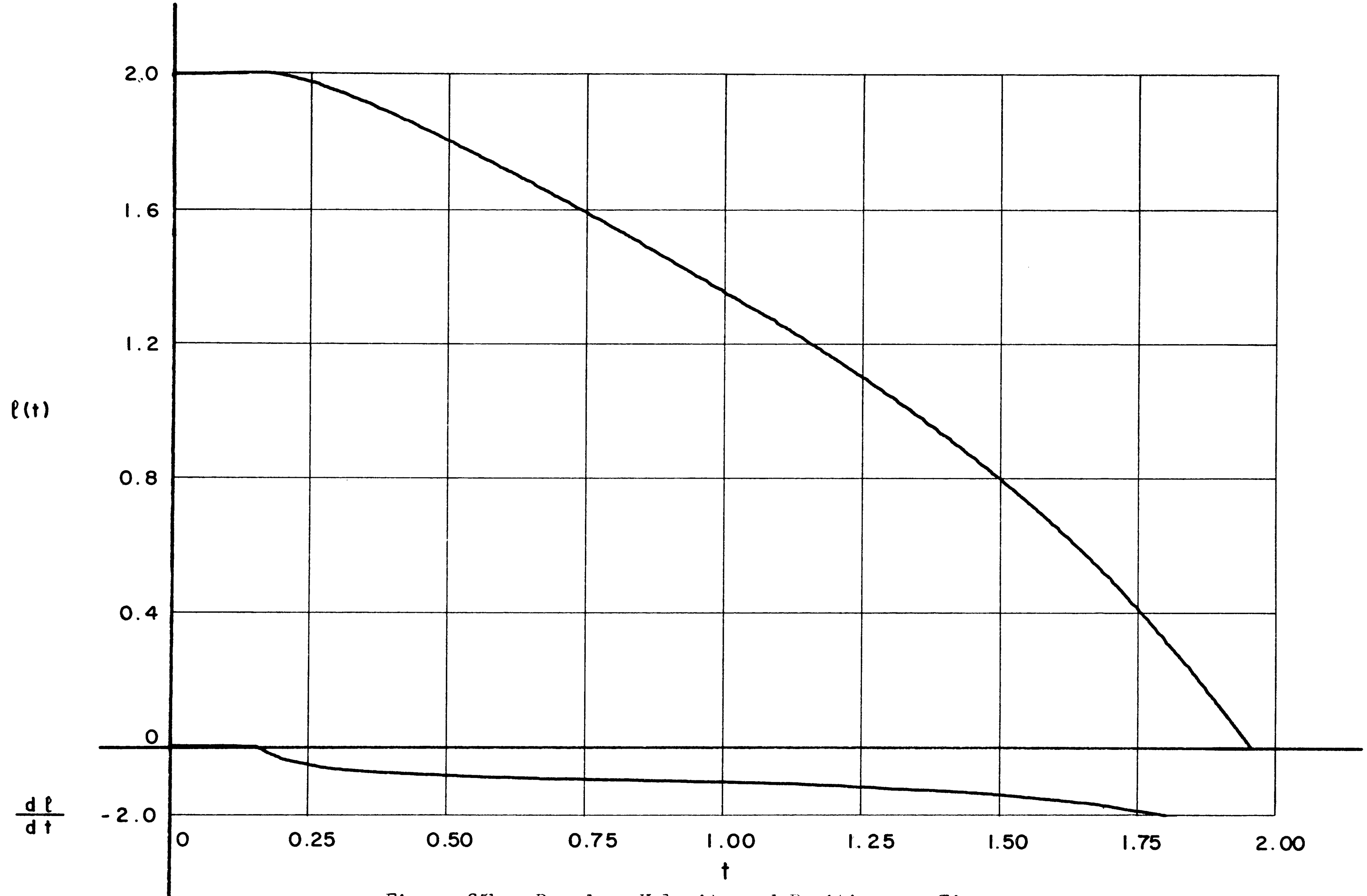


Figure G5b. Boundary Velocity and Position vs. Time.
ℓ (C) = 2.C; g_n = -1.C; H = 2.C; r = 6
Linear Station Distribution in $\hat{x}$.

