A mechanism design approach to decentralized resource allocation in wireless and large-scale networks: Realization and implementation

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Electrical Engineering: Systems)
in The University of Michigan
2009

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To my grandparents, parents, and all my teachers

ACKNOWLEDGEMENTS

I still remember the day when my advisor Professor Demosthenis Teneketzis had told me, "You will have to believe me about the potential of this field". Today when I see the vast scope of research in mechanism design and its potential to create an impact on a huge number of applications, I do not have enough words to thank him for introducing me to this field. His vision, patience, and dedication to educate his students enabled me to pursue my research in a field that I could have never imagined before the graduate school. I express my humble gratitude to him for his friendly and ever inspiring mentorship.

I am grateful to my thesis committee for their expert feedback on my thesis. I express my gratitude to Professor Yan Chen and Professor Tilman Borgers for giving me extra time for discussions and providing me valuable insights on economic concepts. I also thank Professor Achilleas Anastasopoulos for his stimulating discussions that helped me appreciate the application of our work within the context of wireless networks. I am grateful to Professor Mingyan Liu for her excellent teaching in the course "Communication Networks" that intrigued my interest to pursue research in networks.

I extend special thanks to Professor Volodymyr Babich and Professor Mark Van Oyen for giving me an opportunity to work with them and explore the application of mechanism design in the field of supply-chain. Their insightful discussions helped me build intuition behind complicated mathematics and provide interpretation of results in the context of real systems.

I express sincere thanks to my mentors at IBM T.J. Watson Research Center, Drs. Asser Tantawi, Michael Spreitzer and Malgorzata Steinder for providing me with an exposition to a number of real world problems during my internship. My discussions with them helped me appreciate the practical significance of my work.

I thank all my instructors at the University of Michigan, Professors Serap Savari, Petar Momcilovic, Wayne Stark, Jessy Grizzle, Romesh Saigal, Marina Epelman, Michael Wellman, Yusufcan Masatlioglu, and Emre Ozdenoren for enriching my academic experience. I also thank Professors David Neuhoff, Alfred Hero, Stéphane Lafortune, Kim Winick, and Alexander Ganago for enriching my pedagogical skills with their experience during my Graduate Student Instructorship. I am also thankful to the Graduate Student Mentor program at the University of Michigan and its coordinator Dr. Tershia Pinder-Grover for providing me a broad exposure to learning and teaching techniques.

I thank my colleagues Dr. Aditya Mahajan, David Shuman, Ashutosh Nayyar, Ali Kakhbod and Dr. Abhishek Dubey for many useful discussions and suggestions to improve my presentations. I also thank my friend and colleague Ankit Jain for his critical remarks that helped improve the written presentation of my work.

I express my sincere gratitude to Ms. Rebecca Turanski, Beth Lawson, Karen Liska, Ann Pace and Nancy Goings for their administrative support that helped me concentrate on my studies.

I thank my roommates Pratibha Permandla, Sonia Gupta and Lynn Zwinck, my friends Gayathri Seenumani, Krithika Srinivasan, Nabanita Datta, Sujay Phadke, Anindya Bhadra and Ramashis Das, all my dinner group friends, and the members of Indian Classical Music and Dance Group, Tau Beta Pi, and Eta Kappa Nu at the

University of Michigan for making my stay in Ann Arbor an enjoyable experience.

I express my heartfelt gratitude to my uncle Dr. Ajit Shenoy and aunt Dr. Uma Shenoy for providing me with a homely atmosphere in Ann Arbor and for all their help with my initial settlement in Ann Arbor.

Finally, I express my respectful gratitude to my family who have always provided me unconditional love, encouragement, and support. My parents' and grandparents' hard work, integrity and dedication to teaching has always been an inspiration for me. My grandfather's blessings to become a successful academic and my grandmother's all-rounded personality and achievements as a women have been great inspiration for me. I thank my parents for their loving support in letting me be the first person in the family to pursue science and engineering. Last but not the least I thank my brother Abhinav Sharma for always being a best friend of mine and all my dear cousins for always cheering up my life.

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ABSTRACT

In this thesis we present a mechanism design approach to decentralized resource allocation in wireless and large-scale networks. First, for wireless networks we study the problem of power allocation where each user's transmissions create interference to all network users, and each user has only partial information about the network. We investigate the problem under two scenarios; the realization theory scenario and the implementation theory scenario. Under the realization theory scenario, we formulate the power allocation problem as an allocation problem with externalities, and develop a decentralized optimal power allocation algorithm that (i) preserves the private information of the users; and (ii) converges to the optimal centralized power allocation. Under the implementation theory scenario, we formulate the power allocation problem as a public good allocation problem, and develop a game form that (i) implements in Nash equilibria the optimal allocations of the corresponding centralized power allocation problem; (ii) is individually rational; and (iii) results in budget balance at all Nash equilibria and off equilibria. Then, we generalize the wireless network model to study resource allocation in large-scale networks where the actions of each user affect the utilities of an arbitrary subset of network users. This generalization is motivated by several applications including power allocation in large-scale wireless networks where the transmissions of each user create interference to only a subset of network users. We develop a formal model to study resource allocation problems in large-scale networks with above characteristics. We formu-

late two resource allocation problems for the large-scale network model; one for the realization theory scenario, and the other for the implementation theory scenario. For the realization theory scenario we develop a decentralized resource allocation algorithm (using the principles of mechanism design) that (i) preserves the private information of the users; and (ii) converges to the optimal centralized resource allocation. For the implementation theory scenario we develop a game form that (i) implements in Nash equilibria the optimal allocations of corresponding centralized resource allocation problem; (ii) is individually rational; and (iii) results in budget balance at all Nash equilibria and off equilibria.

CHAPTER 1

Introduction

1.1 Motivation

Networks exist in a vast variety of real world systems. They have played an important role in the social and technological growth of our society. Some prominent examples of networked systems are urban & transportation systems, military systems, political/social networks, production and consumer markets, supply-chains, energy markets, internet, web data centers, electronic commerce systems, sensor networks and telecommunication systems. Because of the diversity of network applications, networks are studied in a wide range of professional and academic domains including engineering, business management and social science.

In electrical and systems engineering wireless communication networks have received a significant research focus as they form part of many systems that are of interest to these disciplines. For example, wireless networks form the basis of bluetooth and Wi-Fi networks, mobile cellular networks, satellite communication, sensor networks, surveillance networks, and military communication networks.

Irrespective of the diversity of applications, a fundamental similarity in all of the above networks is that, (i) the network consists of multiple agents that interact with and influence each other; (ii) each agent has different characteristics and a different individual role in the network; and (iii) the actions of individual agents together with their interactions determine the function/performance of the network. An alternative term for networks that captures the above characteristics is multi-agent systems. In many applications such as electronic commerce, artificial intelligence and social networks, the use of the term multi-agent system is more common. In this thesis we will use the term network or multi-agent system interchangeably to describe a network.

Apart from the fundamental similarities in the structure and function of networks described by the above mentioned features, an identical objective in the design of all networks is their efficient operation. This requires optimization of network performance measures. As mentioned above, a network's performance is determined by the collective actions of network agents. Actions that are critical in determining a network performance are – consumption/generation of resources by network agents and their decisions regarding network tasks. Therefore, for a network to achieve its performance objective, proper allocation of the network's resources and coordination of the network agents' decisions are extremely important. With the technological and social advancement, many networks such as the internet, energy markets and e-commerce systems are expanding at a very fast pace. The resources that are required for the operation of these networks, e.g. bandwidth, fossil fuels, and web server resources often do not increase at the same rate. Therefore, in these cases resource allocation and utilization become even more crucial for efficient network operation.

In the context of wireless communication networks some of the important resources are bandwidth, energy, coding schemes, relay routes, and the physical space available for the network. The important performance measures are data communication rate, probability of error, communication delay, battery life, interference, mobility of agents, ability to dynamically adjust to varying channel conditions, etc. The requirement to efficiently utilize the above resources to achieve desirable performance gives rise to several challenging resource allocation problems in wireless networks. Examples of such problems are spectrum/rate/code allocation that govern throughput and delay, power and code allocation that govern interference and battery life, admission control that governs the number of agents in the network, topology control that governs the placement and interconnections of network agents, and dynamic resource allocation that looks at the above aspects under dynamic situations. The numerous applications and the technical challenge of these resource allocation problems make them an important and exciting area of wireless networks research. This motivated us to investigate some of these problems in this thesis.

1.2 Key issues and challenges in resource allocation

An inherent characteristic of many real world networks that makes resource allocation challenging is the *decentralization of information*. Information decentralization arises due to the following reasons: One, the large-scale nature of the networks. To have a centralized control in such networks requires the communication of enormous amounts of data to a central controller and this is practically infeasible. For example, in a sensor network with thousands of sensors, it is difficult to keep a centralized record of the data collected at all sensors. Two, the presence of selfish/competitive network agents who do not want to reveal their private information and thus make centralized control infeasible. For example, cell phone manufacturers may not want to reveal their chip technologies for competitive reasons; thus, the service provider in a cellular network cannot have complete information on the signal processing technologies of the cell phones in the network.

Even in networks where centralized resource allocation is feasible, it is often undesirable from an implementation point of view, because, (i) centralized allocation is computationally intensive; (ii) it is not scalable; and (iii) a single failure in the central control may disrupt the entire network operation.

For all the reasons described above, it is highly desirable to have decentralized resource allocation in networks. Allocating resources in a decentralized way requires a completely different philosophical framework from that of centralized allocation methods. Some prominent features of decentralized resource allocation problems that make them substantially different from centralized ones are the following. In decentralized resource allocation there are multiple decision makers unlike centralized allocation where the allocation decisions are taken by a single central agent. In a decentralized system each decision maker makes its decision based on some partial information about the network, whereas the decision maker in a centralized system has complete network information. Because the decisions are based on partial network information, a decentralized resource allocation may not achieve network performance similar to a centralized allocation. In order to achieve a performance which is equivalent to a centralized one, the agents in the decentralized system must communicate and exchange information with one another. They must then make decisions based on their private information and the information exchanged. Thus, communication among the agents is an essential component of any decentralized allocation method that aims to achieve centralized performance without complete revelation of the agents' private information. This is contrary to centralized methods where, either communication is not required at all, or, the central agent is able to gather *all* network information before it determines the allocations.

The individual agents' behavior is critical in determining the performance of a decentralized resource allocation process. Therefore, decentralized resource allocation problems can be classified into two classes based on the characteristics of the network agents. The first class represents scenarios in which the network agents cooperate to achieve the network objective. Because the agents are cooperative, they obediently follow the rules of any decentralized resource allocation mechanism that is designed to achieve the network objective. Therefore, for this class of problems, the challenge in achieving a desirable performance lies in designing an appropriate set of rules that tell the network agents “what” to communicate and “what” action to take based on their information. The second class represents scenarios in which the network agents are selfish or competitive and whose individual objectives differ from the network objective. Under such a scenario, the network agents may not be willing to follow the rules of a decentralized resource allocation mechanism if the mechanism requires the agents to take actions that are not aligned with their individual objectives. Therefore, for this class of problems, the challenge lies not only in decentralizing the resource allocation process but also in providing appropriate incentives to the agents that induce them to take actions that lead to achieving the network objective. In this thesis we study problems from both classes of decentralized resource allocation scenarios.

As is evident from the above discussion, addressing decentralized resource allocation problems requires a framework that can provide a systematic methodology for the design of decentralized resource allocation mechanisms by harnessing the decentralized information characteristics of the networks and the behavioral characteristics of the agents. One such framework for the systematic study of decentralized resource allocation problems is provided by the theory of *mechanism design* which is a branch of mathematical economics [36].

In this thesis we follow the philosophy of mechanism design to address decentralized resource allocation problems in wireless and large-scale networks. Therefore, in the next section we present a brief discussion on the mechanism design approach to resource allocation.

1.3 Mechanism design

Mathematical economists have studied decentralized resource allocation problems for a long time. These studies have been motivated by the characteristics of economic systems which are in general multi-agent systems with decentralized information. An example of a typical economic system is a consumer good market that consists of multiple decision makers – the consumers and the manufacturers of the good. In this market the information is decentralized as the manufacturer of a good does not know a consumer's personal preference for various goods, and a consumer does not know the cost of production of various goods for a manufacturer.

The theory of mechanism design was developed by mathematical economists to provide a formal treatment of decentralized resource allocation problems [20, 21, 24]. In particular the theory was developed to provide guidelines for the design of decentralized mechanisms that can obtain centralized allocations. To appreciate the approach provided by mechanism design, we briefly present the conceptual ideas behind this theory.

In mechanism design a resource allocation problem is described by the triple $(\mathcal{E}, \mathcal{D}, \gamma)$: the *environment space* $\mathcal{E}$, the *action space* $\mathcal{D}$ and the *goal correspondence* γ.

The environment space $\mathcal{E}$ is defined to be the space of all possible environments $\boldsymbol{e}$ of the resource allocation problem, where an environment refers to a set that specifies for each network agent the infrastructure and the information available to it as well as its preference for the allocations/outcomes in the network. The action space $\mathcal{D}$ is defined to be the set of all actions/resource allocations that are feasible in the network. Thus, the environment space and the action space together specify a network model. The specification of a network in this form provides a formal way to describe (through the agents' environments) the information decentralization of the network.

A centralized performance objective for the network described by $(\mathcal{E}, \mathcal{D})$ is specified by the goal correspondence γ. Specifically, γ assigns for every environment $\boldsymbol{e} \in \mathcal{E}$, the set of allocations $\gamma(\boldsymbol{e}) \subset \mathcal{D}$ that meet some pre-specified network wide performance objective. The specification of allocations $\gamma(\boldsymbol{e})$ in terms of complete network information $\boldsymbol{e}$ allows the description of any centralized objective with this formulation.

In an informationally decentralized network each agent knows only a part of the network environment $\boldsymbol{e}$; hence, no network agent can directly compute the centralized allocations $\gamma(\boldsymbol{e})$. As discussed in Section 1.2, any allocation mechanism for these networks that aims to achieve the performance of a centralized allocation scheme $(\mathcal{E}, \mathcal{D}, \gamma)$ must consist of (i) an information exchange process among the network agents; and (ii) an allocation process based on the outcome of information exchange. Mechanism design focusses on developing rules for such information exchange and allocations so as to achieve the centralized performance specified by γ.

The theory of mechanism design is divided into two components: *realization theory* and *implementation theory/theory of incentives*. These components address the two classes of decentralized resource allocation problems described in Section 1.2.

Realization theory addresses the first class of problems in which the network agents cooperate to achieve the network objective. In realization theory, a decentralized mechanism is specified in terms of the following components: (i) A *message space* for each agent that specifies the set of messages the agent can use to communicate with other agents. (ii) A *message communication rule* for each agent that specifies how the agent should generate a message in its message space based on its information about the network. To obtain an optimal centralized allocation the message exchange process usually needs to be iterative. In such a case, the message communication rule also specifies how to take into account the messages received from the other agents in past iterations when generating one's new message. An equilibrium message correspondence specifies the set of equilibrium messages resulting from the message exchange process. (iii) An *outcome function* that specifies the rules to determine allocations based on equilibrium messages. The designer of a mechanism must simultaneously choose the above three components so as to achieve the network objective.

Implementation theory addresses the second class of decentralized resource allocation problems in which the network agents are selfish/competitive [25, 41, 38] and their individual objectives differ from the network objective. Under this scenario, any mechanism that aims to optimize the network performance criterion and relies on prescribed communication rules to achieve it may fail to obtain its targeted outcome. This is because the agents can attempt to divert the outcome of the mechanism in their own favor by communicating messages that do not correspond to their true environment. Because each agent's environment is usually its private information, the other agents or the mechanism designer cannot check if an agent follows the prescribed communication rule to generate its message. To overcome this problem, a decentralized mechanism in implementation theory is specified only in terms of the message space and the outcome function. A mechanism with this structure is called a *game form*. Since the message space and the outcome function are externally specified, they can be made common knowledge. Therefore, it can easily be verified if the agents generate their messages in the specified message space, and, given the equilibrium messages, if they compute the allocations as specified by the outcome function. The specification of a decentralized mechanism as a game form allows the selfish users to strategically choose messages from their message space; thus, it induces a game among the users. Depending on the agents' information about the network environment, there are appropriate equilibrium concepts for the induced game. For example, for games of complete information the equilibrium concepts are Nash equilibrium, subgame perfect equilibrium, dominant strategy equilibrium, rationalizability, etc. For games of incomplete information the equilibrium concepts are Bayesian Nash equilibrium, Perfect Bayesian equilibrium, etc. Given an equilibrium concept, the specific equilibria that an induced game can attain are governed by the design of the game form. Thus, in implementation theory, a game form along with an equilibrium concept indirectly specifies the (equilibrium) message communication rule. In order to (indirectly) drive the induced game to attain equilibria that result in the optimization of the network performance criterion, the outcome function must provide appropriate incentives to the agents to align their individual objectives with the network objective. Because the agents' behavior is mainly controlled through the outcome function, the complexity of the design of an implementation mechanism lies in its outcome function. This is different from a realization mechanism where the complexity of the design usually lies in the communication rule as the agents' behavior is directly controlled by the communication rule.

As can be seen from the discussion so far, mechanism design utilizes the fundamental nature of networks to build a logical structure for the design of decentralized resource allocation mechanisms. Such an approach can provide insights into the basic characteristics of any decentralized allocation problem to be solved and therefore, has a potential to provide a fundamental framework for the study of decentralized resource allocation problems. For these reasons in this thesis we chose to the philosophy and approach of mechanism design to address decentralized resource allocation problems.

In the next section we state the contributions of this thesis.

1.4 Contribution of the thesis

The main contribution of this thesis is the formulation and solution of decentralized resource allocation problems arising in wireless communication networks and other large-scale networks within the context of mechanism design.

Initially, in the context of wireless networks we address the problem of decentralized power allocation for systems where the transmissions of every agent create interference to every other agent. The mechanism design approach helped us to classify this problem as a problem of public good allocation (alternatively, as an allocation problem in the presence of externalities). Based on this classification, we developed decentralized mechanisms for two power allocation problems: one that addresses the realization theory scenario, and the other that addresses the implementation theory scenario.

The exercise of formulating the power allocation problems as public good allocation problems (alternatively, allocation problems with externalities) not only helped us to obtain solutions to these specific power allocation problems but also provided us with insights on the considerations that are required to address decentralized resource allocation problems in general. In particular, the approach of mechanism design helped us to characterize the features of network (centralized) objective achieving decentralized mechanisms based on the network structure and the behavior of the network agents. In the second half of the thesis we discuss these features for both realization theory and implementation theory scenarios, and use them as guidelines for the design of decentralized resource allocation mechanisms for generic large-scale networks.

Specifically, by large-scale networks we refer to networks where the actions of each agent affect the utility of an arbitrary subset of network agents, and each agent knows only that part of the network that either affects it or is affected by it. This network model resembles network of local public goods, and is motivated by several applications including large-scale wireless networks where the transmissions of each user create interference to a subset of network users. For this network model we formulate decentralized resource allocation problems from both the realization theory and implementation theory perspectives, and develop decentralized resource allocation mechanisms that obtain optimal centralized allocations. To the best of our knowledge the formulation of these resource allocation problems and their solutions is the first attempt to analyze the large-scale network model in the framework of realization theory or implementation theory. Therefore, we believe that the formulation and solution of these problems are not only contributions to the engineering literature but also contributions to the state of the art in mechanism design.

Below we discuss the specific contributions of the thesis in each of the problems we investigate for wireless networks and large-scale networks.

- *Power allocation in wireless networks: A realization perspective*
  - A novel formulation of a power allocation problem for wireless networks

with interference as an allocation problem with externalities.

- The specification of an iterative decentralized power allocation algorithm for the above problem that has the following properties:
  (i) It preserves the private information of each agent.
  (ii) It guarantees convergence to the network optimal power allocation (optimal centralized allocation).

• *Power allocation in wireless networks: An implementation perspective*

- A novel formulation of a power allocation problem for single cell uplink network with interference and strategic users as a public good allocation problem.
- The specification of a decentralized power allocation mechanism (game form) for the above problem that possesses the following properties:
  (i) All Nash equilibria (NE) of the game induced by the mechanism result in allocations that are optimal solutions of the corresponding centralized uplink problem (Nash implementation, cf Section 3.2.1).
  (ii) All users voluntarily participate in the allocation process specified by the mechanism (individual rationality, cf Section 3.2.1).
  (iii) Budget balance at all NE and off equilibrium.

• *Resource allocation in large-scale networks: A realization perspective*

- The formulation of a decentralized resource allocation problem for large-scale networks (where the actions of each agent affect a subset of network agents) in the framework of realization theory.
- The specification of an iterative decentralized resource allocation algorithm for the above problem that has the following properties:

(i) Each agent in the network needs to communicate only with those agents that either affect it or are affected by it.

(ii) It preserves the private information of each agent.

(iii) It guarantees convergence to the network optimal resource allocation (optimal centralized allocation).

- *Resource allocation in large-scale networks: An implementation perspective*

  – The formulation of a decentralized resource allocation problem for large-scale networks (where the actions of each agent affect a subset of network agents) in the framework of implementation theory.

  – The specification of a decentralized resource allocation mechanism (game form) for the above problem that possesses the following properties:

  (i) All Nash equilibria (NE) of the game induced by the mechanism result in allocations that are optimal solutions of the corresponding centralized resource allocation problem (Nash implementation).

  (ii) All users voluntarily participate in the allocation process specified by the mechanism (individual rationality).

  (iii) The mechanism results in budget balance at all NE and off equilibrium.

1.5 Organization of the thesis

This thesis is organized as follows: In Chapter 2 we present the problem of power allocation in wireless networks from the realization perspective. In Chapter 3 we present the problem of power allocation in wireless networks from the implementation perspective. In Chapter 4 we present the problem of resource allocation in large-scale networks from the realization perspective. In Chapter 5 we present the problem of resource allocation in large-scale networks from the implementation perspective. We conclude in Chapter 6.

CHAPTER 2

Power allocation in wireless networks: A realization perspective

Wireless communication applications have seen a tremendous growth in demand over past several years. This has made efficient utilization of resources extremely important for these networks. A characteristic of wireless networks that makes it different from wired networks is that all users share the *same* communication channel. Therefore, the signal transmitted by a source is not only received by its intended destination but also other unintended destinations; as a result these unintended destinations experience interference to the reception of their desired signals. Hence, interference control is an important issue in wireless networks. In this chapter we study transmission power allocation to control interference and optimize the network performance.

We consider a wireless network where the transmission of a user creates interference to all other users and directly affects their utilities. The network has multiple interference temperature constraints to control interference. We consider a decentralized information scenario, in which each user knows only its own utility and the channel gains from the transmitters of other users to its own receiver. For the above network we address the power allocation problem from the realization theory perspective, and propose a decentralized power allocation algorithm that has the following properties: (i) It preserves the private information of each agent. (ii) It guarantees convergence to the network optimal power allocation.

The chapter is organized as follows: In Section 2.1.1 we present the network model. In Section 2.1.2 we present the power allocation problem. We present a literature survey in Section 2.1.3 and discuss our motivation to investigate the problem presented in this chapter in Section 2.1.4. We state our contributions in Section 2.1.5. In Section 2.2.1 we formulate the power allocation problem of Section 2.1.2 as an allocation problem with externalities. In Section 2.2.2 we present a decentralized power allocation algorithm based on the externalities formulation and we discuss the properties of the algorithm. We prove these properties in Appendix 2.A. In Section 2.2.3 we show how the decentralized resource allocation problem and its solution presented in this chapter fit within the framework of realization theory. We conclude in Section 2.3 with numerical results for two practical examples.

2.1 The power allocation problem

We begin this section with a description of the wireless network model and the assumptions we make for its analysis. We also discuss scenarios that motivate the model. We then formulate a power allocation problem for the above model.

2.1.1 The model (M.2)

We consider a wireless network consisting of N transmitter-receiver pairs (connected by solid arrows in Fig. 2.1). We call each transmitter-receiver pair a user [2.1], and we denote the set of all users in the network by $\mathcal{N} := \{1, 2, \dots, N\}$. The users,

[2.1] In the rest of the chapter, any action by a user associated with the transmission of the signal means that it is done by the corresponding transmitter, and any action/computation associated with the reception of a signal means that it is done by the corresponding receiver.

in other words the transmitters and the receivers, can be arbitrarily located in the network. This model captures a wide variety of scenarios, e.g. when the transmitters and receivers are located anywhere in the network, the model can represent a wireless ad hoc network or a segment of a wireless mesh network; if the transmitters are co-located it can represent a cellular downlink network; and if the receivers are co-located it can represent a cellular uplink network. User $i, i \in \mathcal{N}$, transmits with

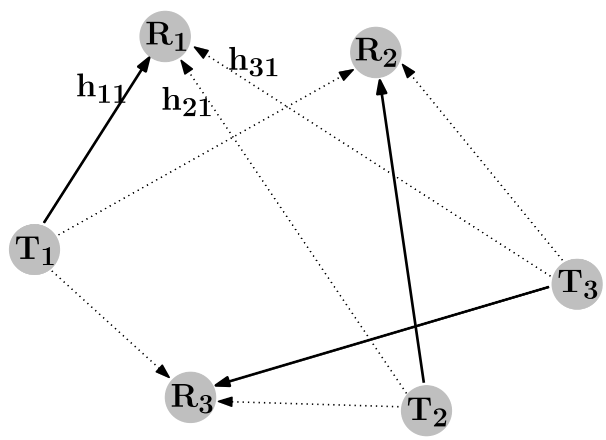

Figure 2.1: An example of a wireless mesh network with three users (pairs of nodes); T_i and R_i denote the transmitter and the receiver of user i respectively, and h_{ij} are the channel gains from T_i to R_j.

power p_i. We assume that

Assumption 2.1. *Every user's transmission creates interference (shown by the dashed arrows in Fig. 2.1) to all other users in the network i.e. the graph in Fig. 2.1 is fully connected.*

Assumption 2.1 implies that the interference to user i, $i \in \mathcal{N}$, depends on the transmission power $p_j, j \in \mathcal{N}, j \neq i$, of all other users.

To control interference the system has Interference Temperature Constraints (ITCs) at K different Measurement Centers (MCs), $MC_1, MC_2, \ldots, MC_K$. Interference Temperature (IT) is defined in [1] as the net Radio Frequency (RF) power measured at a receiving antenna per unit bandwidth; ITC is a constraint that puts an upper limit on the IT. To keep the RF noise floor in a wireless network below a safe threshold it is desirable that the network satisfies an ITC. In our model we consider multiple ITCs because multiple ITCs can ensure a balanced interference threshold throughout the network. Each ITC is governed by one of the MCs. The MCs can be installed either at the receivers of the users or there can be separate stations acting as MCs. To simplify the notation, we refer to MC_k as user $0_k, k \in \mathcal{K} := \{1, 2, \ldots, K\}$, and we denote the set of MCs by $0_{\mathcal{K}} := \{0_1, 0_2, \ldots, 0_K\}$. The ITCs are given by

$$\sum_{i=1}^{N} p_i h_{i0_k} \leq P_k, \qquad k \in \mathcal{K} \tag{2.1}$$

where h_{i0_k} is the channel gain from user i's transmitter to the kth measurement center. We assume that

Assumption 2.2. *Each measurement center $MC_k, k \in \mathcal{K}$, can measure the channel gains h_{i0_k}, $i \in \mathcal{N}$, hence it knows these channel gains* [2.2]. *However, MC_k need not know the channel gains h_{i0_z}, $i \in \mathcal{N}, z \neq k$, to other MCs.*

Because of the presence of interference, the quality of service of a user in such a network depends not only on the power received from its own transmitter but also, on the total interference, which depends on the transmission powers of all other users. Hence to quantify the users' performance, we associate with each user $i, i \in \mathcal{N}$, a utility function $u_i(p_1, p_2, \ldots, p_N)$ which is a function of the transmission power of all the users. We assume that

Assumption 2.3. *The transmission power p_i of user $i, i \in \mathcal{N}$, lies in $\mathcal{P}_i = [0, P_i^{max}]$, and the set $\mathcal{P}_i$ is user i's private information i.e. it is known only to user i and nobody else in the system. Furthermore, a set $\mathcal{P} = [0, P^{max}] \supset \cup_{i \in \mathcal{N}} \mathcal{P}_i$ is common knowledge to all the users (including the MCs).*

[2.2] We assume that the users in the network are cooperative (Assumption 2.9). Therefore, the transmitters can periodically transmit pilot signals with known amplitudes; the receivers at the MCs can then measure the received amplitudes from which they can compute the channel gains.

Unlike active users $i \in \mathcal{N}$ the MCs do not receive any personal benefits from the power allocation, therefore, we assume that the MCs have zero utilities,

Assumption 2.4. $u_{0_k}(p) = 0, \quad \forall\, k \in \mathcal{K},$

$$\textit{where} \;\; p := (p_1, p_2, \ldots, p_N).$$

For users $i \in \mathcal{N}$ we assume that,

Assumption 2.5. *$u_i(p), i \in \mathcal{N}$, from $\mathbb{R}^N$ into $\mathbb{R}$ is a strictly concave, continuous function of p.*

In Appendix 2.B we present an example of a utility function that satisfies Assumption 2.5.

We also assume that,

Assumption 2.6. *Each user's utility function is its own private information.*

The above assumption captures a variety of scenarios. One such scenario is a multimedia wireless communication network where different users run different applications, each application, with a different utility associated with it which is known only to the user that runs the application. Another possible scenario is where the received data is processed/decoded by different users using different technologies which are not public information. In either scenario, we assume that it is not feasible for the users on informational grounds to communicate their exact utility functions to other users in the network. [2.3]

Because of Assumption 2.1 every user can hear every other user in the network, therefore we assume that,

[2.3] Since each user's utility function is concave, it is generally parameterized by an infinite number of parameters. Furthermore, since certain regularity conditions must be satisfied by the communication rules employed by the users (see [23]), the dimension (see [23]) of the message space (the space used for message exchange) must be infinite.
Thus, communication of the users' utility functions is infeasible on informational grounds.

Assumption 2.7. *The number of participating users (including the MCs) $N + K$ is common knowledge. We also assume that the set of active users $\mathcal{N}$ remains fixed during a power allocation period.*

If the time scale in which a power allocation is determined is sufficiently small, the system can be assumed to be static for an allocation period. Therefore, we assume that

Assumption 2.8. *The channel gains $h_{i0_k}, i \in \mathcal{N}, k \in \mathcal{K}$, and the utilities of the users remain constant during a power allocation period.*

Finally, we make the following assumption about the users' behavior.

Assumption 2.9. *All users in the network obediently follow the rules that any mechanism specifies to determine their power allocations.*

Examples of situations where Assumption 2.9 holds are the following: (i) Networks which are owned/managed by a single network operator. For example, a sensor network installed by an operator, or a satellite communication network owned by a communication service provider. (ii) Networks in which all users have a common objective which is also the network objective. For example, a military communication network.

In the next section we formulate the resource allocation problem for the network model (M.2).

2.1.2 The power allocation problem

For the wireless network model (M.2), the objective is to determine the users' transmission powers under the constraints imposed by the model so as to maximize the sum of users' utilities. We formally write this optimization problem, that we call

Problem (P.2), as follows:

Problem (P.2)

$$\max_{p} \sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} u_i(p) \tag{2.2}$$

$$\text{s.t. Assumptions 2.1–2.8,} \tag{2.3}$$

$$p \in \mathcal{D} := \{p \mid \sum_{i=1}^{N} p_i h_{i0_k} \leq P_k,\ k \in \mathcal{K},\ p_i \in \mathcal{P}_i,\ \forall\ i \in \mathcal{N}\}. \tag{2.4}$$

Because of Assumptions 2.2,2.3, 2.6 and 2.7, Problem (P.2) is a decentralized optimization problem i.e. none of the users in the network has complete information of *all* the parameters that describe Problem (P.2). Our objective is to develop an algorithm which satisfies the above informational constraints of Problem (P.2) and obtains optimal solutions of the corresponding centralized problem where one of the users/a center has complete information of *all* the parameters that describe Problem (P.2). The centralized counterpart of Problem (P.2) is,

$$\max_{p} \sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} u_i(p) \tag{2.5}$$

$$\text{s.t. } p \in \mathcal{D}, \text{ and Assumptions 2.1, 2.4, 2.5, 2.7 and 2.8.} \tag{2.6}$$

It should be noted that the centralized counterpart of Problem (P.2) is a strictly concave optimization problem and hence it has a unique optimum solution. The solution of this centralized problem is the ideal power allocation that we would like to obtain. If there exists an entity that has centralized information about the network, i.e. it knows all the utility functions $u_i, i \in \mathcal{N}$, all feasible power sets $\mathcal{P}_i, i \in \mathcal{N}$, and all ITCs $\sum_{i=1}^{N} p_i h_{i0_k} \leq P_k, k \in \mathcal{K}$, then, that entity can compute the ideal power allocation by solving the above centralized p-problem. Therefore, we call the solution of the centralized counterpart of Problem (P_C.2) the optimal centralized power allocation. In the network described by Model (M.2), there is no entity that knows perfectly all the parameters that describe the centralized counterpart of Problem (P_C.2). Therefore, we need to develop a decentralized mechanism that allows the network users to communicate with one another and that leads to the optimal solution of the centralized counterpart of Problem (P_C.2). Because we assume that the users obediently follow the rules specified by a mechanism (2.9), the outcome of any mechanism we design will be the same as that predicted by the mechanism.

In the next section we present a literature survey on previous works on decentralized power allocation in wireless networks, and we discuss our motivation to investigate the power allocation problem presented in this section.

2.1.3 Literature survey

Decentralized mechanisms for power allocation/control in wireless networks have received considerable attention in the literature. These mechanisms can be classified according to their application and the structure of the underlying network. Wireless networks can be broadly classified into two types; networks with hierarchical structure and networks without hierarchical structure. In networks with hierarchical structure, users communicate with each other via one or more central entities (called base stations in wireless cellular networks) and these central entities often play an important role in determining the power allocation. The hierarchical structure in cellular networks can further be classified as uplink or downlink based on whether the communication is from the users to a base station or vice versa. In networks without hierarchy, users communicate with one another directly without any central entity. Examples of such networks are ad hoc or mesh networks. Below we present a brief summary of the existing work on decentralized power allocation for voice and data networks. Within each category we present existing results for hierarchical and non-hierarchical network structures.

One of the most well-known decentralized algorithms for power control in fixed data rate cellular voice network was proposed by Foschini and Miljanic in [13]. The algorithm proposed in [13] requires only local measurements, and achieves the desired minimum Signal to Interference Ratio (SIR) for each user with exponentially fast convergence if there exist power levels that meet these requirements for the SIRs. Later in [58] the power control problem similar to that of [13] was formulated as a utility maximization problem by replacing the hard SIR constraints of [13] with sigmoid utilities. With this modification, the algorithm proposed in [58] overcomes the difficulty of divergence which occurs in the algorithm of [13] when there are no feasible power levels that can attain the desired SIRs. The algorithm of [58] also has the flexibility to be tuned for both voice and data services. Lately, in [18] the Foschini-Miljanic algorithm of [13] was generalized for time varying channels and ad-hoc networks.

For wireless data networks, results on decentralized mechanisms for uplink power control can be found in [46, 12, 28, 47, 2]. In [12] the problem of uplink power control in a single cell Code Division Multiple Access (CDMA) data network was formulated as a utility maximization problem. An uplink problem similar to that of [12] with SIR based utilities was also investigated in [28]; in this paper the existence of an equilibrium was shown and an algorithm for solving the decentralized power control problem was suggested. The problem formulated in [12] was reinvestigated in [47] using pricing; it was shown that pricing results in multiple equilibria which are Pareto superior to the equilibria obtained in [12] and [28]. Pricing-based analysis of the uplink power control problem was also done in [2], by introducing user specific parametric utility functions. The authors of [2] proposed two decentralized algorithms, the parallel update and the random update algorithms, that converge to the unique equilibrium of the problem. Work on pricing for downlink CDMA data networks can be found in [32, 59, 30]. In [32] and [59], optimal resource allocation strategies were determined for a single class CDMA system for the case when the utility functions of the users are common knowledge (see [4, 57] for the definition of common knowledge). In [30], the downlink power allocation problem for multi-class CDMA networks was studied by a decentralized mechanism based on dynamic pricing and partial cooperation between mobiles and the base station. This mechanism achieves a partial-cooperative optimal power allocation which was shown to be close to a globally-optimal power allocation. In [19] pricing ideas were used for power allocation in wireless CDMA data networks having a mesh structure. The authors studied power allocation under an ITC, and proposed two auction-based power allocation mechanisms. Under certain conditions the SIR auction of [19] achieves a power allocation arbitrarily close to a Pareto optimal one, and the power auction achieves an allocation arbitrarily close to the socially optimal one. These conditions however require in essence, that the manager should know the users' utility functions.

Having provided an overview of the existing works in the literature, we now present our motivation for studying the power allocation problem presented in Section 2.1.2.

2.1.4 Motivation

A wireless network is said to have externalities when: (i) each user's transmission creates interference to other users; and (ii) each user's utility is directly affected by the interference. Thus, the power allocation problem presented in Section 2.1.2 is an allocation problem with externalities.

As described in Section 2.1.3 power allocation problems in wireless networks where externalities are present were previously considered in [30, 12, 2, 47, 28] and [19]. In [12, 2, 47, 28] power allocation problem is formulated as a non-cooperative game and in [19, 30] power allocation problem is formulated as a social welfare maximization problem. The solution approach in all of the above references is based on different variations of pricing mechanisms. The pricing mechanisms employed in [19, 12, 47, 28] do not achieve globally optimal allocations; the pricing mechanism in [2] does not achieve optimal allocations unless the users vary their utilities according to their target signal to interference ratios; and the pricing mechanism proposed in [30] obtains close to globally optimal allocations.

The reason why pricing mechanism proposed in [30] results in a globally optimal allocation is the following. The authors of [30] introduce a constraint on the total power transmitted by the base station. Due to this constraint the original problem, where each user's utility depends on everyone's transmission power, reduces to one where each user's utility depends only on the power allocated to it. Thus, the externalities due to interference, that are present in the original problem formulated in [30], disappear, and the pricing mechanism proposed in [30] results in efficient allocations. For the cases where the system has either no maximum power constraint or has multiple power constraints, the above-stated reduction is not possible.

In general, in decentralized resource allocation problems with externalities (i.e. in problems where the resources allocated to each user directly affect the utility of every other user) pricing mechanisms fail to obtain globally optimal allocations. This fact is well known in the Economics literature (see [36, Chapter 10]) and has also been identified by the authors of [47] in the context of power allocation in wireless networks. Decentralized resource allocation problem for an economy with externalities was studied by Reichelstein in [44]. Under the assumption that the users of the system cooperate and obey the rules of the mechanism, Reichelstein determined a lower bound on the dimensionality of the message space [2.4] required by any mechanism so as to achieve globally optimal allocations. For a system with N users, this lower bound is of the order $O(N^2)$. In game-theoretic formulations of problems with externalities, the message space required by any mechanism to obtain globally optimal allocations is of even higher dimension (see discussion in [41, 37, 52]). On the other hand, the dimensionality of the message space of pricing mechanisms (including those in the aforementioned communication networks literature) is of the order $O(N)$. Thus, the information exchanged among the users in the mechanisms proposed in the aforementioned literature is not sufficient (rich enough) to lead to globally optimal allocations. The failure of pricing mechanisms to produce globally optimal solutions of power allocation problems where the users' utilities are directly affected by the interference provides the key motivation for the formulation and solution methodology presented in this chapter for power allocation in wireless networks.

In the next section we state the contributions of this chapter.

2.1.5 Contribution of the chapter

The key contributions of this chapter are:

- The formulation of power allocation problem for wireless networks with interference as an allocation problem with externalities;
- The specification of an iterative decentralized power allocation algorithm for the above problem that has the following properties:
  - (i) It preserves the private information of each agent.
  - (ii) It guarantees convergence to the network optimal power allocation.

[2.4] i.e. the space of communication language used by the users to communicate with one another.

Our formulation properly captures and directly addresses the effect of transmission power externalities on the system performance. Our problem formulation and the proposed power allocation algorithm are distinctly different from all previous studies of utility based power allocation problems, because the previous studies employ pricing mechanisms. The message space of the proposed algorithm has dimension N^2 and thus, has the same order as Reichelstein's lower bound ($O(N^2)$). This means that the information exchanged among the users in the proposed algorithm is very close to the minimum information exchange required by any mechanism that achieves globally optimal power allocations.

In the following section, we formulate Problem (P.2) as an externality problem and present a decentralized algorithm (which we call the externality algorithm) that obtains optimal solutions of the centralized counterpart of Problem (P.2).

2.2 Solution of the power allocation problem

2.2.1 Formulation as an externality problem

From each user's perspective, we divide the allocation variables into two classes. One consisting of allocations for which the user is responsible, and the other consisting of the rest of the allocation variables for which the user is not directly responsible.

Specifically, we associate with user i a variable p_i which is the power allocated to/transmitted by user i. We also associate with user i an *external environment* p_{-i} which consists of the powers allocated to/transmitted by all other users $j \neq i$. Mathematically, the external environment of user i is defined to be the vector

$$p_{-i} := (p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N). \tag{2.7}$$

As is clear, p_i is under user i's control whereas the variables in p_{-i} are controlled by other users. We define a *power profile* for the network of N users to be the full

N-dimensional vector

$$p := (p_1, p_2, \ldots, p_N). \tag{2.8}$$

By Assumption 2.3, p_i is constrained to lie in the set $\mathcal{P}_i$. In the absence of an exact knowledge of the constraint sets $\mathcal{P}_j, j \neq i$, of other users, the set of possible external environments of user i as perceived by i will be

$$\mathcal{P}^{N-1} = \{p_{-i} \mid p_j \in \mathcal{P},\ j \in \mathcal{N}\backslash\{i\}\,\}. \tag{2.9}$$

It should be noted that some of the elements of $\mathcal{P}^{N-1}$ may never actually exist as an external environment of user i because $\mathcal{P}^{N-1}$ also contains elements outside $\prod_{j\in\mathcal{N}\backslash\{i\}} \mathcal{P}_j$, which cannot be used by other users. However, since $\mathcal{P}$ is common knowledge, user i knows that its external environment must lie within $\mathcal{P}^{N-1}$. Since $\mathcal{P}^{N-1}$ is a product set of $N-1$ convex and compact sets, it is also convex and compact.

To see the effect of a user's external environment on the choices of p_i it can transmit, we first note that the presence of other users does not prohibit user i to use any power in its possible range $\mathcal{P}_i$. We call p_i to be *technically possible* for user i if, given the technical constraints of its device and the externalities, it is possible for it to use power p_i. Thus, for a given external environment p_{-i}, any $p_i \in \mathcal{P}_i$ is technically possible for user i. By combining the possible external environments of user i with the corresponding technically possible choices of p_i we define the set of power profiles that are technically feasible for user i, $i \in \mathcal{N}$, as

$$\mathcal{D}_i := \{p \mid p_i \in \mathcal{P}_i,\ \ p_{-i} \in \mathcal{P}^{N-1}\}. \tag{2.10}$$

As can be seen, $\mathcal{D}_i$ is a product set of two convex and compact sets, hence $\mathcal{D}_i$ is also convex and compact. The elements in the intersection $\bigcap_{i\in\mathcal{N}} \mathcal{D}_i$ are the only power profiles which are technically feasible for all the users in the system. Hence a feasible solution of Problem (P.2) must lie in this intersection. However, it should be noted that the technically feasible power profiles in (2.10) do not ensure that the ITCs in (2.1) are satisfied.

The responsibility of making sure that the power profiles satisfy the ITCs is given to the MCs (users 0_k, $k \in \mathcal{K}$). Since by Assumption 2.2 user 0_k knows the channel gains $h_{i0_k}, i \in \mathcal{N}$, exactly, it can check whether or not a given power profile satisfies the corresponding ITC in (2.1). We call a power profile that satisfies the k^{th} ITC as k – *constraint-feasible*. We associate the set $\mathcal{D}_{0_k}$ of k – constraint-feasible power profiles with user 0_k, $k \in \mathcal{K}$ as follows:

$$\mathcal{D}_{0_k} := \{p \mid \sum_{i=1}^{N} p_i h_{i0_k} \leq P_k, \;\; p_i \in \mathcal{P} \; \forall \, i\}. \tag{2.11}$$

Since the sets $\mathcal{D}_{0_k}$, $k \in \mathcal{K}$, are intersections of halfspaces and are bounded, they are convex and compact. We call a power profile to be *constraint-feasible* if it is k – constraint-feasible for all $k \in \mathcal{K}$. It can be seen from (2.11) that a constraint-feasible profile is acceptable for the system in terms of satisfying the ITCs but it may not be technically feasible for all users $i \in \mathcal{N}$ because it does not necessarily satisfy the technical feasibility condition $p_i \in \mathcal{P}_i, \; \forall \, i$. For a constraint to be fully *feasible* for the system, it must be both constraint-feasible as well as technically feasible for all users $i \in \mathcal{N}$. Mathematically, the set of feasible power profiles can be defined as $\mathcal{D} := \bigcap_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \mathcal{D}_i$. It should be noted that $\mathcal{D}$ is a non-empty set since $p = \mathbf{0}$ is an element of $\mathcal{D}_i \; \forall \; i \in \mathcal{N} \cup 0_{\mathcal{K}}$. Furthermore, $\mathcal{D}$ is also convex and compact since it is an intersection of convex and compact sets. Going back to the optimization problem in Section 2.1.2 it can be seen that the set $\mathcal{D}$ we have just defined is the same as the set defined in (2.4) over which the objective function in Problem (P.2) has to be optimized. But now, by separating the external environment and private information of each user from those of other users, we have decomposed $\mathcal{D}$ into a number of sets, $\mathcal{D}_i,\ i \in \mathcal{N} \cup 0_\mathcal{K}$, each of which can be associated with an individual user. Furthermore, each set $\mathcal{D}_i$ is such that user i, $i \in \mathcal{N} \cup 0_\mathcal{K}$, has complete knowledge of the parameters required to completely describe $\mathcal{D}_i$. With this decomposition we are now ready to present an algorithm for solving the power allocation problem presented in Section 2.1.2.

