ERRATA SHEET NO. 1

UNIVERSITY OF MICHIGAN

WILLOW RUN RESEARCH CENTER

UMM-37

Pages 34 and 35 (figures 111 - 119) Δ should be d in the numerator throughout.

University of Michigan

Memorandum No. 37

The Stability and Response of an Aerodynamic Missile in Plane Motion

Prepared by

H. G. Mazurkiewicz *Richard M. Spath*

Approved By

F. W. Ross

Supervisor Aerodynamics Section

Project MX-794

(USAF Contract No. W33-038-ac-14222)

December 1949

ABSTRACT

Complete and systematic stability analyses based upon evaluation of the solutions of the equations of motion are seldom feasible. This is due primarily to the computational difficulties inherent in current techniques. Available mathematical criteria yield information regarding only the nature of the roots of the characteristic equation of the system. Examination of the solutions of the equations, however, gives additional knowledge as to the character of the motion, the relative importance of the variables involved, and the practical significance of any instabilities which may arise. It is the purpose of this report to present a method for obtaining such knowledge, which lends itself readily to facilitating numerical analyses because of the simplified forms of the solutions and the use of their invariant characteristics.

In achieving this goal, La Place definite integral methods are an important asset. The utility of the La Place transformation is further enhanced by its close correlation with algebraic methods used in controls design work. The algebraic properties of the transformation are especially useful in demonstrating the properties of linear systems. Missile response is determined to a quite general forcing function, requiring only a simple integration for any specific application. The frequency response of the system is derived heuristically from the general response integral. Certain invariant properties of the solutions are defined. Amplitude and phase invariants are then calculated in terms of the roots of the characteristic equation. These results are all derived in terms of the aerodynamic pitch plane problem. Included is a preface, written in general terms, to serve as an introduction for persons unfamiliar with the specific notions and terminology employed.

Several applications are demonstrated for cases of interest in preliminary design work.

PREFACE

A basis for undertaking a study of the stability of systems lies in the practical consideration that certain configurations shall remain stable over a range of specified performance requirements. It is often further required that these configurations shall retain their stable characteristics in the presence of an arbitrary drive. The existence of this drive then requires that a study be made of the response of the configuration to the drive. In the absence of the drive, a stable configuration is said to possess inherent dynamic stability. The work contained in this report is based upon these premises.

The results presented later are in terms of application to the aerodynamic pitch plane problem. The approach, however, need not be so limited. To illustrate, in a more general way, the concepts involved, assume a configuration composed of a complete interceptor defense system and a ballistic target. In the sense implied above, this configuration must remain stable. If the interceptor system is completely specified, the requirements for stability are theoretically definable in terms of the target parameters. Definition of a physically realizable limiting process will then statistically insure that interception will take place, and will describe the stable phase of the configuration. If the limitations upon the target parameters are known, it is also theoretically possible to determine the tactical limitations of the interceptor system. This problem could be well labelled one in inherent dynamic stability. Suppose, however, that the target is no longer of the ballistic type, but is now capable of taking evasive action. This implies that the maneuvers it executes to avoid interception are not completely predictable. Such a property of the target requires that an arbitrary drive be imposed upon the configuration and that the response of the configuration to this drive be studied. If it can be shown that interception will occur in the presence of a specified drive, then the configuration is said to have a stable response; that is, it is stable in the presence of this drive. Insofar as the configuration may remain stable for only a certain class of drives, it is then possible to define the tactical limitations of the interceptor system.

Additional practical considerations restrict studies to considerably less general configurations. The restrictions are due not only to the difficulties of explicitly defining the configuration, but also to the computational problem involved. It is a fair presumption that such considerations preclude any particularly formidable generalizations.

Detailed analysis is usually limited to regions where dynamic phenomena are sensibly linear. Configurations which are presumed to lie in such linear regions can be analyzed by well-known methods. The knowledge of such methods does not mean that they enable one to determine a workable component configuration, nor do they yield conclusive information directly useful in terms of the overall configurations. Furthermore, the data which are determined can be applied with confidence only within these regions of linearity. The process of linearization in the study of more general dynamical systems introduces the attendant difficulty of determining the extent of such regions. This in turn leads to the likelihood that, in practical applications, the specific linearization will vary under different dynamical conditions.

The restriction of a dynamic stability study to these linear regions gives rise to several notions. Representative of these are controls-locked stability, automatic stabilization, and response to controls, which arise in the aerodynamic stability problem. The instability of a configuration in any such region does not imply that the configuration is unstable except in the particular region under consideration. Thus, the stability of configurations in these regions of linearity represents a sufficient condition for the stable behavior of the configurations which will be considered.

TABLE OF CONTENTS

INTRODUCTION

In the field of missile design, the stability of systems has several important applications. The problem resolves itself, generally speaking, into the stability of various components and finally of the entire missile system. The problem discussed here is but one of several in this class. The subject matter is restricted to motions which are reasonably functions of missile external dynamics alone.

Definite integral methods are used to solve the linearized pitch plane problem. The algebraic properties of the La Place transform are particularly useful in emphasizing characteristics of linear dynamical systems. These characteristics are of considerable interest in this application. The convolution property of the transform is well adapted to determining, in simple integral form, the missile response to a quite general forcing function.

The formal development is carried out in some detail in the belief that this might enhance the usefulness of particular sections applicable to pertinent special problems. Since solutions are obtained for quite general forcing functions, control driven configurations are not evaluated in terms of impressed trigonometric functions. Such functions might be used as a point of departure for obtaining a reasonably general response theory. Actually, the results obtained here are a natural extension of such a finite interval theory.

The invariant properties of the solutions are discussed. Amplitude and phase invariants are calculated in terms of the roots of the characteristic equation. The following definitions are used throughout the development:

1. A configuration is considered to possess inherent dynamic stability if, upon being subjected to small disturbances, the configuration tends to, and in the limit does, return to the predisturbed state without undergoing large departures from the predisturbed state.

2. Missile response to controls means here the perturbation motions due to the application of scheduled drives.

3. Disturbing forces are regarded as being impulsive; that is,

$$\hat{F} = \Delta (m\dot{x}) = \lim_{t \to t_o} \int_{t_o}^{t} m\ddot{x}dt$$

The plane equations of motion are linearized by considering only small deviations from the undisturbed state of the missile. Practical limitations require that the motion be defined by the displacement and velocity derivatives. Available theoretical and experimental data indicate that such an assumption may be made reasonably. The stability coefficients considered are those which are determined by static parameters and a term involving the aerodynamic and jet damping. These, together with the control force and moment coefficients, may be regarded constant for small deviations from a given undisturbed state. The separation of the linearized pitch plane problem into two problems, of (1) inherent aerodynamic stability and (2) missile response to scheduled control forces and moments, is justifiable within the limitations of the linearized perturbation equations. This natural separation of the solutions is emphasized because of the basic difference in the application of these results to the art of determining suitable stability and response characteristics.

DEVELOPMENT

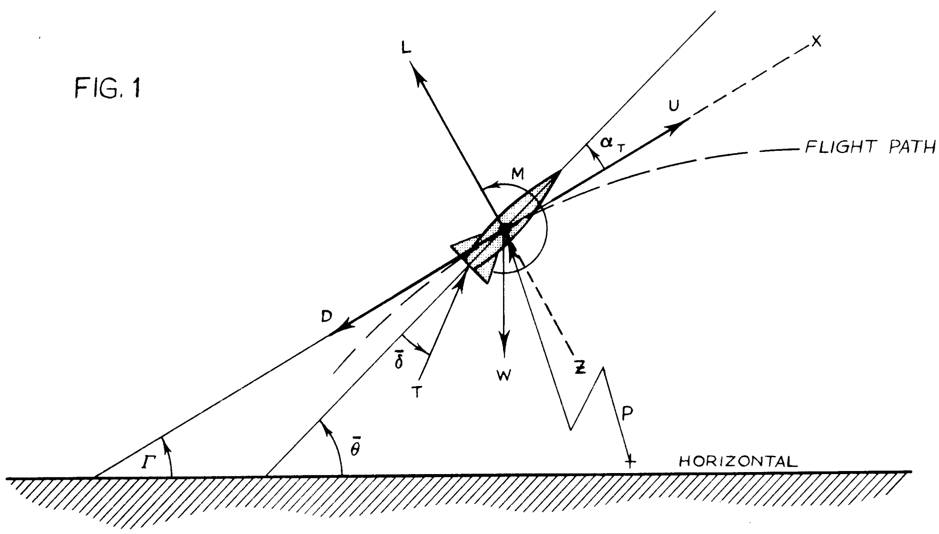
The rocket propelled missile is a non-closed dynamical system. The motions of the rigid body representation of this missile may be described by six differential equations involving translation along and rotation about three mutually perpendicular axes. The non-closed aspects of the system are easily accommodated in the equations as forces and moments applied to the missile. The mass and mass-moment characteristics of the missile may be regarded as functions of time.

The equations describing such a system may be linearized by the small perturbation technique. Assumption of a plane of symmetry then permits separation of these simplified equations into two independent systems of three simultaneous equations. Of interest here, the symmetric system represents motion in the plane of symmetry. The remaining, or asymmetric, system represents motion out of the plane of symmetry. These two systems of equations may be solved independently and the results summed linearly.

Equations of Motion in the Pitch Plane

The equations of motion are written for a rigid body moving along a path parallel to a fixed plane containing the gravity vector.

FIG. 1



Using the assumption of symmetry about the pitch plane, the small perturbation equations may be obtained from the following equations:

$$X = mU \tag{1}$$

$$Z = -mU\dot{\Gamma} \tag{2}$$

$$M = I\ddot{\bar{\theta}} \tag{3}$$

where equations (1) and (2) are the summation of forces tangent and normal to the missile flight path, and equation (3) is the summation of moments about the missile mass center.

For this plane, curvilinear motion, the wind axes, as shown in figure 1, form a convenient reference frame.

Linearizing the Equations

To aid in the formal linearization, the following relations are defined:

$$\begin{array}{lll} U - U_o = u & \Gamma - \Gamma_o = \gamma & X - X_o = dX \\ \alpha_T - \alpha_o = \alpha & \bar{\delta} - \delta_o = \delta & Z - Z_o = dZ \\ \bar{\theta} - \theta_o = \theta & P - P_o = p & M - M_o = dM \end{array} \tag{4}$$

The subscript zero indicates the values of the particular variables in the undisturbed condition. The right hand terms are the perturbation variations of the parameters of the missile motions. The perturbation variables are assumed to be mutually independent. The equations of the perturbation motions are found to be:

$$m\dot{u} = dX \tag{5}$$

$$mU_o\dot{\gamma} = -dZ \tag{6}$$

$$I\ddot{\theta} = dM \tag{7}$$

where products of the perturbation variables are neglected.

The aerodynamic forces acting on the missile are assumed to be those which would act in the instantaneous static situation. The aerodynamic moment is assumed to be the static moment except for a damping factor which may be estimated from static parameters alone. This latter representation may be improved if warranted by the circumstances. Any arbitrary thrust fluctuations will be treated as disturbances.

