SI 563 Lecture 6

Normal Form Games of Incomplete Information

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Some material in this lecture drawn Maggie Levenstein, Ross School of Business

Agenda

– Games of incomplete information

– Random events and incomplete information

– Risk and incentives in contracting

– Bayesian Nash equilibrium

– Lemons and Auctions

Games of Incomplete Information

- To say that a game is of *incomplete information* is to say something about what is known about the circumstances under which the game is played
- Games having *moves of nature* that generate asymmetric information between the players
- *Type*: different moves of nature that a single player privately observes

Examples

- **Online auctions**
  - unrealistic to assume that bidders know the other bidders' valuations or risk attitudes
- **Russian roulette**
  - Both players have an incentive to pretend to be more reckless than they actually are
- **Oligopoly**
  - Unrealistic to assume that one firm knows the cost structure of the other firm

Main Tools

- Harsanyi's theory of incomplete information offers a means to get a handle on such matters: a technique for completing a structure in which information is incomplete
- Main technique: expected utility calculation

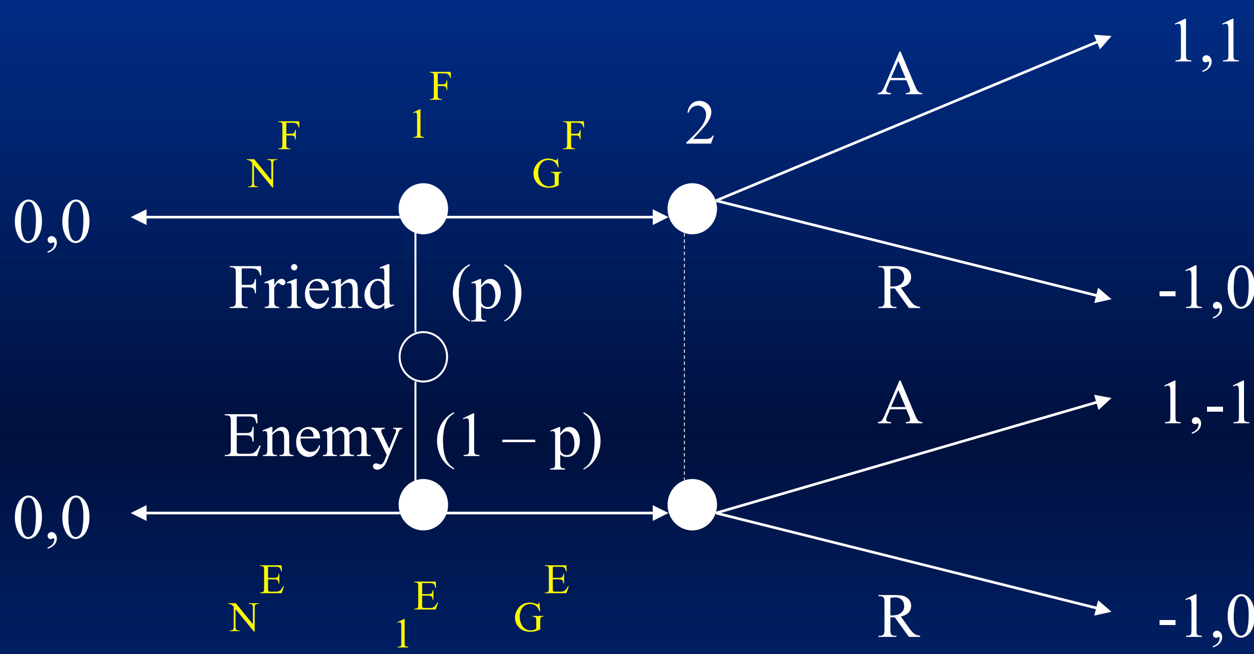
Solution Concepts: A Comparison

	Normal Form Games	Extensive Form Games
Complete Information	**Nash Equilibrium**	**Subgame Perfect Nash Equilibrium**
Incomplete Information	**Bayesian Nash Equilibrium**	**(Perfect Bayesian Equilibrium)**

Random Events and Incomplete Information

(Watson Chapter 24)

Example: The Gift Game



Chance node: nature's decision node;
Nature determines player 1's type: Friend (with probability p) or Enemy (1-p);
Player 1 observes Nature's move, so he knows his own type;
Player 2 does not observe player 1's type.

The Gift Game in Bayesian Normal Form

1 \ 2	A	R
$G^F G^E$	**1,2p-1**	**-1,0**
$G^F N^E$	**p,p**	**-p,0**
$N^F G^E$	**1-p,p-1**	**p-1,0**
$N^F N^E$	**0,0**	**0,0**

Strategy: If F, G; if E, G

In games of incomplete info, rational play require a player who knows his own type to think about what he would have done had he been another type.

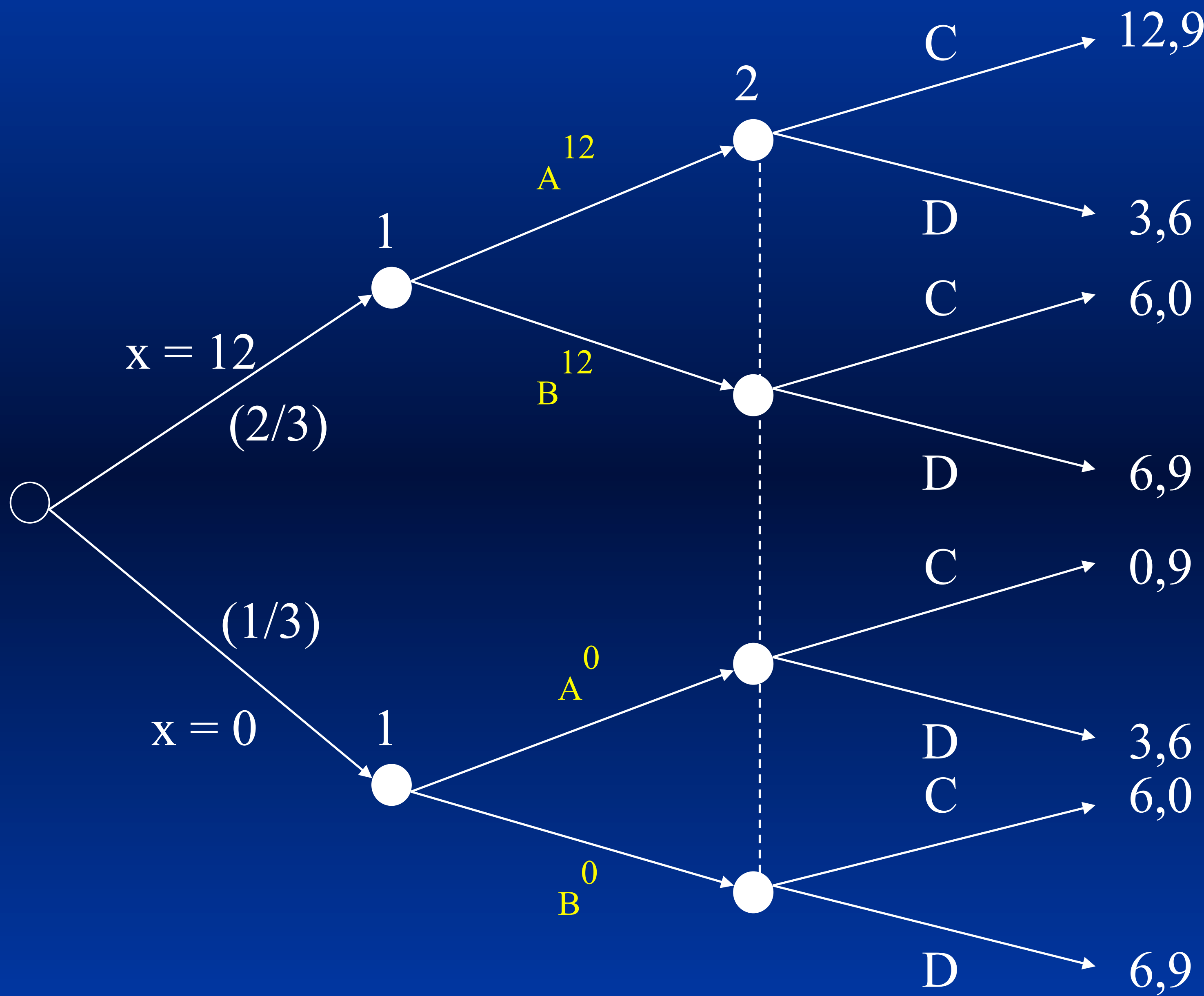
Example: A Game of Incomplete Information

1 \ 2	C	D
A	**x,9**	**3,6**
B	**6,0**	**6,9**

$$x = \begin{cases} 12 \text{ with probability } 2/3 \\ 0 \text{ with probability } 1/3 \end{cases}$$

Player 1's payoff number x is private information;
Player 2 knows only that x=12 with probability 2/3 and x=0 with probability 1/3.
This matrix is not the true normal form of the game.