2.2.2 A decentralized optimal power allocation algorithm

We present a synchronous [2.5] iterative process, which we call the externality algorithm, that satisfies the informational constraints of Problem (P.2) and leads to an optimal solution of the centralized counterpart of Problem (P.2).

The externality algorithm (EA):

0) Before the start of the iterative process all users (including $0_1, 0_2, \ldots, 0_K$) agree upon a common power profile. This profile can be any arbitrary $p^{(0)} \in \{p \mid p_i \in \mathcal{P}\ \forall i\}$ that need not necessarily be a constraint-feasible or technically feasible one.

   Before the start of the iterative process the users also agree upon [2.6] a sequence $\{\tau^{(n)}\}_{n=1}^{\infty}$ of modification parameters that will be used in the algorithm. The

[2.5] In each iteration the message update is done synchronously by all the users.

[2.6] Since the users have a common objective, they can communicate with one another before the iterative process/algorithm begins, and determine $\{\tau^{(n)}\}_{n=1}^{\infty}$ and $p^{(0)}$ that will be used in the algorithm. Alternatively, $\{\tau^{(n)}\}_{n=1}^{\infty}$ as well as $p^{(0)}$ can be given to the users by the system designer.

sequence $\tau^{(n)}$ is chosen to satisfy the following three properties:

$$0 < \tau^{(n+1)} \leq \tau^{(n)} \leq 1, \ \forall\, n \geq 1, \tag{2.12}$$

$$\lim_{n\to\infty} \tau^{(n)} = 0, \tag{2.13}$$

$$\lim_{n\to\infty} \sigma^{(n)} = \infty, \tag{2.14}$$

$$\text{where,} \quad \sigma^{(n)} := \sum_{t=1}^{n} \tau^{(t)}. \tag{2.15}$$

For instance, $\tau^{(n)} = \frac{1}{n}, \ n = 1, 2, 3, \ldots,$ can be chosen as the sequence.

The counting variable n is set to 0.

1) At the nth iteration each user $i \in \mathcal{N}$ (respectively MC_k, $k \in \mathcal{K}$) maximizes its nth stage payoff on its technically feasible set $\mathcal{D}_i$ (respectively the kth – constraint-feasible set $\mathcal{D}_{0_k}$). Specifically, user i, $i \in \mathcal{N}$, solves

$$\hat{p}_i^{(n+1)} = \arg\max_{p\in\mathcal{D}_i} \left\{ u_i(p) - \frac{1}{\tau^{(n+1)}} \|p - p^{(n)}\|^2 \right\}. \tag{2.16}$$

and MC_k, $k \in \mathcal{K}$, solves

$$\hat{p}_{0_k}^{(n+1)} = \arg\max_{p\in\mathcal{D}_{0_k}} \ -\frac{1}{\tau^{(n+1)}} \|p - p^{(n)}\|^2. \tag{2.17}$$

The optimal answers [2.7] $\hat{p}_i^{(n+1)}$, $\forall\, i \in \mathcal{N} \cup 0_{\mathcal{K}}$, are broadcast to all the users in the system.

2) Upon receiving $\hat{p}_i^{(n+1)}$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, the users compute the average of all these power profiles,

$$p^{(n+1)} = \frac{1}{N + K} \sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} \hat{p}_i^{(n+1)}. \tag{2.18}$$

[2.7] Since $\mathcal{D}_i$ is a compact set and $u_i(\cdot)$ is strictly concave, a unique maximum exists for every i.

Each user i, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, also computes the following weighted average:

$$\hat{w}_i^{(n+1)} = \frac{1}{\sigma^{(n+1)}} \sum_{t=1}^{n+1} \tau^{(t)} \hat{p}_i^{(t)}, \tag{2.19}$$

$$\text{where,} \quad \sigma^{(n+1)} = \sum_{t=1}^{n+1} \tau^{(t)} = \sigma^{(n)} + \tau^{(n+1)}. \tag{2.20}$$

The counter n is increased to $n+1$ and the process repeats from Step 1).

At the $(n+1)$th iteration the average calculated in (2.18) is used as a reference power profile for maximization in (2.16) and (2.17). The new modification parameter, $\tau^{(n+2)}$, for the $(n+1)$th iteration is selected from the predefined sequence chosen in Step 0).

As stated in Section 2.1.1 the network model considered in this chapter can represent both hierarchical as well as non-hierarchical networks. For networks that have hierarchy such as a single cell cellular uplink or downlink network in which all the users and the MCs communicate with one base station that is responsible for power allocation, the externality algorithm is modified as follows. After computing $\hat{p}_i^{(n+1)}$ in Step 1), all users and measurement centers send their respective $\hat{p}_i^{(n+1)}$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, to the base station. In Step 2) of the algorithm, the base station computes $p^{(n+1)}$ and $\hat{w}_i^{(n+1)}$ for some given i, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$; this i remains fixed throughout the algorithm. [2.8] Then, the base station announces $p^{(n+1)}$ back to the users; $p^{(n+1)}$ is used by the users as a reference power profile for optimization in Step 1) of the next iteration. With this modification a big part of computations are done at the base station and each user or measurement center needs to compute only $\hat{p}_i^{(n+1)}$ at each iteration of the algorithm.

The externality algorithm has the following feature:

[2.8] It is sufficient for the base station to compute $\hat{w}_i^{(n+1)}$ only for one i since all the sequences $\hat{w}_i^{(n+1)}$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, converge to the same limit (see Theorem 2.1) which is the optimal solution of the centralized counterpart of Problem (P.2).

Theorem 2.1. *The sequences* $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, *obtained by the externality algorithm converge to the unique global optimum of the centralized counterpart of Problem (P.2).*

$\boxtimes$

It should be noted that the dimensionality of the message space required by the externality algorithm for both hierarchical as well as non-hierarchical networks is N^2 which is same as the order $O(N^2)$ of the lower bound ([44]) on the dimensionality of the message space required by any mechanism so as to achieve globally optimal power allocations.

The proof of Theorem 2.1 is given in Appendix 2.A. Below we present a discussion that explains the intuition behind the externality algorithm.

As stated in Section 2.1.2 our objective in developing the externality algorithm is to come up with a decentralized iterative process that satisfies the constraints (posed by the network model (M.2)) on the information available to different users and, obtains an optimal solution of the centralized counterpart of Problem (P.2) in which one of the users (or a center) has complete system information. To accomplish the above objective the design of the iterative process requires that at each step, every user must solve an individual optimization problem based only on the information available to it at that step. [2.9] Based on the outcome of individual optimization, every user should then send a message which can be used by other users as additional information in the following iterations. Thus designing an appropriate iterative process breaks down to designing appropriate "individual optimization problems" and "message exchange rules" that lead to the maximization of the system objective function. Below we discuss how the externality algorithm accomplishes these goals.

[2.9] This consists of information available to the user at the beginning of the iterative process and the information gathered by it during the course of the iterative process till that particular iteration.

Because individual utility functions of the users are conflicting due to interference, letting users maximize their respective utilities will not lead to the maximization of the system objective function (the sum of all users' utilities). Therefore, the objective function for individual optimization problems must be some modification of the users' utility functions that can capture the effect of externalities. The norm square terms in (2.16) and (2.17) serve this purpose. The norm square term puts a penalty on user i in proportion to its deviation from the average of everybody's proposal for the optimal power profile; thus, it "pulls" user i's decision towards the other users' evaluations of i; the evaluations of these users incorporate the externalities that i generates to them. Furthermore, since the norm square term contains only those variables that are known [2.10] to user i, and the set $\mathcal{D}_i$ over which the optimization is performed is also known to i, the optimization problem described by (2.16) is well defined for each user $i \in \mathcal{N}$. Similarly, since the set $\mathcal{D}_{0_k}$ is known to MC_k (user 0_k), the optimization problem described by (2.17) is well defined for each user $0_k \in 0_{\mathcal{K}}$. The results $\hat{p}_i^{(n)}, i \in \mathcal{N} \cup 0_{\mathcal{K}}$, of the individual optimization problems described by (2.16)-(2.17), which are announced at the end of each iteration convey how each user valued everybody's transmissions. The average $p^{(n)}$ of everybody's optima conveys the average system valuation of users' transmissions and hence is used as a reference for the next iteration.

A desirable property for any iterative process to be useful is its convergence. This is achieved in the externality algorithm by reducing the value of the modification parameter $\tau^{(n)}$ in each iteration so as to increase the penalty of individual user deviation from the average, given by (2.18), of the optima of previous iteration. Thus, as the algorithm progresses, the power profile $\hat{p}_i^{(n)}$ proposed by user $i, i \in \mathcal{N} \cup 0_{\mathcal{K}}$, gets closer

[2.10] Note: After the announcement of users' optimal power profile proposals at the nth iteration, every user knows $p^{(n)}$ at the $(n+1)$th iteration.

and closer to those proposed by other users and eventually everybody agrees upon a "common" power profile. It should be noted however that the objective is not only the convergence of the iterative process but also, the maximization of the system objective function at the point of convergence. As can be noted from (2.16) and (2.17), the power profile which the users optima $\hat{p}_i^{(n)}, i \in \mathcal{N} \cup 0_{\mathcal{K}}$, converge to, need not be a maximizer of the system objective function. The reason is the following. Towards the end of the algorithm the norm square terms in (2.16) dominate (since $\tau^{(n)} \to 0$) the utility terms in the individual optimization problems. Thus, for very large n the outcome of each individual optimization problem is very close to the average proposal of the previous iteration. However, these outcomes are not representative of the users' utilities that form the system objective function; thus, even though in the limit the outcomes are equal, the limit point is not optimal. The contribution of the users' utilities is accounted for by the weighted time average $\hat{w}_i^{(n)}, i \in \mathcal{N} \cup 0_{\mathcal{K}}$. By taking a weighted average of the individual optima over the entire run of the algorithm, the two contributing components to the system objective are taken into account simultaneously: the individual utilities, which are more prominent in the individual optimization towards the beginning of the iterative process (when $\tau^{(n)}$ is comparatively large); and, the externalities, whose effect becomes more prominent in the individual optimization towards the end of the algorithm (when $\tau^{(n)}$ approaches 0). The decreasing weights $\tau^{(n)}$ facilitate convergence of each sequence $\hat{w}_i^{(n)}$ and provide appropriate balance between the contributions of the above two parameters in the point of convergence; thus, making the common point of convergence the global optimum of the system objective function.

Recall that algorithm (EA) has been designed to address decentralized power allocation under the scenario where the users obediently follow the rules of the algorithm

(Assumption 2.9). As mentioned in the introduction of this thesis (cf. (Section 1.3)), our approach to the solution of decentralized resource allocation problems under the above scenario is based on the principles of realization theory which is a component of mechanism design. In the next section we show how algorithm (EA) can be viewed as a solution approach of realization theory.

2.2.3 Relating algorithm (EA) with the solution approach of realization theory

To see the relation of algorithm (EA) with the solution approach of realization theory we first present a brief introduction to the realization theory approach to decentralized resource allocation. We then show how algorithm (EA) can be related with this framework.

Realization theory is a branch of the theory of mechanism design developed by mathematical economists. It provides a systematic methodology for the design of decentralized resource allocation mechanisms for systems that consist of agents whose objectives are aligned with the network objective. It focuses on the design of decentralized mechanisms that can achieve some pre specified objective, e.g. maximizing some network-wide/social welfare function.

In the mechanism design framework a centralized resource allocation problem is described by the triple $(\mathcal{E}, \mathcal{D}, \gamma)$: the *environment space* $\mathcal{E}$, the *action/allocation space* $\mathcal{A}$ and the *goal correspondence* γ. This is shown in Fig. 2.2.

$$\mathcal{E} \xrightarrow{\gamma} \mathcal{D}$$

Figure 2.2: A centralized resource allocation problem.

The environment $\boldsymbol{e}$ of a resource allocation problem, centralized or decentralized, is defined to be the set of resources and technologies available to all the users, their utilities, and any other information available to them, taken together. These are circumstances that cannot be changed either by the users in the network or by the designer of the resource allocation mechanism. For the network described by Model (M.2), the environment $\boldsymbol{e_i}$ of user i, $i \in \mathcal{N}$, consists of the set $\mathcal{P}_i$ of its feasible transmission powers, its utility function u_i, and the common knowledge about the set of users $\mathcal{N}$ as well as the fact that the set of users, their utilities and the channel gains remain fixed throughout a power allocation period. For user $0_k, k \in \mathcal{K}$, the environment $\boldsymbol{e_{0_k}}$ consists of its utility function u_{0_k}, the channel gains $h_{i0_k}, i \in \mathcal{N}$, and the aforementioned common knowledge. The environments of all the users collectively define the system environment $\boldsymbol{e} := (\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_N}, \boldsymbol{e_{0_1}}, \boldsymbol{e_{0_2}}, \ldots, \boldsymbol{e_{0_K}})$. The set of all possible environments $\boldsymbol{e_i}$ of a user defines its environment space $\mathcal{E}_i$. The environment spaces of all the users collectively define the environment space $\mathcal{E} := (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N, \mathcal{E}_{0_1}, \mathcal{E}_{0_2}, \ldots, \mathcal{E}_{0_K})$ of the system/problem.

The action / allocation space $\mathcal{D}$ of a resource allocation problem, centralized or decentralized, is defined to be the set of all possible resource allocation / exchange actions that can be taken by the users. For the network described by Model (M.2), the action space is the set $\mathcal{D}$ of all feasible power profiles p.

The goal correspondence γ of a centralized resource allocation problem is a map from $\mathcal{E}$ to $\mathcal{D}$ which assigns for every environment $\boldsymbol{e} \in \mathcal{E}$, the set of allocations in $\mathcal{D}$ that are solutions to the centralized resource allocation problem according to some pre-specified system goal. For the centralized counterpart of the power allocation problem (P.2), the system goal is the maximization of the sum $\sum_{i \in \mathcal{N} \cup 0_\mathcal{K}} u_i(p)$ of users' utilities, and γ is a mapping that maps every environment $\boldsymbol{e} \in \mathcal{E}$, defined in the previous paragraph, to the set of solutions of the centralized counterpart of Problem (P.2). Since in a centralized scenario one of the users has complete system information, i.e. it knows $\boldsymbol{e}$, it can determine optimal allocations $\gamma(\boldsymbol{e})$ in $\mathcal{D}$ corresponding to any given $\boldsymbol{e}$ using centralized optimization methods (such as mathematical programming or dynamic programming).

In an informationally decentralized system as the one described by Model (M.2), no one completely knows $\boldsymbol{e}$, therefore optimal centralized allocations $\gamma(\boldsymbol{e})$ can not be determined by methods similar to those for the centralized problems. Therefore, for resource allocation in a decentralized system, it is desirable to devise a communication/message exchange process among the users that does not require them to reveal their private information, yet leads to sufficient information exchange that eventually enables them to determine optimal centralized allocations. In the context of mechanism design, a formal treatment of the design of such communication and allocation rules is provided by realization theory.

In realization theory, a decentralized allocation mechanism is described by the triple $(\mathcal{M}, \mu, f)$ as shown in Fig. 2.3.

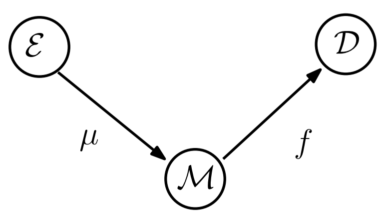

Figure 2.3: A decentralized resource allocation mechanism in realization theory.

In the decentralized mechanism, $\mathcal{M} := \prod_{i=1}^{N} \mathcal{M}_i$ is the *message space* which specifies for each $i \in \mathcal{N}$ the set of messages $\mathcal{M}_i$ that user i can communicate to other users. Each user $i \in \mathcal{N}$ generates messages from its message space $\mathcal{M}_i$ based on the communication rules specified by the mechanism. The message exchange process is normally iterative, therefore it is desirable that this process reaches an equilibrium, where any further update of messages by the users results in the same set of messages. μ is the *equilibrium message correspondence* that determines, for every problem instance (specified by the system environment), the set of equilibrium messages resulting from the specified communication rule. Once an equilibrium is attained, the resource allocations corresponding to the equilibrium message are determined by the *outcome function* f.

The following property characterizes the decentralized resource allocation mechanisms that can obtain optimal centralized allocations.

Definition 2.1 (Realization of goal correspondence)**.** *A decentralized mechanism* $(\mathcal{M}, \mu, f)$ *is said to "realize" the goal correspondence* γ *if,*

$$f(\mu(\boldsymbol{e})) \subset \gamma(\boldsymbol{e}) \quad \forall\, \boldsymbol{e} \in \mathcal{E},$$

i.e., for any given environment, the set of allocations resulting from the equilibrium messages specified by $(\mathcal{M}, \mu, f)$ *is a subset of the set of allocations* $\gamma(\boldsymbol{e})$ *that are optimal solutions of the corresponding centralized problem* $(\boldsymbol{e}, \mathcal{D}, \gamma)$*.*

The decentralized resource allocation model we discussed so far emphasizes on the equilibrium property of the mechanism. It does not provide details of communication rules or iterative message exchange process. In realization theory, these details are left to be designed according to the system under consideration. In the externality algorithm (EA) we explicitly describe the communication rules and the iterative message exchange process for the power allocation problem. Because these rules cannot be directly described in terms of the realization theory model, therefore, for clarity of presentation we presented the externality algorithm in Section 2.2.2 without referring to the realization model. However, at this point we can show how the equilibrium property of Algorithm (EA) can be described in terms of the realization theory model presented in this section.

In the discussion above we have already defined the environment, the action space, and the goal correspondence for the power allocation problem of Section 2.1.2. The decentralized mechanism $(\mathcal{M}, \mu, f)$ corresponding to Algorithm (EA) can be constructed as follows. Define the message space for each user $i \in \mathcal{N} \cup 0_{\mathcal{K}}$ to be $\mathcal{M}_i = \mathcal{D}_i \times \mathcal{D}_i$. Define the communication rule for the users as follows. In each iteration $n = 1, 2, 3, \ldots$, let each user $i \in \mathcal{N} \cup 0_{\mathcal{K}}$ generate from its message space $\mathcal{M}_i$ (defined above) the vector $(\hat{p}_i^{(n)}, \hat{w}_i^{(n)})$ that it obtains from (2.16), (2.17) and (2.19) [2.11].

The communication rule defined above implies that the equilibrium message correspondence μ is a function that maps the system environment to the equilibrium message vector $((\hat{p}_1^*, \hat{w}_1^*), (\hat{p}_2^*, \hat{w}_2^{(n)}), \ldots, (\hat{p}_N^*, \hat{w}_N^*), (\hat{p}_{0_1}^*, \hat{w}_{0_1}^*), (\hat{p}_{0_2}^*, \hat{w}_{0_2}^*), \ldots, (\hat{p}_{0_K}^*, \hat{w}_{0_K}^*)) \in \prod_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \mathcal{M}_i$. This vector is obtained at the point of convergence of Algorithm (EA). [2.12]

Finally, define the outcome function $f : \prod_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \mathcal{M}_i \to \mathcal{D}$ to be the function that maps the above equilibrium message vector to the vector $w^* := \frac{1}{(N+K)} \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \hat{w}_i^* \in \mathcal{D}$. As stated in Theorem 2.1 and proved in Appendix 2.A, for each $i \in \mathcal{N} \cup 0_{\mathcal{K}}$ the point of convergence $\hat{w}_i^*$ of the sequence $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ is the unique optimum of the centralized counterpart of Problem (P.2). Hence, the average w^* of $\hat{w}_i^*, i \in \mathcal{N} \cup 0_{\mathcal{K}}$, is also the unique optimum of the centralized counterpart of Problem (P.2). Thus, the mechanism $(\mathcal{M}, \mu, f)$ (equivalently, Algorithm (EA)) realizes the goal correspondence defined by the solution of the centralized counterpart of Problem (P.2).

In the next section we present numerical results that demonstrate the performance of the externality algorithm.

[2.11] Note that since for each $t = 1, 2, \ldots, \hat{p}_i^{(t)} \in \mathcal{D}_i$, the convex combination $\hat{w}_i^{(n)} = \frac{1}{\sigma(n)} \sum_{t=1}^{n} \hat{p}_i^{(t)} \in \mathcal{D}_i$

[2.12] Theorem 2.1 establishes that for each $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, the sequence $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ is a converging sequence.

2.3 Numerical Results

In this section we study the impact of various parameters on the performance of the externality algorithm. We study the performance of the algorithm for two types of cellular CDMA systems. We first consider an uplink CDMA system where the Base Station (BS) employs Minimum Mean Squared Error (MMSE) Multi-User Detection (MUD) to decode the signal of each mobile user. We take the utility of a user to be the negative of the Mean Squared Error (MSE) corresponding to the user less the power loss incurred due to signal transmission. Thus, the utility of user $i, i \in \mathcal{N}$, is given by (see [55]),

$$
\begin{aligned}
u_i(p) &= - \min_{\boldsymbol{z}_{\boldsymbol{i}}^T \in \mathbb{R}^{1\times N}} E[\, \|b_i - \boldsymbol{z}_{\boldsymbol{i}}^T \boldsymbol{y}\|^2] - 0.1\, p_i \\
&= -\big[(\boldsymbol{I} + \frac{2}{N_0}\boldsymbol{S}\boldsymbol{X}\boldsymbol{S})^{-1}\big]_{ii} - 0.1\, p_i
\end{aligned}
\tag{2.21}
$$

where, b_i is the transmitted data symbol of user i, $\boldsymbol{y}$ is the output of the matched filter employed by the BS, $\boldsymbol{I}$ is the identity matrix of size $N \times N$, $N_0/2$ is the two sided power spectral density of thermal noise, $\boldsymbol{S} := \mathrm{diag}(S_1, S_2, \ldots, S_N)$ is the diagonal matrix consisting of the received amplitudes of users 1 through N, and $\boldsymbol{X}$ is the cross-correlation matrix of the users' signature waveforms. The second term in the utility expression represents the input power loss incurred when user i transmits its signal at power p_i. For small cross correlation of user's waveforms, the utility functions given by (2.21) are close to concave (see Appendix 2.B) and therefore, the externality algorithm is expected to converge to the optimum centralized power allocation for the above system.

To test the performance of the algorithm, we ran simulations as explained below. For each test case, we distributed the users uniformly in a circular area of radius 5 around the BS. We placed three Measurement Centers (MCs) in the system; the first coinciding with the BS and the other two symmetrically placed along a diameter of the circle so that each MC is two-thirds radius away from the BS. We assumed that the maximum transmission power limit of each user is anywhere between 2W and 5W and the power loss due to propagation is determined by the inverse squared distance between the transmitter and the receiver. We assumed that each user uses a normalized bipolar signature waveform of dimension 6, and arbitrarily picks a waveform from all possible signature waveforms for its data transmission. Thus the users create interference to each other when their chosen waveforms have the same polarity in one or more signal dimensions. In the simulation results that follow, each point in the plots is obtained by averaging 50 identical independent simulation runs.

In Fig. 2.4, we compare the performance of the externality algorithm for different modification parameter sequences $\{\tau^{(n)}\}$. We choose the sequence $\{\tau^{(n)}\}$ to be of the form $\tau^{(n)} = (1/n)^{\delta}$ for varying δ. We plot the sum of users' utilities for a three user network against the number of iterations. As can be seen from Fig. 2.4, for $\delta = 0.001$ the algorithm shows best convergence. For values of δ smaller than 0.001, the modification term remains small compared to the users' utilities in the individual optimization; therefore, it takes longer time for the users to agree with other users' proposals and reach the global optimum. On the other hand, for values of δ greater than 0.001, the modification term dominates the users' utilities in the individual optimization; therefore, it takes the users longer to move away from the initial point $p^{(0)}$ and reach the global optimum. Thus for any system, there exists an optimum sequence $\{\tau^{(n)}\}$ that provides best convergence. The optimum sequence would however depend on the numerical value range of the user's utility functions under consideration. Therefore, to obtain best convergence, a sequence suitable for the given system must be used.

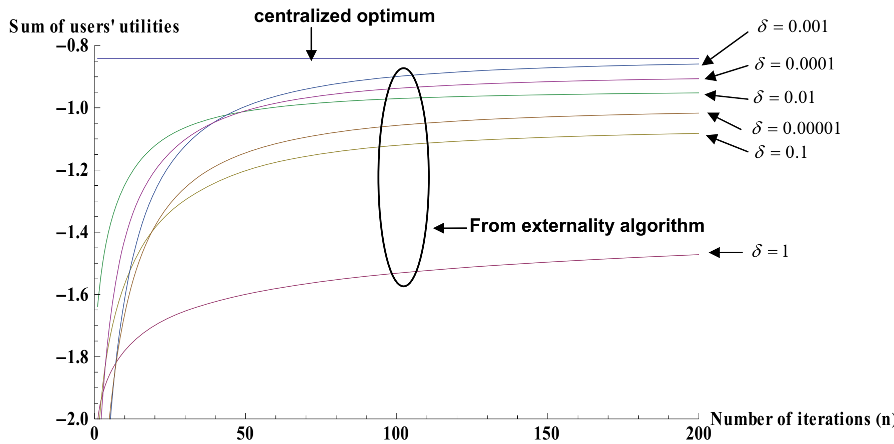

Figure 2.4: Sum of users' utilities vs. number of iterations for different modification parameter sequences $\{\tau^{(n)}\}$. Uplink cellular network with three users employing MMSE-MUD; $\tau^{(n)} = (1/n)^{\delta}$, $2P^{max}/N_0 = 15dB$.

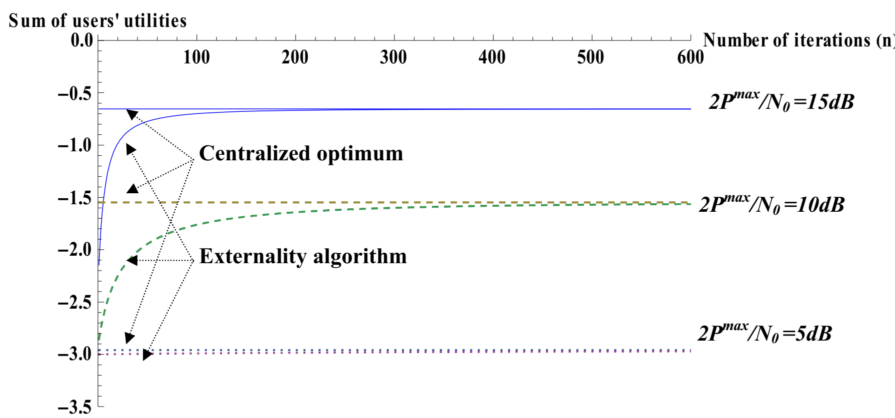

Figure 2.5: Sum of users' utilities vs. number of iterations for different values of $2P^{max}/N_0$. Uplink cellular network with three users employing MMSE-MUD; $\tau^{(n)} = (1/n)^{0.001}$.

In Fig. 2.5, we compare the performance of the externality algorithm for different SNRs ($2P^{max}/N_0$). For a three user network we plot the sum of users' utilities against the number of iterations for various SNRs. As can be seen, the algorithm converges to the centralized optimum for all SNR values. However, since different SNRs result in different numerical values of users' utility functions, the convergence time to the centralized optimum varies in each case and depends on the distance of the initial point from the optimum.

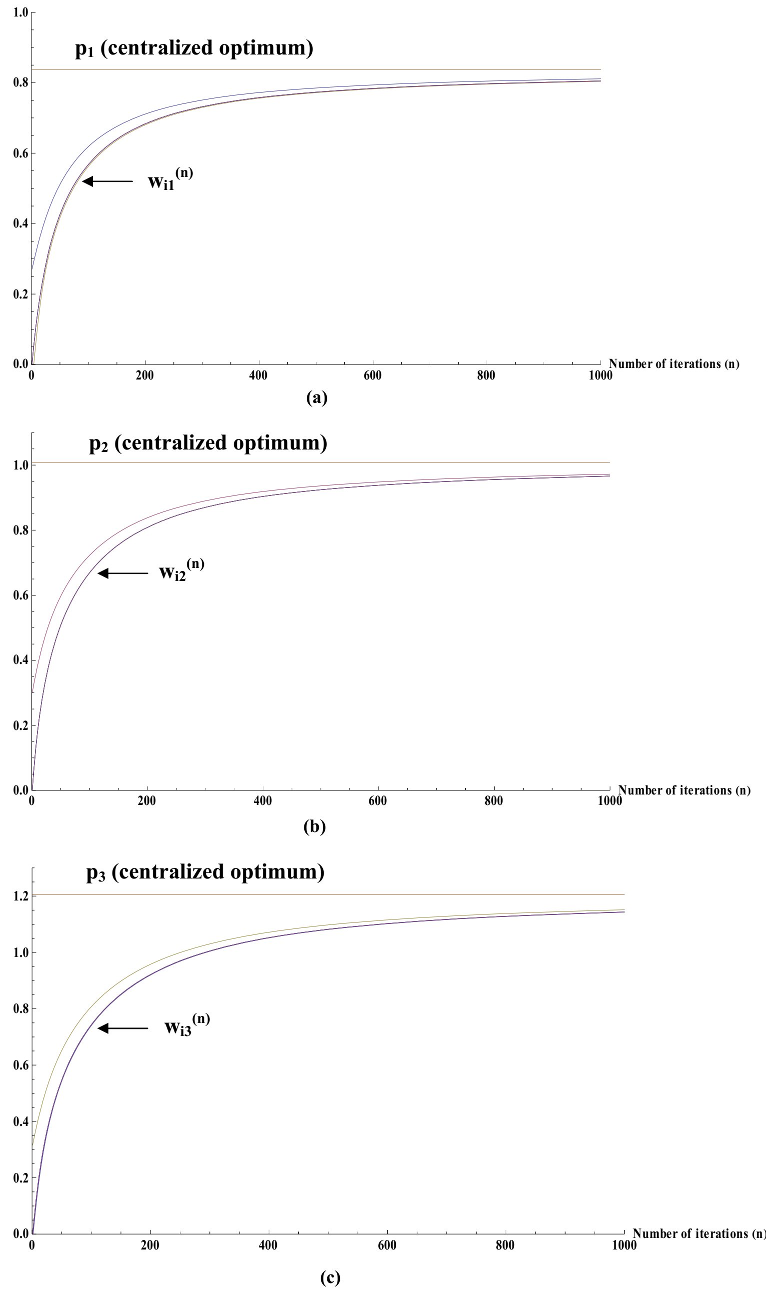

Figure 2.6: Sequence $\{\hat{w}_i^{(n)}\}$ vs. number of iterations n for users $i \in \{1,2,3\}$ and MCs $j \in \{0_1, 0_2, 0_3\}$. Uplink cellular network with three users employing MMSE-MUD; $\tau^{(n)} = (1/n)^{0.001}$, $2P^{max}/N_0 = 15$dB.

For $\tau^{(n)} = (1/n)^{0.001}$ and $2P^{max}/N_0 = 15$dB that result in best convergence in Fig. 2.4 and Fig. 2.5, we plot the sequences $\{\hat{w}_i^{(n)}\}$ corresponding to each user $i \in \{1,2,3\}$ and each MC $j \in \{0_1, 0_2, 0_3\}$ in Fig. 2.6. The three components of vectors $\hat{w}_i^{(n)}, n = 1, 2, \ldots$ corresponding to users 1, 2 and 3 are plotted on three separate plots. On the same plots, we also plot the optimum centralized power allocation (p_1, p_2, p_3). It is clear from the plots that the sequences $\{\hat{w}_i^{(n)}\}$ converge to the centralized optimum which illustrates the statement of Theorem 1.

To compare the performance of the externality algorithm with increasing interference, we plot in Fig. 2.7 the average utility per user resulting from the externality algorithm against the number of iterations for different number of users. On the same plot we also show the optimum centralized value of the average utility per user for each case. As can be seen, the algorithm converges to the centralized optimum for 3 and 4 user case, whereas for 6 and 8 users, it does not converge to the optimum value in given number of iterations. The reason for this is as follows. When the number of users is small compared to the signal dimensions, the cross-correlation between the users' waveforms is small; hence the users' utility functions are close to concave as mentioned in the beginning of Section 2.3. As the number of users starts overshooting the available signal dimensions which is 6 in our simulations, the cross-correlation between the users' waveforms increases and the utility functions are not guaranteed to be close to concave. Therefore, even though the externality algorithm improves the value of the objective function, it does not converge completely to the global optimum.

Next we consider an uplink CDMA system where the Base Station (BS) employs simple matched filtering to decode the signal of each mobile user. In this case, we take the utility of a user to be its SINR at the BS receiver, i.e.,

$$u_i(p) = \frac{p_i h_{i0}}{\frac{N_0}{2} + \frac{1}{N}\sum_{j \in \mathcal{N}, j \neq i} p_j h_{j0}}, \tag{2.22}$$

where, h_{j0} is the channel gain from user $j, j \in \mathcal{N}$, to the BS, and N is the number of

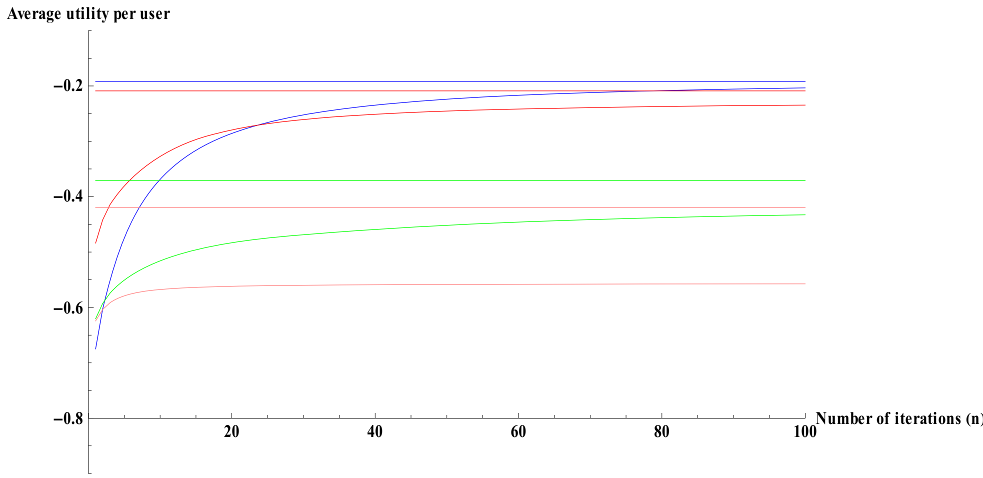

Figure 2.7: Sequence $1/N \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} u_i(w^{(n)})$ vs. number of iterations n for 3, 4, 6 and 8 users. Legend: Blue→N=3, Red→N=4, Green→N=6, Pink→N=8. Uplink cellular network employing MMSE-MUD; $\tau^{(n)} = (1/n)^{0.001}$, $2P^{max}/N_0 = 15$dB.

dimensions of the users' waveforms. For this system, we set up a simulation scenario in the same way as done for the MMSE-MUD system. We use the same transmission power limits for the users and use the same parameters for the circle radius and propagation loss.

For the SINR utility, The impact of different sequences $\{\tau^{(n)}\}$ on the convergence of the externality algorithm is similar to that discussed for MMSE-MUD utility. However, the optimum sequence of the form $\tau^{(n)} = (1/n)^{\delta}$ in this case has $\delta = 0.5$. Using this modification sequence, we compare the impact of varying SNR on convergence in Fig. 2.8. In this figure we plot two curves representing the average utility per user for an 8 user network. The upper curve shows the optimum centralized value of per-user utility corresponding to each SNR, and the lower curve shows the per-user utility for that SNR at the point of convergence of the externality algorithm. As can be seen, the externality algorithm converges close to the centralized optimum for negative SNR values. For positive SNRs, as the SNR increases, the gap between the centralized optimum and the solution of the externality algorithm increases. The reason for this deviation is that for negative SNRs, the SINR utility is close to concave but as SNR increases in positive direction, the SINR utility becomes more and more non-concave, hence the externality algorithm does not guarantee convergence to the centralized optimum.

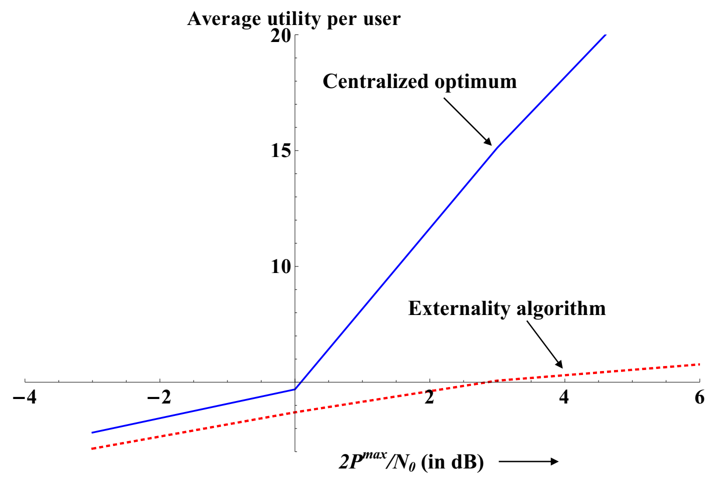

Figure 2.8: Average utility per user vs. $2P^{max}/N_0$. Uplink cellular network of 8 users with SINR utility; $\tau^{(n)} = (1/n)^{0.5}$.

The two examples from cellular CDMA communication presented in this section show that the externality algorithm results in optimum centralized power allocation when the networks operate under conditions where users' utilities are close to concave. The modification parameter sequence plays a critical role in determining the convergence speed of the algorithm. Other factors such as SNR and signal dimension impact convergence if they are critical in determining the shape of the utility functions.

2.A Proof of Theorem 2.1

Key ideas of the proof of Theorem 2.1

There are two key steps in the proof of Theorem 2.1. We first note that each sequence of allocations $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, generated by the externality algorithm is in a compact set and therefore, each sequence $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ has a convergent subsequence. In the first step (Claim 2.1 and Claim 2.2) we consider such a converging subsequence $\{\hat{w}_i^{(n')}\}$ of a given user i, and the corresponding subsequences $\{\hat{w}_j^{(n')}\}$, $j \in (\mathcal{N} \cup 0_{\mathcal{K}})\backslash\{i\}$, of all other users. We show that the subsequences $\{\hat{w}_j^{(n')}\}$, $j \in (\mathcal{N} \cup 0_{\mathcal{K}})\backslash\{i\}$, are also converging subsequences and that all of them converge to the same limit $\hat{w}_i^*$ as the subsequence $\{\hat{w}_i^{(n')}\}$. Furthermore, we show that this common limit is a feasible solution of Problem (P.2). In the second step (Claim 2.2 and Claim 2.3) we show that the aforementioned common limit is an optimal solution of the centralized counterpart of Problem (P.2). Since the centralized counterpart of Problem (P.2) has a unique optimal solution (as it is a concave optimization problem), and any arbitrarily chosen converging subsequence $\{\hat{w}_i^{(n')}\}$ of an arbitrarily chosen user i is shown to converge to the optimal solution, for every $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, all converging subsequences of $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ converge to the optimal solution. This in turn implies that for every $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, the sequences $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ themselves converge to the optimal solution.

Proof of Theorem 2.1

Let $i \in \mathcal{N} \cup 0_{\mathcal{K}}$ be a given user. Since the set $\mathcal{D}_i$ is convex and $\hat{p}_i^{(t)} \in \mathcal{D}_i,\ \forall\ t$, it follows that the convex combination of $\hat{p}_i^{(t)}$, $\hat{w}_i^{(n)} = \frac{1}{\sigma^{(n)}} \sum_{t=1}^{n} \tau^{(t)} \hat{p}_i^{(t)} \in \mathcal{D}_i,\ \forall\ n \in \{1, 2, 3, \dots\}$. Since $\mathcal{D}_i$ is compact, there exists a subsequence $\{\hat{w}_i^{(n')}\}$ of $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ that converges to a limit $\hat{w}_i^*$ in $\mathcal{D}_i$.

Define,

$$w^{(n)} := \frac{1}{\sigma^{(n)}} \sum_{t=0}^{n-1} \tau^{(t+1)} p^{(t)} \tag{2.23}$$

In the following claim we show that the subsequence $\{w^{(n')}\}$ of $\{w^{(n)}\}_{n=1}^{\infty}$ that is defined by the same set of indices as those of $\{\hat{w}_i^{(n')}\}$, converges to $\hat{w}_i^*$. Using this result we show that the subsequences $\{\hat{w}_j^{(n')}\}, j \in (\mathcal{N} \cup 0_{\mathcal{K}})\backslash\{i\}$, corresponding to users other than i that are specified by the same set of indices as those of $\{\hat{w}_i^{(n')}\}$ also converge to the same limit, i.e., $\hat{w}_j^{(n')} \to \hat{w}_j^* = \hat{w}_i^*, \;\; \forall\, j \in (\mathcal{N} \cup 0_{\mathcal{K}})\backslash\{i\}$.

Claim 2.1. *Let for some* $i \in \mathcal{N} \cup 0_{\mathcal{K}}, \quad \lim_{n'\to\infty} \hat{w}_i^{(n')} = \hat{w}_i^*$. *Then,*

(i) $\lim_{n'\to\infty} \|\hat{w}_i^{(n')} - w^{(n')}\|^2 = 0$, *i.e.,* $\lim_{n'\to\infty} w^{(n')} = \hat{w}_i^*$.

(ii) $\lim_{n'\to\infty} \|\hat{w}_j^{(n')} - w^{(n')}\|^2 = 0$, *i.e.,* $\lim_{n'\to\infty} \hat{w}_j^{(n')} = \hat{w}_i^*, \; \forall\, j \in (\mathcal{N} \cup 0_{\mathcal{K}})\backslash\{i\}$.

(iii) The common limit $\hat{w}_i^*$ *of the subsequences* $\{\hat{w}_j^{(n')}\}$, $j \in \mathcal{N} \cup 0_{\mathcal{K}}$, *is a feasible solution of Problem (P.2).*

Proof:

(i) We must show that

$$\forall \;\; \epsilon > 0, \; \exists\, n_0^{'} : \; \forall \;\; n' \geq n_0^{'}, \quad \|\hat{w}_i^{(n')} - w^{(n')}\|^2 \leq \epsilon. \tag{2.24}$$

Since $\|\cdot\|^2$ is a convex function, for any n',

$$\|\hat{w}_i^{(n')} - w^{(n')}\|^2 \leq \frac{1}{\sigma^{(n')}} \sum_{t=0}^{n'-1} \tau^{(t+1)} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2. \tag{2.25}$$

By (2.12) we have for any $n_0 < n'$,

$$\begin{aligned} &\frac{1}{\sigma^{(n')}} \sum_{t=0}^{n'-1} \tau^{(t+1)} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \\ \leq\; &\frac{1}{\sigma^{(n')}} \tau^{(1)} \sum_{t=0}^{n_0-1} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 + \tau^{(n_0)} \frac{1}{\sigma^{(n')}} \sum_{t=n_0}^{n'-1} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2. \end{aligned} \tag{2.26}$$

In Claim 2.2 we show that there exists a constant $C_p \in (0, \infty)$ independent of n' such that

$$\frac{1}{\sigma^{(n')}} \sum_{t=n_0}^{n'-1} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \leq C_p\,. \tag{2.27}$$

Assuming (2.27) to be true, and given any $\epsilon > 0$, we can choose n_0 (by (2.13) and [45, Definition 3.1, p.41]) such that

$$\tau^{(n_0)} \leq \frac{\epsilon}{2C_p}. \tag{2.28}$$

Since $\mathcal{D}_i$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, is compact and $\hat{p}_i^{(t+1)} \in \mathcal{D}_i$, there exist constants $C_{\mathcal{D}_i}$ independent of t ([45, Theorem 2.41, p.35]) such that

$$\|\hat{p}_i^{(t+1)}\| \leq C_{\mathcal{D}_i}, \quad i \in \mathcal{N} \cup 0_{\mathcal{K}}. \tag{2.29}$$

Therefore the sum A_{0_i} defined below is finite for any $n_0 < \infty$ and in particular, for n_0 chosen in (2.28),

$$A_{0_i} := \sum_{t=0}^{n_0-1} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 < \infty. \tag{2.30}$$

By (2.14) $\sigma^{(n')} \to \infty$ as $n' \to \infty$, therefore we can choose an n'_{0_i} large enough such that

$$\sigma^{(n'_{0_i})} \geq \frac{2\tau^{(1)} A_{0_i}}{\epsilon}. \tag{2.31}$$

Then,

$$\forall\, n' \geq n'_0 := \max_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} n'_{0_i}, \quad \sigma^{(n')} \geq \sigma^{(n'_0)} \geq \sigma^{(n'_{0_i})}, \quad \forall\, i \in \mathcal{N} \cup 0_{\mathcal{K}}. \tag{2.32}$$

Substituting (2.27) and (2.30) in (2.26) and using (2.32) implies that

$$\|\hat{w}_i^{(n')} - w^{(n')}\|^2 \;\leq\; \frac{1}{\sigma^{(n'_0)}}\tau^{(1)} A_{0_i} + \tau^{(n_0)} C_p \;\leq\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon, \quad \forall\, n' \geq n'_0. \tag{2.33}$$

The second inequality in (2.33) follows from (2.28) and (2.31). Since $\{\hat{w}_i^{(n')}\}$ is a converging subsequence with limit $\hat{w}_i^*$, (2.33) implies that the subsequence $\{w^{(n')}\}$ also converges, and has the limit $\hat{w}_i^*$. $\square$

(ii) Replacing i by j in (2.25)–(2.33) we obtain for each $j \in (\mathcal{N} \cup 0_{\mathcal{K}}) \backslash \{i\}$ that,

$$\|\hat{w}_j^{(n')} - w^{(n')}\|^2 \;\leq\; \frac{1}{\sigma^{(n'_0)}}\tau^{(1)} A_{0_j} + \tau^{(n_0)} C_p \;\leq\; \epsilon. \tag{2.34}$$

Since by part (i) $\{w^{(n')}\}$ is a converging subsequence with limit $\hat{w}_i^*$, it follows from (2.34) that for each $j \in (\mathcal{N} \cup 0_{\mathcal{K}})\backslash\{i\}$, $\{\hat{w}_j^{(n')}\}$ is also a converging subsequence with the limit $\hat{w}_i^*$. □

(iii) Since each set $\mathcal{D}_j$, $j \in \mathcal{N} \cup 0_{\mathcal{K}}$, is compact, the limit of each subsequence $\{\hat{w}_j^{(n')}\}$, $j \in \mathcal{N} \cup 0_{\mathcal{K}}$, lies in the respective set $\mathcal{D}_j$, $j \in \mathcal{N} \cup 0_{\mathcal{K}}$. By part (i) and part (ii) we know that $\forall\, j \in \mathcal{N} \cup 0_{\mathcal{K}}$, the subsequences $\{\hat{w}_j^{(n')}\}$ converge to the same limit $\hat{w}_i^*$. Therefore, by above argument $\hat{w}_i^* \in \mathcal{D}_j$, $\forall\, j \in \mathcal{N} \cup 0_{\mathcal{K}}$. It follows that $\hat{w}_i^* \in \mathcal{D} = \bigcap_{j \in \mathcal{N} \cup 0_{\mathcal{K}}} \mathcal{D}_j$ and hence $\hat{w}_i^*$ is a feasible solution of Problem (P.2). □

To complete the proof of Claim 2.1 we need to prove (2.27). This is done in Claim 2.2.