The forces dX, dZ, and the moment dM are calculated by Taylor series:

$$dX = \left(\frac{\partial X}{\partial U}\right)_{U_o} u + \left(\frac{\partial X}{\partial \Gamma}\right)_{\Gamma_o} \gamma + \left(\frac{\partial X}{\partial \alpha_T}\right)_{\alpha_o} \alpha \tag{8}$$

$$dZ = \left(\frac{\partial Z}{\partial U}\right)_{U_o} u + \left(\frac{\partial Z}{\partial \Gamma}\right)_{\Gamma_o} \gamma + \left(\frac{\partial Z}{\partial \alpha_T}\right)_{\alpha_o} \alpha \tag{9}$$

$$dM = \left(\frac{\partial M}{\partial U}\right)_{U_o} u + \left(\frac{\partial M}{\partial \dot{\theta}}\right)_{\dot{\theta}_o} \dot{\theta} + \left(\frac{\partial M}{\partial \alpha_T}\right)_{\alpha_o} \alpha \tag{10}$$

where products of the perturbation variables are again neglected.

The term $\frac{\partial M}{\partial \Gamma}$ does not appear among the terms for the moment because its contribution is zero. This is reasonable since Γ is regarded as an angle determining the orientation of the velocity vector with respect to a horizontal reference line. The stability force derivatives corresponding to this variable then will be functions of only the inclination of the velocity vector and the gravity vector.

To simplify the calculations we define:

$$X_u = \frac{\left(\frac{\partial X}{\partial U}\right)_{U_o}}{m} \quad X_\gamma = \frac{\left(\frac{\partial X}{\partial \Gamma}\right)_{\Gamma_o}}{m} \quad X_\alpha = \frac{\left(\frac{\partial X}{\partial \alpha_T}\right)_{\alpha_o}}{m} \quad X_\delta = \frac{\left(\frac{\partial X}{\partial \delta}\right)_{\delta_o}}{m}$$

$$Z_u = \frac{\left(\frac{\partial Z}{\partial U}\right)_{U_o}}{mU_o} \quad Z_\gamma = \frac{\left(\frac{\partial Z}{\partial \Gamma}\right)_{\Gamma_o}}{mU_o} \quad Z_\alpha = \frac{\left(\frac{\partial Z}{\partial \alpha_T}\right)_{\alpha_o}}{mU_o} \quad Z_\delta = \frac{\left(\frac{\partial Z}{\partial \delta}\right)_{\delta_o}}{mU_o} \tag{11}$$

$$M_u = \frac{\left(\frac{\partial M}{\partial U}\right)_{U_o}}{I} \quad M_{\dot{\theta}} = \frac{\left(\frac{\partial M}{\partial \dot{\theta}}\right)_{\dot{\theta}_o}}{I} \quad M_\alpha = \frac{\left(\frac{\partial M}{\partial \alpha_T}\right)_{\alpha_o}}{I} \quad M_\delta = \frac{\left(\frac{\partial M}{\partial \delta}\right)_{\delta_o}}{I}$$

In evaluating these coefficients it is assumed that lift, drag, and moment coefficients are known for the complete configuration, as well as their variation with angle of attack and Mach Number. These data may be obtained from test results or from available theory. Once the data are obtained, they may be directly substituted into the forms derived here.

Further assumptions made to derive these coefficients are:

1. In the neutral position, the rocket thrust acts through the missile mass center.

2. The thrust remains constant in magnitude, and its point of application is always at the center of the base of the missile.

3. The missile is in uniform motion in the pitch plane in the atmosphere.

4. The missile is symmetric about the pitch plane including the arrangement of any external control surfaces.

5. Aerodynamic forces and moments for a given configuration are uniquely determined by the magnitude of the resultant velocity vector and its direction with respect to the missile longitudinal axis. These forces and moments are not dependent upon missile attitude with respect to the ground, e.g.,

$$C_D = C_D(\alpha_T, U) \qquad C_L = C_L(\alpha_T, U) \qquad C_M = C_M(\alpha_T, \dot{\bar{\theta}}, U)$$

For the system as shown in figure 1, the forces and moments acting upon the missile may be described by:

$$X = T\cos(\alpha_T + \delta_o) - C_D \frac{\rho}{2} U^2 S - W \sin\Gamma + f(\delta) \tag{12}$$

$$Z = -T\sin(\alpha_T + \delta_o) - C_L \frac{\rho}{2} U^2 S + W\cos\Gamma + h(\delta) \tag{13}$$

$$M = C_M \frac{\rho}{2} U^2 S d + j(\delta) \tag{14}$$

where $f(\delta)$ and $h(\delta)$ are the X and Z components respectively of the applied scheduled control, and $j(\delta)$ is the moment due to this applied scheduled control. The stability and control coefficients may be readily calculated from the above equations directly, with the exception of the derivative $M_{\dot{\theta}}$ which may estimated from any one of several existing theories. We note that $\left(\frac{\partial}{\partial U}\right)_{U_o} \cong \frac{1}{a}\left(\frac{\partial}{\partial M_\infty}\right)_{M_\infty}$ where M_∞ is the free stream Mach Number. The coefficients are then found to be:

$$X_u = -\frac{\rho U_o S}{m}\left[C_{D_o} + \frac{M_\infty}{2}\left(\frac{\partial C_D}{\partial M_\infty}\right)_{M_\infty}\right]$$

$$X_\gamma = -g \cos \Gamma_o$$

$$X_\alpha = -\frac{q_o S}{m}\left[C_{T_o} \sin(\alpha_o + \delta_o) + \left(\frac{\partial C_D}{\partial \alpha}\right)_{\alpha_o}\right]$$

$$X_\delta = \frac{1}{m}\left[\frac{\partial f(\delta)}{\partial \delta}\right]_{\delta_o}$$

$$Z_u = -\frac{\rho S}{m}\left[C_{L_o} + \frac{M_\infty}{2}\left(\frac{\partial C_L}{\partial M_\infty}\right)_{M_\infty}\right]$$

$$Z_\gamma = -\frac{g}{U_o} \sin \Gamma_o$$

$$Z_\alpha = -\frac{\rho U_o S}{2m}\left[C_{T_o} \cos(\alpha_o + \delta_o) + \left(\frac{\partial C_L}{\partial \alpha}\right)_{\alpha_o}\right]$$

$$Z_\delta = \frac{1}{mU_o}\left[\frac{\partial h(\delta)}{\partial \delta}\right]_{\delta_o}$$

$$M_u = \frac{\rho U_o S d}{I}\left[C_{M_o} + \frac{M_\infty}{2}\left(\frac{\partial C_M}{\partial M_\infty}\right)_{M_\infty}\right]$$

$$M_{\dot{\theta}} = \frac{q_o S d}{I}\left(\frac{\partial C_M}{\partial \dot{\theta}}\right)_{\dot{\theta}_o}$$

$$M_\alpha = \frac{q_o S d}{I}\left(\frac{\partial C_M}{\partial \alpha}\right)_{\alpha_o}$$

$$M_\delta = \frac{1}{I}\left[\frac{\partial j(\delta)}{\partial \delta}\right]_{\delta_o}$$

When a swivelling jet is used for control purposes, the controls coefficients X_δ, Z_δ, and M_δ will include terms showing the change in the components of thrust due to its altered position, and should also include the effect of this new jet position upon the aerodynamic parameters. If an aerodynamic surface is used for control, these controls coefficients will include only the effect of the change of aerodynamic coefficients due to the altered control position, and will be of the form:

$$X_\delta = \frac{1}{m}\left[\frac{\partial f(\delta)}{\partial \delta}\right]_{\delta_o} = -\frac{q_o S}{m}\left(\frac{\partial C_D}{\partial \delta}\right)_{\delta_o}$$

$$Z_\delta = \frac{\left[\frac{\partial h(\delta)}{\partial \delta}\right]_{\delta_o}}{mU_o} = -\frac{\rho U_o S}{2m}\left(\frac{\partial C_L}{\partial \delta}\right)_{\delta_o}$$

$$M_\delta = \frac{\left[\frac{\partial j(\delta)}{\partial \delta}\right]_{\delta_o}}{I} = \frac{q_o S d}{I}\left(\frac{\partial C_M}{\partial \delta}\right)_{\delta_o}$$

The system of equations is now defined as:

$$\dot{u} - X_u u - X_\gamma \gamma - X_\alpha \alpha = X_\delta \delta \qquad (15)$$

$$\dot{\gamma} + Z_u u + Z_\gamma \gamma + Z_\alpha \alpha = - Z_\delta \delta \qquad (16)$$

$$\ddot{\alpha} + \ddot{\gamma} - M_{\dot{\theta}} (\dot{\alpha} + \dot{\gamma}) - M_\alpha \alpha - M_u u = M_\delta \delta \qquad (17)$$

Solution of the Small Perturbation Equations

The solutions of the system of equations describing the pitch plane perturbation motions are now obtained. If equations (15), (16), and (17) are multiplied by $e^{-st}\,dt$ ($R(s) > \epsilon > 0$) and integrated from $t = 0$ to $t = \infty$, while holding the stability and control (or differential) coefficients constant, we obtain the transformed equations:

$$(s - X_u)\tilde{u}(s) - X_\gamma\,\tilde{\gamma}(s) - X_\alpha\,\tilde{\alpha}(s) = X_\delta\,\tilde{f}(s) + u(o) \qquad (18)$$

$$Z_u\,\tilde{u}(s) + (s + Z_\gamma)\tilde{\gamma}(s) + Z_\alpha\,\tilde{\alpha}(s) = - Z_\delta\,\tilde{f}(s) + \gamma(o) \qquad (19)$$

$$-M_u\tilde{u}(s) + (s^2 - M_{\dot{\theta}}s)\tilde{\gamma}(s) + (s^2 - M_{\dot{\theta}}s - M_\alpha)\tilde{\alpha}(s) = M_\delta\tilde{f}(s) + \dot{\theta}(o) + (s - M_{\dot{\theta}})\theta(o) \qquad (20)$$

where the La Place transforms of the solutions being determined are represented by:

$$\tilde{u}(s) = \int_0^\infty e^{-st}\, u\,dt \qquad \tilde{\gamma}(s) = \int_0^\infty e^{-st}\, \gamma\,dt \qquad \tilde{\alpha}(s) = \int_0^\infty e^{-st}\, \alpha\,dt \quad \text{and where}$$

$$\tilde{f}(s) = \int_0^\infty e^{-st}\, \delta\,dt$$ is the La Place transform of the scheduled drive $\delta(t)$.

Here $\gamma(0)$, $u(0)$, $\theta(0)$, $\dot{\theta}(0)$, are the initial values of the perturbation variables. These values represent the symmetric force and moment contributions of the disturbing elements. It is noted that $\theta(0) = \alpha(0) + \gamma(0)$ and $\dot{\theta}(0) = \dot{\alpha}(0) + \dot{\gamma}(0)$.

