

**SINGULAR SYSTEMS OF DIFFERENTIAL
EQUATIONS AS DYNAMIC MODELS
FOR CONSTRAINED ROBOT SYSTEMS**

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Abstract

There are many robot system configurations where external contact forces on a robot play an important role in the system dynamics. Such contact forces arise due to constraints on the motion of the robot. Mathematical models of such robot systems naturally give rise to a mathematical system of differential equations and algebraic equations, that can be viewed as a singular system of differential equations. Such singular models are developed for several robot system configurations. A general reduction approach is presented that can, in principle, form the basis for the development of planning and feedback control schemes for such robot system configurations.

1. INTRODUCTION

There has been considerable research on the dynamics and control of robots. However, much of that research has been based on the assumption that the robot does not interact with its environment in any significant way. In some cases, the presence of external contact forces on the robot have been taken into account but, usually, that has been done in a simple way by viewing contact forces as disturbances. In fact a careful treatment of robot dynamics, taking into account contact forces that arise as a consequence of motion constraints, is quite difficult. Development of general principles for planning and feedback control of such robot system configurations is in its infancy, in large part due to a lack of understanding of the robot dynamics in cases where contact forces are important.

If robots are to be used to carry out more complicated manufacturing tasks, a deeper knowledge of the dynamics of robot interactions with its environment is required. Typical tasks of this type include: grinding, cutting, drilling, insertion, fastening, joining, contour following, deburring, scribing, drawing, sweeping, as well as all assembly related tasks. Recent papers motivated by the applications of robots to performance of such tasks include [1-2, 10, 12-15, 23, 27-29, 31]; but in no case was a mathematical model for the robot system dynamics, taking into account relevant contact forces, used in the development.

In this paper certain nonclassical mathematical models of robot systems are proposed, where the robot system can be viewed as a decomposition into individual robot subsystems and environmental subsystems. These models are proposed as being suitable for the characterization of robot dynamics strongly constrained by environmental interactions [6, 19], where such strong interactions are produced by contact forces through physical contact with objects in the workspace or physical contact with other robots. These contact constraints may give rise to the presence of closed kinematic chains in typical cases. The focus in this paper is on the presentation of a class of mathematical models, expressed as singular systems of differential equations; this framework is felt to be the proper formulation for a subsequent careful consideration of the associated problems in robot planning and feedback control.

There are two objectives of this paper. First, it is intended to demonstrate that a number of interesting and important robot system configurations can be modelled using singular systems of differential equations; such configurations include a single robot with end effector gripping an inertial load, a single robot whose end effector is in contact with a rigid surface, and two robots with end effectors gripping a common inertial load. The latter two cases involve the existence of closed kinematic chains, and thus are characterized by substantial dynamic complexity. Second, it is intended to propose a fundamental reduction procedure whereby a singular system of differential equations, of the form considered, can be represented by an equivalent reduced system of differential equations that is more mathematically tractable. The reduction procedure is

introduced, and its advantages and disadvantages are discussed.

2. CONSTRAINED ROBOT SYSTEM CONFIGURATIONS

In this section, three different robot configurations are presented; in each case mathematical equations, in singular form, are derived from the physical assumptions. Although the first configuration is, in fact, a commonly studied case, a mathematical model, in singular form, seems not to have been previously used. The latter two robot configurations are more complicated; but mathematical models, in singular form, are naturally obtained.

Throughout the presentation, \dot{x}, \ddot{x} denote the first and second time derivatives of x ; and C^T denotes the transpose of a matrix C .

One Robot with End Effector Gripping an Inertial Load

A simple schematic diagram of a single articulated arm robot with n links is shown in Figure 1; it is assumed that the end effector of the robot rigidly grips an inertial load as shown. Assuming that the load can be described as a point mass, let $p \in R^m$ denote the position vector of the load, in a fixed workspace coordinate system. Let $q \in R^n$ denote the vector of robot joint angles. The equations of motion are most easily obtained by consideration of the free-body diagrams shown in Figure 2, where $f \in R^m$ denotes the contact force vector, in workspace coordinates, between the robot and the load.