Claim 2.2. *There exists a constant* $0 < C_p < \infty$ *such that*

$$\frac{1}{\sigma^{(n)}} \sum_{t=0}^{n-1} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \leq C_p\,, \qquad \forall\, n.$$

Proof:

Since $\hat{p}_i^{(t+1)}$ is the optimal solution of Step (1) of the algorithm, it follows from [31, Theorem 1.6] that [2.13]

$$\begin{aligned} &\tau^{(t+1)} u_i(\hat{p}_i^{(t+1)}) - \|\hat{p}_i^{(t+1)} - p\|^2 + \|p^{(t)} - p\|^2 - \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \\ &\geq \tau^{(t+1)} u_i(p), \qquad \forall\, p \in \mathcal{D}_i, \quad i \in \mathcal{N} \cup 0_{\mathcal{K}}. \end{aligned} \tag{2.35}$$

Adding inequality (2.35) over all i implies

$$\begin{aligned} &\tau^{(t+1)} \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} u_i(\hat{p}_i^{(t+1)}) - \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \|\hat{p}_i^{(t+1)} - p\|^2 + (N+K)\|p^{(t)} - p\|^2 \\ &- \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \;\geq\; \tau^{(t+1)} \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} u_i(p), \qquad \forall\, p \in \mathcal{D} := \bigcap_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \mathcal{D}_i. \end{aligned} \tag{2.36}$$

By convexity of $\|\cdot\|^2$,

$$\|p^{(t+1)} - p\|^2 \leq \frac{1}{N+K} \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \|\hat{p}_i^{(t+1)} - p\|^2. \tag{2.37}$$

[2.13] by taking $\|\cdot\|^2$ as function $J_1(\cdot)$ and $u_i(\cdot)$ as function $J_2(\cdot)$ in Theorem 1.6 of [31].

Replacing the second term in (2.36) using (2.37), adding (2.36) over $t = 0, 1, \ldots, n-1$, and dividing by $N + K$ we get,

(2.38)
$$\frac{1}{N+K}\sum_{t=0}^{n-1}\tau^{(t+1)}\sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} u_i(\hat{p}_i^{(t+1)}) - \|p^{(n)} - p\|^2 - \frac{1}{N+K}\sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}}\sum_{t=0}^{n-1}\|\hat{p}_i^{(t+1)} - p^{(t)}\|^2$$
$$\geq \frac{\sigma^{(n)}}{N+K}\sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} u_i(p) - \|p^{(0)} - p\|^2, \quad \forall\, p \in \mathcal{D}.$$

By concavity of $u_i(p)$ in p,

$$\frac{1}{N+K}\sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}}\sum_{t=0}^{n-1}\tau^{(t+1)} u_i(\hat{p}_i^{(t+1)}) \leq \frac{\sigma^{(n)}}{N+K}\sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} u_i(\hat{w}_i^{(n)}). \tag{2.39}$$

Substituting (2.39) in (2.38) and multiplying by $(N + K)/\sigma^{(n)}$ we obtain

$$\begin{aligned} &\sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} u_i(\hat{w}_i^{(n)}) - \frac{N+K}{\sigma^{(n)}}\|p^{(n)} - p\|^2 - \sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}}\frac{1}{\sigma^{(n)}}\sum_{t=0}^{n-1}\|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \\ &\geq \sum_{i\in\mathcal{N}\cup 0_{\mathcal{K}}} u_i(p) - \frac{N+K}{\sigma^{(n)}}\|p^{(0)} - p\|^2, \quad \forall\, p \in \mathcal{D}. \end{aligned} \tag{2.40}$$

Since $\mathcal{D}_i$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, and $\mathcal{D}$ are compact, the numerators of the second terms on both the LHS and the RHS of (2.40) are bounded. From (2.14), $\sigma^{(n)} \to \infty$ as $n \to \infty$. Therefore,

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{\sigma^{(n)}}\|p^{(n)} - p\|^2 = 0, \\ \text{and} \quad &\lim_{n\to\infty}\frac{1}{\sigma^{(n)}}\|p^{(0)} - p\|^2 = 0. \end{aligned} \tag{2.41}$$

Furthermore, since $\mathcal{D}_i$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, and $\mathcal{D}$ are compact, $\hat{w}_i^{(n)} \in \mathcal{D}_i$, $p \in \mathcal{D}$, and $u_i(\cdot)$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, are continuous functions on $\mathbb{R}^{\mathcal{N}}$, there exist constants $0 < C_{US_i} < \infty$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, independent of n such that

$$u_i(p) \leq C_{US_i}, \quad \text{and} \quad u_i(\hat{w}_i^{(n)}) \leq C_{US_i}, \quad i \in \mathcal{N} \cup 0_{\mathcal{K}}. \tag{2.42}$$

Then (2.40) together with (2.41)–(2.42) imply that for an appropriate constant $0 < C_p < \infty$,

$$\frac{1}{\sigma^{(n)}}\sum_{t=0}^{n-1}\|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \leq C_p\,, \quad \forall\, n. \tag{2.43}$$

This completes the proof [2.14] of Claim 2.2 and, therefore, the proof of Claim 2.1.

□

The arguments used in the proof of Claim 2.2, specifically those leading to inequality (2.40) together with Claim 2.1 allow us to prove that the limit $\hat{w}_i^*$ is an optimal power allocation.

Claim 2.3. *The limit point $\hat{w}_i^*$ of the subsequences $\{\hat{w}_j^{(n')}\}$, $j \in \mathcal{N} \cup 0_{\mathcal{K}}$, is an optimal solution of the centralized counterpart of Problem (P.2).*

Proof:

From Claim 2.1 we have that the subsequence $\hat{w}_j^{(n')} \to \hat{w}_i^*$, $\forall\, j \in \mathcal{N} \cup 0_{\mathcal{K}}$. Therefore as $n' \to \infty$, the first term on the LHS of (2.40) converges [2.15] to the value of the objective function (in (2.5)) at $\hat{w}_i^*$; this can be compared with the value of the objective function at any point $p \in \mathcal{D}$ if the limits of the other three terms in (2.40) are known. From (2.41), the second terms on both the LHS and the RHS of (2.40) converge to 0. Since $\|\cdot\|^2$ is convex, $\forall\, i \in \mathcal{N} \cup 0_{\mathcal{K}}$,

$$\begin{aligned} \|\hat{w}_i^{(n')} - w^{(n')}\|^2 &\le \frac{1}{\sigma^{(n')}} \sum_{t=0}^{n'-1} \tau^{(t+1)} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2 \\ &\le \frac{1}{\sigma^{(n')}} \sum_{t=0}^{n'-1} \|\hat{p}_i^{(t+1)} - p^{(t)}\|^2, \quad \because\ \tau^{(t+1)} \le 1,\ \forall\, t \ge 0. \end{aligned} \tag{2.44}$$

Substituting (2.44) in (2.40) implies that

$$\begin{aligned} &\sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} u_i(\hat{w}_i^{(n')}) - \frac{N+K}{\sigma^{(n')}} \|p^{(n')} - p\|^2 - \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} \|\hat{w}_i^{(n')} - w^{(n')}\|^2 \\ \ge &\sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} u_i(p) - \frac{N+K}{\sigma^{(n')}} \|p^{(0)} - p\|^2, \quad \forall\, p \in \mathcal{D}. \end{aligned} \tag{2.45}$$

Taking the limit $n' \to \infty$ in (2.45) and using (2.33), (2.34) and (2.41) we obtain

$$\sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} u_i(\hat{w}_i^*) \ge \sum_{i \in \mathcal{N} \cup 0_{\mathcal{K}}} u_i(p), \quad \forall\, p \in \mathcal{D}. \tag{2.46}$$

[2.14] It should be noted that the result of Claim 2.1 has not been used in the proof of Claim 2.2. The two claims are presented in the given order only to facilitate the flow of the proof of Theorem 2.1

[2.15] We only consider the subsequence $\{n'\}$ of $\{n\}$ here for which $\{\hat{w}_i^{(n')}\}$ converges.

□

Claim 2.4. *The sequences* $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$, $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, *generated by the externality algorithm converge to the optimal solution of the centralized counterpart of Problem (P.2).*

Proof:

In Claims 2.1–2.3 we have shown that if we consider any arbitrary converging subsequence $\{\hat{w}_i^{(n')}\}$ of an arbitrary user i, this subsequence converges to an optimal solution $\hat{w}_i^*$ of the centralized counterpart of Problem (P.2). Since the centralized counterpart of Problem (P.2) is a concave maximization problem, it has a unique optimal solution w^* which must be equal to $\hat{w}_i^*$. Since the user and the corresponding subsequence are arbitrarily chosen in Claims 2.1–2.3, the results of Claims 2.1–2.3 hold for all the users and all converging subsequences of each user. This means that for every $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, all converging subsequences of $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ must converge to the unique optimal solution w^*. Since each sequence $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}, i \in \mathcal{N} \cup 0_{\mathcal{K}}$, lies in a compact set $\mathcal{D}_i \subset \mathbb{R}^N$, and since for each $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, all converging subsequences of $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ converge to the same limit w^* (in other words, each sequence $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}, i \in \mathcal{N} \cup 0_{\mathcal{K}}$, has exactly one point of accumulation [2.16]), by [17, Corollary, p.53] each sequence $\{\hat{w}_i^{(n)}\}_{n=1}^{\infty}$ for $i \in \mathcal{N} \cup 0_{\mathcal{K}}$, itself converges to the optimal solution w^*. This completes the proof of Claim 2.4 and establishes the assertion of Theorem 2.1. □

2.B Concavity of the MMSE-MUD utility function

One crucial assumption in Section 2.1.1 that is required to prove the convergence of the externality algorithm is that the users' utilities are concave functions of power profiles. In this section we show the conditions under which the MMSE-MUD utility function studied in Section 2.3 is close to concave.

[2.16] See [17, Lemma, p.52]

Suppose there are N users in a network, and all the users use MMSE-MUD receivers to decode the received data. The minimum mean square error at the output of user i's ($i \in \mathcal{N}$) receiver is then given by (see [55, Chapter 6])

$$\min_{\boldsymbol{z_i}^T \in \mathbb{R}^{1\times N}} E[\,\|b_i - \boldsymbol{z_i}^T \boldsymbol{y_i}\|^2] = \left[(\boldsymbol{I} + \frac{2}{N_0}\boldsymbol{S}\boldsymbol{X}\boldsymbol{S})^{-1}\right]_{ii}, \tag{2.47}$$

where, b_i is the transmitted data symbol of user i, $\boldsymbol{y_i}$ is the output of user i's matched filter corresponding to its input received data, $\boldsymbol{I}$ is the identity matrix of size $N \times N$, $N_0/2$ is the two sided power spectral density of the thermal noise, $\boldsymbol{S} := \mathrm{diag}(S_1, S_2, \ldots, S_N)$ is the diagonal matrix consisting of the received amplitudes of users 1 through N, and $\boldsymbol{X}$ is the cross-correlation matrix of the users' signature waveforms. For simplicity of analysis and for analytical tractability we consider the case of two users below.

For the two-user ($N = 2$) case, the expression for the MMSE in (2.47) becomes

$$MMSE_i = \frac{\frac{N_0}{2}}{\frac{N_0}{2} + p_i h_{ii}\left(1 - \frac{\rho^2 (p_j h_{ji})}{(N_0/2 + p_j h_{ji})}\right)}, \quad i, j \in \{1, 2\},\ j \neq i. \tag{2.48}$$

where, $p_i h_{ii} = S_i^2$ and $p_j h_{ji} = S_j^2$, $i, j \in \{1, 2\}$, $j \neq i$; h_{ii} and h_{ji} are the channel gains from transmitters T_i and T_j respectively to the receiver R_i; and ρ is the cross correlation between the signature waveforms of users 1 and 2.

We take the users' utility functions to be

$$u_i(p) = -MMSE_i(p), \quad i \in \{1, 2\}. \tag{2.49}$$

Below we investigate the properties of function u_i defined in (2.49). From (2.48) and (2.49) we see that for a given PSD $N_0/2$ of the thermal noise, the channel gains h_{11}, h_{21}, the cross correlation ρ, and the transmission power p_2 of user 2, the function u_1 is of the form $\frac{-1}{c_1 + c_2 p_1}$ for some constants c_1 and c_2. Thus u_1 is concave in p_1. On the other hand, for a given p_1, if ρ is very small which is usually the case in practical wireless systems, the coefficient ρ^2 in the denominator of (2.48) makes the variation of u_1 with p_2 very small. Thus p_1 dominantly determines the curvature of function u_1. To illustrate this we plot $u_1(p)$ vs. (p_1, p_2) in Fig. 2.9. It can clearly be seen from Fig. 2.9 that u_1 is a nice concave function of p_1 and varies very little with p_2. Therefore, it is close to concave in $p = (p_1, p_2)$. To check the utility of user 2, we use

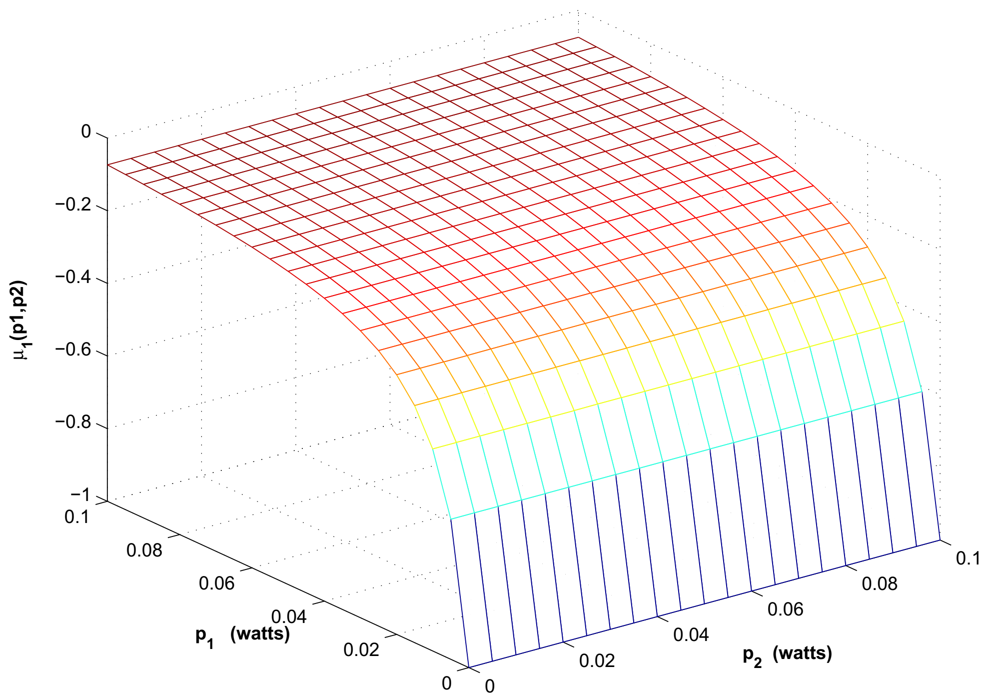

Figure 2.9: $u_1(p_1, p_2)$ vs. (p_1, p_2) for $N_0/2 = 10^{-1.2}$, $h_{11} = 0.5$, $h_{21} = 0.6$, and $\rho = 0.01$.

similar arguments as above by interchanging the indices 1 and 2 and we get that u_2 is also close to concave in p.

For larger networks with $N > 2$, it is difficult to give a general expression for u_i similar to (2.48). However, when the cross correlation among the users' waveforms is small, the curvature of function u_i is dominantly determined by p_i. Similar to the case for $N = 2$, the function u_i is concave in p_i, and varies very little with other components of p, thus, suggesting that it is close to concave in p.

CHAPTER 3

Power allocation in wireless networks: An implementation perspective

In this chapter we consider an implementation theory perspective on power allocation in wireless networks. Specifically, we study power allocation for a single cell wireless Code Division Multiple Access (CDMA) network with interference in the presence of selfish and non-cooperative users. We consider the scenario of decentralized information, where each user knows only its own utility and the channel gain from the base station to itself. For the above network we formulate the uplink power allocation problem as a public good allocation problem, and present a decentralized mechanism (game form) that has the following properties: (i) All Nash equilibria (NE) of the game induced by the game form result in allocations that are optimal solutions of the corresponding centralized uplink problem (Nash implementation, cf Section 3.2.1). (ii) All users voluntarily participate in the allocation process specified by the game form (individual rationality, cf Section 3.2.1). (iii) Budget balance at all NE and off equilibrium.

The chapter is organized as follows: In Section 3.1.1 we present the network model. In Section 3.1.2 we present the power allocation problem. We present a literature survey in Section 3.1.3 and discuss our motivation to investigate the problem presented in this chapter in Section 3.1.4. We state our contributions in Section 3.1.5. In Section 3.2.1 we formulate the power allocation problem of Section 3.1.2 in the framework of implementation theory. In Section 3.2.2 we present a game form for the above problem and we discuss the properties of the game form in Section 3.2.3. We present a discussion on the intuition behind the structure of the proposed game form in Section 3.2.4 and we prove the properties of the game form in Appendices 3.A and 3.B.

Before we present the model, we describe the notation that we will use throughout the chapter.

Notation:

We represent vectors by bold letters and scalars by normal letters. The elements of a vector are represented by subscripting the vector symbol. A bold subscripted-symbol means that the vector-element is also a vector e.g. in $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_N})$, each $\boldsymbol{x_i}$, $i = 1, 2, \ldots, N$, is a vector; in $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$, each x_i, $i = 1, 2, \ldots, N$, is a scalar. Unless otherwise stated, all vectors are treated as column vectors. Bold $\mathbf{0}$ is treated as a zero vector of appropriate size determined by the context. The notation $(x_i, \boldsymbol{x}^*/i)$ (or $(\boldsymbol{x_i}, \boldsymbol{x}^*/i)$) is used to represent the following: $(x_i, \boldsymbol{x}^*/i)$ (or $(\boldsymbol{x_i}, \boldsymbol{x}^*/i)$) is a vector of dimension same as that of $\boldsymbol{x}^*$; the ith element of $(x_i, \boldsymbol{x}^*/i)$ (or $(\boldsymbol{x_i}, \boldsymbol{x}^*/i)$) is x_i (or $\boldsymbol{x_i}$), all other elements of it are the same as the corresponding elements of $\boldsymbol{x}^*$. We represent a diagonal matrix of size $N \times N$ whose diagonal entries are elements of the vector $\boldsymbol{x} \in \mathbb{R}^N$ by $\text{diag}(\boldsymbol{x})$.

3.1 The power allocation problem

In this section we present the wireless network model and the assumptions we make for its analysis. We also discuss scenarios that motivate the model. We then formulate a power allocation problem for the above model.

3.1.1 The model (M.3)

We consider a single cell CDMA wireless data network consisting of a Base Station (BS) and multiple mobile users. In this chapter we focus on the uplink transmission from the mobiles to the BS as shown in Fig. 3.1. Later we briefly discuss how the

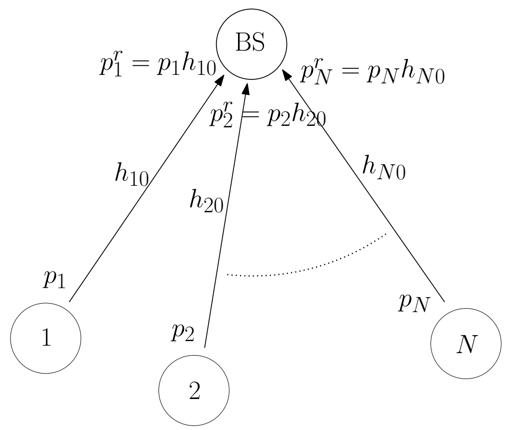

Figure 3.1: An uplink network with N mobile users and one base station

results for the downlink network can be obtained in a similar way. [3.1] We assume that there are N mobile users[3.2], $N \geq 3$, in the network; we denote the set of users by $\mathcal{N} := \{1, 2, \ldots, N\}$. We consider the transmissions of the users in a given carrier frequency; we assume that the signature codes used by the users are not completely orthogonal,[3.3] hence the reception of signals from each user experiences interference at the BS due to other users' transmissions to the BS. Each user $i \in \mathcal{N}$ receives a Quality of Service (QoS) from the data decoded by the BS for user i. Due to interference, the QoS of user $i, i \in \mathcal{N}$, depends not only on the transmission power p_i^t of user i but also, on the power $p_j^t, j \in \mathcal{N}\backslash\{i\}$ of other users' transmissions to the BS. User $i, i \in \mathcal{N}$, is capable of transmitting in the power range $\mathcal{P}_i^t := [0, P_i^{t\,max}]$. We

[3.1] In [51, 49] we treat the problem of downlink transmission from the BS to the mobiles in detail. For this problem we derive results similar to the ones for the uplink problem presented in this chapter.
[3.2] Here onwards we will use the terms "mobiles" and "users" interchangeably to mean mobile users.
[3.3] This helps increase the capacity of the network.

assume that,

Assumption 3.1. *The transmission power range* $\mathcal{P}_i^t$ *is user* i*'s private information.* [3.4]

Due to the path loss from the mobiles to the BS, the QoS of user i actually depends on the power $p_j := p_j^t h_{j0}, j \in \mathcal{N}$, received at the BS from all the users, where h_{j0} is the channel gain from user j to the BS.

The QoS of a user that results from the power transmitted by all the users is quantified by a *utility function*. We denote the utility that user $i \in \mathcal{N}$ obtains when the *power profile* received by the BS is $\boldsymbol{p} := (p_1, p_2, \ldots, p_N)$ by $u_i(\boldsymbol{p})$. The functional form of $u_i : \mathbb{R}^N \to \mathbb{R}$ depends on the technology used by the BS to decode user i's data as well as on the personal preference of (human) user i for the decoded data. We assume that the BS uses a Multi User Detector (MUD) decoder for each user. The BS informs each user a-priori as to which code to use for its data transmission so that the BS can employ an MUD upon receiving the signals from all the users. We note that it is in interest of each user to stick to the code assigned by the BS because otherwise, the BS will not be able to decode their respective data correctly. In Appendix 2.B we present explicitly the utility function of a user when the BS uses an MUD for each user. We show that such a utility function is almost concave in $\boldsymbol{p}$. Hence, we make the following approximation. Let

$$\begin{aligned} \mathcal{D}_i &:= \{\boldsymbol{p} \mid p_i \in \mathcal{P}_i;\ p_j \in \mathbb{R}_+, j \in \mathcal{N}\backslash\{i\}\}, \\ \text{where,} \quad \mathcal{P}_i &:= [0, P_i^{tmax} h_{i0}] =: [0, P_i^{max}]. \end{aligned} \tag{3.1}$$

Assumption 3.2. *For each* $i \in \mathcal{N}$*,* $u_i : \mathbb{R}^N \to \mathbb{R}$ *is concave in* $\boldsymbol{p}$ *for* $\boldsymbol{p} \in \mathcal{D}_i$ *and* $u_i(\boldsymbol{p}) = 0$ *for* $\boldsymbol{p} \notin \mathcal{D}_i$*. Also, the function* u_i *is private information of user* i*.*

[3.4] Private information of a user is defined as the information that is known only to that user and nobody else in the network.

The assumption that $u_i(\boldsymbol{p}) = 0$ for $\boldsymbol{p} \notin \mathcal{D}_i$ is made for the following reason. A power profile $\boldsymbol{p} \notin \mathcal{D}_i$ implies that either $p_i \notin \mathcal{P}_i$, or $p_j \notin \mathbb{R}_+$ for some $j \in \mathcal{N}\backslash\{i\}$. According to user i's knowledge, [3.5] it is not possible for the BS to receive such a power profile because it corresponds to transmission powers that are outside the feasible range (as known to user i) of users' transmission powers. Therefore, a power profile $\boldsymbol{p} \notin \mathcal{D}_i$ cannot provide any QoS to user i and results in zero utility.

We assume that,

Assumption 3.3. *The network users are non-cooperative and selfish. The BS on the other hand does not have any utility associated with the power allocations / transmissions. It acts like an accountant that redistributes taxes (discussed below) according to the specifications of the allocation mechanism.*

Assumption 3.3 implies that the users have an incentive to misrepresent their private information, e.g. a user $i \in \mathcal{N}$ may not want to report to other users or to the BS its true preference for the users' transmissions, if by doing so user i obtains a power allocation in its favor.

We note that each user $i \in \mathcal{N}$ needs to know the channel gain h_{i0} in order to know how the power transmitted by it affects its QoS at the BS. The BS can measure the channel gains $h_{i0}, i \in \mathcal{N}$, and announce them to the respective users if the users send some "pre-specified" pilot signals to the BS. However, because the users are selfish, the BS cannot rely upon the pilot signal transmission from the users. Therefore, we assume that the BS periodically transmits pilot signals to the users so that each user $i \in \mathcal{N}$ can measure the channel gain h_{0i} from the BS to itself. Furthermore, we assume that,

[3.5] Assumption 3.4 that we state later implies that, each user $i \in \mathcal{N}$ knows the channel gain h_{i0} from itself to the BS. As a result it knows the range $\mathcal{P}_i$ as well as the set $\mathcal{D}_i$ exactly. On the other hand, the set $\mathcal{D}_j, j \in \mathcal{N}\backslash\{i\}$ is private information of user j and user i does not know this set. Therefore, user i perceives $\mathcal{D}_i$ to be the set of powers that are feasible for the BS to receive.

Assumption 3.4. *The channel between the BS and the users is symmetric, i.e.* $h_{0i} = h_{i0} \; \forall \; i \in \mathcal{N}$.

Because of Assumption 3.4, each user $i \in \mathcal{N}$ can compute the channel gain h_{i0} from its measurement of h_{0i}. We note that it is in the interest of each user to measure its respective channel gain h_{0i} correctly because this will tell the user correctly the influence of its transmission power on its QoS. We assume that,

Assumption 3.5. *For each* $i \in \mathcal{N}$*, the channel gain* h_{i0} *is user* i*'s private information.*

We would like to mention here that Assumption 3.4 is made only for convenience and that it is not necessary for the power allocation mechanism we present in this chapter to work. We explain the consequence of relaxing this assumption in Section 3.2.3 after we present the power allocation mechanism.

Each user $i \in \mathcal{N}$ pays a *tax* $t_i \in \mathbb{R}$ to the BS. This tax is imposed for the following reasons: (i) For the use of the network by the users. (ii) To provide incentives to the users to transmit powers that result in a network-wide performance objective. The tax for a user can be either positive or negative and is determined by the rules of the power allocation mechanism. With the flexibility of either charging a user (positive tax) or paying compensation/subsidy (negative tax) to a user, it is possible to induce users to behave in such a way that a network wide performance objective is achieved. For example, given the power transmission and interference constraints in the network, we can satisfy all the users by setting “positive tax” for the users that receive power allocations close to those requested by them and paying “compensation” to the users that receive allocations that are not close to their desirable ones. According to Assumption 3.3 the BS does not derive any profit from the above tax and the purpose of the above tax collection is to just redistribute the money among network users. This implies that the tax profile $\boldsymbol{t} := (t_1, t_2, \dots, t_N)$ is determined in a way such that,

$$\sum_{i=1}^{N} t_i = 0 \tag{3.2}$$

To describe the "overall satisfaction" of a user from the QoS it receives from the power profile received by the BS and the tax it pays for this QoS, we define an *aggregate utility function* $u_i^A : \mathbb{R}^{1+N} \to \mathbb{R} \cup \{-\infty\}$ for each user $i \in \mathcal{N}$ as follows:

$$\begin{aligned} u_i^A(t_i, \boldsymbol{p}) &:= -t_i + u_i(\boldsymbol{p}) - \left[\frac{1 - I_{\mathcal{D}_i}(\boldsymbol{p})}{I_{\mathcal{D}_i}(\boldsymbol{p})}\right], \\ \text{where, } I_{\mathcal{D}_i}(\boldsymbol{p}) &= \begin{cases} 1, & \text{if } \boldsymbol{p} \in \mathcal{D}_i \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.3}$$

The last term in (3.3) signifies that an allocation $(t_i, \boldsymbol{p})$ is of no use to user i if $\boldsymbol{p} \notin \mathcal{D}_i$. This is because, based on its knowledge, user i knows that it is not possible for the BS to receive a power profile $\boldsymbol{p} \notin \mathcal{D}_i$. Because of Assumption 3.5 and Assumption 3.1, the set $\mathcal{D}_i$ is user i's private information. This along with Assumption 3.2 implies that for each $i \in \mathcal{N}$, the aggregate utility u_i^A is user i's private information. As stated in Assumption 3.3 users are non-cooperative and selfish. Therefore, *the users are self aggregate utility maximizers.*

In this chapter we restrict attention to static problems. Specifically we make the following assumption:

Assumption 3.6. *The set of users $\mathcal{N}$, their utilities and the channel gains between the BS and the users are fixed in advance and they do not change with time.*

We also assume that before any power allocation period, the BS announces the set of users in the network, therefore,

Assumption 3.7. *The set of users* $\mathcal{N}$ *is* common knowledge. [3.6]

In the following section we formulate the power allocation problem for the network model (M.3).

3.1.2 The uplink power allocation problem

For the network model (M.3) we want to develop a power and tax determination mechanism that works under the constraints imposed by the model and obtains a solution to the following centralized problem corresponding to it.

Problem (P.3)

$$\begin{aligned} &\max_{(\boldsymbol{t},\boldsymbol{p})} \sum_{i\in\mathcal{N}} u_i^A(t_i,\boldsymbol{p}) \\ &\text{s.t.} \sum_{i\in\mathcal{N}} t_i = 0. \end{aligned} \tag{3.4}$$

$$\begin{aligned} \equiv \quad &\max_{(\boldsymbol{t},\boldsymbol{p})\,\in\,\mathcal{D}} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{p}) \\ \text{where,} \quad &\mathcal{D} := \{(\boldsymbol{t},\boldsymbol{p}) \mid \sum_{i\in\mathcal{N}} t_i = 0, \boldsymbol{t}\in\mathbb{R}^N; p_i\in\mathcal{P}_i, i\in\mathcal{N}\}. \end{aligned} \tag{3.5}$$

The optimization problem (3.4) is equivalent to (3.5) because for $(\boldsymbol{t},\boldsymbol{p}) \notin \mathcal{D}$, the objective function in (3.4) is negative infinity by (3.3). Thus $\mathcal{D}$ is the set of feasible solutions of Problem (P.3). Because of Assumption 3.2, the objective function in (3.5) is concave in $\boldsymbol{p}$. Moreover, the sets $\mathcal{P}_i, i \in \mathcal{N}$, are convex and compact. Therefore, there exists an optimal power profile $\boldsymbol{p}^*$ of Problem (P.3). Furthermore, since the objective function in (3.5) does not explicitly depend on $\boldsymbol{t}$, an optimal solution of Problem (P.3) must be of the form $(\boldsymbol{t},\boldsymbol{p}^*)$, where $\boldsymbol{p}^*$ is an optimal power profile and $\boldsymbol{t}$ is any feasible tax profile for Problem (P.3), i.e. a tax profile that satisfies (3.2).

Assumptions 3.1, 3.2 and 3.5 imply that there is no entity in the network that knows perfectly all the parameters that describe Problem (P.3). Therefore, we need

[3.6] See [4, 57] for the definition of common knowledge.

to develop a mechanism that allows the users and the BS to communicate with one another and that leads to optimal allocations for Problem (P.3). Since a key assumption in Model (M.3) is that the users are non-cooperative and selfish, the mechanism we develop must take into account the possible strategic behavior of the users in their communication with the BS.

In the next section we present a literature survey on previous works on decentralized power allocation in wireless networks in the presence of strategic users. After presenting the literature survey we discuss our motivation to investigate the power allocation problem presented in this section.

3.1.3 Literature survey

Decentralized mechanisms for power allocation in cellular networks that study game-theoretic/strategic behavior issues have received considerable attention in the literature. One of the earliest works which introduced an individual utility maximization formulation for uplink power control in a single cell CDMA data network can be found in [12]. An uplink problem similar to that of [12] in which users' utilities are taken to be functions of their respective Signal to Interference Ratio (SIR) was investigated in [28]; in this paper the existence of an equilibrium was shown and a decentralized algorithm for solving the power control problem was suggested. The problem formulated in [12] was re investigated in [47] using pricing; it was shown that pricing results in multiple equilibria which are Pareto superior to the equilibria obtained in [12] and [28]. Pricing-based analysis of the uplink power control problem was also done in [2]; in [2] the authors introduced user specific parametric utility functions and proposed two decentralized algorithms, the parallel update and the random update algorithms, that converge to the unique equilibrium of the problem.

In [46] pricing-based ideas for uplink power control were extended to multi-cell data networks. The authors of [19] studied uplink power allocation under an Interference Temperature Constraint (ITC); they proposed a power auction run by a manager that achieves a power allocation arbitrarily close to the globally optimal one. The conditions under which the power auction achieves an optimal solution however require in essence, that the manager should know the users' utility functions.

Game theoretic study of downlink CDMA data networks can be found in [32, 59, 30]. In [32] and [59], optimal power allocation strategies were determined for a single class CDMA system under the assumption that the utility functions of the users are common knowledge (see [4, 57] for the definition of common knowledge). The authors of [30] studied a downlink power allocation problem for multi-class CDMA networks; they proposed a decentralized mechanism based on dynamic pricing and partial cooperation between the mobiles and the base station that achieves a partial-cooperative optimal power allocation which was shown to be close to a globally-optimal power allocation. In [50, 48] the authors presented a decentralized mechanism for power allocation that works for both uplink and downlink networks, and also takes into account multiple ITCs; the mechanism obtains an optimal power allocation under the assumption that the users are cooperative.

Having provided an overview of the existing works in the literature, we now present our motivation for studying the power allocation problem presented in Section 2.1.2.

3.1.4 Motivation

A network resource is said to be a public good if the presence of the resource simultaneously affects the utilities of all network users without getting divided among them. Thus, the power allocation problem presented in Section 3.1.2 is a problem of public good allocation where the public good is the power vector received by the BS from all the users.

Power allocation problems in cellular wireless networks with interference have been previously considered in the literature cited in Section 3.1.3. The solution approach in all the references [59, 32, 30, 28, 12, 47, 46, 19, 2] is based on different variations of pricing mechanisms where each user pays some money for the power allocated to it.

In general, in decentralized resource allocation problems involving a public good, pricing mechanisms that fix a common price for the public good for all the users, fail to obtain globally optimal allocations. The reason is that in a public good economy the *same* good is simultaneously consumed by users having different valuations of the good; thus, individual valuations of the public good are different from the system's valuation and this results in inefficiency. This explains why the pricing mechanisms employed in [19, 12, 47, 28] do not achieve globally optimal allocations and why the mechanism proposed in [2] does not achieve optimal allocations unless the users vary their utilities according to their target SIRs.

The pricing mechanism proposed in [30] is different from the above references in that it obtains close to globally optimal allocations. The reason for this is the following. The authors of [30] introduce a constraint on the total power transmitted by the BS. Due to this constraint, the original problem, where each user's utility depends on the entire power vector transmitted by the BS, reduces to one where each user's utility depends only on the power transmitted to it. Thus, the problem changes from a public good allocation problem (when explicit interference is present) to a private good allocation problem. This is why the pricing mechanism proposed in [30] results in efficient allocations. In systems where there is no constraint on the maximum sum power, the above-stated reduction is not possible and therefore, pricing mechanisms do not yield optimal allocations. The failure of pricing mechanisms to produce globally optimal power allocations for wireless networks affected by interference, provides the key motivation for the formulation and solution methodology presented in this chapter.

The decentralized power allocation mechanism presented in Chapter 2 appropriately takes into account the externalities (public good effect) due to the interference from other users. The mechanism overcomes the inefficiency of the pricing mechanisms and obtains optimal power allocations. However, the above mechanism is designed for the realization theory scenario where the users obediently follow the rules of the mechanism. The results of Chapter 2 motivated us to explore the design of decentralized optimal power allocation mechanisms for networks in a non-cooperative users setup that we address in this chapter.

In the next section we state the contributions of this chapter.

3.1.5 Contribution of the chapter

The key contributions of this chapter are:

- The formulation of single cell uplink power allocation problem with interference and strategic users as a public good allocation problem;
- The specification of a decentralized power allocation mechanism (game form) for the above problem that possesses the following properties:
  - (i) All Nash equilibria (NE) of the game induced by the mechanism result in allocations that are optimal solutions of the corresponding centralized uplink problem (Nash implementation, cf Section 3.2.1).

(ii) All users voluntarily participate in the allocation process specified by the mechanism (individual rationality, cf Section 3.2.1).

(iii) Budget balance at all NE and off equilibrium.

Our proposed mechanism is distinctly different from the pricing mechanisms studied in the aforementioned literature. Our formulation properly captures the valuation of interference by each individual user as well as the system and hence, the proposed mechanism leads to globally optimal power allocations. Because the valuation of interference has to be properly captured, the complexity of the strategy space (also called message space) of our mechanism is significantly larger than that of pricing mechanisms.

In the next section we present an implementation theory-based solution for the power allocation problem of Section 3.1.2.

3.2 Solution of the uplink power allocation problem

A systematic approach to the development of resource allocation mechanisms for informationally decentralized networks (as the one described by Model (M.3)) where users behave strategically, is provided by *implementation theory*, a branch of Mathematical Economics. In the context of our problem, implementation theory deals with the design of mechanisms that provide rules/guidelines on; (i) how the BS and the mobiles should "communicate" with one another; and (ii) how power allocations and tax allocations should be determined, based on the outcome of communication, so as to induce the desired user/mobile strategic behavior.

In this chapter we use an implementation theory-based approach for the solution of the power allocation problem presented in Section 3.1.2. Therefore, in the next section we provide a brief introduction to implementation theory and set the preliminaries for our solution to the power allocation problem.

3.2.1 Embedding the power allocation problem for Model (M.3) in the framework of implementation theory

Implementation theory is a branch of the theory of mechanism design developed by mathematical economists. It provides a systematic methodology for the design of decentralized resource allocation mechanisms for informationally decentralized systems that consist of selfish/non-cooperative agents. It focuses on the design of decentralized mechanisms that can achieve some pre specified objective, e.g. maximizing some network-wide/social welfare function.

As described in Section 2.2.3, in the mechanism design framework a centralized resource allocation problem is described by the triple $(\mathcal{E}, \mathcal{D}, \gamma)$: the *environment space* $\mathcal{E}$, the *action/allocation space* $\mathcal{D}$ and the *goal correspondence* γ.

To recap, the environment $\boldsymbol{e}$ of a resource allocation problem, centralized or decentralized, is defined to be the set of resources and technologies available to all the users, their utilities, and any other information available to them, taken together. These are circumstances that cannot be changed either by the users in the network or by the designer of the resource allocation mechanism. For the network described by Model (M.3), the environment $\boldsymbol{e_i}$ of user i, $i \in \mathcal{N}$, consists of the channel gains h_{0i} and h_{i0}, its utility function u_i^A, and the common knowledge about the set of users $\mathcal{N}$ as well as the fact that the set of users, their utilities and the channel gains remain fixed throughout a power allocation period. The environments of all the users collectively define the system environment $\boldsymbol{e} := (\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_N})$. The set of all possible environments $\boldsymbol{e_i}$ of a user defines its environment space $\mathcal{E}_i$. The environment spaces of all the users collectively define the environment space $\mathcal{E} := (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N)$ of the system/problem.

The action / allocation space $\mathcal{D}$ of a resource allocation problem, centralized or decentralized, is defined to be the set of all possible resource allocation / exchange actions that can be taken by the users. For the network described by Model (M.3), the action space is the set $\mathcal{D}$ of all tax and received power profiles $(\boldsymbol{t}, \boldsymbol{p})$ that the BS can possibly allocate to the users.

The goal correspondence γ of a centralized resource allocation problem is a map from $\mathcal{E}$ to $\mathcal{D}$ which assigns for every environment $\boldsymbol{e} \in \mathcal{E}$, the set of allocations in $\mathcal{D}$ that are solutions to the centralized resource allocation problem according to some pre-specified system goal. For the centralized power allocation problem (P.3), the system goal is the maximization of the sum $\sum_{i \in \mathcal{N}} u_i^A(t_i, \boldsymbol{p})$ of users' utilities, and γ is a mapping that maps every environment $\boldsymbol{e} \in \mathcal{E}$, defined in the previous paragraph, to the set of solutions of (P.3). Since in a centralized scenario one of the users (or a controller such as the BS) has complete system information, i.e. it knows $\boldsymbol{e}$, it can determine optimal allocations $\gamma(\boldsymbol{e})$ in $\mathcal{D}$ corresponding to any given $\boldsymbol{e}$ using centralized optimization methods (such as mathematical programming or dynamic programming).

In an informationally decentralized system as the one described by Model (M.3), the controller (BS in Model (M.3)) does not completely know $\boldsymbol{e}$, therefore it can not determine optimal centralized allocations $\gamma(\boldsymbol{e})$ by methods similar to those for the centralized problems. Therefore, for resource allocation in a decentralized system, it is desirable to devise a communication/message exchange process among the users and the controller that eventually enables the controller to determine optimal centralized allocations. However, when the users in a system are selfish, they have an incentive to misrepresent their private information while communicating with the controller so as to shift the allocation determined by the controller in their own favor. The users may also choose not to participate in the communication process if they know that the resulting allocation will not be in their favor (or if by not participating they are better off). This may defeat the objective of maximizing the system objective function ($\sum_{i \in \mathcal{N}} u_i^A(t_i, \boldsymbol{p})$ for the power allocation problem). Therefore, for the success of a communication process in leading to desirable outcomes it is required that the allocation rule employed by the controller induces the users to behave in a desirable manner (i.e. it ensures voluntary participation of the users in the communication process and furthermore, it induces the users to communicate information that results in system objective maximizing allocations). In the context of mechanism design, a formal treatment of the design of such communication and allocation rules is provided by implementation theory.

In implementation theory, a decentralized resource allocation mechanism is specified by a *game form*. An N-user game form is defined by the pair $(\mathcal{M}, f)$. $\mathcal{M} := \prod_{i=1}^{N} \mathcal{M}_i$ is the *message space* which specifies for each $i \in \mathcal{N}$ the set of messages $\mathcal{M}_i$ that user i can communicate to other users and the controller. f is the *outcome function* which maps $\mathcal{M} \to \mathcal{D}$; it specifies for each *message profile* $\boldsymbol{m} \in \mathcal{M}$, $\big(\boldsymbol{m} := (\boldsymbol{m_1}, \boldsymbol{m_2}, \ldots, \boldsymbol{m_N}), \boldsymbol{m_i} \in \mathcal{M}_i, i \in \mathcal{N}\big)$, the resulting allocation $f(\boldsymbol{m}) \in \mathcal{D}$.