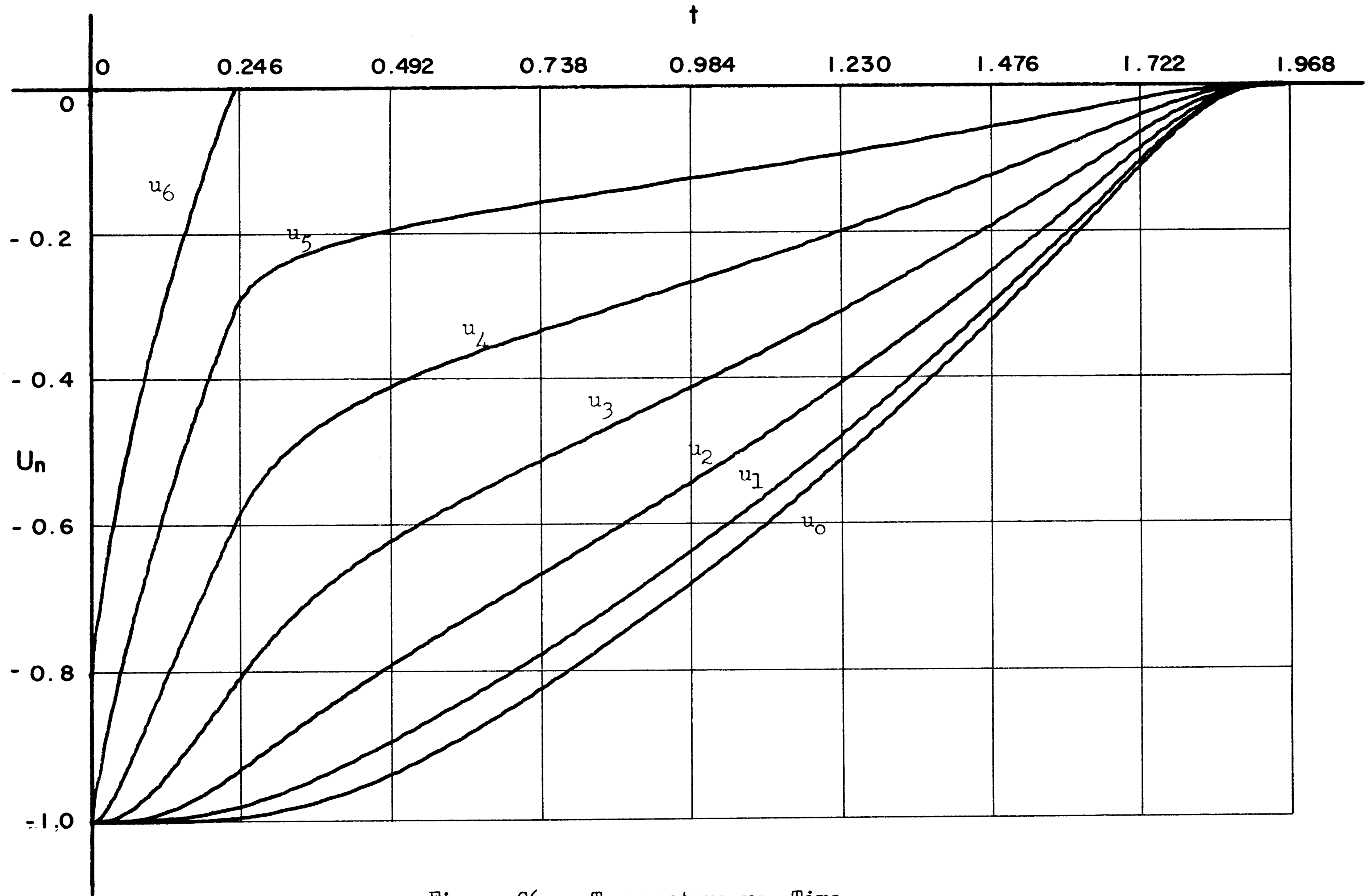


Figure G6a. Temperature vs. Time
$\ell(0) = 2.0$; $g_n = -1.0$; $H = 2.0$; $r = 6$.
Parabolic Distribution of Stations in $\hat{x}$.

