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Control of Constrained Hamiltonian Systems and Applications to Control of Constrained Robots

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ABSTRACT

It is shown that the theory of Hamiltonian control systems can be extended in a very natural manner to a theory of Hamiltonian control systems with constraints. In particular, these problems may be formulated either in terms of the original (symplectic) manifold or in terms of the constraint manifold. The analysis of such problems is shown to be essentially equivalent to the analysis of systems of controlled singular (differential-algebraic) equations. Certain robotic control problems, defined by task constraints, are shown to fall within the defined theoretical framework

1. Introduction

In recent years there has been great interest in Hamiltonian control systems [6,9,20], control systems which preserve the natural Hamiltonian structure. A great number of important mechanical systems are of this type. There has also been much interest recently in the theory of constrained Hamiltonian systems. Such systems have turned out to be of importance in the theory of integrable Hamiltonian systems; for example see [10]. In addition, there has been a recent realization that such problems provide a suitable framework for a number of interesting engineering applications.

In this paper, we develop a framework so that the theories of Hamiltonian control systems and constrained Hamiltonian systems can be combined in a very natural way to produce a theory for constrained Hamiltonian control systems. We show how constrained Hamiltonian control systems may be formulated either on the original symplectic manifold or on the constraint manifold. Control theoretic problems may be formulated in either context, but an explicit formulation in terms of the constraint manifold may be most suitable for purposes of analysis and control design. Decomposition methods for obtaining such a formulation are presented. We briefly mention results on feedback stabilization and optimal control for constrained Hamiltonian control systems which have been developed elsewhere. A controllability result is stated, the proof following from previous controllability results and from the framework established for constrained Hamiltonian systems.

Constrained Hamiltonian control systems can also be viewed as singular sets of differential equations; in this case they are sets of coupled differential and algebraic equations in control theoretic form. The connection of constrained Hamiltonian control systems with a special class of controlled singular systems is thereby established.

Finally, the importance of such theoretical problems in applications is shown. The emphasis is on the critical role of constraints in properly defining an important class of robotic control problems associated with the application of robots to advanced tasks involving interaction with their environment.

2. Controlled Hamiltonian Systems With Constraints

Hamiltonian control systems were first introduced in [6] and developed in [4],[9] and [20], for example. We consider here the case of affine Hamiltonian control systems with constraints.

Let M be a differentiable manifold with local coordinates $(q^1, \cdots, q^n)$ on M and let $T*M$ denote the cotangent bundle of M with local coordinates $(q^1, \cdots, q^n, p^1, \cdots, p^n)$ on $T*M$. $T*M$ is a symplectic manifold and its symplectic form is given locally by

$$\omega = \sum_i dp^i \wedge dq^i. \tag{1}$$

We suppose that we have a Hamiltonian function H: $T*M \rightarrow R$ given by $H(q,p) = T(q,p) + V(q)$ where $T(q,p)$ is a kinetic energy function on $T*M$ that is a positive definite quadratic form in $p^1, \cdots, p^n$ and $V(q)$ is a potential energy function on M. We denote by X_H the Hamiltonian vector field associated with $H(q,p)$.

We also assume that there are functions $G^j : T^*M \rightarrow R$ that define control vector fields X_j on $T^*M, j=1, \cdots, r$, and functions $\phi^j : T^*M \rightarrow R$ that define constraint vector fields Y_j on $T^*M, j=1, \cdots, 2m$. Thus the equations for a constrained Hamiltonian control system are given by

$$\dot{x} = X_H(x) + \sum_i u_i X_i(x) + \sum_i \lambda_i Y_i(x), \tag{2}$$

$$\phi^i(x) = 0, \; i=1, \cdots, 2m, \tag{3}$$

or in local form

$$\dot{q}^i = \frac{\partial H(q,p)}{\partial p^i} + \sum_j u_j \frac{\partial G^j(q,p)}{\partial p^i} + \sum_j \lambda_j \frac{\partial \phi^j(q,p)}{\partial p^i}, \; i=1, \cdots, n, \tag{4}$$