Since the participation of the users in a resource allocation mechanism requires that they be aware of its protocols, it is assumed that the game form is known to all the users in the system. Given the specification of a game form, the selfish users know what allocations their messages would potentially lead to and what utilities they would obtain as a result. Therefore, the agents strategically communicate their messages so as to maximize their respective utilities, and this induces a game. Formally, a game form $(\mathcal{M}, f)$ is said to induce a game $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$ in an environment $\boldsymbol{e} \in \mathcal{E}$ where the users' utilities are $u_i^A, i \in \mathcal{N}$. In this game the players are the users in $\mathcal{N}$, the set of strategies of a user is its respective message space $\mathcal{M}_i, i \in \mathcal{N}$, and the payoff of a user corresponding to a given strategy/message profile $\boldsymbol{m}$ is the utility $u_i^A(f(\boldsymbol{m})), i \in \mathcal{N}$, it obtains from the resulting allocation $f(\boldsymbol{m})$. The property of a game form is studied by analyzing the properties of the allocations that result from various equilibria of the induced game. Depending on the users' information about the system environment, there are appropriate equilibrium concepts for an induced game that specify the equilibrium messages corresponding to the game. For example, for games of complete information the equilibrium concepts are Nash equilibrium, Subgame Perfect equilibrium, dominant strategy equilibrium, rationalizability, etc. ([37, 39, 41, 14]). For games of incomplete information the equilibrium concepts are Bayesian Nash equilibrium, Perfect Bayesian equilibrium, etc. Given an equilibrium concept, the specific equilibria that an induced game can attain are governed by the design of the game form. Thus, in implementation theory, a game form along with an equilibrium concept indirectly specifies the (equilibrium) message correspondence μ. This is shown in Fig. 3.2.

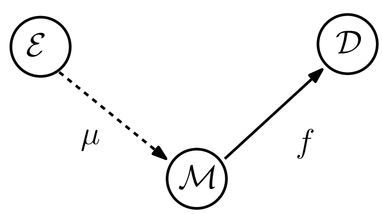

Figure 3.2: A game form $(\mathcal{M}, f)$ inducing the equilibrium message correspondence μ.

In this chapter we consider Nash equilibrium as the equilibrium concept. A Nash Equilibrium (NE) of a game is defined as a message profile $\boldsymbol{m}^*$ such that none of the users finds it profitable to unilaterally deviate to any other message. Mathematically, $\boldsymbol{m}^*$ is a NE of the game $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$ if,

$$u_i^A(f(\boldsymbol{m}^*)) \geq \; u_i^A(f((\boldsymbol{m_i}, \boldsymbol{m}^*/i))), \;\; \forall\, \boldsymbol{m_i} \in \mathcal{M}_i, \; \forall\, i \in \mathcal{N}. \tag{3.6}$$

There are properties that characterize a game form based on whether it can achieve optimal centralized allocations with respect to the NE equilibrium concept. To define these properties let $NE(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$ represent the set of all Nash equilibria of the game $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$, and let

(3.7)

$$\mathcal{D}_{NE}(\mathcal{M}, f, \{u_i^A\}_{i=1}^N) := \Big\{ \boldsymbol{a} \in \mathcal{D} \mid \boldsymbol{a} = f(\boldsymbol{m}) \text{ for some } \boldsymbol{m} \in NE(\mathcal{M}, f, \{u_i^A\}_{i=1}^N) \Big\},$$

that is, $\mathcal{D}_{NE}$ is the set of allocations corresponding to all Nash equilibria of the game.

Now consider a decentralized resource allocation problem. Let $\mathcal{E} = \prod_{i=0}^N \mathcal{E}_i$ be the environment space and $\mathcal{D}$ the allocation space associated with the problem, let $\gamma : \mathcal{E} \to \mathcal{D}$ be a goal correspondence, and let $u_1^A, u_2^A, \ldots, u_N^A$, be the users' utilities corresponding to a given environment $\boldsymbol{e} \in \mathcal{E}$. Then, we have the following:

Definition 3.1 (Implementation in Nash equilibria)**.** *A game form* $(\mathcal{M}, f)$ *is said to "implement in Nash equilibria" the goal correspondence* γ *if,*

$$\mathcal{D}_{NE}(\mathcal{M}, f, \{u_i^A\}_{i=1}^N) \subset \gamma(\boldsymbol{e}) \quad \forall\, \boldsymbol{e} \in \mathcal{E},$$

i.e., for any given environment, the set of allocations resulting (through the outcome function f) from the Nash equilibria of the game $(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$ *is a subset of the set of allocations* $\gamma(\boldsymbol{e})$ *that are optimal solutions of the corresponding centralized problem* $(\boldsymbol{e}, \mathcal{D}, \gamma)$.

Definition 3.1 implies that a game form that implements in NE a goal correspondence, takes into account the users' strategic behavior and obtains centralized solutions, given that the users participate in the message exchange process specified by the game form. However, in order that the users voluntarily participate in a mechanism specified by a game form, the game form must satisfy an additional property defined as follows. Let the *initial endowment* of a user be defined as the amount of resources the user has before participating in a game form; e.g. for the network model (M.3), the initial endowment $\boldsymbol{f_i^0}$ of user $i, i \in \mathcal{N}$, is the tax and transmission power profile before the power allocation mechanism is run, i.e. $\boldsymbol{f_i^0} = (t_i^0, \boldsymbol{p}^0) = (0, \mathbf{0}),\ \forall\, \mathbf{i} \in \mathcal{N}$. We then have the following,

Definition 3.2 (Individual rationality)**.** *A game form $(\mathcal{M}, f)$ is said to be individually rational if $\forall\ i \in \mathcal{N}$, $u_i^A(f(\boldsymbol{m})) \geq u_i^A(\boldsymbol{f_i^0})$ for all $\boldsymbol{m} \in NE(\mathcal{M}, f, \{u_i^A\}_{i=1}^N)$, i.e. at any NE allocation the utility of each user is at least as much as its utility before participating in the game/allocation process.*

Definitions 3.1 and 3.2 imply that a game form that is individually rational and implements in NE a goal correspondence, obtains optimal allocations of the corresponding centralized system by having the users voluntarily participate in the allocation process. These are exactly the properties that we want in a tax and power allocation mechanism for the network model (M.3). Thus the theory of implementation introduced above provides us with a framework to develop the desired decentralized power allocation mechanism for the network model (M.3).

In light of the discussion provided in this section, we now state our objective for the power allocation problem presented in Section 3.1.2.

The objective:

Let $\mathcal{E}$ and $\mathcal{D}$ be respectively the environment space and the allocation space corresponding to the uplink network model (M.3) as defined in Section 3.2.1. Let $\gamma : \mathcal{E} \to \mathcal{D}$ be the goal correspondence for Problem (P.3) as defined in Section 3.2.1. Our objective is to design an individually rational game form $(\mathcal{M}, f)$ that implements in NE the goal correspondence γ.

In the next section, we present a game form that achieves the above objective. However, before we proceed, we present a brief clarification on the interpretation of

NE in the mechanism that we present in the following section. Nash equilibria in general describe strategic behavior of users in games of complete information. This can be seen from (3.6) where, to define the NE, it requires complete information of all users' aggregate utility functions. However, the users in Model (M.3) do not know each other's utilities. Therefore, for any profile of the users' utilities, the game $(\mathcal{M}, f, \{u_i^A\}_{i\in\mathcal{N}})$ induced by the game form we present in the next section is not one of complete information. We can create a game of complete information by increasing the message/strategy space following Maskin's approach [37]. However, such an approach would result in an infinite dimensional message/strategy space for the corresponding game. We do not follow Maskin's approach; instead, we adopt the philosophy of [43]. Specifically, by quoting [43], "we interpret our analysis as applying to an unspecified (message exchange) process in which users grope their way to a stationary message and in which the Nash property is a necessary condition for stationarity."

3.2.2 The game form

In this section we present a game form that provides a decentralized mechanism for solving the uplink power allocation problem presented in Section 3.1.2. To obtain an appropriate game form for the power allocation problem it is useful to observe that in the uplink network, the power profile $\boldsymbol{p} = (p_1, p_2, \dots, p_N)$ received by the BS can be treated as a *public good* [36]. This is because, analogous to a public good in an economy, the same vector $\boldsymbol{p}$ affects the utility of all the users in the network. Furthermore, like a public good, the exact amount of the utility a user obtains from $\boldsymbol{p}$ differs from user to user and depends on its individual function $u_i, i \in \mathcal{N}$ that determines its QoS. Game forms that implement in NE efficient allocation of public goods have been proposed by Groves and Ledyard [16], Hurwicz [22] and Walker [56]. In this section we present a game form for the uplink power allocation problem that is inspired from Hurwicz' mechanism [22]. Below we specify each of the elements of the proposed game form, the message space and the outcome function.

The message space:

Since for the network model (M.3) we are interested in determining the power profile that should be received at the BS and tax that the users should pay, the communication between the users and the BS should contain information that is helpful in determining the optimal amounts of each of commodities. We let each user $i \in \mathcal{N}$ send to the BS a message $\boldsymbol{m_i} \in \mathcal{M}_i := \mathbb{R}_+^N \times \mathbb{R}^N$ that has the following form:

$$\boldsymbol{m_i} := (\boldsymbol{\pi_i}, \boldsymbol{p_i}); \qquad \boldsymbol{\pi_i} \in \mathbb{R}_+^N,\ \boldsymbol{p_i} \in \mathbb{R}^N \tag{3.8}$$

The message $\boldsymbol{m_i}$ consists of two elements: $\boldsymbol{p_i} = (p_{i1}, p_{i2}, \ \ldots, p_{iN})$ which can be interpreted as the received power profile that user $i \, (i \in \mathcal{N})$ suggests to be allocated to all the users $j \in \mathcal{N}$; and $\boldsymbol{\pi_i} = (\pi_{i1}, \pi_{i2}, \ldots, \pi_{iN})$ which can be interpreted as the price that user $i \, (i \in \mathcal{N})$ suggests to be charged to the users $j \in \mathcal{N}$ for using the network.

The outcome function:

Based on the message profile $\boldsymbol{m} = (\boldsymbol{m_1}, \boldsymbol{m_2}, \ldots, \boldsymbol{m_N})$, the BS sets the taxes $\hat{t}_i(\boldsymbol{m}), i \in \mathcal{N}$, and determines powers $\hat{\boldsymbol{p}}(\boldsymbol{m}) = (\hat{p}_1(\boldsymbol{m}), \hat{p}_2(\boldsymbol{m}), \ldots, \hat{p}_N(\boldsymbol{m}))$ to be received from the users as follows:

$$\hat{\boldsymbol{p}}(\boldsymbol{m}) = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{p_i}, \tag{3.9}$$

$$\begin{aligned} \hat{t}_i(\boldsymbol{m}) = \boldsymbol{l_i^T}(\boldsymbol{m})\hat{\boldsymbol{p}}(\boldsymbol{m}) &+ (\boldsymbol{p_i} - \boldsymbol{p_{i+1}})^T \mathrm{diag}(\boldsymbol{\pi_i})(\boldsymbol{p_i} - \boldsymbol{p_{i+1}}) \\ &- (\boldsymbol{p_{i+1}} - \boldsymbol{p_{i+2}})^T \mathrm{diag}(\boldsymbol{\pi_{i+1}})(\boldsymbol{p_{i+1}} - \boldsymbol{p_{i+2}}), \qquad i \in \mathcal{N}, \end{aligned} \tag{3.10}$$

$$\text{where,} \quad \boldsymbol{l_i}(\boldsymbol{m}) = \boldsymbol{\pi_{i+1}} - \boldsymbol{\pi_{i+2}}. \tag{3.11}$$

In (3.10) and (3.11), $i+2 \equiv 1$ for $i = N-1$, and for $i = N$, $i+1 \equiv 1$ and $i+2 \equiv 2$.

The game form defined by (3.8)–(3.11) together with the users' utility functions in (3.3) specify a game. The strategy of user $i, i \in \mathcal{N}$, in this game is its message $\boldsymbol{m_i}$. We note that the message $\boldsymbol{m_i}$ of user $i, i \in \mathcal{N}$, is allowed to take any value (which can be unboundedly large) in the space $\mathbb{R}_+^N \times \mathbb{R}^N$; in particular $\boldsymbol{p_i}$ is not restricted to lie in $\mathcal{D}_i$. Thus, a Nash equilibrium [3.7] of the above game is a message profile $\boldsymbol{m}^*$ from which no user wants to unilaterally deviate (see (3.6)) even when arbitrary deviations are possible by unbounded magnitude of messages.

As discussed in Section 3.2.1, our objective is to develop a game form for which the set of tax and received power allocations obtained at all its NE is the same as the set of optimal tax and received power allocations for the centralized problem (P.3). Below we present theorems that assert that the proposed game form achieves this goal.

3.2.3 Properties of the game form

The main results of this chapter are summarized by Theorems 3.1 and 3.2 below.

Theorem 3.1. *Let $\boldsymbol{m}^*$ be a NE of the game induced by the game form presented in Section 3.2.2 and the users' utility functions (3.3). Let $(\hat{\boldsymbol{t}}(\boldsymbol{m}^*), \hat{\boldsymbol{p}}(\boldsymbol{m}^*)) =: (\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ be the tax and received power allocation at $\boldsymbol{m}^*$ determined by the game form. Then,*

[3.7] See Section 3.2.1 for a discussion on the interpretation of Nash equilibria.

(a) $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ *is individually rational, i.e. all users weakly prefer* $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ *to the initial allocation* $(\mathbf{0}, \mathbf{0})$*. Mathematically,*

$$u_i^A(\hat{t}_i^*, \hat{\boldsymbol{p}}^*) \geq u_i^A(0, \mathbf{0}), \quad \forall\, i \in \mathcal{N}.$$

(b) $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ *is an optimal solution of the centralized problem (P.3).*

⊠

Theorem 3.2. *Let* $\hat{\boldsymbol{p}}^*$ *be an optimum received power profile corresponding to Problem (P.3). Then,*

(a) There exist a set of personalized prices $\boldsymbol{l}_{\boldsymbol{i}}^*$, $i \in \mathcal{N}$, *such that*

$$\arg\max_{\boldsymbol{p} \in \mathcal{D}_i} \left\{ -\boldsymbol{l}_{\boldsymbol{i}}^{*T} \boldsymbol{p} + u_i(\boldsymbol{p}) \right\} \;=\; \hat{\boldsymbol{p}}^*, \quad \forall\, i \in \mathcal{N}.$$

(b) There exists at least one NE $\boldsymbol{m}^*$ *of the game induced by the game form presented in Section 3.2.2 and the users' utility functions (3.3) such that,* $\hat{\mathbf{p}}(\boldsymbol{m}^*) = \hat{\boldsymbol{p}}^*$*. Furthermore, if* $\hat{t}_i^* := \boldsymbol{l}_{\boldsymbol{i}}^{*T} \hat{\boldsymbol{p}}^*, i \in \mathcal{N}$*, the set of all NE* $\boldsymbol{m}^* = (\boldsymbol{m}_1^*, \boldsymbol{m}_2^*, \ldots, \boldsymbol{m}_N^*)$ *(where* $\boldsymbol{m}_i^* = (\boldsymbol{\pi}_{\boldsymbol{i}}^*, \boldsymbol{p}_{\boldsymbol{i}}^*), i \in \mathcal{N}$*) that result in* $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ *is characterized by the solution of the following set of conditions:*

$$\begin{aligned} \frac{1}{N} \sum_{i \in \mathcal{N}} \boldsymbol{p}_{\boldsymbol{i}}^* &= \hat{\boldsymbol{p}}^*, \\ \boldsymbol{\pi}_{\boldsymbol{i+1}}^* - \boldsymbol{\pi}_{\boldsymbol{i+2}}^* &= \boldsymbol{l}_{\boldsymbol{i}}^*, \; i \in \mathcal{N}, \\ (\boldsymbol{p}_{\boldsymbol{i}}^* - \boldsymbol{p}_{\boldsymbol{i+1}}^*)^T diag(\boldsymbol{\pi}_{\boldsymbol{i}}^*)(\boldsymbol{p}_{\boldsymbol{i}}^* - \boldsymbol{p}_{\boldsymbol{i+1}}^*) &= 0, \; i \in \mathcal{N}, \\ \boldsymbol{\pi}_{\boldsymbol{i}}^* &\geq \mathbf{0}, \; i \in \mathcal{N}. \end{aligned}$$

⊠

Because Theorem 3.1 is stated for an arbitrary NE $\boldsymbol{m}^*$ of the game induced by the game form presented in Section 3.2.2 and the users' utility functions (3.3), the assertion of the theorem holds for all NE of this game. Thus, part (a) of Theorem 3.1 establishes that the game form presented in Section 3.2.2 is *individually rational*.

Part (b) of Theorem 3.1 asserts that all NE of the game induced by the game form presented in Section 3.2.2 and the users' utility functions (3.3) result in optimal centralized allocations (solutions of Problem (P.3)). Thus, the set of NE allocations is a subset of the set of centralized allocations. This establishes that the game form presented in Section 3.2.2 *implements in NE* the goal correspondence γ defined by Problem (P.3) (see Section 3.2.1). Because of this property, the game form guarantees to provide a centralized allocation irrespective of which NE is achieved in the game induced by the game form.

The assertion of Theorem 3.1 that establishes the above two properties of the game form is based on the assumption that there exists a NE of the game induced by the game form of Section 3.2.2 and the users' utility functions (3.3). However, Theorem 3.1 does not say anything about the existence of a NE. Theorem 3.2 establishes that NE exist in the above game and also characterizes the set of all NE that result in optimal centralized allocations $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*) = (\boldsymbol{l_i}^{*T}\hat{\boldsymbol{p}}^*, i = 1, 2, \ldots, N, \hat{\boldsymbol{p}}^*)$ where $\boldsymbol{l_i}^*, i = 1, 2, \ldots, N$, are defined in Theorem 3.2 (a).

The proofs of Theorem 3.1 and Theorem 3.2 are given in Appendices 3.A and 3.B. In the next section we provide a brief discussion on the intuition behind the structure of the proposed game form.

Before we proceed to the next section, we note that the game form that presented in Section 3.2.2 determines for the uplink network an optimum power profile that should be "received" at the BS. Once the game form determines an optimum received power profile, each user can determine its respective transmission power that would result in the optimum received power profile since each user knows its respective channel gain $h_{i0}, i \in \mathcal{N}$. Since the optimum received power profile is obtained at the NE of users' messages, no user can gain by unilaterally changing the power received from it at the BS; in other words the user cannot gain by transmitting a power that does not result in the received power determined by the game form. Thus, the game form of Section 3.2.2 not only determines the optimum received powers, but also induces the users to "transmit" with optimum powers.

As we mentioned earlier, Assumption 3.4 is not necessary for the game form proposed in Section 3.2.2 to result in optimal power allocations. Consider the case when the symmetric channel assumption is relaxed. We note that the game form of Section 3.2.2 requires the users to communicate messages in terms of the power vector received at the BS, not the power vector transmitted by the users. Therefore, once the mechanism determines the power vector that should be received at the BS, the BS can announce it to the users. In the absence of the knowledge of uplink channel gains, the users will have to transmit power based on some estimate of the uplink channel gain; if the power received by the BS is not the same as that determined by the mechanism, the BS can send feedback to the users to adjust their transmission powers. As explained in the previous paragraph, it will be in the interest of the users to make the transmission power adjustment so as to match the received power to the optimal one. Thus, the mechanism would result in the same outcome as in the case with the symmetric channel assumption.

3.2.4 Key features of and intuition behind the game form

The key feature of our problem is that the action / transmission power of a user directly affects the utility of every other user. Thus, every user's action creates an externality for every other user. Consequently, we have to view the power allocation problem with strategic users as the decentralized resource allocation of a public good, where the public good is the power profile $\boldsymbol{p} := (p_1, p_2, \dots, p_N)$ received at the BS. Since the users are strategic, the dimensionality of the message space of any "efficient" [3.8] mechanism must be at least as large as the dimensionality of any "efficient" mechanism for non-strategic users [41]. Under the condition that users are non-strategic, the minimum dimensionality of any "efficient" public good mechanism is of the order $O(N^2)$ (See [44]). Therefore, any "efficient" mechanism for our problem must have a message space whose dimensionality is at least of the order $O(N^2)$.

In our mechanism each of the N users announces a $2N$ dimensional message consisting of an N dimensional power profile proposal and an N dimensional price profile proposal. Thus, the dimensionality of the message space of our mechanism/game form is $2N^2$. From the above discussion it is clear that the use of high dimensional mechanism is inevitable if one wants to have full implementation in Nash equilibria.

To understand how the proposed structure of the game form achieves the desired goal, let us now look at the properties the game form induces in its NE. A NE of the game corresponding to the proposed game form can be interpreted as follows: Since the allocated received power profile, given the users' messages $\boldsymbol{m_j}, j \in \mathcal{N}$, is $1/N \sum_{i=1}^{N} \boldsymbol{p_i}$, user i's proposal $\boldsymbol{p_i}$ can be interpreted as the increment user i desires in the power received from each user over the sum of other users' proposals so as to bring the allocated received power profile $\hat{\boldsymbol{p}}(\boldsymbol{m})$ to i's desired value. Thus, if the average of the received power profiles proposed by users other than user i does not lie in $\mathcal{D}_i$, user i can propose an appropriate received power profile and bring the allocated profile within $\mathcal{D}_i$. It should be noted that the flexibility of proposing any received power profile in $\mathbb{R}^N$ gives each user $i \in \mathcal{N}$ the capability to make the

[3.8] We define a mechanism to be "efficient" if it implements in Nash equilibria the solution of the corresponding centralized power allocation problem.

constraint $\boldsymbol{p} \in \mathcal{D}_i$ be satisfied by unilateral deviation. It follows that any NE received power profile must lie in $\cap_{i\in\mathcal{N}}\mathcal{D}_i$. Furthermore, it can be seen from (3.10) that the game form formulation ensures that the allocated tax profile satisfies (3.2) (even at off-NE messages). The above two features imply that all NE allocations $(\boldsymbol{t}, \boldsymbol{p})$ lie in $\mathcal{D}$ and hence are feasible solutions of Problem (P.3).

To see why NE allocations are optimal, let us look at the form of the tax (3.10). The tax for user i consists of three types of terms. Type-1 is $\boldsymbol{l_i}^T(\boldsymbol{m})\hat{\boldsymbol{p}}(\boldsymbol{m})$ that depends on the power proposals of all the users, and the price proposals of users other than user i. Type-2 term is the one that depends on $\boldsymbol{p_i}$ as well as $\boldsymbol{\pi_i}$, and type-3 term is the one that depends only on the messages of users other than user i. Since $\boldsymbol{\pi_i}$ does not affect the received power allocation and affects only the type-2 term in t_i, the NE strategy of user $i, i \in \mathcal{N}$, that minimizes its tax is to propose for each $j \in \mathcal{N}$, $\pi_{ij} = 0$ unless at the NE, $p_{ij} = p_{i+1j}$. Since all the users $i \in \mathcal{N}$ choose the aforementioned strategy at the NE, the type-2 and type-3 terms vanish from every user's tax $t_i, i \in \mathcal{N}$, at the NE. Thus, the tax that users pay at a NE $\boldsymbol{m}^*$ is of the form $\boldsymbol{l_i}^T(\boldsymbol{m}^*)\hat{\boldsymbol{p}}(\boldsymbol{m}^*)$, $i \in \mathcal{N}$. The NE price term $\boldsymbol{l_i}^T(\boldsymbol{m}^*) =: \boldsymbol{l_i}^{*T}$, $i \in \mathcal{N}$, can therefore be interpreted as the "personalized price" [3.9] of the NE received power profile $\hat{\boldsymbol{p}}(\boldsymbol{m}^*) =: \hat{\boldsymbol{p}}^*$ (treated as a public good) for user i; at the NE this price for user i is not controlled by i's message. The above reduction of tax terms in terms of the allocated received power profile implies that, at the NE, the utilities of the users $i \in \mathcal{N}$ effectively depend only on the allocated received power profile. Since each user has the capability (by choosing appropriate $\boldsymbol{p_i} \in \mathbb{R}^N$) to shift the allocated received power profile to its desired value given that the proposals of all other users are fixed, the NE strategy of each user is to propose a power profile that results in

[3.9] In Economics literature, these personalized prices for the public goods are called "Lindahl" prices.

an allocation that maximizes its corresponding utility. Thus, each user maximizes its net utility at the NE, and this results in the maximization of the system objective function at the NE.

It is worth mentioning at this point the significance of type-2 and type-3 terms in the users' tax. As explained above, these terms vanish at NE. However, if these terms are not present in t_i, user $i, i \in \mathcal{N}$, can propose arbitrarily high price for other users in $\boldsymbol{\pi_i}$ as $\boldsymbol{\pi_i}$ would not affect user i's utility at all.[3.10] It is also important that the NE price $\boldsymbol{l_i}$ is not affected by $\boldsymbol{\pi_i}$, otherwise user i may influence its own price in an unfair manner. However, since $\boldsymbol{\pi_i}$ would affect other users' price, it is necessary to prevent user i from proposing unfair prices for other users. Type-2 and type-3 terms in t_i do the above job by imposing a penalty on user i at off-equilibrium messages if user i proposes a high value of $\boldsymbol{\pi_i}$ or if it deviates too much from other users in its power profile proposal.

We divide the proofs of Theorems 3.1 and 3.2 into several claims to organize the presentation.

3.A Proof of Theorem 3.1

Claim 3.1. *If $\boldsymbol{m}^*$ is a NE of the game specified by the game form presented in Section 3.2.2 and the users' utility functions (3.3), then the allocation $(\hat{\boldsymbol{t}}(\boldsymbol{m}^*), \hat{\mathbf{p}}(\boldsymbol{m}^*)) =: (\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is a feasible solution of Problem (P.3), i.e. $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*) \in \mathcal{D}$.*

Proof:

By construction of the game form, the allocated tax (3.10) satisfies (3.2) which implies that the NE tax profile $\hat{\boldsymbol{t}}^*$ also satisfies (3.2). Therefore to prove the claim, we need to show that the NE power profile $\hat{\boldsymbol{p}}^* \in \cap_{i \in \mathcal{N}} \mathcal{D}_i$ (where $\mathcal{D}_i, i \in \mathcal{N}$, is defined

[3.10] Note that $\boldsymbol{l_i}$ depends on $\boldsymbol{\pi_{i+1}}$ and $\boldsymbol{\pi_{i+2}}$ and not $\boldsymbol{\pi_i}$.

by (3.1)). We will prove this by showing that, if $\hat{\boldsymbol{p}}^* \notin \mathcal{D}_i$ for some $i \in \mathcal{N}$, then there exists a profitable unilateral deviation for user i.

Suppose $\hat{\boldsymbol{p}}^* \notin \mathcal{D}_i$ for some $i \in \mathcal{N}$. Then, from (3.3), $u_i^A(\hat{t}_i^*, \hat{\boldsymbol{p}}^*) = -\infty$. Consider $\widetilde{\boldsymbol{m}}_{\boldsymbol{i}} = (\boldsymbol{\pi}_{\boldsymbol{i}}^*, \widetilde{\boldsymbol{p}}_{\boldsymbol{i}})$ where $\boldsymbol{\pi}_{\boldsymbol{i}}^*$ is the NE price profile and $\widetilde{\boldsymbol{p}}_{\boldsymbol{i}}$ ($\widetilde{\boldsymbol{p}}_{\boldsymbol{i}} \in \mathbb{R}^N$) is such that,

$$\hat{\boldsymbol{p}}(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i) = \frac{1}{N}\Big(\sum_{\substack{j \in \mathcal{N} \\ j \neq i}} \boldsymbol{p}_{\boldsymbol{j}}^* + \widetilde{\boldsymbol{p}}_{\boldsymbol{i}}\Big) = \boldsymbol{0} \in \mathcal{D}_i.$$

Then,

$$\begin{aligned} u_i^A\big(\hat{t}_i(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i),\, \hat{\boldsymbol{p}}(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i)\big) &= -\hat{t}_i(\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}, \boldsymbol{m}^*/i) + u_i(\boldsymbol{0}) \\ &> -\infty = u_i^A(\hat{t}_i^*, \hat{\boldsymbol{p}}^*) \end{aligned} \tag{3.12}$$

Thus user i will find it profitable to deviate to $\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}$.

Inequality (3.12) implies that $\boldsymbol{m}^*$ cannot be a NE, which is a contradiction. Therefore we must have that, $\hat{\boldsymbol{p}}^* \in \cap_{i \in \mathcal{N}} \mathcal{D}_i$ and hence, $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*) \in \mathcal{D}$. □

Claim 3.2. *If $\boldsymbol{m}^*$ is a NE of the game specified by the game form presented in Section 3.2.2 and the users' utility functions (3.3), then, the tax $\hat{t}_i(\boldsymbol{m}^*) =: \hat{t}_i^*$ paid by user i, $i \in \mathcal{N}$, at NE $\boldsymbol{m}^*$ is of the form, $\hat{t}_i^* = \boldsymbol{l}_{\boldsymbol{i}}^{*T} \hat{\boldsymbol{p}}^*$, where $\boldsymbol{l}_{\boldsymbol{i}}^* := \boldsymbol{l}_{\boldsymbol{i}}(\boldsymbol{m}^*)$.*

Proof:

Let $\boldsymbol{m}^*$ be a NE described in Claim 3.2. Then, for each $i \in \mathcal{N}$,

$$u_i^A\big(\hat{t}_i(\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{m}^*/i), \hat{\boldsymbol{p}}(\boldsymbol{m}_{\boldsymbol{i}}, \boldsymbol{m}^*/i)\big) \le u_i^A\big(\hat{t}_i^*, \hat{\boldsymbol{p}}^*\big), \quad \forall\, \boldsymbol{m}_{\boldsymbol{i}} \in \mathcal{M}_i. \tag{3.13}$$

Substituting $\boldsymbol{m}_{\boldsymbol{i}} = (\boldsymbol{\pi}_{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{i}}^*)$, $\boldsymbol{\pi}_{\boldsymbol{i}} \in \mathbb{R}_+^N$, in (3.13) and using (3.9) implies that

$$u_i^A\big(\hat{t}_i((\boldsymbol{\pi}_{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{i}}^*), \boldsymbol{m}^*/i), \hat{\boldsymbol{p}}^*\big) \le u_i^A\big(\hat{t}_i^*, \hat{\boldsymbol{p}}^*\big), \quad \forall\, \boldsymbol{\pi}_{\boldsymbol{i}} \in \mathbb{R}_+^N. \tag{3.14}$$

Since u_i^A decreases in t_i (see (3.3)), (3.14) implies that

$$\hat{t}_i((\boldsymbol{\pi}_{\boldsymbol{i}}, \boldsymbol{p}_{\boldsymbol{i}}^*), \boldsymbol{m}^*/i) \ge \hat{t}_i^*, \quad \forall\, \boldsymbol{\pi}_{\boldsymbol{i}} \in \mathbb{R}_+^N. \tag{3.15}$$

Substituting (3.10) in (3.15) implies that

(3.16)

$$\boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}^* + (\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) - (\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*})^T \mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*})$$

$$\geq$$

$$\boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}^* + (\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) - (\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*})^T \mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*}),$$

$$\forall\ \boldsymbol{\pi_i} \in \mathbb{R}_+^N.$$

Canceling the common terms in (3.16) implies

$$(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i} - \boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) \geq 0, \quad \forall\ \boldsymbol{\pi_i} \in \mathbb{R}_+^N. \tag{3.17}$$

Since (3.17) must hold for all $\boldsymbol{\pi_i} \geq \mathbf{0}$, it implies that for each $j \in \mathcal{N}$,

$$\text{either} \ \ p_{i\,j}^* = p_{i+1\,j}^*, \quad \text{or} \ \ \pi_{i\,j}^* = 0. \tag{3.18}$$

From (3.18) it follows that at any NE $\boldsymbol{m}^*$,

$$(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T \mathrm{diag}(\boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) = 0, \quad \forall\ i \in \mathcal{N}. \tag{3.19}$$

Using (3.19) in (3.10) we obtain that any NE tax profile must be of the form

$$\hat{t}_i^* = \boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}^*, \quad \forall\ i \in \mathcal{N}. \tag{3.20}$$

□

Claim 3.3. *The game form given in Section 3.2.2 is individually rational, i.e. for every NE $\boldsymbol{m}^*$ corresponding to it, the allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is weakly preferred by all the users to the initial allocation $(\mathbf{0}, \mathbf{0})$, i.e.,*

$$u_i^A(0, \mathbf{0}) \leq u_i^A(\hat{t}_i^*, \hat{\boldsymbol{p}}^*), \quad \forall\ i \in \mathcal{N}.$$

Proof:

Suppose $\boldsymbol{m}^*$ is a NE of the game specified by the game form presented in Section 3.2.2 and the users' utility functions (3.3). From Claim 3.2 we know the form of the tax at $\boldsymbol{m}^*$. Substituting that from (3.20) into (3.13) we obtain that, for each $i \in \mathcal{N}$,

$$u_i^A(\hat{t}_i((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i), \ \hat{\boldsymbol{p}}((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i)) \leq u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \ \hat{\boldsymbol{p}}^*), \qquad \forall \ \boldsymbol{m_i} = (\boldsymbol{\pi_i}, \boldsymbol{p_i}) \in \mathcal{M}_i. \tag{3.21}$$

Substituting for $\hat{t}_i$ in (3.21) from (3.10) and using equality (3.19) we obtain

$$u_i^A\Big(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i) + (\boldsymbol{p_i} - \boldsymbol{p^*_{i+1}})^T \mathrm{diag}(\boldsymbol{\pi_i})(\boldsymbol{p_i} - \boldsymbol{p^*_{i+1}}), \hat{\boldsymbol{p}}((\boldsymbol{\pi_i}, \boldsymbol{p_i}), \boldsymbol{m}^*/i)\Big) \leq u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \quad \forall \ \boldsymbol{\pi_i} \in \mathbb{R}_+^N, \quad \forall \ \boldsymbol{p_i} \in \mathbb{R}^N. \tag{3.22}$$

In particular, $\boldsymbol{\pi_i} = \boldsymbol{0}$ in (3.22) implies that

$$u_i^A\big(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}((\boldsymbol{0}, \boldsymbol{p_i}), \boldsymbol{m}^*/i), \ \hat{\boldsymbol{p}}((\boldsymbol{0}, \boldsymbol{p_i}), \boldsymbol{m}^*/i)\big) \ \leq \ u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \quad \forall \ \boldsymbol{p_i} \in \mathbb{R}^N. \tag{3.23}$$

Substituting $1/N\big(\boldsymbol{p_i} + \sum_{j \in \mathcal{N} \setminus \{i\}} \boldsymbol{p_j^*}\big) = \overline{\boldsymbol{p}}$ in (3.23) and using the fact that (3.23) holds for all $\boldsymbol{p_i} \in \mathbb{R}^N$ gives

$$u_i^A\left(\boldsymbol{l_i^*}^T \overline{\boldsymbol{p}}, \ \overline{\boldsymbol{p}}\right) \leq u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \quad \forall \ \overline{\boldsymbol{p}} \in \mathbb{R}^N. \tag{3.24}$$

For $\overline{\boldsymbol{p}} = \boldsymbol{0}$, (3.24) implies that

$$u_i^A(0, \boldsymbol{0}) \leq u_i^A(\boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, \hat{\boldsymbol{p}}^*), \quad \forall \ i \in \mathcal{N}. \tag{3.25}$$

$\square$

Claim 3.4. *A NE allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is an optimal solution of the centralized problem (P.3).*

Proof:

For each $i \in \mathcal{N}$, (3.24) can be equivalently written as

$$\begin{aligned}
\hat{\boldsymbol{p}}^* &= \underset{\overline{\boldsymbol{p}} \in \mathbb{R}^N}{\arg\max} \quad u_i^A\left(\boldsymbol{l_i^{*T}}\overline{\boldsymbol{p}},\ \overline{\boldsymbol{p}}\right) \\
&= \underset{\overline{\boldsymbol{p}} \in \mathbb{R}^N}{\arg\max} \quad \left(-\boldsymbol{l_i^{*T}}\overline{\boldsymbol{p}} + u_i(\overline{\boldsymbol{p}}) - \left[\frac{1 - I_{\mathcal{D}_i}(\overline{\boldsymbol{p}})}{I_{\mathcal{D}_i}(\overline{\boldsymbol{p}})}\right]\right) \\
&= \underset{\overline{\boldsymbol{p}} \in \mathcal{D}_i}{\arg\max} \quad \left(-\boldsymbol{l_i^{*T}}\overline{\boldsymbol{p}} + u_i(\overline{\boldsymbol{p}})\right).
\end{aligned} \tag{3.26}$$

Since for each $i \in \mathcal{N}$, $u_i(\overline{\boldsymbol{p}})$ is assumed to be concave in $\overline{\boldsymbol{p}}$ over $\mathcal{D}_i$ and the set $\mathcal{D}_i$ is convex, Karush Kuhn Tucker (KKT) conditions [7, Chapter 11] are necessary and sufficient for $\hat{\boldsymbol{p}}^*$ to be the maximizer in (3.26). Thus, for each $i \in \mathcal{N}$, $\exists\ \boldsymbol{\lambda_1^i} \in \mathbb{R}_+^N$ and $\boldsymbol{\lambda_2^i} \in \mathbb{R}_+^N$ such that, $\hat{\boldsymbol{p}}^*$, $\boldsymbol{\lambda_1^i}$ and $\boldsymbol{\lambda_2^i}$ satisfy the KKT conditions given below:

$$\boldsymbol{l_i^*} - \nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i} = 0 \tag{3.27}$$

$$\boldsymbol{\lambda_1^i}^T \hat{\boldsymbol{p}}^* = 0 \tag{3.28}$$

$$\boldsymbol{\lambda_2^i}^T (\hat{\boldsymbol{p}}^* - P_0^{max}\mathbf{1}) = 0 \tag{3.29}$$

$$\text{where,} \quad \mathbf{1} = (\underbrace{1, 1, \ldots, 1}_{N \text{ times}}) \in \mathbb{R}^{N \times 1}.$$

Combining the KKT conditions of all the users, i.e. summing (3.27) for all $i \in \mathcal{N}$, and using the fact that $\sum_{i \in \mathcal{N}} \boldsymbol{l_i^*} = 0$ (see (3.11)), we obtain

$$\sum_{i \in \mathcal{N}} \Big(- \nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i} \Big) = 0 \tag{3.30}$$

Eq. (3.30) along with (3.28) and (3.29) for all i, and the non-negativity of $\boldsymbol{\lambda_1^i}, \boldsymbol{\lambda_2^i}$, $i \in \mathcal{N}$, specify the KKT conditions (for variable $\boldsymbol{p}$) for (3.5). Since (3.5) is a concave optimization problem, the KKT conditions are necessary and sufficient for its optimum. Since $\hat{\boldsymbol{p}}^*$ satisfies these KKT conditions, it is a maximizer of the objective function in (3.5). Therefore, as described in Section 3.1.2, an optimal solution of Problem (P.3) is of the form $(\boldsymbol{t}, \hat{\boldsymbol{p}}^*)$, where $\boldsymbol{t} \in \mathbb{R}^N$ is any tax profile that satisfies (3.2). Since by construction of the tax the NE allocation $\hat{\boldsymbol{t}}^*$ satisfies (3.2), we conclude that $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ is an optimal solution of (P.3). □

Theorem 3.1 shows that if there exists a NE corresponding to the game of Section 3.2.2, then the allocation at the NE is an optimal solution of the centralized problem (P.3). However, Theorem 3.1 does not guarantee the existence of a NE; in other words, it does not guarantee that a centralized optimum power profile is attainable through NE. This is guaranteed by Theorem 3.2 which is proved next.

3.B Proof of Theorem 3.2

We prove Theorem 3.2 in two steps. In the first step we show that if $\hat{\boldsymbol{p}}^*$ is an optimal power profile for the centralized problem (P.3), there exist a set of personalized prices, one for each user $i \in \mathcal{N}$, such that when every user individually maximizes its own utility taking the above prices as given, then each of them obtains $\hat{\boldsymbol{p}}^*$ as its optimal power profile. In the second step we show that $\hat{\boldsymbol{p}}^*$ and the corresponding set of personalized prices can be used to construct message profiles that are NE of the game induced by the game form of Section 3.2.2 and the users' utility functions (3.3).

Claim 3.5. *If $\hat{\boldsymbol{p}}^*$ is an optimum power profile corresponding to Problem (P.3), there exist a set of personalized prices $\boldsymbol{l}_{\boldsymbol{i}}^*$, $i \in \mathcal{N}$, such that*

$$\underset{\boldsymbol{p} \in \mathcal{D}_i}{\arg\max} \; -{\boldsymbol{l}_{\boldsymbol{i}}^*}^T \boldsymbol{p} + u_i(\boldsymbol{p}) \;=\; \hat{\boldsymbol{p}}^*, \qquad \forall\, i \in \mathcal{N}. \tag{3.31}$$

Proof:

Suppose $\hat{\boldsymbol{p}}^*$ is an optimal power profile corresponding to Problem (P.3). Problem (P.3) does have a solution since it involves maximization of a concave function in $\boldsymbol{p}$ over a convex and compact set in $\boldsymbol{p}$ (The solution in $\boldsymbol{t}$ trivially exists). Writing the optimization problem (P.3) for $\boldsymbol{p}$ we have,

$$\begin{aligned} \hat{\boldsymbol{p}}^* &= \arg\max_{\boldsymbol{p}} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{p}) \\ \text{s.t.} & \quad \boldsymbol{p} \in \mathcal{D}_i, \quad \forall\, i \in \mathcal{N} \end{aligned}$$

An optimal solution of the above problem must satisfy the KKT conditions. Therefore there exist $\boldsymbol{\lambda_1^i} \in \mathbb{R}_+^N$ and $\boldsymbol{\lambda_2^i} \in \mathbb{R}_+^N$, $i \in \mathcal{N}$, such that $\hat{\boldsymbol{p}}^*$, $\boldsymbol{\lambda_1^i}$ and $\boldsymbol{\lambda_2^i}, i \in \mathcal{N}$, satisfy

$$\sum_{i\in\mathcal{N}} \left(-\nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i}\right) = 0, \tag{3.32}$$

$$\boldsymbol{\lambda_1^i}^T \hat{\boldsymbol{p}}^* = 0, \quad \forall\, i \in \mathcal{N}, \tag{3.33}$$

$$\text{and} \quad \boldsymbol{\lambda_2^i}^T (\hat{\boldsymbol{p}}^* - P_0^{max}\mathbf{1}) = 0, \quad \forall\, i \in \mathcal{N}. \tag{3.34}$$

We define for each $i \in \mathcal{N}$,

$$\boldsymbol{l_i^*} := \nabla u_i(\hat{\boldsymbol{p}}^*) + \boldsymbol{\lambda_1^i} - \boldsymbol{\lambda_2^i}. \tag{3.35}$$

Then,

$$\boldsymbol{l_i^*} - \nabla u_i(\hat{\boldsymbol{p}}^*) - \boldsymbol{\lambda_1^i} + \boldsymbol{\lambda_2^i} = 0, \quad \forall\, i \in \mathcal{N}. \tag{3.36}$$

Equations (3.36), (3.33) and (3.34) together imply that for each $i \in \mathcal{N}$, $\hat{\boldsymbol{p}}^*$, $\boldsymbol{\lambda_1^i} \in \mathbb{R}_+^N$ and $\boldsymbol{\lambda_2^i} \in \mathbb{R}_+^N$ satisfy the KKT conditions for the following maximization problem:

$$\max_{\boldsymbol{p}\in\mathcal{D}_i} \; -\boldsymbol{l_i^*}^T \boldsymbol{p} + u_i(\boldsymbol{p}) \tag{3.37}$$

Since (3.37) is a concave optimization problem, KKT conditions are necessary and sufficient for its optimum. Therefore, from (3.33), (3.34) and (3.36) we conclude that

$$\hat{\boldsymbol{p}}^* = \arg\max_{\boldsymbol{p}\in\mathcal{D}_i} \; -\boldsymbol{l_i^*}^T \boldsymbol{p} + u_i(\boldsymbol{p}). \tag{3.38}$$

$\square$

Claim 3.6. *Let $\hat{\boldsymbol{p}}^*$ be an optimal power profile corresponding to Problem (P.3), let $\boldsymbol{l_i^*}$, $i \in \mathcal{N}$, be the personalized prices defined in Claim 3.5, and let $\hat{t}_i^* := \boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^*, i \in \mathcal{N}$. Let $\boldsymbol{m_i^*} := (\boldsymbol{\pi_i^*}, \boldsymbol{p_i^*}), i \in \mathcal{N}$, be a solution to the following set of relations:*

$$\frac{1}{N}\sum_{i \in \mathcal{N}} \boldsymbol{p_i^*} = \hat{\boldsymbol{p}}^*, \tag{3.39}$$

$$\boldsymbol{\pi_{i+1}^*} - \boldsymbol{\pi_{i+2}^*} = \boldsymbol{l_i^*}, \ i \in \mathcal{N}, \tag{3.40}$$

$$(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*})^T diag(\boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p_{i+1}^*}) = 0, \ i \in \mathcal{N}, \tag{3.41}$$

$$\boldsymbol{\pi_i^*} \geq \boldsymbol{0}, \ i \in \mathcal{N}. \tag{3.42}$$

Then, $\boldsymbol{m}^* := (\boldsymbol{m_1^*}, \boldsymbol{m_2^*}, \dots, \boldsymbol{m_N^*})$ is a NE of the game induced by the game form of Section 3.2.2 and the users' utility functions (3.3). Furthermore, $\hat{\boldsymbol{p}}(\boldsymbol{m}^*) = \hat{\boldsymbol{p}}^*$, and for each $i \in \mathcal{N}$, $\boldsymbol{l_i}(\boldsymbol{m}^*) = \boldsymbol{l_i^*}$ and $\hat{t}_i(\boldsymbol{m}^*) = \hat{t}_i^*$.

Proof:

Note that, (3.39)–(3.42) are necessary conditions for any NE $\boldsymbol{m}^*$ corresponding to the game of Section 3.2.2 to result in the allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$ (This follows from (3.9),(3.11) and (3.19)). Therefore, the set of solutions of (3.39)–(3.42), if one exists, is a superset of the set of all NE that result in $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$. Below we show that the solution set of (3.39)–(3.42) is in fact exactly the set of NE that result in $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$.