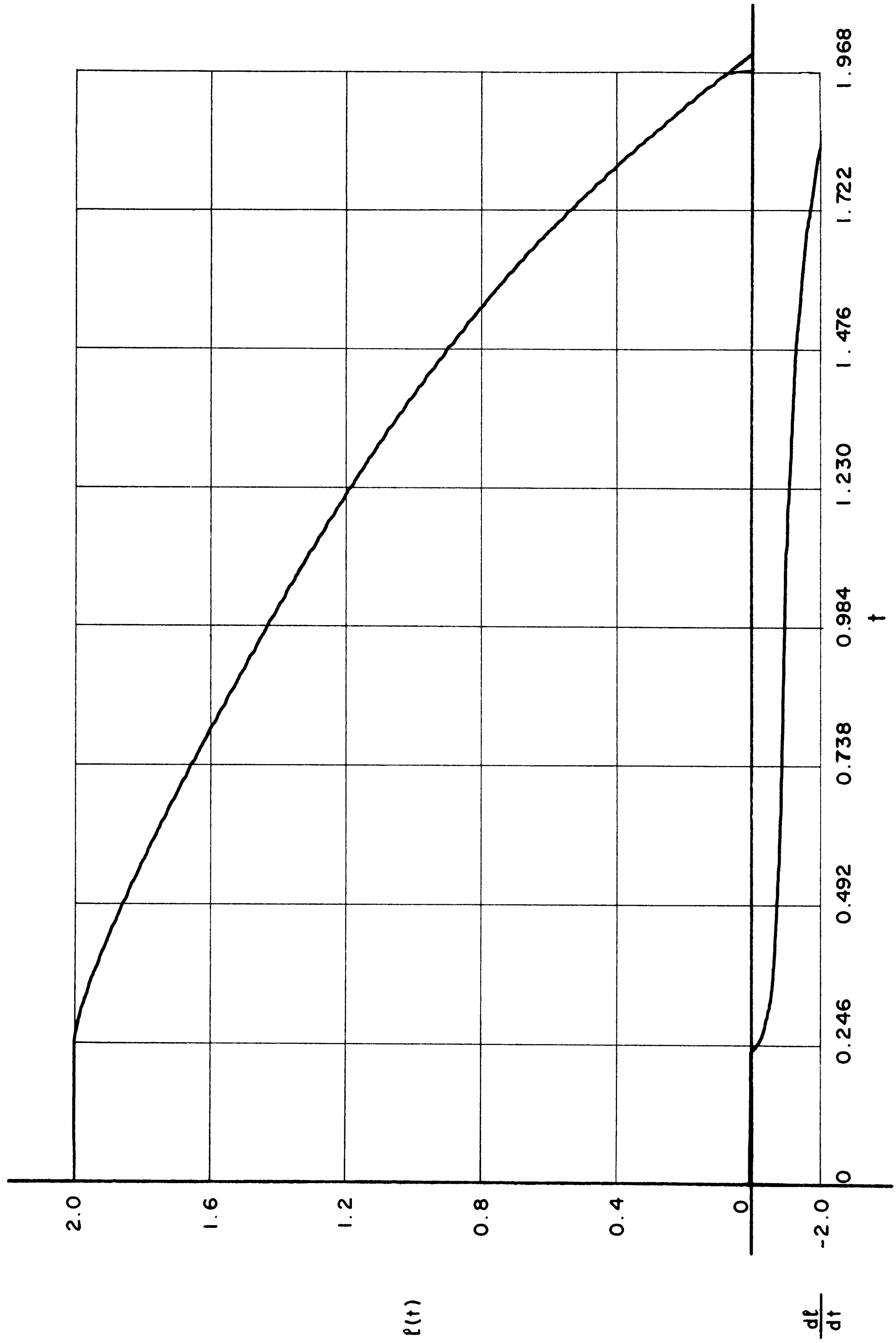


Figure G6b. Boundary Velocity and Position vs. Time. $\ell(0) = 2.0$; $g_m = -1.0$; $H = 2.0$; $r = 6$. Parabolic Distribution of Stations in $\hat{x}$.