$$\dot{p}^i = -\frac{\partial H(q,p)}{\partial q^i} - \sum_j u_j \frac{\partial G^j(q,p)}{\partial q^i} - \sum_j \lambda_j \frac{\partial \phi^j(q,p)}{\partial q^i}, \; i=1, \cdots, n, \tag{5}$$

$$\phi^i(q,p) = 0, \; i=1, \cdots, 2m. \tag{6}$$

The $(u_1, \cdots, u_r)$ are the control inputs and the $(\lambda_1, \cdots, \lambda_{2m})$ are the multipliers corresponding to the constraints. Note that the control inputs and the constraint multipliers enter the equations in precisely the same manner, although their meanings are quite different; the control inputs are viewed as arbitrarily specified external inputs whereas the multipliers are viewed as implicitly specified by the requirement that the constraints be satisfied.

Notice that this formulation is very much in the spirit of Dirac's theory of constrained Hamiltonian systems. In [11] the system is driven by a total Hamiltonian

$$H_T(q,p,\lambda) = H(q,p) + \sum \lambda_i \phi^i(q,p). \tag{7}$$

In our more general context the system is driven by a total Hamiltonian

$$H_T(q,p,\lambda,u) = H_C(q,p,u) + \sum \lambda_i \phi^i(q,p) \tag{8}$$

where

$$H_C(q,p,u) = H(q,p) + \sum u_i G^i(q,p) \tag{9}$$

is the drift Hamiltonian together with the control Hamiltonians.

We now distinguish between holonomic and nonholonomic constraints as follows:

Definition 1. A set of constraints ψ^j, $j=1, \cdots, m$ on $T*M$ is holonomic if the forms $d\psi^j$, $j=1, \cdots, m$ define an integrable distribution on $T*M$. Otherwise, the constraints are called nonholonomic.

Note that the definition is a slight generalization of the classical definition of holonomic constraints: where the ψ^j are functions of the variables q^i, $i=1, \cdots, n$, only. In this case $\phi^j = \psi^j$, $j=1, \cdots, m$ and $\phi^j = \dot{\psi}^j$, $j=m+1, \cdots, 2m$.

3. Decomposition Of Controlled Hamiltonian Systems With Constraints

In this section we show how the control of a Hamiltonian system with constraints may be viewed as the control of a Hamiltonian system on the constraint manifold.

Suppose that the zero set of the constraints defines a submanifold N of $T{*}M$. We assume that the matrix $[\{\phi^i, \phi^j\}]$ is nondegenerate, where $\{F,G\}$ is the Poisson bracket on $T{*}M$. Thus N is symplectic.

Now we can generalize the arguments in [14] to the control situation as follows. Let Z denote the Hamiltonian vector field corresponding to $H_C(q,p,u) = H(q,p) + \sum u_i G^i(q,p)$. Since N is a symplectic submanifold of $T{*}M$ at every point n of N, the tangent space of $T{*}M$ at n can be decomposed as

$$T\,T{*}M_n = T\,N_n \oplus TN_n^{\perp} \tag{10}$$

where $\perp$ denotes orthogonal complement. Then the vector field Z has the decomposition

$$Z = Z^N \oplus Z^{N\perp}. \tag{11}$$

If v is any tangent vector to N then

$$\omega(Z^N, v) = \omega(Z, v) = \langle dH_C, v\rangle = \langle dH_{C|N}, v\rangle. \tag{12}$$

Hence Z^N is the Hamiltonian vector field on N corresponding to $H_{C|N}$, the restriction of H_C to N, relative to the restricted symplectic form $\omega_{|N}$.

Now let Y_j denote the Hamiltonian vector field corresponding to ϕ^j, j=1, $\cdots$,2m. Since the $\phi^j = 0$ on N, the restricted control vector fields Y_j^N satisfy $Y_j^N = 0$, and the Y_j, j=1, $\cdots$, 2m, thus form a basis of $TN^{\perp}$. Hence $z^N \perp = \sum \lambda_i$, for some scalars $\lambda_1,...,\lambda_{2m}$, and $Z^N = Z^t$ where Z^T is the Hamiltonian vector field corresponding to $H_T\left(q,p,\lambda,u\right)$. Now along N, $Z^T \phi^K = \{H_T, \phi^k\} = 0$. Hence

$$\{H_C, \phi^j\} = \sum \lambda_i \{\phi^i, \phi^j\}, \; j=1, \cdots, 2m. \tag{13}$$

Since the matrix of Poisson brackets $[\{\phi^i, \phi^j\}]$ is nonsingular at each point of N, there is a unique solution for $\lambda_1, \cdots, \lambda_{2m}$. Thus we have the following.