Based on the usual Lagrangian formulation [9], the equations of motion of the robot, taking into account the contact force, are given by

$$M(q)\ddot{q} + G(q, \dot{q}) = T + J^T(q)f \quad (1)$$

where $M(q)$ denotes the inertial matrix function and $G(q, \dot{q})$ is a vector function which characterizes the Coriolis, centrifugal and gravitational effects of the robot. $T \in R^n$ is the input torque vector at the joints.

The equations of motion of the pure inertial load, taking into account the contact and gravitational forces, are

$$m\ddot{p} = -f - mg \quad (2)$$

where the scalar m denotes the mass of the load and g denotes the gravity vector.

Assuming that the load is rigidly gripped by the robot end effector, the additional constraint

$$p = H(q) \quad (3)$$

must also be satisfied, where $H(q)$ is a vector function representing the direct kinematic relations of the robot. The proposed view is that equation (3) represents an *equation of constraint* rather than a *definition* of kinematic relations.

The matrix function $J(q)$ in equation (1) can be shown to be given by the Jacobian matrix

$$J(q) = \frac{\partial H(q)}{\partial q} \quad (4)$$

The above equations can be conveniently represented in the vector form

$$\begin{bmatrix} M(q) & 0 & 0 \\ 0 & mL & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{p} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} T - G(q, \dot{q}) + J^T(q)f \\ -f - mg \\ H(q) - p \end{bmatrix} \quad (5)$$

which are $n + m$ differential equations and m algebraic equations in the $n + 2m$ unknowns q, p and f . The differential equations and the algebraic equations are intrinsically coupled. Since the matrix of coefficients multiplying the derivatives is necessarily singular, these equations are referred to as a singular system of differential equations.

One Robot with End in Contact with a Rigid Surface

In this configuration the end of a single articulated arm robot with n links is assumed to be in contact with a rigid constraint surface as shown in Figure 3; it is assumed that contact between the end of the robot and the constraint surface occurs at a point. Let $p \in R^m$ denote the position vector of the contact point, in a fixed workspace coordinate system. Let $q \in R^n$ denote the vector of robot joint angles and let $f \in R^m$ denote the contact force vector, in workspace coordinates, between the end of the robot and the constraint surface.

The equations of motion of the robot, taking into account the contact force, are given by

$$M(q)\ddot{q} + G(q, \dot{q}) = T + J^T(q)f \quad (6)$$

where $M(q)$ denotes the inertial matrix function and $G(q, \dot{q})$ is a vector function which characterizes the Coriolis, centrifugal and gravitational effects of the

robot. $T \in R^n$ is the input torque vector at the joints.

As indicated in Figure 3, it is assumed that the constraint surface can be defined in algebraic terms by

$$\Phi(p) = 0 \quad (7)$$

where $\Phi: R^m \rightarrow R^1$ is a given scalar function. Assuming that the constraint surface is rigid and frictionless and that $\Phi(p)$ has a continuous gradient, it can be shown that the contact force vector has the form [9]

$$f = D^T(p)\pi \quad (8)$$

where π is a scalar multiplier for the constraint function and $D(p)$ is the gradient of the constraint function

$$D(p) = \frac{\partial \Phi(p)}{\partial p} \quad (9)$$

As in the first configuration, the kinematic relation is $p = H(q)$ so the matrix function $J(q)$ in equation (6) is defined by the Jacobian matrix

$$J(q) = \frac{\partial H(q)}{\partial q} \quad (10)$$

The above differential and algebraic equations can be represented in the vector form

$$\begin{bmatrix} M(q) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} -G(q, \dot{q}) + T + J^T(q)D^T(H(q))\pi \\ \Phi(H(q)) \end{bmatrix} \quad (11)$$

which are n differential equations and 1 algebraic equation in the $n + 1$ unknowns q and π . These differential equations and algebraic equations are intrinsic

sically coupled. Since the matrix of coefficients multiplying the derivatives is necessarily singular, these equations are also a singular system of differential equations.

Two Robots with End Effectors Gripping a Common Inertial Load

The final robot system configuration considered consists of two robots as shown in the schematic diagram in Figure 4. It is assumed that the end effector of each robot rigidly grips a pure inertial load as shown. Assuming that the load can be described as a point mass, let $p \in R^m$ denote the position vector of the load, in a fixed workspace coordinate system. Let $q_1 \in R^{n_1}$ denote the vector of robot joint angles for the first robot and let $q_2 \in R^{n_2}$ denote the vector of robot joint angles for the second robot. The equations of motion are most easily obtained by considering the free-body diagrams as shown in Figure 5, where $f_1 \in R^m$ denotes the contact force vector, in workspace coordinates, between the first robot and the load; similarly, $f_2 \in R^m$ denotes the contact force vector, in workspace coordinates, between the second robot and the load.