Extensive-form Representation



Normal-form Representation

1 \ 2	C	D
$A^{12}A^{0}$	**8,9**	**3,6**
$A^{12}B^{0}$	**10,6**	**4,7**
$B^{12}A^{0}$	**4,3**	**5,8**
$B^{12}B^{0}$	**6,0**	**6,9**

Player 1's decision:
(1) whether to select A or B after observing x=0;
(2) whether to select A or B after observing x=12.

Risk and Incentives in Contracting

Nature moves at the end of the game … - the simplest case

(Watson Chapter 25)

Background Definitions

- More than one possible ***outcome*** can occur.
- ***Probability*** refers to the likelihood that an outcome will occur.
  - Objective probabilities
  - Subjective probabilities
- The ***expected value*** (average, mean) of a random variable is a weighted average of the values of all possible outcomes, with the probabilities of each outcome used as the weights .
- ***Variance*** and ***standard deviation*** are measures of dispersion of individual outcomes from the mean.

Expected Value and Variance

Expectation:

$$E(X) = \Sigma\, p_i(X_i)$$

$$= p_1(X_1) + p_2(X_2) + \ldots + p_n(X_n),$$

where p_i = probability of outcome i,

X_i = value of random variable associated with outcome i,

and $p_1 + p_2 + \ldots + p_n = 1.$

Variance:

$$\sigma^2 = \sum_{i=1}^{n} p_i[X_i - E(X_i)]^2$$

Standard Deviation: σ

Attitude Toward Risk

- ***Risk neutral*** **individuals maximize** ***expected value***
- ***Everyone*** **maximizes** ***expected utility***
- ***St Petersburg paradox:***
  - **A gamble of consecutive tossing of a fair coin**
  - **Payoff doubles for every consecutive heads that appears**

Expected Utility Example

- The Bet:
  - 50% chance of winning $100
  - 50% chance of winning $400
- Risk Neutral Player:
  - EV = (.5)(100) + (.5)(400) = 250
  - Bet is equivalent to having $250 for sure
  - Player would be willing to pay up to $250 for a lottery ticket with these odds
  - Player would be willing to pay up to $250 for insurance rather than assume the risk of any bet that is worse than this bet

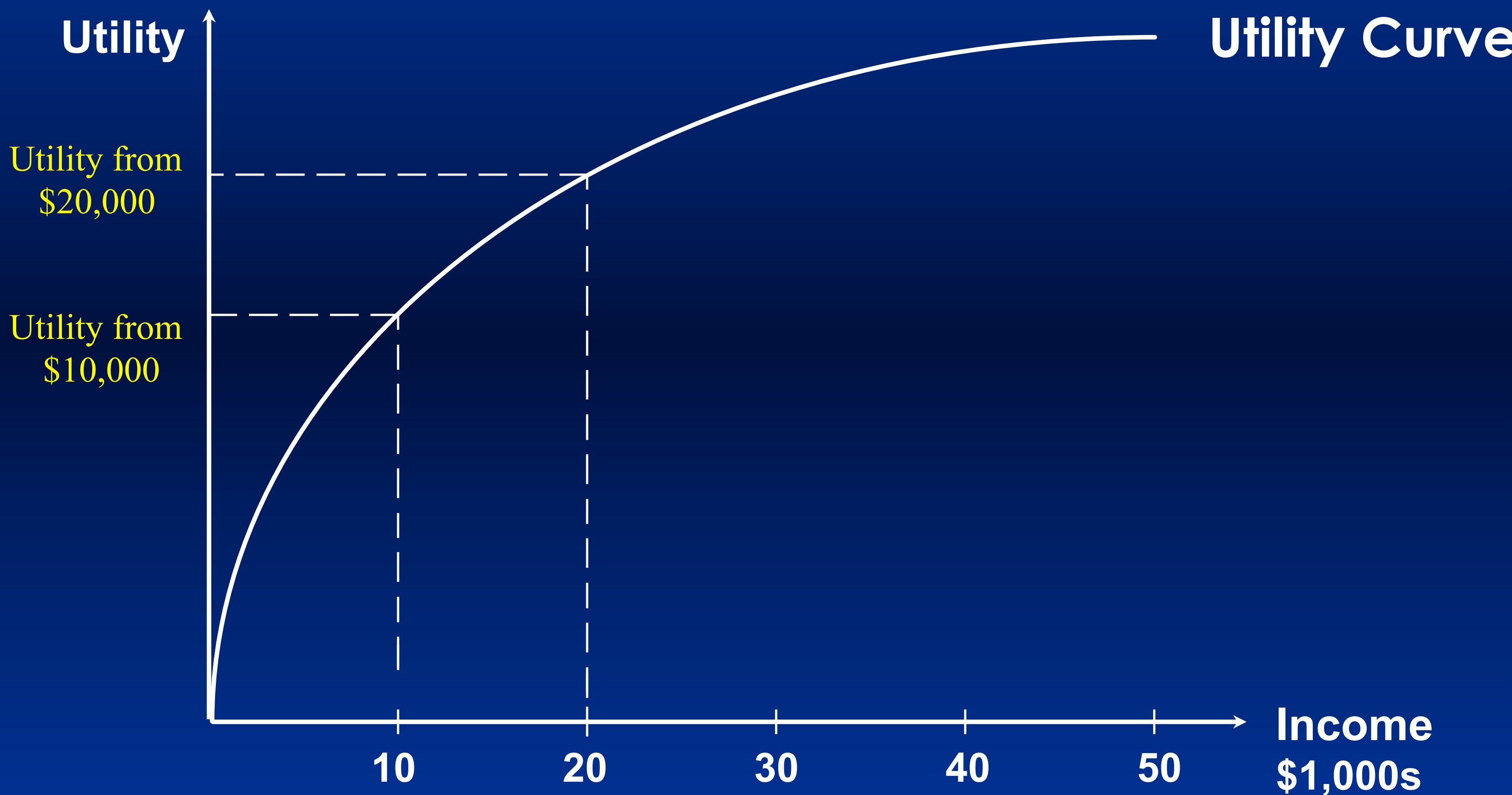
Expected Utility Example

- The Bet:
  - 50% chance of winning $100
  - 50% chance of winning $400
- Risk Averse Player might have $U = (I)^{½}$
  - $EU = (.5)(100)^{½} + (.5)(400)^{½} = 15$
    - » What income will give him U = 15 for sure?
    - » $15 = (I)^{½} \Rightarrow I = 225$
  - Bet is equivalent to having $225 for sure
  - Player would be willing to pay up to $225 for a lottery ticket with these odds
  - Player would be willing to pay up to $225 for insurance rather than assume the risk of any bet that is worse than this bet

Risk Aversion and Expected Utility

If most of us are risk averse (in most situations), then we are obviously not trying to maximize the expected dollar return.