Define the determinant

$$\Delta(s) = \begin{vmatrix} s-X_u & -X_\gamma & -X_\alpha \\ Z_u & s + Z_\gamma & Z_\alpha \\ -M_u & s^2 - M_{\dot{\theta}} s & s^2 - M_{\dot{\theta}} s - M_\alpha \end{vmatrix} \tag{21}$$

and the column matrix of the impressed forces and the instantaneous disturbing forces and moments as:

$$C = \begin{Bmatrix} u(0) + X_\delta \tilde{f}(s) \\ \gamma(0) - Z_\delta \tilde{f}(s) \\ [s - M_{\dot{\theta}}]\, \theta(0) + \dot{\theta}(0) + M_\delta \tilde{f}(s) \end{Bmatrix} \tag{22}$$

Then, with the aid of Cramer's Rule

$$\tilde{u}(s) = \frac{\begin{vmatrix} & -X_\gamma & -X_\alpha \\ C & s + Z_\gamma & Z_\alpha \\ & s^2 - M_{\dot{\theta}} s & s^2 - M_{\dot{\theta}} s - M_\alpha \end{vmatrix}}{\Delta(s)} \tag{23}$$

$$\tilde{\gamma}(s) = \frac{\begin{vmatrix} s - X_u & & -X_\alpha \\ Z_u & C & Z_\alpha \\ -M_u & & s^2 - M_{\dot{\theta}} s - M_\alpha \end{vmatrix}}{\Delta(s)} \tag{24}$$

$$\tilde{\alpha}(s) = \frac{\begin{vmatrix} s - X_u & -X_\gamma & \\ Z_u & S + Z_\gamma & C \\ -M_u & s^2 - M_{\dot{\theta}} s & \end{vmatrix}}{\Delta(s)} \tag{25}$$

These equations may be written in the form:[1]

$$\tilde{u}(s) = \frac{\begin{vmatrix} u(0) & -X_\gamma & -X_\alpha \\ \gamma(0) & s + Z_\gamma & Z_\alpha \\ (s - M_{\dot{\theta}})\theta(0) + \dot{\theta}(0) & s^2 - M_{\dot{\theta}} s & s^2 - M_{\dot{\theta}} s - M_\alpha \end{vmatrix}}{\Delta(s)} + \tilde{f}(s) \frac{\begin{vmatrix} X_\delta & -X_\gamma & -X_\alpha \\ -Z_\delta & s + Z_\gamma & Z_\alpha \\ M_\delta & s^2 - M_{\dot{\theta}} s & s^2 - M_{\dot{\theta}} s - M_\alpha \end{vmatrix}}{\Delta(s)}$$

$$= \frac{\Delta_u}{\Delta} + \tilde{f} \frac{d_u}{\Delta} \tag{26}$$

$$\tilde{\gamma}(s) = \frac{\begin{vmatrix} s - X_u & u(0) & -X_\alpha \\ Z_u & \gamma(0) & Z_\alpha \\ -M_u & (s - M_{\dot{\theta}})\theta(0) + \dot{\theta}(0) & s^2 - M_{\dot{\theta}} s - M_\alpha \end{vmatrix}}{\Delta(s)} + \tilde{f}(s) \frac{\begin{vmatrix} s - X_u & X_\delta & -X_\alpha \\ Z_u & -Z_\delta & Z_\alpha \\ -M_u & M_\delta & s^2 - M_{\dot{\theta}} s - M_\alpha \end{vmatrix}}{\Delta(s)}$$

$$= \frac{\Delta_\alpha}{\Delta} + \tilde{f} \frac{d_\alpha}{\Delta} \tag{27}$$

---

[1]From theory of determinants:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} + \beta_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} + \beta_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} + \beta_{33} \end{vmatrix} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & \beta_{23} \\ \alpha_{31} & \alpha_{32} & \beta_{33} \end{vmatrix}$$

$$\tilde{\alpha}(s)\ \frac{\begin{vmatrix} s-X_u & -X_\gamma & u(0) \\ Z_u & s+Z_\gamma & \gamma(0) \\ -M_u & s^2-M_{\dot\theta}s & (s-M_{\dot\theta})\theta(0)+\dot\theta(0) \end{vmatrix}}{\Delta(s)} + \tilde{f}(s)\ \frac{\begin{vmatrix} s-X_u & -X_\gamma & X_\delta \\ Z_u & s+Z_\gamma & -Z_\delta \\ -M_u & s^2-M_{\dot\theta}s & M_\delta \end{vmatrix}}{\Delta(s)}$$

$$= \frac{\Delta_\alpha}{\Delta} + \tilde{f}\,\frac{d_\alpha}{\Delta} \qquad (28)$$

The first terms to the right of the equality signs are the basis for an inherent dynamic stability study. $\Delta(s) = 0$ then is the characteristic equation of the system. Since solutions of the two particular problems are additive, there is no loss of generality if the study considers first the system with no impressed forces, and then the system free of disturbances.

Inherent Dynamic Stability-Controls Locked

Setting $\Delta(s) = 0$, we obtain the characteristic equation:

$$\Delta(s) = As^4 + Bs^3 + Cs^2 + Ds + E = 0 \qquad (29)$$

where

$$A = 1$$

$$B = -\left[Z_\alpha + M_{\dot\theta} + X_u - Z_\gamma\right]$$

$$C = -\left[M_\alpha + M_{\dot\theta}(Z_\gamma - X_u) + X_u Z_\gamma - Z_\alpha(M_{\dot\theta} + X_u) + Z_u(X_\alpha - X_\gamma)\right]$$

$$D = -\left[M_\alpha(Z_\gamma - X_u) - M_{\dot\theta}\left\{X_u(Z_\gamma - Z_\alpha) - Z_u(X_\gamma - X_\alpha)\right\} + M_u X_\alpha\right]$$

$$E = -\left[M_\alpha(X_\gamma Z_u - X_u Z_\gamma) + M_u(X_\alpha Z_\gamma - X_\gamma Z_\alpha)\right]$$

The necessary and sufficient condition that the missile undergo no divergent or constant amplitude motions is that the roots of this equation have no positive or zero real parts. This requirement is met if, and only if, the following conditions are satisfied:

1. The coefficients A, B, C, D, E, are all positive.

2. $BCD - AD^2 - B^2E > 0$.

These are the Routh conditions for stability (Ref. 3).

The transforms of the solutions required here are, from equations (26), (27), and (28):

$$\tilde{u}(s) = \frac{\Delta_u}{\Delta} \tag{30}$$

$$\tilde{\gamma}(s) = \frac{\Delta_\gamma}{\Delta} \tag{31}$$

$$\tilde{\alpha}(s) = \frac{\Delta_\alpha}{\Delta} \tag{32}$$

Assuming that the roots of equation (29) are all different, these solutions are found to be:[1]

$$u(t) = \sum_{i=1}^{4} \frac{\Delta_u(R_i)}{\Delta'(R_i)} e^{R_i t} = \sum_{i=1}^{4} u_i e^{R_i t} \tag{33}$$

$$\gamma(t) = \sum_{i=1}^{4} \frac{\Delta_\gamma(R_i)}{\Delta'(R_i)} e^{R_i t} = \sum_{i=1}^{4} \gamma_i e^{R_i t} \tag{34}$$

$$\alpha(t) = \sum_{i=1}^{4} \frac{\Delta_\alpha(R_i)}{\Delta'(R_i)} e^{R_i t} = \sum_{i=1}^{4} \alpha_i e^{R_i t} \tag{35}$$

where

$$\Delta'(R_i) = \left(\frac{d\Delta}{ds}\right)_{R_i}$$

---

[1]Given the roots R_i of $\Delta(s) = 0$, the required solutions may be obtained also in the following manner. $\Delta(s)$ may be written in the factored form:

$$\Delta(s) = \prod_{1}^{4} (s-R_i) = (s-R_1)(s-R_2)(s-R_3)(s-R_4) \tag{f1}$$

Equations (26), (27), and (28) may then be written as:

$$\tilde{u}(s) = \frac{\Delta_u}{\prod_{1}^{4} (s-R_i)} \tag{f2}$$

$$\tilde{\gamma}(s) = \frac{\Delta_\gamma}{\prod_{1}^{4} (s-R_i)} \tag{f3}$$

$$\tilde{\alpha}(s) = \frac{\Delta_\alpha}{\prod_{1}^{4} (s-R_i)} \tag{f4}$$

(footnote continued next page)

Missile Response to Controls

The problem of missile response to scheduled control forces and moments is next to be considered. Assuming the system to be free of disturbances, it is found from equations (26), (27) and (28):

$$\tilde{u}(s) = \tilde{f}\,\frac{d_u}{\Delta} \tag{36}$$

$$\tilde{\gamma}(s) = \tilde{f}\,\frac{d_\gamma}{\Delta} \tag{37}$$

$$\tilde{\alpha}(s) = \tilde{f}\,\frac{d_\alpha}{\Delta} \tag{38}$$

Inverting, the solutions to this problem are found to be:

$$u(t) = \sum_{i=1}^{4} \frac{d_u(R_i)}{\Delta'(R_i)} \int_0^t e^{R_i\beta}\,\delta(t-\beta)\,d\beta = \sum_{i=1}^{4} \frac{d_u(R_i)}{\Delta'(R_i)} \int_0^t e^{R_i(t-\beta)}\,\delta(\beta)\,d\beta \tag{39}$$

$$\gamma(t) = \sum_{i=1}^{4} \frac{d_\gamma(R_i)}{\Delta'(R_i)} \int_0^t e^{R_i\beta}\,\delta(t-\beta)\,d\beta = \sum_{i=1}^{4} \frac{d_\gamma(R_i)}{\Delta'(R_i)} \int_0^t e^{R_i(t-\beta)}\,\delta(\beta)\,d\beta \tag{40}$$

(footnote continued from previous page)

Working with equation (f4)

$$\tilde{\alpha}(s) = \frac{\Delta_\alpha}{\prod_1^4 (s-R_i)} = \frac{a_1}{s-R_1} + \frac{a_2}{s-R_2} + \frac{a_3}{s-R_3} + \frac{a_4}{s-R_4} \tag{f5}$$

The a_i are easily calculated to be:

$$a_1 = \frac{\Delta_\alpha(R_1)}{(R_1-R_2)(R_1-R_3)(R_1-R_4)} \qquad a_2 = \frac{\Delta_\alpha(R_2)}{(R_2-R_1)(R_2-R_3)(R_2-R_4)}$$

$$a_3 = \frac{\Delta_\alpha(R_3)}{(R_3-R_1)(R_3-R_2)(R_3-R_4)} \qquad a_4 = \frac{\Delta_\alpha(R_4)}{(R_4-R_1)(R_4-R_2)(R_4-R_3)} \tag{f6}$$

The inverse of equation (f4) is:

$$\alpha(t) = \sum_{i=1}^{4} a_i e^{R_i t} \tag{f7}$$

Similarly, b_i and c_i may be found by substitution of γ or u for α in the previous expressions, yielding:

(footnote continued next page)

$$\alpha(t) = \sum_{i=1}^{4} \frac{d_\alpha(Ri)}{\Delta'(Ri)} \int_0^t e^{R_i\beta}\,\delta(t-\beta)d\beta = \sum_{i=1}^{4} \frac{d_\alpha(Ri)}{\Delta'(R_i)} \int_0^t e^{Ri(t-\beta)}\,\delta(\beta)d\beta \qquad (41)$$

where the R_i are again the roots of $\Delta(s) = 0$.