The equations of motion of the first robot, taking into account the contact force on it, are given by

$$M_1(q_1)\ddot{q}_1 + G_1(q_1, \dot{q}_1) = T_1 + J_1^T(q_1)f_1 \quad (12)$$

where $M_1(q_1)$ denotes the inertial matrix function and $G_1(q_1, \dot{q}_1)$ is a vector function which characterizes the Coriolis, centrifugal and gravitational effects of the first robot. Here $T_1 \in R^{n_1}$ denotes the input torque vector at the joints of the first robot.

The equations of motion of the second robot, taking into account the contact force on it, are given by

$$M_2(q_2)\ddot{q}_2 + G_2(q_2, \dot{q}_2) = T_2 + J_2^T(q_2)f_2 \quad (13)$$

where $M_2(q_2)$ denotes the inertial matrix function and $G_2(q_2, \dot{q}_2)$ is a vector function which characterizes the Coriolis, centrifugal and gravitational effects of the second robot. Here $T_2 \in R^{n_2}$ denotes the input torque vector at the joints of the second robot.

The equations of motion of the pure inertial load, taking into account the contact and gravitational forces, are

$$m\ddot{p} = -f_1 - f_2 - mg \quad (14)$$

where the scalar m denotes the mass of the load and g denotes the gravity vector.

Assuming that the load is rigidly gripped by the robot end effectors, the additional constraints

$$p = H_1(q_1) \quad (15)$$

and

$$p = H_2(q_2) \quad (16)$$

must also be satisfied, where $H_1(q_1)$ and $H_2(q_2)$ are vector functions representing the direct kinematic relations of the two robots. As previously, the proposed view is that equations (15) and (16) represents *equations of constraint* rather than *definitions* of kinematic relations. The matrix functions $J_1(q_1)$ and $J_2(q_2)$

in equations (12) and (13) are given by the Jacobian matrices

$$J(q_1) = \frac{\partial H_1(q_1)}{\partial q_1} \quad (17)$$

and

$$J(q_2) = \frac{\partial H_2(q_2)}{\partial q_2} \quad (18)$$

The above equations can be conveniently represented in the vector form

$$\begin{bmatrix} M_1(q_1) & 0 & 0 & 0 & 0 \\ 0 & M_2(q_2) & 0 & 0 & 0 \\ 0 & 0 & ml & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{p} \\ \dot{f}_1 \\ \dot{f}_2 \end{bmatrix} = \begin{bmatrix} T_1 - G_1(q_1, \dot{q}_1) + J_1^T(q_1)f_1 \\ T_2 - G_2(q_2, \dot{q}_2) + J_2^T(q_2)f_2 \\ -f_1 - f_2 mg \\ H_1(q_1) - p \\ H_2(q_2) - p \end{bmatrix} \quad (19)$$

which are $n_1 + n_2 + m$ differential equations and $2m$ algebraic equations in the $n_1 + n_2 + 3m$ unknowns q_1, q_2, p, f_1 and f_2 . These differential equations and algebraic equations are intrinsically coupled. Since the matrix of coefficients multiplying the derivatives is necessarily singular, these equations are also a singular system of differential equations.

Other Configurations

The above three robot configurations are illustrative of the fact that explicit algebraic constraints may form an integral part of a mathematical model for the dynamics of a robot system. Other cases where singular systems of differential equations are suitable as mathematical models include robots with

linkages which are designed to include closed kinematic chains [33] and robot biped locomotion [11]. Other examples involving two interacting robots are given in [34].

3. SINGULAR SYSTEMS OF DIFFERENTIAL EQUATIONS

Special Singular Form

The above three robot system configurations have been shown to be described by mathematical models, directly derived from physical principles, expressed as a nonlinear, multivariable singular system of differential equations. In particular, the mathematical models developed above are each a special case consisting of an ordinary vector differential equation of the form

$$M(x)\dot{x} + G(x,\dot{x}) = u + B^T(x)\lambda \quad (20)$$

and a vector algebraic equation of the form

$$\Theta(x) = 0 \quad (21)$$

where the constraint function $\Theta: R^n \rightarrow R^m$, with $m < n$ and

$$B(x) = \frac{\partial \Theta(x)}{\partial x} \quad (22)$$

$x \in R^n$, $\lambda \in R^m$, and $u \in R^n$.

