SI 563 Lecture 2

Dominance and Nash Equilibrium

Professor Yan Chen
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Some material in this lecture drawn from http://gametheory.net/lectures/level.pl

Agenda

- Dominance and best response
  - » Dominance
  - » Best response
  - » Dominant strategy equilibrium
- Rationalizability and iterated dominance
  - » Dominance-solvable equilibrium
- Nash equilibrium
  - » Pure strategy Nash equilibrium
  - » Mixed strategy Nash equilibrium

Dominance and Best Response

(Watson Chapter 6)

Example: Prisoners' Dilemma

Tchaikovsky

Conductor

	Confess	Not Confess
Confess	**-5, -5**	**0, -15**
Not Confess	**-15, 0**	**-1, -1**

Solving a strategic form game: Best response (reply)

- A strategy is a *best response (reply)* to a particular strategy of another player, if it gives the highest payoff against that particular strategy
- How to find best responses
  - Discrete strategy space: for each of opponent's strategy, find strategy yielding best payoff
  - Continuous strategy space: use calculus

Best Response: Prisoners' Dilemma

Tchaikovsky

Conductor

	Confess	Not Confess
Confess	-5, -5	0, -15
Not Confess	-15, 0	-1, -1

Best response: Battle of Sexes

1 \ 2	A	B
A	**$\underline{1}$, $\underline{3}$**	**0, 0**
B	**0, 0**	**$\underline{3}$, $\underline{1}$**

Solving strategic form games: Dominance

- Confess is a best reply regardless of what the other player chooses
- Strategy s_1 *strictly dominates* another strategy s_2, if the payoff to s_1 is strictly greater than the payoff to s_2, regardless of which strategy is chosen by the other player(s). Or
  $u_i(s_1, t) > u_i(s_2, t)$, for all t.
- Strategy s_2 : *strictly dominated* strategy

Examples of dominance:

1 \ 2	L	R
U	**<u>2</u>, 3**	**<u>5</u>, 0**
D	**1, 0**	**4, 3**

For player 1, U strictly dominates D.

Weak Dominance

- **Strategy s_1 *weakly dominates* another strategy s_2, if the payoff to s_1 is at least as good as the payoff to s_2, regardless of which strategy is chosen by the other player(s). Or**

≥

$$u_i(s_1, t) \geq u_i(s_2, t) \text{ for all } t, \text{ and}$$
$$u_i(s_1, t') > u_i(s_2, t'), \text{ for some } t'$$

- **In this case, strategy s_2 is called a *weakly dominated* strategy.**

Example of strict and weak dominance:

1 \ 2	L	C	R
U	**8, 3**	**0, 4**	**4, 4**
M	**4, 2**	**1, 5**	**5, 3**
D	**3, 7**	**0, 1**	**2, 0**

For player 1, M strictly dominates D,
U weakly dominates D.
Player 2: C weakly dominates R.

Example of dominance:

1 \ 2	L	R
U	**4, 1**	**0, 2**
M	**0, 0**	**4, 0**
D	**1, 3**	**1, 2**

Randomize between U and M dominates D, or D is dominated by the mixed strategy (½, ½, 0).

Dominant strategy equilibrium

- **If every player has a dominant strategy, the game has a *dominant strategy equilibrium* (solution).**
- **Dominant strategy axiom: if a player has a dominant strategy, she will use it.**
- **Problem with dominant strategy equilibrium: in many games there does not exist one**

DSE:
Prisoners' Dilemma

Tchaikovsky

Conductor

	Confess	Not Confess
Confess	**-5, -5**	**0, -15**
Not Confess	**-15, 0**	**-1, -1**

(Confess, Confess) is a dominant strategy equilibrium.

Is this a good outcome?

- So, both confess
- Question: They both would be better off with (Not, Not). So why don't they play "Not Confess"?

Tchaikovsky

Conductor

	Confess	Not Confess
Confess	-5, -5	0, -15
Not Confess	-15, 0	-1, -1

Efficiency and equilibrium

- **Game equilibrium is a characterization of the outcome of *individually rational behavior***
  - **Because of strategic interactions, rational behavior** does not always **lead to outcomes that are mutually the best**
- **Dominant strategy equilibrium in Prisoner's Dilemma: (Confess, Confess)**
- **But this is not socially efficient: both players are better off with (Not Confess, Not Confess)**
- **Many applications**
  - **Arms race**
  - **Tragedy of commons**

Pareto Optimality

- A solution is **Pareto optimal** if and only if there is no other solution that is
  (1) Better for at least one agent
  (2) No worse for everyone else
- A mild (weak) criterion for social efficiency
- The Prisoner's Dilemma solution is *not* Pareto optimal

Example: (Low, Low) is DSE

Firm B

Firm A

	Low price	High price
Low price	0, 0	50, -10
High price	-10, 50	10, 10

What to do when equilibrium is inefficient?