To prove this we first show that the set of relations (3.39)–(3.42) do have a solution. Notice that by setting $\boldsymbol{p_i^*} = \hat{\boldsymbol{p}}^* \ \ \forall \ i \in \mathcal{N}$, equations (3.39) and (3.41) are satisfied. Notice also that the right hand side of (3.40) sums to 0 by taking the sum over $i \in \mathcal{N}$. Therefore, (3.40) has a solution in $\boldsymbol{\pi_i^*}, i \in \mathcal{N}$. Furthermore, for any solution $\boldsymbol{\pi_i^*}, i \in \mathcal{N}$, of (3.40), $\boldsymbol{\pi_i^*} + c, i \in \mathcal{N}$, where c is some constant, is also a solution of (3.40). Therefore by appropriately choosing c, we can select a solution of (3.40) such that (3.42) is satisfied.

It is clear from above that (3.39)–(3.42) have multiple solutions. We now show that the set of solutions $\boldsymbol{m}^*$ of (3.39)–(3.42) is the set of NE that result in the given centralized solution. From Claim 3.5, (3.31) can be equivalently written as

$$\begin{aligned}\hat{\boldsymbol{p}}^* &= \underset{\boldsymbol{p}\in\mathbb{R}^N}{\arg\max} \quad -\boldsymbol{l_i^*}^T\boldsymbol{p} + u_i(\boldsymbol{p}) - \left[\frac{1-I_{\mathcal{D}_i}(\boldsymbol{p})}{I_{\mathcal{D}_i}(\boldsymbol{p})}\right] \\ &= \underset{\boldsymbol{p}\in\mathbb{R}^N}{\arg\max} \quad u_i^A(\boldsymbol{l_i^*}^T\boldsymbol{p},\, \boldsymbol{p}), \qquad \forall\, i\in\mathcal{N}.\end{aligned} \tag{3.43}$$

A change of variable $N\boldsymbol{p} - \sum_{\substack{j\in\mathcal{N}\\ j\neq i}} \boldsymbol{p_j^*} = \boldsymbol{p_i}$ in (3.43) gives

$$\boldsymbol{p_i^*} = \underset{\boldsymbol{p_i}\in\mathbb{R}^N}{\arg\max}\ u_i^A\left(\boldsymbol{l_i^*}^T\frac{1}{N}\Big(\boldsymbol{p_i} + \sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\boldsymbol{p_j^*}\Big), \frac{1}{N}\Big(\boldsymbol{p_i} + \sum_{\substack{j\in\mathcal{N}\\ j\neq i}}\boldsymbol{p_j^*}\Big)\right) \tag{3.44}$$

Because of (3.41) Eq. (3.44) also implies the following:

$$\begin{aligned}(\boldsymbol{\pi_i^*},\boldsymbol{p_i^*}) = \underset{(\boldsymbol{\pi_i},\,\boldsymbol{p_i})\in\mathbb{R}_+^N\times\mathbb{R}^N}{\arg\max}\ u_i^A\Big(&\boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i) - (\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*})^T \\ &\mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*}),\ \hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i)\Big)\end{aligned} \tag{3.45}$$

Furthermore, since u_i^A is strictly decreasing in the tax (see (3.3)), Eq. (3.45) also implies the following:

$$\begin{aligned}&(\boldsymbol{\pi_i^*},\boldsymbol{p_i^*}) = \\ &\underset{(\boldsymbol{\pi_i}\boldsymbol{p_i})\in\mathbb{R}_+^N\times\mathbb{R}^N}{\arg\max}\ u_i^A\Big(\boldsymbol{l_i^*}^T\hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i) + (\boldsymbol{p_i} - \boldsymbol{p_{i+1}^*})^T\mathrm{diag}(\boldsymbol{\pi_i})(\boldsymbol{p_i} - \boldsymbol{p_{i+1}^*}) \\ &\quad - (\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*})^T\mathrm{diag}(\boldsymbol{\pi_{i+1}^*})(\boldsymbol{p_{i+1}^*} - \boldsymbol{p_{i+2}^*}), \hat{\boldsymbol{p}}((\boldsymbol{\pi_i},\boldsymbol{p_i}),\boldsymbol{m}^*/i)\Big), \qquad i\in\mathcal{N}.\end{aligned} \tag{3.46}$$

Eq. (3.46) implies that, if the message exchange and allocation is done according to the game form defined in Section 3.2.2, then user $i, i\in\mathcal{N}$, maximizes its utility at $\boldsymbol{m_i^*}$ given that all other users $j\in\mathcal{N}\backslash\{i\}$ use their respective messages $\boldsymbol{m_j^*}, j\in\mathcal{N}\backslash\{i\}$. This implies that a message profile $\boldsymbol{m}^*$ that is a solution to (3.39)–(3.42) is a NE corresponding to the aforementioned game. Furthermore, it follows from (3.39)–

(3.42) that the allocation at $\boldsymbol{m}^*$ is

$$\hat{\boldsymbol{p}}(\boldsymbol{m}^*) = \frac{1}{N}\sum_{i\in\mathcal{N}} \boldsymbol{p_i^*} = \hat{\boldsymbol{p}}^*, \tag{3.47}$$

$$\boldsymbol{l_i}(\boldsymbol{m}^*) = \boldsymbol{\pi^*_{i+1}} - \boldsymbol{\pi^*_{i+2}} = \boldsymbol{l_i^*}, \quad \forall\, i \in \mathcal{N}, \tag{3.48}$$

$$\begin{aligned} \hat{t}_i(\boldsymbol{m}^*) &= \boldsymbol{l_i}^T(\boldsymbol{m}^*)\hat{\boldsymbol{p}}(\boldsymbol{m}^*) + (\boldsymbol{p_i^*} - \boldsymbol{p^*_{i+1}})^T \mathrm{diag}(\boldsymbol{\pi_i^*})(\boldsymbol{p_i^*} - \boldsymbol{p^*_{i+1}}) \\ &\quad - (\boldsymbol{p^*_{i+1}} - \boldsymbol{p^*_{i+2}})^T \mathrm{diag}(\boldsymbol{\pi^*_{i+1}})(\boldsymbol{p^*_{i+1}} - \boldsymbol{p^*_{i+2}}) \\ &= \boldsymbol{l_i^*}^T \hat{\boldsymbol{p}}^* = \hat{t}_i^*. \end{aligned} \tag{3.49}$$

It follows from (3.47)–(3.49) that the set of solutions $\boldsymbol{m}^*$ of (3.39)–(3.42) is exactly the set of NE corresponding to the game of Section 3.2.2 that result in the allocation $(\hat{\boldsymbol{t}}^*, \hat{\boldsymbol{p}}^*)$. This completes the proof of Claim 3.6 and hence the proof of Theorem 3.2.

$\square$

CHAPTER 4

Resource allocation in large-scale networks: A realization perspective

In this chapter we consider a generalization of the problem presented in Chapter 2. Specifically, we consider a situation where each agent's actions affect only a subset of the network agents. Such a situation arises in various applications including large-scale wireless networks where each user creates interference to only a subset of the network users. We develop a generic model (so it can be used to study resource allocation problems arising in a number of network applications) to capture the above situation and formulate a resource allocation problem from the realization theory perspective. For this problem we propose a decentralized resource allocation mechanism that has the following properties: (i) Each agent in the network needs to communicate only with those agents that either affect it or are affected by it. (ii) The mechanism preserves the private information of each agent. (iii) It guarantees convergence to the network optimal resource allocation.

The chapter is organized as follows: In Section 4.1.1 we present the network model and discuss motivating applications in Section 4.1.2. In Section 4.1.3 we formulate the resource allocation problem. We state our contributions in the formulation and solution of the above problem in Section 4.1.5. In Section 4.2.1 we develop ideas for the construction of decentralized resource allocation mechanism for the problem formulated in Section 4.1.3, and follow that with the specification of a decentralized mechanism in Section 4.2.2. We discuss the properties of the proposed mechanism in Section 4.2.3 and we prove these properties in Appendix 4.A. In Section 4.2.4 we conclude with a discussion on how the decentralized resource allocation problem and its solution presented in this chapter fit within the framework of the realization theory component of mechanism design.

Before we present the model, we describe the notation that we will use throughout the chapter.

Notation:

We use bold font to represent vectors and regular font for scalars. We represent the element of a vector by a subscript on the vector symbol, and the element of a matrix by double subscript on the matrix symbol. To denote the vector whose elements are all x_i such that $i \in \mathcal{S}$ for some set $\mathcal{S}$, we use the notation $(x_i)_{i\in\mathcal{S}}$ and we abbreviate it as $\boldsymbol{x}_{\mathcal{S}}$. We treat bold $\mathbf{0}$ as a zero vector of appropriate size which is determined by the context. We represent a diagonal matrix of size $N \times N$ whose diagonal entries are elements of the vector $\boldsymbol{x} \in \mathbb{R}^N$ by $\text{diag}(\boldsymbol{x})$.

4.1 The network resource allocation problem

In this section we present a network model and consider a resource allocation problem for it. We first present the network model as an abstract generic model. We describe the components of the model and the assumptions we make on the properties of the network. We then discuss applications that motivate such a model. At the end of the section we present a resource allocation problem and formulate it as an optimization problem.

4.1.1 The model (M.4)

We consider a network consisting of N users. We denote the set of users by $\mathcal{N} := \{1, 2, \ldots, N\}$. Each user $i \in \mathcal{N}$ has to take an action $a_i \in \mathcal{A}_i$ where $\mathcal{A}_i$ is the space that specifies the set of user i's feasible actions. In a real network, the actions of a user can be consumption/generation of resources or decisions regarding various tasks. We assume that

Assumption 4.1. *For each $i \in \mathcal{N}$, $\mathcal{A}_i$ is a convex and compact set in $\mathbb{R}$,* [4.1] *and $\mathcal{A}_i$ is user i's private information (i.e. $\mathcal{A}_i$ is known only to user i and nobody else in the network). Furthermore, for each $i \in \mathcal{N}$, a set $\overline{\mathcal{A}}_i \supset \mathcal{A}_i$ is common knowledge among the users whose performance (discussed below) is affected by the actions of user i.*

Because of the users' interactions in the network, the actions taken by a user directly affect the performance of other users in the network. Thus, the performance of the network is determined by the collective actions of all users. In this chapter we assume that the network is large-scale, thus, every user's actions directly affect only a subset of network users in $\mathcal{N}$. This is depicted in the directed graph in Fig. 4.1. In the graph, an arrow from j to i indicates that user j affects user i; we represent the same in the text as $j \to i$. We assume that $i \to i$ for all $i \in \mathcal{N}$.

Mathematically, we denote the set of users that affect user i by $\mathcal{R}_i := \{k \in \mathcal{N} \mid k \to i\}$. Similarly, we denote the set of users that are affected by user j by $\mathcal{C}_j := \{k \in \mathcal{N} \mid j \to k\}$. We call sets $\mathcal{R}_i$ and $\mathcal{C}_i$ the neighbor sets of user i. We represent the interactions of all network users together by a graph matrix

[4.1] In this chapter we assume the sets $\mathcal{A}_i, i \in \mathcal{N}$, to be in $\mathbb{R}$ for simplicity. The decentralized mechanism and the results we present in this chapter can be easily generalized to the scenario where for each $i \in \mathcal{N}$, $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ is a convex and compact set in a higher dimensional space $\mathbb{R}^{n_i}$. Furthermore, for each $i \in \mathcal{N}$, the space $\mathbb{R}^{n_i}$ can be of a different dimension n_i.

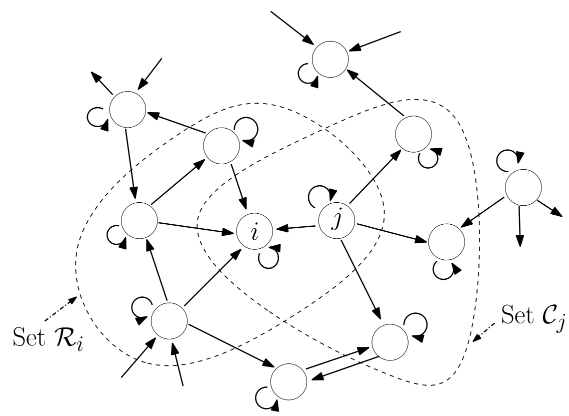

Figure 4.1: A large scale network depicting the neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_j$ of users i and j respectively.

$\boldsymbol{G} := [g_{ij}]_{N\times N}$. The matrix $\boldsymbol{G}$ consists of 0's and 1's where $g_{ij} = 1$ represents that user i is affected by user j, i.e. $j \in \mathcal{R}_i$, and $g_{ij} = 0$ represents no influence of user j on user i, i.e. $j \notin \mathcal{R}_i$. Note that $\boldsymbol{G}$ is not necessarily a symmetric matrix. However, $g_{ii} = 1$ for all $i \in \mathcal{N}$ because $i \to i$. We assume that

Assumption 4.2. *The sets $\mathcal{R}_i, \mathcal{C}_i, i \in \mathcal{N}$, are independent of users' action profile $\boldsymbol{a}_{\mathcal{N}} := (a_k)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}} \mathcal{A}_k$.*

Assumption 4.2 implies that the graph matrix $\boldsymbol{G}$ does not depend on users' actions. There are applications (for example see [27, Chapter 6]) where this assumption does not hold. However, we do not consider such scenarios in this chapter. Applications where Assumption 4.2 is valid are discussed in Section 4.1.2.

We quantify the performance that a user $i \in \mathcal{N}$ achieves as a result of the actions of users in its neighbor set $\mathcal{R}_i$ by a utility function. Let

$$\mathcal{D}_i := \{\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|} \mid a_i \in \mathcal{A}_i; a_j \in \overline{\mathcal{A}}_j, j \in \mathcal{R}_i\backslash\{i\}\}. \tag{4.1}$$

We denote the utility of user i that results from the action profile $\boldsymbol{a}_{\mathcal{R}_i} := (a_k)_{k\in\mathcal{R}_i} \in \mathcal{D}_i$ by $u_i(\boldsymbol{a}_{\mathcal{R}_i})$. We assume that

Assumption 4.3. *For all $i \in \mathcal{N}$, the utility function $u_i : \mathcal{D}_i \to \mathbb{R}$ is strictly concave in $\boldsymbol{a}_{\mathcal{R}_i}$. The function u_i is private information of user i.*

The assumptions that u_i is concave and is private information of user i are reasonable as evidenced by the applications described in Section 4.1.2. We define the domain of u_i as $\mathcal{D}_i$ because user i knows from its information about the network (see Assumption 4.1) that any feasible action profile $\boldsymbol{a}_{\mathcal{R}_i}$ must lie within $\mathcal{D}_i$. Furthermore, as user i does not know the exact sets $\mathcal{A}_j, j \in \mathcal{R}_i \backslash \{i\}$, of other users, it cannot distinguish between feasible and infeasible action profiles within $\mathcal{D}_i$.

In this chapter we restrict attention to static problems. Specifically, we make the following assumption:

Assumption 4.4. *The set $\mathcal{N}$ of users, the graph $\boldsymbol{G}$, the users' action spaces $\mathcal{A}_i, i \in \mathcal{N}$, and their utility functions $u_i, i \in \mathcal{N}$, are fixed in advance and they do not change during the time period of interest.*

Assumption 4.4 is restrictive. Ideally, we would like to address dynamic problems where $\mathcal{N}$, $\boldsymbol{G}$, $\mathcal{A}_i, i \in \mathcal{N}$, and $u_i, i \in \mathcal{N}$, change over time. At this point we are unable to handle dynamic problems, and for this reason we restrict attention to static problems.

We also assume that,

Assumption 4.5. *Every user $i \in \mathcal{N}$ knows the set $\mathcal{R}_i$ of users that affect it as well as the set $\mathcal{C}_i$ of users that are affected by it.*

In networks where the sets $\mathcal{R}_i$ and $\mathcal{C}_i$ are not known to the users beforehand, Assumption 4.5 is still reasonable for the following reason. As the graph $\boldsymbol{G}$ does not change during the time period of interest (Assumption 4.4), the information about the neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_i, i \in \mathcal{N}$, can be passed to the respective users once before the users determine their actions. [4.2] Thus, Assumption 4.5 can hold true for the rest

[4.2] In a real network, the exact method by which the information about the neighbor sets is passed to the users depends on network characteristics. We discuss these methods for the networks described in Section 4.1.2.

of the action determination process.

Finally, we make the following assumption about the users' behavior.

Assumption 4.6. *All network users cooperate to achieve the network performance objective, i.e., they obediently follow the rules of any mechanism that is designed to achieve the network performance objective.*

Examples of situations where Assumption 4.6 holds are the following: (i) Networks which are owned/managed by a single network operator that has complete control over the agents (devices) in the network. For example, a sensor network installed by an operator, or a satellite communication network owned by a communication service provider. In these networks the network operator can install the programs in the devices that dictate their actions according to the network performance criterion. (ii) Networks in which all users have a common objective which is also the network objective. For example, a military communication network.

In the next section we present an example from real world applications that motivate Model (M.5).

4.1.2 Application: Allocation of Central Processing Unit (CPU) computation power on web servers

Consider a web-service management system that allows service providers to offer and manage Service Level Agreements (SLAs) for multiple web services. According to the SLA, each type of web service may be offered in different grades that specify different targets for the average response time of the service. For example, an SLA may say that the customers will pay \$10 per month for a service if they want an average service time of 2 s, and they will pay \$5 per month if they want an average service time of 5 s.

The system classifies incoming service requests into *clusters* that specify the type of the service as shown in Fig. 4.2. The system consists of C clusters and the set

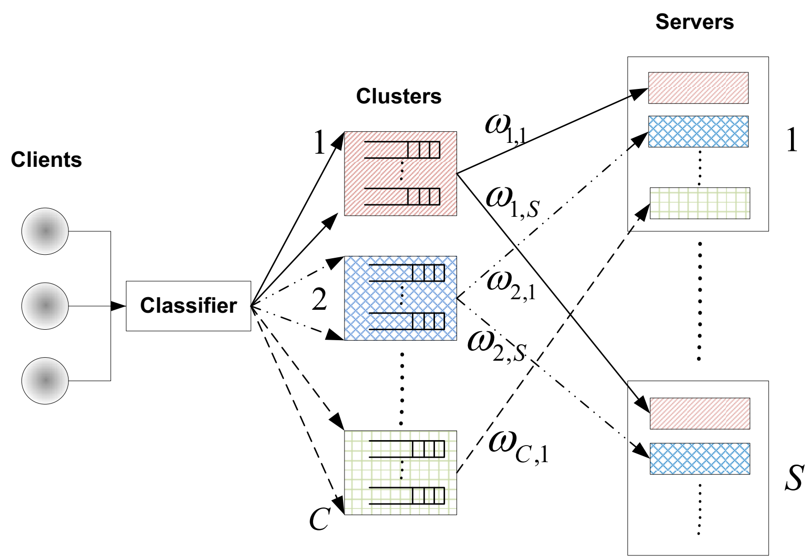

Figure 4.2: System architecture of the web-service management system

of clusters is denoted by $\mathcal{C} := \{1, 2, \ldots, C\}$. The requests belonging to each cluster are further classified into *flows* based on the grade of the service. We assume that cluster $c \in \mathcal{C}$ has F_c flows; the set of these flows is denoted by $\mathcal{F}_c := \{c_1, c_2, \ldots, c_{F_c}\}$. There are S *servers* (physical machines) that run the service applications for the clusters. The set of all servers is denoted by $\mathcal{S} := \{1, 2, \ldots, S\}$. Because of the hardware requirements of applications (in particular memory requirements) and the hardware constraints on physical machines, each server can run only some specific service applications at any given time. Thus, each cluster is served by a subset of servers in $\mathcal{S}$. We assume that the system under consideration has a given *application placement*, i.e., the service applications that each server will run has already been determined.

The application placement can be represented by a graph with $N = C + S$ nodes similar to one shown in Fig. 4.1. In the graph each node would represent either a cluster or a server. An arrow from node c to s would represent that server s is running the service application for cluster c. Note that in this graph, no arrows would emerge out of the server nodes. On the other hand multiple arrows may emerge from a cluster node that point towards different server nodes that serve the cluster. The latter characteristic is a generalization of the case considered in Fig 4.1. This generalization is mentioned in Footnote 4.1. Specifically, in the context of the above model, we can treat the arrows emerging from a cluster as multi-dimensional action taken by the cluster each element of which affects a different server. We discuss the interpretation of these actions in the following paragraphs. Because the application placement has been predetermined, each server would know which clusters it is serving. Similarly, each cluster $c \in \mathcal{C}$ can know beforehand which servers run its service application. Thus, the system under consideration satisfies Assumption 4.5 of Model (M.4).

Given an application placement, the average service time that can be delivered to a cluster depends on the total CPU computation power of the servers available to the cluster. Each server $s \in \mathcal{S}$ has a CPU computation power [4.3] capacity Ω_s which is its private information. This capacity must be divided among the applications that run on the server. If $\omega_{c,s}$ denotes the CPU power available to cluster c on server s, then, the CPU capacity constraints of the servers can be written as

$$\sum_{c \in \mathcal{C}} \omega_{c,s} \quad \leq \quad \Omega_s. \tag{4.2}$$

Note that for each $c \in \mathcal{C}$, $\omega_{c,s}$ can be non zero only if $c \to s$. The CPU power vector $(\omega_{c,s})_{s \in \mathcal{C}_c}$ can be interpreted as the multi-dimensional action taken by cluster c. The

[4.3] Henceforth we will use the term CPU power to mean CPU computation power.

feature that the application placement does not change with CPU power allocation is modeled by Assumption 4.2 in Model (M.4).

Each cluster $c \in \mathcal{C}$ has a utility $u_c(\sum_{s\in\mathcal{C}_c} \omega_{c,s})$ which represents the satisfaction of cluster c from the average service response time it obtains when the total CPU power available to it is $\sum_{s\in\mathcal{C}_c} \omega_{c,s}$. Below we show how the utility function u_c relates the CPU power $\sum_{s\in\mathcal{C}_c} \omega_{c,s}$ to the average service time delivered to cluster c.

For each flow $c_f \in \mathcal{F}_c$ of each cluster $c \in \mathcal{C}$, let T_{c_f} be the target service response time specified by the SLA. The utility of flow $c_f \in \mathcal{F}_c$ is defined in terms of its target response time T_{c_f} and the actual response time t_{c_f} delivered to the flow;

$$u_{c,f}(t_{c_f}) \quad := \quad \frac{T_{c_f} - t_{c_f}}{T_{c_f}}, \quad c_f \in \mathcal{F}_c,\ c \in \mathcal{C}. \tag{4.3}$$

The actual response time t_{c_f} of each flow is calculated from a closed queueing model corresponding to each flow as shown in Fig. 4.3. The number of clients M_{c_f} and the

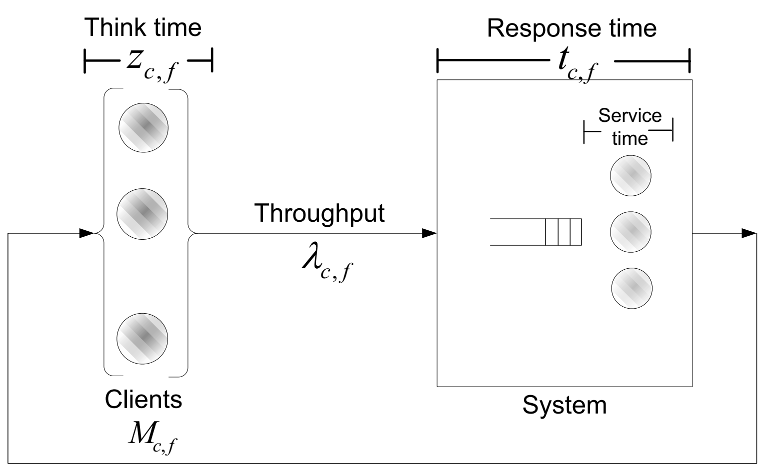

Figure 4.3: Queueing model for deriving the response time of requests for flow $c_f \in \mathcal{F}_c$, $c \in \mathcal{C}$.

average think time z_{c_f} of each client in the queueing model are estimated based on the performance measurements on this model [54]. The queueing model with the estimated M_{c_f} and z_{c_f} is then used to calculate the CPU power required by each cluster so as to achieve some given utility for each of its flows. For the utility function given by (4.3), the response time t_{c_f} required to guarantee utility $u_{c,f}$ for flow $c_f \in \mathcal{F}_c$ is

$$t_{c_f} = (1 - u_{c,f})T_{c_f}. \tag{4.4}$$

For the queueing model of Fig. 4.3, the throughput achieved by flow $c_f \in \mathcal{F}_c$ when the response time for this flow is t_{c_f} is

$$\lambda_{c_f} = \frac{M_{c_f}}{(z_{c_f} + t_{c_f})}. \tag{4.5}$$

For each flow, a work profiler estimates the work factor [4.4] of the flow as proposed in [40]. If α_{c_f} is the estimated work factor, the CPU power required to sustain throughput λ_{c_f} for flow $c_f \in \mathcal{F}_c$ is

$$\omega_{c_f} = \alpha_{c_f}\lambda_{c_f} \tag{4.6}$$

The total CPU power required by cluster $c \in \mathcal{C}$ to guarantee utility u_c for each flow $c_f \in \mathcal{F}_c$ in cluster c is obtained by taking the sum of (4.6) over $c_f \in \mathcal{F}_c$, i.e.,

$$\omega_c(u_c) = \sum_{c_f \in \mathcal{F}_c} \alpha_{c_f} \frac{M_{c_f}}{(z_{c_f} + (1 - u_c)T_{c_f})} \tag{4.7}$$

The utility function of each cluster $c \in \mathcal{C}$ is obtained by inverting the function $\omega_c(u)$ in (4.7). The utility function $u_c(\omega_c)$ thus obtained is concave in its argument ω_c. Hence, the cluster utility $u_c(\sum_{s \in \mathcal{C}_c} \omega_{c,s})$ as a function of the CPU power allocated to the cluster on the servers is a concave function of the vector $(\omega_{c,s})_{s \in \mathcal{C}_c}$.

[4.4] Work factor of a flow is defined to be the average number of CPU cycles required to complete the service of one request belonging to that flow.

Note that in Model (M.4) each node obtains a utility from the actions associated with the arrows pointing towards the node. This is not true in the CPU power allocation model described above, because the servers do not obtain any utility from the CPU power allocated to the clusters served by the servers. However, the web server system of this section can be modeled by Model (M.4) as follows. Since the CPU power capacity Ω_s (given in (4.2)) of each server s is its private information, we define a utility function for each server as

$$
\begin{aligned}
& u_s((\omega_{c,s})_{c\in\mathcal{R}_s}) = I_{\mathcal{O}_s}((\omega_{c,s})_{c\in\mathcal{R}_s}), \\
& \text{where,}\quad \mathcal{O}_s := \{(\omega_{c,s})_{c\in\mathcal{R}_s} \mid \sum_{c\in\mathcal{C}} \omega_{c,s} \leq \Omega_s\} \\
& I_{\mathcal{O}_s}(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} \in \mathcal{O}_s \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{4.8}
$$

The utility of server $s \in \mathcal{S}$ constructed above depends exactly on the actions associated with the applications that are served by the server and therefore, it can be modeled by Model (M.4). Note that the utility given by (4.8) is concave in its argument. It has been shown earlier that the cluster utilities are also concave functions. Furthermore, these utility functions do not change if the application placement remains fixed. These features are modeled by Assumptions 4.3 and 4.4 in Model (M.4).

Finally, a web server system such as one described in this section may be operated by a single service provider. Thus, it can control the behavior of all the machines in the system by pre installing the algorithm for decentralized resource allocation. Such a scenario gives rise to Assumption 4.6 in Model (M.4), and generates resource allocation problems under the realization scenario.

In the next section we formulate the resource allocation problem for the network model (M.4).

4.1.3 The resource allocation problem ($P_D.4$)

For the network model (M.4) we want to develop a mechanism to determine the users' action profile $\boldsymbol{a}_{\mathcal{N}} := (a_1, a_2, \dots, a_N)$. We want the mechanism to work under the decentralized information constraints imposed by the model and to lead to a solution to the following centralized problem.

Problem ($\boldsymbol{P_C}$.4)

$$\begin{aligned} & \max_{\boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}} \quad \sum_{i \in \mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}) \\ & \text{where,} \quad \mathcal{D} := \{\boldsymbol{a}_{\mathcal{N}} \in \mathbb{R}^N \mid a_i \in \mathcal{A}_i \ \forall\, i \in \mathcal{N}\}. \end{aligned} \tag{4.9}$$

$\mathcal{D}$ is the set of feasible solutions of Problem (P_C.4). Because of Assumption 4.3, the objective function in (4.9) is strictly concave in $\boldsymbol{a}_{\mathcal{N}}$. Moreover, the sets $\mathcal{A}_i, i \in \mathcal{N}$, are convex and compact. Therefore, there exists a unique optimal action profile $\boldsymbol{a}^*_{\mathcal{N}}$ for Problem (P_C.4).

The solution of Problem (P_C.4) is the ideal action profile that we would like to obtain. If there exists an entity that has centralized information about the network, i.e. it knows all the utility functions $u_i, i \in \mathcal{N}$, and all action spaces $\mathcal{A}_i, i \in \mathcal{N}$, then that entity can compute the above ideal profile by solving Problem (P_C.4). Therefore, we call the solution of Problem (P_C.4) the optimal centralized action profile. In the network described by Model (M.4), there is no entity that knows perfectly all the parameters that describe Problem (P_C.4). This is indicated by Assumptions 4.1 and 4.3. Therefore, we need to develop a mechanism that allows the network users to communicate with one another and that leads to the optimal solution of Problem (P_C.4).

4.1.4 Literature survey

The problem formulated in Section 4.1.3 has the nature of local public goods allocation problem. A resource is said to be a local public good is it is accessible to and influences the utilities of users in a particular locality within a big network. Thus, the action of each user in the network model (M.4) can be treated as a local public good. There is a large literature on local public goods within the context of local public good provision by various municipalities that follows the seminal work of [53]. These works mainly consider network formation problems in which individuals choose where to locate based on their knowledge of the revenue and expenditure patterns (on local public goods) of various municipalities. For Model (M.4) we consider the problem of determining the levels of local public goods (actions of network agents) for a fixed network; thus, the problem formulation of Section 4.1.3 is distinctly different from those in the above literature. Recently, in [8] a public good network model similar to Model (M.4) was investigated. In the model considered in [8] any pair of users that are linked in the network affect each other's utilities; thus, it is a special case of Model (M.4) where the influence of users' actions on their neighbors' utilities can be either unidirectional or bidirectional. In [11] the model of [8] is generalized to consider directed links between the users; thus, the network structure considered in [11] is similar to Model (M.4). Both [8] and [11] study the problem of local public good provision for a given network structure. However, the problems addressed in [8, 11] are game theoretic and analyze the incentives of users to provide local public goods in such a network. The resource allocation problem (P_D.4) formulated in Section 4.1.3 is non-game theoretic, and hence is different from the problems in [8, 11]. To the best of our knowledge Problem (P_D.4) and its solution that we present in Section 4.2.2 is the first attempt to analyze Model (M.4) in the framework of the realization theory component of mechanism design. In the next section we state our contributions in the problem formulation and solution presented in this chapter.

4.1.5 Contribution of the chapter

The key contributions of this chapter are:

- The formulation of a decentralized resource allocation problem for Model (M.4) in the framework of the realization theory component of mechanism design.
- The specification of an iterative decentralized resource allocation mechanism for the above problem that has the following properties:
  - (i) Each agent in the network needs to communicate only with those agents that either affect it or are affected by it.
  - (ii) The mechanism preserves the private information of each agent.
  - (iii) It guarantees convergence to the network optimal resource allocation (the optimal centralized action profile).

In the next section we develop ideas for the design of decentralized resource allocation mechanisms for Problem (P_D.4), and present a decentralized mechanism that follows these ideas and achieves the problem objectives (the optimal centralized action profile).

4.2 A decentralized resource allocation mechanism

In this section we present a decentralized resource allocation mechanism for Model (M.4). We first develop some ideas/guidelines for the design of decentralized allocation mechanisms for systems having the structure of Model (M.4). We then specify a decentralized mechanism based on these ideas. We conclude the section with a discussion on the properties of the decentralized mechanism. These properties are summarized in Theorem 4.1 the proof of which appears in Appendix 4.A.

4.2.1 Design of decentralized mechanism for Problem (P_D.4)

As stated in Section 4.1.3 our objective is to develop a decentralized mechanism that works under the informational constraints of Model (M.4), and obtains a solution to the centralized Problem (P_C.4). To directly obtain the solution of Problem (P_C.4) one needs complete system information, i.e., one must know the utilities of all the users and the sets $\mathcal{A}_i, i \in \mathcal{N}$, of their feasible actions. One way to obtain complete system information is to let each user communicate all its private information (its utility and set of feasible actions) to some common entity in the network. However, it may not be feasible for the users on informational grounds to communicate all their private information. Furthermore, there may be privacy issues due to which the users may not want to share their private information even if sharing this information is feasible on informational grounds. For the above reasons, we want to develop a mechanism that is capable of achieving the optimal solution without requiring the users to directly share their private information.

A decentralized mechanism that preserves the users' private information must have the following features: (i) It must allow each user to communicate with other users using a message space that is smaller and simpler than the space of its private information. Since the users do not communicate all of their private information, the information conveyed by the users in a single message exchange is not sufficient to determine the centralized allocation. Therefore, the mechanism must consist of an iterative communication process. (ii) It must determine the users' actions based on their private information and their communication with one another.

In the network model (M.4) the information available to a user $i \in \mathcal{N}$ is its utility function u_i, the set of its feasible actions $\mathcal{A}_i$, the set of its neighbors $\mathcal{R}_i$ and $\mathcal{C}_i$, and an estimate $\overline{\mathcal{A}}_j, j \in \mathcal{R}_i$, of the set of feasible actions of its neighbors. Therefore, one way to construct an iterative message update process for Model (M.4) is to let each user solve an individual optimization problem constructed with the above information, and let the user communicate the outcome of this optimization to other users. Thus, designing a decentralized mechanism for Problem (P_D.4) reduces to defining appropriate individual optimization problems for each user that eventually lead to the optimal centralized allocation. Since an allocation consists of all users' actions, the individual optimizations must provide information about each user's perspective on optimal actions. One possible way to accomplish this is to let each user $i \in \mathcal{N}$ propose for each of its neighbors in $\mathcal{R}_i$, a set of actions ${}^i a_j, j \in \mathcal{R}_i$, that maximize its own utility $u_i(\boldsymbol{a}_{\mathcal{R}_i})$. Since each user's optimization must be based only on its own information, the proposal/message of user i must be generated from the set $\mathcal{D}_i := \{\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|} \mid a_i \in \mathcal{A}_i; a_j \in \overline{\mathcal{A}}_j, j \in \mathcal{R}_i \backslash \{i\}\}$ which represents user i's information about the set of feasible actions of its neighbors in $\mathcal{R}_i$. However, the above construction of individual optimization problems may lead to the following difficulty. Since each user $i \in \mathcal{C}_j$ makes a proposal for user j's action, and each of these users has a different utility, the above message communication process may not lead to an agreement among the users' proposals. This difficulty can be addressed by modifying each user's individual optimization problem as follows.

We add a penalty term to each user's utility function. This penalty term must lower the net value of a user's objective function if its proposal deviates from other users' proposals, and the amount of this penalty must increase with the increase in deviation. With each iteration the penalty term of each user should be updated to reflect the new information gathered from the message exchange among the users. The updates in penalty terms should be done in a way so as to eventually bring all the users to an agreement with respect to their action proposals. Furthermore, the series of updated optimizations should allow the users to eventually determine the optimal centralized allocation.

In the next section we present a decentralized resource allocation mechanism that possesses all the desirable features discussed above.

4.2.2 The decentralized resource allocation mechanism

In this section we present a decentralized mechanism for the resource allocation problem formulated in Section 4.1.3. The proposed mechanism consists of an iterative process. Each iteration of this mechanism consists of the following steps: (i) each user solves an optimization problem based on its own information about the network; and (ii) based on the individual optimization, each user *synchronously* updates its message to its neighbors. For the convergence of the iterative process it is important that the message updates are synchronized. We assume that it is possible to synchronize these message updates using synchronization methods. Below we provide the details of the mechanism.

The decentralized mechanism (DM.4):

0) Before the start of the iterative process all users agree upon a common initial action profile. This can be any arbitrary action profile $\boldsymbol{a}_{\mathcal{N}}^{(0)} \in \prod_{i \in \mathcal{N}} \overline{\mathcal{A}}_i$ and does not need to be a feasible one.

   Before the start of the iterative process the users also agree upon a sequence $\{\tau^{(n)}\}_{n=1}^{\infty}$ of modification parameters that will be used in the mechanism. [4.5]

[4.5] Since the users have a common objective, they can communicate with one another before the iterative process begins, and determine $\{\tau^{(n)}\}_{n=1}^{\infty}$ and $\boldsymbol{a}_{\mathcal{N}}^{(0)}$ that will be used in the mechanism. Alternatively, $\{\tau^{(n)}\}_{n=1}^{\infty}$ as well as $\boldsymbol{a}_{\mathcal{N}}^{(0)}$ can be given to the users by the system designer.

The sequence $\{\tau^{(n)}\}_{n=1}^{\infty}$ is chosen to satisfy the following three properties:

$$0 < \tau^{(n+1)} \le \tau^{(n)} \le 1 \quad \forall\, n \ge 1, \tag{4.10}$$

$$\lim_{n\to\infty} \tau^{(n)} = 0, \tag{4.11}$$

$$\lim_{n\to\infty} \sigma^{(n)} = \infty, \tag{4.12}$$

$$\text{where,} \quad \sigma^{(n)} := \sum_{t=1}^{n} \tau^{(t)}, \quad n \ge 1. \tag{4.13}$$

For instance, $\tau^{(n)} = \frac{1}{n}$, $n = 1, 2, 3, \ldots$, can be chosen as the sequence. The counting variable n is set to 1.

1) At the nth iteration, each user $i \in \mathcal{N}$ maximizes its nth stage modified utility function over the set of i-feasible action profiles, and obtains its individual optimum as follows:

(4.14)

$$^{i}\boldsymbol{a}_{\mathcal{R}_i}^{(n)} := ({}^{i}a_j^{(n)})_{j\in\mathcal{R}_i} = \arg\max_{\boldsymbol{a}_{\mathcal{R}_i}\in\mathcal{D}_i} \left\{ u_i(\boldsymbol{a}_{\mathcal{R}_i}) - \frac{1}{\tau^{(n)}} \|\boldsymbol{a}_{\mathcal{R}_i} - \boldsymbol{a}_{\mathcal{R}_i}^{(n-1)}\|^2 \right\}, \quad i \in \mathcal{N},$$

We call ${}^{i}a_j^{(n)}$ the nth stage action proposal of user $i, i \in \mathcal{N}$, for user $j, j \in \mathcal{R}_i$. After the optimization, each user $i \in \mathcal{N}$ sends its action proposal [4.6] ${}^{i}a_j^{(n)}$ to its respective neighbor $j \in \mathcal{R}_i$.

Each user $i \in \mathcal{N}$ also computes a weighted average of its action proposals over all iterations up to the nth one:

$$^{i}\boldsymbol{w}_{\mathcal{R}_i}^{(n)} = \frac{1}{\sigma^{(n)}} \sum_{t=1}^{n} \tau^{(t)}\, {}^{i}\boldsymbol{a}_{\mathcal{R}_i}^{(t)}, \quad i \in \mathcal{N}. \tag{4.15}$$

2) Each user $j \in \mathcal{N}$ receives the action proposals from all its neighbors $k \in \mathcal{C}_j$, and computes the average of all these proposals:

$$a_j^{(n)} = \frac{1}{|\mathcal{C}_j|} \sum_{k\in\mathcal{C}_j} {}^{k}a_j^{(n)}, \quad j \in \mathcal{N}. \tag{4.16}$$

[4.6] Since u_i is strictly concave in $\boldsymbol{a}_{\mathcal{R}_i}$ and $\mathcal{D}_i$ is compact, a unique maximum ${}^{i}a_j^{(n)}$ exists for every $i \in \mathcal{N}$ and $j \in \mathcal{R}_i$.

After computing the average proposal $a_j^{(n)}$, user j announces this proposal to all its neighbors $k \in \mathcal{C}_j$.

3) Each user $i \in \mathcal{N}$ hears the average proposal $a_j^{(n)}$ from all its neighbors $j \in \mathcal{R}_i$, and forms a reference action profile $\boldsymbol{a}_{\mathcal{R}_i}^{(n)} = (a_j^{(n)})_{j \in \mathcal{R}_i}$ for the $(n+1)$th iteration. The counter n is increased to $n+1$ and the process repeats from Step 1). The modification parameter $\tau^{(n+1)}$ for the $(n+1)$th iteration is selected from the predefined sequence chosen in Step 0).

In the next section we discuss the properties of the above mechanism.

4.2.3 Properties of the decentralized mechanism (DM.4)

We begin this section with an intuitive discussion on how the decentralized mechanism (DM.4) presented in Section 4.2.2 achieves optimal centralized allocations. We then formalize the results in Theorem 4.1.

As discussed in Section 4.2.1, in order for a decentralized mechanism to achieve the objective in Problem (P_D.4), it should have the following features: (i) It should be iterative in nature; (ii) In each iteration, each user should perform an individual optimization based only on its own information; and (iii) the iterative process should converge to the optimal centralized allocation.

From the description of the decentralized mechanism (DM.4) in Section 4.2.2, it is clear that it has Feature (i).

To see Feature (ii) observe from (4.14) that for each $i \in \mathcal{N}$, the individual optimization problem for user i is constructed only with the information available to i. Specifically, in the objective function in (4.14), user i knows its own utility function u_i, it knows the sequence $\{\tau^{(n)}\}_{n=1}^{\infty}$, and at the beginning of each iteration it also gets information about the reference point $\boldsymbol{a}_{\mathcal{R}_i}^{(n-1)}$ that completely defines the norm square function for that iteration. Furthermore, user i performs its optimization over the set of actions $\mathcal{D}_i$ which is also completely known to it. Thus, the decentralized mechanism (DM.4) possesses Feature (ii).

As discussed in Section 4.2.1, Feature (iii) requires that in the decentralized mechanism the individual objective function of each user should consist of the user's utility and an updating penalty term that drives the sequence of optimizations to the optimal centralized allocation. In mechanism (DM.4), the norm square term in the objective function of each user serves as the aforementioned penalty term. For each $i \in \mathcal{N}$, the norm square term puts a penalty on user i in proportion to its deviation from the average proposal for the action profile $\boldsymbol{a}_{\mathcal{R}_i}$. Thus, it pulls user i's decision towards the other users' evaluations of the actions $\boldsymbol{a}_{\mathcal{R}_i}$. The individual optima ${}^i\boldsymbol{a}_{\mathcal{R}_i}^{(n)}, i \in \mathcal{N}$, which are announced at the end of each iteration convey how each user values the actions of users that affect its utility. For each user $j \in \mathcal{N}$, the average $a_j^{(n)}$ of its neighbors' optima conveys the average system valuation of user j's actions. Hence for the next iteration, $a_j^{(n)}$ is used as a reference in the individual optimizations of each of user j's neighbors. As stated before, we want the sequence of optima to converge to the optimal allocation. This is achieved in mechanism (DM.4) by reducing the value of the modification parameter $\tau^{(n)}$ in each iteration. The reduction in $\tau^{(n)}$ increases the penalty of deviation for each user. Thus, as the iterative process proceeds, the action profile ${}^i\boldsymbol{a}_{\mathcal{R}_i}^{(n)}$ proposed by user $i, i \in \mathcal{N}$, gets closer and closer to the average profile $\boldsymbol{a}_{\mathcal{R}_i}^{(n-1)}$, and eventually for each $j \in \mathcal{N}$, all the users $k \in \mathcal{C}_j$ agree upon a common action a_j^*. For all n each user's optimum ${}^i\boldsymbol{a}_{\mathcal{R}_i}^{(n)}$ lies in its corresponding set $\mathcal{D}_i$, therefore, if for each $j \in \mathcal{N}$, the optima of all the users $k \in \mathcal{C}_j$ converge to a common action a_j^*, the action a_j^* must lie in the set $\left(\cap_{k \in \mathcal{C}_j \setminus \{j\}} \overline{\mathcal{A}}_j\right) \cap \mathcal{A}_j = \mathcal{A}_j$. Consequently, the action profile $\boldsymbol{a}_{\mathcal{N}}^*$ must lie in the set $\cap_{i \in \mathcal{N}} \mathcal{D}_i = \mathcal{D}$ which is the set of feasible actions for Problem (P_C.4). Thus, the point of convergence $\boldsymbol{a}^*_{\mathcal{N}}$ of the users' optima is a feasible solution of Problem (P_C.4). However, for the reasons explained below, $\boldsymbol{a}^*_{\mathcal{N}}$ need not be a maximizer of the system objective function in Problem (P_C.4). Note that for large n as $\tau^{(n)} \to 0$, the outcomes of individual optimizations are dominantly determined by the norm square terms which force each individual optima to be very close to the average proposal for the respective actions. Since the users' utilities are suppressed in these optimizations, the resulting optima are not representative of the utility functions that form the system objective function. Thus, even though in the limit the users' proposals for the action profile $\boldsymbol{a}_{\mathcal{N}}$ are in agreement, the limit point may not be optimal. The contribution of the users' utilities is accounted for by the weighted average ${}^i\boldsymbol{w}^{(n)}_{\mathcal{R}_i}, i \in \mathcal{N}$. By taking a weighted average of the individual optima over the entire run of the mechanism, the two contributing components to the system objective are taken into account simultaneously: the individual utilities, which are more prominent in the individual optimizations towards the beginning of the iterative process (when $\tau^{(n)}$ is comparatively large); and, the conflicts in the users' evaluations of one another's actions, whose effect becomes more prominent in the individual optimizations towards the end of the mechanism (when $\tau^{(n)}$ approaches 0). The decreasing weights $\tau^{(n)}$ facilitate convergence of each sequence ${}^i\boldsymbol{w}^{(n)}_{\mathcal{R}_i}$ and provide appropriate balance between the contributions of the above two factors at the point of convergence. As a result, the common point of convergence is the optimal solution of the centralized problem (P_C.4).