Here, $M(x)$ is a matrix function and $G(x,\dot{x})$ is a vector function. Such differential and algebraic equations can be represented in the form of a singular system of differential equations given by

$$\begin{bmatrix} M(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} u - G(x, \dot{x}) + B^T(x)\lambda \\ \Theta(x) \end{bmatrix} \quad (23)$$

Such mathematical equations are a special form of the general singular system of differential equations considered in [4-5]. They are also referred to as implicit systems of differential equations, descriptor equations and generalized state space equations. An extensive summary of the existing literature related to singular systems of differential equations has been give in [5], including numerous references. The singular system (23) is a particular singular system of index three [7,8]. Such higher index singular systems are known to involve substantial difficulty. General higher index singular systems, and equation (23) in particular, have not been extensively studied.

The special form of singular system of differential equations given by (23) is such that a mathematical theory can be developed, so that such singular systems of differential equations can be effectively used as control theoretic models for robot system configurations, such a those described. A basic reduction transformation that has both theoretical and computational implications is now described.

Reduction Transformation

The most natural approach to the analysis of a singular system of differential equations of the form given by (23) is to rewrite that system in an equivalent but more mathematically tractable form.

One approach would be to use the algebraic equation (21) to solve for m of the variables in terms of the other $n - m$ variables of the vector x . An ordinary differential equation in the reduced variables only can be developed from (20). Such an approach is effective only for certain special forms of algebraic constraints.

Although the above approach to obtaining a reduced form may be suitable in special cases, there is need for a more generally applicable approach. One reduction approach has been developed for the special singular system of differential equations given by (23); the approach is quite general and is presented here as a fundamental approach for dealing with singular systems of differential equations of the form (23). It has previously been used in a robotics context in [11, 20] and in the context of numerical integration in [8, 25]. This reduction approach is to find an equivalent representation for (23) of the form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} m(x, \dot{x}, u, \lambda) \\ \lambda - n(x, \dot{x}, u) \end{bmatrix}$$

A singular system of differential equations in this form is said to be reduced.

Thus the algebraic equations can be used to eliminate λ to obtain

$$\ddot{x} = m(x, \dot{x}, u, n(x, \dot{x}, u))$$

which effectively describes the dynamics of the system in the manifold defined by the equality constraints [25].

Suppose that $\Theta(x) = 0$;

then $\dot{\Theta}(x) = B(x)\dot{x} = 0$

and
$$\ddot{\Theta}(x) = \dot{B}(x)\dot{x} + B(x)\ddot{x} = 0$$

where $\dot{B}(x)$ denotes the time derivative of the matrix function $B(x)$. Assuming that $M(x)$ is always symmetric and positive definite it follows that

$$\dot{x} = M(x)^{-1} \left\{ u + B^T(x)\lambda - G(x, \dot{x}) \right\}$$

Substituting into the previous expression for $\ddot{\Theta}(x)$ obtain

$$B(x)M(x)^{-1} \left\{ u + B^T(x)\lambda - G(x, \dot{x}) \right\} + \dot{B}(x)\dot{x} = 0$$

or equivalently

$$A(x)\lambda = B(x) \left\{ M(x)^{-1} \left[G(x, \dot{x}) - u \right] \right\} - \dot{B}(x)\dot{x} \quad (24)$$

where the $m \times m$ matrix $A(x)$ is defined by

$$A(x) = B(x)M(x)^{-1}B^T(x) \quad (25)$$

Assuming that the Jacobian matrix $B(x)$ is always full rank, it follows that $A(x)$ is symmetric and positive definite so that the multiplier λ can be determined by solving the linear algebraic equation (24) to obtain

$$\lambda = \left[B(x)M(x)^{-1}B^T(x) \right]^{-1} \left\{ B(x) \left\{ M(x)^{-1} \left[G(x, \dot{x}) - u \right] \right\} - \dot{B}(x)\dot{x} \right\} \quad (26)$$

$$\lambda := \Lambda(x, \dot{x}, T)$$

Thus an explicit algebraic relation for λ , expressed in terms of x, \dot{x}, u , has been obtained that can be used to simplify the singular equation (23). In particular, (23) can be written in the equivalent form.