- Can't always be improved (arms race not an easy problem!)
- Opportunities:
  - Collude / cooperate (sometimes illegal!)
    - OPEC
    - marriage
    - Might involve side payments if not win-win
  - Design systems to increase trust
  - Repeated interactions
    - Build trust
    - Or create opportunities for punishment!

Rationalizability and Iterated Dominance

(Watson Chapter 7)

Dominance Solvability

- **In some games, there might not be a dominant strategy, but there are** *dominated strategies* **(i.e., bad)**
- **If we can reach a unique strategy vector by** iterated elimination of dominated strategies**, the game is said to be** ***dominance solvable*****.**

Example:
Playing mind games

1 \ 2	X	Y	Z
A	**3,3**	**0,5**	**0,4**
B	**0,0**	**3,1**	**1,2**

If you are player 1, which strategy should you play?

FIGURE 7.2 (a)
Iterative removal of strictly dominated strategies.

1 \ 2	X	Y	Z
A	**3,3**	**0,5**	**0,4**
B	**0,0**	**3,1**	**1,2**

FIGURE 7.2 (b)
Iterative removal of strictly dominated strategies.

1 \ 2	X	Y	Z
A	**~~3,3~~**	**~~0,5~~**	**~~0,4~~**
B	**~~0,0~~**	**3,1**	**1,2**

FIGURE 7.2 (c)
Iterative removal of strictly dominated strategies.

1 \ 2	X	Y	Z
A	**3,3**	~~**0,5**~~	~~**0,4**~~
B	**0,0**	**3,1**	**1,2**

Rationalizable Strategies

- **The set of strategies that survive iterated dominance is called the** *rationalizable strategies*
- **Logic of rationalizability depends on**
  - **Common knowledge of rationality**
  - **Common knowledge of the game**

Example: rationalizability/iterated dominance

1 \ 2	L	C	R
U	**5,1**	**0,4**	**1,0**
M	**3,1**	**0,0**	**3,5**
D	**3,3**	**4,4**	**2,5**

L is strictly dominated by (0, ½, ½), etc.
Set of rationalizable strategies is {(M, R)}.

Strategic Uncertainty

- Rationalizability requires players' beliefs and behavior be consistent with common knowledge of rationality
- It does not require that their beliefs be correct
- It does not help solve the strategic uncertainty in coordination games

Strategic Uncertainty: Battle of the Sexes

1 \ 2	Opera	Movie
Opera	**2, 1**	**0, 0**
Movie	**0, 0**	**1, 2**

Coordination game: want to go to an event together, with slightly different preferences

Any dominant strategies?
Any dominated strategies?

Example:
Stag hunt

1 \ 2	Stag	Hare
Stag	**5,5**	**0,4**
Hare	**4,0**	**4,4**

Any dominant strategies?
Any dominated strategies?
Pareto optimal outcomes?

Facilitate Coordination

- **Focal point**
  - **Schelling : *The strategy of conflict***
  - **Rome**
- **Institutions, rules, norms**
- **Communication**

Nash Equilibrium

(Watson Chapters 9, 11)

Pure Strategy Nash Equilibrium

- **A set of strategies forms a** ***Nash equilibrium*** **if the strategies are** ***best replies to each other***
- **Recall: A strategy is a** ***best reply*** **to a particular strategy of another player, if it gives the highest payoff against that particular strategy**

Hawk-Dove

- In this situation, the players can either choose aggressive (hawk) or accommodating strategies
- From each player's perspective, preferences can be ordered from best to worst:
  - Hawk – Dove
  - Dove – Dove
  - Dove – Hawk
  - Hawk – Hawk
- The argument here is that two aggressive players wipe out all surplus

Hawk-Dove Analysis

- We can draw the game table as:
- Best Responses:
  - Reply Dove to Hawk
  - Reply Hawk to Dove
- Equilibrium
  - There are two equilibria
  - (Hawk, Dove)
  - (Dove, Hawk)

	Hawk	Dove
Hawk	0, 0	<u>4</u>, <u>1</u>
Dove	<u>1</u>, <u>4</u>	2, 2

FIGURE 9.2 (1)
Equilibrium and rationalizability in the classic normal forms

1 \ 2	H	T
H	**1,-1**	**-1,1**
T	**-1,1**	**1,-1**

Matching Pennies

FIGURE 9.2 (2)
Equilibrium and rationalizability in the classic normal forms

1 \ 2	Not Confess	Confess
Not Confess	**2,2**	**0,$\underline{3}$**
Confess	**$\underline{3}$,0**	**$\underline{1}$,$\underline{1}$**