Below we summarize the property of mechanism (DM.4) in the form of Theorem 4.1.

Theorem 4.1. *Let* $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ *be a sequence of* $N \times N$ *matrices defined as follows:*

$$\boldsymbol{W}^{(n)} := [W_{ij}^{(n)}]_{N\times N},$$

$$\text{(4.17)} \qquad \textit{where,} \quad W_{ij}^{(n)} = \begin{cases} {}^{i}w_j^{(n)}, & \textit{if } j \in \mathcal{R}_i, i \in \mathcal{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

Then,

(i) The sequence $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ *converges and has the limit* $\lim_{n\to\infty} \boldsymbol{W}^{(n)} = \boldsymbol{W}^{*}$, *where* $\boldsymbol{W}^{*} := [W_{ij}^{*}]_{N\times N}$ *has the following property:*

$$\text{(4.18)} \qquad W_{ij}^{*} = \begin{cases} w_j^{*} \textit{ for some } w_j^{*} \in \mathcal{A}_j, & \textit{if } i \in \mathcal{C}_j, j \in \mathcal{N}, \\ 0, & \textit{otherwise.} \end{cases}$$

(ii) The vector $\boldsymbol{w}^{*} := (w_j^{*})_{j\in\mathcal{N}}$ *is the optimal centralized action profile corresponding to Problem* (P_C.4).

⊠

As stated at the beginning of this chapter and in the introduction of the thesis (Section 1.3)), our solution approach for the decentralized resource allocation problem presented in this chapter is based on the principles of the realization theory component of mechanism design. In the next section we show how the decentralized mechanism (DM.4) can be related with the solution approach of realization theory.

4.2.4 Relating mechanism (DM.4) with the solution approach of realization theory

In Section 2.2.3 we presented a detailed description of the components that describe a resource allocation problem and a decentralized resource allocation mechanism in the framework of realization theory. In this section we illustrate how the resource allocation problem (P_D.4) and the decentralized mechanism (DM.4) can be represented in terms of the components of the realization theory framework.

First we define the resource allocation problem $(\mathcal{E}, \mathcal{D}, \gamma)$ corresponding to Problem $(P_D.4)$. For the network model (M.4) the environment $\boldsymbol{e}_i$ of user $i, i \in \mathcal{N}$, consists of the set $\mathcal{A}_i$ of its feasible actions, its utility function u_i, its information about its neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_i$, and its (common) knowledge about the facts described by Assumptions 4.2, 4.4, 4.5 and 4.6. The environment space $\mathcal{E}_i$ of user i is the space of all possible environments $\boldsymbol{e}_i$, i.e., it consists of the following: the space of all convex and compact sets $\mathcal{A}_i \subset \mathbb{R}$, the space of all concave functions $u_i : \mathcal{D}_i \to \mathbb{R}$, the space of all finite subsets $\mathcal{R}_i$ and $\mathcal{C}_i$ of the set of natural numbers, and the common knowledge mentioned above.

The action space $\mathcal{D}$ for Problem $(P_D.4)$ is the space of all feasible action profiles $\boldsymbol{a}_{\mathcal{N}}$ as defined in (4.9).

The goal correspondence γ for Problem $(P_D.4)$ is a correspondence that maps each environment $\boldsymbol{e} \in \mathcal{E}$ to the optimal action profile $\boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}$ of Problem $(P_C.4)$.

The components $(\mathcal{M}, \mu, f)$ corresponding to the decentralized mechanism (DM.4) are defined as follows. The message space for each user $i \in \mathcal{N}$ is $\mathcal{M}_i = \mathcal{D}_i \times \mathcal{D}_i$. The communication rule for the users is the following. In each iteration $n = 1, 2, 3, \ldots,$ each user $i \in \mathcal{N}$ generates the vector $({}^i\boldsymbol{a}^{(n)}_{\mathcal{R}_i}, {}^i\boldsymbol{w}^{(n)}_{\mathcal{R}_i}) \in \mathcal{M}_i$ which it obtains from (4.14) and (4.15). [4.7]

The communication rule defined above implies that the equilibrium message correspondence μ for the mechanism (DM.4) is a function that maps the system environment to the equilibrium message vector $(({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{w}^*_{\mathcal{R}_i})_{i \in \mathcal{N}}) \in \prod_{i \in \mathcal{N}} \mathcal{M}_i$. This vector is obtained at the point of convergence of mechanism (DM.4) given by (4.17) and (4.18).

[4.7] Note that since for each $t = 1, 2, \ldots,$ ${}^i\boldsymbol{a}^{(t)}_{\mathcal{R}_i} \in \mathcal{D}_i$, the convex combination ${}^i\boldsymbol{w}^{(n)}_{\mathcal{R}_i} = \frac{1}{\sigma^{(n)}} \sum_{t=1}^{n} {}^i\boldsymbol{a}^{(t)}_{\mathcal{R}_i} \in \mathcal{D}_i$.

Finally, the outcome function $f : \prod_{i\in\mathcal{N}} \mathcal{M}_i \to \mathcal{D}$ is the following:

$$f_j(({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{w}^*_{\mathcal{R}_i})_{i\in\mathcal{N}}) = \frac{1}{|\mathcal{C}_j|}\sum_{i\in\mathcal{C}_j} {}^i\boldsymbol{w}^*_j = w^*_j \in \mathcal{A}_j, \quad j \in \mathcal{N}. \tag{4.19}$$

The second equality in (4.19) follows from Theorem 4.1 part (i) which states that the point of convergence of mechanism (DM.4) satisfies ${}^i\boldsymbol{w}^*_j = w^*_j,\ \forall\, i \in \mathcal{C}_j, j \in \mathcal{N}$. Since by Theorem 4.1 part (ii) the vector $\boldsymbol{w}^* := (w^*_j)_{j\in\mathcal{N}}$ is the optimal solution of Problem (P_C.4), it follows that the outcome function defined in (4.19) results in the optimal centralized allocation at the equilibrium message vector. Thus, the decentralized mechanism $(\mathcal{M}, \mu, f)$ which represents mechanism (DM.4) in the framework of the realization theory component of mechanism design "realizes" the goal correspondence γ defined by the solution of Problem (P_C.4).

4.A Proof of Theorem 4.1

Claim 4.1. *There exist constants* $0 \le K_{Qij} \le \infty$, $j \in \mathcal{N}, i \in \mathcal{C}_j$, *such that*

$$\frac{1}{\sigma^{(n)}}\sum_{t=0}^{n-1} \|{}^i a^{(t+1)}_j - a^{(t)}_j\|^2 \le K_{Qij}, \qquad \forall\, n \in \{1, 2, \dots\}. \tag{4.20}$$

Proof:

Since ${}^i\boldsymbol{a}^{(t+1)}_{\mathcal{R}_i}$ is the optimal solution in (4.14), it follows from [31, Theorem 1.6] that [4.8]

$$\begin{aligned}&\tau^{(t+1)}u_i({}^i\boldsymbol{a}^{(t+1)}_{\mathcal{R}_i}) - \|{}^i\boldsymbol{a}^{(t+1)}_{\mathcal{R}_i} - \boldsymbol{a}_{\mathcal{R}_i}\|^2 + \|\boldsymbol{a}^{(t)}_{\mathcal{R}_i} - \boldsymbol{a}_{\mathcal{R}_i}\|^2 - \|{}^i\boldsymbol{a}^{(t+1)}_{\mathcal{R}_i} - \boldsymbol{a}^{(t)}_{\mathcal{R}_i}\|^2 \\ &\ge\ \tau^{(t+1)}u_i(\boldsymbol{a}_{\mathcal{R}_i}), \qquad \forall\, \boldsymbol{a}_{\mathcal{R}_i} \in \mathcal{D}_i.\end{aligned} \tag{4.21}$$

By writing the squared vector norms in (4.21) as the sum of squared scalar norms we get

$$\begin{aligned}&\tau^{(t+1)}u_i({}^i\boldsymbol{a}^{(t+1)}_{\mathcal{R}_i}) - \sum_{j\in\mathcal{R}_i}\|{}^i a^{(t+1)}_j - a_j\|^2 + \sum_{j\in\mathcal{R}_i}\|a^{(t)}_j - a_j\|^2 \\ &- \sum_{j\in\mathcal{R}_i}\|{}^i a^{(t+1)}_j - a^{(t)}_j\|^2\ \ge\ \tau^{(t+1)}u_i(\boldsymbol{a}_{\mathcal{R}_i}), \qquad \forall\, \boldsymbol{a}_{\mathcal{R}_i} \in \mathcal{D}_i.\end{aligned} \tag{4.22}$$

[4.8] by taking $\|\cdot\|^2$ as function $J_1(\cdot)$ and $u_i(\cdot)$ as function $J_2(\cdot)$ in Theorem 1.6 of [31, Theorem 1.6].

Adding (4.22) over all $i \in \mathcal{N}$ we obtain

$$\begin{aligned}\tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i({}^i\boldsymbol{a}_{\mathcal{R}_i}^{(t+1)}) - \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} \|{}^i a_j^{(t+1)} - a_j\|^2 + \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} \|a_j^{(t)} - a_j\|^2 \\ - \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \ \geq \ \tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}.\end{aligned} \tag{4.23}$$

From the construction of the graph matrix $\mathcal{G}$ and the sets $\mathcal{R}_i$ and $\mathcal{C}_j$, $i, j \in \mathcal{N}$, the sum $\sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} (\cdot)$ is equal to the sum $\sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j} (\cdot)$. Therefore, we can rewrite (4.23) as

$$\begin{aligned}\tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i({}^i\boldsymbol{a}_{\mathcal{R}_i}^{(t+1)}) - \sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j} \|{}^i a_j^{(t+1)} - a_j\|^2 + \sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j} \|a_j^{(t)} - a_j\|^2 \\ - \sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \ \geq \ \tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}.\end{aligned} \tag{4.24}$$

By convexity of $\|\cdot\|^2$,

$$\|a_j^{(t+1)} - a_j\|^2 \leq \frac{1}{|C_j|} \sum_{i\in\mathcal{C}_j} \|{}^i a_j^{(t+1)} - a_j\|^2. \tag{4.25}$$

Using (4.25) to replace the second term on the Left Hand Side (LHS) of (4.24) gives

$$\begin{aligned}\tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i({}^i\boldsymbol{a}_{\mathcal{R}_i}^{(t+1)}) - \sum_{j\in\mathcal{N}} |\mathcal{C}_j|\ \|a_j^{(t+1)} - a_j\|^2 + \sum_{j\in\mathcal{N}} |\mathcal{C}_j|\ \|a_j^{(t)} - a_j\|^2 \\ - \sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \ \geq \ \tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}.\end{aligned} \tag{4.26}$$

Adding (4.26) over $t = 0, 1, 2, \ldots, n-1$, gives

$$\begin{aligned}\sum_{t=0}^{n-1} \tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i({}^i\boldsymbol{a}_{\mathcal{R}_i}^{(t+1)}) + \sum_{j\in\mathcal{N}} |\mathcal{C}_j| \left(\|a_j^{(0)} - a_j\|^2 - \|a_j^{(n)} - a_j\|^2 \right) \\ - \sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j} \sum_{t=0}^{n-1} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \ \geq \ \sum_{t=0}^{n-1} \tau^{(t+1)} \sum_{i\in\mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}.\end{aligned} \tag{4.27}$$

By concavity of u_i, $i \in \mathcal{N}$,

$$\sum_{i\in\mathcal{N}} \frac{1}{\sigma^{(n)}} \sum_{t=0}^{n-1} \tau^{(t+1)} u_i({}^i\boldsymbol{a}_{\mathcal{R}_i}^{(t+1)}) \leq \sum_{i\in\mathcal{N}} u_i({}^i\boldsymbol{w}_{\mathcal{R}_i}^{(n)}). \tag{4.28}$$

Substituting (4.28) in (4.27), and dividing by $\sigma^{(n)}$, we obtain

$$\begin{aligned}\sum_{i\in\mathcal{N}} u_i({}^i\boldsymbol{w}_{\mathcal{R}_i}^{(n)}) + \frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}} |\mathcal{C}_j| \left(\|a_j^{(0)} - a_j\|^2 - \|a_j^{(n)} - a_j\|^2\right) \\ - \frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\sum_{t=0}^{n-1} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \ \geq\ \sum_{i\in\mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}.\end{aligned} \tag{4.29}$$

Because $\mathcal{D}$ and $\overline{\mathcal{A}}_j, j \in \mathcal{N}$, are compact, and $\boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}$, $\boldsymbol{a}_{\mathcal{N}}^{(0)} \in \prod_{j\in\mathcal{N}} \overline{\mathcal{A}}_j$, and $a_j^{(n)} = \frac{1}{|\mathcal{C}_j|}\sum_{k\in\mathcal{N}} {}^k a_j^{(n)} \in \overline{\mathcal{A}}_j$, $j \in \mathcal{N}$, the numerators of the second and third terms on the LHS of (4.29) are bounded. Furthermore, from (4.12), $\sigma^{(n)} \to \infty$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}} |\mathcal{C}_j| \left(\|a_j^{(0)} - a_j\|^2 - \|a_j^{(n)} - a_j\|^2\right) = 0 \tag{4.30}$$

Since $\mathcal{D}_i, i \in \mathcal{N}$, and $\mathcal{D}$ are compact, ${}^i\boldsymbol{w}_{\mathcal{R}_i}^{(n)} \in \mathcal{D}_i$, $i \in \mathcal{N}$, $\boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}$, and $u_i, i \in \mathcal{N}$, are continuous functions on $\mathbb{R}^{|\mathcal{R}_i|}$, there exist constants K_{Ui}, $i \in \mathcal{N}$, independent of n such that

$$|u_i({}^i\boldsymbol{w}_{\mathcal{R}_i}^{(n)})| \leq K_{Ui}, \quad \text{and} \quad |u_i(\boldsymbol{a}_{\mathcal{R}_i})| \leq K_{Ui}, \quad i \in \mathcal{N}. \tag{4.31}$$

Inequality (4.29) together with (4.30) and (4.31) imply that for appropriate constants $0 \leq K_{Qij} \leq \infty$, $j \in \mathcal{N}, i \in \mathcal{C}_j$,

$$\frac{1}{\sigma^{(n)}}\sum_{t=0}^{n-1} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \leq K_{Qij}, \quad \forall\, n \in \{1, 2, \dots\}. \tag{4.32}$$

□

Claim 4.2. *Let $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ be a sequence of $N \times N$ matrices defined as follows:*

$$\begin{aligned}\boldsymbol{W}^{(n)} &:= [W_{ij}^{(n)}]_{N\times N}, \\ \textit{where,} \quad W_{ij}^{(n)} &= \begin{cases} {}^i w_j^{(n)}, & \textit{if } j \in \mathcal{R}_i, i \in \mathcal{N}, \\ 0, & \textit{otherwise.} \end{cases}\end{aligned} \tag{4.33}$$

Define for each $n = 1, 2, 3, \dots,$

$$w_j^{(n)} := \frac{1}{\sigma^{(n)}} \sum_{t=1}^{n} \tau^{(t)} a_j^{(t-1)}, \quad j \in \mathcal{N}, \tag{4.34}$$

and let $\{\overline{\boldsymbol{W}}^{(n)}\}_{n=1}^{\infty}$ *be a sequence of* $N \times N$ *matrices defined as follows:*

$$\begin{aligned} &\overline{\boldsymbol{W}}^{(n)} := [\overline{W}_{ij}^{(n)}]_{N\times N}, \\ &\textit{where,} \quad \overline{W}_{ij}^{(n)} = \begin{cases} w_j^{(n)}, & \textit{if } j \in \mathcal{R}_i, i \in \mathcal{N}, \\ 0, & \textit{otherwise.} \end{cases} \end{aligned} \tag{4.35}$$

Then,

(i) $\lim_{n\to\infty} \|\boldsymbol{W}^{(n)} - \overline{\boldsymbol{W}}^{(n)}\|^2 = 0$*, where* $\|\cdot\|^2$ *is the Euclidean matrix norm.*

(ii) There exists a converging subsequence $\{\boldsymbol{W}^{(n')}\}$ *of* $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ *such that* $\lim_{n'\to\infty} \boldsymbol{W}^{(n')} = \boldsymbol{W}^* \in \mathbb{R}^{N\times N}$.

(iii) The point of convergence $\boldsymbol{W}^* = [W_{ij}^*]_{N\times N}$ *has the following property:*

$$W_{ij}^* = \begin{cases} w_j^* \textit{ for some } w_j^* \in \mathcal{A}_j, & \textit{if } i \in \mathcal{C}_j, j \in \mathcal{N}, \\ 0, & \textit{otherwise.} \end{cases} \tag{4.36}$$

Furthermore, the vector $\boldsymbol{w}^* := (w_j^*)_{j\in\mathcal{N}}$ *is the optimal centralized action profile, i.e., the solution of Problem* (P_C*.4).*

Proof:

(i) We must show that

$$\forall\, \epsilon > 0,\ \exists\, n_0 :\ \forall\, n \geq n_0,\ \|\boldsymbol{W}^{(n)} - \overline{\boldsymbol{W}}^{(n)}\|^2 \leq \epsilon. \tag{4.37}$$

From the definition of $\boldsymbol{W}^{(n)}$ and $\overline{\boldsymbol{W}}^{(n)}$ we have

$$\begin{aligned} \|\boldsymbol{W}^{(n)} - \overline{\boldsymbol{W}}^{(n)}\|^2 &= \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \|W_{ij}^{(n)} - \overline{W}_{ij}^{(n)}\|^2 \\ &= \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} \|{}^i w_j^{(n)} - w_j^{(n)}\|^2 \\ &= \sum_{j\in\mathcal{N}} \sum_{i\in\mathcal{C}_j} \|{}^i w_j^{(n)} - w_j^{(n)}\|^2. \end{aligned} \tag{4.38}$$

The first equality in (4.38) follows from the definition of the Euclidean matrix norm, and the last equality follows because $\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i}(\cdot) = \sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}(\cdot)$. Since $\|\cdot\|^2$ is a convex function, it follows from (4.15) and (4.34) that for each $n = 1, 2, 3, \ldots,$

$$\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\|{}^iw_j^{(n)} - w_j^{(n)}\|^2 \;\le\; \sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\frac{1}{\sigma^{(n)}}\sum_{t=0}^{n-1}\tau^{(t+1)}\|{}^ia_j^{(t+1)} - a_j^{(t)}\|^2. \tag{4.39}$$

By (4.10) we have for any $n_1 < n$,

$$\begin{aligned}&\frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\sum_{t=0}^{n-1}\tau^{(t+1)}\|{}^ia_j^{(t+1)} - a_j^{(t)}\|^2 \;\le\\ &\frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\tau^{(1)}\sum_{t=0}^{n_1-1}\|{}^ia_j^{(t+1)} - a_j^{(t)}\|^2 + \frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\tau^{(n_1)}\sum_{t=n_1}^{n-1}\|{}^ia_j^{(t+1)} - a_j^{(t)}\|^2.\end{aligned} \tag{4.40}$$

Note that for all $j \in \mathcal{N}$, $\mathcal{A}_j$ and $\overline{\mathcal{A}}_j$ are compact. Furthermore, for each $j \in \mathcal{N}$, ${}^ia_j^{(t+1)} \in \mathcal{A}_j \;\forall\, t$ if $i = j$, and ${}^ia_j^{(t+1)} \in \overline{\mathcal{A}}_j \;\forall\, t$ if $i \in \mathcal{C}_j$. Therefore, there exist constants $0 < K_{aij} < \infty$ such that

$$\forall\, t, \quad \|{}^ia_j^{(t+1)}\| \le K_{aij}, \quad i \in \mathcal{C}_j, j \in \mathcal{N}. \tag{4.41}$$

Since for each $j \in \mathcal{N}$, $a_j^{(t)}$ is an average of ${}^ia_j^{(t)}, i \in \mathcal{C}_j$, (see (4.16)), it follows from (4.41) that there exist constants $0 < K_{aj} < \infty$ such that

$$\forall\, t, \quad \|a_j^{(t)}\| \le K_{aj}, \quad j \in \mathcal{N}. \tag{4.42}$$

From (4.41) and (4.42) we have that,

$$K_1 := \sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\tau^{(1)}\sum_{t=0}^{n_1-1}\|{}^ia_j^{(t+1)} - a_j^{(t)}\|^2 < \infty. \tag{4.43}$$

From Claim 4.1 we also have,

$$\frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\sum_{t=n_1}^{n-1}\|{}^ia_j^{(t+1)} - a_j^{(t)}\|^2 \le K_Q, \quad \forall\, n. \tag{4.44}$$

Substituting (4.43) and (4.44) in (4.40) gives for each $n > n_1$,

$$\frac{1}{\sigma^{(n)}}\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}\sum_{t=0}^{n-1}\tau^{(t+1)}\|{}^ia_j^{(t+1)} - a_j^{(t)}\|^2 \le \frac{1}{\sigma^{(n)}}K_1 + \tau^{(n_1)}K_Q. \tag{4.45}$$

Because of (4.11), for any given $\epsilon > 0$ we can choose an n_1 large enough (by (4.11) and [45, Definition 3.1, p.41]) such that

$$\tau^{(n_1)} \le \frac{\epsilon}{2}K_Q. \tag{4.46}$$

Furthermore, because of (4.12), for any given $\epsilon > 0$ and for the n_1 chosen in (4.46), we can choose an n_2 large enough (see [45]) such that

$$\sigma^{(n_2)} \ge \frac{2}{\epsilon}K_1. \tag{4.47}$$

Define $n_0 := \max(n_1, n_2)$. Then, for all $n \ge n_0$,

$$\frac{1}{\sigma^{(n)}}K_1 + \tau^{(n_1)}K_Q \le \frac{1}{\sigma^{(n_2)}}K_1 + \tau^{(n_1)}K_Q \le \frac{\epsilon}{2} + \frac{\epsilon}{2} \ = \ \epsilon. \tag{4.48}$$

From (4.38), (4.39), (4.45) and (4.48) we conclude that $\forall\, n \ge n_0$,

$$\|\boldsymbol{W}^{(n)} - \overline{\boldsymbol{W}}^{(n)}\|^2 \le \epsilon. \tag{4.49}$$

$\square$

(ii) From (4.14) we know that for all t, ${}^ia_j^{(t)} \in \mathcal{A}_j$ if $i = j, j \in \mathcal{N}$, and ${}^ia_j^{(t)} \in \overline{\mathcal{A}}_j$ if $i \in \mathcal{C}_j\backslash\{j\}, j \in \mathcal{N}$. From Assumption 4.1 we also know that for each $j \in \mathcal{N}$, $\mathcal{A}_j$ and $\overline{\mathcal{A}}_j$ are convex. Therefore, for each $n = 1, 2, 3, \ldots$, the convex combination ${}^iw_j^{(n)} = \frac{1}{\sigma^{(n)}}\sum_{t=1}^{n}{}^ia_j^{(t)} \in \mathcal{A}_j$ if $i = j, j \in \mathcal{N}$, and ${}^iw_j^{(n)} \in \overline{\mathcal{A}}_j$ if $i \in \mathcal{C}_j\backslash\{j\}, j \in \mathcal{N}$. This implies that for each $n = 1, 2, 3, \ldots$, the matrix $\boldsymbol{W}^{(n)}$ lies in the following product space:

$$\boldsymbol{W}^{(n)} \in \bigotimes_{i\in\mathcal{N}}\bigotimes_{j\in\mathcal{N}}\mathcal{D}_{ij}, \qquad \text{where,} \quad \mathcal{D}_{ij} = \begin{cases} \mathcal{A}_j, & \text{if } i = j, j \in \mathcal{N}, \\ \overline{\mathcal{A}}_j, & \text{if } i \in \mathcal{C}_j, j \in \mathcal{N}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.50}$$

Since for each $j \in \mathcal{N}$, $\mathcal{A}_j$ and $\overline{\mathcal{A}}_j$ are compact (cf. Assumption 4.1), the product space $\bigotimes_{i\in\mathcal{N}} \bigotimes_{j\in\mathcal{N}} \mathcal{D}_{ij}$ is also compact. Thus, because of (4.50), the sequence of matrices $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ lies in the compact space $\bigotimes_{i\in\mathcal{N}} \bigotimes_{j\in\mathcal{N}} \mathcal{D}_{ij}$. Therefore, by [33, p.40] there exists a subsequence $\{\boldsymbol{W}^{(n')}\}$ of the sequence $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ such that $\lim_{n'\to\infty} \boldsymbol{W}^{(n')} = \boldsymbol{W}^* \in \bigotimes_{i\in\mathcal{N}} \bigotimes_{j\in\mathcal{N}} \mathcal{D}_{ij} \subset \mathbb{R}^{N\times N}$. $\square$

(iii) From Claim 4.2 part (i), we know that the sequence $\{\|\boldsymbol{W}^{(n)} - \overline{\boldsymbol{W}}^{(n)}\|^2\}_{n=1}^{\infty}$ converges to 0. Therefore, any subsequence of this sequence also converges to 0 (cf. [17]). In particular, the subsequence of the above sequence that is generated by the same set of indices as those of the converging subsequence $\{\boldsymbol{W}^{(n')}\}$ (defined in Claim 4.2 part (ii)), also converges to 0, i.e.,

$$\lim_{n'\to\infty} \|\boldsymbol{W}^{(n')} - \overline{\boldsymbol{W}}^{(n')}\|^2 = 0. \tag{4.51}$$

Since $\{\boldsymbol{W}^{(n')}\}$ is a converging subsequence with the limit $\lim_{n'\to\infty} \boldsymbol{W}^{(n')} = \boldsymbol{W}^*$, (4.51) implies that $\{\overline{\boldsymbol{W}}^{(n')}\}$ is also a converging subsequence and has the limit

$$\lim_{n'\to\infty} \overline{\boldsymbol{W}}^{(n')} = \boldsymbol{W}^*. \tag{4.52}$$

Consequently, because of (4.52), for each $j \in \mathcal{N}$ and $i \in \mathcal{C}_j$, $\lim_{n'\to\infty} \overline{W}_{ij}^{(n')} = W_{ij}^*$. Since for each n' and each $j \in \mathcal{N}$, $\overline{W}_{ij}^{(n')} = w_j^{(n')}$, $\forall\, i \in \mathcal{C}_j$,

$$W_{ij}^* = \lim_{n'\to\infty} \overline{W}_{ij}^{(n')} = \lim_{n'\to\infty} \overline{W}_{jj}^{(n')} = W_{jj}^*,\ \forall\, i \in \mathcal{C}_j, j \in \mathcal{N}. \tag{4.53}$$

Moreover, since for each $j \in \mathcal{N}$ we also have $W_{jj}^* = \lim_{n'\to\infty} W_{jj}^{(n')}$, and the sequence $\{W_{jj}^{(n')}\}$ lies in the compact set $\mathcal{A}_j$ (see (4.50)), the limit W_{jj}^* must lie in $\mathcal{A}_i$. Define

$$w_j^* := W_{jj}^*, \quad j \in \mathcal{N}. \tag{4.54}$$

Then, (4.53) gives

$$W_{ij}^* = w_j^* \in \mathcal{A}_j, \quad \forall\, i \in \mathcal{C}_j, j \in \mathcal{N}. \tag{4.55}$$

Furthermore, from the definition of $\boldsymbol{W}^{(n)}$ in (4.33) we know that for all n', $W_{ij}^{(n')} = 0$ if $i \notin \mathcal{C}_j, j \in \mathcal{N}$. Therefore,

$$\lim_{n' \to \infty} W_{ij}^{(n')} = W_{ij}^* = 0, \quad \forall\, i \notin \mathcal{C}_j, j \in \mathcal{N}. \tag{4.56}$$

Rewriting (4.29) for the indices $\{n'\}$ corresponding to the converging subsequence $\{\boldsymbol{W}^{(n')}\}$ we get

$$\begin{aligned} &\sum_{i \in \mathcal{N}} u_i({}^i\boldsymbol{w}_{\mathcal{R}_i}^{(n')}) + \frac{1}{\sigma^{(n')}} \sum_{j \in \mathcal{N}} |\mathcal{C}_j| \left(\|a_j^{(0)} - a_j\|^2 - \|a_j^{(n')} - a_j\|^2 \right) \\ &- \frac{1}{\sigma^{(n')}} \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{C}_j} \sum_{t=0}^{n'-1} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \ \geq \ \sum_{i \in \mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}. \end{aligned} \tag{4.57}$$

Since $\|\cdot\|^2$ is convex and (4.10) holds,

$$\begin{aligned} \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{C}_j} \|{}^i w_j^{(n')} - w_j^{(n')}\|^2 &\leq \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{C}_j} \frac{1}{\sigma^{(n')}} \sum_{t=0}^{n'-1} \tau^{(t+1)} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2 \\ &\leq \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{C}_j} \frac{1}{\sigma^{(n')}} \sum_{t=0}^{n'-1} \|{}^i a_j^{(t+1)} - a_j^{(t)}\|^2. \end{aligned} \tag{4.58}$$

Substituting (4.58) in (4.57) we get

$$\begin{aligned} &\sum_{i \in \mathcal{N}} u_i({}^i\boldsymbol{w}_{\mathcal{R}_i}^{(n')}) + \frac{1}{\sigma^{(n')}} \sum_{j \in \mathcal{N}} |\mathcal{C}_j| \left(\|a_j^{(0)} - a_j\|^2 - \|a_j^{(n')} - a_j\|^2 \right) \\ &- \sum_{j \in \mathcal{N}} \sum_{i \in \mathcal{C}_j} \|{}^i w_j^{(n')} - w_j^{(n')}\|^2 \ \geq \ \sum_{i \in \mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}. \end{aligned} \tag{4.59}$$

Taking the limit $n' \to \infty$ in (4.59) and using (4.30), (4.51) and (4.55) we get

$$\sum_{i \in \mathcal{N}} u_i(\boldsymbol{w}_{\mathcal{R}_i}^*) \geq \sum_{i \in \mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}), \quad \forall\, \boldsymbol{a}_{\mathcal{N}} \in \mathcal{D}. \tag{4.60}$$

From (4.55) we know that the vector $\boldsymbol{w}^* := (w_j^*)_{j \in \mathcal{N}} \in \mathcal{D}$. Therefore, (4.60) implies that $\boldsymbol{w}^*$ is an optimal solution of Problem (P_C.4). $\square$

Claim 4.3. *The sequence* $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ *converges, and* $\lim_{n \to \infty} \boldsymbol{W}^{(n)} = \boldsymbol{W}^*$, *where* $\boldsymbol{W}^*$ *is the limit defined in Claim 4.2.*

Proof:

By its formulation, Problem (P_C.4) is a strictly concave optimization problem. Therefore, it has a unique optimal solution. Because of Claim 4.2, part (iii), this optimal solution is the vector $\boldsymbol{w}^* = (w_i^*)_{i \in \mathcal{N}} := (W_{ii}^*)_{i \in \mathcal{N}}$ where W_{ii}^*, $i \in \mathcal{N}$, are the diagonal elements of the matrix $\boldsymbol{W}^*$ which is the limit of the converging subsequence $\{\boldsymbol{W}^{(n')}\}$ of $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$.

Since the subsequence $\{\boldsymbol{W}^{(n')}\}$ considered in Claim 4.2 is arbitrary, from Claim 4.2 and the uniqueness of the solution of Problem (P_C.4) it follows that all converging subsequences $\{\boldsymbol{W}^{(n')}\}$ of $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ have limit equal to $\boldsymbol{W}^*$. Since all converging subsequences of $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ have the same limit, by [17, Corollary, p.53] the sequence $\{\boldsymbol{W}^{(n)}\}_{n=1}^{\infty}$ converges to $\boldsymbol{W}^*$. Consequently, mechanism (DM.4) results in an optimal solution/allocation of Problem (P_C.4). $\square$

CHAPTER 5

Resource allocation in large-scale networks: An implementation perspective

In this chapter we consider an implementation theory perspective on resource allocation for the large-scale network model developed in Chapter 4. Specifically, for the above model we formulate a decentralized resource allocation problem with strategic agents, and develop a game form that possesses the following properties: (i) It implements in Nash equilibria (NE) the optimal solutions of a centralized resource allocation problem. (ii) It is individual rational. (iii) It is budget-balanced at all NE and off equilibrium.

The chapter is organized as follows: In Section 5.1.1 we present the network model and discuss motivating applications in Section 5.1.2. In Section 5.1.3 we formulate the resource allocation problem. We state our contributions in the formulation and solution of the above problem in Section 5.1.5. In Section 5.2.1 we discuss how the problem formulated in Section 5.1.3 can be addressed with an implementation theory approach. We develop ideas for the construction of game form for this problem in Section 5.2.2 and follow that with the specification of a game form in Section 5.2.3. We discuss the properties of the proposed game form in Section 5.2.4 and we present their proofs in Appendices 5.A and 5.B. We conclude with a discussion on implementation aspects of the proposed mechanism in Section 5.2.5.

Before we present the model, we describe the notation that we will use throughout the chapter.

Notation:

We use bold font to represent vectors and normal font for scalars. We use bold uppercase letters to represent matrices. We represent the element of a vector by a subscript on the vector symbol, and the element of a matrix by double subscript on the matrix symbol. To denote the vector whose elements are all x_i such that $i \in \mathcal{S}$ for some set $\mathcal{S}$, we use the notation $(x_i)_{i\in\mathcal{S}}$ and we abbreviate it as $\boldsymbol{x}_{\mathcal{S}}$. We treat bold $\mathbf{0}$ as a zero vector of appropriate size which is determined by the context. We use the notation $(x_i, \boldsymbol{x}^*/i)$ to represent a vector of dimension same as that of $\boldsymbol{x}^*$, whose ith element is x_i and all other elements are the same as the corresponding elements of $\boldsymbol{x}^*$. We represent a diagonal matrix of size $N \times N$ whose diagonal entries are elements of the vector $\boldsymbol{x} \in \mathbb{R}^N$ by $\text{diag}(\boldsymbol{x})$.

5.1 The network resource allocation problem

In this section we present a network model and consider a resource allocation problem for it. We first present the network model as an abstract generic model. We describe the components of the model and the assumptions we make on the properties of the network. We then discuss applications that motivate such a model. At the end of the section we present a resource allocation problem and formulate it as an optimization problem.

5.1.1 The model (M.5)

We consider a network consisting of N users and one network operator. We denote the set of users by $\mathcal{N} := \{1, 2, \ldots, N\}$. Each user $i \in \mathcal{N}$ has to take an action $a_i \in \mathcal{A}_i$ where $\mathcal{A}_i$ is the space that specifies the set of user i's feasible actions. In a real network, a user's actions can be consumption/generation of resources or decisions regarding various tasks. We assume that,

Assumption 5.1. *For all $i \in \mathcal{N}$, $\mathcal{A}_i$ is a convex and compact set in $\mathbb{R}$ that includes 0.* [5.1] *Furthermore, for each user $i \in \mathcal{N}$, the set $\mathcal{A}_i$ is its private information, i.e. $\mathcal{A}_i$ is known only to user i and nobody else in the network.*

Because of the users' interactions in the network, the actions taken by a user directly affect the performance of other users in the network. Thus, the performance of the network is determined by the collective actions of all users. In this chapter we assume that the network is large-scale, thus, every user's actions directly affect only a subset of network users in $\mathcal{N}$. This is depicted in the directed graph in Fig. 5.1. In the graph, an arrow from j to i indicates that user j affects user i; we represent the same in the text as $j \to i$. We assume that $i \to i$ for all $i \in \mathcal{N}$.

Mathematically, we denote the set of users that affect user i by $\mathcal{R}_i := \{k \in \mathcal{N} \mid k \to i\}$. Similarly, we denote the set of users that are affected by user j by $\mathcal{C}_j := \{k \in \mathcal{N} \mid j \to k\}$. We represent the interactions of all network users together by a graph matrix $\boldsymbol{G} := [G_{ij}]_{N \times N}$. The matrix $\boldsymbol{G}$ consists of 0's and 1's, where $G_{ij} = 1$ represents that user i is affected by user j, i.e. $j \in \mathcal{R}_i$ and $G_{ij} = 0$ represents no influence of user j on user i, i.e. $j \notin \mathcal{R}_i$. Note that $\boldsymbol{G}$ is not necessarily a symmetric matrix. However, $G_{ii} = 1$ for all $i \in \mathcal{N}$ because $i \to i$. We assume that,

[5.1] In this chapter we assume the sets $\mathcal{A}_i, i \in \mathcal{N}$, to be in $\mathbb{R}$ for simplicity. However, the decentralized mechanism and the results we present in this chapter can be easily generalized to the scenario where for each $i \in \mathcal{N}$, $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ is a convex and compact set in higher dimensional space $\mathbb{R}^{n_i}$. Furthermore, each space $\mathbb{R}^{n_i}$ can be of a different dimension n_i for different $i \in \mathcal{N}$.

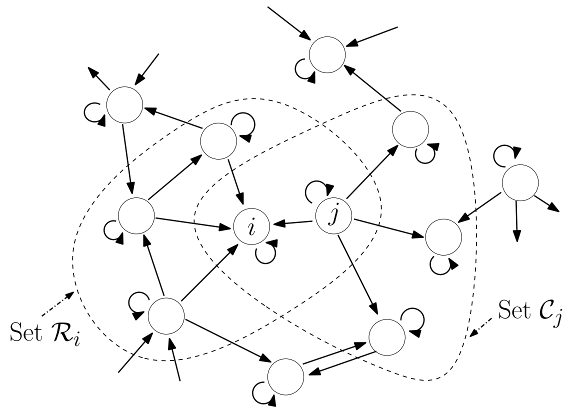

Figure 5.1: A large scale network depicting the neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_j$ of users i and j respectively.

Assumption 5.2. *The sets $\mathcal{R}_i, \mathcal{C}_i, i \in \mathcal{N}$, are independent of the users' action profile $\boldsymbol{a}_{\mathcal{N}} := (a_k)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}} \mathcal{A}_k$.*

Assumption 5.2 implies that the graph matrix $\boldsymbol{G}$ does not depend on users' actions. There are applications (for example see [27, Chapter 6]) where this assumption does not hold; we do not consider such scenarios in this chapter. Applications where Assumption 5.2 is valid are discussed in Section 5.1.2.

The performance of a user that results from actions taken by the users affecting it is quantified by a utility function. We denote the utility of user $i \in \mathcal{N}$ resulting from the action profile $\boldsymbol{a}_{\mathcal{R}_i} := (a_k)_{k\in\mathcal{R}_i}$ by $u_i(\boldsymbol{a}_{\mathcal{R}_i})$. We assume that,

Assumption 5.3. *For all $i \in \mathcal{N}$, the utility function $u_i : \mathbb{R}^{|\mathcal{R}_i|} \to \mathbb{R}$ is concave in $\boldsymbol{a}_{\mathcal{R}_i}$ and $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = 0$ for $a_i \notin \mathcal{A}_i$.* [5.2] *The function u_i is user i's private information.*

The assumptions that u_i is concave and is user i's private information are reasonable as evidenced by the applications described in Section 5.1.2. The assumption, $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = 0$ for $a_i \notin \mathcal{A}_i$, is made for the following reason. Because $\mathcal{A}_i$ is the set of user i's feasible actions and user i knows this set (Assumption 5.1), it also knows that any action profile $\boldsymbol{a}_{\mathcal{R}_i}$ in which $a_i \notin \mathcal{A}_i$, is not possible to occur. Therefore,

[5.2] Note that a_i is always an element of $\boldsymbol{a}_{\mathcal{R}_i}$ because $i \to i$ and hence $i \in \mathcal{R}_i$.

such an action profile $\boldsymbol{a}_{\mathcal{R}_i}$ does not provide any utility to user i.

We assume that,

Assumption 5.4. *Each network user $i \in \mathcal{N}$ is strategic and non-cooperative/selfish.*

Assumption 5.4 implies that the users have an incentive to misrepresent their private information, e.g. a user $i \in \mathcal{N}$ may not want to report to other users or to the network operator its true preference for the users' actions, if this results in an action profile in its own favor.

Each user $i \in \mathcal{N}$ pays a tax $t_i \in \mathbb{R}$ to the network operator. This tax can be imposed for the following reasons: (i) For the use of the network by the users. (ii) To provide incentives to the users to take actions that achieve a network-wide performance objective. The tax is set according to the rules specified by a mechanism and it can be either positive or negative for a user. With the flexibility of either charging a user (positive tax) or paying compensation/subsidy (negative tax) to a user, it is possible to induce the users to behave in a way such that a network-wide performance objective is achieved. For example, in a network with limited resources, we can set "positive tax" for the users that receive resources close to the amounts requested by them and we can pay "compensation" to the users that receive resources that are not close to their desirable ones. Thus, with the available resources, we can satisfy all the users and induce them to behave in a way that leads to a resource allocation that is optimal according to a network-wide performance criterion. We assume that,

Assumption 5.5. *The network operator does not have any utility associated with the users' actions or taxes. It does not derive any profit from the users' taxes and acts like an accountant that redistributes the tax among the users according to the* *specifications of the allocation mechanism.*

Assumption 5.5 implies that the tax is charged in a way such that

$$\sum_{i \in \mathcal{N}} t_i = 0. \tag{5.1}$$

To describe the "overall satisfaction" of a user from the performance it receives from all users' actions and the tax it pays for it, we define an "aggregate utility function" $u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) : \mathbb{R}^{|\mathcal{R}_i|+1} \to \mathbb{R} \cup \{-\infty\}$ for each user $i \in \mathcal{N}$ as follows:

$$\begin{aligned} u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) &:= -t_i + u_i(\boldsymbol{a}_{\mathcal{R}_i}) - \left[\frac{1 - I_{\mathcal{A}_i}(a_i)}{I_{\mathcal{A}_i}(a_i)}\right] \\ \text{where,} \quad I_{\mathcal{A}_i}(a_i) &:= \begin{cases} 1, & \text{if } a_i \in \mathcal{A}_i \\ 0, & \text{otherwise} \end{cases}. \end{aligned} \tag{5.2}$$

The third term in the definition of u_i^A in (5.2) indicates that an allocation $(\boldsymbol{a}_{\mathcal{R}_i}, t_i)$ is of no significance to user i if $a_i \notin \mathcal{A}_i$. This is because, as mentioned earlier, user i knows that an allocation $(\boldsymbol{a}_{\mathcal{R}_i}, t_i)$ in which $a_i \notin \mathcal{A}_i$ is not possible to occur as i cannot take an action outside $\mathcal{A}_i$. Because u_i and $\mathcal{A}_i$ are user i's private information (Assumptions 5.1 and 5.3), the aggregate utility u_i^A is also user i's private information. As stated in Assumption 5.4, users are non-cooperative and selfish. Therefore, *the users are self aggregate utility maximizers.*

In this chapter we restrict attention to static problems. Specifically, we make the following assumption:

Assumption 5.6. *The set $\mathcal{N}$ of users, the graph $\boldsymbol{G}$, users' action spaces $\mathcal{A}_i, i \in \mathcal{N}$, and their utility functions $u_i, i \in \mathcal{N}$, are fixed in advance and they do not change during the time period of interest.*

Assumption 5.6 is restrictive. Ideally, we would like to address dynamic problems where $\mathcal{N}$, $\boldsymbol{G}$, $\mathcal{A}_i, i \in \mathcal{N}$, and $u_i, i \in \mathcal{N}$, change over time. At this point we are unable to handle dynamic problems, and for this reason we restrict attention to static problems.

We also assume that,

Assumption 5.7. *Every user $i \in \mathcal{N}$ knows the set $\mathcal{R}_i$ of users that affect it as well as the set $\mathcal{C}_i$ of users that are affected by it. The network operator knows $\mathcal{R}_i$ and $\mathcal{C}_i$ for all $i \in \mathcal{N}$.*

In networks where the sets $\mathcal{R}_i$ and $\mathcal{C}_i$ are not known to the users beforehand, Assumption 5.7 is still reasonable because of the following reason. As the graph $\boldsymbol{G}$ does not change during the time period of interest (Assumption 5.6), the information about the neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_i, i \in \mathcal{N}$, can be passed to the respective users by the network operator before the users determine their actions. Alternatively, the users can themselves determine the set of their neighbors before determining their actions. [5.3] Thus, Assumption 5.7 can hold true for the rest of the action determination process.

In the next section we present some applications that motivate Model (M.5).

5.1.2 Applications

5.1.2.1 Application A: Power allocation in cellular networks

Consider a single cell downlink wireless data network consisting of a Base Station (BS) and N mobile users as shown in Fig. 5.2. The BS uses Code Division Multiple Access (CDMA) technology to transmit data to the users and each mobile user uses Minimum Mean Square Error Multi-User Detector (MMSE-MUD) receiver to decode its data. The signature codes used by the BS are not completely orthogonal as this helps increase the capacity of the network. Because of non-orthogonal codes, each

[5.3] The exact method by which the users get information about their neighbor sets in a real network depends on the network characteristics. We discuss these methods for the networks described in Section 5.1.2.