Economist's view: We maximize *Expected Utility*



- **Risk aversion implies a "concave" utility function, or "diminishing marginal utility of money."**

Source: Maggie Levenstein, Ross School of Business

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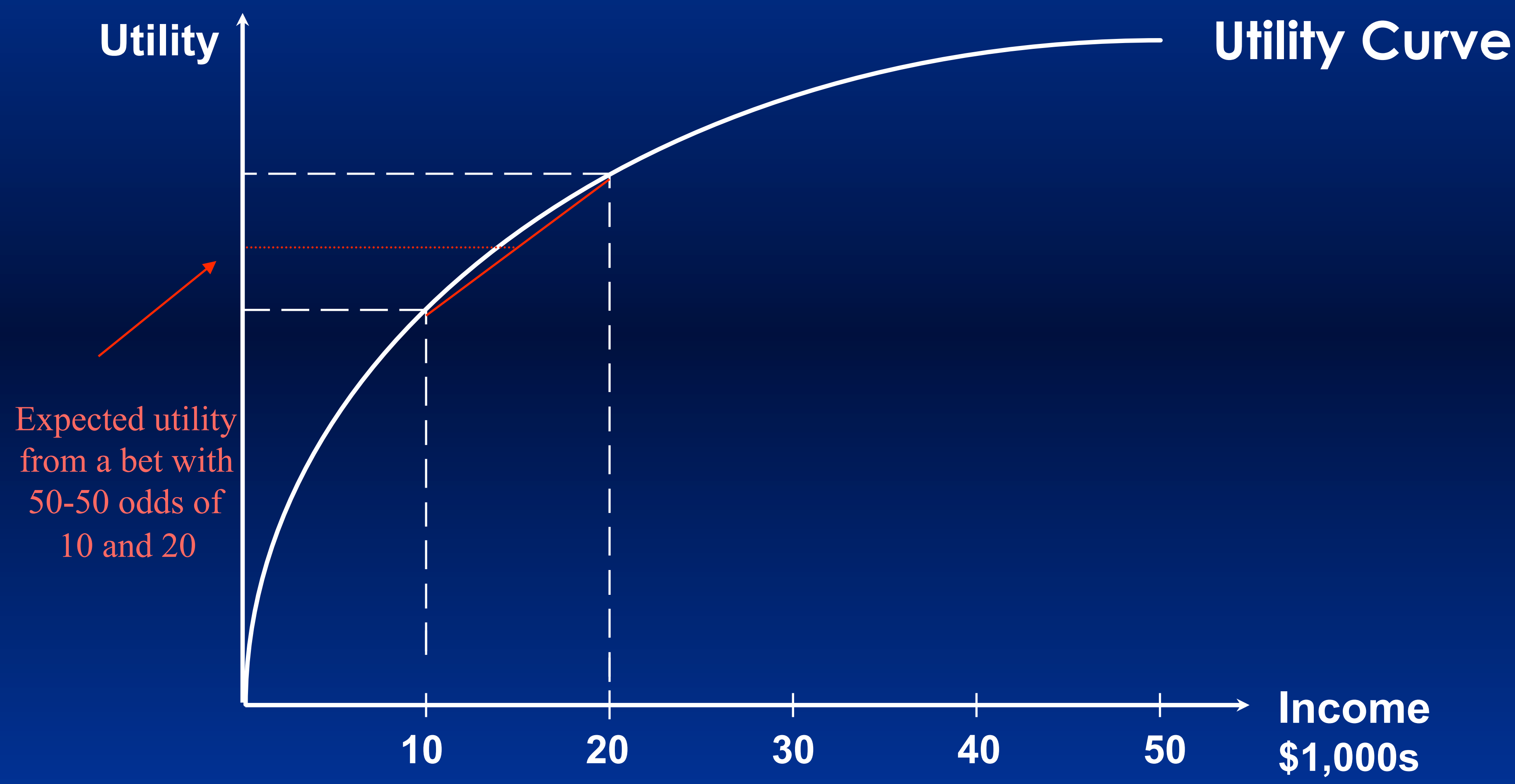

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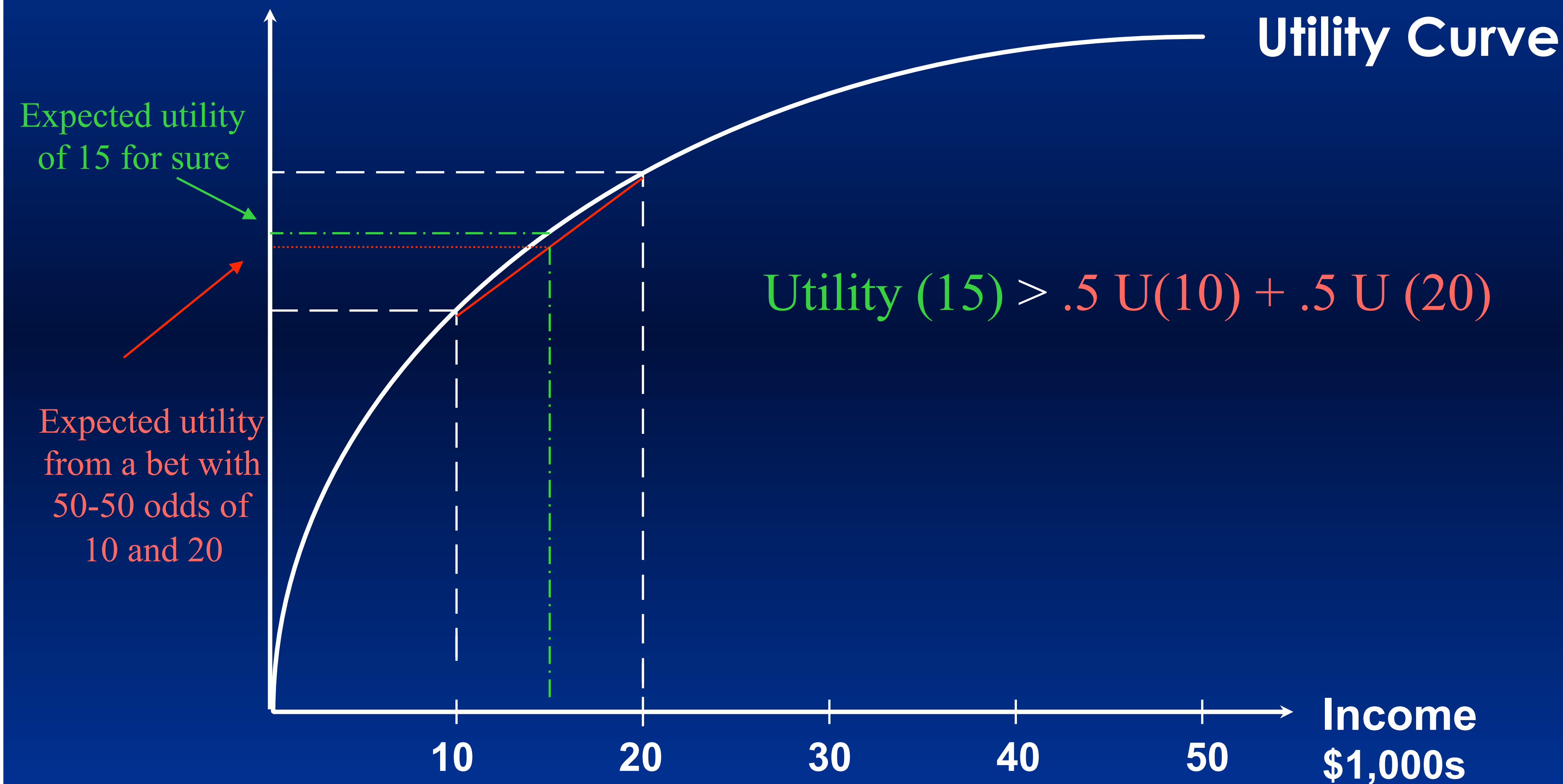