$$M(x)\ddot{x} + G(x, \dot{x}) = u + B^T(x)\lambda(x, \dot{x}, T) \quad (27)$$

$$\lambda = \Lambda(x, \dot{x}, T) \quad (28)$$

Equations (27) and (28) are mathematically equivalent to equations (20) and (21), under the stated assumptions. However, the form of (27) and (28) is considerable simplified. In particular, the vector differential equation (27) does not depend explicitly on the multiplier λ so that the singular nature is avoided. The methods for nonsingular ordinary differential equations can be directly applied to equation (27).

Hence, if equations (20) and (21) represent a mathematical model for a particular robot system configuration, such as were considered previously, a reduction transformation can be carried out by solving a set of symmetric linear algebraic equations, namely (24), to obtain simplified algebraic and differential equations in the form of (27) and (28). However, note that the function $\Lambda(x, \dot{x}, u)$ appearing in equation (27) is a quite complicated function of its arguments.

4. APPLICATION TO ROBOT SYSTEM CONFIGURATIONS

In this section the above reduction transformation is used to obtain nonsingular models equivalent to the singular models developed previously. The transformed nonsingular models could form the basis for an analysis of the dynamics and control issues associated with the robot configurations. In each case the reduction procedure is exactly as indicated in the previous section; hence the details are omitted.

One Robot with End Effector Gripping an Inertial Load

The singular system (5) describing the dynamics of one robot gripping an inertial load can be transformed into the following simplified form.

The contact force vector $f \in R^m$ satisfies the linear equation

$$\left[m^{-1}I + J(q)M(q)^{-1}J^T(q) \right] f = J(q)M(q)^{-1} \left[G(q, \dot{q}) - T \right] - \dot{J}(q)\dot{q} - g \quad (29)$$

If the matrix

$$\left[m^{-1}I + J(q)M(q)^{-1}J^T(q) \right]$$

is nonsingular, the contact force is of the form $f = F(q, \dot{q}, T)$ where $F(q, \dot{q}, T)$ is the unique function satisfying (29)

Consequently, the robot dynamics can be described by the differential equation

$$M(q)\ddot{q} + G(q, \dot{q}) = T + J^T(q)F(q, \dot{q}, T) \quad (30)$$

and the equations of motion of the inertial load, taking into account the contact forces, can be described by the differential equation

$$m\ddot{p} = -F(q, \dot{q}, T) - mg \quad (31)$$

Equations (30) and (31) can be viewed as a nonsingular system of differential equations for the robot and load dynamics, with the contact force given as the solution of equation (29). If initial conditions, consistent with satisfaction of the constraints (3), are specified,

$$\begin{aligned}\Phi(H(q(t_0))) &= 0 \\ D(H(q(t_0)))J(q(t_0))\dot{q}(t_0) &= 0\end{aligned}\tag{32}$$

equations (30), (31), (32) define a well-posed initial value problem on the constraint manifold if the usual smoothness assumptions are made.

As described previously, either the singular system (5) or the reduced form (29), (30), (31) could be used to describe the dynamics of a robot gripping an inertial load. Although the singular system (5) is expressed in terms of functions arising directly from the physical development, direct methods of analysis for such equations are not available. The reduced form (29), (30), (31) can be analyzed using methods based on ordinary differential equations, but the functional forms involved are complicated functions of their arguments.

One Robot with End in Contact with a Rigid Surface

The singular system (11) describing the dynamics for one robot in contact with a rigid surface can be transformed into reduced form. The multiplier $\pi \in R^1$ satisfies the linear equation

$$\left[D(H(q))J(q)M(q)^{-1}J^T(q)D^T(H(q)) \right] \pi = D(H(q)) \left\{ J(q)M(q)^{-1} \left[G(q, \dot{q}) - T \right] + \dot{J}(q)\dot{q} \right\} \tag{33}$$

If the scalar

$$D(H(q))J(q)M(q)^{-1}J^T(q)D^T(H(q))$$

is nonzero, the multiplier is of the form

$$\pi = \Pi(q, \dot{q}, T)$$

where $\Pi(q, \dot{q}, T)$ is the unique function satisfying (33).