Prisoners' Dilemma

FIGURE 9.2 (3)
Equilibrium and rationalizability in the classic normal forms

1 \ 2	Opera	Movie
Opera	**$\underline{2},\underline{1}$**	**0,0**
Movie	**0,0**	**$\underline{1},\underline{2}$**

Battle of the Sexes

FIGURE 9.2 (4)
Equilibrium and rationalizability in the classic normal forms

1 \ 2	H	D
H	**0,0**	**3,1**
D	**1,3**	**2,2**

Hawk-Dove/Chicken

FIGURE 9.2 (5)
Equilibrium and rationalizability in the classic normal forms

1 \ 2	A	B
A	**1,1**	**0,0**
B	**0,0**	**1,1**

Coordination

FIGURE 9.2 (6)
Equilibrium and rationalizability in the classic normal forms

2
1

	A	B
A	**$\underline{2},\underline{2}$**	**0,0**
B	**0,0**	**$\underline{1},\underline{1}$**

Pareto Coordination

FIGURE 9.2 (7)
Equilibrium and rationalizability in the classic normal forms

S \ D	P	D
P	**4,2**	**$\underline{2}$,$\underline{3}$**
D	**$\underline{6}$,-1**	**0,$\underline{0}$**

Pigs

FIGURE 9.3 (a)
Determining Nash equilibria.

1 \ 2	X	Y	Z
J	**5,6**	**3,7**	**0,4**
K	**8,3**	**3,1**	**5,2**
L	**7,5**	**4,4**	**5,6**
M	**3,5**	**7,5**	**3,3**

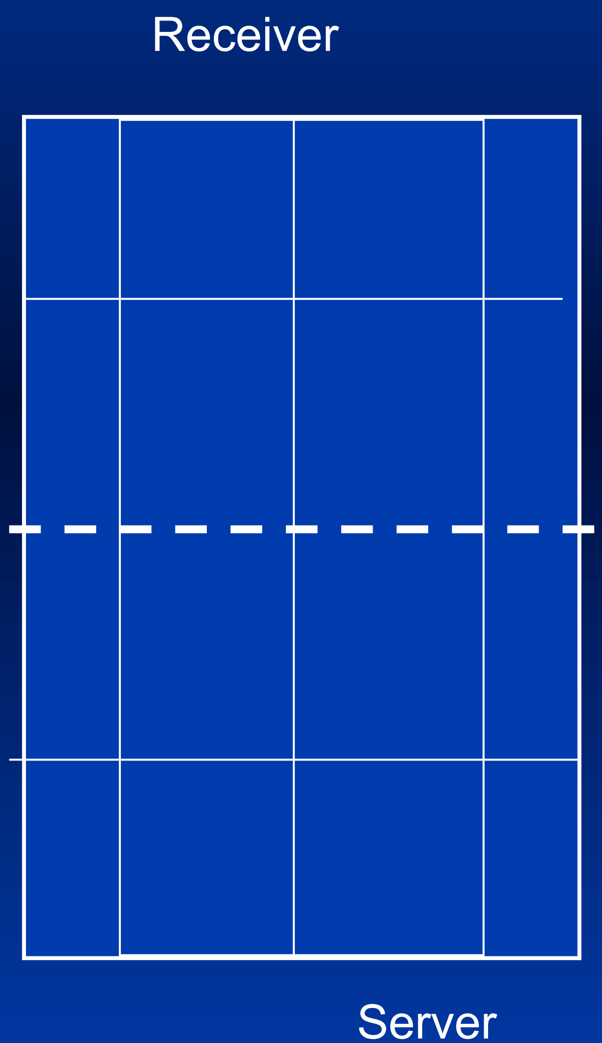
FIGURE 9.3 (b)
Determining Nash equilibria.

1 \ 2	X	Y	Z
J	**5,6**	**3,$\underline{7}$**	**0,4**
K	**$\underline{8}$,$\underline{3}$**	**3,1**	**$\underline{5}$,2**
L	**7,5**	**4,4**	**$\underline{5}$,$\underline{6}$**
M	**3,$\underline{5}$**	**$\underline{7}$,$\underline{5}$**	**3,3**