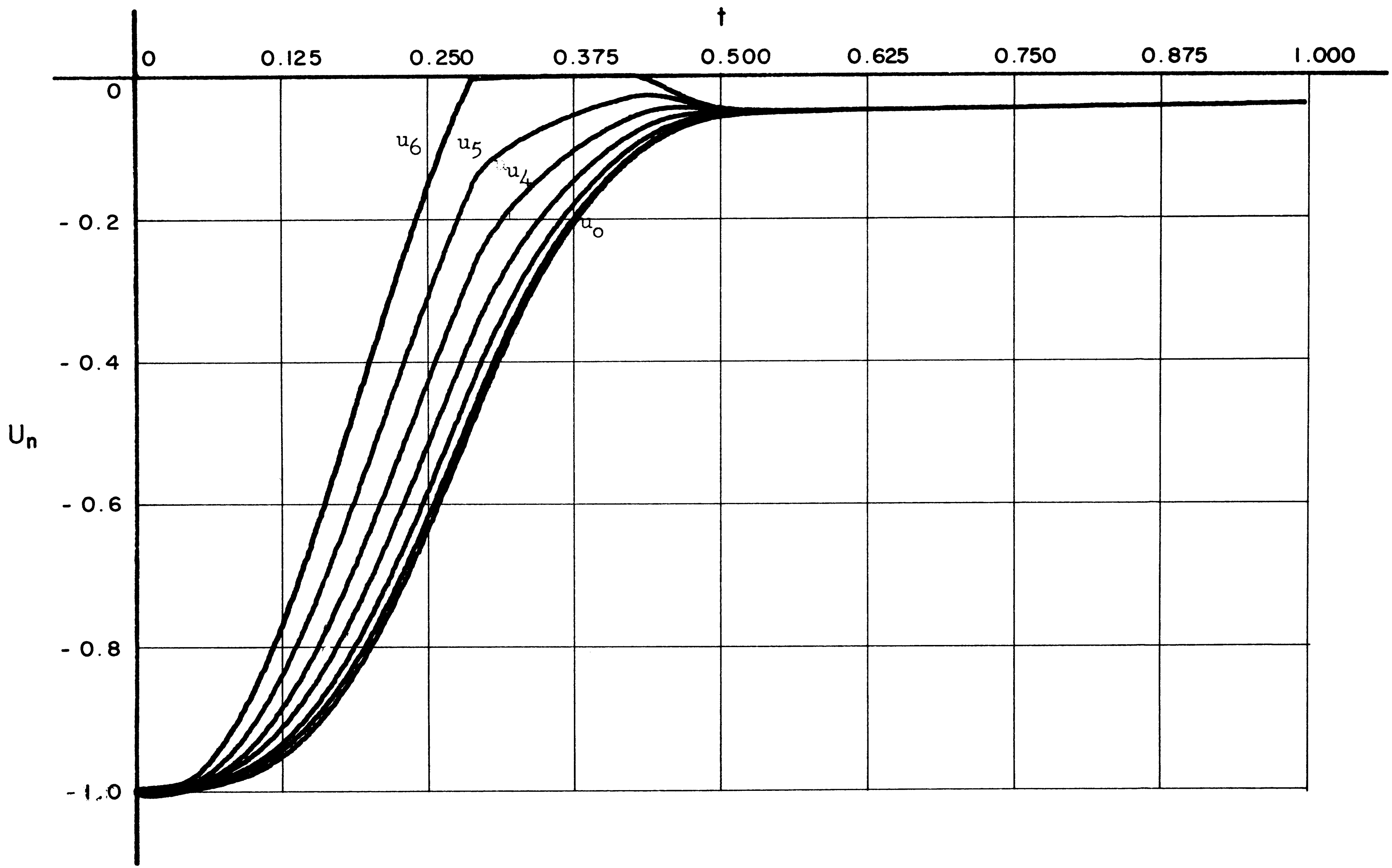


Figure G7a. Temperature vs. Time.
$\ell(0) = 0.5$; $g_n = -1.0$; $H = 1-\cos 4\pi t$; $r = 6$
Linear Station Distribution in $\hat{x}$.