Mathematical Review

It may be of interest to state the general development in more detail. Beginning with the equations of the perturbation motions, (equations 15, 16, and 17, where

$$\dot{u} - X_u u - X_\gamma \gamma - X_\alpha \alpha = X_\delta \delta$$

$$Z_u u + \dot{\gamma} + Z_\gamma \gamma + Z_\alpha \alpha = - Z_\delta \delta \qquad (42)$$

and

$$- M_u u + \ddot{\gamma} - M_{\dot{\theta}}\dot{\gamma} + \ddot{\alpha} - M_{\dot{\theta}}\dot{\alpha} - M_\alpha \alpha = M_\delta \delta \qquad),$$

(footnote continued from previous page)

$$\gamma(t) = \sum_{i=1}^{4} b_i e^{R_i t} \qquad (f8)$$

$$u(t) = \sum_{i=1}^{4} c_i e^{R_i t} \qquad (f9)$$

It can be easily shown that the a_i, b_i, and c_i obtained here are the α_i, γ_i, and u_i previously determined. Examining a_1:

$$\frac{a_1}{\Delta_\alpha(R_1)} = \frac{1}{(R_1-R_2)(R_1-R_3)(R_1-R_4)} = \lim_{s\to R_1} \frac{(s-R_1)}{\Delta(s)} \qquad (f10)$$

by L'Hospital's Rule:

$$\lim_{s\to R_1} \frac{(s-R_1)}{\Delta(s)} = \lim_{s\to R_1} \frac{1}{\Delta'(s)} \qquad (f11)$$

where

$$\Delta'(s) = \frac{d\Delta}{ds}$$

Then

$$\alpha(t) = \sum_{i=1}^{4} \alpha_i e^{R_i t} = \sum_{i=1}^{4} a_i e^{R_i t} \qquad (f12)$$

$$\gamma(t) = \sum_{i=1}^{4} \gamma_i e^{R_i t} = \sum_{i=1}^{4} b_i e^{R_i t} \qquad (f13)$$

(footnote continued next page)

the dependent variables and their derivatives with respect to time, t, are operated upon as follows:

1. Multiply each term of equation (42) by $e^{-st}dt$.

2. Integrate with respect to t from 0 to ∞. This results in three algebraic equations in three unknowns, $\bar{u}(s)$, $\bar{\gamma}(s)$, and $\bar{\alpha}(s)$. These are equations (18), (19), and (20).

It is to be noted that in performing this integration the following is obtained:

$$\int_0^\infty e^{-st}\dot{u}dt = e^{-st}u]_0^\infty + s\int_0^\infty e^{-st}udt = -u(0) + s\,\tilde{u}(s)$$

The variables $\ddot{\alpha}$ and $\ddot{\gamma}$, for example, imply making use of this result twice.

(footnote continued from previous page)

$$u(t) = \sum_{i=1}^{4} u_i e^{R_i t} = \sum_{i=1}^{4} c_i e^{R_i t} \qquad (f14)$$

The solutions may be verified in the following straightforward manner. By definition of the transform:

$$\tilde{\alpha}(s) = \int_0^\infty e^{-st}\alpha(t)dt = \sum_{i=1}^{4}\alpha_i \int_0^\infty e^{-(s-R_i)t}dt \qquad (f15)$$

Performing the indicated integration yields

$$\tilde{\alpha}(s) = \sum_{i=1}^{4}\left[-\frac{\alpha_i}{s-R_i}e^{-(s-R_i)t}\right]_0^\infty = \sum_{i=1}^{4}\frac{\alpha_i}{s-R_i} \qquad (f16)$$

This is precisely equation (f4). $\gamma(t)$ and u(t) may, of course, be similarly verified.

The resulting algebraic equations are then:

$$(s-X_u)\,\tilde{u}(s) - X_\gamma\tilde{\gamma}(s) - X_\alpha\tilde{\alpha}(s) = X_\delta\tilde{f}(s) + u(0)$$

$$Z_u\tilde{u}(s) + (s + Z_\gamma)\tilde{\gamma}(s) + Z_\alpha\tilde{\alpha}(s) = - Z_\delta\tilde{f}(s) + \gamma(0) \qquad (43)$$

$$- M_u\tilde{u}(s) + (s^2 - M_{\dot\theta}s)\tilde{\gamma}(s) + (s^2 - M_{\dot\theta}s-M_\alpha)\tilde{\alpha}(s) = M_\delta\tilde{f}(s) + (s-M_{\dot\theta})\theta(0) + \dot\theta(0)$$

The use of the transform leads directly to several fundamental results that may be established more easily with its aid. Aside from the then relative familiarity of the tools required to establish these basic results there is the further argument of the direct manner in which solutions involving a quite general forcing function may be found.

Returning to the transforms $\tilde{u}(s)$, $\tilde{\gamma}(s)$ and $\tilde{\alpha}(s)$ of equations (43), we may evidently solve for these transforms of the required solutions. These are then functions of the parameter s, the stabil'ty and control coefficients, and the initial conditions. Here, this was done using Cramer's rule from the theory of determinants. The solutions were then separated into those corresponding to the two basic problems being treated. The result is given in equations (26), (27) and (28) and is repeated here using the abbreviated notation.

$$\tilde{u}(s) = \frac{\Delta_u}{\Delta} + \tilde{f}\,\frac{d_u}{\Delta}$$

$$\tilde{\gamma}(s) = \frac{\Delta_\gamma}{\Delta} + \tilde{f}\,\frac{d_\gamma}{\Delta} \qquad (44)$$

$$\tilde{\alpha}(s) = \frac{\Delta_\alpha}{\Delta} + \tilde{f}\,\frac{d_\alpha}{\Delta}$$

The remaining task is the inversion or determination of the required solution. The procedure that has been followed here is:

1. Assume no applied forces $\quad X_\delta = Z_\delta = M_\delta = 0$

2. Assume no disturbances $\quad \theta(0) = \gamma(0) = u(0) = \dot\theta(0) = 0$

Assuming no applied forces, equation (44) reduces to:

$$\tilde{u}(s) = \frac{\Delta_u}{\Delta} \qquad \tilde{\gamma}(s) = \frac{\Delta_\gamma}{\Delta} \qquad \tilde{\alpha}(s) = \frac{\Delta_\alpha}{\Delta} \qquad (45)$$

Then, assuming no disturbances, equation (44) reduces to:

$$\tilde{u}(s) = \tilde{f}\,\frac{d_u}{\Delta} \qquad \tilde{\gamma}(s) = \tilde{f}\,\frac{d_\gamma}{\Delta} \qquad \tilde{\alpha}(s) = \tilde{f}\,\frac{d_\alpha}{\Delta} \tag{46}$$

The components $\frac{\Delta_u}{\Delta}$, $\frac{\Delta_\gamma}{\Delta}$, $\frac{\Delta_\alpha}{\Delta}$ and $\frac{d_u}{\Delta}$, $\frac{d_\gamma}{\Delta}$, $\frac{d_\alpha}{\Delta}$ are the ratios of polynomials in s. It was shown for $\tilde{\alpha}(s)$ [equations (f15) and (f16)] that the transform of $e^{R_i t}$ is $\frac{1}{s-R_i}$. This result is of immediate interest here. Using partial fractions,the expressions for the components may be written in the form

$$\frac{A_1}{s-R_1} + \frac{A_2}{s-R_2} + \frac{A_3}{s-R_3} + \frac{A_4}{s-R_4}$$

In the controls locked problem of equation (45) we then simply invert term by term.

$$A_1 e^{R_1 t} + A_2 e^{R_2 t} + A_3 e^{R_3 t} + A_4 e^{R_4 t}$$

The solution for the response to a scheduled control force from equation (46) requires a few more steps. The components are further multiplied by the transform of the applied force. If $\tilde{f}(s)$ were a polynomial or had certain simple properties, we might apply partial fractions again. The results may be obtained more easily with fewer restrictions if the following result is utilized. The integral $\int_0^t F_1(t-\lambda)F_2(\lambda)d\lambda$ transforms into $\tilde{f}_1(s)\tilde{f}_2(s)$ (Ref. 2, p.37).

The solution of the response problem up to this point is of the form:

$$\frac{B_1}{s-R_1}\tilde{f}(s) + \frac{B_2}{s-R_2}\tilde{f}(s) + \frac{B_3}{s-R_3}\tilde{f}(s) + \frac{B_4}{s-R_4}\tilde{f}(s)$$

Making use of the above integral, the required solutions are of the type:

$$B_i \int_0^t e^{R_i \lambda}\,\delta(t-\lambda)d\lambda \qquad \text{or} \qquad B_i \int_0^t e^{R_i(t-\lambda)}\delta(\lambda)d\lambda$$

the two integrals being equivalent. The required solutions are given in equations (39), (40) and (41).

Invariant Characteristics of the Solutions[1]

The solutions describing the pitch plane perturbation motions have certain useful invariant characteristics. The more important and useful of these results which will be derived may be readily extended to systems of n linear simultaneous ordinary differential equations with constant coefficients.

The solutions obtained for the inherent dynamic stability problem are recalled to be equations (33), (34) and (35) :

$$u(t) = \sum_{i=1}^{4} u_i e^{R_i t} \qquad \gamma(t) = \sum_{i=1}^{4} \gamma_i e^{R_i t} \qquad \alpha(t) = \sum_{i=1}^{4} \alpha_i e^{R_i t}$$

Let any one of these solutions be labelled x(t), and either of the remaining solutions y(t). It can be shown that the ratio $\frac{x(t)}{y(t)} = \lambda_o$ (where λ_o is a constant) holds if and only if,

$$x_i = \lambda_o y_i \qquad (i = 1, 2, 3, 4) \tag{47}$$

The condition (47) must hold for all i. If, however, λ_o is defined for a particular i, the equality will be violated for at least one of the other i. It is therefore necessary to consider the ratios of the components of x(t) and y(t) evaluated at given i. In determining these ratios, the following property of determinants will be used:

Let

$$\Delta(s) = \det(a_{ij})$$

Then, the adjoint of $\det(a_{ij})$ is defined as $\det(A_{ij})$, where the A_{ij} are the cofactors of the elements a_{ij}. It is then possible to prove that (Ref.1, p. 30)

$$\alpha_{ij} = a_{ij}\det(a_{ij}) \tag{48}$$

[1]The authors are indebted to Mr. Lester S. Hecht for suggesting the utility of the key theorem in the argument of this section.

where the α_{ij} are the cofactors of the elements A_{ij}.