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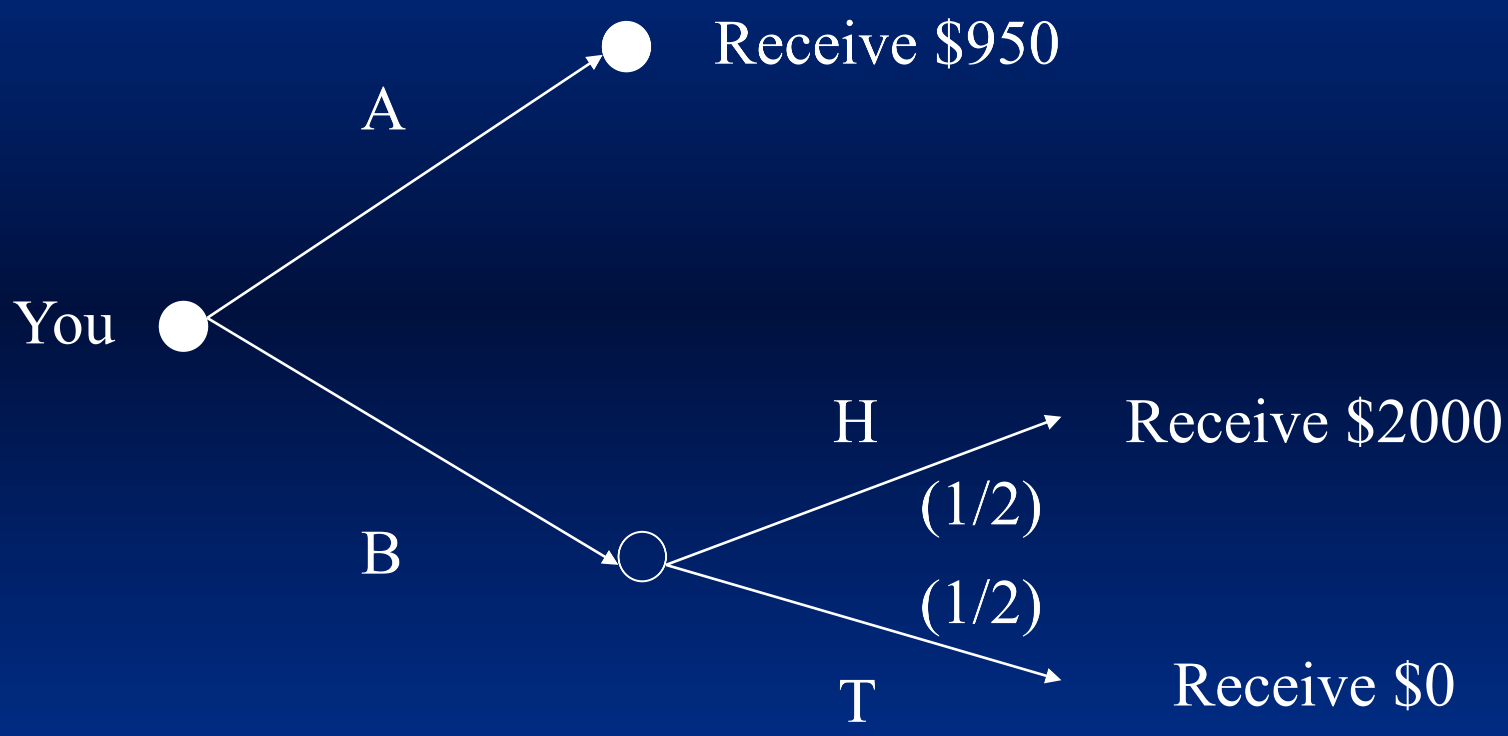
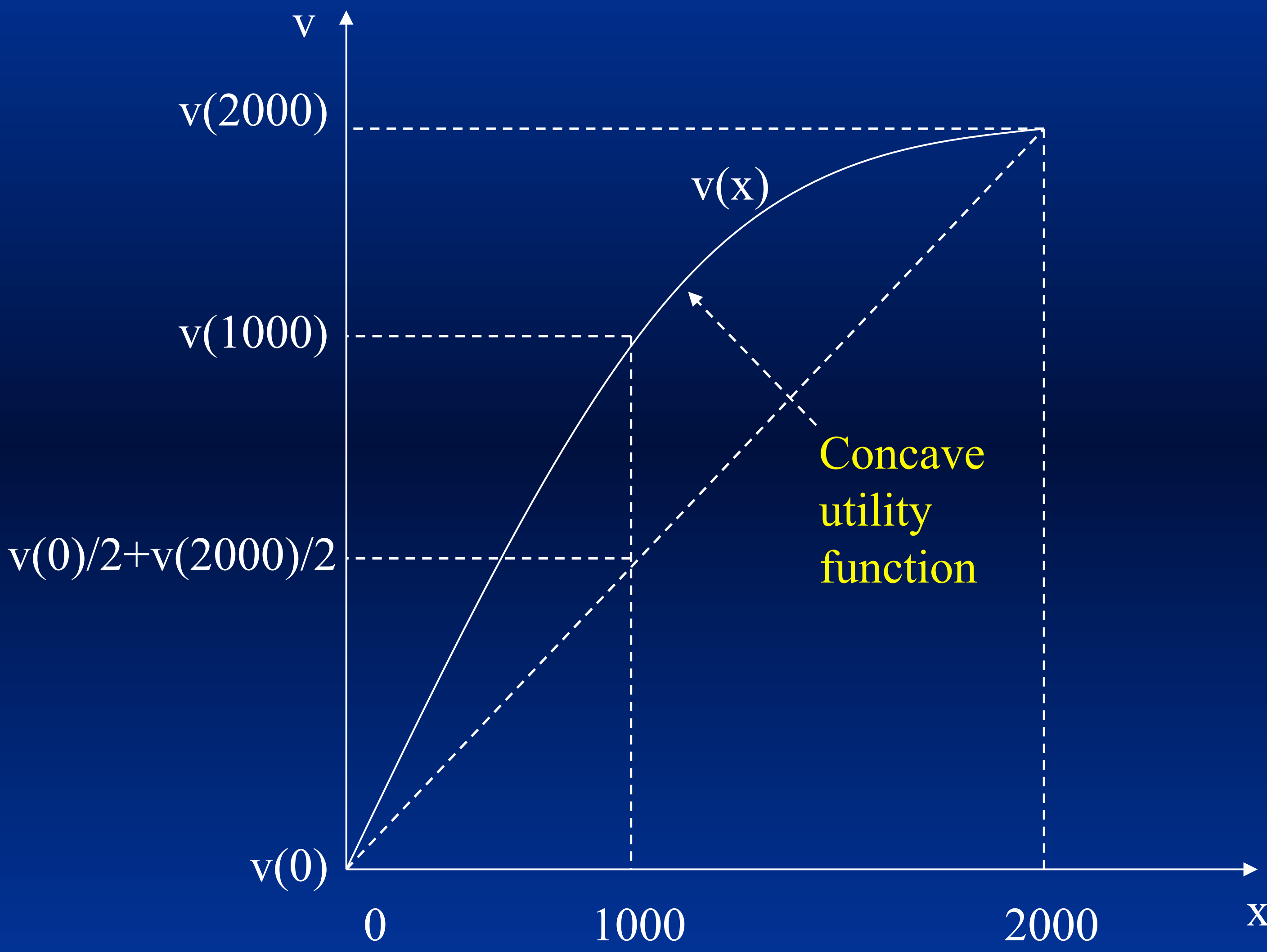
Example: Lottery or Sure Thing?