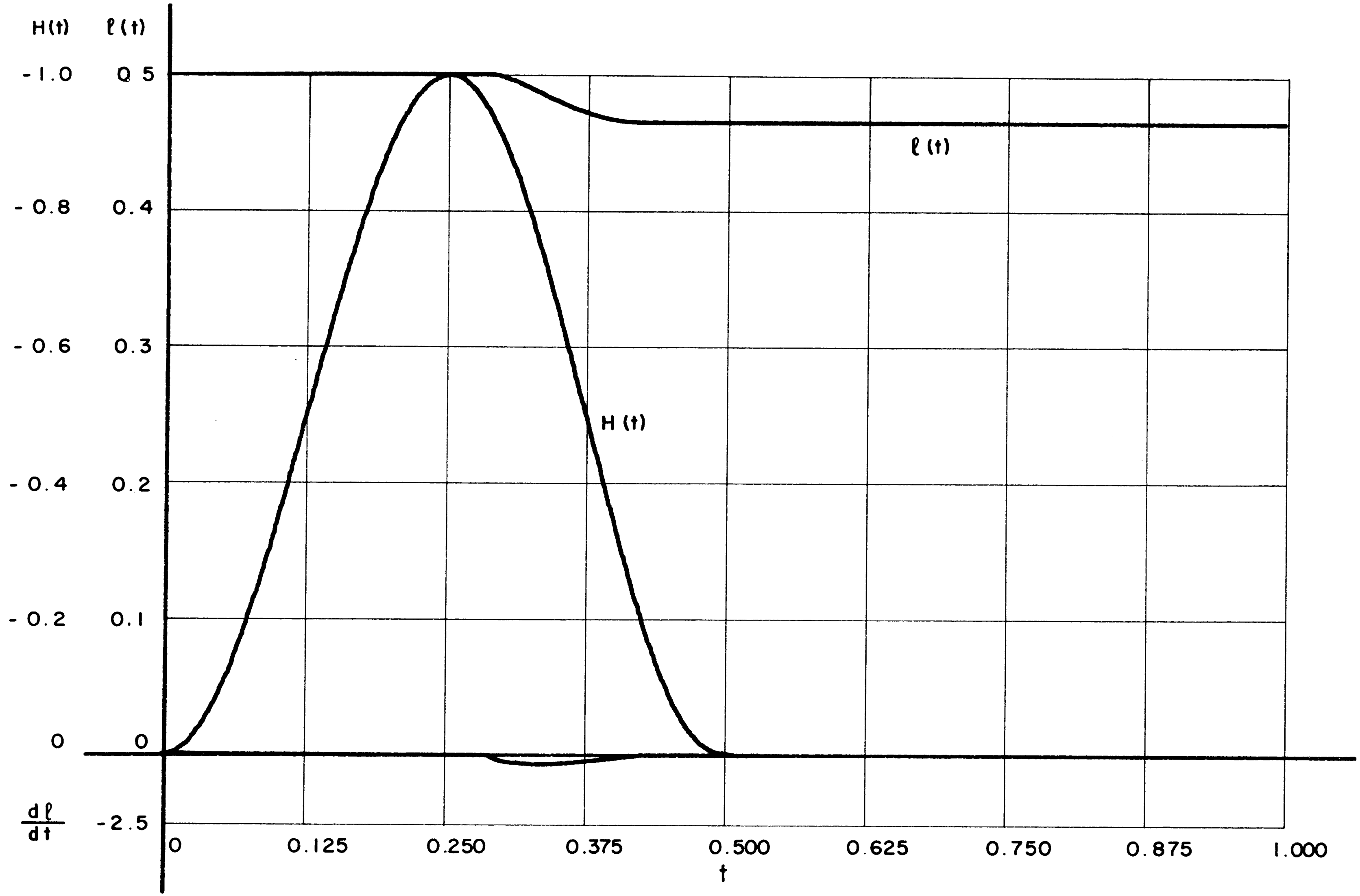


Figure G7b. Boundary Velocity and Position and Heat Pulse vs. Time.
$\ell(0) = 0.5$; $g_n = -1.0$; $H = 1 - \cos 4\pi t$; $r = 6$
Linear Station Distribution in $\hat{x}$.