The invariants of the solutions can now be calculated. Let:

$$\eta_1 = \frac{\Delta_u(R_i)}{\Delta_\gamma(R_i)} \qquad \eta_2 = \frac{\Delta_u(R_i)}{\Delta_\alpha(R_i)} \qquad \eta_3 = \frac{\Delta_\alpha(R_i)}{\Delta_\gamma(R_i)} = \frac{\eta_1}{\eta_2}$$

These are the amplitude ratios of the components of the solutions $u(t)$, $\gamma(t)$ and $\alpha(t)$ for any given root. The number η_1 may be obtained from equations (26) and (27), assuming that $X_\delta = Z_\delta = M_\delta = 0$.

$$\eta_1 = \frac{X\,A_{11} + Y\,A_{21} + Z\,A_{31}}{X\,A_{12} + Y\,A_{22} + Z\,A_{32}} \tag{49}$$

where $X = u(0)$, $Y = \gamma(0)$, $Z = (R_i - M_{\dot\theta})\theta(0) + \dot\theta(0)$.

Now from equation (48), $\alpha_{ij}(R_i) = 0$; that is, the cofactors of the adjoint of $\det(a_{ij})$ are all zero. The following conditions are then satisfied:

$$\frac{A_{11}}{A_{12}} = \frac{A_{21}}{A_{22}} = \frac{A_{31}}{A_{32}} = K_1(R_i) \tag{50}$$

$$\frac{A_{11}}{A_{13}} = \frac{A_{21}}{A_{23}} = \frac{A_{31}}{A_{33}} = K_2(R_i) \tag{51}$$

$$\frac{A_{12}}{A_{13}} = \frac{A_{22}}{A_{23}} = \frac{A_{32}}{A_{33}} = K_3(R_i) = \frac{K_2(R_i)}{K_1(R_i)} \tag{52}$$

From equation (50), $\eta_1 = K_1(R_i)$. Similarly $\eta_2 = K_2(R_i)$ and $\eta_3 = K_3(R_i)$. The numbers η_1, η_2, η_3 are independent of initial conditions and time. These are the amplitude ratios of interest for the components of the solution due to the real roots. In the event that it is desired to calculate the invariants for the components of the solutions due to a pair of conjugate complex roots, the following will be the ratios to be calculated:

$$\xi_1 = \frac{\dfrac{\Delta_u(R_i)}{\Delta'(R_i)}\, e^{R_i t} + \dfrac{\Delta_u(\bar R_i)}{\Delta'(\bar R_i)}\, e^{\bar R_i t}}{\dfrac{\Delta_\gamma(R_i)}{\Delta'(R_i)}\, e^{R_i t} + \dfrac{\Delta_\gamma(\bar R_i)}{\Delta'(\bar R_i)}\, e^{\bar R_i t}}$$

$$\xi_2 = \frac{\dfrac{\Delta_u(R_i)}{\Delta'(R_i)}\, e^{R_i t} + \dfrac{\Delta_u(\bar{R}_i)}{\Delta'(\bar{R}_i)}\, e^{\bar{R}_i t}}{\dfrac{\Delta_\alpha(R_i)}{\Delta'(R_i)}\, e^{R_i t} + \dfrac{\Delta_\alpha(\bar{R}_i)}{\Delta'(\bar{R}_i)}\, e^{\bar{R}_i t}}$$

$$\xi_3 = \frac{\dfrac{\Delta_\alpha(R_i)}{\Delta'(R_i)}\, e^{R_i t} + \dfrac{\Delta_\alpha(\bar{R}_i)}{\Delta'(\bar{R}_i)}\, e^{\bar{R}_i t}}{\dfrac{\Delta_\gamma(R_i)}{\Delta'(R_i)}\, e^{R_i t} + \dfrac{\Delta_\gamma(\bar{R}_i)}{\Delta'(\bar{R}_i)}\, e^{\bar{R}_i t}}$$

The expression for ξ_1 may be simplified to

$$\xi_1 = \frac{\Delta'(\bar{R}_i)\Delta_u(R_i)e^{j\beta_i t} + \Delta'(R_i)\Delta_u(\bar{R}_i)e^{-j\beta_i t}}{\Delta'(\bar{R}_i)\Delta_\gamma(R_i)e^{j\beta_i t} + \Delta'(R_i)\Delta_\gamma(\bar{R}_i)e^{-j\beta_i t}} \tag{53}$$

where $j\beta_i$ is the imaginary part of the complex root. But, it has been previously shown that $\Delta_u(R_i) = \eta_1\Delta_\gamma(R_i) = K_1(R_i)\Delta_\gamma(R_i)$. Equation (53) may therefore be written in the form:

$$\xi_1 = \frac{K_1(R_i)\Delta_\gamma(R_i)\Delta'(\bar{R}_i)e^{j\beta_i t} + K_1(\bar{R}_i)\Delta_\gamma(\bar{R}_i)\Delta'(R_i)e^{-j\beta_i t}}{\Delta_\gamma(R_i)\Delta'(\bar{R}_i)e^{j\beta_i t} + \Delta_\gamma(\bar{R}_i)\Delta'(R_i)e^{-j\beta_i t}} \tag{54}$$

Now let

$$\Delta'(R_i) = r_i\, e^{j\psi_i} \qquad K_1(R_i) = r_{K_1}\, e^{j\psi_{K_1}}$$

$$\Delta_u(R_i) = r_u\, e^{j\psi_u} \qquad K_2(R_i) = r_{K_2}\, e^{j\psi_{K_2}}$$

$$\Delta_\gamma(R_i) = r_\gamma\, e^{j\psi_\gamma} \qquad K_3(R_i) = r_{K_3}\, e^{j\psi_{K_3}}$$

$$\Delta_\alpha(R_i) = r_\alpha\, e^{j\psi_\alpha}$$

The ratio ξ_1 is then found to be:

$$\xi_1 = r_{K_1}\frac{\cos(\psi_\gamma + \beta_i t - \psi_i + \psi_{K_1})}{\cos(\psi_\gamma + \beta_i t - \psi_i)} \tag{55}$$

where

$$R_{K_1} = \text{modulus } K_1(R_i)$$

$$\psi_{K_1} = \tan^{-1} \frac{\text{Imag } K_1(R_i)}{\text{Real } K_1(R_i)}$$

where r_{K_1} and ψ_{K_1} are independent of both initial conditions and time. By similar calculations

$$\xi_2 = r_{K_2} \frac{\cos(\psi_\alpha + \beta_i t - \psi_i + \psi_{K_2})}{\cos(\psi_\alpha + \beta_i t - \psi_i)} \tag{56}$$

$$\xi_3 = r_{K_3} \frac{\cos(\psi_\gamma + \beta_i t - \psi_i + \psi_{K_3})}{\cos(\psi_\gamma + \beta_i t - \psi_i)} \tag{57}$$

The numbers r_{K_1}, r_{K_2}, r_{K_3}, ψ_{K_1}, ψ_{K_2}, ψ_{K_3} are invariant with time and choice of initial conditions. The nature of the invariance is more restricted if the ratios of the components at unequal roots are considered. The calculation of a particular ratio will suffice to bring out the pertinent details.

Consider the ratio

$$E\left(\frac{u_i}{u_j}\right) = \frac{\dfrac{\Delta_u(R_i)}{\Delta'(R_i)} e^{R_i t} + \dfrac{\Delta_u(\overline{R}_i)}{\Delta'(\overline{R}_i)} e^{\overline{R}_i t}}{\dfrac{\Delta_u(R_j)}{\Delta'(R_j)} e^{R_j t} + \dfrac{\Delta_u(\overline{R}_j)}{\Delta'(\overline{R}_j)} e^{\overline{R}_j t}} \tag{58}$$

where $R_j \neq R_i$ and also $R_j \neq \overline{R}_i$. After expanding and simplifying, this becomes:

$$E\left(\frac{u_i}{u_j}\right) = \frac{\Delta'(R_j)\Delta'(\overline{R}_j)}{\Delta'(R_i)\Delta'(\overline{R}_i)} \cdot$$

$$\left\{\frac{\Delta'(\overline{R}_i)(XA_{11}+YA_{21}+ZA_{31})e^{R_i t}+\Delta'(R_i)(\overline{XA}_{11}+\overline{YA}_{21}+\overline{ZA}_{31})e^{\overline{R}_i t}}{\Delta'(\overline{R}_j)(xB_{11}+yB_{21}+zB_{31})e^{R_j t}+\Delta'(R_j)(\overline{xB}_{11}+\overline{yB}_{21}+\overline{zB}_{31})e^{\overline{R}_j t}}\right\} \tag{59}$$

This equation is a function of time as well as initial conditions.

It is noted that

$$X = \overline{X} = x = \overline{x} = u(0)$$

$$Y = \overline{Y} = y = \overline{y} = \gamma(0)$$

$$Z = (R_i - M_{\dot{\theta}})\theta(0) + \dot{\theta}(0)$$

$$z = (R_j - M_{\dot{\theta}})\theta(0) + \dot{\theta}(0)$$

Evidently, $E\left(\frac{u_i}{u_j}\right)$ does not depend upon initial conditions if it is calculated for the four conditions $u(0)$, $\gamma(0)$, $\theta(0)$, $\dot{\theta}(0)$ different from zero one at a time. With this restriction, the amplitude invariants may be calculated as functions of time, but independent of the initial conditions one at a time.

APPLICATIONS

A Simple Disturbance

For a given missile, flying under given conditions, solutions may of course be summed linearly.[1] Solutions might be obtained with but one of the initial conditions $u(0)$, $\gamma(0)$, $\theta(0)$, $\dot{\theta}(0)$ different from zero. Such solutions could then be varied and combined to include several causes of disturbance, and variation of the location of these disturbing elements with respect to the missile. Sharp-edged gusts, exploding shells, and instantaneous variations of thrust are possible disturbances which might be handled in this fashion.