Consequently, the robot dynamics can be described by the differential equations

$$M(q)\ddot{q} + G(q, \dot{q}) = T + J^T(q)D^T(H(q))\Pi(q, \dot{q}, T) \quad (34)$$

Equation (34) can be viewed as a nonsingular system of differential equations for the constrained robot, with the contact force as the solution of (33). If initial conditions, consistent with satisfaction of the constraints (7), are specified,

$$\begin{aligned} \Phi(H(q(t_0))) &= 0 \\ D(H(q(t_0)))J(q(t_0))\dot{q}(t_0) &= 0 \end{aligned}$$

equations (34), (35) define a well-posed initial value problem on the constraint manifold if the usual smoothness assumptions are made.

As described previously, either the singular system (11) or the reduced form equations (33) and (34) could be used to describe the dynamics of a robot with its end effector in contact with a rigid surface. Although the singular system (11) is expressed in terms of functions arising directly from the physical development, direct methods of analysis are not available. The reduced form equations (33), (34) can be analyzed using methods based on ordinary differential equations, but the functional forms involved are complicated functions of their arguments.

Two Robots with End Effectors Gripping a Common Inertial Load

The singular system (19) describing the dynamics of two robots gripping a common inertial load can also be transformed into a reduced form. The contact force vectors f_1 and f_2 satisfy the linear equations

$$\begin{aligned} & \left[I + mJ_1(q_1)M_1(q_1)^{-1}J_1^T(q_1) \right] f_1 + f_2 \\ & = m \left\{ J_1(q_1)M_1(q_1)^{-1} \left[G_1(q_1, \dot{q}_1) - T_1 \right] - \dot{J}_1(q_1)\dot{q}_1 - g \right\} \\ f_1 + & \left[I + mJ_2(q_2)M_2(q_2)^{-1}J_2^T(q_2) \right] f_2 \\ & = m \left\{ J_2(q_2)M_2(q_2)^{-1} \left[G_2(q_2, \dot{q}_2) - T_2 \right] - \dot{J}_2(q_2)\dot{q}_2 - g \right\} \end{aligned} \quad (36)$$

If the matrix

$$\begin{bmatrix} I + mJ_1(q_1)M_1(q_1)^{-1}J_1^T(q_1) & I \\ I & I + mJ_2(q_2)M_2(q_2)^{-1}J_2^T(q_2) \end{bmatrix}$$

is nonsingular, the contact force vectors are of the form

$$\begin{aligned} f_1 & = F_1(q_1, \dot{q}_1, T_1, q_2, \dot{q}_2, T_2) \\ f_2 & = F_2(q_1, \dot{q}_1, T_1, q_2, \dot{q}_2, T_2) \end{aligned}$$

where $F_1(q_1, \dot{q}_1, T_1, q_2, \dot{q}_2, T_2), F_2(q_1, \dot{q}_1, T_1, q_2, \dot{q}_2, T_2)$ are the unique functions satisfying (36).

Consequently, the dynamics of the two robots can be described by the differential equations

$$\begin{aligned} M_1(q_1)\ddot{q}_1 + G_1(q_1, \dot{q}_1) & = T_1 + J_1^T(q_1)F_1(q_1, \dot{q}_1, T_1, q_2, \dot{q}_2, T_2) \\ M_2(q_2)\ddot{q}_2 + G_2(q_2, \dot{q}_2) & = T_2 + J_2^T(q_2)F_2(q_2, \dot{q}_2, T_2, q_1, \dot{q}_1, T_1) \end{aligned} \quad (37)$$

The equations of motion of the inertial load can be described by the differential

equations

$$m\ddot{p} = -F_1(q_1, \dot{q}_1, T_1, q_2, \dot{q}_2, T_2) - F_2(q_1, \dot{q}_1, T_1, q_2, \dot{q}_2, T_2) - mg \quad (38)$$

Equations (37) and (38) can be viewed as nonsingular system of differential equations for the dynamics of the two robots and the load, with the contact forces given as the solution of (36). If initial conditions, consistent with satisfaction of the constraints (15), (16), are specified,

$$\begin{aligned} p(t_0) &= H_1(q_1(t_0)) & p(t_0) &= H_2(q_2(t_0)) \\ \dot{p}(t_0) &= D_1(q_1(t_0))\dot{q}_1(t_0) & \dot{p}(t_0) &= D_2(q_2(t_0))\dot{q}_2(t_0) \end{aligned} \quad (39)$$

equations (37), (38), (39) define a well-posed initial value problem on the constraint manifold if the usual smoothness assumptions are made.