Mixed Strategies

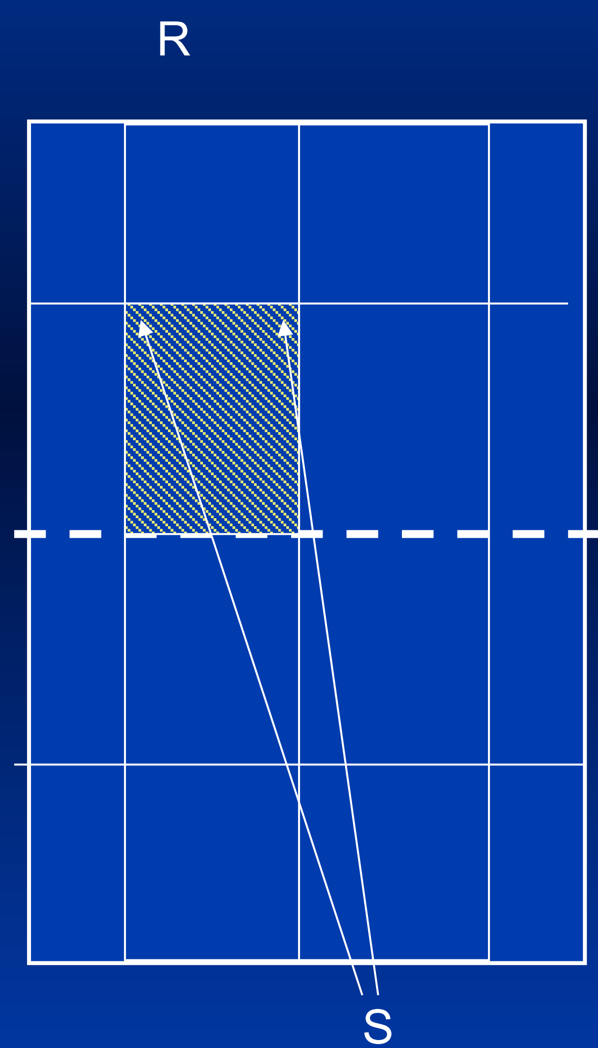
(Watson Chapter 11)

Tennis Anyone?



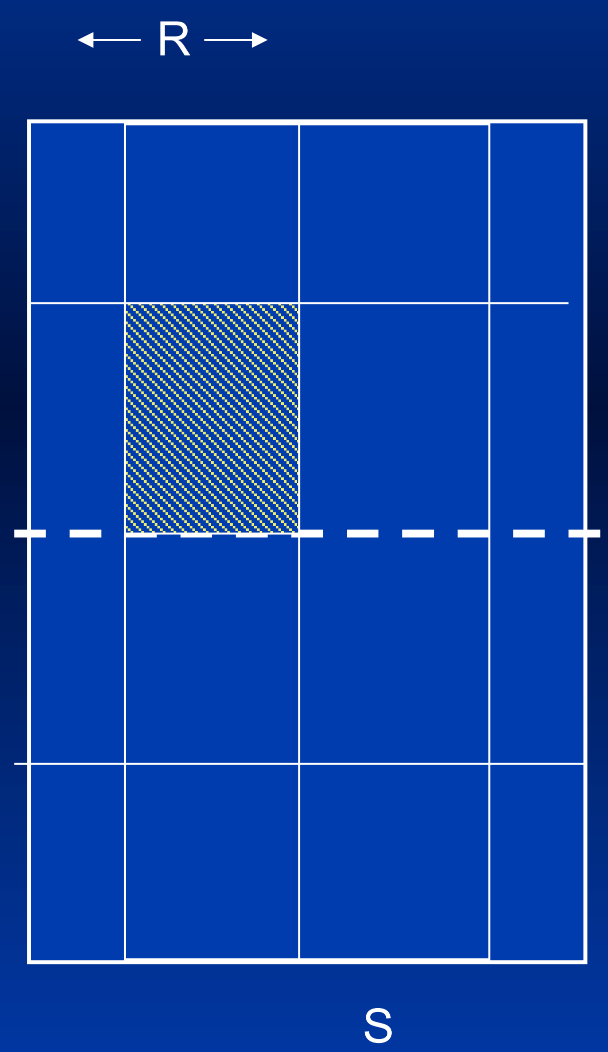
Source: John Morgan, gametheory.net

Serving



Source: John Morgan, gametheory.net

Serving



Source: John Morgan, gametheory.net

The Game of Tennis

- Server chooses to serve either left or right
- Receiver defends either left or right
- Better chance to get a good return if you defend in the area the server is serving to

Source: John Morgan, gametheory.net

Game Table

	Receiver		
		Left	Right
Server	Left	¼, ¾	¾, ¼
	Right	¾, ¼	¼, ¾

Source: John Morgan, gametheory.net

Game Table

	Receiver		
		Left	Right
	Left	¼, ¾	¾, ¼
Server	Right	¾, ¼	¼, ¾

For server: Best response to defend left is to serve right
Best response to defend right is to serve left
For receiver: Just the opposite

Source: John Morgan, gametheory.net

Nash Equilibrium

- Notice that there are *no* mutual best responses in this game.
- This means there are no Nash equilibria in pure strategies
- But games like this always have at least one Nash equilibrium
- What are we missing?