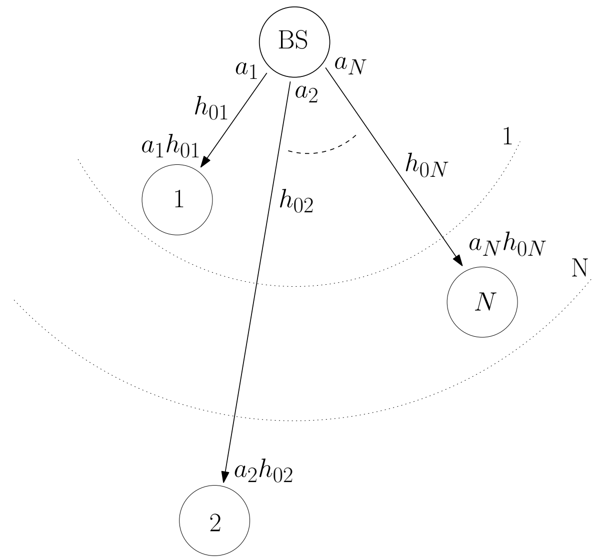

Figure 5.2: A downlink network with N mobile users and one base station

user experiences interference due to the BS transmissions intended for other users. However, as the users in the cell are at different distances from the BS, and the power transmitted by the BS undergoes propagation loss, not all transmissions by the BS create interference to every user. For example, let us look at arcs 1 and N shown in Fig. 5.2 that are centered at the BS. Suppose the radius of arc 1 is much smaller than that of arc N. Then, the signal transmitted by the BS for users inside circle 1 (that corresponds to arc 1) will become negligible when it reaches outside users such as user N or user 2. On the other hand, the BS signals transmitted for user N and user 2 will be received with significant power by the users inside circle 1. This asymmetric interference relation between the mobile users can be depicted in a graph similar to one shown in Fig. 5.1. In the graph an arrow from j to i would represent that the signal transmitted for user j also affects user i. Note that since the signal transmitted for user i must reach i, the assumption $i \to i$ made in Section 5.1.1 holds in this case. If the users do not move very fast in the network, the network topology can be assumed to be fixed for small time periods. Therefore, if the BS transmits some pilot signals to all network users, the users can figure out which signals are creating interference to their signal reception. Thus, each user would know its (interfering) neighbor set as assumed in Assumption 5.7. Note that if the power transmitted by the BS to the users change, it may result in a change in the set of interfering neighbors of each user. This is different from Assumption 5.2 in Model (M.5). However, if the transmission power fluctuations resulting from a power allocation mechanism are not large, the set of interfering neighbors can be treated to be fixed, and this can be approximated by Assumption 5.2.

The Quality of Service (QoS) that a user receives from decoding its data is quantified by a utility function. Due to interference, the utility $u_i(\cdot)$ of user $i, i \in \mathcal{N}$, is a function of the vector $\boldsymbol{a}_{\mathcal{R}_i}$, where $a_j, j \in \mathcal{R}_i$, is the transmission power used by the BS to transmit signals (to the users $j \in \mathcal{R}_i$) that reach user i. Note that in this case all transmissions, in other words the actions $a_i, i \in \mathcal{N}$, are carried out by the BS unlike Model (M.5) where each user $i \in \mathcal{N}$ takes its own action a_i. However, as we discuss below, the BS is only an agent that executes the outcome of the mechanism that determines these transmission powers. Thus, we can embed the downlink network scenario into Model (M.5) by treating each a_i as a decision "corresponding" to user $i, i \in \mathcal{N}$, which is executed by the BS for i. Since each user uses an MMSE-MUD receiver, a measure of user i's ($i \in \mathcal{N}$) utility can be the negative of the MMSE at the output of its receiver, [5.4] i.e.,

$$\begin{aligned} u_i(\boldsymbol{a}_{\mathcal{R}_i}) &= -MMSE_i \\ &= -\min_{\boldsymbol{z_i}^T \in \mathbb{R}^{1\times N}} E[\|b_i - \boldsymbol{z_i}^T \boldsymbol{y_i}\|^2] \qquad (5.3) \\ &= -\big[(\boldsymbol{I} + \frac{2}{N_{0i}} \boldsymbol{S}_i \boldsymbol{X}_{\mathcal{R}_i} \boldsymbol{S}_i)^{-1}\big]_{ii}, \quad i \in \mathcal{N}. \end{aligned}$$

In (5.3) b_i is the transmitted data symbol for user i, $\boldsymbol{y_i}$ is the output of user i's

[5.4] See [55] for the derivation of (5.3).

matched filter generated from its received data, $\boldsymbol{I}$ is the identity matrix of size $N \times N$, $N_{0i}/2$ is the two sided power spectral density (PSD) of thermal noise, $\boldsymbol{X}_{\mathcal{R}_i}$ is the cross-correlation matrix of signature waveforms corresponding to the users $j \in \mathcal{R}_i$, and $\boldsymbol{S}_i := \operatorname{diag}((\boldsymbol{S}_{ij})_{j \in \mathcal{R}_i})$ is the diagonal matrix consisting of the signal amplitudes $S_{ij}, j \in \mathcal{R}_i$, received by user i. S_{ij} is related to a_j as $S_{ij}^2 = a_j h_{0i}$, $j \in \mathcal{R}_i$, where h_{0i} is the channel gain from the BS to user i which represents the power loss along this path. As shown in [50, 51], the utility function given by (5.3) is close to concave in $\boldsymbol{a}_{\mathcal{R}_i}$. Thus, Assumption 5.3 in Model (M.5) can be thought of as an approximation to the downlink network scenario.

Note that to compute user i's utility given in (5.3), knowledge of N_{0i}, $\boldsymbol{X}_{\mathcal{R}_i}$, and h_{0i} is required. The BS knows $\boldsymbol{X}_{\mathcal{R}_i}$ for each $i \in \mathcal{N}$ as it selects the signature waveform for each user. On the other hand, user $i, i \in \mathcal{N}$, knows the PSD N_{0i} of thermal noise and the channel gain h_{0i} as these can be measured only at the respective receiver. Consider a network where the mobile users are selfish and non cooperative. Then, these users may not want to reveal their measured values N_{0i} and h_{0i}. On the other hand if the network operator that owns the BS does not have a utility and is not selfish, then, the BS can announce the signature waveforms it uses for each user. Thus, each user $i \in \mathcal{N}$ would know its corresponding cross correlation matrix $\boldsymbol{X}_{\mathcal{R}_i}$ and consequently, its utility function u_i. However, since N_{0i} and h_{0i} are user i's private information, the utility function u_i is private information of i which is similar to Assumption 5.3 in Model (M.5). If the wireless channel conditions vary slowly compared to the time period of interest, the channel gains and hence the users' utility functions can be assumed to be fixed. As mentioned earlier, for slowly moving users the network topology and hence the set of interfering neighbors can also be assumed to be fixed. These features are captured by Assumption 5.6 in Model (M.5).

In the presence of limited resources, the provision of desired QoS to all network users may not be possible. To manage the provision of QoS under such a situation the network operator (BS) can charge tax to the users and offer them the following tradeoff. It charges positive tax to the users that obtain a QoS close to their desirable one, and compensates the loss in the QoS of other users by providing a subsidy to them. Such a redistribution of money among users through the BS is possible under Assumption 5.5 in Model (M.5).

5.1.2.2 Application B: Building departmental libraries

Consider a university that has several academic departments. The university wants each department to build its own library. Each departmental library should exclusively have the collection of books related to the department discipline. Since in each department, the training of students as well as research collaboration requires library resources from many other disciplines, building a departmental library would not only benefit the affiliated department, but also several other departments. However, focussing on only one discipline to set up its own library reduces the required effort and organization for each department. A network of departmental libraries thus established is similar to the network of Model (M.5).

To represent the library network with the notation used to describe the network Model (M.5), let us assume that there are N different departments. Let us denote each department and its affiliated library by an index $i \in \mathcal{N}$. Then, each department i benefits from a subset $\mathcal{R}_i$ of libraries in the network of N libraries, and each departmental library j benefits a subset $\mathcal{C}_j$ of N departments. Each department knows from its academic programs the sets $\mathcal{R}_i$ and $\mathcal{C}_i$ of its collaborating departments, and does not need any information about the rest of the departments in the university.

Suppose each department $i \in \mathcal{N}$ has to make a decision about the number of volumes a_i it should have in its library. Each department may have some constraint on the size of its library that can be represented as $a_i \leq A_i^{max}$. This constraint may arise due to the budget constraint of the department or the space limitation to store the books, and may be private information of the department. However, for each department $i \in \mathcal{N}$, an estimate $\overline{A}_i^{max} \supset A_i^{max}$ of its capacity may be common knowledge. Such an estimate may be based on external knowledge; e.g. the knowledge of the department building layout. Note that in this case, the actions $a_i, i \in \mathcal{N}$, of the departments can take only natural number values. Therefore, Assumption 5.1 of Model (M.5) can be thought of as an approximation to this case. With such approximation and we can define the action space for each department to be $\mathcal{A}_i := [0, A_i^{max}]$.

Suppose each department incurs a cost $c_i(a_i)$ for building a library of size a_i, and obtains a benefit $b_i(\boldsymbol{a}_{\mathcal{R}_i})$ if the departments it collaborates with have libraries of sizes $a_j, j \in \mathcal{R}_i$. Suppose further that for each $i \in \mathcal{N}$, c_i is convex in a_i and b_i is concave in $\boldsymbol{a}_{\mathcal{R}_i}$ and that these costs and benefits are the respective departments' private information. This scenario can be modeled by Model (M.5) if we define a utility function for each department as $u_i(\boldsymbol{a}_{\mathcal{R}_i}) := b_i(\boldsymbol{a}_{\mathcal{R}_i}) - c_i(a_i)$.

Since each department benefits from multiple libraries, each department contributes to building these libraries by donating money to a pool of library fund. According to the requirements of each department, the money is reallocated to the departments from this fund and reallocation is done in such a way that no money is left unused in the fund. This is modeled by (5.1) in Model (M.5). In this case the university authorities act as a network operator; they collect and redistribute the money among the departments.

Suppose each department is strategic and contributes to the library network to maximize its own net payoff. With the strategic departments and the university authorities that help coordinate their decisions on building the libraries, this set up gives rise to Model (M.5).

In the next section we formulate the resource allocation problem for the network model (M.5).

5.1.3 The resource allocation problem (P_D.5)

For the network model (M.5) we wish to develop a mechanism to determine the users' action profile $\boldsymbol{a}_{\mathcal{N}} := (a_1, a_2, \ldots, a_N)$ and tax profile $\boldsymbol{t}_{\mathcal{N}} := (t_1, t_2, \ldots, t_N)$. We want the mechanism to work under the decentralized information constraints imposed by the model and to lead to a solution to the following centralized problem.

Problem ($\boldsymbol{P_C}$.5)

$$\begin{aligned} \max_{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}})} \quad & \sum_{i \in \mathcal{N}} u_i^A(\boldsymbol{a}_{\mathcal{R}_i}, t_i) \\ \text{s.t.} \quad & \sum_{i \in \mathcal{N}} t_i = 0 \end{aligned} \tag{5.4}$$

$$\begin{aligned} \equiv \max_{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \in \mathcal{D}} \quad & \sum_{i \in \mathcal{N}} u_i(\boldsymbol{a}_{\mathcal{R}_i}) \\ \text{where, } \mathcal{D} := \{(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \in \mathbb{R}^{2N} \mid a_i \in \mathcal{A}_i \ \forall\, i \in \mathcal{N}; & \sum_{i \in \mathcal{N}} t_i = 0\} \end{aligned} \tag{5.5}$$

The optimization problem (5.4) is equivalent to (5.5) because for $(\boldsymbol{a}_{\mathcal{N}}, \boldsymbol{t}_{\mathcal{N}}) \notin \mathcal{D}$, the objective function in (5.4) is negative infinity by (5.2). Thus $\mathcal{D}$ is the set of feasible solutions of Problem (P_C.5). Since by Assumption 5.3, the objective function in (5.5) is concave in $\boldsymbol{a}_{\mathcal{N}}$ and the sets $\mathcal{A}_i$, $i \in \mathcal{N}$, are convex and compact, there exists an optimal action profile $\boldsymbol{a}_{\mathcal{N}}^*$ for Problem (P_C.5). Furthermore, since the objective function in (5.5) does not explicitly depend on $\boldsymbol{t}_{\mathcal{N}}$, an optimal solution of Problem (P_C.5) must be of the form $(\boldsymbol{a}_{\mathcal{N}}^*, \boldsymbol{t}_{\mathcal{N}})$, where $\boldsymbol{t}_{\mathcal{N}}$ is any feasible tax profile for Problem (P_C.5), i.e. a tax profile that satisfies (5.1).

The solutions of Problem (P_C.5) are ideal action and tax profiles that we would like to obtain. If there exists an entity that has centralized information about the network, i.e. it knows all the utility functions $u_i, i \in \mathcal{N}$, and all action spaces $\mathcal{A}_i, i \in \mathcal{N}$, then that entity can compute the above ideal profiles by solving Problem (P_C.5). Therefore, we call the solutions of Problem (P_C.5) optimal centralized allocations. In the network described by Model (M.5), there is no entity that knows perfectly all the parameters that describe Problem (P_C.5) (Assumptions 5.1 and 5.3). Therefore, we need to develop a mechanism that allows the network users to communicate with one another and that leads to optimal solutions of Problem (P_C.5). Since a key assumption in Model (M.5) is that the users are strategic and non-cooperative, the mechanism we develop must take into account the users' strategic behavior in their communication with one another. To address all of these issues we take the approach of implementation theory [5.5] for the solution of Problem (P_D.5).

In the next section we present a literature survey on related work.

5.1.4 Literature survey

It was discussed in Chapter 4 that the large-scale network model (M.4) resembles a local public good network. Because Model (M.5) is similar to Model (M.4) in terms of the users' influence on each other's utilities, Model (M.5) also has the nature of a local public good network. As mentioned in Section 4.1.4, local public good network models that have network structures similar to Model (M.4) and hence Model (M.5) were investigated in [8, 11]. Both of these works analyze the influence of selfish users' behavior on the provision of local public goods in a network with fixed links among the users. In particular, the authors of [8] show that the selfish behavior of users can lead to *specialization* in local public good provision at Nash equilibria (NE).

[5.5] Refer to Section 3.2.1 for an introduction to implementation theory.

Specialization means that only a subset of individuals contribute to the public goods and others free ride. The authors also show that specialization can benefit the society as a whole because among all Nash equilibria, the ones that are "specialized" result in the highest social welfare. However, it is shown in [8] that none of the NE can result in a local public goods provision that achieves the maximum possible social welfare. In [11] the work of [8] is extended to directed networks where the externality effects of users' actions on each other's utilities can be unidirectional or bidirectional. Thus, the network model in [11] is same as Model (M.5). The authors of [11] investigate the relation between the structure of directed networks and the existence and nature of Nash equilibria in those networks. For that matter they introduce a notion of ergodic stability to study the influence of perturbation of users' equilibrium efforts on the stability of NE. However, none of the NE of the game analyzed in [11] result in a public goods provision that achieves optimum social welfare. The problem (P_D.5) formulated in Section 5.1.3 is different from those in [8, 11] because our objective is to develop a decentralized resource allocation mechanism for Model (M.5) that can achieve the optimum solutions of Problem (P_C.5) (i.e. achieve optimum social welfare). To the best of our knowledge Problem (P_D.5) and its solution that we present in Section 5.2 is the first attempt to analyze Model (M.5) in the framework of the implementation theory component of mechanism design. In the next section we state our contributions in the problem formulation and solution presented in this chapter.

5.1.5 Contribution of the chapter

The key contributions of this chapter are:

- The formulation of a decentralized resource allocation problem for Model (M.5) in the framework of the implementation theory component of mechanism design.

- The specification of a game form for the above problem that possesses the following properties:

  (i) It implements in Nash equilibria [5.6] the optimal solution of Problem (P_C.5).

  (ii) It is individually rational. [5.7]

  (iii) It results in budget balance at all NE and off equilibrium.

In the next section we formulate the resource allocation problem (P_D.5) in the framework of implementation theory, and present a game form that achieves the above properties.

5.2 A decentralized resource allocation mechanism

Because we use the approach of implementation theory to address Problem (P_D.5), we begin this section by stating Problem (P_D.5) in the language of implementation theory. We then discuss an approach on how to construct a game form (decentralized allocation mechanism) for this problem and follow that discussion with the specification of the proposed game form. We conclude the section by stating the properties of the proposed game form. These properties are summarized in Theorems 5.1 and 5.2 the proofs of which appear in the appendices.

5.2.1 Embedding Problem (P_D.5) of Section 5.1.3 in implementation theory framework

As discussed in Section 3.2.1, in the implementation theory framework a resource allocation problem is described by specifying a triple $(\mathcal{E}, \mathcal{D}, \gamma)$. The *environment*

[5.6]Refer to Section 3.2.1 for the definition of "implementation in Nash equilibria."
[5.7]Refer to Section 3.2.1 for the definition of "individual rationality."

space $\mathcal{E}$ and the *action space* $\mathcal{D}$ characterize the problem model, and the goal correspondence $\gamma : \mathcal{E} \to \mathcal{D}$ characterizes the desirable centralized allocations for the problem.

There are N users in the network model (M.5); therefore the environment space of Problem (P_D.5) is a product space of N environment spaces, one corresponding to each user. The environment $\boldsymbol{e}_i$ of user i, $i \in \mathcal{N}$, consists of the set $\mathcal{A}_i \times \mathbb{R}$ of its feasible actions and taxes, its utility function u_i, its information about its neighbor sets $\mathcal{R}_i$ and $\mathcal{C}_i$, and its (common) knowledge about the facts described by Assumptions 5.2, 5.4, 5.5, 5.6 and 5.7. The environment space $\mathcal{E}_i$ of user i is the space of all possible environments $\boldsymbol{e}_i$, i.e., it consists of the following: the space of all sets $\mathcal{A}_i \times \mathbb{R} \subset \mathbb{R}^2$ such that $\mathcal{A}_i \subset \mathbb{R}$ is convex and compact and $0 \in \mathcal{A}_i$, the space of all concave functions $u_i : \mathbb{R}^{|\mathcal{R}_i|} \to \mathbb{R}$ such that $u_i(\boldsymbol{a}_{\mathcal{R}_i}) = 0$ for $a_i \notin \mathcal{A}_i$, the space of all finite subsets $\mathcal{R}_i$ and $\mathcal{C}_i$ of the set of natural numbers, and the common knowledge mentioned above.

The action space $\mathcal{D}$ of Problem (P_D.5) is the space of all feasible action and tax profiles $(\boldsymbol{a}_\mathcal{N}, \boldsymbol{t}_\mathcal{N})$ as defined in (5.5).

The goal correspondence γ for Problem (P_D.5) maps each environment $\boldsymbol{e} \in \mathcal{E}$ to the set of action and tax profiles $(\boldsymbol{a}_\mathcal{N}, \boldsymbol{t}_\mathcal{N}) \in \mathcal{D}$ that are solutions to Problem (P_C.5).

Having described Problem (P_D.5) in the framework of implementation theory, we now look at the specification of a decentralized mechanism from the implementation theory perspective. Recall from Section 3.2.1 that in implementation theory a decentralized resource allocation mechanism is specified in terms of a game form $(\mathcal{M}, f)$, where $\mathcal{M} := \prod_{i\in\mathcal{N}} \mathcal{M}_i$ is the message/strategy space and $f : \mathcal{M} \to \mathcal{D}$ is the outcome function. Therefore, our objective of designing a decentralized allocation mechanism for model (M.5) transforms into designing a game form. For our problem, we want to develop a game form $(\mathcal{M}, f)$ that is *individually rational*, *bud-*

get balanced, and that *implements in Nash equilibria* [5.8] the goal correspondence γ. Implementation in NE guarantees that the allocations corresponding to each NE of the game $(\mathcal{M}, f, \{u_i^A\}_{i\in\mathcal{N}})$ are a subset of the optimal centralized allocations (solutions of Problem (P_C.5)). Individual rationality guarantees voluntary participation of the users in the allocation process specified by the game form, and budget balance guarantees that there is no money left unclaimed/not allocated at the end of the allocation process (i.e. it ensures (5.1)). We present the definition of NE and its interpretation for our current problem at the end of Section 5.2.3. Discussion on how the game form we propose achieves the properties of individual rationality, budget balance, and Nash implementation appears in Section 5.2.4.

In the next section we construct a game form for the resource allocation problem (P_D.5).

5.2.2 Construction of a game form for Problem (P_D.5)

For the network model (M.5) we are interested in determining a game form that has the following properties: (i) It implements in NE the optimal solution of Problem (P_C.5); (ii) It is individually rational; and (iii) It is budget balanced. In this section we first develop a conceptual framework that must guide the construction of game forms which possess the above properties. We then present a game form that is designed within the developed framework.

We begin with a discussion on the construction of the message space. Since an allocation for Problem (P_D.5) consists of the action profile and the tax profile of the users, the message exchange among the users should contain information that is helpful in determining the optimal values of these profiles. Since each user's utility is affected by the actions of a subset of network users, each user should have

[5.8] See Section 3.2.1 for the definition of individual rationality and implementation in Nash equilibria.

a contribution in determining the actions of all its neighbors that affect its utility. Furthermore, a user should make a payment for the actions of all these neighbors because they all contribute to its utility. Since each neighbor's action makes a different contribution to the user's utility, the user may make different payments for each neighbor's actions. One way to take into account the above two factors is to let each user communicate as its message/strategy a proposal that consists of two components: one that indicates what actions the user wants its neighbors to take; and the other that indicates the price the user wants to pay for the actions of each of its neighbors.

We next discuss the construction of the outcome function. The specification of the outcome function is arguably the most important and challenging task in the construction of a game form/decentralized resource allocation mechanism. Since the designer of the mechanism cannot alter the users' utility functions $u_i, i \in \mathcal{N}$, the only way it can achieve the objectives of Nash implementation, individual rationality, and budget balance is through the provision of appropriate tax functions/incentives that induce strategic users to follow the mechanism's operational rules. Below we develop the guidelines for the construction of outcome functions that achieve each of the above objectives.

To achieve implementation in NE, the outcome function must make sure that all NE of the message exchange (that is done according to the discussion presented above) lead to optimal centralized allocations. This suggests that the outcome function must induce price taking behavior for all users at all NE. If price taking behavior is achieved, then, through NE price control, the mechanism can induce users to take actions that are optimal for their own objective and for the centralized problem (P_C.5). As discussed in the previous paragraph, a user should make a payment for the actions of each of its neighbors that affect its utility. In order for the mechanism to induce price taking behavior, the NE price that a user $i \in \mathcal{N}$ pays for its neighbors' actions must depend only on the messages/proposals of users other than i. Thus, the NE tax of user $i, i \in \mathcal{N}$, must be of the form $\sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j$ where $\hat{a}^*_j$ is the NE action of user j and l^*_{ij} is the NE price of this action for user i that is independent of user i's message. With the NE tax form $\sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j$, each user $i \in \mathcal{N}$ can influence its NE aggregate utility only through the actions $\hat{a}^*_j, j \in \mathcal{R}_i$. Since each user's utility is its private information, the utility maximizing actions of a user are known only to that user. Therefore, to allow each user to obtain its utility maximizing actions at given NE prices, the outcome function must provide each user $i \in \mathcal{N}$ an independent control, through its action proposal, over each of the actions $\hat{a}^*_j, j \in \mathcal{R}_i$. In other words, each action $\hat{a}^*_j, j \in \mathcal{N}$, must be independently controlled by each of the users $i \in \mathcal{C}_j$ and this fact should be reflected in the form of the outcome function.

To achieve budget balance, the NE prices $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$, must satisfy

$$\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i} l^*_{ij}\hat{a}^*_j = 0,$$

or, equivalently, [5.9]

$$\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j} l^*_{ij}\hat{a}^*_j = 0. \tag{5.6}$$

One way to satisfy the requirement in (5.6) is to set for each $j \in \mathcal{N}$, $\sum_{i\in\mathcal{C}_j} l^*_{ij} = 0$.

The features of the outcome function discussed so far could lead to price taking behavior and budget balance. However, the construction of an outcome function with the above features only may lead to the following difficulty. Since each user knows that its price proposal does not affect its own tax and hence, its aggregate utility,

[5.9]From the construction of the graph matrix $\mathcal{G}$ and the sets $\mathcal{R}_i$ and $\mathcal{C}_j$, $i,j \in \mathcal{N}$, the sum $\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i}(\cdot)$ is equivalent to the sum $\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}(\cdot)$.

it may propose arbitrary prices for its neighbors in its price proposal. One way to overcome this difficulty without altering price taking behavior and budget balance is to add a penalty to the tax form of each user. To preserve the price taking behavior of the users at NE, this penalty should be imposed only at off NE messages. The penalty should depend on each user's own price proposal and it should increase with the user's price proposal. However, to avoid unnecessary penalties, the penalty of a user should be reduced if its action proposal for its neighbors is in agreement with other users' action proposals. Adding to the tax form a penalty term with the above characteristics may result in an unbalanced budget. To preserve budget balance a third term should be added to the tax of each user. This term must balance the net flow of the money due to the penalty term. Since the penalty is imposed on the users only at off NE messages, this balancing term should be included in the users' tax only at off NE messages. To prevent the balancing term from altering a user's strategic behavior that is governed by the first two terms in the user's tax, the balancing term should be independent of the user's own message.

To achieve individual rationality the outcome function must make sure that at all NE, the utility of each user is at least as much as its initial utility. This property is achieved if the outcome function has the following features discussed earlier in this section: (i) It induces price taking behavior; and (ii) It gives each user an independent control over the actions that affect its utility. Since each user can control the actions that affect its NE utility, for any set of NE prices $l_{ij}^*, j \in \mathcal{R}_i$, a user $i \in \mathcal{N}$ can force all the actions $\hat{a}_j^*, j \in \mathcal{R}_i$, to be 0, thereby also making its NE payment $\sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_j^* = 0$. Thus, with the above features of the outcome function, each user can independently guarantee a minimum of zero utility for itself which is its initial utility.

With the guidelines developed above, we proceed with the construction of a game form in the next section.

5.2.3 The game form

In this section we present a game form for the resource allocation problem presented in Section 5.1.3. We provide explicit expressions of each of the components of the game form, the message space and the outcome function. We assume that the game form is common knowledge among the users and the network operator. The construction of the components of the game form is motivated by the arguments presented in the previous section.

The message space:

We let each user $i \in \mathcal{N}$ send to the network operator a message $\boldsymbol{m}_i \in \mathcal{M}_i := \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}$ that has the following form:

$$\boldsymbol{m}_i := ({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}); \quad {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|},\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}, \tag{5.7}$$

where,

$${}^i\boldsymbol{a}_{\mathcal{R}_i} := ({}^ia_k)_{k\in\mathcal{R}_i} \text{ and } {}^i\boldsymbol{\pi}_{\mathcal{R}_i} := ({}^i\pi_k)_{k\in\mathcal{R}_i}, \quad i \in \mathcal{N}. \tag{5.8}$$

User i also sends the component $({}^ia_k, {}^i\pi_k), k \in \mathcal{R}_i$, of its message to its neighbor $k \in \mathcal{R}_i$. In this message, ${}^ia_k$ is the action proposal for user $k, k \in \mathcal{R}_i$, by user $i, i \in \mathcal{N}$. Similarly, ${}^i\pi_k$ is the price that user $i, i \in \mathcal{N}$, proposes to pay for the action of user $k, k \in \mathcal{R}_i$. A detailed interpretation of these message elements is given in Section 5.2.4.

The outcome function

After the users communicate their messages to the network operator, their actions and taxes are determined as follows. For each user $i \in \mathcal{N}$, the network operator determines the action $\hat{a}_i$ of user i from the messages communicated by its neighbors that are affected by it (set $\mathcal{C}_i$), i.e. from the message profile $\boldsymbol{m}_{\mathcal{C}_i} := (\boldsymbol{m}_k)_{k \in \mathcal{C}_i}$:

$$\hat{a}_i(\boldsymbol{m}_{\mathcal{C}_i}) = \frac{1}{|\mathcal{C}_i|} \sum_{k \in \mathcal{C}_i} {}^k a_i, \quad i \in \mathcal{N}. \tag{5.9}$$

To determine the users' taxes, the network operator assigns indices $1, 2, \ldots, |\mathcal{C}_j|$ in a cyclic order to the users in each set $\mathcal{C}_j$, $j \in \mathcal{N}$. We denote the index of user $i \in \mathcal{N}$ associated with set $\mathcal{C}_j$, $j \in \mathcal{N}$, by $\mathcal{I}_{ij}$. $\mathcal{I}_{ij} \in \{1, 2, \ldots, |\mathcal{C}_j|\}$ if $i \in \mathcal{C}_j$, and $\mathcal{I}_{ij} = 0$ if $i \notin \mathcal{C}_j$. The cyclic order indexing means that, if $\mathcal{I}_{ij} = |\mathcal{C}_j|$, then $\mathcal{I}_{ij} + 1 = 1$, $\mathcal{I}_{ij} + 2 = 2$, and so on. Note that for any user $i \in \mathcal{N}$, and any $j, k \in \mathcal{R}_i$, the indices $\mathcal{I}_{ij}$ and $\mathcal{I}_{ik}$ are different and are independent of each other. Once the indices are assigned to the users in each set $\mathcal{C}_j, j \in \mathcal{N}$, they remain fixed throughout the time period of interest. We denote the user with index $k \in \{1, 2, \ldots, |\mathcal{C}_j|\}$ in set $\mathcal{C}_j$ by $\mathcal{C}_{j(k)}$. Thus, $\mathcal{C}_{j(\mathcal{I}_{ij})} = i$ for $i \in \mathcal{C}_j$, $j \in \mathcal{N}$. In Fig. 5.3 we illustrate the above indexing rule for the set $\mathcal{C}_j$ shown in Fig. 5.1.

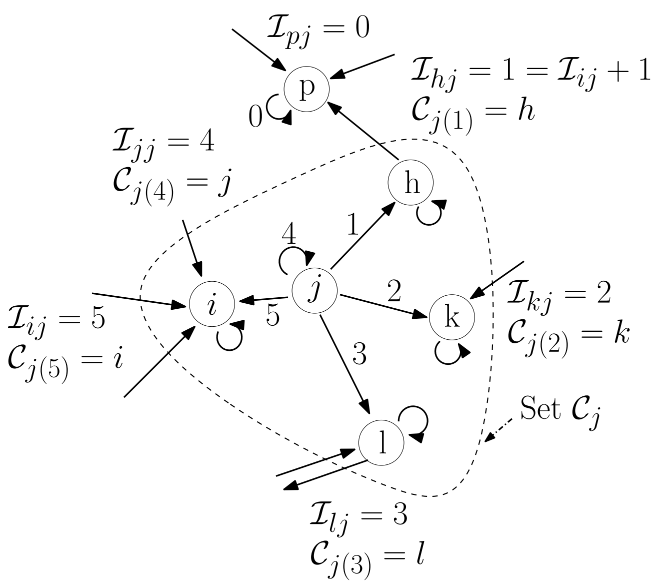

Figure 5.3: Illustration of indexing rule for set $\mathcal{C}_j$ shown in Fig. 5.1. The index $\mathcal{I}_{rj}$ of each user $r \in \mathcal{C}_j$ is indicated on the arrow directed from user r to user j. The notation to denote these indices and to denote the user with a particular index is shown outside the dashed boundary demarcating the set $\mathcal{C}_j$.

Based on the indexing described above, the users' taxes are determined as follows.

For each $i \in \mathcal{N}$, the tax $\hat{t}_i$ is determined from the message profile $(\boldsymbol{m}_{\mathcal{C}_j})_{j \in \mathcal{R}_i}$ as,

$$\begin{aligned}\hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = &\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}) + \sum_{j\in\mathcal{R}_i} {}^{i}\pi_j \left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2 \\ &- \sum_{j\in\mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2, \quad i \in \mathcal{N}\end{aligned} \tag{5.10}$$

where,

$$l_{ij}(\boldsymbol{m}_{\mathcal{C}_j}) = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j, \quad j \in \mathcal{R}_i,\ i \in \mathcal{N}. \tag{5.11}$$

The game form given by (5.7)–(5.11) and the users' aggregate utility functions in (5.2) induce a game $(\mathcal{M}, f, \{u_i^A\}_{i\in\mathcal{N}})$. We define a NE of this game as a message profile $\boldsymbol{m}^*_{\mathcal{N}}$ that has the following property:

$$u_i^A\Big(\big(\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})\big)_{j\in\mathcal{R}_i},\ \hat{t}_i\big((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}\big)\Big) \geq u_i^A\Big(\big(\hat{a}_j(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_j}/i)\big)_{j\in\mathcal{R}_i},\ \hat{t}_i\big((\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big), \quad \forall\, \boldsymbol{m}_i \in \mathcal{M}_i,\ \forall\, i \in \mathcal{N}. \tag{5.12}$$

We interpret the NE defined in (5.12) in the way of [43, 37] as described in Section 3.2.1.

In the next section we show that the allocations obtained by the game form presented in (5.7)–(5.11) at all NE message profiles (satisfying (5.12)), are optimal centralized allocations.

5.2.4 Properties of the game form

We begin this section with an intuitive discussion on how the game form presented in Section 5.2.3 achieves optimal centralized allocations. We then formalize the results in Theorems 5.1 and 5.2.

To understand how the proposed game form achieves optimal centralized allocations, let us look at the properties of NE allocations corresponding to this game form. A NE of the game induced by the game form (5.7)–(5.11) and the users' utility functions (5.2) can be interpreted as follows: Given the users' messages $\boldsymbol{m}_k, k \in \mathcal{C}_i$, the outcome function computes user i's action as $1/|\mathcal{C}_i|\big(\sum_{k\in\mathcal{C}_i} {}^k a_i\big)$. Therefore, user i's action proposal ${}^i a_i$ can be interpreted as the increment over the sum of other users' action proposals for i that i desires so as to bring its allocated action $\hat{a}_i$ to its own desired value. Thus, if the computed action for i based on the neighbors' proposals does not lie in $\mathcal{A}_i$, user i can propose an appropriate action ${}^i a_i$ and bring its allocated action within $\mathcal{A}_i$. The flexibility of proposing any action ${}^i a_i \in \mathbb{R}$ gives each user $i \in \mathcal{N}$ the capability to bring its allocation $\hat{a}_i$ within its feasible set $\mathcal{A}_i$ by unilateral deviation. Therefore, at any NE, $\hat{a}_i \in \mathcal{A}_i,\ \forall\ i \in \mathcal{N}$. By taking the sum of taxes in (5.10) it can further be seen, after some computations, that the allocated tax profile $(\hat{t}_i)_{i\in\mathcal{N}}$ satisfies (5.1) (even at off-NE messages). [5.10] Thus, all NE allocations $\Big((\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}))_{i\in\mathcal{N}},\ (\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}))_{i\in\mathcal{N}}\Big)$ lie in $\mathcal{D}$ and hence are feasible solutions of Problem (P_C.5).

To see further properties of NE allocations, let us look at the tax function in (5.10). The tax of user i consists of three types of terms. The type-1 term is $\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j})$; it depends on all action proposals for each of user i's neighbors $j \in \mathcal{R}_i$, and the price proposals for each of these neighbors by users other than user i. The type-2 term is $\sum_{j\in\mathcal{R}_i} {}^i\pi_j \left({}^i a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2$; this term depends on ${}^i\boldsymbol{a}_{\mathcal{R}_i}$ as well as ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$. Finally, the type-3 term is $-\sum_{j\in\mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2$; this term depends only on the messages of users other than i. Since ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ does not affect the determination of user i's action, and affects only the type-2 term in $\hat{t}_i$, the NE strategy of user $i, i \in \mathcal{N}$, that minimizes its tax is – to propose for each $j \in \mathcal{R}_i$, ${}^i\boldsymbol{\pi}_j = 0$ unless at the NE, ${}^i a_j = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j$. Since the type-2 and type-3 terms in the

[5.10] For details refer to Appendix 5.A.

users' tax are similar across users, for each $i \in \mathcal{N}$ and $j \in \mathcal{R}_i$, all the users $k \in \mathcal{C}_j$ choose the above strategy at NE. Therefore, the type-2 and type-3 terms vanish from every users' tax $\hat{t}_i, i \in \mathcal{N}$, at all NE. Thus, the tax that each user $i \in \mathcal{N}$ pays at a NE $\boldsymbol{m}^*_{\mathcal{N}}$ is of the form $\sum_{j \in \mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$. The NE term $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}), i \in \mathcal{N}, j \in \mathcal{R}_i$, can therefore be interpreted as the "personalized price" for user i for the NE action $\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$ of its neighbor j. Note that at a NE, the personalized price for user i is not controlled by i's own message. The reduction of the users' NE taxes into the form $\sum_{j \in \mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$ implies that at a NE, each user $i \in \mathcal{N}$ has a control over its aggregate utility only through its action proposal. [5.11] If all other users' messages are fixed, each user has the capability of shifting the allocated action profile $\hat{\boldsymbol{a}}_{\mathcal{R}_i}$ to its desired value by proposing an appropriate ${}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$ (See the discussion in the previous paragraph). Therefore, the NE strategy of each user $i \in \mathcal{N}$ is to propose an action profile ${}^i\boldsymbol{a}_{\mathcal{R}_i}$ that results in an allocation $\hat{\boldsymbol{a}}_{\mathcal{R}_i}$ that maximizes its aggregate utility. Thus, at a NE, each user maximizes its aggregate utility for its given personalized prices. By the construction of the tax function, the sum of the users' tax is zero at all NE and off equilibria. Thus, the individual aggregate utility maximization of the users also result in the maximization of the sum of users' aggregate utilities subject to the budget balance constraint which is the objective of Problem (P_C.5).

It is worth mentioning at this point the significance of type-2 and type-3 terms in the users' tax. As explained above, these two terms vanish at NE. However, if for some user $i \in \mathcal{N}$ these terms are not present in its tax $\hat{t}_i$, then, the price proposal ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ of user i will not affect its tax and hence, its aggregate utility. In such a case, user i can propose arbitrary prices ${}^i\boldsymbol{\pi}_{\mathcal{R}_i}$ because they would affect only other users' NE prices. The presence of type-2 and type-3 terms in user i's tax prevent such a

[5.11] Note that user i's action proposal determines the actions of all the users $j \in \mathcal{R}_i$; thus, it affects user i's utility $u_i\Big(\big(\hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})\big)_{j \in \mathcal{R}_i}\Big)$ as well as its tax $\sum_{j \in \mathcal{R}_i} l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$.

behavior as they impose a penalty on user i if it proposes a high value of ${}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}$ or if its action proposal for its neighbors deviates too much from those of other users. Even though the presence of type-2 and type-3 terms in user i's tax is necessary as explained above, it is important that the NE price $l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j}), j \in \mathcal{R}_i$ of user $i \in \mathcal{N}$ is not affected by i's own proposal ${}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}$. This is because, in such a case, user i may influence its own NE price in an unfair manner and may not behave as a price taker. To avoid such a situation, the type-2 and type-3 terms are designed in a way so that they vanish at NE. Thus, this construction induces price taking behavior in the users at NE and leads to optimal allocations.

From all of above discussion it can be seen that the proposed message space, the action function, and the tax function (with three types of terms) satisfy the features, discussed in Section 5.2.2, that are required to achieve the properties of Nash implementation, individual rationality, and budget balance.

The results that formally establish the above properties of the game form are summarized in Theorems 5.1 and 5.2 below.

Theorem 5.1. *Let $\boldsymbol{m}^*_{\mathcal{N}}$ be a NE of the game specified by the game form presented in Section 5.2.3 and the users' utility functions (5.2). Let $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) := (\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}), \hat{\boldsymbol{t}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}})) := \Big((\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}))_{i\in\mathcal{N}},\ (\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}))_{i\in\mathcal{N}}\Big)$ be the action and tax profiles at $\boldsymbol{m}^*_{\mathcal{N}}$ determined by the game form. Then,*

(a) Each user $i \in \mathcal{N}$ weakly prefers its allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i)$ to the initial allocation $(\boldsymbol{0}, 0)$. Mathematically,

$$u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big) \geq u_i^A\Big(\boldsymbol{0},\ 0\Big), \qquad \forall\, i \in \mathcal{N}.$$

(b) $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is an optimal solution of Problem (P_C.5).

⊠

Theorem 5.2. *Let $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ be the optimum action profile corresponding to Problem (P_C.5). Then,*

(a) There exist a set of personalized prices $l^*_{ij}, j \in \mathcal{R}_i, i \in \mathcal{N}$, such that

$$\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} = \underset{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i \backslash \{i\}}}{\arg\max} \; - \sum_{j \in \mathcal{R}_i} l^*_{ij}\, \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \qquad \forall\, i \in \mathcal{N}.$$

(b) There exists at least one NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by the game form presented in Section 5.2.3 and the users' utility functions (5.2) such that, $\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}^*_{\mathcal{N}}) = \hat{\boldsymbol{a}}^*_{\mathcal{N}}$. Furthermore, if $\hat{t}^*_i := \sum_{j \in \mathcal{R}_i} l^*_{ij} \hat{a}^*_j, i \in \mathcal{N}$, the set of all NE $\boldsymbol{m}^*_{\mathcal{N}} = (\boldsymbol{m}^*_i)_{i \in \mathcal{N}} = ({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i})$ that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is characterized by the solution of the following set of conditions:

$$\begin{aligned}
\frac{1}{|\mathcal{C}_i|} \sum_{k \in \mathcal{C}_i} {}^k a^*_i &= \hat{a}^*_i, \quad i \in \mathcal{N}, \\
{}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi^*_j &= l^*_{ij}, \quad j \in \mathcal{R}_i,\ i \in \mathcal{N}, \\
{}^i\pi^*_j \left({}^i a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}} a^*_j \right)^2 &= 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}, \\
{}^i\pi^*_j &\geq 0, \quad j \in \mathcal{R}_i, \quad i \in \mathcal{N}.
\end{aligned}$$

⊠

Because Theorem 5.1 is stated for an arbitrary NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by the game form presented in Section 5.2.3 and the users' utility functions (5.2), the assertion of the theorem holds for all NE of this game. Thus, part (a) of Theorem 5.1 establishes that the game form presented in Section 5.2.3 is *individually rational*, i.e., at any NE allocation, the aggregate utility of each user is at least as much as its aggregate utility before participating in the game/allocation process. Because of this property of the game form, each user voluntarily participates in the allocation process.

Part (b) of Theorem 5.1 asserts that all NE of the game induced by the game form presented in Section 5.2.3 and the users' utility functions (5.2) result in optimal centralized allocations (solutions of Problem (P_C.5)). The set of NE allocations is a subset of the set of centralized allocations. This establishes that the game form presented in Section 5.2.3 *implements in NE* the goal correspondence γ defined by Problem (P_C.5) (see Section 5.2.1). Because of this property, the game form guarantees to provide a centralized allocation irrespective of which NE is achieved in the game induced by the game form.

The assertion of Theorem 5.1 that establishes the above two properties of the game form is based on the assumption that there exists a NE of the game induced by the game form of Section 5.2.3 and the users' utility functions (5.2). However, Theorem 5.1 does not say anything about the existence of NE. Theorem 5.2 asserts that NE exist in the above game, and provides conditions that characterize the set of all NE that result in optimal centralized allocations of the form $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) = (\hat{\boldsymbol{a}}^*_{\mathcal{N}}, (\sum_{j \in \mathcal{R}_i} l^*_{ij} \hat{a}^*_j)_{i \in \mathcal{N}})$, where $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ is any optimal centralized action profile.

In addition to the above, Theorem 5.2 also establishes the following property of the game form. Since the optimal action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ in the statement of Theorem 5.2 is arbitrary, the theorem implies that the game form of Section 5.2.3 can obtain each of the optimum action profiles of Problem (P_C.5) through at least one of the NE of the induced game. This establishes that the above game form is not biased towards any particular optimal centralized action profile.

We present the proofs of Theorem 5.1 and Theorem 5.2 in Appendices 5.A and 5.B respectively.

In the next section we present a discussion on how the game form of Section 5.2.3 can be implemented in a real network and we also discuss the limitations associated with it.

5.2.5 Implementation of the decentralized mechanism

In this section we discuss two aspects of implementation of the decentralized mechanism specified by the game form of Section 5.2.3. First we discuss how the game form itself can be implemented, i.e., how the message communication and the determination of allocations specified by the game form can be carried out in a real system. We then discuss how NE can be achieved in the game induced by the above game form.

We will show below that the presence of a network operator is important for the implementation of the game form. To see this let us first suppose that the network operator is not present in the network. As discussed in Section 5.2.3 the outcome function specifies the allocation $(\hat{a}_i, \hat{t}_i)$ for a user $i \in \mathcal{N}$ based on its neighbors' messages. Since the game form is common knowledge among the users, if each user announces its messages to all its neighbors, every user can have the required set of messages to compute its own allocations. However, with this kind of local communication, the messages required to compute user i's allocation are not necessarily known to users other than i. Therefore, even though the other users know the outcome function for user i, no other user can check if the allocation determined by user i corresponds to its neighbors' messages. Since each user $i \in \mathcal{N}$ is selfish, it cannot be relied upon for the determination of its allocation. Therefore, in large-scale networks such as one represented by Model (M.5), where each user does not hear all other users' messages, the presence of a network operator is extremely important. The network operator's role is twofold. First, according to the specification of the game form (of Section 5.2.3) each user announces its messages to its neighbors as well as to the network operator. The network operator knows the network structure (Assumption 5.7) and the outcome function for each user. Thus, it can compute all the allocations based on the messages it receives, and then it can tell each user its corresponding allocation (or it can check whether the allocation (a_i^*, t_i^*) implemented by user $i, i \in \mathcal{N}$, is the same as that specified by the mechanism). The other role of the network operator that facilitates implementation of the game form is the following. Note that the game form specifies redistribution of money among the users by charging each user an appropriate positive or negative tax (see (5.1)). This means that the tax money must go from one subset of the users to the other subset of users. Since the users do not have complete network information, nor do they know the allocations of other users in the network, they cannot determine the appropriate flow of money in the network. The network operator implements this redistribution of money by acting as an accountant that collects money from the users that have to pay positive tax according to the game form and gives the money back to the users that have to pay negative tax. In the cellular network example (Application A, Section 5.1.2.1), the role of the network operator is performed by the BS whereas, in the library network example (Application B, Section 5.1.2.2) the role of the network operator is performed by the university authorities.