For example, assume a missile to experience a perturbation change in velocity, $u(0) = u_o$ while $\gamma(0) = \theta(0) = \dot{\theta}(0) = 0$. The solution $u(t)$, giving the change in velocity due to this disturbance, is:

$$ {}_1u(t) = \sum_{i=1}^{4} u_i e^{R_i t} \qquad (60)$$

where
$$u_i = \frac{\Delta_u(R_i)}{\Delta'(R_i)}$$

---

[1]
$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & \beta_{23} \\ \alpha_{31} & \alpha_{32} & \beta_{33} \end{vmatrix} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{31} & \alpha_{32} & 0 \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & \beta_{23} \\ \alpha_{31} & \alpha_{32} & 0 \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ \alpha_{31} & \alpha_{32} & \beta_{33} \end{vmatrix}$$

Interpreting a solution is often simplified if one of the boundary conditions is assigned a value and the others set equal to zero. This gives the effect of a single boundary condition upon the missile motion. If this is done for each of the boundary conditions in turn, the solutions express the effect of each condition. Since the equations of motion are linear after application of the small perturbation principle, the effect of two or more boundary conditions is obtained by a simple summation of solutions.

and where $\Delta_u(R_i)$ is determined from:

$$\Delta_u(s) = u_o \begin{vmatrix} s^2 - M_{\dot{\theta}}s - M_\alpha & s^2 - M_{\dot{\theta}}s \\ Z_\alpha & s + Z_\gamma \end{vmatrix} \tag{61}$$

Similarly, the variation in angle of attack is given by:

$$_1\alpha(t) = \sum_{i=1}^{4} \alpha_i e^{R_i t} \tag{62}$$

where

$$\alpha_i = \frac{\Delta_\alpha(R_i)}{\Delta'(R_i)}$$

and where $\Delta_\alpha(R_i)$ is determined from:

$$\Delta_\alpha(s) = u_o \begin{vmatrix} Z_u & s + Z_\gamma \\ -M_u & s^2 - M_{\dot{\theta}}s \end{vmatrix} \tag{63}$$

Finally, the variation of the inclination of the missile velocity vector is:

$$_1\gamma(t) = \sum_{i=1}^{4} \gamma_i e^{R_i t} \tag{64}$$

where

$$\gamma_i = \frac{\Delta_\gamma(R_i)}{\Delta'(R_i)}$$

and where $\Delta_\gamma(R_i)$ is determined from:

$$\Delta_\gamma(s) = -u_o \begin{vmatrix} Z_u & Z_\alpha \\ -M_u & s^2 - M_{\dot{\theta}}s - M_\alpha \end{vmatrix} \tag{65}$$

Solutions for other initial disturbances may be obtained in much the same manner. A solution for a complicated set of initial disturbances may be obtained by adding the solutions for each of the initial conditions alone. Evidently, if $u_1 = Cu_o$, where C is a constant, the solutions become:

$$
\begin{aligned}
{}_2u(t) &= C\ {}_1u(t) \\
{}_2\gamma(t) &= C\ {}_1\gamma(t) \\
{}_2\alpha(t) &= C\ {}_1\alpha(t)
\end{aligned}
\tag{66}
$$

Constant Control Moment

The case in which a constant moment is applied to the missile may be treated with equal facility by several elementary methods. The simplicity of this particular case makes the problem of obtaining solutions just about equally tractable to all approaches. Let

$$
M_\delta \delta = m_o \quad (t > 0) \tag{67}
$$

If the application of the moment $\frac{m_o}{I}$ should involve applied forces of considerable magnitude, one may add to the solution found here solutions yielding the effect of the applied tangential force $X_\delta \delta = X_o$, and the solutions corresponding to the applied normal force $Z_\delta \delta = Z_o$.

In many applications, taking $X_\delta \delta = Z_\delta \delta = 0$ is a legitimate approximation. This assumption is made for the present example. From equations (39), (40) and (41), the solutions for this case are given by:

$$
u(t) = \frac{m_o}{M_\delta} \sum_{i=1}^{4} u_i e^{R_i t} \int_0^t e^{-R_i \beta} d\beta \tag{68}
$$

$$
\gamma(t) = \frac{m_o}{M_\delta} \sum_{i=1}^{4} \gamma_i e^{R_i t} \int_0^t e^{-R_i \beta} d\beta \tag{69}
$$

$$
\alpha(t) = \frac{m_o}{M_\delta} \sum_{i=1}^{4} \alpha_i e^{R_i t} \int_0^t e^{-R_i \beta} d\beta \tag{70}
$$

The above solutions may be also obtained by making use of partial fractions directly upon the response transform. In a sense, the numerical work is somewhat shortened by such an approach in this particular case.

Since $\delta = \frac{m_o}{M_\delta}$, the transform of the applied moment may be written as:

$$\tilde{f}(s) = \frac{m_o}{M_\delta}\frac{1}{s} \tag{71}$$

Substituting this form into equation (36) yields

$$s\,\tilde{u}(s) = \frac{m_o}{M_\delta}\frac{d_u}{\Delta} \tag{72}$$

Then, recalling the result of equation (f11), this becomes

$$\tilde{u}(s) = \frac{m_o}{M_\delta}\sum_{i=0}^{4}\frac{d_u(R_i)}{\Delta(R_i) + R_i\Delta'(R_i)}\;\frac{1}{s-R_i} \tag{73}$$

The R_i here are the roots of $s\,\Delta(s) = 0$. The $\Delta(s)$ is still that of equation (21). Let R_o represent the zero root. The required solution is then:

$$u(t) = \frac{m_o}{M_\delta}\left[\frac{d_u(0)}{\Delta(0)} + \sum_{i=1}^{4}\frac{d_u(R_i)}{R_i\Delta'(R_i)}\,e^{R_i t}\right] \tag{74}$$

Similarly, from equations (37) and (38) the remaining solutions are obtained:

$$\gamma(t) = \frac{m_o}{M_\delta}\left[\frac{d_\gamma(0)}{\Delta(0)} + \sum_{i=1}^{4}\frac{d_\gamma(R_i)}{R_i\Delta'(R_i)}\,e^{R_i t}\right] \tag{75}$$

$$\alpha(t) = \frac{m_o}{M_\delta}\left[\frac{d_\alpha(0)}{\Delta(0)} + \sum_{i=1}^{4}\frac{d_\alpha(R_i)}{R_i\Delta'(R_i)}\,e^{R_i t}\right] \tag{76}$$

In obtaining the above solutions, a missile orientation was assumed as well as a velocity. It is then assumed that a constant moment is applied to the missile. The changes in missile orientation and velocity due to the applied moment are of two kinds. For a stable system, the terms $\sum_1^4 \frac{d_u(R_i)}{R_i\Delta'(R_i)}e^{R_i t}$, $\sum_1^4 \frac{d_\gamma(R_i)}{R_i\Delta'(R_i)}e^{R_i t}$ and $\sum_1^4 \frac{d_\alpha(R_i)}{R_i\Delta'(R_i)}e^{R_i t}$ represent transient effects that disappear with time. The term $\frac{m_o d_\alpha(0)}{M_\delta\,\Delta(0)}$ represents the increment of angle of attack due to the applied moment. The term

$\dfrac{m_o d_\gamma(0)}{M_\delta \, \Delta(0)}$ gives the increment of angle between the horizontal and the missile velocity vector. The term $\dfrac{m_o d_u(0)}{M_\delta \, \Delta(0)}$ represents the increment in missile velocity measured along the new tangent to the missile flight path.

Step Function Control Deflection

The instantaneous application of a constant control deflection at a time t_o, ($t_o > 0$) is considered here. This result is more general than that of the preceding section (constant moment) in that X_δ and Z_δ are no longer considered to be zero. The system is also no longer regarded as disturbance-free. Furthermore, by summing solutions, there is a means of getting at the problems involving constant deflections during finite intervals in time.

The perturbation control deflection is given by

$$\begin{aligned} \delta(t) &= 0 & 0 < t < t_o \\ \delta(t) &= b, \text{ a constant} & t > t_o \end{aligned} \tag{77}$$

The transform of $\delta(t)$ is

$$\tilde{f}(s) = \frac{be^{-t_o s}}{s} \tag{78}$$

The transforms of the required solutions are found by substitution of $\dfrac{be^{-t_o s}}{s}$ in equations (26), (27), and (28). These are:

$$\Delta(s)\tilde{u}(s) = \Delta_u + \frac{be^{-t_o s}}{s} d_u \tag{79}$$

$$\Delta(s)\tilde{\gamma}(s) = \Delta_\gamma + \frac{be^{-t_o s}}{s} d_\gamma \tag{80}$$

$$\Delta(s)\tilde{\alpha}(s) = \Delta_\alpha + \frac{be^{-t_o s}}{s} d_\alpha \tag{81}$$

For certain purposes it is useful to calculate the coefficients as follows. The terms $\frac{\Delta_u}{\Delta}$, $\frac{\Delta_\gamma}{\Delta}$, $\frac{\Delta_\alpha}{\Delta}$ here correspond to the controls-locked solutions obtained in the section on inherent dynamic stability. The required solutions are given by equations (33), (34) and (35). (The subscript "controls locked" is added.) These solutions are

$$u(t)_{\substack{\text{controls}\\ \text{locked}}} \qquad \gamma(t)_{\substack{\text{controls}\\ \text{locked}}} \qquad \alpha(t)_{\substack{\text{controls}\\ \text{locked}}} \tag{82}$$

The remaining terms, yet to be inverted, are

$$\frac{be^{-t_o s} d_\alpha}{s\Delta} \qquad \frac{be^{-t_o s} d_\gamma}{s\Delta} \qquad \frac{be^{-t_o s} d_u}{s\Delta} \tag{83}$$

Expanding these in minors of the column of control coefficients X_δ, Z_δ, M_δ, the first term of equation (83) becomes:

$$\frac{be^{-t_o s} d_\alpha}{s\Delta} = \frac{be^{-t_o s}}{s\Delta}\left[M_\delta \begin{vmatrix} s+Z_\gamma & Z_u \\ -X_\gamma & s-X_u \end{vmatrix} + Z_\delta \begin{vmatrix} s^2-M_{\dot\theta} s & -M_u \\ -X_\gamma & s-X_u \end{vmatrix} + X_\delta \begin{vmatrix} s^2-M_{\dot\theta} s & -M_u \\ s+Z_\gamma & Z_u \end{vmatrix} \right]$$

$$= \frac{be^{-t_o s}}{s\Delta}\left[M_\delta d_{\alpha_{M_\delta}}(s) + Z_\delta d_{\alpha_{Z_\delta}}(s) + X_\delta d_{\alpha_{X_\delta}}(s) \right] \tag{84}$$

Similarly

$$\frac{be^{-t_o s} d_\gamma}{s\Delta} = \frac{be^{-t_o s}}{s\Delta}\left[M_\delta d_{\gamma_{M_\delta}}(s) + Z_\delta d_{\gamma_{Z_\delta}}(s) + X_\delta d_{\gamma_{X_\delta}}(s) \right] \tag{85}$$

$$\frac{be^{-t_o s} d_u}{s\Delta} = \frac{be^{-t_o s}}{s\Delta}\left[M_\delta d_{u_{M_\delta}}(s) + Z_\delta d_{u_{Z_\delta}}(s) + X_\delta d_{u_{X_\delta}}(s) \right] \tag{86}$$

Factoring

$$\frac{be^{-t_o s}}{s\Delta} d_\alpha = be^{-t_o s}\left\{M_\delta \sum_0^4 \frac{d_{\alpha_{M_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i} + \right.$$

$$\left. Z_\delta \sum_0^4 \frac{d_{\alpha_{Z_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i} + X_\delta \sum_0^4 \frac{d_{\alpha_{X_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i}\right\} \quad (87)$$

$$\frac{be^{-t_o s}}{s\Delta} d_\gamma = be^{-t_o s}\left\{M_\delta \sum_0^4 \frac{d_{\gamma_{M_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i} + \right.$$

$$\left. Z_\delta \sum_0^4 \frac{d_{\gamma_{Z_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i} + X_\delta \sum_0^4 \frac{d_{\gamma_{X_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i}\right\} \quad (88)$$

$$\frac{be^{-t_o s}}{s\Delta} d_u = be^{-t_o s}\left\{M_\delta \sum_0^4 \frac{d_{u_{M_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i} + \right.$$

$$\left. Z_\delta \sum_0^4 \frac{d_{u_{Z_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i} + X_\delta \sum_0^4 \frac{d_{u_{X_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)} \frac{1}{s-R_i}\right\} \quad (89)$$