Theorem 1. The full Hamiltonian H_T constraints the Hamiltonian control system to the constraint manifold N, and if the matrix $[\{\phi^i, \phi^j\}]$ is nonsingular on N we can solve uniquely for the multipliers $\lambda_1, \cdots, \lambda_{2m}$.

We see from the above development that the constraint multipliers $\lambda_1, \cdots, \lambda_{2m}$ depend on the drift vector field, the control vector fields and the constraint vector fields. It is this intrinsic coupling that significantly complicates the analysis of control problems. Nevertheless, the Hamiltonian structure is preserved on the constraint manifold, as has been shown.

Now if $T*M = R^{2n}$ we can follow the arguments in [10] to show that the Poisson bracket on the constraint manifold N, $\{F,G\}_N$ is related to the Poisson bracket on R^{2n}, $\{F,G\}$, by

$$\{F,G\}_N = \{F,G\} - \sum_{i,j}\{F,\phi^i\}c_{ij}^{-1}\{\phi^j,G\} \tag{14}$$

where c_{ij}^{-1} is the ij^{th} entry of the inverse of $C = [\{\phi^i,\phi^j\}]$. We thus have

Theorem 2. The equations of motion of the Hamiltonian control system constrained to N are

$$\dot{q}^i = \{q^i, H_T\}, \quad i=1,\cdots,n \tag{15}$$

$$\dot{p}^i = \{p^i, H_T\}, \quad i=1,\cdots,n. \tag{16}$$

or, equivalently,

$$\dot{q}^i = \{q^i, H_C\}_N, \quad i=1,\cdots,n \tag{17}$$

$$\dot{p}^i = \{p^i, H_C\}_N, \quad i=1,\cdots,n. \tag{18}$$

Thus we can regard the system as evolving on R^{2n} under the full Hamiltonian H_T with respect to the original bracket structure or as evolving on the constraint manifold $N \subset R^{2n}$ under the control Hamiltonian H_C with the constraint manifold bracket structure.

We also have the following theorem which follows from [21].

Theorem 3. Suppose ψ^j, $j=1,\cdots,m$ are a set of independent holonomic constraints on $T*M$ with $m < n$. Then there is a local canonical transformation g: $T*M \supset U \rightarrow T*M$ such that the transformed constraint functions $\tilde{\psi}^j = \psi^j \circ g = Q^j$, $j=1,\cdots,m$, if and only if ψ^j, $j=1,\cdots,m$ are in involution, i.e.

$$\{\psi^i, \psi^j\} = 0,\ i,\ j{=}1, \cdots, m. \tag{19}$$

Here Q^j, $j{=}1, \cdots, n$ are the transformed configuration coordinates.

Thus, after a canonical transformation, locally the constraints can be written as

$$Q^1 = Q^2 = \cdots = Q^m = 0. \tag{20}$$

Note that for constraints in the classical form the involution condition is automatically satisfied, and only the linear independence condition is required to satisfy the assumptions of Theorem 3.

For a system on $T{*}M = R^{2n}$, with holonomic constraints $\psi^j(q) = 0$, $j{=}1, \cdots, m$, we can write this transformation quite explicitly as follows. We partition the n-vectors q, p and transformed n-vectors Q, P as:

$$q = \begin{vmatrix} q_1 \\ q_2 \end{vmatrix} \quad p = \begin{vmatrix} p_1 \\ q_2 \end{vmatrix} \quad Q = \begin{vmatrix} Q_1 \\ Q_2 \end{vmatrix} \quad P = \begin{vmatrix} P_1 \\ P_2 \end{vmatrix} \tag{21}$$