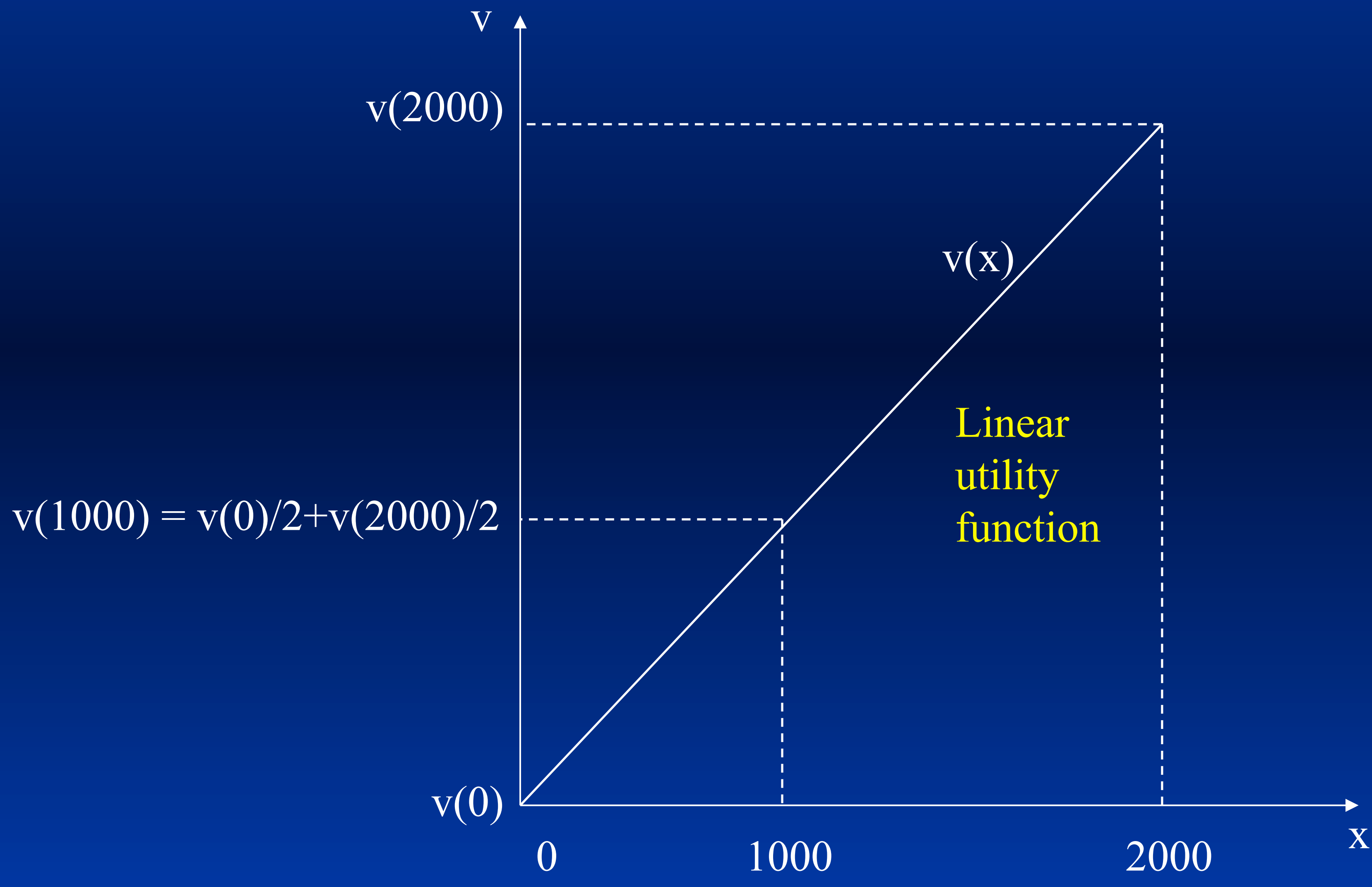
A: sure thing;
B: lottery

Utility functions and concavity

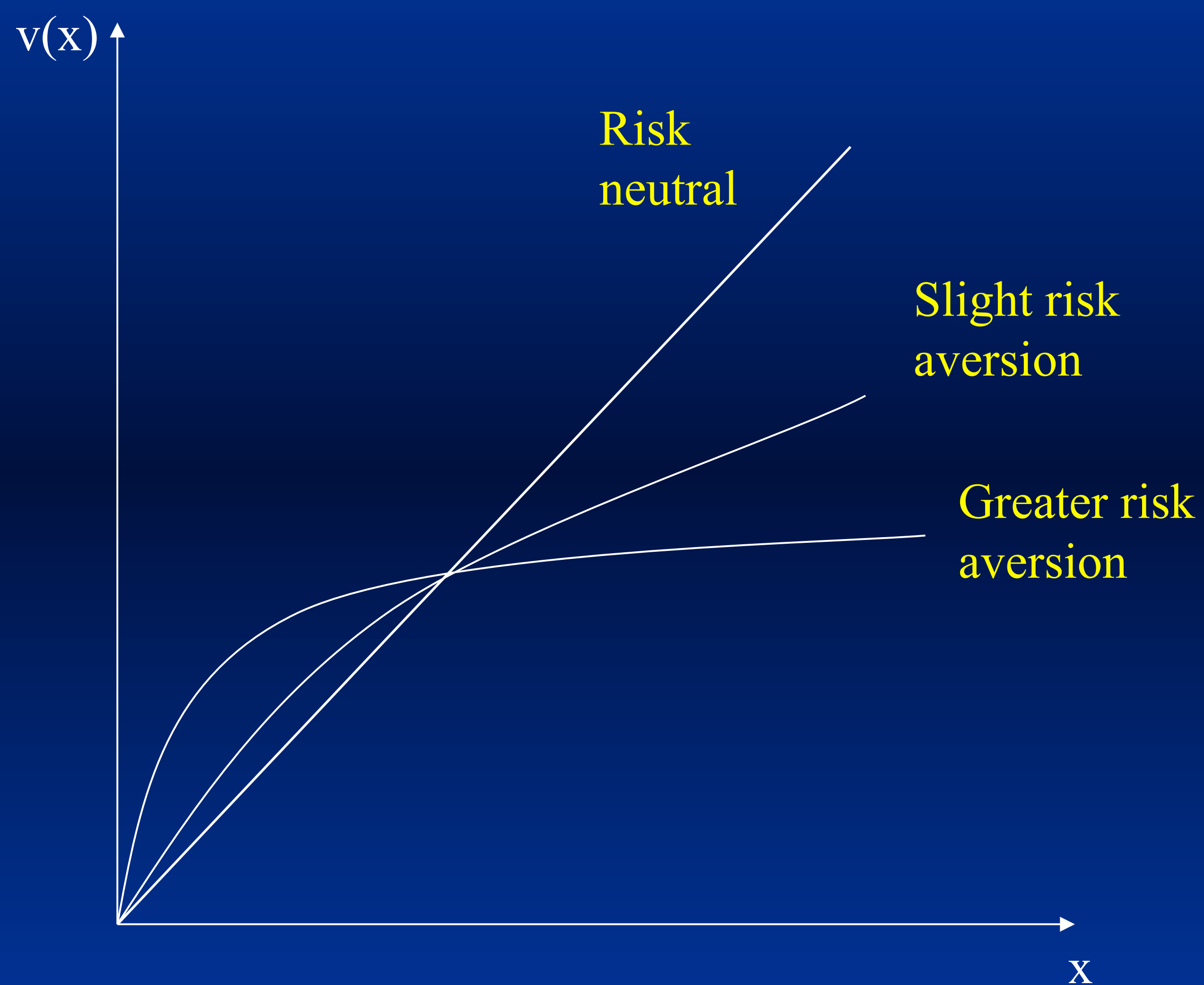


v(x): utility of receiving x dollars

Linear Utility Function: Risk Neutral



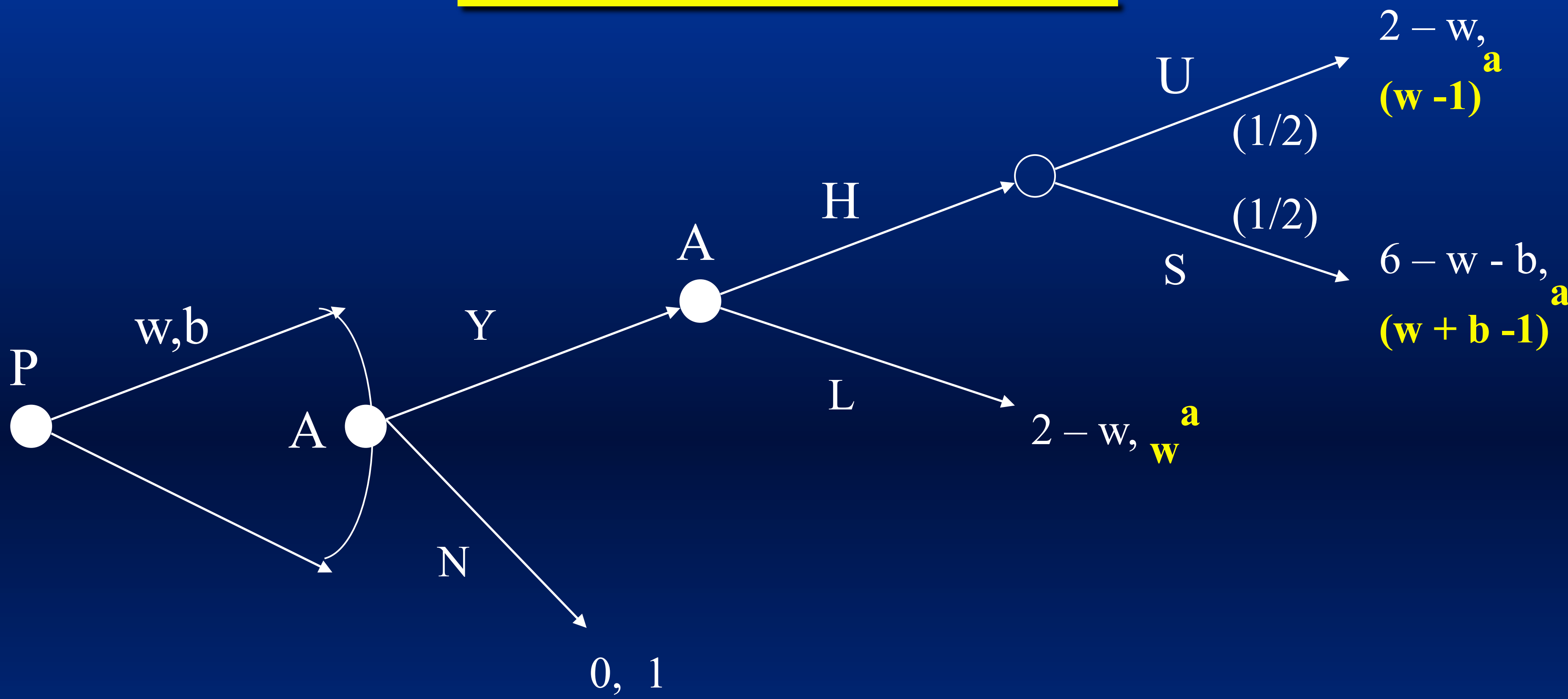
Different Levels of Risk Aversion



A Principal-Agent Game with Moral Hazard

- Set of players: {Pat, Allen}
  - Pat: Principal, risk neutral, $v(x) = x$
  - Allen: Agent, risk averse, $v(x) = x^a$
- Pat: write a contract, (wage, bonus)
- Allen: exert high or low effort
- Success depends on Allen's high effort as well as a random factor

Extensive Form

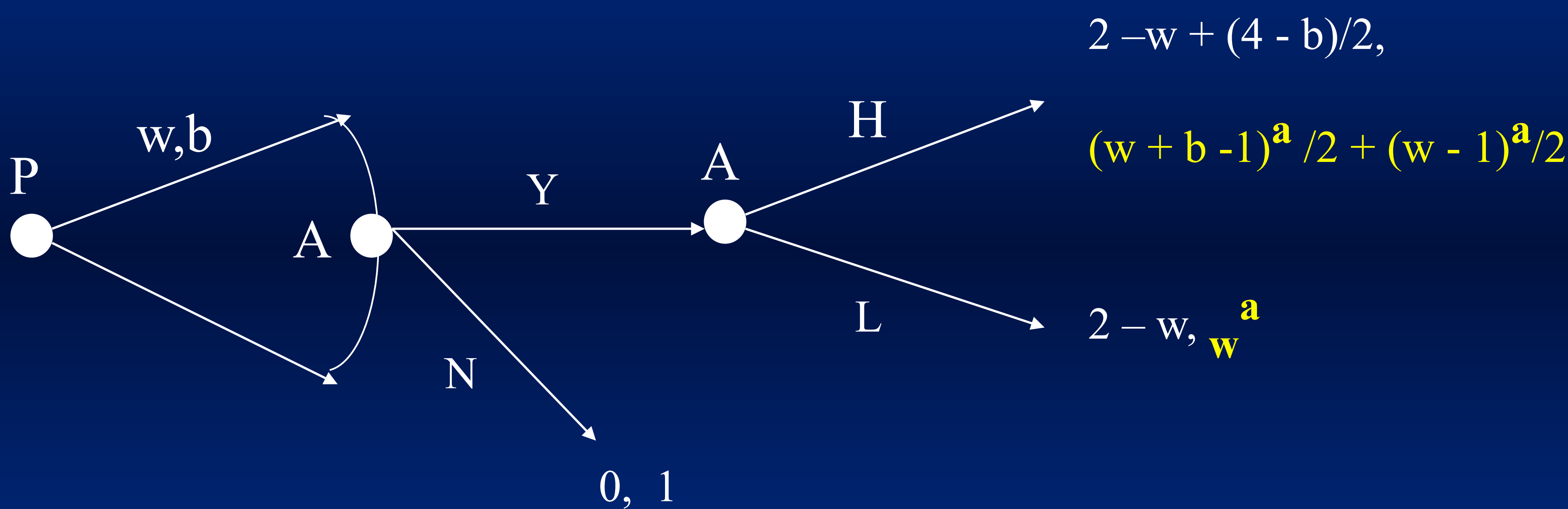


w: wage

b: bonus, paid only if successful

High effort cost: 1

Principle-Agent Game with Expected Payoffs



Pat would like Allen to exert high effort.

Can she write a contract that induces it?

Case 1: No-bonus Contract

- If Pat sets b=0
  - Allen has no incentive to exert high effort
  - The best Pat can do is to offer w = 1
  - Allen is willing to accept
- Best no-bonus contract: w=1, b=0