Inverting and combining with the controls-locked solutions

$$\alpha(t) = \alpha(t)_{\substack{\text{controls}\\ \text{locked}}} + b\left\{M_\delta \sum_0^4 \alpha_{i_{M_\delta}} e^{R_i(t-t_o)} + Z_\delta \sum_0^4 \alpha_{i_{Z_\delta}} e^{R_i(t-t_o)} + X_\delta \sum_0^4 \alpha_{i_{X_\delta}} e^{R_i(t-t_o)}\right\} \quad (90)$$

$$\gamma(t) = \gamma(t)_{\substack{\text{controls}\\ \text{locked}}} + b\left\{M_\delta \sum_0^4 \gamma_{i_{M_\delta}} e^{R_i(t-t_o)} + Z_\delta \sum_0^4 \gamma_{i_{Z_\delta}} e^{R_i(t-t_o)} + X_\delta \sum_0^4 \gamma_{i_{X_\delta}} e^{R_i(t-t_o)}\right\} \quad (91)$$

$$u(t) = u(t)_{\substack{\text{controls}\\ \text{locked}}} + b\left\{M_\delta \sum_0^4 u_{i_{M_\delta}} e^{R_i(t-t_o)} + Z_\delta \sum_0^4 u_{i_{Z_\delta}} e^{R_i(t-t_o)} + X_\delta \sum_0^4 u_{i_{X_\delta}} e^{R_i(t-t_o)}\right\} \quad (92)$$

where the controls-locked terms hold for all t, $(t > 0)$.

Since $\delta(t) \equiv 0 \quad (0 < t < t_o)$, it follows that $M_\delta = Z_\delta = X_\delta = 0$ $(0 < t < t_o)$. The inversion of the transforms (87), (88), and (89) gives an equivalent result.

The coefficients with the double subscripts are also defined in equations (87), (88), and (89); for example,

$$\alpha_{i_{Z_\delta}} = \frac{d_{\alpha_{Z_\delta}}(R_i)}{R_i\Delta'(R_i)+\Delta(R_i)}$$

The extended forms of the required solutions permit direct comparison of tangential and normal force effects as well as moment effects upon the perturbation variables $u(t)$, $\gamma(t)$, $\alpha(t)$.

The motion here again combines the transient effects as well as the limiting value for increments of angle of attack, orientation of the velocity vector, and tangential velocity.

Choosing $u(0) = \gamma(0) = \theta(0) = \dot{\theta}(0) = 0$. Also choosing $X_\delta = Z_\delta = 0$, $bM_\delta = m_o$, $t_o = 0$. These solutions reduce to those obtained for an applied constant control moment, with no arbitrary disturbances introduced into the system.

The problem of a constant control deflection for a finite time can be calculated from these results. Ignoring disturbances due to extraneous effects $u(0) = \gamma(0) = \theta(0) = \dot{\theta}(0) = 0$, the response to a constant control deflection applied at $t = 0$ may be found from equations (90), (91), and (92) if t_o is set equal to zero. From this solution one may subtract the effects of a constant control deflection applied at some time t, $(t > t_o)$. The resultant control deflection is then zero $(t > t_o)$. In the limit a stable system now returns to zero.

Mathematically, the finite time interval problem begins,

$$\begin{aligned} \delta(t) &= b & 0 < t < t_o \\ \delta(t) &= 0 & t > t_o \end{aligned} \tag{93}$$

The transform of $\delta(t)$ here is

$$\tilde{f}(s) = \frac{b}{s}\left[1 - e^{-t_o s}\right] \tag{94}$$

which leads to the results given above.

Solutions to this problem can, in practice, be obtained with considerably less labor. Assuming $(0 < t < t_1)$, the system is free of disturbances; then in equations (90), (91), and (92)

$$\alpha(t)_{\substack{\text{controls}\\\text{locked}}} = \gamma(t)_{\substack{\text{controls}\\\text{locked}}} = u(t)_{\substack{\text{controls}\\\text{locked}}} = 0.$$

Setting $t_o = 0$, write the solutions from these same equations. These solutions are $\alpha_1(t)$, $\gamma_1(t)$, $u_1(t)$; the values at the time $t = t_1$, when the control is returned to its zero position, are $\alpha_1(t_1)$, $\gamma_1(t_1)$, $u_1(t_1)$. To obtain the solutions for $t > t_1$, replace this part of the response by an equivalent disturbance configuration. The solutions to the inherent dynamic stability problem can then be utilized in this response problem. The required solutions for $t > t_1$ are those of equations (33), (34), and (35) if the initial conditions are chosen as

$$\alpha(0) = \alpha_1(t_1),\ \gamma(0) = \gamma_1(t_1),\ u(0) = u_1(t_1),\ \dot{\alpha}(0) = \dot{\alpha}_1(t_1),$$
and $\dot{\gamma}(0) = \dot{\gamma}_1(t_1)$.

Sinusoidal Control Deflection

For this example, assume the deflection to be of the form

$$\delta = \delta_o \sin \omega t \tag{95}$$

where

δ_o = maximum deflection from trim position

ω = frequency of applied drive

Substituting equation (95) into equation (36) gives:

$$u(t) = \sum_{i=1}^{4} u_i\, e^{R_i t}\, \delta_o \int_0^t e^{-R_i\beta} \sin \omega\beta\, d\beta \tag{96}$$

$$u(t) = \sum_{i=1}^{4} \frac{u_i e^{R_i t} \delta_o}{R_i^2+\omega^2} \left[e^{-R_i \beta}(-R_i \sin \omega\beta - \omega \cos \omega\beta)\right]_o^t \tag{97}$$

Then

$$u(t) = \sum_{i=1}^{4} \frac{u_i \delta_o}{R_i^2+\omega^2} [-R_i \sin \omega t - \omega \cos \omega t + \omega e^{R_i t}] \tag{98}$$

$$\gamma(t) = \sum_{i=1}^{4} \frac{\gamma_i \delta_o}{R_i^2+\omega^2} [-R_i \sin \omega t - \omega \cos \omega t + \omega e^{R_i t}] \tag{99}$$

$$\alpha(t) = \sum_{i=1}^{4} \frac{\alpha_i \delta_o}{R_i^2+\omega^2} [-R_i \sin \omega t - \omega \cos \omega t + \omega e^{R_i t}] \tag{100}$$

Writing out equation (98) in detail

$$\begin{aligned} u(t) = &-\delta_o \left[\frac{u_1 R_1}{R_1^2+\omega^2} + \frac{u_2 R_2}{R_2^2+\omega^2} + \frac{u_3 R_3}{R_3^2+\omega^2} + \frac{u_4 R_4}{R_4^2+\omega^2}\right] \sin \omega t \\ &-\delta_o \omega \left[\frac{u_1}{R_1^2+\omega^2} + \frac{u_2}{R_2^2+\omega^2} + \frac{u_3}{R_3^2+\omega^2} + \frac{u_4}{R_4^2+\omega^2}\right] \cos \omega t \\ &+\delta_o \omega \sum_{i=1}^{4} \frac{u_i e^{R_i t}}{R_i^2+\omega^2} \end{aligned} \tag{101}$$

Make the substitution

$$\begin{aligned} -N_1 &= \left[\frac{u_1 R_1}{R_1^2+\omega^2} + \frac{u_2 R_2}{R_2^2+\omega^2} + \frac{u_3 R_3}{R_3^2+\omega^2} + \frac{u_4 R_4}{R_4^2+\omega^2}\right] \\ N_2 &= \omega \left[\frac{u_1}{R_1^2+\omega^2} + \frac{u_2}{R_2^2+\omega^2} + \frac{u_3}{R_3^2+\omega^2} + \frac{u_4}{R_4^2+\omega^2}\right] \end{aligned} \tag{102}$$

Then

$$u(t) = \delta_o (N_1 \sin \omega t - N_2 \cos \omega t) + \omega \delta_o \sum_{i=1}^{4} \frac{u_i e^{R_i t}}{R_i^2+\omega^2} \tag{103}$$

This can be further simplified by defining

$$\sin \eta = \frac{N_2}{\sqrt{(N_1^2+N_2^2)}} \qquad \cos \eta = \frac{N_1}{\sqrt{(N_1^2+N_2^2)}} \tag{104}$$

Equation (103) then becomes

$$u(t) = \delta_o \sqrt{(N_1^2+N_2^2)} \sin(\omega t - \eta) + \omega\delta_o \sum_{i=1}^{4} \frac{u_i e^{R_i t}}{R_i^2+\omega^2} \tag{105}$$

Similarly, from equations (99) and (100) the following solutions are obtained:

$$\gamma(t) = \delta_o \sqrt{M_1^2+M_2^2} \sin(\omega t - \phi) + \omega\delta_o \sum_{i=1}^{4} \frac{\gamma_i e^{R_i t}}{R_i^2+\omega^2} \tag{106}$$

where

$$-M_1 = \left[\frac{\gamma_1 R_1}{R_1^2+\omega^2} + \frac{\gamma_2 R_2}{R_2^2+\omega^2} + \frac{\gamma_3 R_3}{R_3^2+\omega^2} + \frac{\gamma_4 R_4}{R_4^2+\omega^2}\right]$$

$$M_2 = \omega\left[\frac{\gamma_1}{R_1^2+\omega^2} + \frac{\gamma_2}{R_2^2+\omega^2} + \frac{\gamma_3}{R_3^2+\omega^2} + \frac{\gamma_4}{R_4^2+\omega^2}\right] \tag{107}$$

$$\sin \phi = \frac{M_2}{\sqrt{(M_1^2+M_2^2)}} \qquad \cos \phi = \frac{M_1}{\sqrt{(M_1^2+M_2^2)}}$$

and

$$\alpha(t) = \delta_o \sqrt{(L_1^2+L_2^2)} \sin(\omega t - \xi) + \omega\delta_o \sum_{i=1}^{4} \frac{\alpha_i e^{R_i t}}{R_i^2+\omega^2} \tag{108}$$

where

$$-L_1 = \left[\frac{\alpha_1 R_1}{R_1^2+\omega^2} + \frac{\alpha_2 R_2}{R_2^2+\omega^2} + \frac{\alpha_3 R_3}{R_3^2+\omega^2} + \frac{\alpha_4 R_4}{R_4^2+\omega^2}\right]$$

$$L_2 = \omega\left[\frac{\alpha_1}{R_1^2+\omega^2} + \frac{\alpha_2}{R_2^2+\omega^2} + \frac{\alpha_3}{R_3^2+\omega^2} + \frac{\alpha_4}{R_4^2+\omega^2}\right] \tag{109}$$

and

$$\sin \xi = \frac{L_2}{\sqrt{(L_1^2+L_2^2)}} \qquad \cos \xi = \frac{L_1}{\sqrt{(L_1^2+L_2^2)}}$$

Inspection of these solutions shows that if the missile is stable, then terms of the form

$$\omega\delta_0 \sum_{i=1}^{4} \frac{\alpha_i e^{R_i t}}{R_i^2+\omega^2} \tag{110}$$

represent transients which will damp out.