Source: John Morgan, gametheory.net

Extended Game

- Suppose we allow each player to choose *randomizing strategies*
- For example, the server might serve left half the time and right half the time.
- In general, suppose the server serves left a fraction *p* of the time
- What is the receiver's best response?

Source: John Morgan, gametheory.net

Calculating Best Responses

- Clearly if $p = 1$, then the receiver should defend to the left
- If $p = 0$, the receiver should defend to the right.
- The expected payoff to the receiver is:
  - » $p\ \ 3/4 + (1 - p)\ \ 1/4$ if defending left
  - » $p\ \ 1/4 + (1 - p)\ \ 3/4$ if defending right
- Therefore, she should defend left if
  - » $p\ \ 3/4 + (1 - p)\ 1/4 > p\ \ 1/4 + (1 - p)\ \ 3/4$

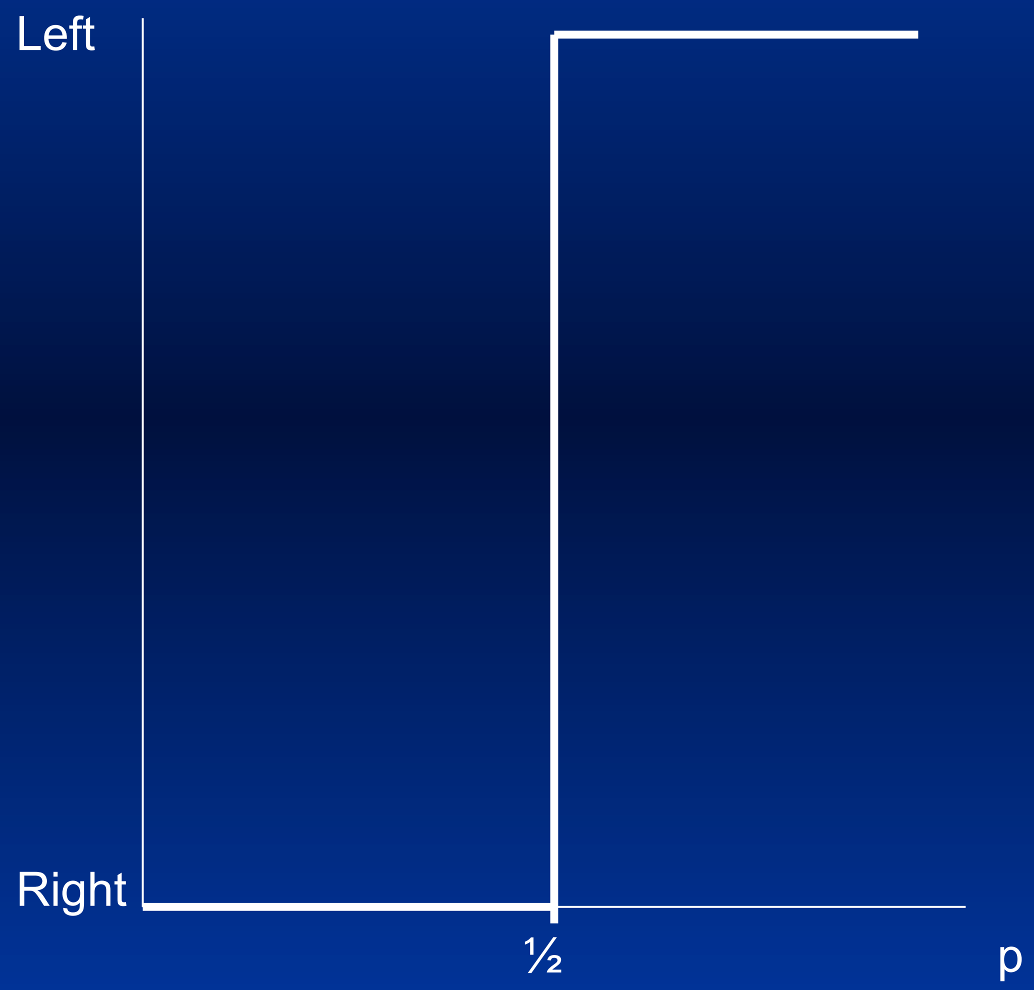
Source: John Morgan, gametheory.net

When to Defend Left

- We said to defend left whenever:
  - » $p \times \frac{3}{4} + (1 - p) \times \frac{1}{4} > p \times \frac{1}{4} + (1 - p) \times \frac{3}{4}$
- Rewriting
  - » $p > 1 - p$
- *Or*
  - » $p > \frac{1}{2}$