The discussion presented above shows how the game form of Section 5.2.3 can be implemented in the presence of a network operator. However, to achieve the properties of the game form described by Theorems 5.1 and 5.2, we need a method to obtain NE of the game induced by this game form. Even though the above game form achieves full implementation in NE, at present we do not have an algorithm for the computation of these equilibria. Therefore, in this chapter, we restrict our focus to equilibrium analysis, and defer the study of equilibrium computation for future work. We comment further on the computation of equilibria in Chapter 6.

In the appendices that follow, we present the proof of Theorems 5.1 and 5.2. We divide the proof into several claims to organize the presentation.

5.A Proof of Theorem 5.1

We prove Theorem 5.1 in four claims. In Claims 5.2 and 5.3 we show that all users weakly prefer a NE allocation (corresponding to the game form presented in Section 5.2.3) to their initial allocations; these claims prove part (a) of Theorem 5.1. In Claim 5.1 we show that a NE allocation is a feasible solution of Problem (P_C.5). In Claim 5.4 we show that a NE action profile is an optimal action profile for Problem (P_C.5). Thus, Claim 5.1 and Claim 5.4 establish that a NE allocation is an optimal solution of Problem (P_C.5) and prove part (b) of Theorem 5.1.

Claim 5.1. *If $\boldsymbol{m}_{\mathcal{N}}^*$ is a NE of the game induced by the game form presented in Section 5.2.3 and the users' utility functions (5.2), then the action and tax profile $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*) := (\hat{\boldsymbol{a}}_{\mathcal{N}}(\boldsymbol{m}_{\mathcal{N}}^*), \hat{\boldsymbol{t}}_{\mathcal{N}}(\boldsymbol{m}_{\mathcal{N}}^*))$ is a feasible solution of Problem (P_C.5), i.e. $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*) \in \mathcal{D}$.*

Proof:

We prove the feasibility of the NE action and tax profiles in two steps. First we prove the feasibility of the NE tax profile, then we prove the feasibility of the NE action profile.

To prove the feasibility of NE tax profile, we need to show that it satisfies (5.1). For this, we first take the sum of second and third terms on the Right Hand Side (RHS) of (5.10) over all $i \in \mathcal{N}$, i.e.

$$\sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{R}_i} \left[{}^{i}\pi_j \left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j \right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j \left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j \right)^2 \right]. \tag{5.13}$$

From the construction of the graph matrix $\mathcal{G}$ and the sets $\mathcal{R}_i$ and $\mathcal{C}_j$, $i, j \in \mathcal{N}$, the sum $\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i}(\cdot)$ is equal to the sum $\sum_{j\in\mathcal{N}}\sum_{i\in\mathcal{C}_j}(\cdot)$. Therefore, we can rewrite (5.13) as

$$\sum_{j\in\mathcal{N}}\left[\sum_{i\in\mathcal{C}_j} {}^{i}\pi_j\left({}^{i}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\right)^2 - \sum_{i\in\mathcal{C}_j} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j\right)^2\right]. \tag{5.14}$$

Note that both the sums inside the square brackets in (5.14) are over all $i \in \mathcal{C}_j$. Because of the cyclic indexing of the users in each set $\mathcal{C}_j$, $j \in \mathcal{N}$, these two sums are equal. Therefore the overall sum in (5.14) evaluates to zero. Thus, the sum of taxes in (5.10) reduces to

$$\sum_{i\in\mathcal{N}} \hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j})\ \hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}). \tag{5.15}$$

Combining (5.11) and (5.15) we obtain

$$\sum_{i\in\mathcal{N}} \hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = \sum_{j\in\mathcal{N}}\left[\sum_{i\in\mathcal{C}_j} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j - \sum_{i\in\mathcal{C}_j} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j\right]\hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}) = 0. \tag{5.16}$$

The second equality in (5.16) follows because of the cyclic indexing of the users in each set $\mathcal{C}_j$, $j \in \mathcal{N}$, which makes the two sums inside the square brackets in (5.16) equal. Because (5.16) holds for any arbitrary message profile $\boldsymbol{m}_{\mathcal{N}}$, it follows that at NE $\boldsymbol{m}^*_{\mathcal{N}}$,

$$\sum_{i\in\mathcal{N}} \hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) = 0. \tag{5.17}$$

To complete the proof of Claim 5.1, we have to prove that for all $i \in \mathcal{N}$, $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \in \mathcal{A}_i$. We prove this by contradiction. Suppose $\hat{a}^*_i \notin \mathcal{A}_i$ for some $i \in \mathcal{N}$. Then, from (5.2), $u^A_i(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i) = -\infty$. Consider $\widetilde{\boldsymbol{m}}_i = (({}^{i}\widetilde{a}_i, {}^{i}\boldsymbol{a}^*_{\mathcal{R}_i}/i), {}^{i}\boldsymbol{\pi}^*_{\mathcal{R}_i})$ where ${}^{i}a^*_k, k \in \mathcal{R}_i\backslash\{i\}$, and ${}^{i}\boldsymbol{\pi}^*_{\mathcal{R}_i}$ are respectively the NE action and price proposals of user i and ${}^{i}\widetilde{a}_i$ is such that

$$\hat{a}_i(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_i}/i) = \frac{1}{|\mathcal{C}_i|}\left({}^{i}\widetilde{\boldsymbol{a}}_i + \sum_{\substack{k\in\mathcal{C}_i \\ k\neq i}} {}^{k}a^*_i\right) \in \mathcal{A}_i. \tag{5.18}$$

Note that the flexibility of user i in choosing any message ${}^{i}\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$ (see (5.7)) allows it to choose an appropriate ${}^{i}\widetilde{a}_i$ that satisfies the condition in (5.18). For the message $\widetilde{\boldsymbol{m}}_i$ constructed above,

$$\begin{aligned} &u_i^A\Big(\big(\hat{a}_k(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\ \ \hat{t}_i\big((\widetilde{\boldsymbol{m}}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \\ &= -\hat{t}_i\big((\widetilde{\boldsymbol{m}}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big) + u_i\Big(\big(\hat{a}_k(\widetilde{\boldsymbol{m}}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i}\Big) > -\infty = u_i^A(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i) \end{aligned} \tag{5.19}$$

Thus if $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \notin \mathcal{A}_i$ user i finds it profitable to deviate to $\widetilde{\boldsymbol{m}}_{\boldsymbol{i}}$. Inequality (5.19) implies that $\boldsymbol{m}^*_{\mathcal{N}}$ cannot be a NE, which is a contradiction. Therefore, at any NE $\boldsymbol{m}^*_{\mathcal{N}}$, we must have $\hat{a}_i(\boldsymbol{m}^*_{\mathcal{C}_i}) \in \mathcal{A}_i\ \forall\ i \in \mathcal{N}$. This along with (5.17) implies that, $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}}) \in \mathcal{D}$. □

Claim 5.2. *If $\boldsymbol{m}^*_{\mathcal{N}}$ is a NE of the game induced by the game form presented in Section 5.2.3 and the users' utility functions (5.2), then, the tax $\hat{t}_i((\boldsymbol{m}^*_{\mathcal{C}_j})_{j\in\mathcal{R}_i}) =: \hat{t}^*_i$ paid by user $i, i \in \mathcal{N}$, at the NE $\boldsymbol{m}^*_{\mathcal{N}}$ is of the form $\hat{t}^*_i = \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j$, where $l^*_{ij} = l_{ij}(\boldsymbol{m}^*_{\mathcal{C}_j})$ and $\hat{a}^*_j = \hat{a}_j(\boldsymbol{m}^*_{\mathcal{C}_j})$.*

Proof:

Let $\boldsymbol{m}^*_{\mathcal{N}}$ be the NE specified in the statement of Claim 5.2. Then, for each $i \in \mathcal{N}$,

$$u_i^A\Big(\big(\hat{a}_k(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\ \ \hat{t}_i\big((\boldsymbol{m}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \le\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big), \quad \forall\ \boldsymbol{m}_i \in \mathcal{M}_i. \tag{5.20}$$

Substituting $\boldsymbol{m}_i = ({}^{i}\boldsymbol{a}^*_{\mathcal{R}_i}, {}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}),\ {}^{i}\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}$, in (5.20) and using (5.9) implies that

$$u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i},\ \ \hat{t}_i\big((({}^{i}\boldsymbol{a}^*_{\mathcal{R}_i}, {}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}),\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \le\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big), \quad \forall\ {}^{i}\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{5.21}$$

Since u_i^A decreases in t_i (see (5.2)), (5.21) implies that

$$\hat{t}_i\Big((({}^{i}\boldsymbol{a}^*_{\mathcal{R}_i}, {}^{i}\boldsymbol{\pi}_{\mathcal{R}_i}),\, \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)_{j\in\mathcal{R}_i}\Big) \ge \hat{t}^*_i, \quad \forall\ {}^{i}\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{5.22}$$

Substituting (5.10) in (5.22) results in

(5.23)
$$\sum_{j\in\mathcal{R}_i}\left[l_{ij}^*\hat{a}_j^* + {}^i\pi_j\left({}^ia_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^*\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2\right]$$
$$\geq \sum_{j\in\mathcal{R}_i}\left[l_{ij}^*\hat{a}_j^* + {}^i\pi_j^*\left({}^ia_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^*\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2\right],$$
$$\forall\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}.$$

Canceling the common terms in (5.23) gives

$$\sum_{j\in\mathcal{R}_i}({}^i\pi_j - {}^i\pi_j^*)\left({}^ia_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 \geq 0,\ \forall\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \tag{5.24}$$

Since (5.24) must hold for all ${}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}$, we must have that

$$\text{for each } j \in \mathcal{R}_i,\ \text{either } {}^i\pi_j^* = 0 \text{ or } {}^ia_j^* = {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*. \tag{5.25}$$

From (5.25) it follows that at any NE $\boldsymbol{m}_{\mathcal{N}}^*$,

$${}^i\pi_j^*\left({}^ia_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 = 0, \quad \forall\ j \in \mathcal{R}_i, \quad \forall\ i \in \mathcal{N}. \tag{5.26}$$

Note that (5.26) also implies that $\forall\ i \in \mathcal{N}$ and $\forall\ j \in \mathcal{R}_i$,

$${}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^*\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2 = 0. \tag{5.27}$$

(5.27) follows from (5.26) because for each $i \in \mathcal{N}$, $j \in \mathcal{R}_i$ also implies that $j \in \mathcal{R}_{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}$. Using (5.26) and (5.27) in (5.10) we obtain that any NE tax profile must be of the form

$$\hat{t}_i^* = \sum_{j\in\mathcal{R}_i} l_{ij}^*\ \hat{a}_j^*, \quad \forall\ i \in \mathcal{N}. \tag{5.28}$$

□

Claim 5.3. *The game form given in Section 5.2.3 is individually rational, i.e. at every NE $\boldsymbol{m}^*_{\mathcal{N}}$ of the game induced by this game form and the users' utilities in (5.2), each user $i \in \mathcal{N}$ weakly prefers the allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i)$ to the initial allocation $(\mathbf{0}, 0)$. Mathematically,*

$$u_i^A\Big(\mathbf{0},\ 0\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \hat{t}^*_i\Big), \qquad \forall\, i \in \mathcal{N}. \tag{5.29}$$

Proof:

Suppose $\boldsymbol{m}^*_{\mathcal{N}}$ is a NE of the game induced by the game form presented in Section 5.2.3 and the users' utility functions (5.2). From Claim 5.2 we know the form of users' tax at $\boldsymbol{m}^*_{\mathcal{N}}$. Substituting that from (5.28) into (5.20) we obtain that for each $i \in \mathcal{N}$,

$$\begin{aligned} &u_i^A\Big(\big(\hat{a}_k(\boldsymbol{m}_i, \boldsymbol{m}^*_{\mathcal{C}_k}/i)\big)_{k\in\mathcal{R}_i},\ \ \hat{t}_i\big((\boldsymbol{m}_i,\, \boldsymbol{m}^*_{\mathcal{C}_j}/i)_{j\in\mathcal{R}_i}\big)\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \boldsymbol{m}_i = ({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}) \in \mathcal{M}_i. \end{aligned} \tag{5.30}$$

Substituting for $\hat{t}_i$ in (5.30) from (5.10) and using (5.27) we obtain,

$$\begin{aligned} &u_i^A\Bigg(\Big(\hat{a}_k\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_k}/i\big)\Big)_{k\in\mathcal{R}_i}, \\ &\qquad \sum_{j\in\mathcal{R}_i}\Big(l^*_{ij}\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) + {}^i\pi_j\big({}^ia_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j\big)^2\Big)\Bigg) \\ &\leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big), \qquad \forall\ {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|},\ \forall\ {}^i\boldsymbol{\pi}_{\mathcal{R}_i} \in \mathbb{R}_+^{|\mathcal{R}_i|}. \end{aligned} \tag{5.31}$$

In particular, ${}^i\boldsymbol{\pi}_{\mathcal{R}_i} = \mathbf{0}$ in (5.31) implies that

$$\begin{aligned} &u_i^A\Bigg(\Big(\hat{a}_k\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, \mathbf{0}), \boldsymbol{m}^*_{\mathcal{C}_k}/i\big)\Big)_{k\in\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i}\Big(l^*_{ij}\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, \mathbf{0}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)\Bigg) \\ &\leq\ u_i^A\Big(\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l^*_{ij}\ \hat{a}^*_j\Big), \qquad \forall\ {}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}. \end{aligned} \tag{5.32}$$

Since (5.32) holds for all ${}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}$, substituting $\frac{1}{|\mathcal{C}_j|}({}^ia_j + \sum_{k\in\mathcal{C}_j\setminus\{i\}} {}^ka_j) = \overline{a}_j$ for all $j \in \mathcal{R}_i$ in (5.32) gives

(5.33)
$$u_i^A\Big(\left(\overline{a}_j\right)_{j\in\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i}\left(l_{ij}^*\ \overline{a}_j\right)\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*, \sum_{j\in\mathcal{R}_i} l_{ij}^*\ \hat{a}_j^*\Big), \qquad \forall\ \overline{a}_{\mathcal{R}_i} := (\overline{a}_j)_{j\in\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}.$$

For $\overline{a}_{\mathcal{R}_i} = \mathbf{0}$, (5.33) implies further that

$$u_i^A\Big(\mathbf{0},\ 0\Big) \leq\ u_i^A\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*, \sum_{j\in\mathcal{R}_i} l_{ij}^*\ \hat{a}_j^*\Big), \qquad \forall\ i \in \mathcal{N}. \tag{5.34}$$

$\square$

Claim 5.4. *A NE allocation $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*)$ is an optimal solution of the centralized problem (P_C).*

Proof:

For each $i \in \mathcal{N}$, (5.33) can be equivalently written as

$$\begin{aligned}
\hat{\boldsymbol{a}}_{\mathcal{R}_i}^* &= \underset{\overline{\boldsymbol{a}}_{\mathcal{R}_i}\in\mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\, u_i^A\Big(\overline{\boldsymbol{a}}_{\mathcal{R}_i}, \sum_{j\in\mathcal{R}_i} l_{ij}^*\ \overline{a}_j\Big) \\
&= \underset{\overline{\boldsymbol{a}}_{\mathcal{R}_i}\in\mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\left\{-\sum_{j\in\mathcal{R}_i} l_{ij}^*\ \overline{a}_j + u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i}) - \left[\frac{1 - I_{\mathcal{A}_i}(\overline{a}_i)}{I_{\mathcal{A}_i}(\overline{a}_i)}\right]\right\} \\
&= \underset{\substack{\overline{a}_i\in\mathcal{A}_i \\ \overline{a}_j\in\mathbb{R},\, j\in\mathcal{R}_i\setminus\{i\}}}{\arg\max}\left\{-\sum_{j\in\mathcal{R}_i} l_{ij}^*\ \overline{a}_j + u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})\right\}
\end{aligned} \tag{5.35}$$

Let for each $i \in \mathcal{N}$, $f_{\mathcal{A}_i}(a_i)$ be a convex function that characterizes the set $\mathcal{A}_i$ as, $a_i \in \mathcal{A}_i \Leftrightarrow f_{\mathcal{A}_i}(a_i) \leq 0$. [5.12]

Since for each $i \in \mathcal{N}$, $u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i})$ is assumed to be concave in $\overline{\boldsymbol{a}}_{\mathcal{R}_i}$ and the set $\mathcal{A}_i$ is convex, the Karush Kuhn Tucker (KKT) conditions [7, Chapter 11] are necessary and sufficient for $\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*$ to be a maximizer in (5.35). Thus, for each $i \in \mathcal{N}\ \exists\ \lambda_i \in \mathbb{R}_+$

[5.12] By [7] we can always find a convex function that characterizes a convex set.

such that, $\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}$ and λ_i satisfy the KKT conditions given below:

$$\begin{aligned}\forall\, j \in \mathcal{R}_i\backslash\{i\}, \quad l^*_{ij} - \nabla_{\overline{a}_j} u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\overline{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} &= 0,\\ l^*_{ii} - \nabla_{\overline{a}_i} u_i(\overline{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\overline{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_i}} + \lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i) \mid_{\overline{a}_i=\hat{a}^*_i} &= 0,\\ \lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) &= 0.\end{aligned} \tag{5.36}$$

For each $i \in \mathcal{N}$, adding the KKT condition equations in (5.36) over $k \in \mathcal{C}_i$ results in

$$\sum_{k\in\mathcal{C}_i} l^*_{ki} - \nabla_{\overline{a}_i} \sum_{k\in\mathcal{C}_i} u_k(\overline{\boldsymbol{a}}_{\mathcal{R}_k}) \mid_{\overline{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} + \lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i) \mid_{\overline{a}_i=\hat{a}^*_i} = 0. \tag{5.37}$$

From (5.11) we have,

$$\sum_{k\in\mathcal{C}_i} l^*_{ki} = \sum_{k\in\mathcal{C}_i} \left({}^{\mathcal{C}_{i(\mathcal{I}_{ki}+1)}}\pi^*_i - {}^{\mathcal{C}_{i(\mathcal{I}_{ki}+2)}}\pi^*_i \right) = 0. \tag{5.38}$$

Substituting (5.38) in (5.37) we obtain [5.13] $\forall\, i \in \mathcal{N}$,

$$\begin{aligned} -\nabla_{\overline{a}_i} \sum_{k\in\mathcal{C}_i} u_k(\overline{\boldsymbol{a}}_{\mathcal{R}_k}) \mid_{\overline{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}^*_{\mathcal{R}_k}} + \lambda_i \nabla_{\overline{a}_i} f_{\mathcal{A}_i}(\overline{a}_i) \mid_{\overline{a}_i=\hat{a}^*_i} &= 0,\\ \lambda_i f_{\mathcal{A}_i}(\hat{a}^*_i) &= 0.\end{aligned} \tag{5.39}$$

The conditions in (5.39) along with the non-negativity of $\lambda_i, i \in \mathcal{N}$, specify the KKT conditions (for variable $\hat{\boldsymbol{a}}_{\mathcal{N}}$) for Problem ($P_C$.5). Since ($P_C$.5) is a concave optimization problem, KKT conditions are necessary and sufficient for optimality. As shown in (5.39), the action profile $\hat{\boldsymbol{a}}^*_{\mathcal{N}}$ satisfies these optimality conditions. Furthermore, the tax profile $\hat{\boldsymbol{t}}^*_{\mathcal{N}}$ satisfies, by its definition, $\sum_{i\in\mathcal{N}} \hat{t}^*_i = 0$. Therefore, the NE allocation $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$ is an optimal solution of Problem (P_C.5). This completes the proof of Claim 5.4 and hence, the proof of Theorem 5.1. □

Claims 5.1–5.4 (Theorem 5.1) establish the properties of NE allocations based on the assumption that there exists a NE of the game induced by the game form of Section 5.2.3 and users' utility functions (5.2). However, these claims do not guarantee the existence of a NE. This is guaranteed by Theorem 5.2 which is proved next in Claims 5.5 and 5.6.

[5.13] The second equality in (5.39) is one of the KKT conditions from (5.36).

5.B Proof of Theorem 5.2

We prove Theorem 5.2 in two steps. In the first step we show that if the centralized problem (P_C.5) has an optimal action profile $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$, there exist a set of personalized prices, one for each user $i \in \mathcal{N}$, such that when each $i \in \mathcal{N}$ individually maximizes its own utility taking these prices as given, it obtains $\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*$ as an optimal action profile. In the second step we show that the optimal action profile $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$ and the corresponding personalized prices can be used to construct message profiles that are NE of the game induced by the game form of Section 5.2.3 and users' utility functions in (5.2).

Claim 5.5. *If Problem* (P_C.5) *has an optimal action profile* $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$*, there exist a set of personalized prices* $l_{ij}^*, j \in \mathcal{R}_i, i \in \mathcal{N}$*, such that*

$$\hat{\boldsymbol{a}}_{\mathcal{R}_i}^* = \underset{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i \setminus \{i\}}}{\arg\max} \; -\sum_{j \in \mathcal{R}_i} l_{ij}^* \, \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \qquad \forall\, i \in \mathcal{N}. \tag{5.40}$$

Proof:

Suppose $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$ is an optimal action profile corresponding to Problem (P_C.5). Writing the optimization problem (P_C.5) only in terms of variable $\hat{\boldsymbol{a}}_{\mathcal{N}}$ gives

$$\begin{aligned} \hat{\boldsymbol{a}}_{\mathcal{N}}^* = \underset{\hat{\boldsymbol{a}}_{\mathcal{N}}}{\arg\max} \; & \sum_{i \in \mathcal{N}} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \\ \text{s.t.} \;\; & \hat{a}_i \in \mathcal{A}_i, \; \forall\, i \in \mathcal{N}. \end{aligned} \tag{5.41}$$

As stated earlier, an optimal solution of Problem (P_C.5) is of the form $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}})$, where $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$ is a solution of (5.41) and $\hat{\boldsymbol{t}}_{\mathcal{N}} \in \mathbb{R}^N$ is any tax profile that satisfies (5.1). Because KKT conditions are necessary for optimality, the optimal solution in (5.41) must satisfy the KKT conditions. This implies that there exist $\lambda_i \in \mathbb{R}_+, i \in \mathcal{N}$, such that for each $i \in \mathcal{N}$, λ_i and $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$ satisfy

$$\begin{aligned} -\nabla_{\hat{a}_i} \sum_{k \in \mathcal{C}_i} u_k(\hat{\boldsymbol{a}}_{\mathcal{R}_k}) \,|_{\hat{\boldsymbol{a}}_{\mathcal{R}_k} = \hat{\boldsymbol{a}}_{\mathcal{R}_k}^*} + \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \,|_{\hat{a}_i = \hat{a}_i^*} = 0, \\ \lambda_i f_{\mathcal{A}_i}(\hat{a}_i^*) = 0, \end{aligned} \tag{5.42}$$

where $f_{\mathcal{A}_i}(\cdot)$ is the convex function defined in Claim 5.4. Defining for each $i \in \mathcal{N}$,

$$
\begin{aligned}
l_{ij}^* &:= \nabla_{\hat{a}_j} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*}, \qquad j \in \mathcal{R}_i \backslash \{i\}, \\
l_{ii}^* &:= \nabla_{\hat{a}_i} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*} - \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \mid_{\hat{a}_i=\hat{a}_i^*},
\end{aligned}
\tag{5.43}
$$

we get $\forall\, i \in \mathcal{N}$,

$$
\sum_{k \in \mathcal{C}_i} l_{ki}^* = \nabla_{\hat{a}_i} \sum_{k \in \mathcal{C}_i} u_k(\hat{\boldsymbol{a}}_{\mathcal{R}_k}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_k}=\hat{\boldsymbol{a}}_{\mathcal{R}_k}^*} - \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \mid_{\hat{a}_i=\hat{a}_i^*} = 0. \tag{5.44}
$$

The second equality in (5.44) follows from (5.42). Furthermore, (5.43) implies that $\forall\, i \in \mathcal{N}$,

$$
\begin{aligned}
\forall\, j \in \mathcal{R}_i \backslash \{i\}, \quad l_{ij}^* - \nabla_{\hat{a}_j} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*} &= 0, \\
l_{ii}^* - \nabla_{\hat{a}_i} u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \mid_{\hat{\boldsymbol{a}}_{\mathcal{R}_i}=\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*} + \lambda_i \nabla_{\hat{a}_i} f_{\mathcal{A}_i}(\hat{a}_i) \mid_{\hat{a}_i=\hat{a}_i^*} &= 0.
\end{aligned}
\tag{5.45}
$$

The equations in (5.45) along with the second equality in (5.42) imply that for each $i \in \mathcal{N}$, $\hat{\boldsymbol{a}}_{\mathcal{R}_i}^*$ and λ_i satisfy the KKT conditions for the following maximization problem:

$$
\max_{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i \backslash \{i\}}} \; - \sum_{j \in \mathcal{R}_i} l_{ij}^* \; \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) \tag{5.46}
$$

Because the objective function in (5.46) is concave (Assumption 5.3), KKT conditions are necessary and sufficient for optimality. Therefore, we conclude from (5.45) and (5.42) that,

$$
\hat{\boldsymbol{a}}_{\mathcal{R}_i}^* = \operatorname*{arg\,max}_{\substack{\hat{a}_i \in \mathcal{A}_i \\ \hat{a}_j \in \mathbb{R},\, j \in \mathcal{R}_i \backslash \{i\}}} \; - \sum_{j \in \mathcal{R}_i} l_{ij}^* \; \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}), \; \forall\, i \in \mathcal{N}.
$$

□

Claim 5.6. *Let* $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$ *be an optimal action profile for Problem* $(P_C.5)$*, let* $l_{ij}^*, j \in \mathcal{R}_i, i \in \mathcal{N}$*, be the personalized prices corresponding to* $\hat{\boldsymbol{a}}_{\mathcal{N}}^*$ *as defined in Claim 5.5, and let* $\hat{t}_i^* := \sum_{j \in \mathcal{R}_i} l_{ij}^* \hat{a}_j^*, i \in \mathcal{N}$*. Let* $\boldsymbol{m}_i^* := ({}^i\boldsymbol{a}_{\mathcal{R}_i}^*, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}^*), i \in \mathcal{N}$*, be a solution to the* *following set of relations:*

$$\frac{1}{|\mathcal{C}_i|}\sum_{k\in\mathcal{C}_i} {}^k a_i^* \;=\; \hat{a}_i^*, \quad i\in\mathcal{N}, \tag{5.47}$$

$$ {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j^* \;=\; l_{ij}^*, \quad j\in\mathcal{R}_i,\; i\in\mathcal{N}, \tag{5.48}$$

$$ {}^i\pi_j^*\left({}^i a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2 \;=\; 0, \quad j\in\mathcal{R}_i, \quad i\in\mathcal{N}, \tag{5.49}$$

$$ {}^i\pi_j^* \;\geq\; 0, \quad j\in\mathcal{R}_i, \quad i\in\mathcal{N}. \tag{5.50}$$

Then, $\boldsymbol{m}_{\mathcal{N}}^* := (\boldsymbol{m}_1^*, \boldsymbol{m}_2^*, \ldots, \boldsymbol{m}_N^*)$ *is a NE of the game induced by the game form of Section 5.2.3 and the users' utility functions (5.2). Furthermore, for each* $i\in\mathcal{N}$*,* $\hat{a}_i(\boldsymbol{m}_{\mathcal{C}_i}^*) = \hat{a}_i^*$*,* $l_{ij}(\boldsymbol{m}_{\mathcal{C}_j}^*) = l_{ij}^*, j\in\mathcal{R}_i$*, and* $\hat{t}_i((\boldsymbol{m}_{\mathcal{C}_j}^*)_{j\in\mathcal{R}_i}) = \hat{t}_i^*$.

Proof:

Note that, the conditions in (5.47)–(5.50) are necessary for any NE $\boldsymbol{m}_{\mathcal{N}}^*$ of the game induced by the game form of Section 5.2.3 and users' utilities (5.2), to result in the allocation $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*)$ (see (5.9), (5.11) and (5.26)). Therefore, the set of solutions of (5.47)–(5.50), if such a set exists, is a superset of the set of all NE corresponding to the above game that result in $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*)$. Below we show that the solution set of (5.47)–(5.50) is in fact exactly the set of all NE that result in $(\hat{\boldsymbol{a}}_{\mathcal{N}}^*, \hat{\boldsymbol{t}}_{\mathcal{N}}^*)$.

To prove this, we first show that the set of relations in (5.47)–(5.50) do have a solution. Notice that (5.47) and (5.49) are satisfied by setting for each $i\in\mathcal{N}$, ${}^k a_i^* = \hat{a}_i^* \;\forall\; k\in\mathcal{C}_i$. Notice also that for each $j\in\mathcal{N}$, the sum over $i\in\mathcal{C}_j$ of the right hand side of (5.48) is 0. Therefore, for each $j\in\mathcal{N}$, (5.48) has a solution in ${}^i\pi_j^*, i\in\mathcal{C}_j$. Furthermore, for any solution ${}^i\pi_j^*, i\in\mathcal{C}_j, j\in\mathcal{N}$, of (5.48), ${}^i\pi_j^* + c, i\in\mathcal{C}_j, j\in\mathcal{N}$, where c is some constant, is also a solution of (5.48). Consequently, by appropriately choosing c, we can select a solution of (5.48) such that (5.50) is satisfied.

It is clear from the above discussion that (5.47)–(5.50) have multiple solutions. We now show that the set of solutions $\boldsymbol{m}_{\mathcal{N}}^*$ of (5.47)–(5.50) is the set of NE that result in $(\hat{\boldsymbol{a}}^*_{\mathcal{N}}, \hat{\boldsymbol{t}}^*_{\mathcal{N}})$. From Claim 5.5, (5.40) can be equivalently written as

$$\begin{aligned}\hat{\boldsymbol{a}}^*_{\mathcal{R}_i} &= \underset{\hat{\boldsymbol{a}}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}}{\arg\max} - \sum_{j \in \mathcal{R}_i} l^*_{ij}\ \hat{a}_j + u_i(\hat{\boldsymbol{a}}_{\mathcal{R}_i}) - \left[\frac{1 - I_{\mathcal{A}_i}(a_i)}{I_{\mathcal{A}_i}(a_i)}\right] \\ &= \underset{\hat{\boldsymbol{a}}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\ u^A_i\Big(\hat{\boldsymbol{a}}_{\mathcal{R}_i}, \sum_{j \in \mathcal{R}_i} l^*_{ij}\ \hat{a}_j\Big), \quad i \in \mathcal{N}.\end{aligned} \tag{5.51}$$

Substituting $\hat{a}_j|\mathcal{C}_j| - \sum_{k \in \mathcal{C}_j \setminus \{i\}} {}^k a^*_j = {}^i a_j$ for each $j \in \mathcal{R}_i$, $i \in \mathcal{N}$, in (5.51) we obtain

$$\begin{aligned}&{}^i\boldsymbol{a}^*_{\mathcal{R}_i} = \underset{{}^i\boldsymbol{a}_{\mathcal{R}_i} \in \mathbb{R}^{|\mathcal{R}_i|}}{\arg\max}\ u^A_i\Bigg(\Big(\frac{1}{|\mathcal{C}_j|}\big({}^i a_j + \sum_{k \in \mathcal{C}_j \setminus \{i\}} {}^k a^*_j\big)\Big)_{j \in \mathcal{R}_i}, \sum_{j \in \mathcal{R}_i} l^*_{ij}\ \frac{1}{|\mathcal{C}_j|}\big({}^i a_j + \sum_{k \in \mathcal{C}_j \setminus \{i\}} {}^k a^*_j\big)\Bigg), \\ &\hspace{30em} i \in \mathcal{N}.\end{aligned} \tag{5.52}$$

Because of (5.49), (5.52) also implies that

$$\begin{aligned}&({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}) = \\ &\quad \underset{\left({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}\right) \in \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}}{\arg\max}\ u^A_i\Bigg(\Big(\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\ \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)_{j \in \mathcal{R}_i}, \\ &\sum_{j \in \mathcal{R}_i} l^*_{ij}\ \hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}),\ \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) - \sum_{j \in \mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a^*_j\right)^2\Bigg), \quad i \in \mathcal{N}.\end{aligned} \tag{5.53}$$

Furthermore, since u^A_i is strictly decreasing in the tax (see (5.2)), (5.53) also implies the following:

$$\begin{aligned}&({}^i\boldsymbol{a}^*_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}^*_{\mathcal{R}_i}) = \\ &\quad \underset{\left({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}\right) \in \mathbb{R}^{|\mathcal{R}_i|} \times \mathbb{R}_+^{|\mathcal{R}_i|}}{\arg\max}\ u^A_i\Bigg(\Big(\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big)\Big)_{j \in \mathcal{R}_i}, \sum_{j \in \mathcal{R}_i} l^*_{ij}\hat{a}_j\big(({}^i\boldsymbol{a}_{\mathcal{R}_i}, {}^i\boldsymbol{\pi}_{\mathcal{R}_i}), \boldsymbol{m}^*_{\mathcal{C}_j}/i\big) \\ &+ \sum_{j \in \mathcal{R}_i} {}^i\pi_j\left({}^i a_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j\right)^2 - \sum_{j \in \mathcal{R}_i} {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi^*_j\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a^*_j - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a^*_j\right)^2\Bigg), \quad i \in \mathcal{N}.\end{aligned} \tag{5.54}$$

Eq. (5.54) implies that, if the message exchange and allocation is done according to the game form presented in Section 5.2.3, then user $i, i \in \mathcal{N}$, maximizes its utility at

$\boldsymbol{m}_i^*$ when all other users $j \in \mathcal{N}\backslash\{i\}$ choose their respective messages $\boldsymbol{m}_j^*, j \in \mathcal{N}\backslash\{i\}$. This, in turn, implies that a message profile $\boldsymbol{m}_\mathcal{N}^*$ that is a solution to (5.47)–(5.50) is a NE of the game induced by the aforementioned game form and the users' utilities (5.2). Furthermore, it follows from (5.47)–(5.50) that the allocation at $\boldsymbol{m}_\mathcal{N}^*$ is

$$
\begin{aligned}
\hat{a}_i(\boldsymbol{m}_{\mathcal{C}_i}^*) &= \frac{1}{|\mathcal{C}_i|}\sum_{k\in\mathcal{C}_i} {}^k a_i^* = \hat{a}_i^*, \quad i \in \mathcal{N},\\
l_{ij}(\boldsymbol{m}_{\mathcal{C}_j}^*) &= {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}\pi_j^* = l_{ij}^*, \quad j \in \mathcal{R}_i,\ i \in \mathcal{N},\\
\hat{t}_i\big((\boldsymbol{m}_{\mathcal{C}_j}^*)_{j\in\mathcal{R}_i}\big) &= \sum_{j\in\mathcal{R}_i} l_{ij}(\boldsymbol{m}_{\mathcal{C}_j}^*)\hat{a}_j(\boldsymbol{m}_{\mathcal{C}_j}^*) + {}^i\pi_j^*\left({}^i a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^*\right)^2\\
&\quad - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}\pi_j^*\left({}^{\mathcal{C}_{j(\mathcal{I}_{ij}+1)}}a_j^* - {}^{\mathcal{C}_{j(\mathcal{I}_{ij}+2)}}a_j^*\right)^2\\
&= \sum_{j\in\mathcal{R}_i} l_{ij}^*\hat{a}_i^* = \hat{t}_i^*, \quad i \in \mathcal{N}.
\end{aligned}
\tag{5.55}
$$

From (5.55) it follows that the set of solutions $\boldsymbol{m}_\mathcal{N}^*$ of (5.47)–(5.50) is exactly the set of NE that result in $(\hat{\boldsymbol{a}}_\mathcal{N}^*, \hat{\boldsymbol{t}}_\mathcal{N}^*)$. This completes the proof of Claim 5.6 and hence the proof of Theorem 5.2. $\square$

CHAPTER 6

Conclusion

6.1 Summary

In this thesis we investigated decentralized resource allocation in wireless and large-scale networks. Initially we studied the problem of power allocation for wireless networks where each user's transmissions create interference to all network users, and each user has only partial information about the network. We investigated the problem under two scenarios; the realization theory scenario and the implementation theory scenario. Under the realization theory scenario, we formulated the power allocation problem as an allocation problem with externalities, and developed a decentralized optimal power allocation algorithm that (i) preserves the private information of the users; and (ii) converges to the optimal centralized power allocation. Under the implementation theory scenario, we formulated the power allocation problem as a public good allocation problem, and we developed a game form that (i) implements in Nash equilibria (NE) the optimal allocations of the corresponding centralized power allocation problem; (ii) is individually rational; and (iii) results in budget balance at all NE and off equilibria. Later we generalized the model investigated in the aforementioned power allocation problems to study resource allocation in large-scale networks where the actions of each user affect the utilities of an arbitrary subset of network users. This generalization was motivated by several applications including power allocation in large-scale wireless networks where the transmissions of each user create interference to only a subset of network users. We developed a formal model to study resource allocation problems in large-scale networks with above characteristics that resemble neither public good allocation nor private good allocation. We formulated two resource allocation problems for the large-scale network model; one for the realization theory scenario, and the other for the implementation theory scenario. For the realization problem we developed a decentralized resource allocation algorithm using the principles of mechanism design. The algorithm has the following properties: (i) it preserves the private information of the users; and (ii) it converges to the optimal centralized resource allocation. For the implementation problem we developed a game form that (i) implements in NE the optimal allocations of the corresponding centralized resource allocation problem; (ii) is individually rational; and (iii) results in budget balance at all NE and off equilibria.

In the following sections we conclude with some reflections on the solution approach and the solution of the resource allocation problems presented in this thesis, and a discussion on possible future directions.

6.2 Reflections

6.2.1 Integrating the literature on decentralized resource allocation

Decentralized decision making (control, resource allocation, etc.) problems have been studied for decades by researchers in various fields; economics, political science, management science, transportation engineering, etc. In electrical and computer engineering, decentralized control received major interest initially in the 70's and then in the 90's with the proliferation of the internet, mobile communication, sensor networks, and electronic commerce. Because of the diversity of applications and the different requirements of different applications, research on decentralized decision making developed quite independently in engineering, social science and management science. With their diverse literature and research developments these fields can gain tremendously from one another. For example, the mechanism design literature from economics can provide methodologies for the design of network objective (social welfare) maximizing decentralized mechanisms; on the other hand the distributed computing and algorithmic game theory literature from computer science can provide fast algorithms to compute equilibria and allocations specified by mechanism design. Such an integration of research efforts from various fields is being done in applications such as electronic commerce. However, an effort to correlate the literatures and learn from the developments in other fields requires a broader understanding of all these fields beyond the specific knowledge of a particular field. In this thesis we provided a step towards relating the ideas from mechanism design with resource allocation problems in wireless communication networks. Specifically, in chapter 2 we illustrated how the power allocation problem in wireless networks with interference can be formulated as an allocation problem with externalities, and in chapter 3 we illustrated how a similar power allocation problem can be formulated as a public good allocation problem. Based on the above formulations we designed decentralized optimal power allocation mechanisms using the principles of mechanism design.

6.2.2 Insights from mechanism design

The approach of mechanism design provided us with insights into the fundamental nature of the resource allocation problems investigated in this thesis. With these insights we harnessed the fundamental characteristics of these problems and developed optimal resource allocation mechanisms for them. Specifically, by identifying the similarities between power allocation in wireless networks with interference and allocation in the presence of externalities or public good allocation, we recognized that any decentralized mechanism in which every user pays the same price for a given power allocation (as in the mechanisms previously proposed in the wireless networks literature) cannot obtain optimal power allocations in the presence of interference. The properties of decentralized mechanisms for public good allocation (or allocation in the presence of externalities) in the mechanism design literature gave us inspiration for the design of decentralized mechanisms for optimal power allocation in wireless networks where each user's transmission affects the utility of all network users. The insights obtained from these mechanisms also helped us to characterize the properties of decentralized resource allocation mechanisms for large-scale networks that we investigated in the second half of the thesis. The basic difference between the power allocation problems we studied in chapters 2 and 3 and the problems of resource allocation in large-scale networks we studied in chapters 4 and 5 is the following. In the large-scale networks each user's utility is directly affected by the actions of only a subset of network users, whereas in the power allocation problems of chapters 2 and 3, each user's utility is affected by the actions of all network users. To develop the decentralized resource allocation mechanisms presented in chapters 4 and 5, we borrowed from public good allocation mechanisms the design principles that capture the interactions among the users. We then applied these principles to appropriately reflect the interactions of the users in the generalized large-scale model.

6.2.3 Contribution to mechanism design

As we discussed in Section 6.2.1 research on decentralized resource allocation in different fields can potentially contribute to the corresponding literature in all of these fields. In this thesis we initially used the principles of mechanism design to address resource allocation problems in engineering networks. In chapters 2 and 3 we derived ideas for the design of decentralized mechanisms from the existing mechanism design literature. Then, in chapters 4 and 5 we extended the ideas of chapters 2 and 3 to develop decentralized resource allocation mechanisms for large-scale networks. To the best of our knowledge, the problem formulations and the solutions presented in chapters 4 and 5 is the first attempt in the engineering literature as well as the mechanism design literature to investigate resource allocation problems for the proposed large-scale network model in the framework of the realization theory and implementation theory components of mechanism design. The large-scale network models studied in these chapters are different from the traditional models studied in mechanism design because in these models, the number of users and the network structure are not common knowledge among all users. Furthermore, the decentralized mechanisms presented in chapters 4 and 5 have features that are different from those of standard mechanism design. Specifically, in the mechanisms presented in chapters 4 and 5, each user communicates its message only to its neighbors (and to the network operator) in the network, whereas in the mechanism design literature a general assumption is that the users broadcast their messages to all the users in the network. [6.1] Therefore, we believe that the formulation of resource allocation problems for the large-scale network models in the framework of realization and implementation theory, and the decentralized resource allocation mechanisms presented

[6.1] One exception is the work of Marshak and Reichelstein [34, 35] but their model, objective and approach to resource allocation is different from ours.

in chapters 4 and 5 are not only new results in the engineering literature but are also a contribution to the mechanism design literature.

6.3 Future directions

In this section we discuss some future directions for research.

- **Computationally efficient algorithms for message exchange:** Message exchange is an important component of decentralized resource allocation mechanisms that aim to achieve optimal centralized allocations. In many systems, specially engineering systems, resource allocations must be determined at very small time scales. Therefore, in order to implement decentralized mechanisms, there should be fast algorithms to generate and communicate messages. Development of such algorithms is an important aspect of research on decentralized resource allocation.

- **Computing Nash equilibria:** In our solution to the resource allocation problems in chapters 3 and 5, we have implementation in NE and we have obtained characterization of the NE. However, at present we do not have an algorithm for the computation of these equilibria. For these problems, best response dynamics do not guarantee convergence to NE because the games induced by the proposed game forms are not, in general, supermodular. For development of efficient mechanisms that can compute NE, there can be two different approaches. (i) The development of algorithms that guarantee convergence to NE of the games constructed in chapters 3 and 5. (ii) The development of alternative mechanisms/game forms that lead to supermodular games. Both of the above problems are open research problems of paramount importance. A relevant work that investigates the latter approach for public good allocation is [10].

- **Dynamic mechanism design:** In this thesis we focused on static resource allocation problems where the system characteristics do not change with time. The development of realization and implementation mechanisms for wireless/large-scale networks under dynamic situations, where the system characteristics change during the determination of resource allocation, are open research problems. Resource allocation mechanisms for these systems must take into account the dynamics of the system and can be addressed using dynamic game theory and dynamic mechanism design. Some important results in this direction can be found in [26, 3, 42, 6, 5, 9].

- **Admission and topology control:** The network models we studied in this thesis assume a given set of network users and a given network topology. In systems such as cognitive radio networks, the set of network users and the network topology must be determined as part of network objective maximization. This generates admission and topology control problems or network formation problems. Many of these problems are open research problems. An exposition to this class of problems and important results are in [27, 15].

- **Networks with multiple network operators:** In this thesis we investigated networks with single network operator. Decentralized resource allocation problems with multiple network operators, e.g. those in wireless networks with multiple cells, are open research problems. With multiple network operators the budget balance conditions of chapters 3 and 5 will change. Furthermore, there may be competition among the network operators; hence, incentive provision for network operators may be required in these systems.

- **Problems with non-convex objectives:** In this thesis we addressed resource allocation problems where the users' utilities and the network objectives are convex (or concave). In many real systems the network objective or the users' utilities are not convex. Problems with non-convex objectives are harder to solve as they do not have a general structure or methodology for the solution. Hence, these problems have not received much attention in the mechanism design literature. Categorizing non-convex problems that can be realized or implemented in various solution concepts is a very important fundamental problem of mechanism design. Developing decentralized mechanisms that realize or implement non-convex network objectives is another problem of fundamental importance. A step in this direction are the results reported in [29].

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