As described previously, either the singular system (19) or the reduced equations (37) and (38) could be used to describe the dynamics of two robot gripping a pure inertial load. Although the singular system (19) is expressed in terms of functions arising directly from the physical development, direct methods of analysis are not available. The reduced equations (37) and (38) can be analyzed using methods based on ordinary differential equations, but the functional forms involved are complicated functions of their arguments.

5. COMMENTS ON ROBOT PLANNING, CONTROL AND SIMULATION

Some general comments are made regarding several fundamental robotics problems, assuming the role of the robot system dynamics is critical.

In the case of constrained robot systems, robot planning involves not only specification of desired robot motions (displacement, velocity and acceleration time histories) of the end effector but it may also involve specification of desired time histories of the contact force on the end effector. Given such specifications which satisfy the imposed constraints, the joint torques necessary to achieve such specifications, in principle, can be determined using the developed mathematical models. The numerical computations required are not significantly different from the unconstrained robot case [17]; there is need to solve the inverse kinematic relations as well as to evaluate the appropriate non-linear inertial, coriolis and gravitational functions. Optimal planning problems, where the robot motions and contact force time histories are to be determined to satisfy an optimality criteria can readily be defined. Such optimization problems, formulated to take into account dynamics of constrained robots, have not received attention to date.

There have been several feedback control schemes that have been proposed for control of robot motions and/or contact forces [18, 21, 24, 26, 30, 32]. Such control schemes can be properly evaluated only in the context of dynamic models which explicitly include the effects of contact forces. At present, there are few theoretical results available. Conditions for closed loop stability of a constrained robot, based on any of the proposed feedback schemes, are not known, even for the simplest cases. It is suggested that the proposed form for singular systems of differential equations constitute the proper mathematical model for carrying out such an investigation.

Computer simulation can be used to evaluate the dynamic performance of a constrained robot system. However, numerical integration of the differential equations, while maintaining satisfaction of the imposed constraints, represents a nontrivial numerical problem. There is a growing literature on the numerical solution of singular systems of differential equations [3, 5, 7-8, 16, 22, 25] but a number of difficulties remain. The numerical simulation of constrained robot systems thus represents a substantial challenge. The presence of constraints typically results in increased computational requirements and decreased accuracy.

6. EXTENSIONS

In the above development, the imposed constraints have been represented as equality constraints. But a more general view of the particular robot system configurations described previously suggests the need to allow the constraints to be either active or inactive. A generalization of the singular form (23) to such a case results in a nonlinear complementarity problem defined by the differential equations

$$M(x)\ddot{x} + G(x, \dot{x}) = u + B^T(x)\lambda \quad (40)$$

with the inequality constraints

$$\theta(x) \geq 0 \quad (41)$$

and the conditions on the multipliers

$$\lambda \geq 0 \quad \text{and} \quad \lambda^T \theta(x) = 0 \quad (42)$$

Such mathematical models are significantly more complicated than the singular systems of differential equations considered previously. In particular, the possibility of impact forces must be considered whenever an inactive constraint becomes active, thus giving rise to possible discontinuities in the velocity variables. One investigation into the numerical simulation of such systems has been described in [16]. The study of mathematical models in this form is only now being initiated, so that our current understanding of the fundamental issues in dynamics, control and simulation of robotic configurations described by such mathematical models is limited.

7. CONCLUSIONS

A class of robot systems has been presented for which mathematical models represented by singular systems of differential equations arise naturally. The use of such mathematical equations is felt to be the correct mathematical formulation for describing the dynamics of constrained robot systems. Three specific robot system configurations have been presented as illustrations, but singular systems of differential equations are suitable representations for any robot configuration where contact forces arise due to motion constraints.