where q_1 is an m-vector and q_2 is an n-m vector, etc. Then it is possible to find a function $\delta(q_2)$ such that $\psi(\delta(q_2), q_2) = 0$ is satisfied locally. Now consider the canonical transformation given by the generating function

$$F(q, P) = (q_1 - \delta(q_2))P_2 + q_1 P_2. \tag{22}$$

Then, the transformation is given by [13]

$$\begin{aligned} Q_1 &= q_1 - \delta(q_2) \qquad & p_1 &= P_1 \\ Q_2 &= q_2 & p_2 &= P_2 - \frac{\partial\delta}{\partial q_2} P_1 \end{aligned} \tag{23}$$

and the transformed equations are

$$\dot{Q}^i = \{Q^i, \tilde{H}_T\},\ i = 1 \cdots n, \tag{24}$$

$$P^i = \{P^i, \tilde{H}_T\},\ i = 1 \cdots n, \tag{25}$$

$$Q^i = 0,\ i = 1, \cdots m, \tag{26}$$

where the transformed Hamiltonian is

$$\tilde{H}_T(Q, P, u, \lambda) = \tilde{H}_C(Q, P, u) + \sum_{i=1}^{m} \lambda_i Q^i \tag{27}$$

and $\tilde{H}_C(Q, P, u) = H_C(q,p,u)$.

Since $Q^i = 0$, $i=1, \ldots, m$ the first m of equations (24) can be used to express P_1 in terms of (P_2, Q_2, u); the first m of equations (25) can be used to express λ in terms of (P_2, Q_2, u). Thus the last n-m of equations (24) and of equations (25) can be written as ordinary differential equations in (P_2, Q_2) and u, which characterize the dynamics on the constraint manifold.

4. Constrained Hamiltonian Systems As Singular Systems

It is clear from the above development that analysis of control problems constrained to a given manifold involves both differential equations and algebraic equations. In fact, these coupled differential and algebraic equations can be represented as a singular set of differential equations.

This can be seen explicitly by writing our constrained Hamiltonian control equations, using vector notation, in local form as

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} \dot{q} \\ \dot{p} \\ \dot{\lambda} \end{vmatrix} = \begin{vmatrix} \frac{\partial H(q,p)}{\partial p} + \sum_j u_j \frac{\partial G^j(q,p)}{\partial p} + \sum_j \lambda_j \frac{\partial \phi^j(q,p)}{\partial p} \\ -\frac{\partial H(q,p)}{\partial q} - \sum_j u_j \frac{\partial G^j(q,p)}{\partial q} - \sum_j \lambda_j \frac{\partial \phi^j(q,p)}{\partial q} \\ \phi(q,p) \end{vmatrix} \tag{28}$$

or equivalently

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} \dot{q} \\ \dot{p} \\ \dot{\lambda} \end{vmatrix} = \begin{vmatrix} \frac{\partial H_T(q,p,\lambda,u)}{\partial p} \\ -\frac{\partial H_T(q,p,\lambda,u)}{\partial q} \\ \frac{\partial H_T(q,p,\lambda,u)}{\partial \lambda} \end{vmatrix} . \tag{29}$$

In this form, we have $2n + 2m$ coupled equations: $2n$ differential equations and $2m$ algebraic equations, in the $2n + 2m$ variables (q,p,λ). These equations are inherently coupled through the constraint multipliers that appear in the differential equations. Note that the constraint multipliers do not appear in the algebraic equations, so the algebraic equations cannot be explicitly used to solve for the multipliers. Thus the differential and algebraic structure of the equations is of an especially complex form.

Such singular systems have been studied in recent years [7,8] but there are few theoretical results that can be applied to the class of systems of interest here. However, it should be mentioned that there have been a number of results obtained for the numerical solution of initial value problems [2,12,18] for singular systems of the class considered here.

From the theoretical analysis of Section 3 and the previous work in [15,17], it becomes clear that there are two distinct approaches to the analysis of control theoretic problems for constrained Hamiltonian control systems expressed in singular form.

One method is to solve, in algebraic form, for the multiplier vector λ from equation (13), and to substitute into equations (15) and (16) to obtain a differential equation on R^{2n}.