  - Payoff vector (1, 1)

Case 2: Bonus Contract (solution via backward induction)

- Last step: Incentive-compatibility constraint
  - Allen's expected payoff from H has to be at least as great as his payoff from L
- Second to last step: Individual rationality (or voluntary participation) constraint
  - Allen's expected payoff from H has to be at least as great as his outside option
- The best contract has to satisfy the IC and IR constraints with equality
- Solution: $w = 1$, $b = 2^{(1/a)}$

Bayesian Nash Equilibrium and Bayesian Rationalizability

(Watson Chapter 26)

Finding BNE

- **Method 1**
  - **Write down Bayesian normal form**
  - **Solve for Nash equilibrium of the normal form: *Bayesian Nash equilibrium***
  - **Or, solve for the set of strategies which survive iterated elimination of dominated strategies: *Bayesian rationalizability***
- **Method 2**
  - **Treat types as separate players (omit)**

Example: A Game of Incomplete Information

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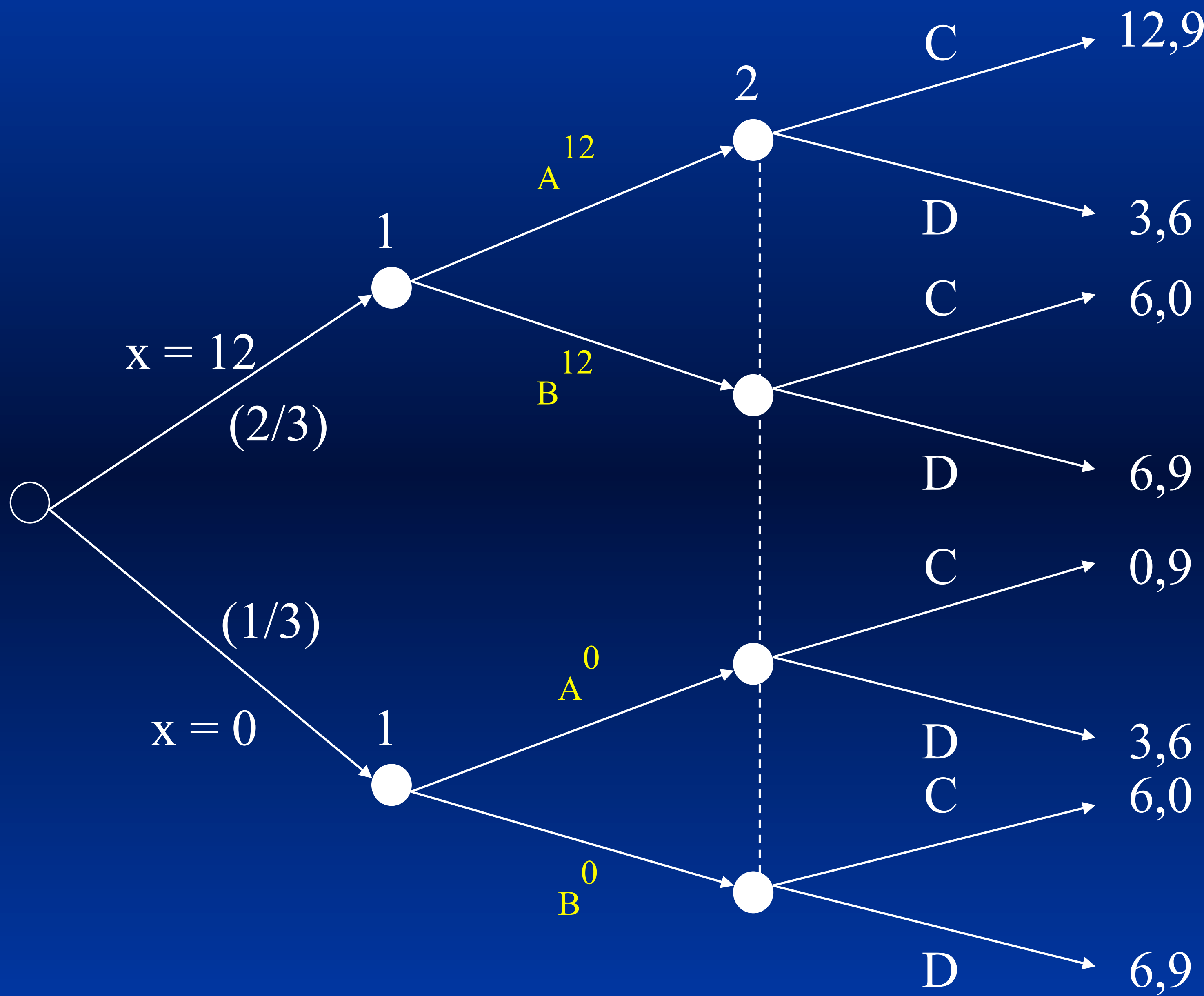