Source: John Morgan, gametheory.net

Receiver’s Best Response



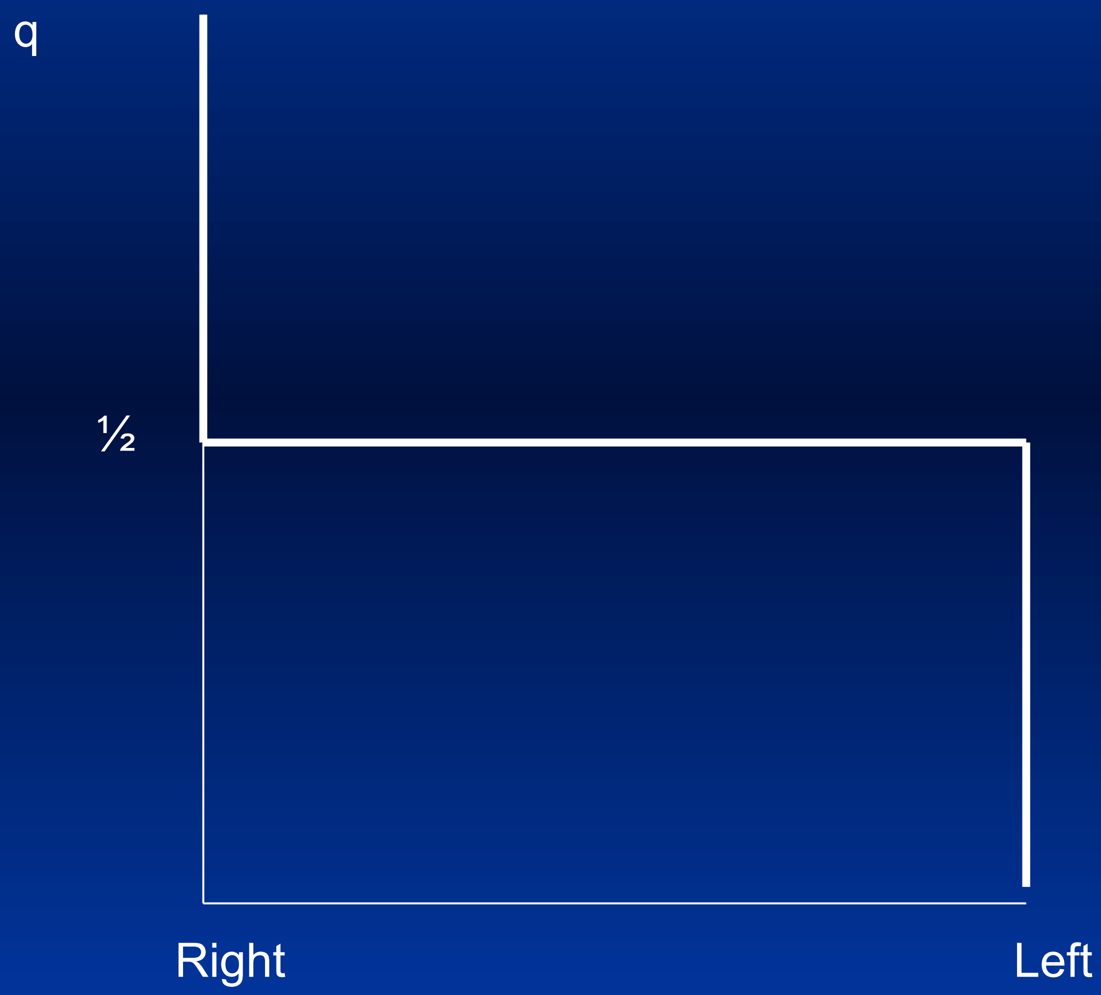
Source: John Morgan, gametheory.net

Server's Best Response

- Suppose that the receiver goes left with probability q.
- Clearly, if $q = 1$, the server should serve right
- If $q = 0$, the server should serve left.
- More generally, serve left if
  - » $\frac{1}{4}\ q + \frac{3}{4}\ (1 - q) > \frac{3}{4}\ q + \frac{1}{4}\ (1 - q)$
- Simplifying, he should serve left if
  - » $q < \frac{1}{2}$