The alternate approach is to decompose the system to obtain nonsingular differential equations which give a representation of the motion on the constraint manifold. Theorem 3 is the key for carrying out this procedure. One makes a canonical transformation so that the constraints are in the simple form given by equation (20). We can thus obtain a set of $2n - 2m$ differential equations on the constraint manifold.

Note that the basic approaches suggested here each depend on the use of the Hamiltonian structure. Our view is that consideration of general nonlinear singular systems is not a tractable approach, but that progress can be made for the class of systems of interest here by exploiting the Hamiltonian relationships.

5. Control Problems For Constrained Hamiltonian Systems

The two approaches to control of constrained Hamiltonian control systems discussed in Section 4 have been shown to be useful in different ways.

The first approach, that of solving for the multipliers, has, for example, proved useful in optimal planning problems for constrained robot manipulators

[15]. There that approach was used to develop a numerical procedure to solve a time optimal control problem for a control system constrained to follow a given path and with a given contact force (multiplier) vector.

On the other hand, the second approach based on canonical transformations to simplify the constraints has been used in the analysis of stability of closed loop constrained robot manipulators [17]. Closed loop stability on the constraint manifold (such that the motion and multipliers satisfy a regulation or tracking property) is guaranteed in terms of conditions on the feedback controller; both global and local modifications of the computed torque controller used in robotics are included. Although the development in these papers was carried out in a Lagrangian formulation, these results can be restated in the Hamiltonian form considered here. Detailed statements of these results can be found in the indicated references.

We state here a controllability result which follows from our previous development. We say that the system (2) and (3) is controllable if for all x and y in $T{*}M$ there exists an admissible control and a time $T > 0$ such that there is a solution of equations (2) and (3) satisfying $x(0) = x$, $x(T) = y$.

Theorem 4. Suppose we are given the constrained Hamiltonian control system defined by equations (2) and (3). Suppose that the control constraints

$$\left|u_j(t)\right| \leq 1,\ i=1, \cdots, r \tag{30}$$

are imposed, and the constraint manifold N is compact. Then the system is controllable on the constraint manifold N if and only if the vector fields $\{X^N, Y_1^N, \cdots, Y_r^N\}$ satisfy the accessibility rank condition, where X^N is the drift Hamiltonian vector field restricted to the manifold N and Y_j^N are the control vector fields restricted to the manifold N, $j=1, \cdots, r$.

Proof: Recall that the accessibility rank condition for $\{X^N, Y_1^N, \cdots, Y_r^N\}$ is that the Lie algebra generated by $\{X^N, Y_1^N, \cdots, Y_r^N\}$ spans the tangent space of N at each point in N. Necessity then follows from the results of Sussman and Jurdjevich in [19] and sufficiency from the results of Bonnard in [5], since, by our earlier analysis, X^N is Hamiltonian on N which is compact and hence the set of points which are Poisson stable with respect to X^N is dense in N. (Poisson stability is discussed in [5].)

6. Control Of Robot Tasks Defined By Holonomic Constraints

Finally we describe several robot tasks which give rise to the imposition of holonomic constraints of the above form. Additional details, including the specific form of the equations in Lagrangian form, are given in [16]. Such holonomic constraints naturally arise in cases where the end effector of the robot interacts with its environment in a way that should be reflected in the dynamics of the robot. We mention three cases here. If the end effector of the robot is to pick up an object (where the dynamics of the object are not neglibible), then the system dynamics are defined by the dynamics of the robot and the dynamics of the object constrained so that the end effector of the robot grasps the object. If the end effector of the robot is to move along a specified (rigid and noncompliant) surface, then the system dynamics should reflect the robot dynamics plus the contact force required to maintain satisfaction of contact between the end effector and the surface. If two robots are to cooperatively grasp a rigid object, then the system dynamics are defined by the dynamics of each robot and by the constraint that they are grasping a common object. These robot tasks are beyond the capability of most current industrial robots; there is no existing method for control of the robots to cause them to carry out the desired task.

It is suggested here that the proper way to view such advanced robot problems is through a formulation involving constrained Hamiltonian control systems.

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