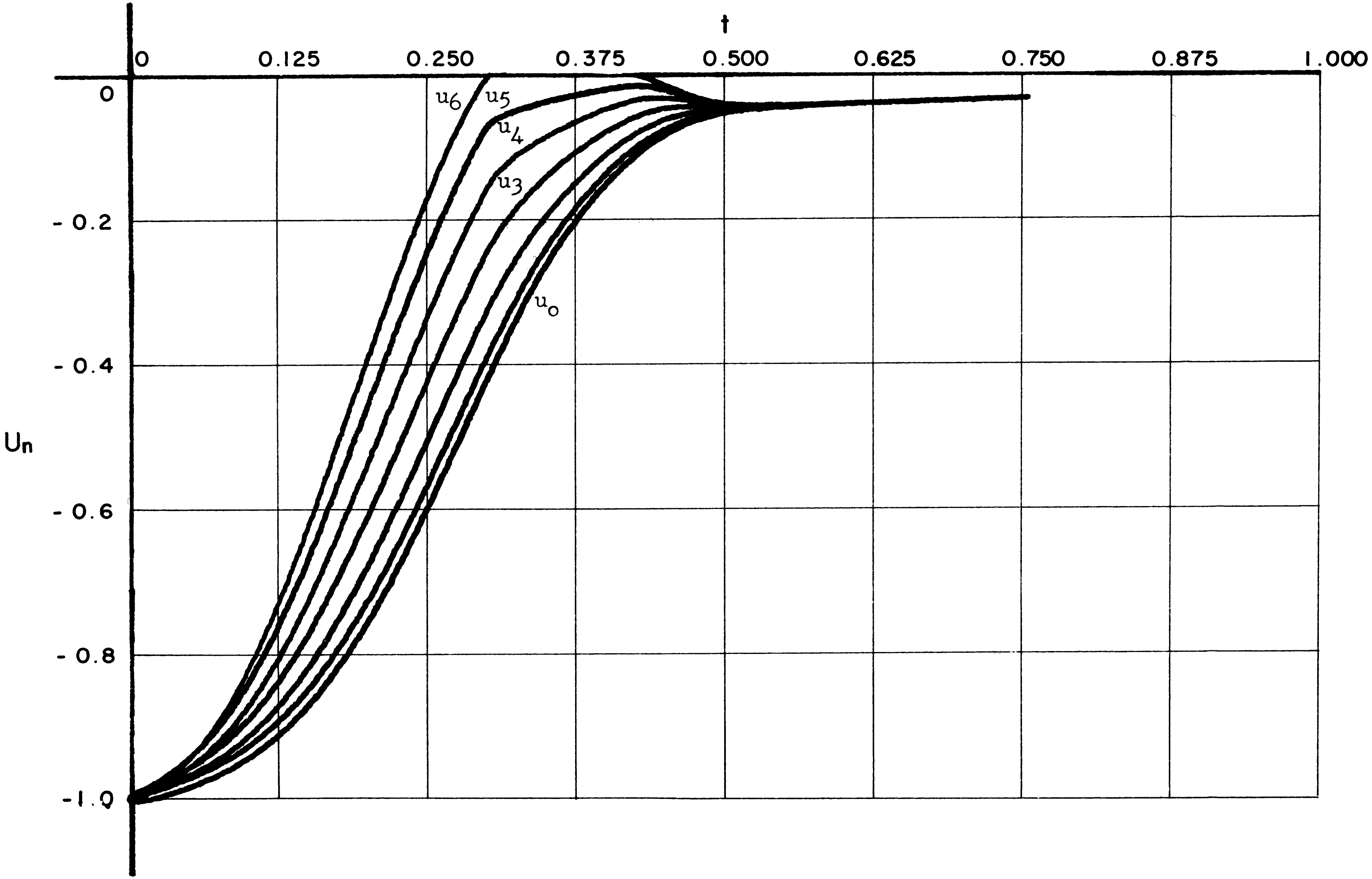


Figure G8a. Temperature va. Time.
$\mathcal{L}(0) = 0.5$; $g_n = -1.0$; $H = 1 - \cos 4\pi t$; $r = 6$
Parabolic Station Distribution in $\hat{x}$.