Frequency Response

A special case of considerable interest is the determination of the frequency response of the missile. In equations (39), (40), and (41), let $\delta = Ae^{i\omega t}$, where A is a complex number. Then

$$u(t) = Ae^{i\omega t} \sum_{i=1}^{4} \frac{\Delta_u(R_i)}{\Delta'(R_i)} \int_0^t e^{(R_i-i\omega)\tau} d\tau \tag{111}$$

$$\gamma(t) = Ae^{i\omega t} \sum_{i=1}^{4} \frac{\Delta_\gamma(R_i)}{\Delta'(R_i)} \int_0^t e^{(R_i-i\omega)\tau} d\tau \tag{112}$$

$$\alpha(t) = Ae^{i\omega t} \sum_{i=1}^{4} \frac{\Delta_\alpha(R_i)}{\Delta'(R_i)} \int_0^t e^{(R_i-i\omega)\tau} d\tau \tag{113}$$

Carrying out the calculation for $\alpha(t)$

$$\alpha(t) = Ae^{i\omega t} \sum_{i=1}^{4} \frac{\Delta_\alpha(R_i)}{\Delta'(R_i)} \frac{\left[e^{(R_i-i\omega)t}-1\right]}{(R_i-i\omega)} \tag{114}$$

Then, for a stable system:

$$\alpha(t) = Ae^{i\omega t} \sum_{i=1}^{4} \frac{\Delta_\alpha(R_i)}{\Delta'(R_i)} \frac{1}{(i\omega-R_i)} \tag{115}$$

for $t > T$

but

$$\sum_{i=1}^{4} \frac{\Delta_\alpha(R_i)}{\Delta'(R_i)(i\omega - R_i)} = \frac{\Delta_\alpha(i\omega)}{\Delta(i\omega)} \qquad (116)$$

The frequency response may therefore be written as:

$$\alpha(t) = \frac{\Delta_\alpha(i\omega)}{\Delta(i\omega)} Ae^{i\omega t} \qquad (117)$$

for $t > T$

Similarly

$$u(t) = \frac{\Delta_u(i\omega)}{\Delta(i\omega)} Ae^{i\omega t} \qquad (118)$$

$$\gamma(t) = \frac{\Delta_\gamma(i\omega)}{\Delta(i\omega)} Ae^{i\omega t} \qquad (119)$$

for $t > T$

Invariants

In this section the application of the invariant characteristics of the solutions is discussed. There are three immediately obvious uses for these invariants; namely,

1. In the calculation of the solutions

2. In the comparison between numerical magnitudes of the components of the solutions

3. In the systematic order analysis of the equations of motion.

A procedure which can be used in each of these instances is outlined here.

To illustrate how the computation of the solutions may be facilitated with the aid of the invariant properties, consider the solutions to the inherent stability problem from equations (33), (34), and (35):

$$u(t) = \sum_{i=1}^{4} u_i e^{R_i t}$$

$$\gamma(t) = \sum_{i=1}^{4} \gamma_i e^{R_i t}$$

$$\alpha(t) = \sum_{i=1}^{4} \alpha_i e^{R_i t}$$

Suppose that only the solution for u(t) is known. The remaining solutions may be written simply as:

$$\gamma(t) = \sum_{i=1}^{4} \frac{u_i}{K_1(R_i)} e^{R_i t} \qquad (120)$$

$$\alpha(t) = \sum_{i=1}^{4} \frac{u_i}{K_2(R_i)} e^{R_i t} \qquad (121)$$

where $K_1(R_i)$ and $K_2(R_i)$ are defined in equations (50) and (51), respectively. This result gives the solutions in terms of the ratios of the components of the solution due to each root of the characteristic equation of the system.

If it is desired to obtain this result in terms of the ratios of the real components of the solutions, slightly different ratios may have to be formed. When all the roots of the characteristic equation are real, the previous result is still applicable. A case more usually encountered in aerodynamic stability problems is the one in which two pairs of conjugate complex roots are found. Presuming that the roots are known to be $R_1 = \alpha_1 + i\beta_1$; $R_2 = \overline{R}_1$; $R_3 = \alpha_2 + i\beta_2$; $R_4 = \overline{R}_3$, then the known solution for u(t) will be of the form:

$$u(t) = A_1 e^{\alpha_1 t}\cos(\beta_1 t - \mu_1) + A_2 e^{\alpha_2 t}\cos(\beta_2 t - \mu_2) \qquad (122)$$

The solutions for $\gamma(t)$ and $\alpha(t)$ may then be written as:

$$\gamma(t) = \frac{A_1}{\xi_1(R_1,\bar{R}_1)} e^{\alpha_1 t}\cos(\beta_1 t - \mu_1) + \frac{A_2}{\xi_1(R_3,\bar{R}_3)} e^{\alpha_2 t}\cos(\beta_2 t - \mu_2) \tag{123}$$

$$\alpha(t) = \frac{A_1}{\xi_2(R_1,\bar{R}_1)} e^{\alpha_1 t}\cos(\beta_1 t - \mu_1) + \frac{A_2}{\xi_2(R_3,\bar{R}_3)} e^{\alpha_2 t}\cos(\beta_2 t - \mu_2) \tag{124}$$

where ξ_1 and ξ_2 are given by equations (55) and (56). This same result could have been obtained from equations (120) and (121) by substituting the complex roots directly into those equations and combining the terms due to the pairs of conjugate complex roots.

In performing a systematic order analysis of the equations of motion, the solutions calculated in terms of the invariants are useful. Taking the results of equations (122), (123), and (124), the small perturbation equation (15), for example, could be written as (assuming $X_\delta = 0$):

$$\begin{aligned}
&+A_1\sqrt{\alpha_1^2+\beta_1^2}\, e^{\alpha_1 t}\cos(\beta_1 t-\mu_1+\phi_1)+A_2\sqrt{\alpha_2^2+\beta_2^2}\, e^{\alpha_2 t}\cos(\beta_2 t-\mu_2+\phi_2) \\
&-X_u A_1 e^{\alpha_1 t}\cos(\beta_1 t-\mu_1)-X_u A_2 e^{\alpha_2 t}\cos(\beta_2 t-\mu_2)-\frac{X_\gamma A_1}{\xi_1(R_1,\bar{R}_1)} e^{\alpha_1 t}\cos(\beta_1 t-\mu_1) \\
&-\frac{X_\gamma A_2}{\xi_1(R_3,\bar{R}_3)} e^{\alpha_2 t}\cos(\beta_2 t-\mu_2)-\frac{X_\alpha A_1}{\xi_2(R_1,\bar{R}_1)} e^{\alpha_1 t}\cos(\beta_1 t-\mu_1) \\
&-\frac{X_\alpha A_2}{\xi_2(R_3,\bar{R}_3)} e^{\alpha_2 t}\cos(\beta_2 t-\mu_2) = 0
\end{aligned} \tag{125}$$

It is to be noted that this same result could have been obtained by making use of the invariant ratio between $u(t)$ and its first derivative, instead of performing the differentiation as was done here. Upon collecting terms, this equation becomes:

$$\begin{aligned}
&-A_1 e^{\alpha_1 t}\left\{\left[\alpha_1 - X_u - \frac{X_\gamma}{\xi_1(R_1,\bar{R}_1)} - \frac{X_\gamma}{\xi_2(R_1,\bar{R}_1)}\right]\cos(\beta_1 t-\mu_1)-\beta_1\sin(\beta_1 t-\mu_1)\right\} \\
&+A_2 e^{\alpha_2 t}\left\{\left[\alpha_2 - X_u - \frac{X_\gamma}{\xi_1(R_3,\bar{R}_3)} - \frac{X_\gamma}{\xi_2(R_3,\bar{R}_3)}\right]\cos(\beta_2 t-\mu_2)-\beta_2\sin(\beta_2 t-\mu_2)\right\} = 0
\end{aligned} \tag{126}$$

Thus, knowing the values for ξ_1 and ξ_2 it is possible to determine which of the coefficients (in this case X_u, X_γ, X_α) has the greatest influence in determining the magnitude of the solution. For example, if ξ_1 and ξ_2 are extremely large, the differential coefficient X_u will be the predominant one. By applying this technique to the remaining equations of motion it might be possible to establish approximations to the solutions which are valid for a reasonable time interval. These would have simplified forms for preliminary estimates of frequency, time to damp to half amplitude, and whatever other pertinent information is desired. Information of this nature is invaluable in preliminary design work.

NOMENCLATURE

a = Velocity of sound (ft/sec)

cg = Mass center

cp = Center of pressure

C_D = Drag coefficient of complete configuration D/qS

C_L = Lift coefficient of complete configuration L/qS

C_M = Moment coefficient of complete configuration M/qSd

C_T = Thrust coefficient T/qS

d = Maximum fuselage diameter (ft)

D = Drag

I = Moment of inertia of the complete missile about an axis through the missile mass center normal to the pitch plane

L = Lift

l_p = Distance from cg to cp, negative when cp is aft of cg (ft)

m = Total mass of missile

M = Total moment about mass center, sense same as M_a

M_a = Aerodynamic moment, positive moment causes increase in α_T

P = Radius of curvature of flight path

q = Dynamic pressure $\rho U^2/2$ (lbs/ft^2)

R_i = Roots of characteristic equation

S = Maximum fuselage cross section area $\pi d^2/4$

t = Time

T = Thrust

U = Total velocity, tangent to flight path

W = Total weight of missile, mg

X = Total force acting tangent to flight path, positive in the direction of the missile velocity vector

Z = Total force acting normal to flight path, positive in direction of center of curvature of flight path

Greek Symbols

$\alpha, \gamma, u, \theta$ = Perturbations in α_T, Γ, U, $\bar{\theta}$

α_T = Angle of attack

Γ = Angle between velocity vector and horizontal

δ = Control deflection

$\Delta(s)$ = Determinant in s, characteristic equation of system when equal to zero

$\bar{\theta}$ = Inclination of missile longitudinal axis from horizontal

ρ = Air density (slugs/ft^3)

ω = Frequency of applied drive

$\Pi(s-R_i)$ = $(s-R_1)(s-R_2)(s-R_3)(s-R_4)$

REFERENCES

1. Bôcher, M., Introduction to Higher Algebra. MacMillan Co., New York, 1947.

2. Churchill, R.V., Modern Operational Mathematics in Engineering. McGraw-Hill, 1944.

3. Routh, E.J., Advanced Rigid Dynamics. MacMillan and Co., London, 1884.

4. Rutkowski, J., Equations of Motion, Dynamic Stability, Stability Coefficients of Missiles. Report No. 4-54-100, Bendix Aviation Corporation.

DISTRIBUTION

Distribution of this report is made in accordance with ANAF-GM Mailing List No. 10, dated January 15, 1950 to include Part A, Part B, and Part C.