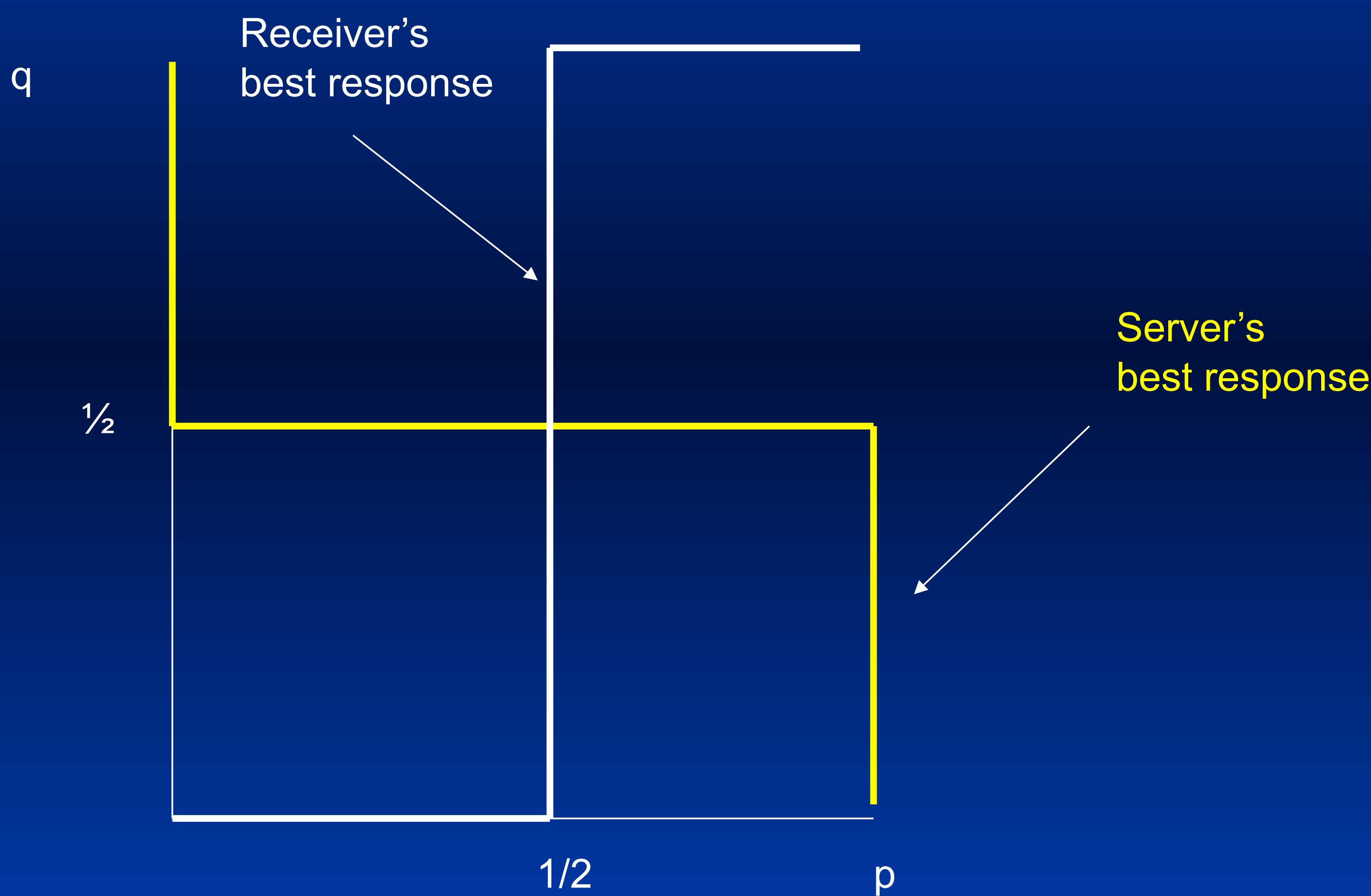
Source: John Morgan, gametheory.net

Server's Best Response



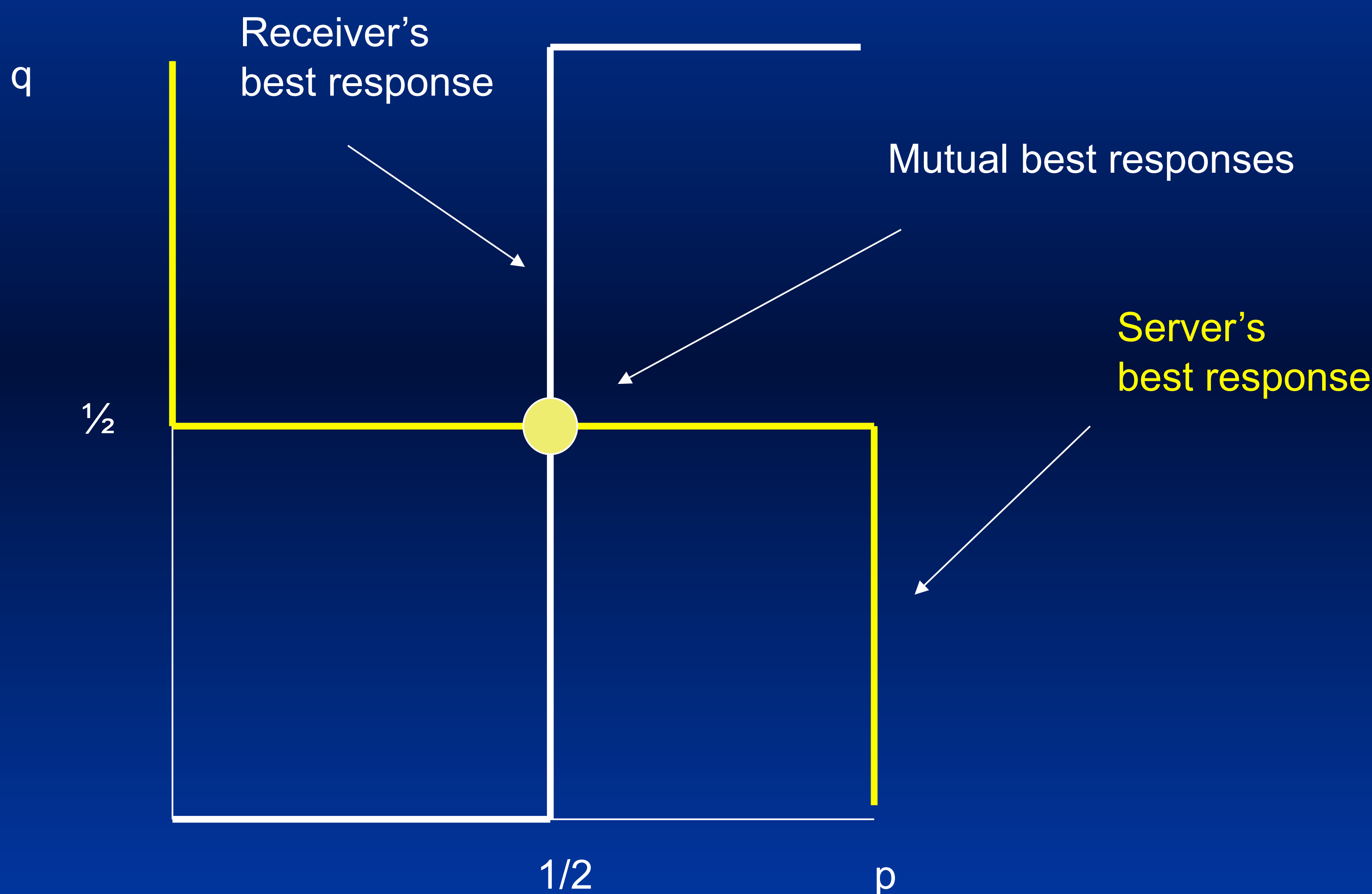
Source: John Morgan, gametheory.net

Putting Things Together



Source: John Morgan, gametheory.net

Equilibrium



Source: John Morgan, gametheory.net

Mixed Strategy Equilibrium

- A mixed strategy equilibrium is a pair of mixed strategies that are mutual best responses
- In the tennis example, this occurred when each player chose a 50-50 mixture of left and right.

Source: John Morgan, gametheory.net

General Properties of Mixed Strategy Equilibria

- A player chooses his strategy so as to make his *rival* indifferent
- A player earns the same expected payoff for each pure strategy chosen with positive probability
- Funny property: When a player's own payoff from a pure strategy goes up (or down), his mixture does not change

Source: John Morgan, gametheory.net

Does Game Theory Work?

- Walker and Wooders (2002)
  - Ten grand slam tennis finals
  - Coded serves as left or right
  - Determined who won each point
- Tests:
  - Equal probability of winning
    - » Pass
  - Serial independence of choices
    - » Fail

Source: John Morgan, gametheory.net

Find all NE:
Battle of the Sexes

	Chris		
		Opera	Boxing
Pat	Opera	3,1	0,0
	Boxing	0,0	1,3

Source: John Morgan, gametheory.net

Find all NE: Hawk-Dove

	Krushchev		
		Hawk	Dove
Kennedy	Hawk	0, 0	4, 1
	Dove	1, 4	2, 2

Source: John Morgan, gametheory.net

Highlights

- Dominance
- Rationalizability and iterated dominance
- Nash equilibrium
  - Pure strategy NE
  - Mixed strategy NE

Homework Assignment

- Chapter 6: #1
- Chapter 7: #1, 2, 3
- Chapter 11: #4, 6