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$B^{12}B^{0}$	6,0	6,9

{$B^{12}B^{0}$, D} is the Bayesian rationalizable set, and the unique BNE.

Iterated elimination of dominated strategies:
(1) $B^{12}B^{0}$ dominates $B^{12}A^{0}$; $A^{12}B^{0}$ dominates $A^{12}A^{0}$.
(2) D dominates C. (3) $B^{12}B^{0}$ dominates $A^{12}B^{0}$.

Lemons and Auctions

(Watson Chapter 27)

Adverse Selection

- This is a problem of *hidden characteristics (*when one side of a transaction knows something about itself that the other does not) and *self-selection*. The uninformed party gets exactly the wrong people trading with it, so we say that the uninformed party gets an *adverse selection* of the informed parties.

Incomplete Information and Adverse Selection

- *worst* risks are the ones most likely to buy insurance, pushing up the price for the best risks
- low-quality products can crowd out high-quality products
  - » There is a "market failure" because sellers of low-quality "lemons" impose a negative externality on the sellers of high-quality products. When low-quality products are offered for sale, they adversely affect the perceived value of high-quality products if buyers cannot differentiate low- and high-quality. Low-quality products prevent the market for high-quality products from functioning properly.

These markets are interesting because there may be indirect ways for the uniformed party to infer what is going on

Adverse Selection

- **Adverse selection is also known as the *lemons problem***
  - **suppose you purchase a new car, drive it for 1 month, and then for reasons entirely out of your control, you must sell it**
    - »**what price do you think you could get for your car? Why?**
    - »**how could you minimize the problem?**

Lemon: An Example

- Jerry is in the market for a used car
- Freddie offers an attractive 15-year old sedan for sale
- Blue book value for the car is p
- The car is a peach with probability q
  - If peach: worth $3000 to Jerry, $2000 to Freddie
  - If lemon: worth $1000 to Jerry, 0 to Freddie
- What is the efficient outcome?

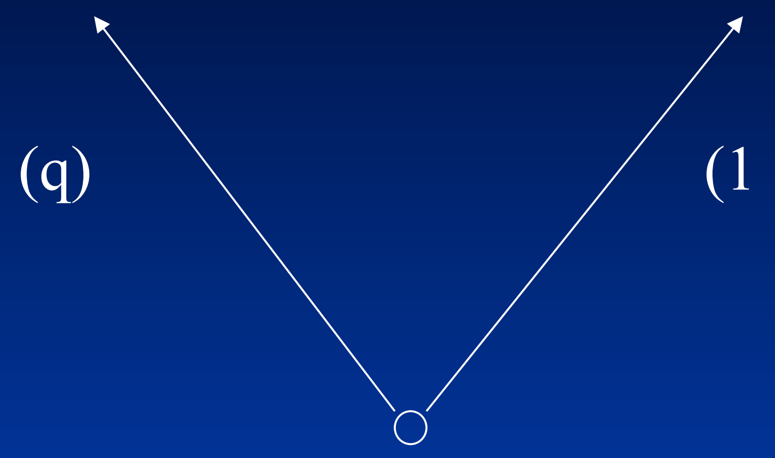
FIGURE 27.1
The lemons problem.

J \ F	T^P	N^P
T	**3000-p, p**	**0,2000**
N	**0, 2000**	**0,2000**

Peach

J \ F	T^L	N^L
T	**1000-p, p**	**0,0**
N	**0,0**	**0,0**

Lemon



Extensive form

- Nature moves first
- Jerry and Freddie then choose their strategies simultaneously

Bayesian Normal Form

- How many strategies does Jerry have?
- How many strategies does Freddie have?
- What is the size of the matrix

Freddie

	T^PT^L	T^PN^L	N^PT^L	N^PN^L
T	2000q +1000-p, p	(3000-p)q, pq	(1000-p)(1-q), 2000q+(1-q)p	0, 2000q
N	0, 2000q	0, 2000q	0, 2000q	0, 2000q

BNE 1: Only Lemons Traded

- (T, $N^P T^L$): two conditions should hold

  (1) $(1000-p)(1-q) \geq 0$,

  or $1000 \geq p$

  (2) $2000q+(1-q)p \geq \max\{p, pq, 2000q\}$

  (non-binding)

- **Intuition:**
  - If price is below \$1000, F would only want to bring lemons to the market
  - Anticipating that only a lemon will be for sale, Jerry is willing to pay no more than \$1000

BNE 2: Both Lemon and Peach Traded

(T, $T^P T^L$): two conditions should hold

(1) $2000q+1000-p \geq 0$,
or $2000q+1000 \geq p$

(2) $p \geq \max\{pq, 2000q+(1-q)p, 2000q\}$
or $p \geq 2000$

(3) Combining both conditions: $q \geq ½$

Intuition:

1. Jerry's expected value from owning the car exceeds its price;
2. Freddie is willing to bring a peach to the market;
3. The probability of a peach should be sufficiently high

Solving the Adverse Selection Problem

Some limited ways to address this

- have a mechanic check over the car
- offer a warranty
- government "lemon laws"
  - » e.g. *Wall Street Journal*, 10/18/96: "California is prohibiting Chrysler Corp. from shipping vehicles into the state for 45 days as punishment for selling defective used vehicles, which allegedly should have been labeled as 'lemons.'. . . In California, a new vehicle is considered a lemon once the owner has attempted to fix a defect four or more times. It is also deemed a lemon after it has spent 30 days or more in the repair shop over a 12-month period. Once it has been acknowledged as a lemon, auto makers are required to buy it back for the original purchase price. . ."
  - » establish a reputation

Some Cures for Adverse Selection in Providing Health Care

- Provide medical policies to entire groups (e.g., through employers),
- Make coverage mandatory
- Refuse coverage for “pre-existing conditions”
- Limit choice

Some Cures for Adverse Selection in Providing Health Care

Example: Suppose a company offers 3 insurance options to employees

- » an HMO at no cost
- » a mid-range plan that has more physician choice & better coverage, but costs each employee $50 per month with higher deductibles
- » a "Cadillac plan" that gives complete choice, wonderful benefits, & no deductibles but costs $150 per month

How might different kinds of employees choose among plans?

Auctions: Background

- **What's so interesting about auctions?**

  An alternative to bargaining for selling a fixed supply of a commodity for which there is no well-established, ongoing market.