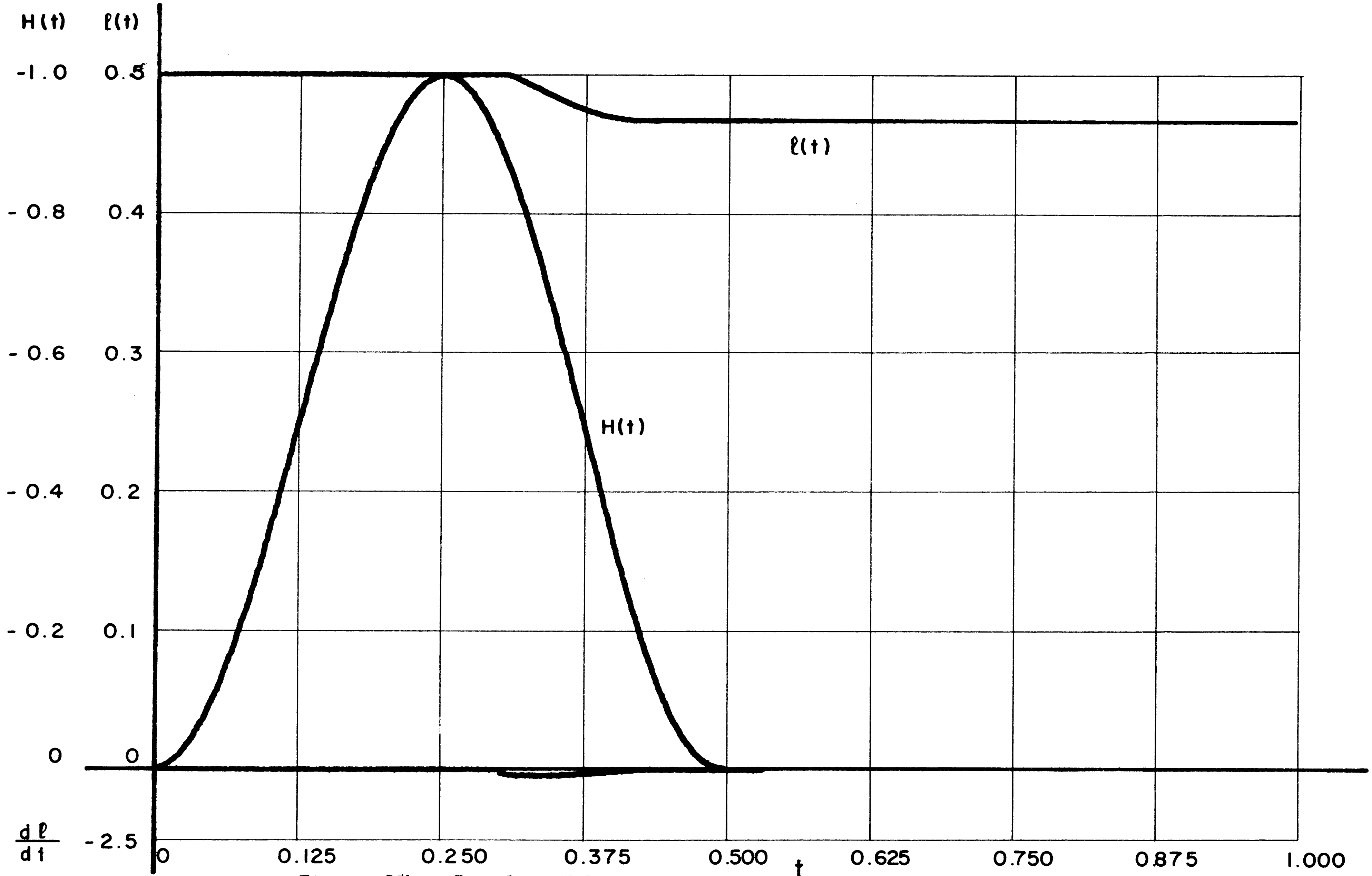


Figure G8b. Boundary Velocity and Position and Heat Pulse vs. Time.
$\ell(0) = 0.5$; $g_n = -1.0$; $H = 1 - \cos 4\pi t$; $r = 6$.
Parabolic Station Distribution in $\hat{x}$.

APPENDIX H

THE EXACT VALUE OF t_m, FINITE SLAB

An equation for the value of the time required to melt the slab, t_m, is derived by Landau in Reference 27 by means of a contour integration about the $(\tilde{x},\tilde{t})$ domain of Figure 15, Chapter IV. The same equation can be obtained, again for a constant heating rate, H, and a uniform initial temperature, g, in the interval $0 \leq x \leq \ell(0)$ by considering the energy required for warming the slab and for melting it. Under these conditions and with unit specific heat, density, and latent heat of fusion as in the nondimensional problem with temperature independent thermal characteristics the pertinent energy quantities per unit surface are are:

$-t_m H$ = total energy supplied during the interval $0 \leq t \leq t_m$;

$-g\ell(0)$ = the energy required to raise the entire slab to zero temperature without melting;

$\ell(0)$ = the energy required to melt the slab without any change in temperature.

The total energy supplied is the sum of the last two quantities. Thus

$$-t_m H = g\,\ell(0) + \ell(0).$$

This equation yields the following expression for t_m:

$$t_m = \frac{g-1}{H}\,\ell(0).$$

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