A careful examination of the planning and feedback control issues for constrained robot systems should be based on use of singular mathematical models. The complexity of the mathematical models is indicative of the practical difficulties that are associated with the applications of robots to advanced manufacturing tasks. This work is an initial step in the process of development of a

framework to deal with those complicated dynamics and control issues.

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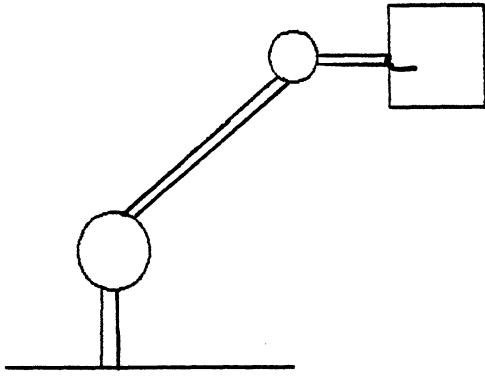


Figure 1: One Robot with End Effector Gripping an Inertial Load

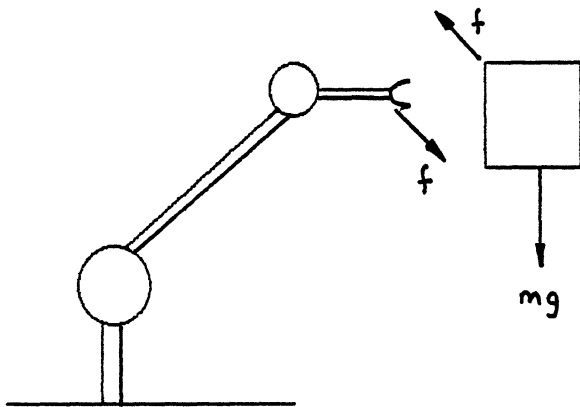


Figure 2: Free Body Diagram for Figure 1

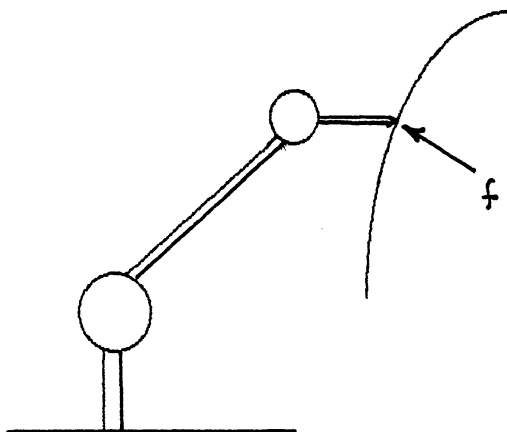


Figure 3: One Robot with End in Contact with Rigid Surface

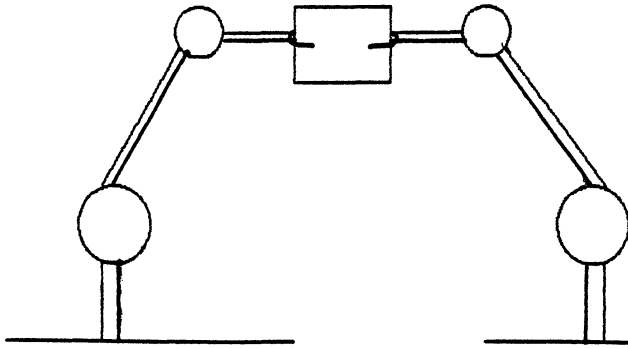


Figure 4: Two Robots with End Effectors
Gripping an Inertial Load

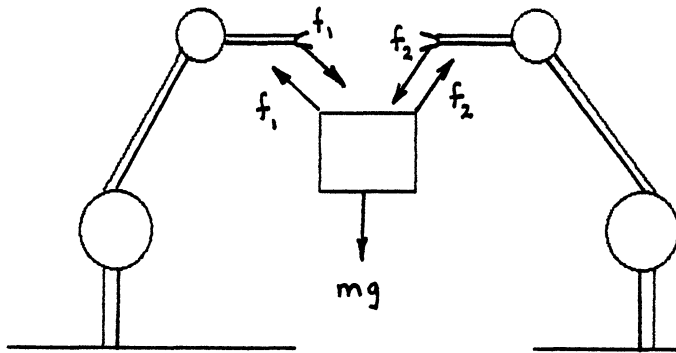


Figure 5: Free Body Diagram for Figure 4

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