- **Applications**
  - **Real estate, art, flowers, oil leases**
  - **Privatization and deregulation**
    - » **Government contracts**
    - » **Electricity**
    - » **Airwaves: FCC spectrum Auctions**
  - **Allocation of common resources**
  - **E-commerce: eBay**

Auctions: Background - cont.

- Auction Institutions
  - English
  - Dutch
  - First Price sealed-bid
  - Vickrey
  - Google Adwords (position)
  - Many other kinds

Types of Auctions

- **Private value auctions**

  Bidders' valuations for the auctioned item(s) are independent from one another and are their private information. e.g., flowers, art, antiques.

- **Common value auctions**

  Bidders are uncertain about the ultimate value of the item, which is the same for all bidders. e.g., oil leases, Olympic broadcast rights.

- **Affiliated (correlated) value auctions**

  Bidders' valuations for the auctioned item(s) are correlated, but not necessarily the same for all. In between private and common value auctions.

Auction Research

- Research Questions
  - Efficiency comparison of auction institutions
  - Revenue comparison
  - Bidder earning comparison
  - Collusion?
  - Transparency?
- What do we know?
  - Single item: well
  - Multiple items: little
    - » Substitutes
    - » Complements

Auction Research – cont.

- Agenda for theoretical research
  - Multi-item auctions
- Agenda for experimental research
  - Test/discriminate among theories
  - Design and test new institutions

English Auction

- Background
  - Oral auctions in English-speaking countries.
  - Originally "Roman."
- Commodities
  - Antiques, artworks, cattle, horses, real estate, wholesale fruits and vegetables, old books, etc.

Rules for Experiment

- Auctioneer first solicits an opening bid from the group.
- Anyone who wants to bid should call out a new price at least $1 higher than the previous high bid.
- The bidding continues until all bidders but one have dropped out.
- The highest bidder gets the object being sold for a price equal to the final bid.
- Winner's profit = Buyer Value – price; Everyone else's profit = 0.
- Your Buyer Value = Last two digits of your SSN

English Auction Outcome

- **Optimal strategy**
  Participate until price = buyer value, then drop out.
- **Equilibrium Outcome**
  The highest bidder gets the object at a price close to the second highest Buyer Value.
- **Comparative statics**
  As *n* increases, the winning bid is closer to the highest BV. The more spread-out the different bidders' valuations are, the larger $|v^{max}-v^{2nd}|$. This means that if there is wide disagreement about the item's value, the winner might be able to get it cheaply.
- **Problems**
  Collusion; bidding rings.

Dutch Auction

- Background
  Wholesale produce, cut-flower markets in the Netherlands.
- Commodities
  - Flowers in the Netherlands
  - Fish market in England and Israel
  - Tobacco market in Canada
- Rules
  - Auctioneer starts with a high price.
  - Auctioneer lowers the price gradually until some buyer shouts “Mine!”
  - The first buyer to shout “Mine!” gets the object at the price the auctioneer just called
  - Winner’s profit = Buyer Value – price; Everyone else’s profit = 0.
  - Your Buyer Value = 100 – Last two digits of your SSN

First-Price Sealed-bid Auctions

- **Background**

  Used to award construction contracts (lowest bidder), real estate, art treasures;

- **Rules**
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer.
  - The auctioneer opens the bid and find the highest bidder.
  - The highest bidder gets the object being sold for a price equal to her own bid.
  - Winner's profit = Buyer Value – price; Everyone else's profit = 0.
  - Your Buyer Value = First and second number of the last four digits of your SSN.

First-Price Sealed-bid Auctions – cont.

- **Set up the problem:**

  In a sealed-bid, first price auction in a private values environment with *n* bidders, each bidder has a private valuation, v_i, which is his private information. The distribution of v_i is common knowledge. Let B_i denote the bid of player *i*. Let π_i denote the profit of player *i*. If $v_i \sim u[0,100]$, what is the Bayesian Nash equilibrium bidding strategy for the players?

- **Optimal bidding strategies: If $B_i \geq v_i$, then $\pi_i \leq 0$. Therefore, $B_i < v_i$, which gives:**

  $\pi_i = 0$, if $B_i \neq \max_j\{B_j\}$, or
  $\pi_i = v_i - B_j$, if $B_i = \max_j\{B_j\}$

  The question is how much below v_i should his bid be? The less B_i is, the less likely he will win the object, but the more profit he makes if he wins the object.

First-Price Sealed-bid Auctions – cont.

- $n = 2$

  You are characterized by the strategy-type two tuple, (B,v). Suppose the other bidder's value is X, and she is characterized by *(αX,X)*, where $\alpha \in (0,1)$. Your expected profit is:

  $E\pi$ = P(Your bid is higher$)\cdot(v-B)+P$(Your bid is lower)*0

  With uniform distribution,
  P(X<B/α) = (1/100)(B/α). Therefore,

  $$E\pi=(1/100)(B/\alpha)(v - B).$$

  assuming risk neutrality, you choose B to:

  $$\max_{B} B(v - B) = Bv - B^2$$

  It follows that $B = v/2$.

First-Price Sealed-bid Auctions – cont.

- With n bidders

  P(Your bid is highest) = $[(1/100)(B/\alpha)]^{n-1}$.

  $\max_B B^{n\text{-}1}(v - B) \Rightarrow B = [(n\text{-}1)/n]v$.

  Note: as n increases, $B \rightarrow v$. i.e., increased competition drives bids close to the valuations.

- Equivalence of Dutch and First-price, sealed-bid auctions: same reduced form.
  - The object goes to the highest bidder at the highest price.
  - A bidder must choose a bid without knowing the bids of any other bidders.
  - Optimal bidding strategies are the same.

Sealed-bid, Second-price Auctions

- Background: Vickrey (1961).
- Commodities
  - stamp collectors' auctions
  - US Treasury's long-term bonds
  - Airwaves auction in New Zealand
  - eBay and Amazon
- Rules
  - Bidders write their bids for the object and their names on slips of paper and deliver them to the auctioneer.
  - The auctioneer opens the bid and finds the highest bidder.
  - The highest bidder gets the object being sold for a price equal to the second highest bid.
  - Winner's profit = Buyer Value – price;

    Everyone else's profit = 0.
  - Your Buyer Value = 100 – First and second number of the last four digits of your SSN.

Sealed-bid, Second-price Auctions – cont.

- **Equilibrium bidding strategy:**

It is a weakly dominant strategy to bid your true value.

Let V be your Buyer Value, let B be your bid, and let X be the highest bid made by anybody else in the auction. We want to show that overbidding or underbidding cannot increase your profit and might decrease it. Let π^t be your profit when $B = V$. Let π be your profit otherwise.

Sealed-bid, Second-price Auctions – cont.

- **Proof:**

  First consider the case of overbidding, $B>V$.

  1. $X>B>V$: You don't get the object either way: $\pi = \pi^t = 0$.
  2. $B>V>X$: $\pi = V - X = \pi^t > 0$.
  3. B>X>V: $\pi = V - X < 0$, but $\pi^t = 0$.

  Next consider the case of underbidding, $B<V$.

  1. $X<B<V$: $\pi = V - X = \pi^t > 0$.
  2. $B<X<V$: $\pi = 0$, but $\pi^t = V - X > 0$.
  3. B<V<X: You don't get the object either way: $\pi = \pi^t = 0$.

- **Equivalence of English and sealed-bid, 2^{nd} Price.**
  - The object goes to the highest bidder.
  - Price is close to the second highest BV.

Google Adwords Auction

- Generalized second-price auction (GSP)
  - Sort bids
  - Top x bids wins
  - Bidder who wins the nth position pays the (n +1)th bids
- Is it VCG?

GSP: Properties

- Google's "unique auction model uses Nobel Prize-winning economic theory to eliminate … that feeling that you've paid too much."†
- GSP is not VCG when x > 1
- Example

† "Maximize Your Revenue From Search Results With Google AdSense", Google 2004. (http://www.google.com/adsense/afs.pdf)

Homework Assignment (For Practice Only)

- Chapter 24: # 3
- Chapter 25: #3, 5
- Chapter 26: #1, 7
- Chapter 27: #3, 4 (a, b)